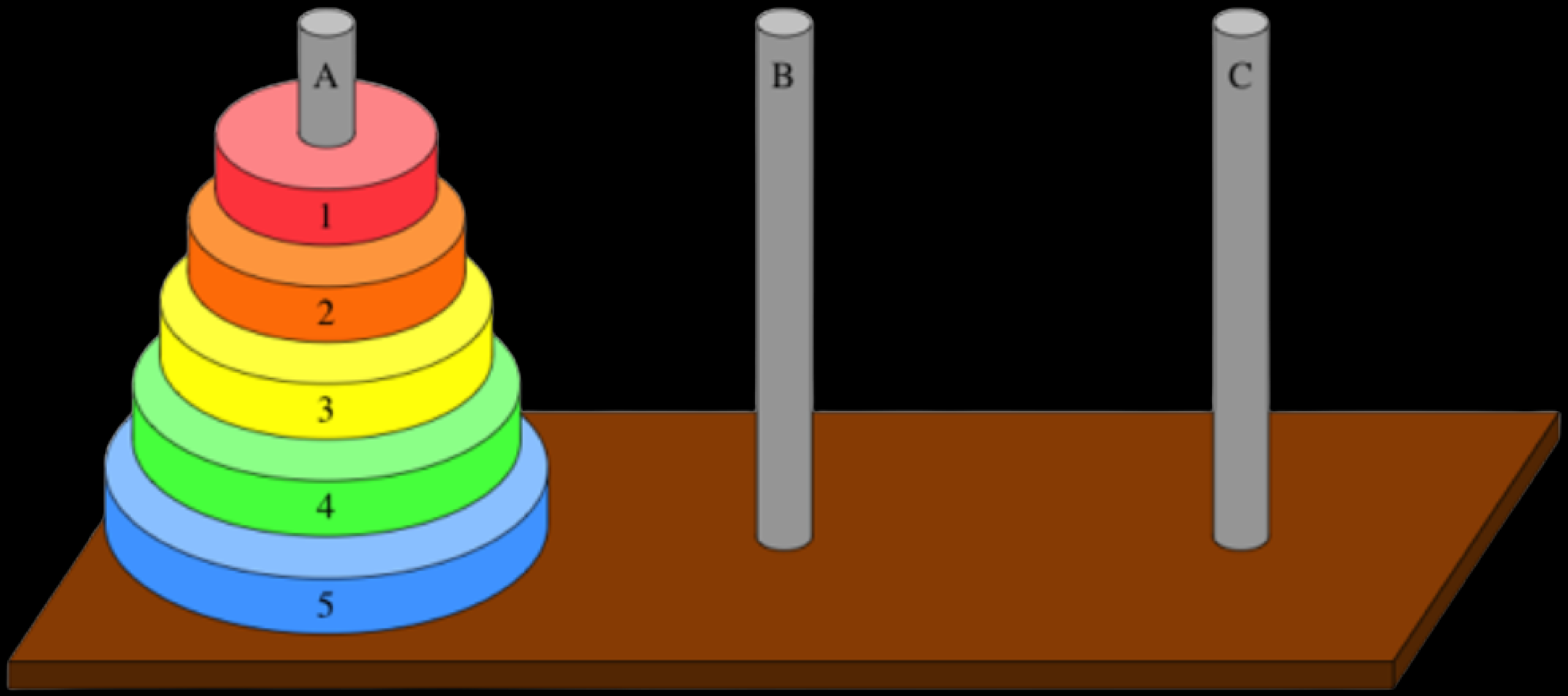
Exploring Difference Equations in the light of Differential Equations

