1. If P is an invertible matrix, prove that $rank(PA) = \underline{rank}(A)$ $P\begin{bmatrix} I_{r\times r} & F \\ 0 & 0 \end{bmatrix}$

dim of $N(A) = n - r$

dim of $N(PA) \geq n - r$.

$PAx = 0$ $\quad Py = 0$ $\quad Ax = 0$

$\Rightarrow \exists$ no $y \neq 0$: $Py = 0$ $\Rightarrow$ dim of $N(PA) \leq n - r$

$\Rightarrow$ dim of $N(PA) = n - r$

2. Let A be a $m \times n$ matrix. If $AB = 0$, show that $rank(A) + rank(B) \leq n$.

rank $(A) +$ $\underline{\dim(N(A))} = n$

Now, colums of $B \in N(A)$.

$\Rightarrow$ rank of $(B) \leq \dim N(A)$

rank of $|A|$ + rank of $B \leq n$

3. Show that $Ax = b$ has multiple solutions if and only if $b \in Col(A)$ and the dimension of $Null(A)$ is non-zero.

$Ax = b$ has multiple solutions.

A has some solution $\Rightarrow \exists x : Ax = b$, we can get b some lin comb of cols of C.

Let $Ax_1 = Ax_2 = b$, $x_1 \neq x_2$.

$\Rightarrow A(x_1 - x_2) = 0$, $\exists$ $A\tilde{x} = 0$, $\tilde{x} = x_1 - x_2 \neq 0$

$\tilde{x} \in N(A)$

Let $\tilde{x} \in N(A)$, Let $\boxed{Ax = b}$,

$Ax = b$,

$A\tilde{x} = 0$, $\Rightarrow A(x + \tilde{x}) = b$

$\Rightarrow$ $x + \tilde{x}$ also satisfies eqn.

4. Find the basis for vector space $\{(x, y, z): \; 2x + 3y + 4z = 0\}$

Let $x = a, \; y = b, \; z = \frac{-2a - 3b}{4}$

$\underline{\text{Soln}} = \begin{bmatrix} a \\ b \\ \frac{-2a - 3b}{4} \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -3/4 \end{bmatrix}$

Basis is $\begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3/4 \end{bmatrix}$

6. Consider the subspace of cubic polynomials, $p(x)$ such that $p(5) = p(7) = 0$.

a. Show that it is a vector space.

b. Find the basis for the vector space.

a) Let $p_1, p_2 \in V$ $\quad \left.\begin{array}{l} \alpha p_1(5) + \beta p_2(5) = 0 = p(5) \\ \alpha p_1(7) + \beta p_2(7) = 0 = p(7) \end{array}\right\} \Rightarrow p \in V$

b) $p(x) = Ax^3 + Bx^2 + Cx + D$

$\begin{bmatrix} 1 & 5 & 25 & 125 \\ 1 & 7 & 49 & 343 \end{bmatrix} \begin{pmatrix} D \\ C \\ B \\ A \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We want nullspace of $\begin{bmatrix} 1 & 5 & 25 & 125 \\ 0 & 2 & 24 & 218 \end{bmatrix}$

$\begin{array}{r} 545 \\ 125 \\ \hline 420 \end{array}$

$\begin{bmatrix} 1 & 5 & 25 & 125 \\ 0 & 1 & 12 & 109 \end{bmatrix} \sim \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \; \underbrace{\begin{matrix} -35 & -420 \\ 12 & 109 \end{matrix}}_{F} \right)$

$(I \; F)\begin{pmatrix} +F \\ -I \end{pmatrix}$

$\begin{pmatrix} D \\ C \\ B \\ A \end{pmatrix} \rightarrow \begin{bmatrix} 35 & 420 \\ -12 & -109 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{array}{r} 109 \\ 7 \\ \hline 763 \end{array}$

$35 - 12(5) + 1(25) = 0$

$35 - 84 + 49 = 0$

$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -F \\ +I \end{bmatrix}$

$420 - \cancel{5}45 + 125 = 0$

$420 - 763 + 343$

Basis is $\boxed{\begin{matrix} x^2 - 12x + 35 \\ x^3 - 109x + 420 \end{matrix}}$

7. Check if the following set of vectors are linearly independent.

a. $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right\}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \to \text{Dependent}$$

b. $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 13 \\ 12 \\ 28 \end{bmatrix} \right\}$

$2[8] - 3(32) + 8(10)$

$80 + 16 - 96$

$= 0$

Dependent

8. Let $\mathcal{A} = \{A_1, A_2, A_3\}$ and $\mathcal{B} = \{B_1, B_2, B_3\}$ be two sets of basis vectors for the vector space $\mathcal{V}$. Assume that $A_1 = 4B_1 - B_2$, $A_2 = -B_1 + B_2 + B_3$ and $A_3 = B_2 - 2B_3$. If the co-ordinate vector of v with respect to $\mathcal{A}$ is $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, find the co-ordinate vector v with respect to $\mathcal{B}$.

$A = 3A_1 + 4A_2 + A_3$

$= 12B_1 - 3B_2 - 4B_1 + 4B_2 + 4B_3 + B_2 - 2B_3$

$= \boxed{8B_1 + 2B_2 + 2B_3}$

$\begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

9. State whether the following statements are True or False and give reasons.

(a) Let $S = \{v_1, v_2, \cdots, v_n\}$. If $\mathcal{V} = \text{span}\{S\}$, S is a basis for $\mathcal{V}$. False, might not be independent

(b) If $\{v_1, v_2, \cdots, v_n\}$ are a set linearly independent vectors in $\mathcal{V}$, they form a basis for $\mathcal{V}$. False (Not span)

(c) Let $\{w_1, w_2, \cdots, w_m\}$ be linear combinations of the vectors $\{v_1, v_2, \cdots, v_n\}$, with $m > n$. If the vectors $\{v_1, v_2, \cdots, v_n\}$ are linearly independent, then $\{w_1, w_2, \cdots, w_m\}$ are also linearly independent. False There has to be some dependency

(d) If a matrix B is obtained from A by performing elementary row operations, $\text{rank}(B) = \text{rank}(A)$.

True