> Haricharan

Signals and Systems

3rd May

Signal

Some definitions:

Signal: Mathematical function that depends on some variables

Classification

- Continuous or Discrete time
- Digital or analog signal: has to do with the amplitude of the signal
  - Digital: amplitude can take discrete values. With "M" different values, it's called an M-ary signal
  - Analog: amplitude can take any value
- Even or odd signal
- Conjugate symmetry x(t) = x*(t)
  - Real part is even, Imaginary part is odd
- Periodic and aperiodic:
  - x(t) = x(t + T0), $\omega = \frac{2\pi}{T_0}$
  - x[n] = x[n + N], $\Omega = \frac{2\pi}{N}$
- Causal Signal:
  - x(t) = 0 for t < 0
  - Anticausal: x(t) = 0 for t >= 0
- Deterministic or Random
  - Deterministic can be predicted
  - Random cannot be predicted

Power and Energy

For current and resistor:

$$P(t) = i(t)v(t)$$

$$E = \int_{t_1}^{t_2} P(t)dt$$

Analogously, for any signal x(t):

$$P(t) = |x(t)|^2$$

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt$$

$$P_{\text{avg}} = \frac{E}{t_2 - t_1}$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t \qquad E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t \qquad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

If energy is finite, power is zero and is called **energy signal**.
Eg.

1. x(t) is 1 if 0 < t < 1, 0 otherwise

   $$E_\infty = 1, P_\infty = 0$$

   If energy is infinite, power is finite, it is called **power signal**
   Eg.
2. x[n] = 2 for all n

   $$P_\infty = 4, E_\infty = \infty$$

We can have neither power nor energy signal also (Ramp signal).
Eg.
3. x(t) = t

$$E_\infty = \infty, P_\infty = \infty$$

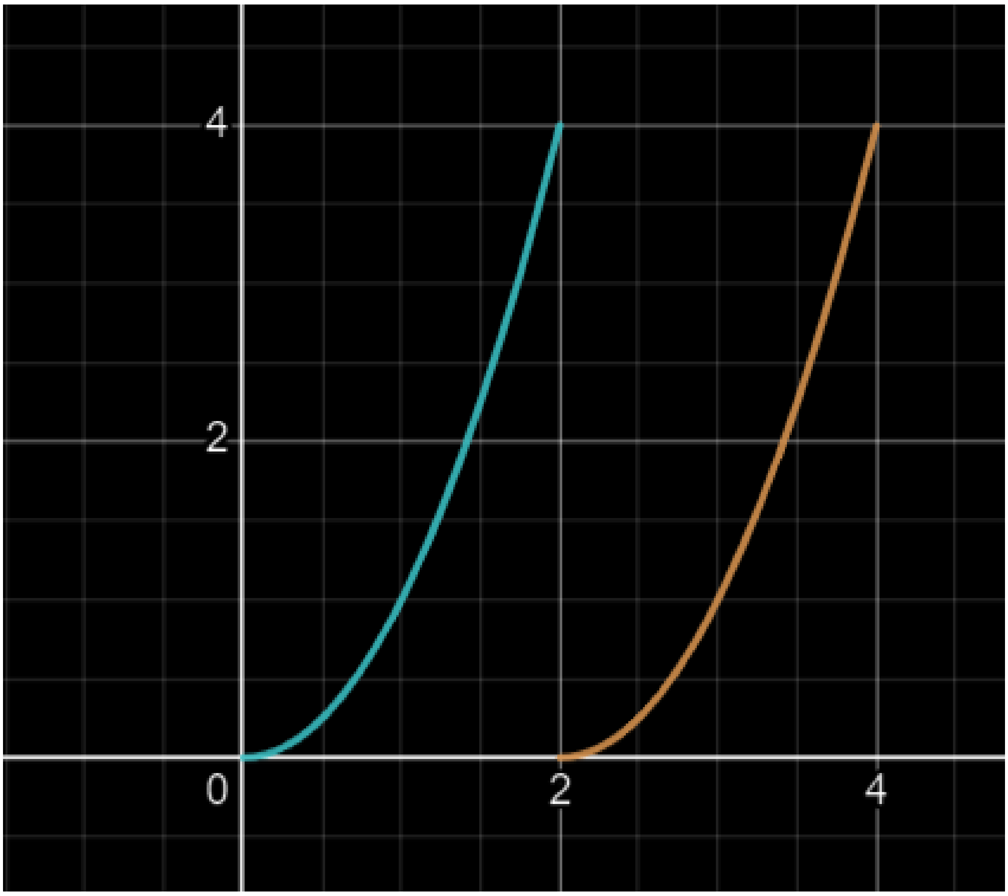
Signal Transformation

Time shifting

x(t) → x(t - T)
T units delayed

x(t) → x(t + T)
T units advanced

Eg.



Time Scaling

CT

- x(a t) squishes the signal by a times
- x(t / a) stretches the signal by a times
- x(-t) → reflection about y axis

DT:

- x (at) squishes the signal, but they are losing information. For eg, x [3 n]: x[1], x[2], x[4], x[5], x[6], etc. are thrown away.
- x (at) stretches the signal. For eg, x[n/2] would stretch the signal.
    - However, what do we do for intermediate values? We *Interpolate* the signal.

