Complex Variables

Symmetry of partial derivatives

Divergence Theorem

$\iint \nabla \cdot \vec{A} dS = \oint \vec{A} \cdot d\vec{\rho}$

Now, choose $A = (v, -u)$

$\iint v_x - u_y dS = \oint \vec{(}v, -u) \cdot (dy, -dx) = \oint udx + vdy$

Green's theorem proved!!

Green's Theorem

$\iint (v_x - u_y) dA = \oint udx + vdy$

Now, choose $u = f_x, v = f_y$

$\iint f_{yx} - f_{xy} dA = \oint \nabla F \cdot \vec{dl} = 0$

If this is true for all areas, $f_{yx} = f_{xy}$!!

Multi-valued Function

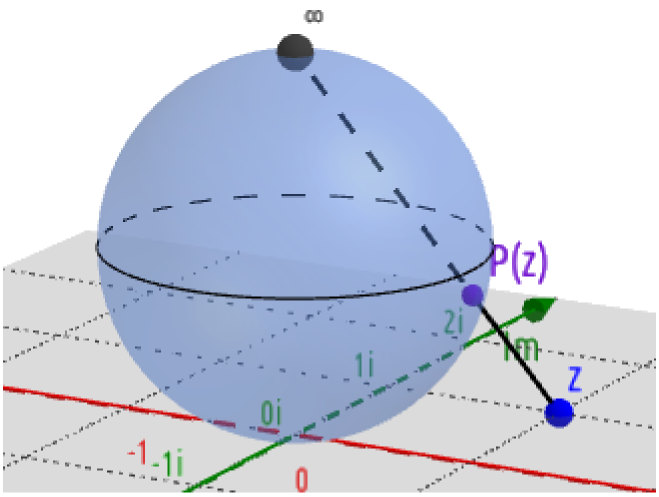
- Eg.
  - $\sqrt{z} \rightarrow \pm \text{roots}$
  - If we go through the origin, we incur a $\Delta\theta = 2\pi$
    - Order is how many rotations it takes to return to a particular value
  - Eg. $z^{\frac{1}{n}}$
    - $f(z) = |z|^{\frac{1}{n}} e^{\frac{i\theta p}{n}} e^{\frac{2\pi m}{n}} \forall m$
    - So, order is n
  - For stuff like $\ln(z)$, there are infinite order
  - Also stuff like $z^{\frac{1}{\sqrt{3}}}$ has infinite order

- Branch Cut → If we have a multivalued function, we have to restrict θ to something.
  - For instance, for $\sqrt{z}$, we can restrict $0 \leq \theta < 2\pi$
  - Likewise, for $\log z$, we can restrict θ similarly

Complex function→ $f(z) = u + iv$, $f : \mathbb{C} \to \mathbb{C}$

Riemann Sphere

Note that the definition of ∞ is not clear. To remove any ambiguity, we define a Riemann sphere, such that the stereo-graphic projection of the Riemann sphere yields the complex plane. The complex plane is extended to yield the *extended complex plane*, which is $\mathbb{C}' = \mathbb{C} \cup \{\infty\}$



Derivative of a function

If $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ exists, and is unique, the function is said to be *differentiable*.

It doesn't matter which path we take, the derivative should be the same. Let's choose the direction to be along x-axis, which yields:

$f'(z) = u_x + iv_x$

Cauchy-Riemann conditions for existence of derivative at a point

If f(z) is differentiable at a point iff partial derivatives at u and v are continuous and satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, and $u(x,y)$ and $v(x,y)$ are real differentiable

Proof:

- Choose $\Delta z = \Delta x$, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$
- Choose $\Delta z = \Delta y$, $f'(z) = \frac{1}{i}(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}) = (-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y})$
- Equate real and imaginary parts!

