1. (a) Let V and W be vector spaces and $V^\perp$ and $W^\perp$ be the corresponding complementary orthogonal spaces. If V is orthogonal to W (if $v \in V$ and $w \in W$, $v^T w = 0$), is $V^\perp$ orthogonal to $W^\perp$?

(b) Given an $m \times n$ matrix A and $m \times 1$ vectors y and b, such that $A^T y = 0$ and $y^T b \neq 0$. Does the system of equations $Ax = b$ have a solution?

1) a) No take $V = \{0\}$, $W = \{0\}$,

$W^\perp = V^\perp = U$ [the universal set],

But $\boxed{W^\perp \not\perp V^\perp}$

b) let there exist some x

$x^T A^T = b^T$

$x^T A^T y = (b^T y)$

$\underbrace{x^T A^T y}_{= 0}$, $\underbrace{b^T y}_{\neq 0}$ Contradiction

2. Find the projection matrix P_1 that projects a matrix along the vector $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and the projection matrix P_2 that projects onto a line perpendicular to a. Compute $P_1 + P_2$ and $P_1 P_2$.

$$\boxed{P = A (A^T A)^{-1} A^T}$$

$$P_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)^{-1} \begin{pmatrix} 1 & 3 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 1 \\ 3 \end{pmatrix} (1\ 3) = \boxed{\left(\frac{1}{10}\right) \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}}$$

$a^\perp (-3, 1)$

$$P_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \left(\begin{bmatrix} -3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -3 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \end{bmatrix}}_{10} = \frac{\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}}{10}$$

$$P_1 + P_2 = \frac{\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}}{10} = \boxed{I}$$

$$P_1 P_2 = 0$$

3. A 200 mg sample of radioactive polonium-210 is observed as it decays. The mass remaining at various times is as follows

Time (days)	Mass (mg)
0	200
30	172
60	148
90	128

$\rightarrow$

Use an exponential model $m(t) = ce^{kt}$ and do a least square fit to find the half-life of polonium-210.

$$\boxed{\log m(t) = \log c + kt} \leq kt + c$$

5.29	0
5.14	30
4.99	60
4.85	90

$$\begin{bmatrix} 0 & 1 \\ 30 & 1 \\ 60 & 1 \\ 90 & 1 \end{bmatrix} \begin{bmatrix} k \\ c \end{bmatrix} = \begin{bmatrix} 5.29 \\ 5.14 \\ 4.99 \\ 4.89 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 30 & 60 & 90 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 30 & 1 \\ 60 & 1 \\ 90 & 1 \end{bmatrix} \begin{bmatrix} k \\ c \end{bmatrix} = \begin{bmatrix} 0 & 30 & 60 & 90 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5.25 \\ 5.14 \\ \cdots \\ \cdots \end{bmatrix}$$

$$\begin{bmatrix} 12600 & 180 \\ 180 & 4 \end{bmatrix} \begin{bmatrix} k \\ c \end{bmatrix} = \begin{bmatrix} 890.1 \\ 20.27 \end{bmatrix}$$

$c = 5.288$ $\qquad$ $k = -4.9 \times 10^{-3}$

$$\frac{0.693}{4.9 \times 10^{-3}} = \boxed{141.42\text{ day}}$$

4. Find the best fit (least squares) cubic polynomial ($y = c_0 + c_1x + c_2x^2 + c_3x^3$) for the following data.

$$(x, y) = [(-1, -2), (-\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{7}{4}), (\frac{1}{2}, \frac{13}{4})]$$

A $m \times n$ $\qquad$ $A_{n \times m}$ $\qquad$ $P^2 = P$ $\qquad$ If $Pv = v$, $v \in \text{col}$, $\Rightarrow Pv \neq 0$, $v \notin N$

$\text{rank} + N(A) = n$

5. If P is an $n \times n$ projection matrix, show that Range(P) $\oplus$ Null(P) $= \mathbb{R}^n$.

Let A^T be vector space on which we are projecting

Now, $\text{col}(P) = \text{col}(A^T)$.