Amplitude Scaling:

- Stretch about y axis
-
    - x(t) flip about x axis

Elementary Signals

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1. Exponential

$\bar{A}e^{\bar{s}t} = Ae^{j\phi}e^{(\sigma+j\omega)t}$
$\bar{s}$ → complex frequency

Real Signal = x(t) + x*(t)*
Imaginary Signal = x(t) - x(t)

s can be thrown into the complex plane

Depending on sign of σ:

- Anything in the left-half plane is decaying exponential
- Anything on imaginary axis is neither growing nor decaying, i.e. it is pure sinusoid
- Anything on right half plane is growing exponential

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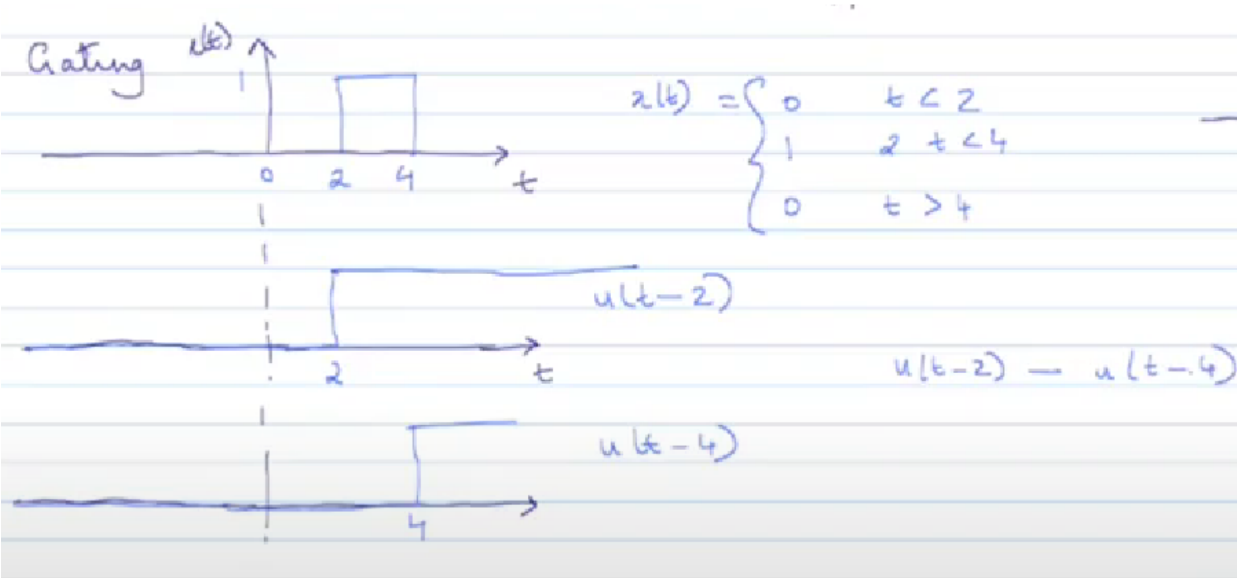
2. Unit step function or heavyside step function
CT: u(t) = 1 for t > 0, 0 for t < 0
DT: u[n] = 1 for t $>=$ 0, 0 for t < 0

- Make any signal into a causal signal: x(t) u(t) is causal.
- Gating:



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3. Unit impulse or Unit Delta or Dirac delta function: a generalized function

- $\delta(x) = \delta(-x)$
- Sifting property: $\int_{-\infty}^{\infty} f(t)\delta(t-T)dt = f(T)$

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4. Unit Ramp function
r(t) = t u(t)

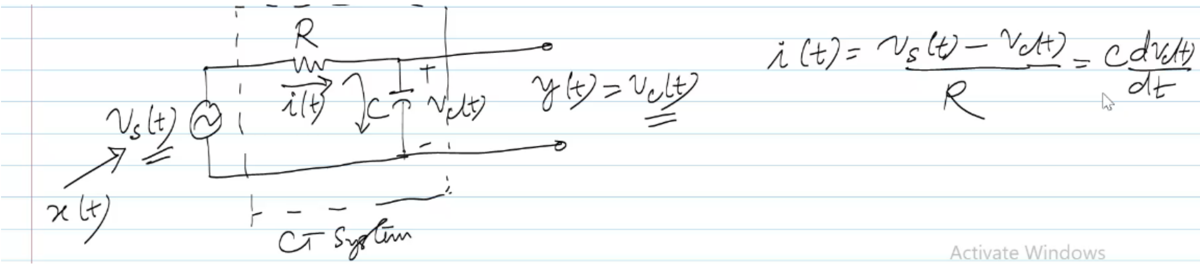
$$\delta(t) = \frac{du(t)}{dt} = \frac{d^2r(t)}{dt^2}$$

System

Continuous Time System

Input Signal → System → Output Signal
x (t) y(t)

Eg. Some electric Circuit:



$V_s = x(t)$

$$i = \frac{V_s - V_c}{R} = C\frac{dV_c}{dt}$$

$$\frac{dV_c}{dt} + \frac{V_c}{RC} = \frac{V_s}{RC}$$

Eg. Cart: This is similar to cart with friction: x(t) is the external force, a y(t) corresponds to friction.

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

This is first-order system!

