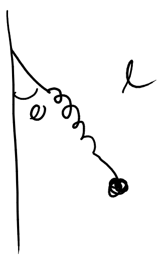
1. One of the characteristic of an inertial frame of reference is that space is *isotropic*. Show that the isotropy of space leads to the conservation of angular momentum for a closed system.

$\vec{\delta\phi} \rightarrow \quad \delta\vec{r} = \delta\vec{\phi} \times \vec{r} \}$

2. Consider a stretchable plane pendulum, described by a mass m suspended from a spring of spring constant k and relaxed (unstretched) length l_0, constrained to move in a vertical plane. Write down the Lagrangian and obtain the Euler-Lagrange equations.



$$T = \frac{1}{2} m \left(\dot{l}^2 + l^2 \dot{\theta}^2\right)$$

$$U = -mg\, l \cos\theta + \frac{1}{2} k (l - l_0)^2$$

$$\mathcal{L} = \frac{1}{2} m \left(\dot{l}^2 + l^2 \dot{\theta}^2\right) + mg\, l \cos\theta + \frac{1}{2} k (l - l_0)^2$$

$$\frac{d}{dt}\left(m l^2 \dot{\theta}\right) = -mgl \sin\theta$$

$$\frac{d}{dt}\left(l^2 \dot{\theta}\right) = -gl \sin\theta$$

$$l^2 \ddot{\theta} + 2 l \dot{l} \dot{\theta} = -gl \sin\theta$$

$$\boxed{l \ddot{\theta} + 2 \dot{l} \dot{\theta} = -g \sin\theta}$$

$$m l \dot{\theta}^2 + mg \cos\theta + k (l - l_0)$$

$$= \frac{d}{dt}(m \dot{l}) = m \ddot{l}$$

$$\boxed{\ddot{l} = l \dot{\theta}^2 + g \cos\theta + \frac{k}{m}(l - l_0)}$$

3. Consider a bead of mass m slides without friction in a uniform gravitational field on a vertical circular hoop of radius R. The hoop is constrained to rotate at a fixed angular velocity Ω about is vertical diameter. Let θ be the position of the bead on the hoop measured from its lowest point (see figure).

(a) Write down the Lagrangian $L(\theta, \dot{\theta})$.

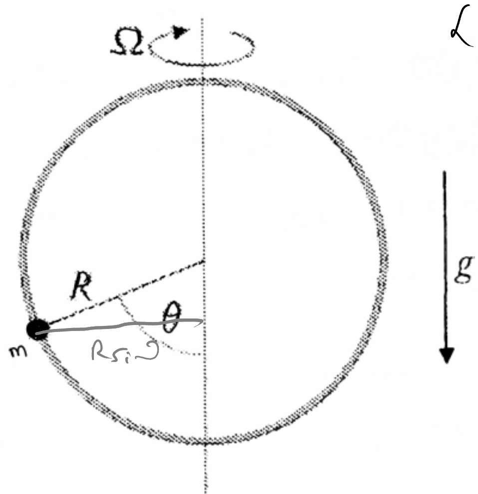
(b) Deduce the Euler-Lagrange equation of motion for the bead.

(c) Find how the equilibrium values of θ depends on Ω.
(Hint: Equilibrium occurs where $\ddot{\theta} = 0$).

$T = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2\theta\, \omega^2$

$V = -mgl\cos\theta$

$\mathcal{L} = \frac{1}{2} m R^2 \dot{\theta} + \frac{1}{2} m R^2 \sin^2\theta\, \omega^2 + mgl\cos\theta$



$-mgl\sin\theta + mR^2\omega^2 \sin\theta\cos\theta$

$= mR^2 \ddot{\theta}$

$-\frac{gl}{R^2}\sin\theta + \omega^2 \sin\theta\cos\theta = \ddot{\theta}$

$\ddot{\theta} = \omega^2 \sin\theta \left(\frac{-gl}{\omega^2 R^2} + \cos\theta \right)$