Forays

Mathematics Club, CFI, IITM

Towers of Hanoi Problem



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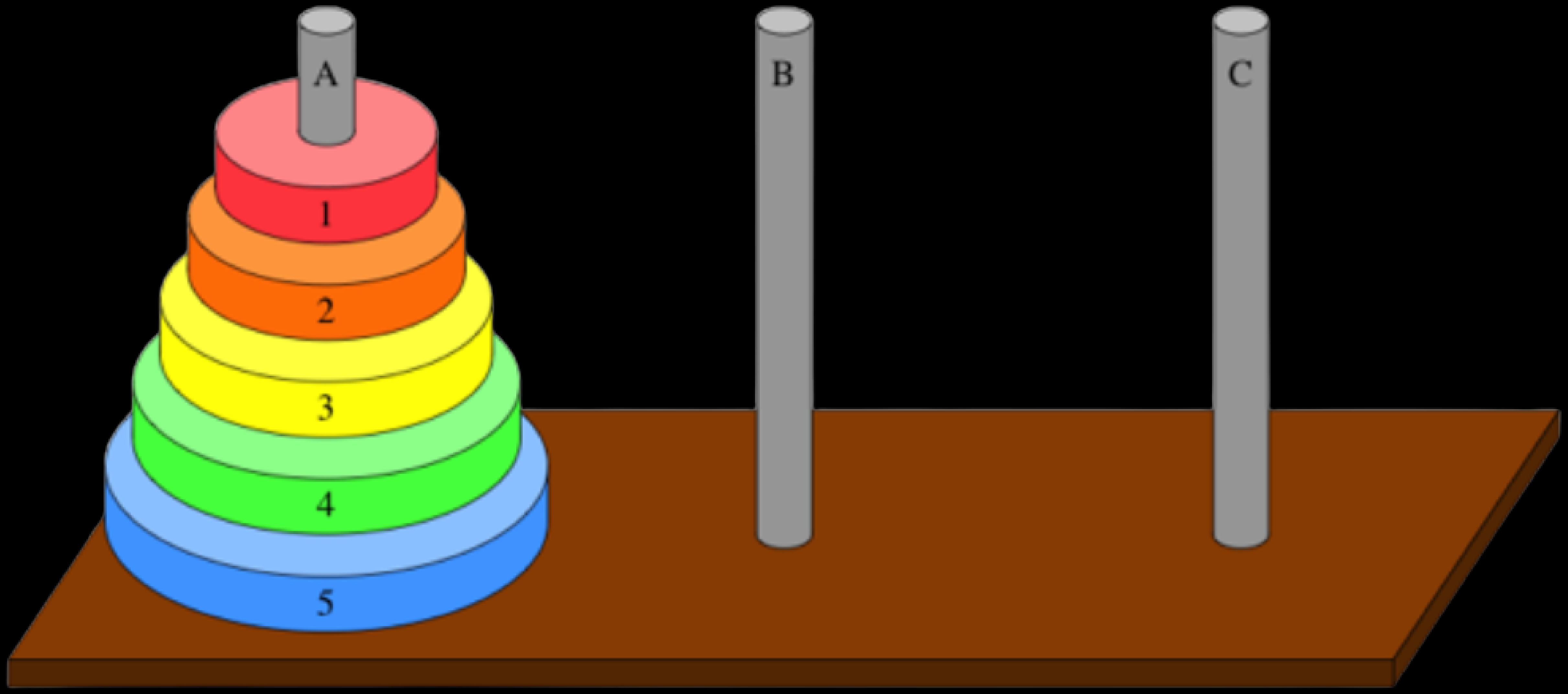

How many moves does it take to move a tower of height n?

Differential Equations

$$\frac{d^2x(t)}{dt^2} + x^2(t) = 0$$

Harmonic Oscillator

Exponential Growth

$$\frac{dy(x)}{dx} = y(x)$$

$$\nabla^2 f(x, y, z) = 0$$

Laplace's Equation

Constant Coefficient DEs

Equations like this might not be solvable in general

$$y^3 \sin x \frac{dy}{dx} + xy \frac{d^2x}{dt^2} = 0$$

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We will look at simpler equations

$$a_3\frac{d^3y}{dx^3} + a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = \sin(x)$$

$$a_0, a_1, a_2, a_3 \in \mathbb{R}$$

Homogeneous Constant Coefficient DEs

$$a_3\frac{d^3y}{dx^3} + a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = y(x)$$

$$a_3\frac{d^3y}{dx^3} + a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = 0$$

RHS should be 0

Formalizing Homogeneous CC DEs

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$a_0, a_1, \ldots a_n \in \mathbb{R}$$

How do we solve this?