Discrete Time system

Input Signal → System → Output signal
x[n] y[n]
$n = 0, \pm 1, \pm 2, \ldots$

Eg. Bank monthly balance:

- y[n] is monthly balance
- x[n] is net amount deposited = Deposit - Withdrawal
- 1% Interest also

$$y[n] = x[n] + 1.01y[n-1]$$

This is a difference equation

Eg. For the same cart, it can be represented as DT also, where time would be like Δn.

$\frac{dy(t)}{dt} + ay(t) = bx(t)$ becomes

$$\frac{v[n] - v[n-1]}{\Delta n} + \frac{\rho v[n]}{m} = \frac{f[n]}{m}$$

For CT → Differential equation, For DT → Difference equation

> **Imp**
>
> If we come up with methods to solve one class of system, we can do same thing for other systems in same class.

Classification of systems

Interconnection of subsystems, devices, to transform input into output

- CT and DT systems
- Systems with and without memory
    - Without memory:
        - Resistor system: v(t) = i (t) R
        - Output depends only on present value
    - With memory:
        - Depends on present/future
        - It "stores" energy!
        - capacitor system: $v(t) = \frac{\int_{-\infty}^{t} i(t)dt}{C}$
        - summer: $y[n] = \sum_{k=-\infty}^{n} x[k]$ → computes running average
- Invertible and non-invertible
    - Invertible: If there exists S2 for S1, such that x(t) → S1 → y(t) → S2 → x(t), S1 is said to be invertible
        - Eg. y(t) = 2 x(t)
    - Non-invertible
        - Eg. y(t) = 0
- Causal and Non-causal *system* (*not signal*)
    - Causal:
        - Output at any time depends on values from present or past times *not future*
        - Eg: y(t) = x(t) cos (t + 1)

- Non-causal
    - Eg. y[n] = x[n] - x[n + 1]
- Stability
  - Stable system:
    - Small input leads to responses which do not diverge
    - If |x(t)| < B_x < ∞ implies |y(t)| < B_y < ∞
    - BIBO → Bounded input, bounded output
- Time invariance
  - Time invariant: If behaviour and characteristics do not vary with time
    - $x(t) \to y(t) \implies x(t - T_0) \to y(t - T_0)$
    - x(t) → S → Delay → y(t - T_0) = x(t) → Delay → x(t - T_0) → S → y(t - T_0)
    - Eg. y(t) = sin (x(t))
    - No explicit time dependence
  - Time variant
    - Eg. y(t) = n (x(t))
    - Explicit Time dependence
- Linearity
  - Linear system:
    - a x1(t) + b x2(t) → S → a y1 (t) + b y2(t)
    - Eg. y(t) = t x(t)
  - Non-linear
    - Eg. y(t) = m x(t) + c

We're worried only about LTI systems in this course.

11th May

Convolution of two functions:
$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$

Imagine a DT system that is linear

Sifting property

$$\sum_{-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

Consider LTI system

- x[k] → S → y[k]
- $\delta[n-k]$ → S → $h_k[n] = h[n-k]$
- $x[n] = \sum_{-\infty}^{\infty} x[k]\delta[k] \to \sum_{-\infty}^{\infty} x[k]h[n-k] = y[n]$
  This is called the superposition sum or convolution sum → $y[n] = \sum_{-\infty}^{\infty} x[n]h[n-k]$
  where $h[n]$ is the unit impulse response.

$y[n] = x[n] * h[n]$

CT LTI system

> **Wow!**
>
> The response of the system is determined by unit impulse response
>
> - We can give an unit impulse, see what it does, then convolve x[n] to figure out y[n]!

$\delta_{\Delta}(t-k_0) \to h_{k\Delta}(t)$
$x(t) = \sum_{-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k_0)\Delta$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

Using Linearity:
$\sum_{\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k) \to \sum_{\infty}^{\infty} x(k\Delta)h_{k\Delta}(t)$

Using time invariance:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(\tau) * h(t)$$

17th May

Properties:

- **Commutative property**
  x[n] *h[n] = h[n]* x[n]

$y[n] = \sum_{-\infty}^{\infty} x[k]x[n-k] = \sum_{-\infty}^{\infty} x[n-k]x[k]$
(Property of reversing summation)

- **Distributive property**
  $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
  Proof:
  $y[n] = \sum_{-\infty}^{\infty} x[k](h_1[n-k] + h_2[n-k]) == \sum_{-\infty}^{\infty} x[k]h_1[n-k] + \sum_{-\infty}^{\infty} x[k]h_2[n-k] = y_1[n]$ -
- **Associative property**
  $(x[n] * h1[n]) * h2[n] = x[n] * (h1[n] * h2[n])$

x(t) → S1 → y1(t) → S2 → y2(t)
$\delta[n] \to h_1[n]$, $\delta[n] \to h_2[n]$

Using commutativity,
$(x[n] * h_1[n]) * h_2[n] = (x[n] * h_2[n]) * h_1[n]$

If we're passing a signal x(t) through many LTI systems, order doesn't matter.