Some implications

- If f is analytic, $\frac{\partial f}{\partial z^*} = 0$. This is because, in the $u_x = v_y$, $\frac{df}{dz^*} = u_x + iv_x = i(u_y + iv_y) = -v_y + iu_y. \implies u_x = v_x = 0$
- The curves u and v are orthogonal
  - $\nabla u = (u_x, u_y), \nabla v = (v_x, v_y)$
  - $\nabla u \cdot \nabla v = u_x v_x + u_y v_y = 0$

<u>Polar version:</u>

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Polar Derivative:

$$\boxed{f'(z) = e^{-i\theta}\frac{\partial f}{\partial r}}$$

Analytic Function

<u>Analytic Function:</u> A function is said to be analytic at a point if it is differentiable at that point and some points in a region around it A function is said to be analytic over a region if it satisfies Cauchy-Riemann conditions over that region

- $\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} \implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
  - Real and Imaginary parts of analytic functions are harmonic functions
    - Analytic functions $\iff$ harmonic pieces
      - Forward is lite
      - Backward
        - Given u is harmonic, u is the real part of some analytic function

    - Let $\nabla^2 u = 0$
    - Now, choose a field $g(z) = u_x - iu_y$
    - Now, let f be the antiderivative of g
    - $f(z) = U + iV$
    - $f'(z) = U_x + iV_x$
    - Now, f is differentiable (we just differentiated it)
    - $f'(z) = U_x - iU_y = g(z)$
    - $U_x = u_x$, therefore $u = U + c$, likewise $v = V + c$
  - If u is harmonic over a region U_1, v is harmonic over a region U_2, z is analytic over $U_1 \cap U_2$

Entire function: If a function is harmonic over the entire plane, it is called *entire function.* Eg. $e^z, e^{-z}. \cosh(z), \sinh(z)$, etc.

- Analytic Functions are infinitely differentiable, i.e. derivative is also analytic
  - This implies Taylor series is valid around that point
- Taylor Series→ $f(z) = f(z_0) + f'(z_0)(z - z_0) + \ldots$

Finding derivative

- Chain rule, product rule, quotient rule works

Recovering function from $u(x, y)$: Lite, just use Cauchy-Riemann, and integrate

Contour integration

Consider the contour integral $\int f(z)dz = \int (u + iy)(dx + idy)$

Consider $\oint \frac{1}{z^{n+1}} dz$. Let's integrate it along a circle of radius r around the point of singularity a = 0

$\oint \frac{1}{z^{n+1}} dz = \int_0^{2\pi} \frac{ie^{i\phi}}{e^{i(n+1)\phi}} d\phi = 2\pi i \delta_{n,0}$

Cauchy's integral theorem

For a function with no singularities, the closed loop integral is zero

- $\oint f(z)dz = 0$
- Proof: Just use Stokes' theorem on real and imaginary parts, we're done!

Converse

- If $f(z)$ is continuous and $\oint f(z)dx = 0 \forall C$ in a region, $f(z)$ is analytic

Cauchy's integral Formula

$\oint_{\text{Contour containing z0}} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

Proof:

- Let $z - z_0 = re^{i\theta}$
- $\oint_{\text{Contour containing z0}} \frac{f(z)}{z-z_0} dz = \lim_{r\to 0} \oint \frac{f(z_0+re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta = f(z_0)2\pi i$

Differentiation under the integral sign

Differentiate with respect to z_0

$$\boxed{f'(z_0) = \oint \frac{f(z)}{(z-z_0)^2} dz}$$

Laurent Series

$f(z) = \sum_{-=\infty}^{\infty} c_i (z-a)^i$

- It's like a Taylor series + a reciprocal series
- Laurent Series is good, as the point around which we are evaluating could be a singularity!
- Finding coefficients:
  - $c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$

Singularities

Order "m" of a singularity: If $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, where $\phi(z)$ is analytic and $\phi(z_0) \neq 0$

