Classical Dynamics

- Lagrangian motion
- Conservation laws
- Integration of equations of motion
- Small oscillations
- Motion of rigid body
- Canonical transforms
- Louiville theorem
- TI perturbation theory
- Special Relativity

$\vec{v} = \frac{d\vec{r}}{dt}$

$\vec{a} = \frac{d\vec{v}}{dt}$

$\vec{p} = m\vec{v}$

$\vec{F} = \frac{d\vec{p}}{dt}$

Conservation of momentum:

- If $\vec{F} = 0, \vec{p} = \text{constant}$

Principle of Extremization

Snell’s Law (Fermat’s Principle)

Time is minimized!

$t_{AB}(x) = \frac{\sqrt{h_1^2+x^2}}{v_1} + \frac{\sqrt{h_2^2+(x-a)^2}}{v_2}$

$t_{AB}\text{'}(x) = \frac{x}{v_1\sqrt{(h_1^2+x^2)}} + \frac{x-a}{v_2\sqrt{(h_2^2+(x-a)^2)}} = 0$

$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$

Variational Calculus

$J = \int_{x_1}^{x_2} f(y(x), y'(x), x)dx$

J is a functional here

y(x) should satisfy:

- $y(x_1) = a$
- $y(x_2) = b$
- $y(x)$ should be continuous
- $y'(x)$ should be piecewise differentiable

$J(\alpha) = \int_{x_1}^{x_2} f(y(\alpha, x), y'(\alpha, x), x)dx$

$\frac{dJ(\alpha)}{d\alpha}\Big|_{\alpha=0} = 0$

Let $y = y + \alpha\eta(x)$

$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}dx = \int_{x_1}^{x_2}(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'})\eta(x)dx = 0$ (Note: $\frac{\partial x}{\partial \alpha} = 0$)

$\implies \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$

The Brachistochrone problem

$v = \sqrt{2gx}$ (from conservation of energy)

$t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gx}} = \int_{x_1=0}^{x_2} (\frac{1+y'^2}{2gx})^{1/2}dx$

Here, $f = (\frac{1+y'^2}{2gx})^{1/2}$

From Euler-Lagrange Equation, $c = \frac{y'}{\sqrt{1+y'^2}\sqrt{2gx}} = \frac{1}{2a}$

$y = \int \frac{x}{\sqrt{2ax-x^2}}dx$

Let $x = a(1-\cos\theta), dx = a\sin\theta$

$y = a\int(1-\cos\theta)d\theta = a(\theta - \sin\theta) + C$

Using boundary conditions, C = 0.

So, our curve is $x = a(1-\cos\theta), y = a(\theta - \sin\theta)$

Geodesics: Shortest path between two points on a sphere

Constraints

- Holonomic constraint: can write down constraint as $f(\vec{x}, t) = 0$
- Explicit Time dependence?
  - Rheonomic Constraint: Explicit time dependence of constraint
  - Scleronomic: No explicit time dependence

Lagrangian formulation: Extremization of action

$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$

Action is extremized across all paths, i.e. the system follows the phase space trajectory for which action is extremized.

$\implies \delta S = 0$ (δ is known as the variation)

Let q be a minimum path, so that implies, $q(t) = q(t) + \delta q(t)$ will not be minimum, but $\delta q(t_1) = \delta q(t_2) = 0$

$\delta S = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q})dt = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d(\delta q)}{dt})dt = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})\delta q dt.$
Now, this is true for all t_1, t_2. $\implies \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$.

- This is a connection between coordinates, velocity, and acceleration.

Some properties of Lagrangian:

- Lagrangian can be scaled
- Additivity of Lagrangian as distance between two system tends to infinity implies that either of the two systems cannot have quantities pertaining to the other part
- It can be specified only up to a time derivative

Inertial frame

Frame in which Newton’s laws are valid

Homogeneity of Space and Time and Isotropy of Space

For an Inertial frame

- We can choose the origin of space and time, i.e. the space and time are translationally invariant
  - For a free particle, $\frac{\partial L}{\partial x_i} = 0 \implies \frac{\partial L}{\partial \dot{x}_i} = c$. In other words, velocity is constant, which is called law of inertia!!

- Isotropy of Space: Orientation of space doesn't matter, we can choose which angle we want to look at space
    - For a free particle, in a similar way, total angular momentum is conserved!