- **For memoryless systems**
  $h[n] = A\delta[n]$
  $y[n] = \sum_{-\infty}^{\infty} x[k]A\delta[n-k] = Ax[n]$

- **Invertibility**
  $x[n] * (h_1[n] * h_2[n]) = x[n] = x[n] * \delta[n]$
  $\implies h_1[n] * h_2[n] = \delta[n]$

- **Causality**
  $h[n] < 0 \text{ for } n < 0$
  Reason→ Unit impulse function is 0 for n < 0, so as it is causal, unit impulse response is also 0 for n < 0

- **Stability**
  Consider giving a Bounded input x[n] < B.
  $y[n] = \sum_{-\infty}^{\infty} x[k]h[n-k] < B\sum_{\infty}^{\infty} |h[k]|$
  $\sum\limits_{-\infty}^{\infty} |h[k]| < \infty$

18th May

Unit Step Response:

u(t) → System → s(t)

s(t) = u(t) *h(t)*
= *h(t)* u(t)

s[n] = h[n] * u[n]
= $\sum_{-\infty}^{\infty} h[k]u[n-k]$
= $\sum_{-\infty}^{n} h[k]$

h[n] = s[n] - s[n - 1]

$s(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$
$h(t) = \frac{ds(t)}{dt}$

19th May

> **FOURIER LESGOO**
>
> Lesgoo

CT system
$x(t) = e^{st}$
$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{-st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = e^{st}H(s)$

e^{st} → Eigenfunction of system
H(s) → Eigenvalue of the system → System function or Transfer function

DT system
$x[n] = z^n$
$y[n] = \sum_{-\infty}^{\infty} h[k]z^n z^{-k} = z^n \sum_{-\infty}^{\infty} h[k]z^{-k}$
z^n is the Eigenfunction of system
$H(z)$ is Eigenvalue of system

If I can decompose x(t):
$x(t) = \sum_{-\infty}^{\infty} a_k e^{s_k t} \to \sum_{-\infty}^{\infty} a_k H(s_k)e^{sk}$

For real functions

- Synthesis equation

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jw_o t}$$

- Analysis equation

$$a_k = \frac{\int_0^T x(t)e^{-jkw_o t}}{T}$$

$x(t) = a_o + \sum_{k=1}^{\infty}(2B_k \cos(w_o kt) - 2C_k \sin(w_o kt)) = a_o + 2A_k \cos(w_o kt + \theta_k)$

Dirichlet conditions for convergence:

- Over a period, $\int_T |x(t)|dt < \infty$
- Finite maxima and minima
- Finite interval of time → Finite number of discontinuities

May 24

Properties of Fourier series

- **Linearity:** $ax(t) + by(t) \leftrightarrow aa_k + bb_k$
  Proof: Just throw this into the integral and separate stuff
- **Time shifting:** $x(t-t_0) \leftrightarrow a_k e^{-jkw_0 t_0}$
  Proof: $a_{k'} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-t_0)e^{-kw_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}$
- **Frequency Shifting:** $e^{jMw_0 t}x(t) \leftrightarrow a_{k-M}$
- **Conjugate Input:** $x^*(t) \leftrightarrow a^*_{-k}$

- **Time reversal:** $x(-t) \leftrightarrow a_{-k}$
  Just integrate and substitute
- **Time Scaling:** $x(at) \leftrightarrow a_k$

$$x(\alpha t) \longrightarrow ?$$

$$a_k' = \frac{1}{T/\alpha}\int_{-T/\alpha}^{T/\alpha} x(\alpha t)\, e^{-jk\alpha w_0 t}\, dt \qquad \boxed{\alpha t = u}$$

$$= \frac{1}{T/\alpha}\int_{-T}^{T} x(u)\, e^{-jkw_0 u}\, \frac{du}{\alpha}$$

$$= \frac{1}{T}\int_{-T}^{T} x(u)\, e^{-jkw_0 u}\, du$$

$$= a_k$$

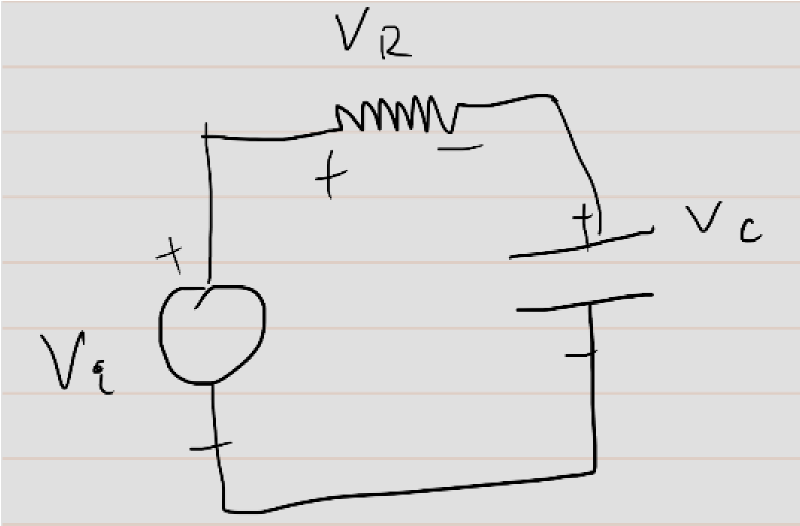
- **Differentiation:** $\frac{dx(t)}{dt} \leftrightarrow jkw_0 a_k$
- **Integration:** $\int_{-\infty}^{\infty} x(t) \leftrightarrow \frac{a_k}{jkw_0}$
- **Conjugate symmetry for real x(t):** $a_k = a^*_{-k}$
- **Fourier Series of x(t) y(t)**
  $c_n = a_n * b_n$