Pole strength: $\phi(z_0)$

- <u>Removable Singularity:</u> Has a finite limit at the point of singularity
  - Just make function equal to its limiting value to remove it!
- <u>Pole</u>: Point z_0 such that $f(z_0) = \infty$
  - Note that we could have orders of poles

        - $f(z) = \frac{c_{-1}}{(z-a)^n} + \text{ stuff}$ has a pole of order n at $z = a$
- Essential Singularity: Neither of the above two, i.e. $\lim_{z\to a} f(z), \lim_{z\to a} \frac{1}{f(z)}$ don't exist
    - Another way to put this is: Laurent Series expanded around that point has all the c_- coefficients non-zero
    - Eg. $f(z) = \sin(\frac{1}{z})$
        - Method 1: $u \equiv \frac{1}{z}, \sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \ldots, f(z) = \frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - \ldots$
- Residue: c_{-1} in Laurent series is called the *residue* of the function.

Residue Theorem

- This is a very nice result, as in general, it $\frac{1}{2\pi i}\oint_C f(z)dz = \sum_a Res_{z=a}\{f(z)\}$, where the summation is over all the residues enclosed by C

Finding Residues

- Just evaluate the Laurent Series
- For simple pole, $c_{-1} = \lim_{z\to a} f(z)(z-a)$
- For pole of order n
    - Multiply both sides by $(z-z_0)^m, m \geq n$
    - Differentiate $m-1$ times and divide by $(m-1)!$
    - You get the residue!
    - *Proof:* trivial

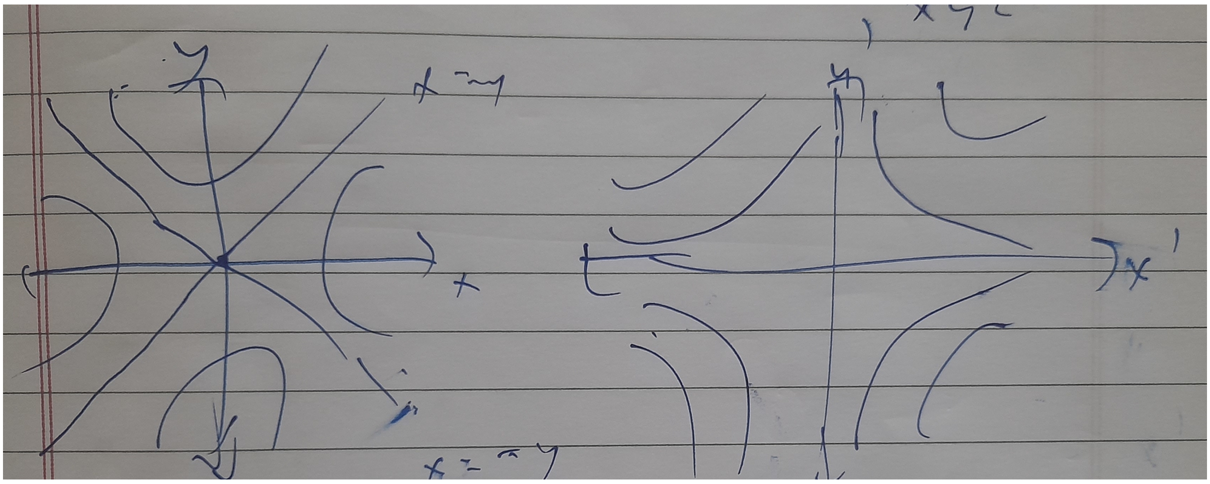
Fluid Mechanics Problem

Level set of u(x, y)

$\{x, y : u(x,y) = \text{constant}\}$

$u = y^2 - x^2$

Given u, by CR, $f = -z^2 + c$



For a point of intersection, level sets of u and v intersect orthogonally

$\frac{\partial v}{\partial x_t}\Delta x_t + \frac{\partial v}{\partial x_p}\Delta x_p$

Now, if gradients are perpendicular, curves are orthogonal!!