More general Lagrangian:

💡 $L = \frac{1}{2}\sum_{i,j}\alpha(q)\dot{q}_i\dot{q}_j - U(q,t)$

Conserved Quantities

Integrals of Motion

- Some function of $(q_i, \dot{q}_i)$, that is constant throughout motion and only depends on initial conditions
- Additive Integrals of motion are *conserved*
- Conserved doesn't mean something is constant, it means something is the same throughout the trajectory

Conservation of Energy

For a closed system, time is homogeneous, i.e. the Lagrangian has no explicit time dependence, $\implies \frac{\partial L}{\partial t} = 0$

$\frac{dL}{dt} = \sum_j(\frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial q}\dot{q}) = \sum_j(\frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\dot{q}) = \sum_j \frac{d}{dt}(\dot{q}\frac{\partial L}{\partial \dot{q}})$.

$\implies \frac{d}{dt}\sum_j \dot{q}\frac{\partial L}{\partial \dot{q}} - L = \frac{d}{dt}\sum_j \dot{q}\dot{p} - L = 0$

- $H = \dot{q}\dot{p} - L = c$

Homogeneity of time implies energy conservation. Such systems are conservative systems.

$\sum_j \dot{q}\frac{\partial L}{\partial \dot{q}} = 2T$

Conservation of Linear Momentum

Let $\mathbf{r} = \mathbf{r} + \delta\mathbf{r}$

Now, $\delta L = 0 = \sum_i \frac{\partial L}{\partial \mathbf{r}} \cdot \delta\mathbf{r} \implies \sum_i \frac{\partial L}{\partial \mathbf{r}} = 0 = -\frac{\partial U}{\partial \mathbf{r}}$

Total Linear momentum is conserved!

Conservation of Angular Momentum

- Likewise, due to isotropy of space, angular momentum is conserved

Consider rotation $\delta\phi$, where magnitude is $||\delta\phi||$, whose axis of rotation is in the direction of $\delta\phi$

$\delta\mathbf{r} = \delta\phi \times \mathbf{r}$

$\delta\mathbf{v} = \delta\phi \times \mathbf{v}$

> If these expressions are substituted in the condition that the Lagrangian is unchanged by the rotation:
>
> $$\delta L = \sum_a \left(\frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta \mathbf{r}_a + \frac{\partial L}{\partial \mathbf{v}_a} \cdot \delta \mathbf{v}_a \right) = 0$$
>
> and the derivative $\partial L / \partial \mathbf{v}_a$ replaced by $\mathbf{p}_a$, and $\partial L / \partial \mathbf{r}_a$ by $\dot{\mathbf{p}}_a$, the result is
>
> $$\sum_a (\dot{\mathbf{p}}_a \cdot \delta\boldsymbol{\phi} \times \mathbf{r}_a + \mathbf{p}_a \cdot \delta\boldsymbol{\phi} \times \mathbf{v}_a) = 0$$
>
> or, permuting the factors and taking $\delta\boldsymbol{\phi}$ outside the sum,
>
> $$\delta\boldsymbol{\phi} \sum_a (\mathbf{r}_a \times \dot{\mathbf{p}}_a + \mathbf{v}_a \times \mathbf{p}_a) = \delta\boldsymbol{\phi} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_a \mathbf{r}_a \times \mathbf{p}_a = 0.$$

$\mathbf{L} = \sum_\alpha r_\alpha \times p_\alpha$

Generalized momentum: $p_j = \frac{\partial L}{\partial \dot{q}_j}$

Now, if q_i doesn't appear in the Lagrangian, $\implies p_i = c$, i.e. the canonical momentum is conserved, and that coordinate is called a cyclic coordinate.

Nice Question

PROBLEM 3. Which components of momentum **P** and angular momentum **M** are conserved in motion in the following fields?

(a) the field of an infinite homogeneous plane, (b) that of an infinite homogeneous cylinder, (c) that of an infinite homogeneous prism, (d) that of two points, (e) that of an infinite homogeneous half-plane, (f) that of a homogeneous cone, (g) that of a homogeneous circular torus, (h) that of an infinite homogeneous cylindrical helix.

22 *Conservation Laws* §10

SOLUTION. (a) P_x, P_y, M_z (if the plane is the xy-plane), (b) M_z, P_z (if the axis of the cylinder is the z-axis), (c) P_z (if the edges of the prism are parallel to the z-axis), (d) M_z (if the line joining the points is the z-axis), (e) P_y (if the edge of the half-plane is the y-axis), (f) M_z (if the axis of the cone is the z-axis), (g) M_z (if the axis of the torus is the z-axis), (h) the Lagrangian is unchanged by a rotation through an angle $\delta\phi$ about the axis of the helix (let this be the z-axis) together with a translation through a distance $h\delta\phi/2\pi$ along the axis (h being the pitch of the helix). Hence $\delta L = \delta z\ \partial L/\partial z + \delta\phi\ \partial L/\partial\phi = \delta\phi(h\dot{P}_z/2\pi + \dot{M}_z) = 0$, so that $M_z + hP_z/2\pi =$ constant.