Parseval's Theorem:
Power = $\sum_{k=-\infty}^{\infty} |a_k|^2$

May 26

Filter Design and Analysis

- Frequency reshaping
- Frequency selecting



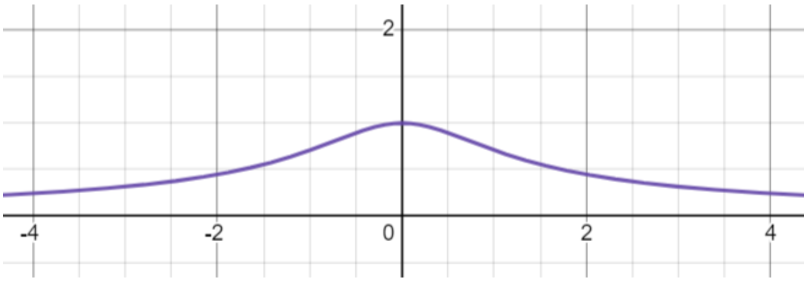
Taking output over V_c, we get low pass filter!

$RC\frac{dV_o}{dt} + V_o = V_i$

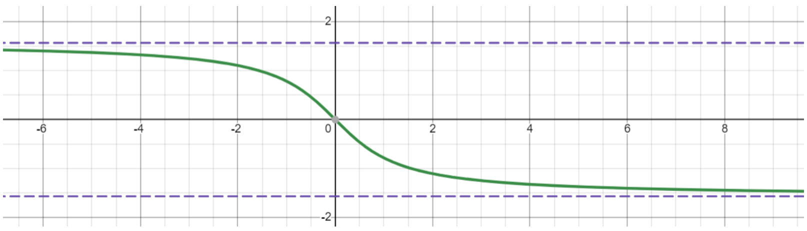
$RCjwH(jw)e^{jwt} + H(jw)e^{jwt} = e^{jwt}$

$H(jw) = \frac{1}{1+RCjw}$

$|H(jw)|$ vs w



$\angle H(jw)$ vs w

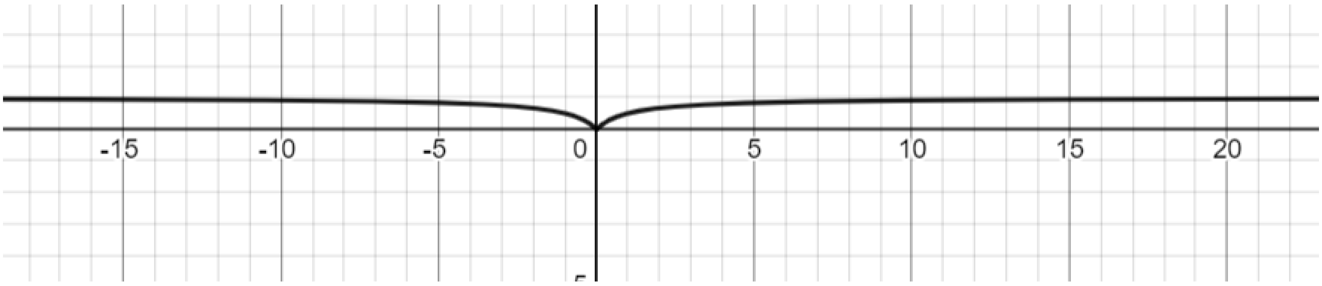


**Taking output over V_r we get High pass filter,

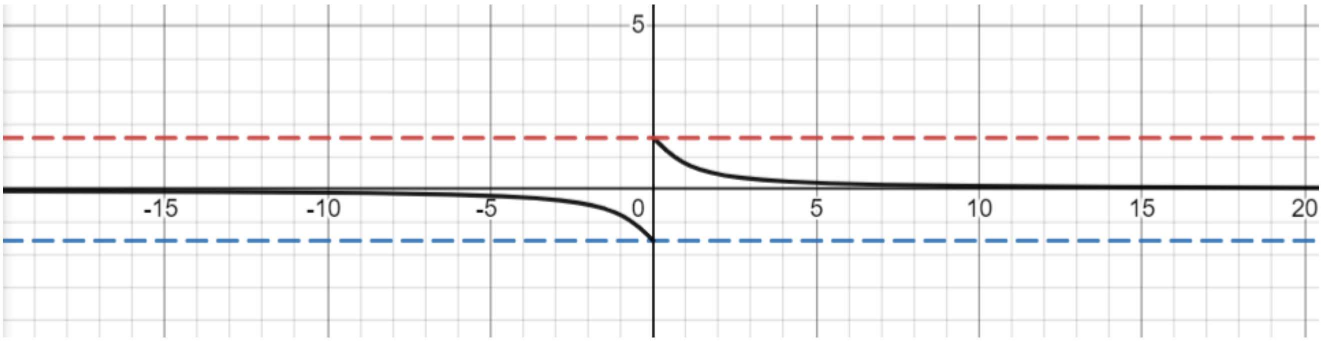
$V_r = V_s - V_c = V_i - H(s)V_i = (1 - \frac{1}{1+RCjw})V_i$

$H(jw) = \frac{jwRC}{1+jwRC}$

$|H(jw)|$ vs w



$\angle H(jw)$ vs w



June 6

Continuous time Fourier transform

- A aperiodic signal is periodic with $T \to \infty$
- $w \to 0$

$\tilde{x}(t) = \sum_{-\infty}^{\infty} a_k e^{jkw_0 t}$
$a_k = \frac{1}{T} \int_{-\infty}^{\infty} \tilde{x}(t) e^{-jkw_0 t} dt$
Spectrum$= a_k T = X(jkw_0) = \int_{-\infty}^{\infty} x(t) e^{-jkw_0 t} dt$
$\tilde{x}(t) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} X(jkw_0) e^{jkw_0 t} w_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw) e^{jwt} dw = x(t)$

