Problem Set 2

Mathematics Club, IITM

27th November 2022

1 Problems

1.1 A General Problem

Five-digit numbers are formed using digits 1, 4, 5, 6 and 9 without repetition. What is the sum of all such possible numbers?

1.2 Co-prime integers

Define the function $\phi(n)$ to be the number of integers co-prime to and lesser than n.

- What is $\phi(p)$, where p is prime?
- What is $\phi(z)$, such that $z = p^k$ for some prime p?
- Come up with the general formula for $\phi(n)$ for arbitrary n. Is this even possible?

1.3 L_p metrics

Background: Consider 2-D Euclidean space, with the distance between two points to be defined as $D((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Now, a circle is the set of all points equidistant from a given point (called the *centre*). The value of π is, of course, the ratio of the circumference to the diameter, which happens to be 3.14.

The distance between two points in L^p space is defined to be:

$$\|x\|_p = (|x_1 - x_2|^p + |y_1 - y_2|^p)^{\frac{1}{p}}$$

Indeed, our familiar Euclidean space is just the case $p = 2$.

If $p = 1$,

$$D((x_1, y_1), (x_2, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

Using our new definition of the distance, **what figure would be the set of all points equidistant from a given point? What would be the value of π_p (i.e. the ratio of circumference to the diameter for the new figure) in this case?**

It is certainly possible to come up (numerically) with a value of the circumference to the diameter, for every value of π_p for every p in the L^p metric. **What interesting observations do we get in that case?**

Wikipedia Article

1.4 Similarity between addition and multiplication

Consider the set of Real numbers, $\mathbb{R}$ under addition. We can observe a few nice properties:

1. If you add two real numbers, you end up with a real number
2. $a + (b + c) = (a + b) + c$
3. There is a real number 0, such that $a + 0 = 0 \forall a$
4. There is an additive inverse for every element, i.e. for every a, there exists some b such that $a + b = 0$. b is, of course, the negative of a.

This is the notion of a mathematical group. A mathematical group is a set which satisfies four conditions, under some operation:

1. Closure
2. Associativity
3. Existence of identity
4. Existence of inverse

Prove that $\mathbb{R}_+$ is a group under multiplication.

Further, notice a few interesting properties:

1. $f(x) = e^x$ maps $\mathbb{R}$ to $\mathbb{R}_+$
2. $\mathbb{R}$ is a group under addition
3. $\mathbb{R}_+$ is a group under multiplication
4. $e^{x+y} = e^x \times e^y$

Imagine you have the set $\mathbb{R}$, with addition as the operation. If you apply the function $f(x) = e^x$ on the group, we end up with R_+, and addition *converts to multiplication*.

This is the idea of a Group Isomorphism.

A group isomorphism is a function f from a group G with an operation $+$ to a group H with an operation $\times$, such that $f(u + v) = f(u) \times f(v)$

Prove that the group $\mathbb{R}$ under addition is isomorphic to $\mathbb{R}_+$ under multiplication.

2 A Footnote

We'll sign off with a hint about the integration bee: Learn about the Gaussian Integral!