Mechanical Similarity

$L = T - V$

Scaling Lagrangian doesn't change dynamics of the system

Consider the transformation $(r_1, r_2, ... r_n) \rightarrow (\eta r_1, \eta r_2, ... \eta r_n), (t) \rightarrow (\beta t)$

$V(\eta \vec{r}_\alpha) = \eta^k V(\vec{r}_\alpha)$, k is the degree of homogeneity of the equation

Consider changes in v, KE, PE

$\vec{v}' = \frac{\eta}{\beta}\vec{v}$

$T \rightarrow (\frac{\eta}{\beta})^2$

$V \rightarrow \eta^k$

Choose η, β, such that $(\frac{\eta}{\beta})^2 = \eta^k$

$L' \rightarrow \eta^k$

- Equation of motion is unaltered now
- Shape of the trajectory is similar
- Homogeneous function of degree k, the equation of motion permits a series of geometrically similar paths

1. $\frac{t'}{t} = (\frac{l'}{l})^{1-k/2}$

2. $\frac{v'}{v} = (\frac{l'}{l})^{k/2}$
3. $\frac{E'}{E} = (\frac{l'}{l})^{k}$
4. $\frac{L'}{L} = (\frac{l'}{l})^{1+k/2}$ → L is angular momentum

Virial Theorem

- Assumptions:
  - Motion is bounded
  - V is homogeneous function

Time averages:

- If a function f(t) is the time derivative of another function
  - $< f(t) >= \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} f(t)dt = \lim_{\tau\to\infty} \frac{F(\tau)-F(0)}{\tau} = 0$

Define $G = \sum_{\alpha} \vec{p}_{\alpha} \cdot r_{\alpha}$

$\frac{dG}{dt} = \sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \mathbf{r}_{\alpha} + \sum_{\alpha} \mathbf{p}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha} = 2T + \sum_{\alpha} \mathbf{F} \cdot \mathbf{r}_{\alpha}$

$< \frac{dG}{dt} >= 0$

$< T >= \frac{-1}{2} < \sum_{\alpha} \mathbf{F}_{\alpha} \cdot \mathbf{r}_{\alpha} >$

Further, if U is homogeneous,

$< T >= \frac{-1}{2} < \sum_{\alpha} \boldsymbol{\nabla}\mathbf{U}_{\alpha} \cdot \mathbf{r}_{\alpha} >= \frac{n+1}{2} < U(\mathbf{r}) >$

- $< V >= \frac{2E}{k+2}, < K >= \frac{kE}{k+2}$

Ideal Gas law follows from Virial theorem

- $dF = PdA\hat{n}$
- $\frac{-1}{2}\sum_{\alpha} \mathbf{F}_{\alpha} \cdot \mathbf{r}_{\alpha} = \frac{P}{2}\int \mathbf{r} \cdot \hat{n} dA = \frac{3}{2}PV$
- $\frac{3k_B TN}{2} = \frac{3PV}{2}$
- $PV = Nk_B T$

Adding a differentiable function to Lagrangian

$L' = L + \frac{dF(q_1, q_2 \ldots q_n, \dot{q}_1, \dot{q}_2 \ldots \dot{q}_n, t)}{dt}$

$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}} + \frac{d}{dt}\frac{\partial F}{\partial \dot{q}}) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$

- $\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}$

- $\frac{\partial}{\partial \dot{q}} \frac{dF}{dt} = \frac{\partial F}{\partial q}$

$\frac{\partial L'}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\frac{dF}{dt} = \frac{\partial L}{\partial q} + \frac{d}{dt}\frac{\partial F}{\partial q} = \frac{\partial L}{\partial q} + 0$

Central Forces

Consider a particle of reduced mass μ in a central force field

- Prove: Spherical symmetry $\implies$ central force

Properties:

- $\vec{r}, \vec{p} \perp \vec{L}$

$L(r, \theta) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$

For two particles interacting with each other, treat them as if it's one particle around a central potential with mass μ

As θ is cyclic, $p_\theta = \mu r^2\dot{\theta}$ is constant

Kepler's second law: $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu} = c$

Solving Equation:

- $E = T + U$

$\dot{r} = \sqrt{\frac{2}{\mu}(E - U(r)) - \frac{l^2}{\mu^2 r^2}}$

r as a function of t is pain, use r as a function of θ

💡 Just refer Marion or something.