$\theta_c = \cos^{-1}\left(\frac{gl}{\omega^2 R^2}\right), 0, \pi$

Take $\theta = 0$

4. Consider a system consisting of n particles. If the kinetic energy is expressed in fixed rectangular coordinate, such as,

$$T = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{i}^{2} m_\alpha \dot{x}_{\alpha,i}^2$$

$m_\alpha \dot{x}_\alpha^2$

then show that

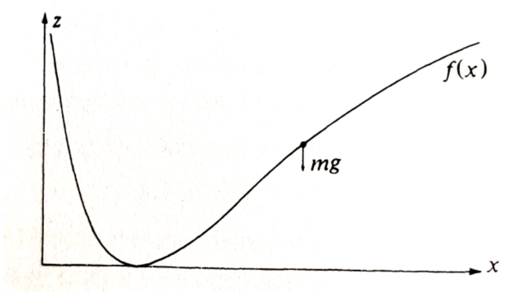
$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

Consider $\sum_{i=1,2,3}^{N} \dot{x}_{i\alpha} \frac{dT}{d\dot{x}_{i\alpha}} = \sum m_\alpha \dot{x}_\alpha^2 = 2T$

$z = f(x)$

$\dot{z} = f'(x)\dot{x}$

5. Find the Lagrangian of the motion of a bead of mass m sliding smoothly on a wire in the shape $z = f(x)$, the z and x-axis being respectively vertical and horizontal (see figure).



$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{z}^2\right)$

$V = mgz$

$$\boxed{\mathcal{L} = \frac{1}{2} m\left(\dot{x}^2 + f'(x)^2\dot{x}^2\right) - mg\,f(x)}$$

7. Consider a region of space divided by a plane. The potential energy of a particle in region 1 is U_1 and in region 2 is U_2. A particle of mass m and with speed v_1 in region 1 passes from region 1 to region 2. Determine the change in the direction of motion of the particle.

$\frac{dE}{dt} = -\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow E$ is constant

$U_1 + \frac{1}{2} m v_1^2 = U_2 + \frac{1}{2} m v_2^2$

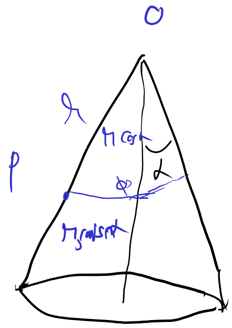
$U_1 - U_2 = \frac{1}{2} m (v_2^2 - v_1^2)$

$v_1 \sin\theta_1 = v_2 \sin\theta_2$

$\frac{2}{m}(U_1 - U_2) = v_2^2 - v_1^2$

$V_2 = \sqrt{v_1^2 + \frac{2}{m}(U_1 - U_2)}$

8. A particle P of mass m slides on the smooth inner surface of a circular cone of semi-angle α. The axis of symmetry of the cone is vertical with the vertex O pointing downwards. Take as generalised coordinates r , the distance OP, and ϕ, the azimuthal angle about the vertical through O. Obtain Lagrange's equations. Show that ϕ is a cyclic coordinate and identify the conserved momentum p_ϕ.



$$\left(r\sin\alpha\cos\phi,\ r\sin\alpha\sin\phi,\ r\cos\alpha \right)$$

$$\sin\alpha\left(\dot r\cos\phi - r\sin\phi\,\dot\phi\right),\ \sin\alpha\left[\dot r\sin\phi + r\cos\phi\,\dot\phi,\ \dot r\cos\alpha\right)$$

$$T = \frac{1}{2}m\left[\sin^2\alpha\left[\dot r^2 + r^2\dot\phi^2\right] + \dot r^2\cos^2\alpha\right]$$

$$= \frac{1}{2}m\left[\dot r^2 + r^2\sin^2\alpha\,\dot\phi^2\right] \qquad U = -mgr\cos\alpha$$

$$L = \frac{1}{2}m\left(\dot r^2 + r^2\sin^2\alpha\,\dot\phi^2\right) + mgr\cos\alpha$$

$$\boxed{m r^2\sin^2\alpha\,\dot\phi = \text{const}}$$

1. One of the characteristic of an inertial frame of reference is that space is *isotropic*. Show that the isotropy of space leads to the conservation of angular momentum for a closed system.

