1. Let $A = \begin{bmatrix} A_1 & A_2 & \cdots A_n \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$, where A_i and B_i are used to denote the columns and rows of the matrices A and B respectively. Use the fact that the columns of C are linear combinations of columns of A to show that the product $C = AB$ can be written as the sum of outer products as follows.

$$C = \sum_{i=1}^{n} A_i B_i$$

Let $C = AB$

$$(C)_{i\text{th col}} = B_{1i} A_1 + B_{2i} A_2 \text{ --- } B_{ni} A_n$$

$$C_{ji} = B_{1i} A_{j1} + B_{2i} A_{j2} \text{ ——— } B_{ni} A_{jn}$$

$$= \sum_{k=1}^{n} A_{jk} B_{kj} = \sum_{k=1}^{n} (A_k B_k)_{ji}$$

2. Using the fact that the columns of AB are linear combination of columns of A, show

$$(AB)^T = B^T A^T$$

Let $A_1 \ldots A_n$ indicate colums of A. Let $(AB)_i$ indicate ith colm

$$(AB)_i = B_{1i} A_1 + B_{2i} A_2 \text{ ——— } + B_{in} A_n$$

$$((AB)_i)^T = B_{1i} A_1^T + B_{2i} A_2^T + \text{ ——— } + B_{in} A_n^T$$

Here, $A_1^T, A_2^T \ldots A_n^T$ are rows vectors.
B_{1i}, B_{2i}, scales the rows in appropriate manner.

So the ith row of $(AB)^T$ is linear comb of $A_1^T \text{———} A_n^T$, scaled by $B_{1i}, \ldots, B_{ni}$. This is taking scaling factors from B^T,

$\therefore (AB)^T = B^T A^T$

3. Without partial pivoting, find the LU decomposition for the following matrix A

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{bmatrix}$$

LU decomposition

$$\begin{bmatrix} 1 & 0 & 0 \\ +2 & 1 & 0 \\ +3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

4. For an LU decomposition with partial pivoting, find all the permutation and lower triangular matrices for the following matrix.

Exchange $R_1 R_2$

$PA = LU$

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 4 & 8 & 12 & -8 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{bmatrix}$$

$R_4 + \frac{3}{4} R_3$

$4 -$

$1 - 9 - -6 + 12$

$-3 - 12 \quad -8 \quad -16$

$-1 + 6$

$$\begin{bmatrix} 4 & 8 & 12 & -8 \\ 1 & 2 & -3 & 4 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ +1/4 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ -3/4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 & 12 & -8 \\ 0 & 0 & -6 & 6 \\ 0 & -1 & -4 & 5 \\ 0 & 5 & 10 & -10 \end{bmatrix}$$

[Exchange R_2, R_4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 1/4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 & 12 & -8 \\ 0 & 5 & 10 & -10 \\ 0 & -1 & -4 & 5 \\ 0 & 0 & -6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ 1/2 & -1/5 & 1 & 0 \\ 1/4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 & 12 & -8 \\ 0 & 5 & 10 & -10 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -6 & 6 \end{bmatrix}$$

$\downarrow$ (Exchange R_3 & R_4)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/2 & -1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 & 12 & -8 \\ 0 & 5 & 10 & -10 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

$R_3 - \frac{R_4}{3}$

$\downarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/2 & -1/5 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 & 12 & -8 \\ 0 & 5 & 10 & -10 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We do exchange $R_1 \leftrightarrow R_2$, $R_2 \leftrightarrow R_4$, $R_3 \leftrightarrow R_4$,

So, $P =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

5. (a) Consider two vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$. Write a proof if the statement is True otherwise provide a counter example.

   1. $\mathcal{V}_1 \cap \mathcal{V}_2$ is a vector space
   2. $\mathcal{V}_1 \cup \mathcal{V}_2$ is a vector space

(b) Are the following sets vector spaces?

   1. $\left\{\begin{bmatrix} x \\ y \end{bmatrix} : x = y, x \in \mathbb{R}, y \in \mathbb{R}\right\}$ Yes
   2. $\left\{\begin{bmatrix} x \\ y \end{bmatrix} : x^2 = y^2, x \in \mathbb{R}, y \in \mathbb{R}\right\}$ No: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, but $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ &
   3. $\left\{\begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 > 0, x \in \mathbb{R}, y \in \mathbb{R}\right\}$ No [Does not contain $(0,0)$]
   4. $\{\underline{\boldsymbol{u}} : u_i \geq 0\ \forall i, \underline{\boldsymbol{u}} \in \mathbb{R}^n\}$ No, $-\vec{u}$ does not belong

a) 1. Let

$x, y \in V_1 \cap V_2$.

$\Rightarrow x, y \in V_1, \quad x, y \in V_2$

$\Rightarrow ax + by \in V_1, \quad ax + by \in V_2$ [Linearity in V_1 & V_2]

$\Rightarrow \boxed{ax + by \in V_1 \cap V_2}$

$\Rightarrow$ True

2. $V_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}, a \in \mathbb{C}, \quad V_2 = \begin{bmatrix} 0 \\ b \end{bmatrix}$ $b \in \mathbb{C}$ are vector spaces.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V_1, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V_2 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V_1 \cup V_2,$

but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin V_1 \cup V_2$

6. For the following systems $Ax = b$, transform it to $Rx = \hat{b}$ where R is the RREF. Find solution(s) if they exist.

a.

$$x + 2y + 3z = 4$$
$$3x + 4y + z = 5$$
$$2x + y + 3z = 6$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 5 \\ 2 & 1 & 3 & 6 \end{array}\right] \rightarrow \text{Augmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -8 & -7 \\ 0 & -3 & -3 & -2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 7/2 \\ 0 & -3 & -3 & -2 \end{array}\right]$$

$$-2 + \frac{7}{4} \cdot 3$$

$$\frac{21}{4} - 2 = \frac{13}{4}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 7/2 \\ 0 & 0 & 9 & 17/2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 7/2 \\ 0 & 0 & 1 & 17/18 \end{array}\right]$$

$$4 - \frac{13}{12}$$

$$\frac{7}{4} - 4 \cdot \frac{13}{36}$$

$$\frac{7}{4} - \frac{13}{9} = \frac{63 - 52}{36}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3/18 \\ 0 & 1 & 0 & -5/18 \\ 0 & 0 & 1 & 17/18 \end{array}\right]$$

$\frac{7}{2} - \frac{4 \cdot 17}{18}$ $\quad 4 - \frac{39}{36}$

$\frac{63-68}{18}$ $\quad \frac{31 \cdot 4 - 39}{36}$

$3 - \frac{51}{18}$

105
22
83

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 13/18 \\ 0 & 1 & 0 & -5/18 \\ 0 & 0 & 1 & 17/18 \end{array}\right] \begin{array}{l} \to x \\ \to y \\ \to z \end{array}$$

b)

$$x + 2y + 3z = 4$$
$$2x + y = 2$$
$$x + 5y + 8z = 10$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 2 \\ 1 & 5 & 8 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -6 \\ 0 & 3 & 5 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & 5 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \longleftarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{x = 0, \quad y = 2, \quad z = 0}$$

c)

$$x - 2y + z = 1$$
$$2x - 5y + 3z = 4$$
$$2x - 3y + z = 0$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & -5 & 3 & 4 \\ 2 & -3 & 1 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & +1 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\downarrow \qquad 1 - E_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Last equation is trivial, $\therefore$ $(0 = 0)$

The equations say

$$x - z = -3$$
$$y - z = -2$$

$\Rightarrow$ Solutions are of the form

$$\begin{bmatrix} r-3 \\ r-2 \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$$