Strategy Wars

Mathematics Club, $C\Phi$

September 14th 2023

1

a) If B is 1, A can be any number from 2 till 6. If B is 2, A can be any number from 3 till 6. Keep going on, until we say that B is 6, there's no valid A. Summing up, we get the required probability to be $\frac{15}{36}$.

b) Required probability is 1 - probability of getting a sum of 3, 4, 5 $= 1 - \frac{10}{216} = \frac{206}{216}$.

c) The statement is false. The former probability is $\frac{1}{216}$, while the latter is a bit more (you can calculate it as an exercise).

d) For the condition to be true, my first throw and my second throw both need to be an even number. The probability that this happens is $\frac{1}{4}$.

2

a) The expected score is $\frac{1}{6}(4+5+6) + \frac{3}{6}(3.5) = 4.25$.

b) In two rolls, your expected score is 4.25. So in the first roll, if you get a number > 4.25, stick with it. Else, roll again. Now if you did reroll, in this second throw, if you get a number > 3.5, stick with it, otherwise re-roll.

c) The expected value is $\frac{1}{6}(5+6) + \frac{4}{6}(4.25) = 4.666\ldots$.

d) By the same token, $n = 4$ gives us an expected value of ≈ 4.94, while for $n = 5$ we get a nice expected value of $n = 5.12$. So, $n = 5$ is the answer!

3

The strategy goes like this: if the number is a multiple of 4, if Krishna removes i stones, Skandan removes $4 - i$ stones, so eventually Skandan will remove the last stone. If the number is not a multiple of 4, Krishna can remove enough stones to make the new number divisible by 4, so Krishna wins.

So, the general solution is that for n is divisible by 4, Skandan wins, else Krishna wins.

4

The strategy is similar to the third question. Let us define *good* states as states which the first player wins if he starts from it and *bad* states as states which the first player loses if he starts from it. We observe that if the first player can make the second player reach a *bad* state, he wins and therefore he is in a *good* state. We also observe that every state is either *good* or *bad*. Now, let's try to identify these states. Clearly 1, 3, 4 and 5 are *good* states and 2 is a *bad* state. Now, we can recursively label the other states. n is a good state if one of $n-1$, $n-3$, $n-4$ and $n-5$ is a bad state.

2, 8, 10, 16, 18 etc are bad states. So if a state is of the form $8k+2$ or $8k$, it is a bad state and other states are good states.

Observe that a pattern is formed and this pattern repeats every 8 states.

5

Check out this link!

Based on this, the solutions are as follows:

a) Karthikeya

b) Atreya

c) Karthikeya

d) Atreya

e) Karthikeya

f) Karthikeya

g) Atreya

6

If there is 1 dacoit only, he will rob the merchant because he himself will become a merchant and put peace. The merchant will be robbed.

If there are 2 dacoits, both of them knows that if one robs the merchant, the other dacoit will rob him. So, the merchant will not be robbed.

Going by the same pattern, if there are even number of dacoits the merchant will survive, else he will be robbed.

7

If you notice carefully, the game is the same as the nim game shown above! Once you observe this, our job becomes peaceful. Based on this, the solutions are just to take the bitwise xor of the given values as follows:

a) Gorlaid

b) Gorlaid

c) Lex

8

This is an amazing problem in game theory, you can read about it here: Wythoff's Game

Based on this, the positions $(1, 2), (2, 1), (3, 5), (5, 3)$ will be won by 2C, any other position within the range will be won by Sreejaa.

9

We want to roll until the expected sum after rolling is less than our current sum. Let C be our current sum and S be the set of faces that have been rolled already. Then we should stop if

$$\frac{|S|}{6} \cdot 0 + \sum_{i \notin S} \frac{1}{6}(C+i) < C.$$

Using the fact that $\sum_{i \in S} i = C$, this inequality simplifies to

$$C(|S|+1) > 21.$$

After our first roll, $|S| = 1$ and $C < 6$ so $C(|S|+1) \leq 12$. Hence we should roll again. After our second roll, $|S| = 2$, so we should stop if $C > 7$. After our third roll, $|S| = 3$, so we should stop if $C > \frac{21}{4}$, that is, if $C \geq 6$. However, if we have made it to our third roll, C must be at least 6, and so we should stop at this point.

Thus: Roll twice. If the second roll is not the same as the first, and the sum is less than 7, roll again and stop; otherwise, stop.

10

A strategy in this game is merely a rule for deciding whether the first roll should be the "tens" digit or the "ones" digit. If the first roll is a 6, then it must go in the "tens" digit, and if it's a 1, then it must go in the "ones" digit. This leaves us with what to do with 2,3,4 and 5. If the first roll is b, then using it as the "ones" digit results in an expected number of $\frac{7}{2} \cdot 10 + b$. Using it as the "tens" digit results in an expected number of $10b + \frac{7}{2}$. So, when is $10b + \frac{7}{2} > \frac{7}{2} \cdot 10 + b$? When $b \geq 4$. Thus, if the first roll is 4, 5 or 6, the player should use it for the "tens" digit. With this strategy, the expected value of the number is

$$\frac{1}{6}(63.5 + 53.5 + 43.5 + 38 + 37 + 36) = 45.25.$$