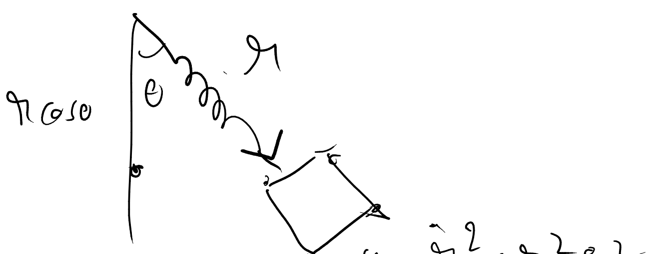
Consider θ, ϕ as coordinates.

Consider $\mathcal{L}$. Now, $\theta \to \theta + \delta\theta$, $\dot{\theta} \to \theta$

$$\delta L = \left(\frac{\partial \mathcal{L}}{\partial \theta}\right) \delta\theta = \underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)}\delta\theta = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \text{constant} \rightarrow \text{Angular momentum}$$

2. Consider a stretchable plane pendulum, described by a mass m suspended from a spring of spring constant k and relaxed (unstretched) length l_0, constrained to move in a vertical plane. Write down the Lagrangian and obtain the Euler-Lagrange equations.



$v = \dot{r}^2 + r^2\dot{\theta}^2$

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - \frac{1}{2}k(r-l)^2$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgr\sin\theta = \frac{d}{dt}\left[r^2\dot{\theta}\right] = r^2\ddot{\theta} + 2r\dot{\theta}\dot{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} = mg\cos\theta - k(r-l) + \frac{1}{2}m\dot{\theta}^2 r = m\ddot{r}$$

3. Consider a bead of mass m slides without friction in a uniform gravitational field on a vertical circular hoop of radius R. The hoop is constrained to rotate at a fixed angular velocity Ω about is vertical diameter. Let θ be the position of the bead on the hoop measured from its lowest point (see figure).

(a) Write down the Lagrangian $L(\theta, \dot{\theta})$.

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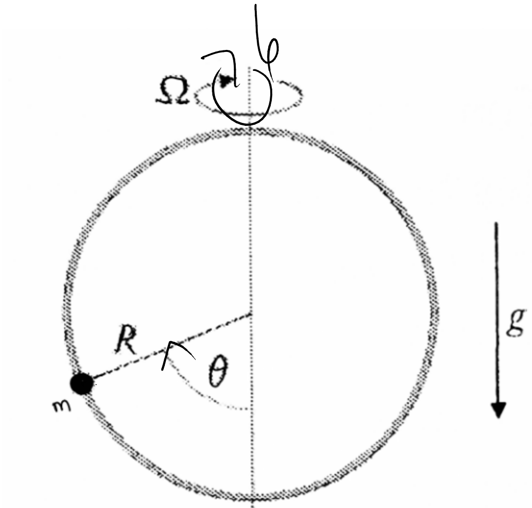
(c) Find how the equilibrium values of θ depends on Ω.
(Hint: Equilibrium occurs where $\ddot{\theta} = 0$).