FOURIER TRANSFORM FORMULAE

Fourier transform
$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$
Inverse Fourier Transform
$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw) e^{jwt} dw$

Conditions:

- Absolutely integrable
- Finite number of maxima and minima in finite interval of time
- Finite number of discontinuities in finite interval of time

AWESOME DUAL

> Rectangular pulse in time domain → Sinc in frequency domain
> $X(jw) = \frac{2}{\omega}\sin(wT)$
> Rectangular pulse in frequency domain → Sinc in time domain
> $x(t) = \frac{1}{\pi t}\sin(Wt)$

June 2

- $x(t) \leftrightarrow X(jw)$
- $y(t) \leftrightarrow Y(jw)$
- $h(t) \leftrightarrow H(jw)$

$$Y(jw) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-jwt}dt = \int_{-\infty}^{\infty} x(\tau)\left(\int_{\infty}^{\infty} h(t-\tau)e^{-jwt}dt\right)d\tau$$

$$Y(jw) = \int_{-\infty}^{\infty} x(\tau)e^{-jw\tau}H(jw)d\tau = H(jw)X(jw)$$

To get output from a system, multiplication is easier than convolution, so we can take Fourier transforms of h(t) and x(t), multiply, then take inverse Fourier transform to get y(t)

Properties of Fourier Transform

1. Linearity $ax_1 + bx_2 \leftrightarrow aX_1(jw) + bX_2(jw)$
   Proof: Trivial
2. Time shifting $x(t-t_0) \leftrightarrow e^{-jwt_0}X(jw)$
   Proof: $X(jw) = \int_{-\infty}^{\infty} x(t-t_0)e^{-jwt}dt = \int_{-\infty}^{\infty} x(u)e^{-jw(u+t_0)}du$
3. Frequency Shifting $e^{jw_0t}x(t) \leftrightarrow X(w-w_0)$
   Proof: Same as above
4. Conjugation $x^*(t) = X^*(-jw)$
   Proof: $\int_{-\infty}^{\infty} x^*(t)e^{-jwt}dt = \left(\int_{-\infty}^{\infty} x(t)e^{jwt}dt\right)^* = X^*(-jw)$
5. Time and frequency scaling $x(at) = \frac{X(\frac{jw}{a})}{|a|}$
   Proof: $\int_{-\infty}^{\infty} x(at)e^{-jwt}dt = \int_{-\infty}^{\infty} x(u)e^{\frac{-jwu}{a}}\frac{du}{|a|}$
6. Time reversal $x(-t) \leftrightarrow X(-jw)$
   Proof: Trivial
7. Convolution in time domain $x(t) * y(t) \leftrightarrow X(jw)Y(jw)$
   Proof:

$$F\left(x(t) * y(t) \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) \, y(t-\tau) \, d\tau \, e^{-jw\tau} e^{jw\tau} \, e^{-jwt} \, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-jw\tau} d\tau \int_{-\infty}^{\infty} y(t-\tau) e^{-jw(t-\tau)} dt$$

$$t - \tau = u$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-jw\tau} d\tau \int_{-\infty}^{\infty} y(u) e^{-jwu} du$$

$$= X(jw) \, Y(jw)$$

8. Multiplication $x(t)y(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)Y(j(w-\theta))d\theta$

$$F^{-1}\left(X(jw) * Y(jw) \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(j\theta) \, Y(w-j\theta) \, d\theta \, e^{jwt} \, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(j\theta) \, Y(w-j\theta) \, e^{jwt} e^{j\theta t} e^{-j\theta t} \, d\theta \, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) e^{j\theta t} d\theta \int_{-\infty}^{\infty} Y(w-j\theta) e^{j(w-\theta)t} dw$$

$$= 2\pi \, x(t) \, y(t)$$

9. Integration $\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{X(jw)}{jw} + \pi X(0)\delta(w)$

Proof:

Integration in Fourier Transform

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw}$$

$$e^{at}u(-t) \qquad \frac{1}{a-jw}$$

$$e^{-at}u(t) - e^{at}u(-t) \longleftrightarrow \frac{-2jw}{a^2+w^2}$$

Limit $a \to 0$

$$u(t) - u(-t) \longleftrightarrow \frac{-2j}{w}$$

$$= sgn(t)$$

$$u(t) = \frac{1+sgn(t)}{2}$$

$$u(t) \longleftrightarrow \frac{1}{2}\left(2\pi\,\delta(0) - \frac{2j}{w}\right)$$

$$= \pi\,\delta(0) + \frac{1}{jw}$$

$$\int_{-\infty}^{t} x(t)\,dt = x(t) * u(t) \longleftrightarrow \frac{X(jw)}{jw} + \pi X(0)\delta(0)$$

10. Differentiation $\frac{dx(t)}{dt} \leftrightarrow jwX(jw)$

    Proof:

$$\frac{d}{dt}X(t) = \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\,e^{jwt}\,dw$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(X(jw)e^{jwt}\right)dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\,jw\,e^{jwt}\,dw$$