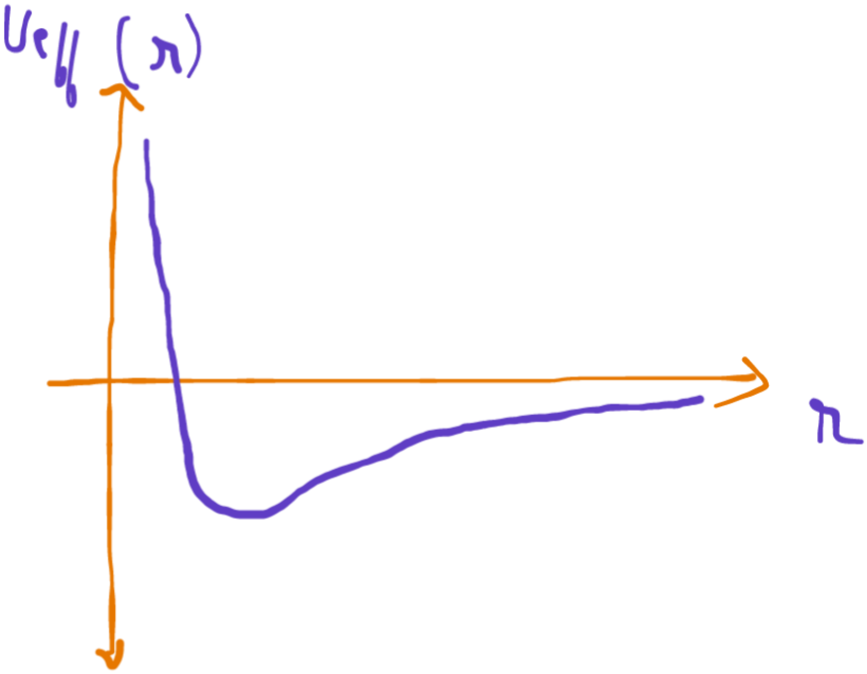
Bertrand's theorem: Finite motions in closed orbit are $\frac{k}{r}, kr^2$ potentials

Kepler's Problem: $U(r) = \frac{-k}{r}$

Solving the equation

Centrifugal Potential Energy = $\frac{l^2}{2\mu r^2}$

$U_{eff} = U + \frac{l^2}{2\mu r^2}$



Effective Potential vs r graph, depicting what happens to motion

- $E > 0$: Hyperbolic
- $E = 0$: Parabolic
- $E_{min} < E < 0$: Elliptical
- $E = E_{min}$: Sits at the center
- $E < E_{min}$: No motion

$U_{eff,min} = \frac{-\mu k^2}{2l^2}$

$\alpha = \frac{l^2}{\mu k}$

$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta$, where $\varepsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}}$

Clearly, it follows a conic section orbit:

- ε is the eccentricity
- 2α is the Latus Rectum

Laplace-Runge-Lenz vector (LRL vector)

- Conserved quantity for stable orbits (for 1/r potentials)
- r_{min} remains same even if orbit changes

For any central motion,

$\dot{\mathbf{p}} = f(r)\hat{r}$

$\dot{\mathbf{p}} \times \mathbf{L} = \frac{mf(r)}{r}(\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})) = \frac{mf(r)}{r}(\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - (r^2\dot{\mathbf{r}}))$

$\frac{d}{dt}(\mathbf{p} \times L) = -mf(r)r^2 \frac{d}{dt}\frac{\mathbf{r}}{r}$

$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$

Kepler's problem

$\frac{d\mathbf{A}}{dt} = 0$

- It is confined to the plane as it is orthogonal to L

$Ar\cos\theta = L^2 - mkr$

> 💡 Given r as a function of θ,
> $\frac{\partial^2}{\partial\theta^2}\frac{1}{r} + \frac{1}{r} = -\mu\frac{r^2}{l^2}F$

> 💡 **Stable Orbits condition**
>
> $$\boxed{\frac{F'(\rho)}{F(\rho)} + \frac{3}{\rho} > 0}$$

Scattering

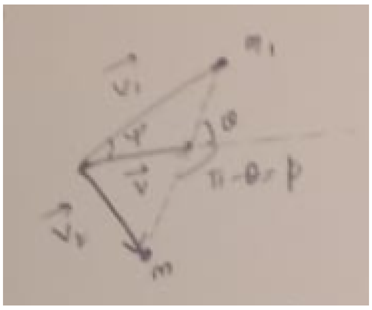
> 💡 Refer Marion

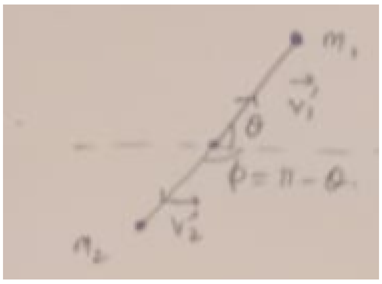
Frame of reference:

- Lab frame: Some momentum is there
- COM frame: Momentum is zero

ψ →Scattering angle in lab frame

θ → Scattering angle in COM frame





$$\boxed{\tan\psi = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}}$$

- ψ can be measured, θ cannot be measured easily
- If $m_1 <<< m_2, \psi \approx \theta$
- If $m_1 = m_2, \psi = \theta/2$

b → impact parameter

Intensity, $I = \frac{\text{number of particles}}{\text{Area}\times\text{time}}$

$\sigma(\Omega)d\Omega = \frac{\text{number of particles scattered in a solid angle}}{\text{incident intensity}}$

σ → Differential scattering cross section

$l = m_1 u_1 b = b\sqrt{2mT_0}$

$E = T_0 = \frac{1}{2}m_1 u_1^2$

$I 2\pi b|db| = I\sigma(\theta)2\pi\sin\theta|d\theta|$

$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$

$\Theta(r) = \int_{r_{min}}^{\infty}\frac{br^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{U}{T_0}}}dr$

$\theta = \pi - 2\Theta$

Rutherford Scattering

$\theta = \int_{r_{min}}^{\infty} \frac{\frac{b}{r^2}\, dr}{\sqrt{r^2 - \frac{Cr}{T_0'} - b^2}}$

$b = k \cot \frac{\theta}{2}$

$\sigma(\theta) = \frac{C^2}{(4T_0)'^2} \frac{1}{\sin^4(\frac{\theta}{2})}$

Oscillations

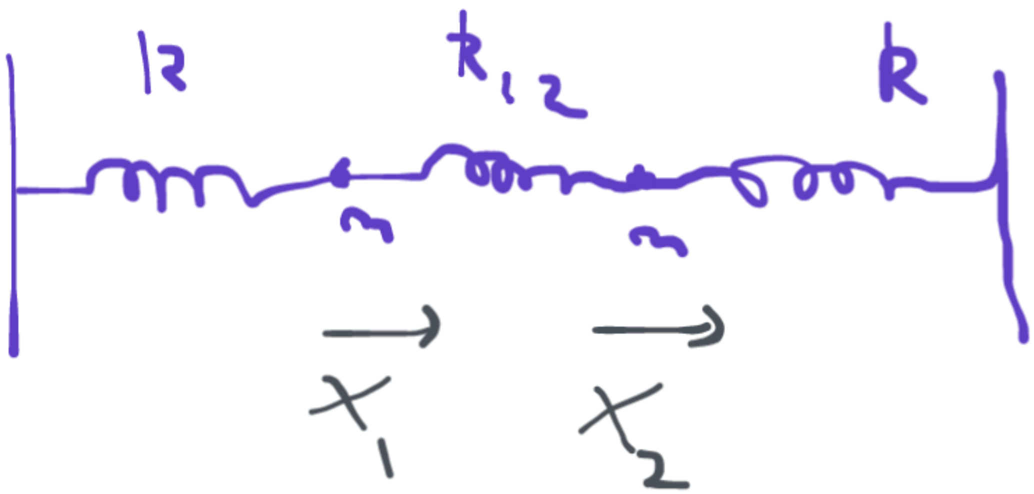
One-dimensional Form

- $U \approx \frac{1}{2}k(x - x_0)^2$, where x_0 is the stable equilibrium point
- Here, $k = \frac{d^2U}{dx^2}|_{x_0}$
- $x(t) = a\cos(wt + \phi)$
- Or, $x(t) = Re\{Ae^{iwt}\}$

More dimensions

Same idea

Coupled Oscillators



- Force on $m_1 = -kx_1 - k_{12}(x_1 - x_2)$
- Force on $m_2 = -kx_2 - k_{12}(x_2 - x_1)$

Assume $x_1 = A_1 e^{iwt}, x_2 = A_2 e^{iwt}$

$w_1 = \sqrt{\frac{k+2k_{12}}{M}}, w_2 = \sqrt{\frac{k}{M}}$

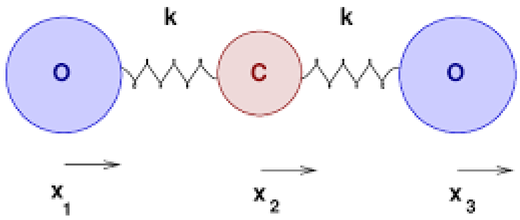
Normal Coordinates

- Choose coordinates $\eta_1(x_1), \eta_2(x_2)$

  Usually, anti-symmetric frequency > symmetric frequency

- The new set of coordinates would be *normal coordinates*
- $\ddot{\Theta}_\alpha + w_\alpha \Theta_\alpha = 0$
- Choose $Q = \sqrt{m_\alpha}\Theta_\alpha$, so that coefficient is $\frac{1}{2}$

