Parameterized Algorithms

CS6101

* q - SAT is NP-complete for $q \geq 3$
* <u>ETH</u> [Roughly]
  $\nexists$ $2^{o(n)}$ - algo for 3-SAT
* <u>SETH</u>:
  $\delta_q \to$ infimum of constants c s.t. $\exists$ an algo. running in time $O^*(2^{cn})$ for q-sat
  $\boxed{\lim_{q \to \infty} \delta_q = 1}$

[Roughly] $\nexists$ $O^*((2-\varepsilon)^n)$, $\varepsilon > 0$ algo for CNF-SAT.

<u>Recall</u>: $O^*((2-\varepsilon)^n) = O^*(2^{\delta n})$, $\varepsilon > 0$, $0 < \delta < 1$

Implications of ETH & SETH
Intuitively, ETH gives asymptotics
SETH gives exact details

<u>We will prove (under SETH)</u>

$\nexists$ exist an algo which solves hitting set in $O^*((2-\varepsilon)^n)$-time, $\varepsilon > 0$.

Recap

Hitting Set:

Given: Universe $\mathcal{U}$, Collection of Subsets $\mathcal{F}$, integer k

Find: A set $X \subseteq \mathcal{U}$, s.t. $|X \cap F| \geq 1 \; \forall F \in \mathcal{F}$, $|X| \leq k$. (or refute)

$|\mathcal{U}| = n$, $|\mathcal{F}| = m$.

Example:

$\mathcal{U} = \{e_1 \;\; e_2 \;\; e_3\}$ $\qquad X = \{e_1\}$

$\mathcal{F} = \{\{e_1, e_2\}, \{e_1\}, \{e_1, e_2, e_3\}\}$

CNF - SAT

Given: Vars = $\{x_1, x_2, \ldots, x_n\}$, Clauses $C_1, C_2 \ldots\ldots C_m$

Find: An assignment φ : Vars $\rightarrow$ {True, False}, s.t. every Clause is set to True, or correctly conclude $\nexists$ such an assignment.

q - SAT

Given: Vars = $\{x_1, x_2, \ldots, x_n\}$, Clauses $C_1, C_2 \ldots\ldots C_m$, each Clause contains at most q distinct variables.

Find: An assignment φ : Vars $\rightarrow$ {True, False}, s.t. every Clause is set to True, or correctly conclude $\nexists$ such an assignment.

Example

Bad Reduction (CNF-SAT → HITTING SET)

x_i (n vars) $\longrightarrow$ $\xrightarrow{U}$ $x_i^T \quad x_i^F$ (2n elems)

$\mathcal{F} \to \{x_i^T, x_i^F\}$ $\qquad k := n$

$C = x_1 \vee \overline{x_2} \vee x_3 \qquad \mathcal{F} \to \{x_1^T, x_2^F, x_3^T\}$

CNF-SAT	HITTING SET
$\text{Vars} = \{x_1, x_2, \ldots, x_n\}$	$U = \{x_1^T, x_1^F, x_2^T, x_2^F, \ldots, x_n^T, x_n^F\}$
Clauses $C_1, C_2, \ldots, C_m$	$\mathcal{F} := \{\{x_1^T, x_1^F\}, \{x_2^T, x_2^F\}, \ldots, \{x_n^T, x_n^F\}\}$
$C_i := x_{i1} \vee \overline{x_{i2}} \vee x_{i3}$	$\cup \{\{x_{i1}^T, x_{i2}^F, x_{i3}^T\}, \ldots\}$
	$\boxed{k = n}$
	$\lvert U \rvert = 2n = n'$
	$\lvert \mathcal{F} \rvert = n + m$

If there was a $\boxed{O^*((\sqrt{2}-\varepsilon)^{n'})}$-algo for Hitting set, then CNF-SAT runs in $O^*((\sqrt{2}-\varepsilon)^{2n}) = \boxed{O^*((2-\varepsilon')^n)}$ time, violating SETH. $[\varepsilon, \varepsilon' > 0]$

REDUCTION (CNF-SAT → HITTING SET)

Some preliminaries:

→ Let $n' = |U|$ in the reduced instance. $n' = 2n$ is NOT fine.

