1. A particle P of mass m moves under the simple harmonic force field,

$$\mathbf{F} = -(m\Omega^2 r)\hat{\mathbf{r}},$$

where Ω is a positive constant. Obtain the radial motion equation and show that all orbits of P are bounded.

$V = \frac{1}{2} m\Omega^2 r^2$ $\qquad L = \frac{1}{2} m\dot{r}^2 - \frac{1}{2} m \Omega^2 r^2$

$L = \sum_{i=1,2,3} \frac{1}{2} m \dot{x}_i^2 - \frac{1}{2} m\Omega^2 x_i^2$

$m\ddot{x}_i = -m\Omega^2 x_i$

$\ddot{x}_i + \Omega^2 x_i = 0 \qquad x_i = A\sin\Omega t + B\cos\Omega t$

$\vec{r} = \sum x_i e_i$

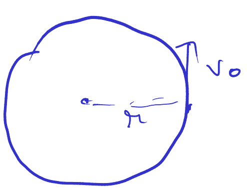
$= \boxed{\vec{A}\sin\Omega t + \vec{B}\cos\Omega t} \leq |A| + |B|$

Bonded

2. Two particles move about each other in circular orbits under the influence of gravitational forces (inverse square force). Let the time time period be T. Imagine that the motion of both the particles is suddenly stopped and they are released. Naturally, the particles will fall into each other. Prove that they collide after a time $T/4\sqrt{2}$.

$\frac{mv^2}{r} = \frac{GMm}{r^2}$

$\boxed{v = \sqrt{\frac{GM}{r}}}$



$v = r\omega$

$\omega = \frac{v}{r} = \sqrt{\frac{GM}{r^3}}$

$\ddot{r} = -\frac{GM}{r}$ $\qquad \dot{r} = GM \ln(r)$

$\int \ddot{r}\frac{\dot{r}}{dr} = \int -\frac{GM}{r}\dot{r}\, dr$

$\frac{\dot{r}^2}{2} = -GM\ln(r)$

1. A particle P of mass m moves under the simple harmonic force field,

$$\mathbf{F} = -(m\Omega^2 r)\hat{\mathbf{r}},$$

where Ω is a positive constant. Obtain the radial motion equation and show that all orbits of P are bounded.

$\vec{F} = -m\Omega^2 \vec{r}$

$U(r) = \boxed{\frac{1}{2} m\Omega^2 r^2}$

$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} m\Omega^2 (x^2 + y^2 + z^2)$

$\frac{\partial \mathcal{L}}{\partial x} = -\frac{1}{2} m\Omega^2 2x = m\ddot{x}$

$m\ddot{x} = -m\Omega^2 x$

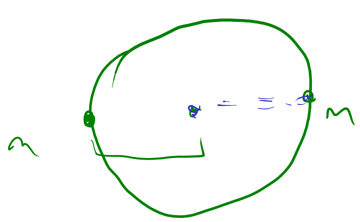
$\ddot{x} + \Omega^2 x = 0 \qquad x = A\sin(\Omega t) + B\cos\Omega t$

Likewise for y & z.

$\vec{r} = \vec{A}\sin\Omega t + \vec{B}\cos\Omega t$

$|\vec{r}| \le |\vec{A}| + |\vec{B}|$ Bounded

2. Two particles move about each other in circular orbits under the influence of gravitational forces (inverse square force). Let the time time period be T. Imagine that the motion of both the particles is suddenly stopped and they are released. Naturally, the particles will fall into each other. Prove that they collide after a time $T/4\sqrt{2}$.