CO_2 Molecule



$U = \frac{k}{2}(x_2 - x_1 - b)^2 + \frac{k}{2}(x_3 - x_2 - b)^2$

- Let Equilibrium distances be x_{10}, x_{20}, x_{30}
- $\eta_j = x_j - x_{j0}$
- $U = \frac{1}{2} k_{jk} \eta_j \eta_k$
- $\mathbf{k} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$
- $T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_3^2) + \frac{M}{2}\dot{x}_2^2 = \frac{m}{2}(\dot{\eta}_1^2 + \dot{\eta}_3^2) + \frac{M}{2}\dot{\eta}_2^2$
- $m = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}$
- $\det(k - w^2 m) = 0$

- Solving this, $w_1 = 0, w_2 = \sqrt{\frac{k}{m}}, w_3 = \sqrt{\frac{k}{m}(1 + \frac{2m}{M})}$
- Eigenvalues
  - For $w_1, a_{11} = a_{21} = a_{31}$, but we have to normalize it, so $\vec{a}_1 = \frac{1}{\sqrt{M+2m}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- Coupled two oscillators
- $\ddot{x} + w_0^2 x = \alpha y$

Rigid body motion

If we have a system on n-particles, $r_{ij} = |r_i - r_j| = \text{constant} \forall i, j$

- Consider motion wrt COM
- We need three angles, Euler angles, ϕ, θ, ψ
- $\sum_{i=1}^{3} \cos\theta_{lm'} \cos\theta_{lm'} = \delta_{m,m'}$
- It can be represented using a transformation matrix, which is a like Euler rotation matrix
- $A = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Euler angles

- $\phi \rightarrow \text{counterclockwise about z axis}$
- $\theta \rightarrow \text{counterclockwise about x axis}$
- $\psi \rightarrow \text{counterclockwise about z' axis}$

Properties of orthogonal matrices

- Determinant = ± 1
- $AA^T = I$
- Modulus of eigenvalue = 1

Euler's theorem

The general displacement of a rigid body with one point fixed is a rotation

Another way to put it is: if $\exists Rx = x$, we're done!

- Throw it into Characteristic equation
- $(A - I)A^T = I - A$
- $\det(A - I)\det(A^T) = \det(I - A)$
- $\det A^T = +1$
- $\implies \det(A - I) = 0, \implies$ 1 is eigenvalue!!
- Hence proved!!
- Note that it's true only for odd dimensions

Vector changing in space-fixed and time-fixed coordinate system

- $d\vec{G}_{space} = d\vec{G}_{body} + d\vec{G}_{rot}$
- $d\vec{G}_{rot} = d\vec{\Omega} \times \vec{G}$

$$\boxed{\frac{d\vec{G}_{space}}{dt} = \frac{d\vec{G}_{body}}{dt} + \vec{\omega} \times \vec{G}}$$

$v_s = v_r + \omega \times r$

$\vec{a}_s = \vec{a}_r + (\vec{\omega} \times \vec{v}_r) + (\vec{\omega} \times \vec{v}_r) + \vec{\omega} \times (\vec{\omega} \times \vec{r})$

$F = m\vec{a}_r$

$F_{eff} = F - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m(\vec{\omega} \times \vec{v}_r)$

Force Centripetal Coriolis

- Foucault pendulum

Chasles' Theorem

Any rigid body motion can be decomposed into translation and rotation with axis of rotation parallel to that axis

$\vec{l} = m\vec{r}' \times \vec{v}' = m_i(\vec{R} + \vec{r}) \times (\vec{V} + \vec{\omega} \times \vec{r}) = m_i(\vec{R} \times \vec{V} + \vec{R} \times (\vec{\omega} \times \vec{r}) + \vec{r} \times \vec{V} + \vec{r} \times (\vec{\omega} \times \vec{r}))$

Choose origin of second frame to be the center of mass, so $\sum m_i \vec{r}_i = 0$

$\vec{l} = \vec{R} \times \vec{P} + \sum m_i \vec{r} \times (\vec{\omega} \times \vec{r})$

Intrinsic angular momentum $\vec{L} = \sum m\vec{r} \times (\vec{\omega} \times \vec{r})$