$n' = n + \underbrace{O(\log n)}_{o(n)}$ or similar might be fine

$^{1000}C_5$

$\rightarrow$ k needs to be $\simeq n'/2$

$n'! \approx \left(\frac{n'}{e}\right)^{n'}$

$$\binom{n'}{n'/2} = \frac{n'^{n'}}{e^{n'}\left(\frac{n'}{2e}\right)^{n'}} \approx \boxed{2^{n'}}$$

Need to try

$\binom{n'}{k} = \binom{n'}{n'/2} \simeq 2^{n'}$ sets $\rightarrow$ good

$\rightarrow$ Otherwise, say $k < 0.49n'$

Need to try

$$\binom{n'}{k} = \binom{n'}{0.49n'} = \frac{1}{(0.49)^{0.49n'}(0.51)^{0.51n'}} = \boxed{1.996^{n'}}$$

Easy to check!

$\rightarrow$ Likewise for $k > 0.51n'$

Challenge: Encoding clauses also with the remaining space

Reduction

q & α are given $\alpha > 0$

q-SAT $\xrightarrow{\hspace{2cm}}$ Hitting set of size $\underbrace{(1+\alpha)n}_{= n'}$

n variables

Running Time: Exp. in q and $1/\alpha$

Lemma 1: If the above reduction exists, then $\exists$ any algo for hitting set running in $O^*(2^{\delta n'})$ time. $\delta < 1$

Proof:

Pick α s.t. $\boxed{\delta' = \delta(1+\alpha) < 1}$

Reduce the instance

Run the algo for Hitting set

$O^*(2^{\delta' n})$ time

Thus, we will get $\delta_q < \delta' \quad \forall q$

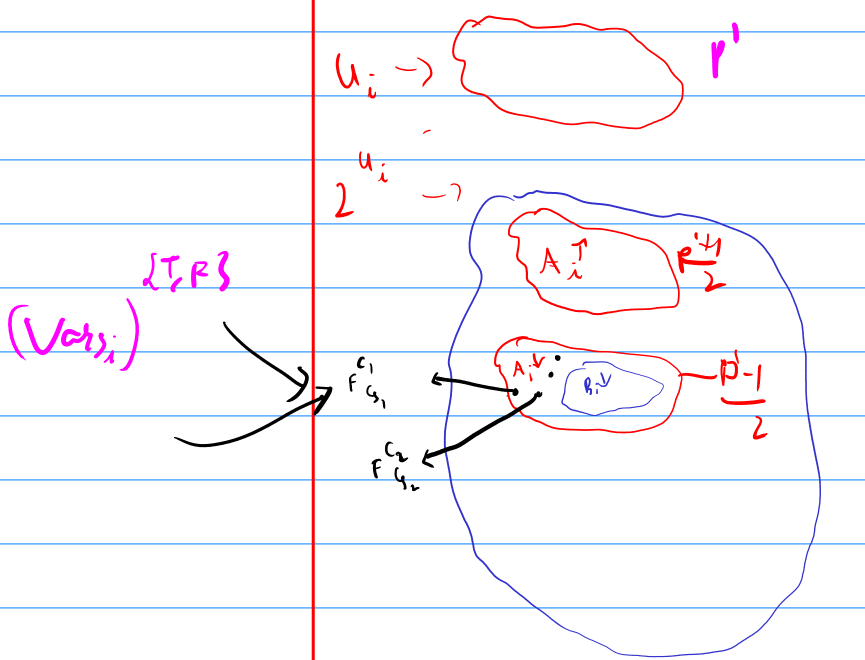
$\Rightarrow \exists \left(\lim_{q \to \infty} \delta_q\right) \leq \delta' < 1$

$\boxed{\text{Violates SETH}}$

$n' = (1+\alpha)n = n + O(\log n)$ $\alpha \simeq O\left(\frac{\log n}{n}\right)$