$\nabla u = -2x, 2y$, $\nabla v = -2y, -2x$

$\nabla u \cdot \nabla v = 0$, hence they are orthogonal

Ideal fluid

- Zero velocity
- Steady state: $\frac{\partial}{\partial t} = 0$
- Incompressible $\frac{d\rho}{dt} = 0 = \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0$
- Irrotational $\nabla \times v = 0$
- Mass conservation $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 = \frac{\partial \rho}{\partial t} + \rho \nabla \cdot v + \nabla \rho \cdot v$
  - $\implies \nabla \cdot v = 0$
  - Velocity is solenoidal!

Lots of definitions:

- $v = \nabla \phi$
- Velocity $v = \nabla \times \psi$
- Curves for $\psi = \text{const}$ are called **streamlines**
- $\phi \rightarrow$ Velocity potential

- ψ → Stream function

Now, $v = \nabla \times \psi, v = \nabla\phi$

$\nabla^2\psi = 0, \nabla^2\phi = 0$

Define $\Omega = \phi + i\psi$

$\nabla^2\Omega = 0$

$\boxed{\Omega\text{'} = v_x - iv_y}$

Flow past an obstacle

Asymptotic behavior

- For $\frac{r}{a} \gg 1, \vec{v} \approx v_o\hat{x}, \Omega \approx V_0 z$
- For $\frac{r}{a} \to 1, v \approx \vec{v}_\theta\hat{\theta}, \Omega \approx 0$

The solution:

Guess $\Omega = v_0 z + \frac{v_0}{z}a^2$ → It has to be the only solution by uniqueness theorem

$\Omega\text{'}(z) = v_0(1 - \frac{a^2}{z^2})$

$v_x = v_0(1 - \frac{a^2}{r^2}\cos 2\theta)$

$v_y = v_0(\frac{a^2}{r^2}\sin 2\theta)$

Verifying asymptotic behavior

- As $\frac{r}{a} \to \infty$, $v_x = v_0, v_y = 0$
- As $\frac{r}{a} \to 1$, $v_x = 2v_0\sin^2\theta, v_y = 2v_0\cos\theta\sin\theta$, so it also satisfies!!

Curious Problem

Where is the field $\vec{A} = \frac{\hat{\theta}}{r}$ conservative?

At $r \neq 0$, $\nabla \cdot \vec{A} = 0, \nabla \times \vec{A} = 0$

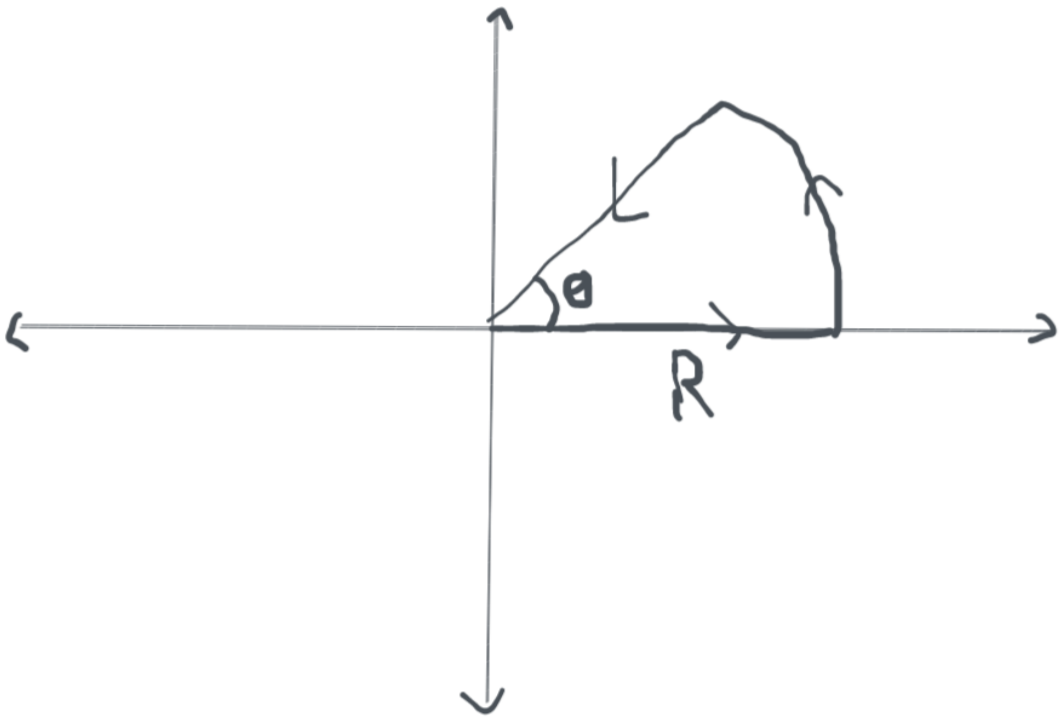
$\Omega = \phi + i\psi$, such that $\nabla\phi = \vec{A}, \nabla \times \psi\hat{z} = \vec{A}$

