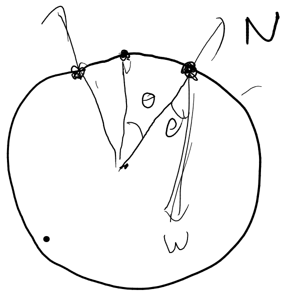
1. Two beads of mass m are positioned at the top of a frictionless hoop of mass M and radius R, which stands vertically on the ground. The beads are given tiny kicks, and they slide down the hoop, one to the right and the other to the left. What is the smallest value of m/M for which the hoop will rise up off the ground, at some point during the motion?



$$\boxed{2mgR[1-\cos\theta] = \frac{1}{2}mv^2}$$

$$2N\cos\theta = Mg$$

$$mv^2 = -Rmg\cos\theta + NR$$

$$2gR(1-\cos\theta) = gR\cos\theta + \frac{NR}{m}$$

$$NR = \quad \frac{n}{R} = 2mg - 3g\cos\theta$$

$$2gR - 3gR\cos\theta$$

$$Mg = 2N\cos\theta$$

$$= 2(2mg - 3mg\cos\theta)\cos\theta$$

$$\frac{M}{m} = 2\cos\theta(2 - 3\cos\theta)$$

$$2\cos\theta - 3\cos^2\theta$$

$$-2\sin\theta + 3(2)\sin\theta\cos\theta$$

$$\sin\theta = 0$$

or $\cos\theta = \frac{1}{3}$

$$\boxed{\theta = \cos^{-1}\frac{1}{3}}$$

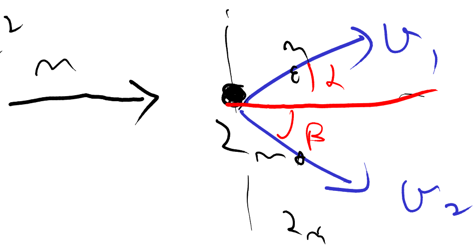
$$\cos\theta = \frac{1}{3}$$

$$= \frac{2}{3}\left[2 - 3\cdot\frac{1}{3}\right]$$

$$= \boxed{\frac{2}{3}}$$

2. Consider a moving mass m collides elastically with a stationary mass $2m$. Let their resulting velocities be $\mathbf{v_1}$ and $\mathbf{v_2}$ respectively. Show that $\mathbf{v_2}$ must be perpendicular to $2\mathbf{v_1}+\mathbf{v_2}$.

$\sqrt{V_1^2+2V_2^2}$



Note WLOG, planarity is assumed

$$2V_2 \cdot V_1 + |V_2|^2 = (2)\cos(\alpha+\beta)V_1 + V_2 = 0$$

$$V_1\cos\alpha + 2V_2\cos\beta = \sqrt{V_1^2+2V_2^2}$$

$$V_1\sin\alpha = 2V_2\sin\beta$$

$$\frac{V_1}{V_2} = \frac{2\sin\beta}{\sin\alpha}$$

$$V_1^2\cos^2\alpha + 4V_2^2\cos^2\beta + 4V_1V_2\cos\alpha\cos\beta = V_1^2+2V_2^2$$

$$2\cos(\alpha+\beta)V_1V_1 + V_2V_2$$

$$= 2\cos\alpha\cos\beta V_1V_2 - 2\sin\alpha\sin\beta V_1V_2 + V_2^2$$

$$= 2\cos\alpha\cos\beta V_1V_2 - 4\sin^2\beta V_2^2 + V_2^2$$

$$= \frac{1}{2}\left[4\cos\alpha\cos\beta V_1V_2 - 8\sin^2\beta V_2^2 + 2V_2^2\right]$$

$$= \frac{1}{2}\left[+V_1^2\sin^2\alpha + 2V_2^2 - 4V_2^2\cos^2\beta - 8\sin^2\beta V_2^2 + 2V_2^2\right]$$

$$= \frac{1}{2}\left[4V_2^2\sin^2\beta - 4V_2^2\cos^2\beta - 8\sin^2\beta V_2^2\right)$$

$$= (0)$$

3. We explained in the PH1010 course that *a conservative force is one for which the work done in moving from one point to another is path-independent.* It is equivalent to say that the work done around a closed path vanishes. Stokes' theorem relates the line integral of a vector around a closed path to the integral of the curl of the vector over the area enclosed by that path:

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \oint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{s}$$

Thus, any conservative force must have $\boldsymbol{\nabla} \times \mathbf{F} = 0$ everywhere to ensure the left side vanishes for an arbitrary loop C. Calculate the curl to determine which of the following force fields is conservative. For any that are conservative, find the potential energy U(r)

(a) $F_x = ayz + bx + c$, $F_y = axz + bz$, $F_z = axy + by$ (consider entire 3D space)

(b) $F_x = -ze^{-x}$, $F_y = \ln z$, $F_z = e^{-x} + \frac{y}{z}$ (consider the region $z > 0$)