$(A(A^TA)^{-1}A^T)x = 0$

Also, for $A^Tx = 0$ $\Rightarrow Px = 0$.

$\Rightarrow \dim N(A^T) \leq \dim N(P)$

Also, $\dim C(A) + \dim N(A^T) = n$

$\boxed{n \leq \dim(\text{col } P) + \dim(N(A^T))}$ $\qquad$ $\mathbb{R}^n$

6. If x^* is the minimum-norm solution to $Ax = b$, show that ${x^*}^T y = 0$ where $y \in Null(A)$.

$$b^T (AA^T)^{-1} Ay = 0$$

$$x^T = -\frac{1}{2} A^T \lambda$$

$$x = A^T (AA^T)^{-1} b$$

$$\lambda = -2(AA^T)^{-1} b$$

7. Let $f(x)$ denote a scalar function and $\mathbf{f}(x)$ denote a system of m equations of a vector $x = [x_1, x_2, \cdots x_n]^T$. Then ∇f and $\nabla \mathbf{f}$ are defined as follows

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}, \quad \nabla \mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Using these definitions, find the gradient of the following functions (a) $\mathbf{f} = Ax$ (b) $\mathbf{f} = x^T A$ (c) $f = x^T x$ (d) $f = x^T Ax$ and (e) $f = \lambda^T Ax$. In all cases, start with the definition and compute the gradient.

a)

$$\nabla f = \nabla \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n \\ \vdots \\ a_{m1}x_1 \cdots a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ & & & & \vdots \\ & & & & a_{mn} \end{bmatrix}$$

$$= \boxed{A^T}$$

b)

$$f = x^T A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ & & & \\ a_{n1} & & & a_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{21}x_2 + a_{31}x_1 + \cdots a_{n1}x_n & \cdots & a_{1m}x_1 \\ & & a_{2m}x_2 \\ & & \vdots \\ & & a_{nm}x_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & & a_{nm} \end{bmatrix}$$

$$= A$$

c) $\frac{\partial}{\partial x}\left(x^T x\right) = 2x$

d) $x^T A x$

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & & & & \\ a_{n1} & & \cdots & & a_{nn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right)$$

$$= x_1 a_{11} x_1 + x_1 \underline{a_{12}} x_2 + \cancel{x_1 a_{12} x} \cdots\cdots x_1 a_{1n} x_n$$

$$+ \cdots$$

$$+ x_n a_{n1} x_1 + \cdots\cdots x_n a_{nn} x_n$$

$$[\nabla f]_1$$

$$= a_{11} x_1 + a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots a_{1n} x_n$$

$$+ a_{21} x_2 + \cdots\cdots a_{n1} x_n$$

∇f

$$= Ax + A^T x = \boxed{(A + A^T) x}$$

(e) $f = \lambda^T A x.$

$$= d_1 a_{11} x_1 + d_1 a_{12} x_2 + \cdots \quad d_1 a_{1n} x_n$$

$$\vdots$$

$$d_n a_{n1} x_1 \quad \cdots \quad d_n a_{nn} x_n$$

$$(\nabla f)_{x_1} = d_1 a_{11} + d_2 a_{21} + \cdots \quad d_n a_{n1}$$

$$(\nabla f) = A^T \lambda$$

8. A is a $m \times n$ matrix, with $m > n$ and rank n. C is a $p \times q$ matrix with $p < q$ and rank p. $b \notin \text{col}\{A\}$ is a $m \times 1$ vector. Use the method of Lagrange multipliers and find the equations that need to be solved in order to determine x in the following optimization problem.

$$\begin{aligned} \text{Minimize} \quad & (Ax - b)^T(Ax - b) \\ \text{Subject to} \quad & Cx = d \end{aligned}$$

$$f(x) = (Ax - b)^T(Ax - b) + d^T(Cx - d)$$

$$\nabla f = 0$$

$$= 2\nabla(Ax - b) + C^T \lambda + 0$$

$$= \boxed{2A^T + C^T d = 0} \longrightarrow \boxed{A + d^T C = 0}$$