$p \geq 3$ is an odd number s.t. $\frac{2\lceil \log_2 p \rceil}{p} \leq \frac{\alpha}{3}$

Divide variables into blocks of size p



$p' \equiv p + 2\lceil \log_2 p \rceil$ → Odd Num

$p \geq 3$

Extra dummy vars

$Vars = \{x_1, x_2, \ldots, x_{\tilde{n}}, \boxed{x_{\tilde{n}+1}, \ldots, x_n}\}$

$\mathcal{U} = \bigcup_{i=1}^{n/p} U_i$

$Vars_1, Vars_2, \ldots, Vars_{n/p}$

$Vars_i \rightsquigarrow \boxed{2^p \text{ assign}} \longrightarrow U_i, \quad |U_i| = p' = p + 2\lceil \log_2 p \rceil$

$$\boxed{\eta_i \longrightarrow \text{Map from } (Vars_i)^{\{T,F\}}_{2^p} \xrightarrow[\text{Arbitrary}]{} A_i^{\downarrow}{}_{2^{p'}} = \binom{U_i}{\frac{p'-1}{2}}}$$

All subsets of U_i size $\frac{p'-1}{2}$

$\mathcal{F}$

- $A_i^{\uparrow} = \binom{U_i}{\frac{p'+1}{2}}$ → Has $\geq \frac{2^{p'}}{1+p'}$ elements
- $B_i = A_i^{\downarrow} \setminus Im(\eta_i)$ $2^{p'} - 2^p$

Clauses

"C" Let $Vars_{C_1}, Vars_{C_2}, \ldots Vars_{C\,r_c}$ be variable groups containing vars in C.

$Vars_C = \bigcup_{t=1}^{r_c} Vars_{C_i}$ $\boxed{\text{ap elements}}$ 2^{qp}

$Vars_{C_1}$ T T T
$Vars_{C_2}$ T F T
$Vars_{C_3}$ F F F

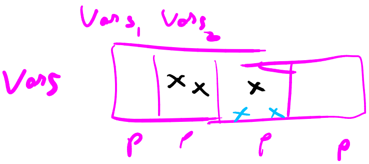
$\mapsto \eta_{C_1}(TTT) \cup \eta_{C_2}(TFT) \cup \eta_{C_3}(FFF)$

Z_C → set of assign of $(Vars_C)^{\to\{T,F\}}$ where C is not satisfied.

$\forall\, \zeta \in Z_C$

$r_c\left(\frac{p'-1}{2}\right)$

$\boxed{F^C_\zeta} = \bigcup_{i=1}^{r_c} \eta_{C_i}\left(\zeta \big|_{Vars_{C_i}} \right)$



Correctness

$\Rightarrow$ Let ζ a satisfying assignment,
$\varphi : Vars \to \{T, F\}$

$$\boxed{k = \left(\frac{n}{p}\right) \cdot \left(\frac{p'+1}{2}\right)}$$

$$|U| = \frac{n}{p} p'$$

$$\leq n\left(1 + \alpha/3\right)$$

We added $\leq p-1$ variables
$\leq \frac{\alpha}{3} \tilde{n}$ (Can be proved)

$$|U| \leq \tilde{n}\left(1+\frac{\alpha}{3}\right)\left(1+\frac{\alpha}{3}\right) \leq \boxed{(1+\alpha)\tilde{n}}$$

Orig instance

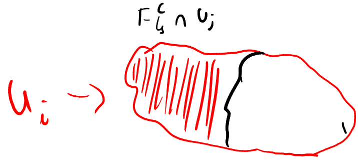
$X = \bigcup^{n/p} U_i \setminus \eta_i(\varphi(vars_i))$

$U_i \rightarrow$ 

Size bound :

$\rightarrow S \in A_i^{?}$ has $\frac{p'+1}{2}$ elements

$\rightarrow S \in B_i$ has $\frac{p'-1}{2}$ elements



$\to$ For some clause C

G was some assignment of $Vars_C \to \{T,F\}$ s.t. C is not satisfied.