$L_i = \sum m(w_i|\vec{r}|^2 + \vec{r}_i(\vec{r} \cdot \vec{\omega}))$

$L_i = \sum m(\delta_{ik}r^2 - r_i r_k)w_k$

Define $I_{ik} = \delta_{ik}r^2 - r_i r_k$

- I is a real, symmetric matrix
- $I = \begin{bmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2+y^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(y^2+z^2) \end{bmatrix}$
- Moment of inertia along principal axis, $I_p = \begin{bmatrix} \sum m(y^2+z^2) & 0 & 0 \\ 0 & \sum m(x^2+y^2) & 0 \\ 0 & 0 & \sum m(y^2+z^2) \end{bmatrix}$
- Shifting axes: If we shift axes, $I \to I - M(\text{diag terms})$

I_1, I_2, I_3 are eigenvalues

Spherically Symmetric Top

$I_1 = I_2 = I_3$

- Like Sphere, Cube, etc.

$\vec{L} = I\vec{w}$

Angular velocity and Angular momentum along same axis!!

Symmetric Top

$I_1 = I_2 \neq I_3$

Like a cylinder

Asymmetric Top

$I_1 \neq I_2 \neq I_3$

Kinetic Energy of rotating body

$T = \sum \frac{1}{2}mv^2 = \sum \frac{1}{2}m(\vec{V} + \vec{\omega} \times \vec{r})^2 = \frac{1}{2}\sum mV^2 + \sum m\vec{V} \cdot (\vec{\omega} \times \vec{r}) + \frac{1}{2}\sum m(\vec{\omega} \times \vec{r})^2$

$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum m\vec{r} \cdot (\vec{V} \times \vec{\omega}) + \frac{1}{2}\sum m(\vec{v} \times \vec{\omega})^2$

3rd Term:

$\frac{1}{2}\sum m(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r}) = \frac{1}{2}\sum m\epsilon_{ijk}w_j r_k \epsilon_{ilm} w_l r_m$
$= \frac{1}{2}\sum m(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})w_j r_k w_l r_m = \frac{1}{2}\sum m(\delta_{jl}r^2 - r_j r_l)w_j w_l = \frac{1}{2}I_{jl}w_j w_l$

- Therefore, $T = \frac{1}{2}mV^2 + \frac{1}{2}w_j I_{jl} w_l$
- $\frac{dL}{dt}_S = \frac{dL}{dt}_b + \omega\times\vec{L} = \vec{N}$ → Torque
- $I_1\dot{w}_1 - w_2 w_3(I_2 - I_3) = N_1$

Hamiltonian Mechanics

- Legendre transforms

Lagrangian is a function of $(q, \dot{q}, t)$. This is sometimes problematic, so we consider $H(q,p,t)$, where $p = \frac{\partial L}{\partial \dot{q}}$.

So, we should be able to go from the Lagrangian to Hamiltonian using a Legendre transform!

- Legendre transform of L → $L - \frac{\partial L}{\partial \dot{q}}\dot{q}$. Now, we could let this be the Lagrangian, but we instead choose the negative of that to be the Hamiltonian (with n variables, by extension)

$$\boxed{H = (\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i) - L = \sum_i p_i\dot{q}_i - L}$$

The first term is like 2T!

So, $H = 2T - (T - U) = \boxed{T + U}$

- $\dot{q}_i = \frac{\partial H}{\partial p}$
- $\dot{p}_i = -\frac{\partial H}{\partial q}$
- $-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$

Hamiltonian Formulation

Example: Simple Pendulum

- Say Hamiltonian is time-independent,
  - $\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \dot{p}\dot{q} - \dot{q}\dot{p} = 0$
  - Hamiltonian is a constant of motion!

- Consider a quantity $F(q,p,t)$
  - $\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial F}{\partial t} = \{F,H\} + \frac{\partial F}{\partial t}$

Hamiltonian Formulation from Lagrangian action minimization

$\delta\int Ldt = 0$

$\delta\int p_i\dot{q}_i - Hdt = 0$

Now, the integrand is like a function f, which satisfies Euler Lagrange equations!! The integrand is a function of p_i, q_i.