$$\mathcal{L} = \frac{1}{2} m \left[R^2 \dot{\theta}^2 + \Omega^2 R^2 \sin^2\theta \right] + mgR\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2} m \, \Omega^2 R^2 \, 2 \sin\theta \cos\theta - mgR\sin\theta$$

$$= mR^2 \ddot{\theta}$$

c) $\cancel{\tfrac{1}{2}} m \Omega^2 R^2 \sin\theta\cos\theta = mgR\sin\theta$

$$\Omega^2 R\cos\theta = g$$

$$\boxed{\theta = \cos^{-1} \frac{g}{\Omega^2 R}}$$

4. Consider a system consisting of n particles. If the kinetic energy is expressed in fixed rectangular coordinate, such as,

$$T = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{i}^{3} m_\alpha \dot{x}_{\alpha,i}^2$$

then show that

$$\sum_{j=1}^{3n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

$$\frac{\partial T}{\partial \dot{x}_{(\alpha,i)}} = m_\alpha \dot{x}_{(\alpha,i)}$$

$$\sum_{\alpha=1}^{n}\sum_{i=1}^{3} \dot{x}_{\alpha,i} \frac{\partial T}{\partial \dot{x}_{(\alpha,i)}} = 2\sum_{\alpha=1}^{n}\sum_{i=1}^{3} \frac{1}{2} m_\alpha x_{\alpha,i}^2 = 2T.$$

Now, go from $(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \cdots x_{N1}, x_{N2}, x_{N3})$

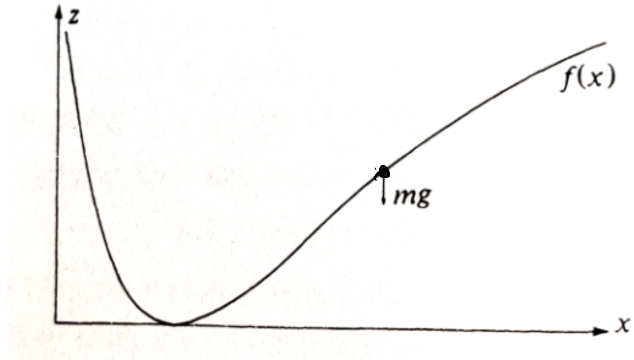
$$\Downarrow$$

$(q_1, q_2, q_3 \quad \cdots \quad q_{3N})$

$2T$ is a scalar, invariant under coord transform.

$$\Rightarrow \sum_{\alpha=1}^{n}\sum_{i=1}^{3} \dot{x}_{\alpha,i} \frac{\partial T}{\partial \dot{x}_{(\alpha,i)}} = \boxed{\sum_{j=1}^{3n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T}$$

5. Find the Lagrangian of the motion of a bead of mass m sliding smoothly on a wire in the shape $z = f(x)$, the z and x-axis being respectively vertical and horizontal (see figure).



$z = f(x)$

$\dot{z} = \frac{dz}{dt} = \frac{df(x)}{dt}$

$= f'(x)\dot{x}$

$\mathcal{L}(x, \dot{x}) = \frac{1}{2} m (\dot{x}^2 + \dot{z}^2) - (mgz)$

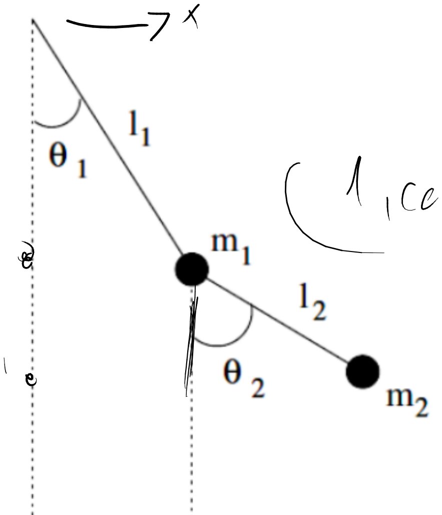
$= \frac{1}{2} m (\dot{x}^2 + (f'(x))^2 \dot{x}^2) - mg f(x)$

$= \frac{1}{2} m \dot{x} [1 - f'(x)^2] - mg f(x)$

6. A double pendulum consisting of two simple pendula is shown in figure. If both the pendula are confined to move in the same plane, find the Lagrange's equation of motion for the system. (Neglect the mass of the rods)