$T^2 \propto r^3$, $\quad T' = \frac{T}{2\sqrt{2}}$, $\quad T_{req} = \frac{T'}{2} = \boxed{\frac{T}{4\sqrt{2}}}$

3. Consider a particle moving in an elliptical orbit in a central inverse-square-law force field. By explicitly calculating the time averages (i.e. the average over one complete period) of potential energy and kinetic energy, verify the virial theorem.

$$\frac{\alpha}{r} = 1+\varepsilon\cos\theta \qquad \boxed{r = \frac{\alpha}{1+\varepsilon\cos\theta}} \qquad \boxed{\alpha = \frac{\ell^2}{mk}}$$

$$\dot\theta = \frac{\ell}{mr^2} = \frac{\ell}{m}\left(\frac{1+\varepsilon\cos\theta}{\alpha}\right)^2 \qquad \dot\theta = \frac{d\theta}{dt} \qquad dt = \frac{d\theta}{\dot\theta}$$

$$K.E = \frac{1}{2} m \frac{\alpha\,\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}$$

$$\langle T\rangle = \int_0^{2\pi} \frac{1}{2} m\alpha \frac{\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}\,\frac{d\theta\;\alpha^2 m}{\ell(1+\varepsilon\cos\theta)^2}$$

$1+\varepsilon\cos\theta = u$

$-\varepsilon\sin\theta\, d\theta = du$

$-u^{-4}$

$$= \int_{1+\varepsilon}^{1-\varepsilon} \frac{1}{2}\frac{m^2\alpha^3}{\ell}\,\frac{-du}{u^4} = \frac{1}{2}\frac{m^2\alpha^3}{\ell}\,\frac{1}{3}\left[\frac{1}{u^3}\right]_{1+\varepsilon}^{1-\varepsilon}$$

$$\langle U\rangle = \int_0^{2\pi} -k\left(\frac{1+\varepsilon\cos\theta}{\alpha}\right)\frac{d\theta}{\dot\theta} = \int_0^{2\pi} -k\,\frac{\cancel{(1+\varepsilon\cos\theta)}}{\alpha}\,\frac{d\theta\;\alpha^2 m}{\ell(1+\varepsilon\cos\theta)^{\cancel{2}}}$$

$$= -\frac{k\alpha m}{\ell}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta} = \frac{k\alpha m}{\ell}\,\frac{1}{\varepsilon}\int_0^{2\pi}\frac{d\theta}{\frac{1}{\varepsilon}+\cos\theta}$$

$$= \frac{k\alpha m}{\ell}\,\frac{2\pi}{\sqrt{1-\varepsilon^2}} = \frac{k\alpha m}{\ell}\,2\pi\,\frac{1}{\sqrt{1-\varepsilon^2}}$$

5. Find the force law for a central-force field, that allows a particle to move in a spiral orbit given by, $r = k\theta^2$, where k is a constant.

$$\sqrt{\frac{r}{k}} = \theta$$

$$\left(\sqrt{\frac{1}{k}}\right)\frac{\dot{r}}{2\sqrt{r}} = \dot{\theta} = \frac{L}{\mu r^2}$$

$$\dot{r} = 2\sqrt{k}\,\frac{L\sqrt{r}}{\mu r^2} = \left(\frac{2\sqrt{k}L}{\mu}\right) r^{-3/2}$$

$$\frac{dr}{dt} = \dot{r} = \alpha\, r^{-3/2}$$

$$\int r^{3/2}\,dr = \alpha\int dt$$

$$\frac{r^{5/2}\,2}{5} = \alpha(t + c)$$

$$r = \left(\frac{5}{2}\alpha(t+c)\right)^{\frac{2}{5}}$$

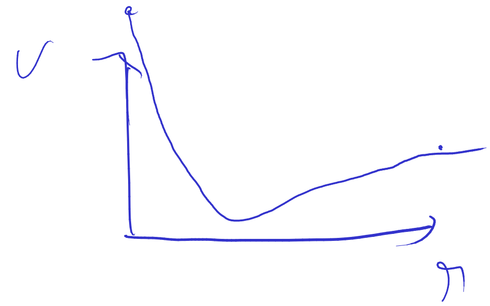
$$t^{-3/5}$$

$$\ddot{r} = \left(\frac{5}{2}\alpha\right)^{2/5} \frac{2}{5}\left(\frac{-3}{5}\right)(t+c)^{-8/5}$$