$\dot{p}_j = -\frac{\partial H}{\partial q_j}$

$\dot{q}_j = \frac{\partial H}{\partial p_j}$

Poisson Brackets properties

$\{A,B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$

- $\{A,B\} = -\{B,A\}$
- $\{A+B,C\} = \{A,C\} + \{B,C\}$
- $\{\alpha A,B\} = \alpha\{A,B\}$
- $\{A,\alpha\} = 0$
- $\{A,BC\} = B\{A,C\} + C\{A,B\}$
- Jacobi Identity: $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$

Symplectic Geometry

- By Hamilton's equations,
  - $\dot{q}_i = \frac{\partial H}{\partial p}$
  - $\dot{p}_i = -\frac{\partial H}{\partial q}$
- Consider a vector $\vec{x} = (q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n)$
- $\dot{x} = J\nabla_{\vec{x}}H$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$
- Properties of J:
  - $J^{-1} = J$
  - $J^2 = -I$

- $\{A, B\} = (\nabla A)^T J(\nabla B)$ → Symplectic Dot product
- Standard Canonical Poisson Bracket relations
  - $\{q_k, q_l\} = \{p_k, p_l\} = 0$
  - $\{q_k, p_l\} = \delta_{kl}$

Canonical Transformations

Consider autonomous system, $H(q_i, p_i)$

Consider some transformation T, such that every coordinate is cyclic.

$\dot{p}_j = 0, p_j = \text{constant}$

That means, $p_j = \alpha_j$, where we can find α_j depends on initial condition

$H(\alpha_i)$ →. Hamiltonian is a function only of initial conditions

$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = w_i(\alpha_1, \alpha_2, \ldots \alpha_n)$

$q_i = w_i t + \beta_i$, where β_i depends on initial conditions

Transformation

Consider the coordinate transformation, $q_i, p_i \to Q_i, P_i$

We want the transformation to be invertible

$H(q_i, p_i, t) \to K(Q_i, P_i, t)$

Conditions

1. For the transformation to be invertible, $\det(\frac{\partial(Q,P)}{\partial(q,p)}) = 1$
2. Canonical Poisson brackets are true:
   a. $\{Q_i, Q_j\} = 0 = \{P_i, P_j\}$
   b. $\{Q_i, P_j\} = \delta_{ij}$

One always canonical transformation:

$Q = p, P = -Q$

- Momentum and position lose their meaning in the Hamiltonian framework

Generating Function

We know that we can shift the Lagrangian by at most a time derivative of a function

$\delta \int (P_i \dot{Q}_i - K) dt = 0 = \delta \int (p_i \dot{q}_i - H) dt$

$\implies p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF}{dt}$

Here, F is called a generating function. It is useful only half of the variables are the new variables

Consider $F_1(q, Q)$

$\implies p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF_1}{dt} = P_i\dot{Q}_i - K + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}$

Equating coefficients of $\dot{q}_i, \dot{Q}_i$ to 0:

- $p_i = \frac{\partial F_1}{\partial q_i}$
- $P_i = -\frac{\partial F_1}{\partial Q_i}$
- $K = H + \frac{\partial F_1}{\partial t}$

$F_2(q, P, t) = F_1(q, Q, t) + P_i Q_i$

Example

$F_2 = q_i P_i$ → Identity transformation

Harmonic Oscillator

$H(p, q) = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{1}{2m}(p^2 + m^2w^2q^2)$

Consider the transformation $p = f(P)\cos Q, q = \frac{f(P)}{mw}\sin Q$

Consider the generating function $F_1(q, Q) = \frac{mwq^2}{2}\cot Q$

$q = \sqrt{\frac{2P}{mw}}\sin Q, p = \sqrt{2Pmw}\cos Q$

$f(P) = \sqrt{2mwP}$

- $K = wP$, which is cyclic in Q
- $Q = wt + \alpha$
- $P = \frac{E}{w}$
- $q = A\cos(wt + \alpha)$
- $p = Awsin(wt + \alpha)$
- We got SHM equations!!

Liouville's theorem

Consider a volume V in phase space, q, p

Consider $x = (q_1, q_2, \dots q_n, p_1, p_2, \dots p_n)$

$dV(t + \Delta t) = dx_1 dx_2 \dots dx_{2n}$

$dx^{2n}(t + \Delta t) = \det(J) dx^{2n}(t_0)$, $J = \frac{\partial x(t+\Delta t)}{\partial x(t_0)}$

Now, what is J?

$J = \frac{\partial x_k(t+\Delta t)}{\partial x_m(t_0)} = \delta_{km} + \frac{\partial f_k}{\partial x_m} dt$

$\det(J) = 1 + tr(A)dt = 1 + \nabla \cdot f$

$V = \int d^{2n}x(t + dt) = (\int 1 + \nabla \cdot f dt) d^{2n}x$

$\frac{dV}{dt} = \int \nabla \cdot f d^{2n}x(t_0) = \frac{\partial}{\partial q_1} \frac{\partial H}{\partial p_1} + \dots = 0$

V is constant!!!

So, phase flow is actually an in-compressible flow through a Hamiltonian space