$\exists$ some j depending on C, s.t.

$$G(vars_j) \neq \varphi(vars_j)$$

$$\boxed{\Rightarrow F_G^c \cap U_j \neq U_j \setminus X}$$

Look at

$\underline{F_G^c \cap U_j}$ and $X \cap U_j$

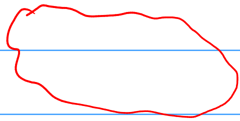
$\frac{p'-1}{2}$ $\qquad$ $\frac{p'+1}{2}$

$\Leftarrow$ Assume $\exists$ a hitting set $X \subseteq U$, $|X| \le k$.

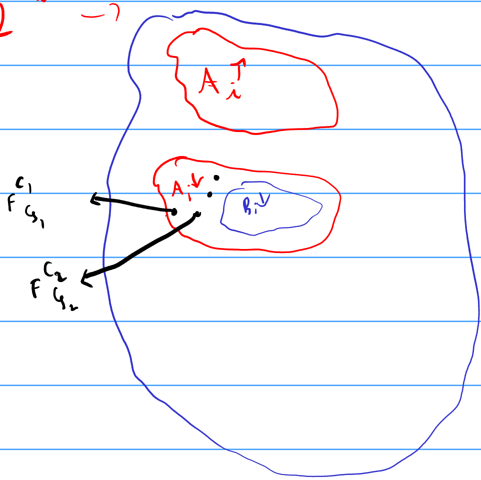
$|X \cap U_i| \geq \frac{p'+1}{2}$ to hit A_i^T

$\Rightarrow |X \cap U_i| = \frac{p'+1}{2}$

$U_i \to$



$2^{U_i} \to$



Let $\boxed{\varphi_i = \eta_i^{-1}(U_i \setminus X)}$

Take a clause "C" not satisfied, by contradiction

$\gamma = \varphi|_{vars_C}$, since for γ

C is false, $\gamma \in Z_C$.

For $F \in B_i$, $F \cap X \neq \phi$

The only elements we can ignore $\in Im(\eta_i)$ for some i.

$U_i \setminus X \in Im(\eta_i)$

Look at F_C^{γ}

$X \cap F_C^{\gamma} \cap U_j \neq \phi$ for some j [Hitting set]

$F_{\gamma}^{C} \cap U_j = \eta_j(\gamma(vars_j))$

$= \boxed{\eta_j(\varphi[vars_j])}$

$F_{\gamma}^{C} \cap U_j = U_j \setminus X$

Contradiction

The hardness stems from the actual SETH, not the relaxed version where we deny the existence of $O^*((2-\varepsilon)^n)$ for CNF-SAT.

Other problems :

→ SET-SPLITTING
→ NAE-SAT

$\left.\right\}$ No $O^*((2-\varepsilon)^n)$ algo

SET-COVER

Given : A set of 'n' elements E, A collection $\bar{C} = \{C_1, C_2, \dots, C_m\}$, $C_i \subseteq E \; \forall i$, int. k

Find : $X \subseteq \bar{C}$, $|X| \leq k$, $\boxed{\bigcup_{c \in X} c = E}$

SET-COVER & HITTING SET

$E = \{s_1, s_2, s_3\}$ $\Longleftrightarrow$ $U = \{x_1, x_2, x_3\}$

$\bar{C} = \{\{s_1, s_3\}, \{s_2, s_3\}, \{s_2\}\}$ $\quad$ $\tilde{F} = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_1\}\}$

$O^*(2^n)$ -algo $\Longleftrightarrow$ $O^*(2^m)$ -algo, $m = |\tilde{F}|$

q-SET COVER

Let λ_q be inf. of constants c s.t. $\exists$ an algo running in time $O^*(2^{cn})$ for q-SET COVER. Then, $\boxed{\lim_{q \to \infty} \lambda_q = 1}$

[Roughly] No $O^*((2-\varepsilon)^m)$ algo for Hitting set, $m = |\mathcal{F}|$

Implications

→ Steiner tree no $O^*((2-\varepsilon)^t)$

→ CVC no $O^*((2-\varepsilon)^k)$