1. Determine whether the following mappings are linear transformation

✓ (a) $T: \mathbb{R}^2 \to \mathbb{R}^2$, with

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

$$T\left(\alpha\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a + c \\ \alpha b + d \end{bmatrix}\right) = \begin{bmatrix} \alpha a + c + \alpha b + d \\ \alpha b + d \end{bmatrix} = \alpha\begin{bmatrix} a+b \\ b \end{bmatrix} + \begin{bmatrix} c+d \\ d \end{bmatrix} = \alpha T\begin{pmatrix} a \\ b \end{pmatrix} + T\begin{pmatrix} c \\ d \end{pmatrix}$$

X (b) $T: \mathbb{R}^2 \to \mathbb{R}^2$, with

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \not\to \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Superposition not obeyed

✓ (c) Fix an $m \times n$ matrix A. Then, let $T: M_{lm} \to M_{ln}$, with

$$T(B) = BA$$

$$T(\alpha B + C) = (\alpha B + C)A = \alpha BA + CA = \alpha T(B) + T(C)$$

X (d)

$$T(f) = \begin{bmatrix} f(0) \\ f(1)+1 \end{bmatrix}$$

Let $f = 0$

$$T(f) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Zero not mapped to zero

Not linear

2. Let $T: \overset{(n)}{V} \to \overset{(m)}{W}$ be linear. Prove the following.

Let n be dim(w)

(a) If $\dim(V) < \dim(W)$, ~~T can be one-to-one~~, but not onto.

(b) If $\dim(V) > \dim(W)$, ~~T can be onto~~, but not one-to-one.

$(n) \qquad (m)$

a)

$$\dim(R(T)) + \dim(N(T)) = n < m$$

Now, $\dim N(T) \geq 0$

$$\dim(R(T)) < m = \dim(W)$$

Since $\dim(R(T)) < \dim(W)$, $\exists\, w \in W : T(v \in V) \neq w \;\; \forall v$

b) $\dim R(T) + \dim N(T) = n$ $\qquad \therefore \dim R(T) > m$

$$\dim N(T) = n - \dim R(T) > n - m > 0$$

$$\boxed{\dim N(T) > 0}$$

3. Find the Kernel and Range of the following linear transformations. Indicate whether it is one-to-one, Onto.

(a) $T((x_1, x_2, x_3)) = (x_1, 0, 0)$

(b) $T(a + bx) = ax + \frac{1}{2}bx^2$

a) Kernel $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = 0 \Rightarrow x_1 = 0$, x_2, x_3 arbitrary

$\therefore$ Kernel is span $\{(0,1,0), (0,0,1)\}$

Range $\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \to$ Span $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

Not one-one, as $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ are mapped to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Not onto, as $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$ but $\nexists\, v : T(v) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(b) $T(a + bx) = ax + \frac{1}{2}bx^2$

Kernel $ax + \frac{1}{2}bx^2 = 0 \Rightarrow a = 0, b = 0.$

$T(v) = 0 \Rightarrow v = 0$, $\therefore$ Kernel $= \{0\}$

Range $\{ax + \frac{1}{2}bx^2,\ a, b \in \mathbb{R}\}$

one-one

$\boxed{\text{Onto} \to ?}$

4. Let $T : V \to W$ be defined as $T((x_1, x_2)) = (x_1 + 2x_2, x_1 - x_2)$. Determine the representation matrix if

(a) The basis $\mathcal{B}_V$ and $\mathcal{B}_W$ are the standard basis

(b) $\mathcal{B}_V$ is the standard basis and $\mathcal{B}_W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

a) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

b) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1 + 2x_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (x_1 - x_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$2x_1 + 4x_2 - x_1 + x_2$

$= \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 5x_2 \end{bmatrix}$

$T = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$

5. Given a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$T\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}, \quad T\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$$

Find

$$T\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{1}{2} T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2.5 \\ 3.5 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{2} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$