$\rightarrow x$

$y \downarrow$



$l_1 \sin\theta_1$

$(l_1 \cos\theta_1 + l_2 \cos\theta_2)$

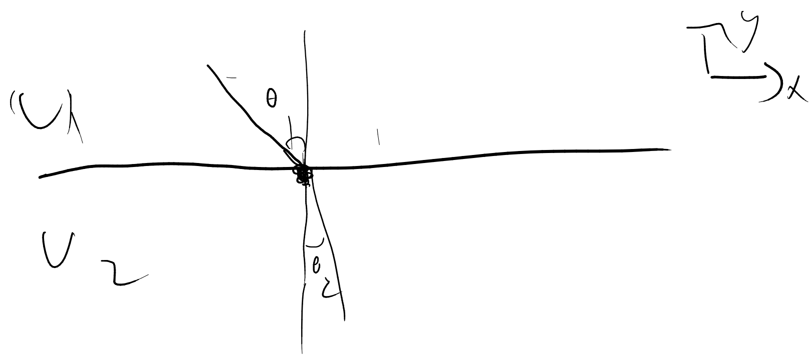
$T = \frac{1}{2} \left[m_1 l_1^2 \dot{\theta}_1^2 + m_2 [l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2]^2 + m_2 (l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 \right.$

$T = \frac{1}{2} [m_1 l_1^2 \dot{\theta}_1^2 + m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))]]$

$U = -m g l_1 \cos\theta_1 - m_2 g l_1 \cos\theta_1 - m g l_2 \cos\theta_2$

$$\mathcal{L} = \frac{1}{2}\left[m_1 l_1^2 \dot{\theta}_1^2 + m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]\right]$$
$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

7. Consider a region of space divided by a plane. The potential energy of a particle in region 1 is U_1 and in region 2 is U_2. A particle of mass m and with speed v_1 in region 1 passes from region 1 to region 2. Determine the change in the direction of motion of the particle.



$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - U(x) \implies \boxed{\dot{y} = \text{constant}}$$

$$m\ddot{x} = -\frac{dU}{dx} \qquad \frac{dU}{dx}$$

$$\implies \boxed{v_1 \sin\theta_1 = v_2 \sin\theta_2}$$

Also, $U_1 + \frac{1}{2} m v_1^2 = U_2 + \frac{1}{2} m v_2^2 = E$

$v_1 \propto E - U_1$

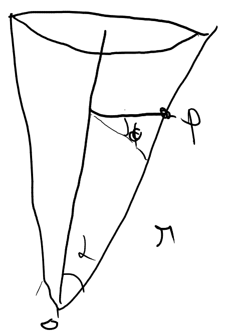
$$U_1 - U_2 = \frac{1}{2} m \left[v_2^2 - v_1^2\right]$$

$$U_1 - U_2 = \frac{1}{2} m \left[\frac{v_1^2 \sin^2\theta_1}{\sin^2\theta_2} - v_1^2\right]$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{E - U_2}{E - U_1}} = \sqrt{\frac{E - U_2}{\frac{1}{2} m v_1^2}}$$

$$\sin\theta_2 = \sin\theta_1 \sqrt{\frac{E - U_2}{\frac{1}{2} m v_2}}$$

8. A particle P of mass m slides on the smooth inner surface of a circular cone of semi-angle α. The axis of symmetry of the cone is vertical with the vertex O pointing downwards. Take as generalised coordinates r , the distance OP, and ϕ, the azimuthal angle about the vertical through O. Obtain Lagrange's equations. Show that ϕ is a cyclic coordinate and identify the conserved momentum p_ϕ.



$$\mathcal{L} = \frac{1}{2} m \left[\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2 \right] - mg r \cos\alpha$$

$m r^2 \sin^2\alpha \, \dot{\phi}$ is constant $= p_\phi$