A is conservative wherever Ω is analytic

$\Omega = -i\ln(z)$, has discontinuity at $z = 0, z = \infty$, for branch $0 \leq \arg(z) < 2\pi$

Optics Problem: Fresnel integral

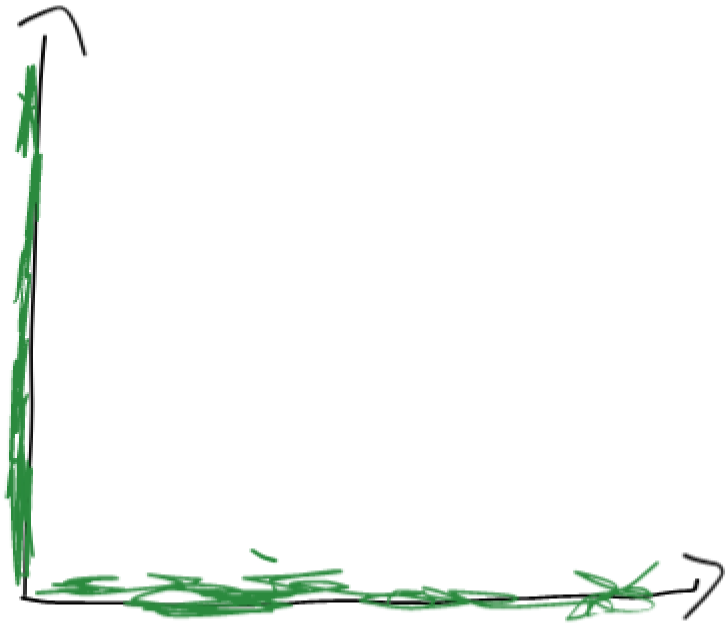
$\int_0^\infty \sin(x^2), \int_0^\infty \cos(x^2)$ is what?



- Choose the given contour above
- There are no residues inside, so $\oint_C e^{-z^2} dz = 0$
- Also, $\oint_C e^{-z^2} dz = \int_0^R e^{-x^2} dx + \int_0^{\pi/4} e^{-R^2 e^{2i\theta}} ie^{i\theta} d\theta + + \int_R^0 e^{ir^2} dre^{i\pi/4}$
- Now, take limit as $R \to \infty$
- $0 = \frac{\sqrt{\pi}}{2} - \int_0^\infty e^{ir^2} dr(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$
- $\int_0^\infty \sin(x^2) = \int_0^\infty \cos(x^2) = \frac{\sqrt{\pi}}{2\sqrt{2}}$

Conformal Mapping

Consider a conformal mapping



$w = z^2$ makes given boundary into straight line

Also note that $\Omega = Aw$ represent complex potential

So, $\Omega = Az^2$ would be the correct complex potential