Homework 06 — Solutions

CS 624, 2024 Spring

1. Problem 26.1-7 (p714)

   Suppose that, in addition to edge capacities, a flow network has vertex capacities. That is, each vertex v has a limit $l(v)$ on how much flow can pass though v. Show how to transform a flow network $G = (V, E)$ with vertex capacities into an equivalent flow network $G' = (V', E')$ without vertex capacities, such that a maximum flow in G' has the same value as a maximum flow in G. How many vertices and edges does G' have?

   Split each node in the original graph into separate "in" and "out" nodes. Each edge into the original node now goes to the "in" node, and each edge out of the original node now comes out of "out" node. Add an edge from the "in" node to the "out" node with the vertex capacity of the original node.

   Specifically: the new flow network $G' = (V', E')$ is defined as follows:

   - For each $v \in V$, V' contains nodes v_i and v_o, and E' contains an edge (v_i, v_o) with capacity $l(v)$.
   - For each edge $(u, v) \in E$ with capacity c, E' contains an edge (u_o, v_i) with capacity c.
   - The source of the new flow network is s_i (the "in" node corresponding to the original source), and the sink is t_o (the "out" node corresponding to the original sink).

   $|V| = 2|V|$, and $|E'| = |E| + |V|$.

2. Exercise 1.4 ([aux16] p7)

   Prove that all problems that are NP-complete are polynomially equivalent, in the sense that if A and B are NP-complete, then $A \leq_P B$ and $B \leq_P A$.

   If A and B are NP-complete, then $A \leq_P B$ and $B \leq_P A$.

   Proof: Suppose A and B are both NP-complete. By the definition of NP-complete, both A and B are in NP, and both A and B are NP-hard. Now we use the definition of NP-hard twice:

   - Since A is NP-hard, for every problem C in NP, $C \leq_P A$. Since B is in NP, we can instantiate $C = B$ to get $B \leq_P A$.
   - Since B is NP-hard, for every problem C in NP, $C \leq_P B$. Since A is in NP, we can instantiate $C = A$ to get $A \leq_P B$.

3. Exercise 2.1 ([aux16] p9)

   Summary: Prove that the problem of satisfiability for expressions in *disjunctive normal form* is in P (that is, it is polynomial-time).

   A formula in disjunctive normal form is satisfiable if any of its clauses is satisfiable, and a DNF clause is satisfiable if it does not contain the both a variable and its negation (that is, $x \wedge y \wedge \bar{x}$ is unsatisfiable). Otherwise, we can build a satisfying truth assignment for that clause by setting each variable that occurs positively to true, each variable that occurs negatively to false, and each unmentioned variable to either value.

So an algorithm can check satisfiability by checking each clause. If it sees a satisfiable clause, it immediately returns true; otherwise after inspecting all of the clauses it returns false. Checking a clause in polynomial time can be done in many different ways. One that requires minimal extra memory is to sort the literals of the clause by their variable number; that way a variable and its negation will appear in adjacent positions, and can be detected in a loop looking at a two-element window. Another solution involves an auxiliary array whose size is the number of variables.

4. Exercise 3.6 ([aux16] p16)

Prove that the following are equivalent:

(a) V_1 is a vertex cover of G.

(b) $V - V_1$ is an independent set in G.

and, continuing, prove that the following are equivalent:

(a) V_2 is an independent set in G.

(b) V_2 is a clique in G^c.

If V_1 is a vertex cover of G, then $V - V_1$ is an independent set in G.
Proof: Suppose V_1 is a vertex cover in G. Pick $u, v \in V - V_1$ where $u \neq v$; we must show there is no edge between them. Suppose for the sake of contradiction that there is an edge between u and v. But $u \notin V_1$ and $v \notin V_1$, so V_1 does not cover that edge, which contradicts the assumption that V_1 is a vertex cover.

If $V - V_1$ is an independent set in G, then V_1 is a vertex cover of G.
Proof: Suppose $V - V_1$ is an independent set in G. Pick an arbitrary edge in G; we must show that one of its endpoints is in V_1. Suppose for the sake of contradition that neither endpoint is in V_1. Then both endpoints of the edge are in $V - V_1$, which contradicts the claim that $V - V_1$ is an independent set.

If V_2 is an independent set in G, then V_2 is a clique in G^c.
Proof: Suppose that V_2 is an independent set in G. We want to show that V_2 is a clique in G^c—that is, for any two vertices in V_2, they are connected by an edge in G^c. Pick any two vertices $u, v \in V_2$ where $u \neq v$. Since V_2 is an independent set in G, there is no edge in G between u and v. So by the definition of G^c, there is an edge between u and v in G^c.

If V_2 is a clique in G^c, then V_2 is an independent set in G.
Proof: Suppose that V_2 is a clique in G^c. We want to show that V_2 is an independent set in G—that is, for any two vertices in V_2, they are not connected by an edge in G. Pick any two vertices $u, v \in V_2$ where $u \neq v$. Since V_2 is a clique in G^c, there is an edge between u and v in G^c. By the definition of G^c, that means there is no edge between u and v in G.