11. Differentiation in frequency domain $-jtx(t) \leftrightarrow \frac{dX(jw)}{dw}$

    Same as above

12. Conjugate symmetry for real signal x(t)

    $X(jw) = X^*(-jw)$

    Real parts are equal

    Imaginary parts are negatives of each other

13. Duality $X(t) \leftrightarrow 2\pi x(-jw)$

$$\mathcal{F}\mathcal{F}(f(t))$$

$$= \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\tau}\, d\tau}_{g(\omega)}\; e^{-j\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} f(\tau) \int_{-\infty}^{\infty} (1)\, e^{-j\omega(t+\tau)}\, d\tau \quad d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\, 2\pi\, \delta(t+\tau)\, d\tau$$

$$= 2\pi f(-t)$$

Fourier transform of periodic function

$X(jw) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(w - w_0)$

June 9th

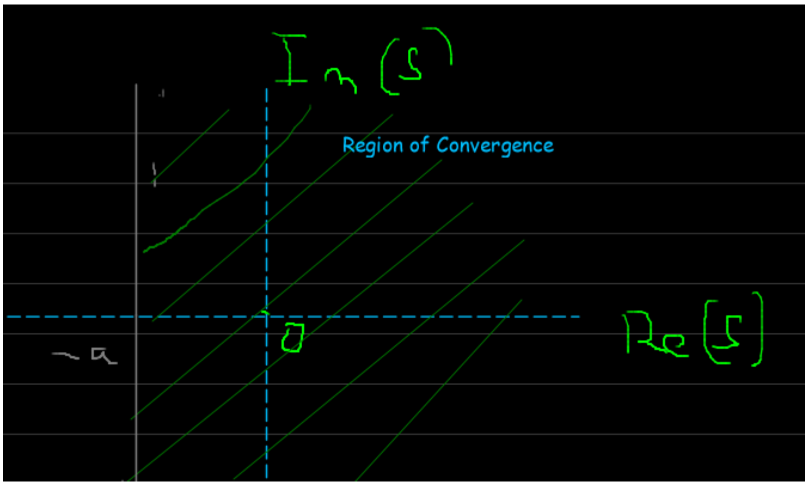
Laplace Transform

For a complex number s,
$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

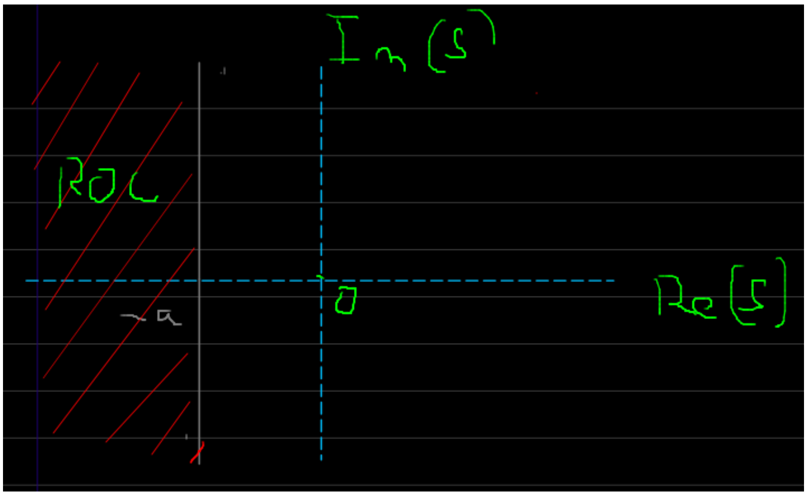
- If $s = jw$, i.e. s is purely imaginary, $X(s) = X(jw)$, so Laplace transform becomes the Fourier transform
- We've transformed one variable t into a complex plane s

Eg. $e^{-at}u(t)$

- Fourier transform $X(jw)$ exists if Re(a) > 0
- $X(s) = \frac{1}{a+s}$



Eg. $x(t) = e^{-at}u(-t)$
$X(s) = \int_{-\infty}^{\infty} e^{-(a+s)t}dt = \frac{1}{a+s}$



Both of above functions have same Laplace transform, so region of convergence has to be specified to identify the function

Pole: s at which $X(s) \to \infty$
Zero: s at which $X(s) \to 0$
Order of pole/zero: Number of times that pole/zero is repeated

$X(s) = \frac{Nr(s)}{Dr(s)}$

- If deg(N) > deg (Dr), Infinity is a pole
- If deg(Nr) = deg (Dr), Infinity is neither pole nor zero
- If def(Nr) < deg(Dr), Infinity is a zero
- Apart from that, there are Nr(s) zeroes and Dr(s) poles

If ROC doesn't cover imaginary axis, Fourier transform doesn't exist
If x(t) is finite over a finite interval, ROC → Entire ℂ s plane

June 14th

Properties of Laplace Transform

1. The ROC of $X(s)$ consists of strips parallel to the jw axis
   Proof: Convergence has nothing to do with the complex part

2. ROC shouldn't contain any pole

3. If $x(t)$ is of a finite duration and integrable, ROC is entire s-plane
   Fourier series integral converges for all s

