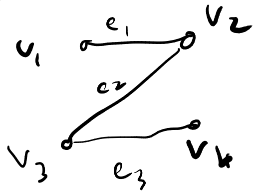
6. $\boxed{\text{10 marks}}$ **(Take Home Question.)** Consider two matroids $\mathcal{M}_1 = (U, \mathcal{I}_1)$ and $\mathcal{M}_2 = (U, \mathcal{I}_2)$ (over the same universe U). The *intersection* of $\mathcal{M}_1$ and $\mathcal{M}_2$ is defined as the pair $\mathcal{M} = (U, \mathcal{I})$, where $\mathcal{I} = \{A \subseteq U \mid A \in \mathcal{I}_1 \text{ and } A \in \mathcal{I}_1\}$.

Prove that (by constructing a small counter example) the intersection of two matroids (over the same universe) is not necessarily a matroid.

G //



On the universe $\mathcal{U} = \{e_1, e_2, e_3\}$, consider the following two independent sets: Let G be as defined as shown.

$\mathcal{I}_1 = \{E \subseteq \mathcal{U}$: At most one edge is incident from E on the vertices v_2 & v_4 in $G\}$

$\mathcal{I}_2 = \{E \subseteq \mathcal{U}$: At most one edge is incident from E on the vertices v_1 & v_3 in $G\}$

$\mathcal{I}_1 = \{\phi, \{e_1\}, \{e_1, e_3\}, \{e_2\}, \{e_2, e_3\}, \{e_3\}\}$

These are all the ind. sets in $M_1 = (\mathcal{U}, \mathcal{I}_1)$, and we can verify that the properties of matroids are satisfied.

Likewise, $\mathcal{I}_2 = \{\phi, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_3\}, \{e_1, e_2\}\}$

Clearly $M_1 \equiv (\mathcal{U}, \mathcal{I}_1)$ & $M_2 \equiv (\mathcal{U}, \mathcal{I}_2)$ are matroids.

Consider $\mathcal{I}_1 \cap \mathcal{I}_2 = \{\phi, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_3\}\}$. This is the set of <u>matchings</u> in G, which is a bipartite graph.

Now, in our 3rd matroid property:
Let $A = \{e_2\}$, $B = \{e_1, e_3\}$. $|A| < |B|$, but $\nexists\, e \in B \setminus A$ s.t. $A \cup B \in \mathcal{I}_1 \cap \mathcal{I}_2$. Thus, $(\mathcal{U}, \mathcal{I}_1 \cap \mathcal{I}_2)$ is NOT a matroid.

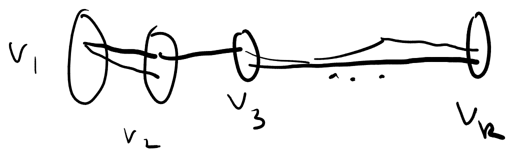
7. $\boxed{\text{10 marks}}$ **(Take Home Question.)** For a graph G, a path P in G is an *induced path* if there are no two distinct vertices $u, v \in V(P)$ such that $\{u, v\} \notin E(P)$ but $\{u, v\} \in E(G)$.

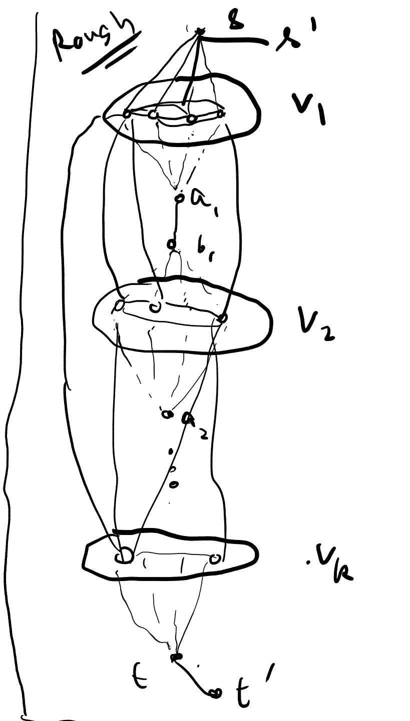
(LIP) In the LONG INDUCED PATH problem we are given a graph G and an integer k, and the goal is to check if G has an induced path on k vertices. Prove that this problem is W[1]-hard when parameterized by k.

