General Runge-Kutta 2

$u = u_k$

$K_1 = f(x(t_k),\ u(t_k + c\Delta t),\ t_k + c\Delta t)$

$K_2 = f(x(t_k) + a \cdot \Delta t \cdot K_1,\ u(t_k + c\Delta t),\ t_k + c\Delta t)$

$x_{k+1} = x(t_k) + \Delta t \sum_{i=1}^{2} b_i \cdot K_i$

0		
c	a	
	b_1	b_2

$e_k = x_{k+1} - x(t_{k+1})$

$u(t) = u_k$

$x(t_{k+1}) = x(t_k) + \Delta t \cdot f(x(t_k), u_k) + \frac{\Delta t^2}{2} \cdot \dot{f}(x(t_k), u_k) + \mathcal{O}(\Delta t^3)$

$K_2 = ?$

$f \cdot \frac{\partial f}{\partial x} = \dot{f}$

$x_{k+1} = x(t_k) + \Delta t \sum_{i=1}^{2} b_i \cdot K_i$

$x_{k+1} = x(t_k) + \Delta t \left(b_1 \cdot K_1 + b_2 \cdot K_2\right)$ → Taylor

$K_{+1} = x(t_k) + \Delta t \left(b_1 f_1 + b_2 f(x(t_k) + a\Delta t K_1)\right)$

$_{+1} = x(t_k) + \Delta t b_1 f + \Delta t b_2 f(x(t_k) + \Delta t a \Delta t K_1)$

$_{+1} = x(t_k) + \Delta t b_1 f + \Delta t b_2 \left(f + \Delta t a \cdot \dot{f} + \mathcal{O}(\Delta t^2)\right)$

$_{+1} = x(t_k) + \Delta t b_1 f + \Delta t b_2 f + \Delta^2 t\, b_2 a \dot{f} + \mathcal{O}(\Delta t^3)$

$e_k = x(t_k) + \Delta t b_1 f + \Delta t b_2 f + \Delta^2 t\, b_2 a \dot{f} - x(t_k) - \Delta t f(x(t_k), u_k)$

$- \frac{\Delta t^2}{2} \cdot \dot{f}(x(t_k), u_k) - \mathcal{O}(\Delta t^3)$

$b_1 = \frac{1}{2},\ b_2 = \frac{1}{2},\ b_2 a = \frac{1}{2}$

$a = 1$

c) $x(t) = x(t_k) + \sum_{i=1}^{n} \frac{a_i}{i!} (t-t_k)^i$

$n, b, c = ?$ $\Rightarrow$ RK2 is exact

$K_1 = f(x(t_k), u(t_k))$

$K_2 = f\left(x_k + \Delta t \sum_{j=1}^{s} a_{ij} K_j, u(t_k + c_2 \Delta t)\right)$

$K_2 = f\left(x_k + \Delta t a_{21} K_1 + \Delta t a_{22} \overset{0}{\cdot} K_2, u(t_k + c_2 \Delta t)\right)$

$K_2 = 1 + a_{21} \Delta t \dot{f} + c \Delta t \dot{u} + O(\Delta t^2)$

$x_{k+1} = x_k + \Delta t \sum_{i=1}^{2} b_i K_i$

$x_{k+1} = x_k + \Delta t b_1 K_1 + \Delta t b_2 K_2$

$x_{k+1} = x_k + \underline{\Delta t b_1 f} + \underline{\Delta t b_2 f} + \underline{\underline{b_2 a_{21} \Delta t^2 \dot{f}}} + \underline{\underline{\Delta t^2 b_2 c \dot{u}}} + O(\Delta t)^3$

$x_{k+1} = x_k + (\Delta t b_1 f + \Delta t b_2 f) + (\Delta t^2 b_2 a_{21} \dot{f}) + (\Delta t^2 b_2 c \dot{u})$

$n=2 \Rightarrow x(t) = x(t_k) + \frac{a_1}{1} (t-t_k) + \frac{a_2}{2} (t-t_k)^2$

$\frac{a_2}{2} = b_2 c \dot{u} + b_2 a_{21} \dot{f}$, $\underline{c=0}$, $\Rightarrow \boxed{b_2 \cdot a_{21} = \frac{1}{2}}$, $\boxed{a_2 = \dot{f}}$

(1) $\underline{a_1} = \underline{(b_1 + b_2) f}$ ($b_1 + b_2$: constant) $\Rightarrow \boxed{b_1 + b_2 = 1}$, $\boxed{f = a}$