4. If $x(t)$ is right-sided function and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} > \sigma_0$ will be in ROC
   Consider the function $f(t)$, let it be 0 for t < 0 (any other starting position can be time shifted to this)

   $$0 \le \left| \int_0^\infty f(t) e^{-\sigma_0 t} dt \right| < \infty$$

   For $\sigma > \sigma_0$

   $$\int_0^\infty f(t) e^{-\sigma_0 t} dt > \int_0^\infty f(t) e^{-\sigma t} dt$$

   $$\int_0^\infty f(t) e^{-\sigma t} dt < \infty \text{ and is in ROC}$$

5. If $x(t)$ is left-sided function and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} < \sigma_0$ will be in ROC
   Proof: Likewise

6. If $x(t)$ is a two-sided function and if $Re\{s\} = \sigma_0$ is within ROC, then the entire ROC will contain a strip of s-plane that includes the line $Re\{s\} = \sigma_0$.
   Two sided function = Left-sided + Right-sided function, say at $x = \sigma_0$. Now, ROC is at least intersection of the two regions. If there is an ROC, it is a strip.

7. If $X(s)$ is rational, then it's ROC is bounded by poles or extends to infinity and no poles of $X(s)$ are contained in ROC

8. If $X(s)$ is rational,
   1. $x(t)$ is right-sided, the ROC is the region to the right side of the rightmost pole
   2. $x(t)$ is left-sided, the ROC is the region to the left side of the leftmost pole

Let $F\{x(t)\}$ converge for some s, $Re\{s\} < Re\{s_0\}$, where s_0 is pole

$X(s_0) = \int_{T_0}^{\infty} x(t) e^{-s_0 t} dt = \infty$

$e^{-s_0 t} < e^{-st}$

$X(s) = \int_{T_0}^{\infty} x(t) e^{st} dt = \infty$

Contradiction

Inverse Laplace Transform

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + jw) e^{(\sigma + jw)t} dw$
$x(t) = \frac{1}{2\pi j} \int_{\sigma - jw}^{\sigma + jw} X(s) e^{-st} ds$
But, this is hard to work with and usually guessing the forward function is easier

Properties of Laplace Transform

1. Linearity
   $ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)$
   ROC: R is at least $R_1 \cap R_2$
2. Time shifting
   $x(t - t_0) \leftrightarrow e^{-st_0} X(s)$
   ROC: No change
3. Frequency shifting
   $x(t)e^{s_0 t} \leftrightarrow X(s - s_0)$
   ROC shifts to right by $Re\{s_0\}$ units
4. Time scaling
   $x(at) \leftrightarrow \frac{1}{|a|} X(s/a)$
   If a is negative, ROC is aR, is changed to the other half plane if a < 0
5. Conjugation
   $x^*(t) \leftrightarrow X^*(s^*)$
   ROC: No change
   If $x(t)$ is purely real, zeroes and poles are conjugates of each other
6. Convolution
   $x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s)$
   ROC at least $R_1 \cap R_2$
7. Differentiation in time domain
   $x'(t) \leftrightarrow sX(s)$
   ROC at least R
8. Differentiation in frequency domain:
   $-tx(t) \leftrightarrow X'(s)$

June 15

Initial and Final value theorem

Initial Value theorem: If $x(t) = 0$ for $t < 0$ and contains no impulse or any higher order singularities at the origin, $x(0^+) = \lim_{x\to 0} sX(s)$
Proof: $\int_{-\infty}^{\infty} x'(t)e^{-st}dt = \int_0^{\infty} x'(t)e^{-st}dt = sX(s) - x(0^-)$
$f(0^+) - f(0^-) + \int_{0^+}^{\infty} f'(t)e^{-st}dt = sX(s) - f(0^-)$
Taking $\lim_{s\to\infty}$,
$f(0^+) = \lim_{s\to\infty} sX(s)$

Final Value theorem: If $x(t) = 0$ for $t < 0$ and if $x(t)$ has finite limit as $\lim_{t\to\infty}$,
$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$
Proof: same as above

Causality: If system is causal, its ROC lies to the right of the rightmost pole. If H(s) ROC lies in the right of the rightmost pole and H(s) is rational, system is causal.
Proof:

- In time domain, $h(t) = 0$ for $t < 0$
  So, h(t) is right sided function, and ROC is to the right of rightmost pole
- If H(s) ROC lies in the right of the rightmost pole and H(s) is rational, h(t) is right sided. Also, if it's a rational function, it can be written as *sum of exponential functions* u(t) *or* -sum of exponential function *u(-t)* in time domain. Since it's right sided also, h(t) is causal.

Stability: ROC must include jw axis
Proof: $\int_{-\infty}^{\infty} |h(t)|dt$ is finite.
$\int_{-\infty}^{\infty} |h(t)|dt <= \int_{-\infty}^{\infty} h(t)dt = H(0)$ exists. So, 0 is in ROC and hence jw-axis is in ROC

Both Stable and Causal: $Re(\text{all poles}) < 0$.

Unilateral Laplace Transform
$L\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt$

$L\{x'(t)\} = sX(s) - x(0^-)$
$L\{x''(t)\} = s(sX(s) - x(0^-)) - x'(0^-) = s^2X(s) - sx(0^-) - x'(0^-)$

Given some differential equation, say some $p(D)y = x(t)$
Zero input response → set $x(t) = 0$, i.e. $X(s) = 0$
Zero state response → set $x(0^-), x'(0^-), x''(0^-)$, etc. to be 0

Final solution = zero input response + zero state response