Multicoloring Ind. Set [MIS] → As defined in class & book.

Consider a red^n from Multicolored Independent Set (MIS) to Long Induced Path. Let the MIS instance be $I = (G, k, (V_1, V_2 \ldots V_k))$ & LIP instance be (G', k').

Start off by assuming $G[V_i]$ is a clique $\forall i \in [k]$, since we're allowed to pick only one vertex in each V_i anyways this does not change the solution set of $(G, k, (V_1, V_2 \ldots V_k))$.





Construct a new graph G' as follows:

→ Have all the vertices & edges in G. Note that each $G[V_i]$, $i \in [k]$ can be made a clique

→ Create a node s & make edges from s to all vertices in V_1. Also create a node s' & the edge s'-s.

→ For every $i \in [k-1]$, create 2 nodes a_i, b_i. Connect edges from all vertices in V_i to a_i, and b_i to all vertices in V_{i+1}. Also add the edge a_i-b_i.

→ Create a node t & make edges from all vertices in V_k to t

→ Create a node t' & an edge $t - t'$

Set the LIP budget to be $k' = 3k + 2$.

Note: we consider a path P a set of vertices.

$\Rightarrow$ Consider an MIS S. We abuse notation & use $S \cap V_i$ to refer to the singular vertex in the set $S \cap V_i$ since it is multicolored set.

Consider the foll. path P

$= \{s', s, S\cap V_1, a_1, b_1, S \cap V_2, a_2, b_2, \ldots\ldots, S \cap V_k, t, t'\}$

Clearly $|P| = 3k+2$, so the size bound is satisfied.

Proof that it is an induced path: $i \in [k-1]$

All the $(S\cap V_i - a_i)$, $(a_i - b_i)$, $(b_i - S\cap V_{i+1})$ edges don't cause any induced path since those vertices are not incident on any other edges. Likewise the vertices s, t, t' also don't cause ind. paths.

The only problematic edges might be edges of type $S\cap V_i - S\cap V_j$, for some i, j. However, since S is an ind set in in G, such an edge does not exist in G'. Thus, P is a valid induced path.

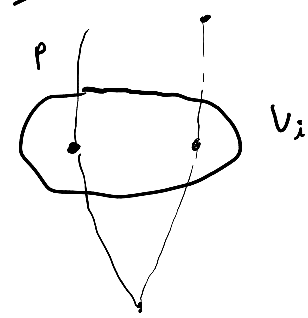
$\Leftarrow$ Let there be a path P of length $\geq 3k+2$. WLOG assume it is exactly length $3k+2$.

Lemma: $|V_i \cap P| = 1 \quad \forall\, i \in [k]$

By contradiction, Let $|V_i \cap P| \geq 1$. Since V_i is a clique, $|V_i \cap P| \leq 2$ [otherwise the path would not be induced].

By the same argument, $|(V_i \cup \{b_{i-1}, a_i\}) \cap P| \leq 3$ [since P is ind. path] $\forall\, i \in \{2, 3, \ldots, k-1\}$ — (1)

Rough



Likewise, $|(V_1 \cup \{s, a_1\}) \cap P| \leq 3$ & $|(V_k \cup \{t, b_{k-1}\}) \cap P| \leq 3$.

Since $|P| = 3k+2$, clearly $s' \in P$ & $t' \in P$.

$s' \in P \Rightarrow s \in P$.

Since $s \in P$, $\exists$ a vertex $v \in P \cap V_1$ and since s' & t' are connected, $a_1 \in P$. Likewise, $b_1 \in P$, likewise exactly one more vertex $\in (P \cap V_2)$ and so on. Applying this argument inductively & using $|P| = 3k+2$, we get $|P \cap V_i| = 1 \; \forall i \in [k]$.

Now consider foll. set, $I = \{v : v \in (P \cap V_i)$ for some $i \in [k]\}$
Clearly $|I \cap V_i| = 1 \; \forall i \in [k]$ in G, so the set is multicolored.

Assume by contradiction this is not an ind set. Then $\exists$ vertices u, v s.t. u & v have an edge between them in G.
But, the same u & v will have an edge in G' also, & thus P' is not an induced path in G' [u & v belong to different partitions].

Thus, I is a valid ind. set.

Since $\exists$ a parameterized redn. from MIS to LIPc we conclude that it is W[1]-hard.