Interesting fact about Exponentials

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

$$\frac{d}{dx^2}e^{ax} = a^2e^{ax}$$

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In general,

$$\frac{d^n}{dx^n}e^{ax} = a^n e^{ax}$$

An Example

$$\frac{d^2y}{dx^2} + y = 0$$

$$\text{In general, } \frac{d^2(e^{rx})}{dx^2} + (e^{rx}) = (r^2 + 1)e^{rx}$$

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$$\text{In general, } \frac{d^2(e^{rx})}{dx^2} + (e^{rx}) = (r^2 + 1)e^{rx}$$

$$r^2 + 1 = 0 \implies r = \pm i$$

Solutions are e^{ix}, e^{-ix}

General Theory of Solutions

$$a_n \frac{d^n(e^{rx})}{dx^n} + a_{n-1} \frac{d^{n-1}(e^{rx})}{dx^{n-1}} + \cdots + a_2 \frac{d^2(e^{rx})}{dx^2} + a_1 \frac{d(e^{rx})}{dx} + a_0(e^{rx}) = 0$$

General Theory of Solutions

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$$a_nr^n(e^{rx})+a_{n-1}r^{n-1}(e^{rx})+\cdots+a_2r^2(e^{rx})+a_1r^1(e^{rx})+a_0(e^{rx})=0$$

$$a_nr^n+a_{n-1}r^{n-1}+\cdots+a_2r^2+a_1r^1+a_0=0$$

So, the "correct" values of r are roots of the above polynomial

An Example

$$\frac{d^2y}{dx^2} + y = 0$$

Solutions are e^{ix}, e^{-ix}

Scaling: If e^{ix} is a solution, Ae^{ix} is also a solution
Superposition: If e^{ix}, e^{-ix} is are solutions, $e^{ix} + e^{-ix}$ are also solutions

An Example

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Scaling: If e^{ix} is a solution, Ae^{ix} is also a solution
Superposition: If e^{ix}, e^{-ix} is are solutions, $e^{ix} + e^{-ix}$ are also solutions

The general solution is $Ae^{ix} + Be^{-ix}$

Linearity of Solutions

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

If $e^{r_1 x}, e^{r_2 x}, \ldots e^{r_n x}$ are solutions,

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If $e^{r_1 x}, e^{r_2 x}, \ldots e^{r_n x}$ are solutions,

$$\boxed{A_1 e^{r_1 x} + A_2 e^{r_2 x} + \cdots + A_n e^{r_n x}}$$

General Solution

Exercise

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

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$$r^3 - 6r^2 + 11r - 6 = 0$$

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$$r^3 - 6r^2 + 11r - 6 = 0$$

$$\boxed{Ae^x + Be^{2x} + Ce^{3x}}$$

A small caveat

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

$$r^2 - 2r + 1 = 0$$

Solutions are e^x, e^x?

Solutions are actually e^x and xe^x

Non-homogeneous DEs

$$\frac{d^2y}{dx^2} + y = e^{2x}$$

Let's try to *guess* a solution $y(x) = Ae^{2x}$

Non-homogeneous DEs

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Let's try to *guess* a solution $y(x) = Ae^{2x}$

$$\implies A4e^{2x} + Ae^{2x} = e^{2x}$$

$$A = \frac{1}{5}, y(x) = \frac{e^{2x}}{5}$$

Non-homogeneous DEs

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$$A = \frac{1}{5}, \boxed{y(x) = \frac{e^{2x}}{5}}$$ Particular Solution

Non-homogeneous DEs

$$\frac{d^2y_p}{dx^2} + y_p = e^{2x}$$

$$\frac{d^2y_h}{dx^2} + y_h = 0$$

$$\frac{d^2(y_p + y_h)}{dx^2} = e^{2x}$$

$$y_p + y_h = \frac{e^{2x}}{5} + Ae^{ix} + Be^{-ix}$$

Initial Conditions

$$\frac{d^2y}{dx^2} + y = e^{2x}$$

$$y_p + y_h = \frac{e^{2x}}{5} + Ae^{ix} + Be^{-ix}$$

If we know $y(0)$ and $y'(0)$, we can find A and B

Difference Equations

Discrete Time Signal Processing

$$y[n] = 2y[n-1], y[0] = 1$$

Towers of Hanoi Time Complexity

$$T[n] = 2T[n-1] + 1$$

Constant Coefficient Homogeneous DE

$$y[n] = 2y[n-1], y[0] = 1$$

$$y[n] = y[n-1] + 1, y[0] = 0$$

Difference Equations

$$y[n] = 2y[n-1], y[0] = 1$$

Difference Equations

$$y[n] = 2y[n-1], y[0] = 1$$

$$y[1] = 2y[0] = 2$$

$$y[2] = 2y[1] = 4$$

$$y[3] = 2y[2] = 8$$

Constant Coefficient Homogeneous DE

$$y[n] = 2y[n-1]$$

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Assume solution of the form a^n

$$a^n = 2a^{n-1} \implies a = 2$$

So, our solution is 2^n

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By linearity, solution is $A\,2^n$