6. Find and comment on the stability of circular orbits in a force field described by the potential

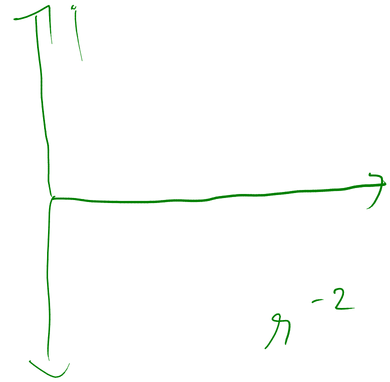
$\frac{1-\frac{r}{a}}{-r} = -\frac{1}{r} - \frac{1}{a}$

$$U(r) = \frac{-k}{r} e^{(-r/a)}$$



where, $k > 0$ and $a > 0$.

$$U_{eff} = -\frac{k}{r} e^{-r/a} + \frac{\ell^2}{2\mu r^2}$$



Turning pt $\rightarrow$ $\frac{\partial U}{\partial r} = 0$

r^{-2}

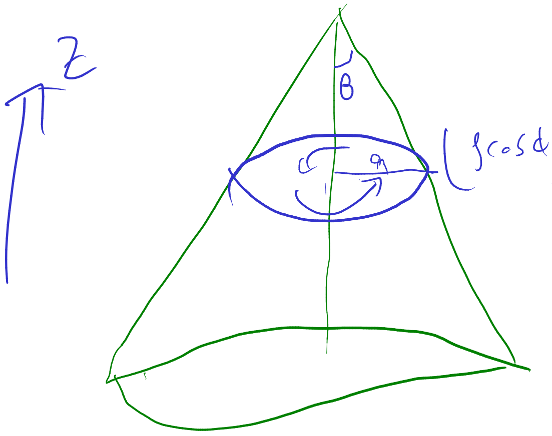
r^{-1}

$= -1$

$$-k\left[\frac{r e^{-r/a}\left(-\frac{1}{a}\right) - e^{-r/a}}{r^2}\right] - \frac{\ell^2}{2\mu}\frac{r}{r^2} = 0$$

$$+k 2\mu\left[+\frac{r}{a} + 1\right] e^{-r/a} - \ell^2 r = 0$$

7. Determine whether a particle moving on the inside surface of a cone under the influence of gravity can have stable circular orbit



$$\frac{\rho}{z} = \tan\theta$$

$$\mathcal{L} = \frac{1}{2} m\left[\rho^2\dot{\phi}^2 + \dot{z}^2\right] - mgz$$

$$z^2 \tan^2\theta\, \dot{\phi}^2 + \dot{z}^2\Big] - mgz$$

8-3. A particle moves in a circular orbit in a force field given by

$$F(r) = -k/r^2$$

Show that, if k suddenly decreases to half its original value, the particle's orbit becomes parabolic.

$$\frac{k}{r} = 1 + e \cos\theta$$

$$U = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$

Now at circular orbit, $\frac{\partial U}{\partial r} = 0$

$$-\frac{k}{r^2} + \frac{l^2}{\mu r^3} = 0$$

$$\frac{k}{\cancel{r^2}} = \frac{l^2}{\mu r^{\cancel{3}}}$$

$$\boxed{r_0 = \frac{l^2}{\mu k}}$$

$$U(r_0) = -\frac{k}{l^2}\mu k + \frac{\cancel{l^2}\,\mu^2 k^2}{2\cancel{\mu}\, l^{\cancel{4}}} = \boxed{-\frac{k^2 \mu}{2 l^2}}$$

$$\frac{mv^2}{\cancel{r}} = \frac{+k}{\cancel{r}} \qquad \boxed{\frac{1}{2} m v^2 = \frac{k}{2}}$$