(c) $\mathbf{F}(\mathbf{r}) = \mathbf{A} \times \mathbf{r}$ (consider entire 3D space), where, $\mathbf{A} = A_x\hat{x} +_y \hat{y} + A_z\hat{z}$.

$$\text{Cx}_z \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ayz+bx+c & axz+bz & axy+by \end{pmatrix} = \vec{i}\left[ax+b-(ax+b)\right] + \vec{j}\left[ay-ay\right] + \vec{k}\left[az-az\right] = \vec{0}$$

$$F_x = ayz + bx + c$$

$$F = -\left(axyz + \frac{bx^2}{2} + cx + \cancel{}\, b\frac{z^2}{2} + \frac{bz^2}{2}\right) + c$$

b) $$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -ze^{-x} & \ln z & e^{-x} + \frac{y}{z} \end{vmatrix} = \vec{i}\left(\frac{1}{z} - \frac{1}{z}\right) + \vec{j}\left[-e^{-x} - (-e^{-x})\right] + \vec{h}\left(0-0\right) = \vec{0}$$

$$\therefore F = ze^{-x} + y\ln z + \cancel{} + c$$

c) $$\nabla \times (\vec{A} \times \vec{r})$$

$$= A \operatorname{div} \vec{r} - \cancel{r \operatorname{div} \vec{A}} + \cancel{(\vec{r} \cdot \nabla)\vec{A}} - (A \cdot \nabla)\vec{r}$$

$$= 3\vec{A} - \vec{A} = 2\vec{A}$$

4. Using extremizing the length, show that the shortest distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in three-dimensional space is a straight line. (You don't have to show it is the shortest, only that it is extremal)

$$\text{Dist} = \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \sqrt{dx_1^2 + dy_1^2 + dz_1^2}$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx$$

$$\boxed{\mathcal{L} = \sqrt{1 + y'^2 + z'^2}}$$

$\frac{\partial \mathcal{L}}{\partial y'}$ is constant

$$\frac{y'}{\sqrt{1 + y'^2 + z'^2}} = c$$

$$y'^2 = c + (cy')^2 + cz'^2$$

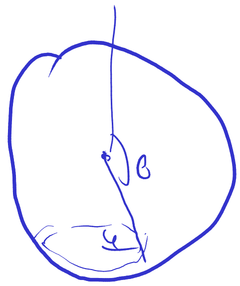
$$z'^2 = d + (dy'^2 + (dz'^2))$$

$$0 = c + (c-1)(y_1^2 + y_2^2)$$

$\Rightarrow y' = c,\ z' = \text{constant}$

$\Rightarrow$ constant

5. A *geodesic* is a line that represents the shortest path between any two points when the path is restricted to a particular surface. Find the geodesic on a sphere.



$$\text{dist} = \sqrt{dr^2 + r^2 d\theta^2 + r^2 d\phi^2 \sin^2\theta}$$

$$= \sqrt{r^2 d\theta^2 + r^2 d\phi^2 \sin^2\theta}$$

$$L = \sqrt{\left[\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta\right]} = \sqrt{\dot\theta^2 + \sin^2\theta}$$

$$f - \dot\theta \frac{\partial f}{\partial \dot\theta} = a$$

$$\sqrt{\dot\theta^2 + \sin^2\theta} - \dot\theta \frac{\dot\theta}{\sqrt{\dot\theta^2 + \sin^2\theta}} = a$$

$$\frac{\sin^2\theta}{\sqrt{\dot\theta^2 + \sin^2\theta}} = a$$

$$a^2 \sin^4\theta = \dot\theta^2 + \sin^2\theta$$

$$\dot\theta = \sqrt{a^2 \sin^4\theta - \sin^2\theta}$$

$$\frac{d\phi}{d\theta} = \frac{1}{\dot\theta} = \frac{1}{\sin\theta \sqrt{a \sin^2\theta - 1}}$$

$$= \frac{a \csc^2\theta}{\sqrt{1 - a^2 \csc^2\theta}}$$

6. A disk of radius R rolls without slipping inside the parabola $y = ax^2$.

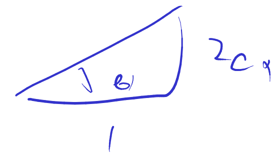
(a) Find the equation of the constraint.

(b) Also find the condition that allows the disk to roll so that it contacts the parabola at one and only one point, independent of its position.

a)

Slope = $2ax$

Slope of Normal = $-\frac{1}{2ax}$



$\frac{1}{\pm 2ax}$ ∠ $mg \sin\theta$

$\tan\theta = \frac{dy}{dx} = 2ax$

$\frac{\pm 1}{2ax}$ ∠ $\frac{mg\, 2ax}{\sqrt{4a^2x^2+1}}$

$\sqrt{4a^2x^2+1}$