Exercise

$$y[n] - 3[y-1] + 2[y-2] = 0$$

Assume solution of the form a^n

Exercise

$$y[n] - 3[y - 1] + 2[y - 2] = 0$$

Assume solution of the form a^n

$$a^2 - 3a + 2 = 0 \implies a = 1, 2$$

Solutions would be $1^n, 2^n$

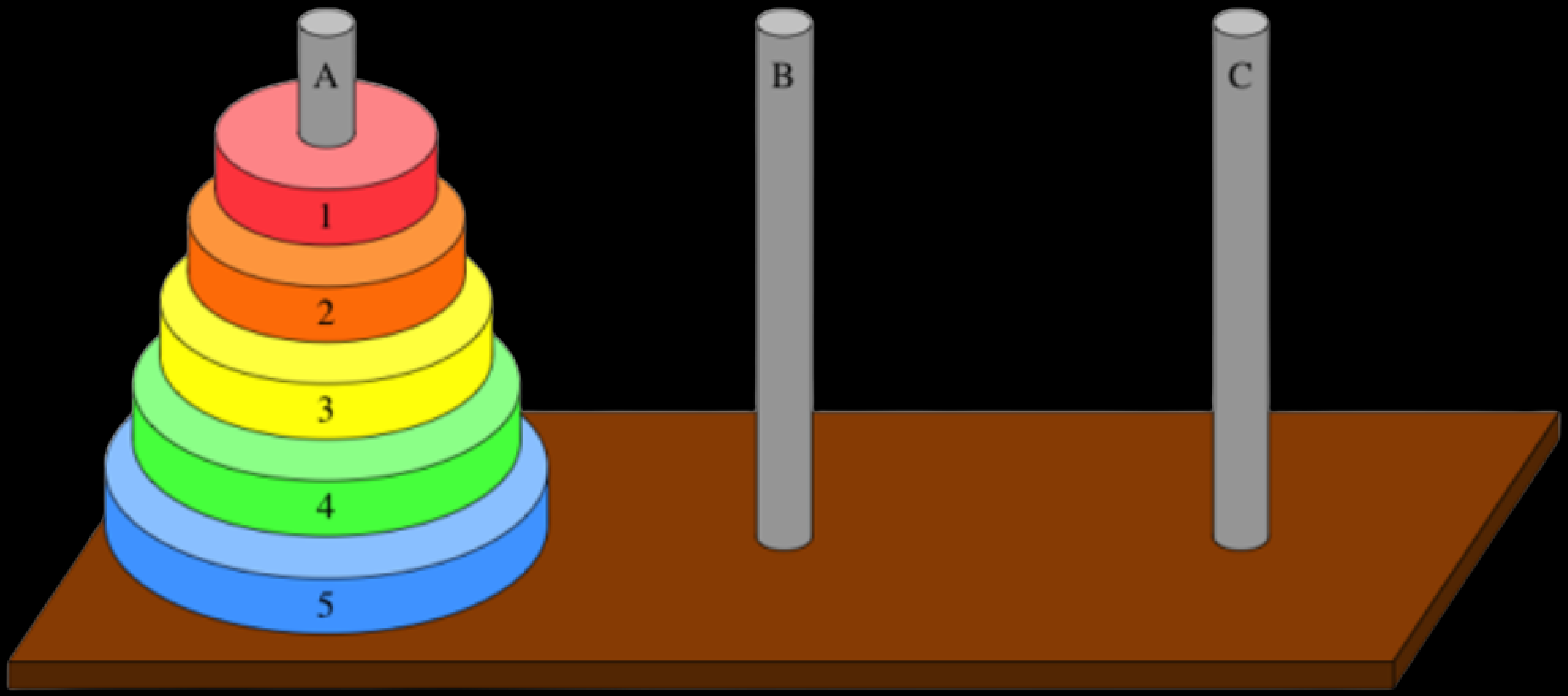
Using Linearity

$$y[n] - 3[y-1] + 2[y-2] = 0$$

Solutions would be $1^n, 2^n$

General Solution is $A1^n + B2^n$

Towers of Hanoi Problem



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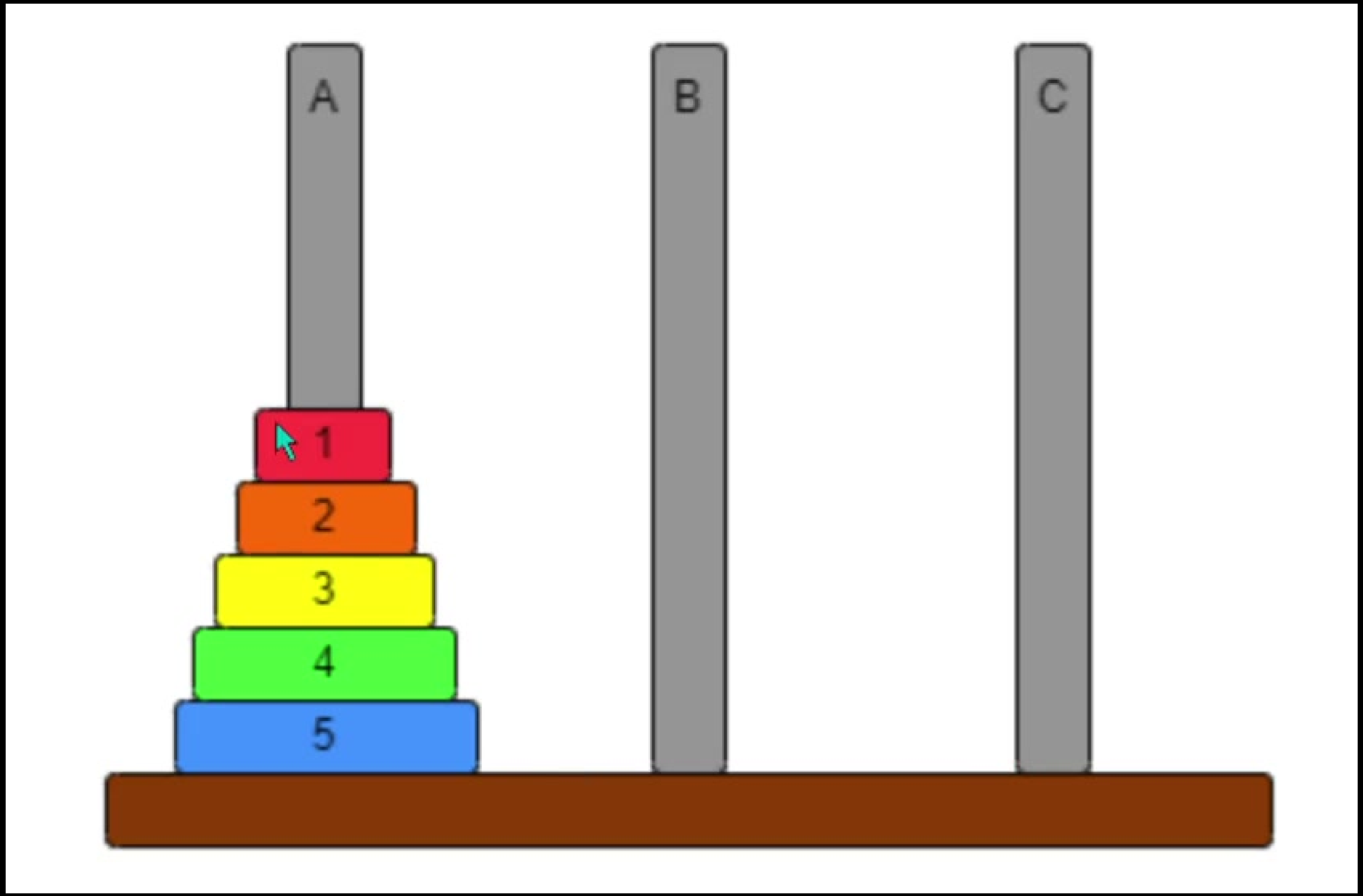

Towers of Hanoi Problem

Algorithm:

To move tower of size n from A to B

1. Move tower of size $n-1$ from A to C
2. Move last block to tower B
3. Move tower of size $n-1$ from C to B

$$T[n] - 2T[n-1] = 1$$

Inhomogeneous DE

$$T[n] - 2T[n-1] = 1$$

Assume solution of the form $T[n] = A \times 1$

$$\implies A = -1$$

$$T[n] = T_p[n] + T_h[n] = -1 + A2^n$$

Towers of Hanoi Problem

Using initial condition: $T[1] = 1$

$$T[n] = 2^n - 1$$

Exercise

Decide what happens if the characteristic polynomial of a difference equation has repeated roots.
For example, what is the solution of $y[n] - 2y[n-1] + y[n-2] = 0$?