Multiple Testing Project

2.2 P-value

1. Distribution of the p-value under the null

1.1 Show that for any α, $c_\alpha = F_H^{-1}(1-\alpha)$

We know that $\alpha = 1 - F_H(c_\alpha)$ from the definition of c_α.

$\alpha = 1 - F_H(c_\alpha)$

$\alpha - 1 = -F_H(c_\alpha)$

$-(\alpha - 1) = F_H(c_\alpha)$

$1 - \alpha = F_H(c_\alpha)$

$F_H^{-1}(1-\alpha) = c_\alpha$

$c_\alpha = F_H^{-1}(1-\alpha)$

Q.E.D.

1.2 Show that the p-value of the test, as a function of the data X used, is given by $p(\mathbf{X}) = 1 - F_H(T(\mathbf{X}))$.

The p-value is defined as $p - value = inf\{\alpha : T(X) \in R_\alpha\}$

The p-value is the *smallest* α for which $T(X)$ is in the region R_α of the probability distribution P_H

The p-value is a value of α, which is defined as $\alpha = 1 - F_H(c_\alpha)$ where c_α is chosen so that the equation is true. Therefore, if we replace c_α with our test statistic $T(\mathbf{X})$, we get a $p(\mathbf{X}) = 1 - F_H(\mathbf{X})$.

This highlights the choice of α sets the minimum p-value for $\mathbf{X}$ as $p(\mathbf{X})$ such that one rejects the hypothesis that X is from the same distribution as Y when $T(\mathbf{X}) > c_\alpha$.

1.3 Show that for any univariate random variable y with continuous distribution function F, the random variables $F(y)$ and $1 - F(y)$ follow the uniform distribution.

We know a function to be uniformly distributed if it has constant probability in the range for which it is defined. Say y_i is some value for which $F_Y(y_i)$ is defined. The set of all y_i is denoted as $\mathbf{Y}$ and the probability of $F_Y(\mathbf{Y})$ is given by $P[F_Y(\mathbf{Y})]$.

F_Y is uniformly distributed when $P[F_Y(y_1)] = P[F_Y(y_2)]$ where $y_1 y_2$ are any 2 values in $\mathbf{Y}$.

$$P[F_Y(\mathbf{Y}) \leq y] = P[F^{-1}(F_Y(\mathbf{Y})) \leq F^{-1}(y)] = P[\mathbf{Y} \leq F^{-1}(y)]$$

We know that F_Y is strictly increasing and $F_Y^{-1}(F_Y(\mathbf{Y})) = y$ by definition of the definition of a quantile function, which is the generalized inverse of a CDF. We use this property to conclude:

$$P[\mathbf{Y} \leq F_Y^{-1}(y)] = F_Y(F_Y^{-1}(y)) = y$$

This is true for all values of $\mathbf{Y}$.

By definition if a function F_Y follows the uniform distribution, so does $1 - F_Y$

1.4 Using the above results, show that the p-value follows the uniform distribution under H.

$\alpha = 1 - F_H(c_\alpha)$

c_α being a continuous random variable, we know $F_H(c_\alpha)$ to have a uniform distribution. If $F_H(c_\alpha)$ follows a uniform distribution, so too will its complement, $1 - F_H(c_\alpha)$.

MT under independence assumptions

2.1 Show that $P_H[\cap_{i=1}^m \{y_i > \alpha\}] = (1-\alpha)^m$

As $y_1, ..., y_m$ are unifrom random variables between 0 and 1, we can state that $p(y_i) > \alpha$ is equivalent to $y_i > \alpha$. As they are independent, the joint distribution of the probability that for every y_i the condition $y_i > \alpha$ holds is nothing but the product of the probabilities,

$$P_H[\cap_{i=1}^m \{y_i > \alpha\}] = \prod_{i=1}^m P(y_i > \alpha)$$

We know from the previous exercises that $P(T(y_i) < \alpha) = \alpha$, or, equivalently, $P(p(y_i) > \alpha) = 1 - \alpha$. In this case, with uniformly distributed y_i, it is equivalent to say that $P(y_i > \alpha) = 1 - \alpha$. Therefore,

$$\prod_{i=1}^m P(y_i > \alpha) = (1-\alpha)^m$$

2.2 Show that the probability of rejecting at significance level α at least one of the independent tests is $1 - (1-\alpha)^m$.

The probability of rejecting at least one of the tests could be expressed as the union,

$P_H[\cup_{i=1}^m \{y_i < \alpha\}]$

And we know that the probability of at least one rejection is the complement of the probability of having no rejections, which we have already determined as being equal to $(1-\alpha)^m$. Thus,

$P_H[\cup_{i=1}^m \{y_i < \alpha\}] = 1 - P_H[\cap_{i=1}^m \{y_i > \alpha\}]$

$P_H[\cup_{i=1}^m \{y_i < \alpha\}] = 1 - (1-\alpha)^m$

2.3 Under the above assumption, show that if we wish that the overall type I error is α, each independent test should be rejected at significance level $1 - (1-\alpha)1/m$

Type I error is rejecting the null when it is actually true. We know from 1.2 that the prbability of type one error for m hypotheses each having been tested at the confidence level α is:

$1 - (1-\alpha)^m$

So if we want the overall type I error to be α for m tests, each test should be rejected at a certain probability p_r that satisfies:

$P_H[\cup_{i=1}^m \{y_i < p_r\}] = 1 - (1-p_r)^m = \alpha$

Solving for p_r to determine the level p_r at which each test should be evaluated:

$1 - (1-p_r)^m = \alpha$

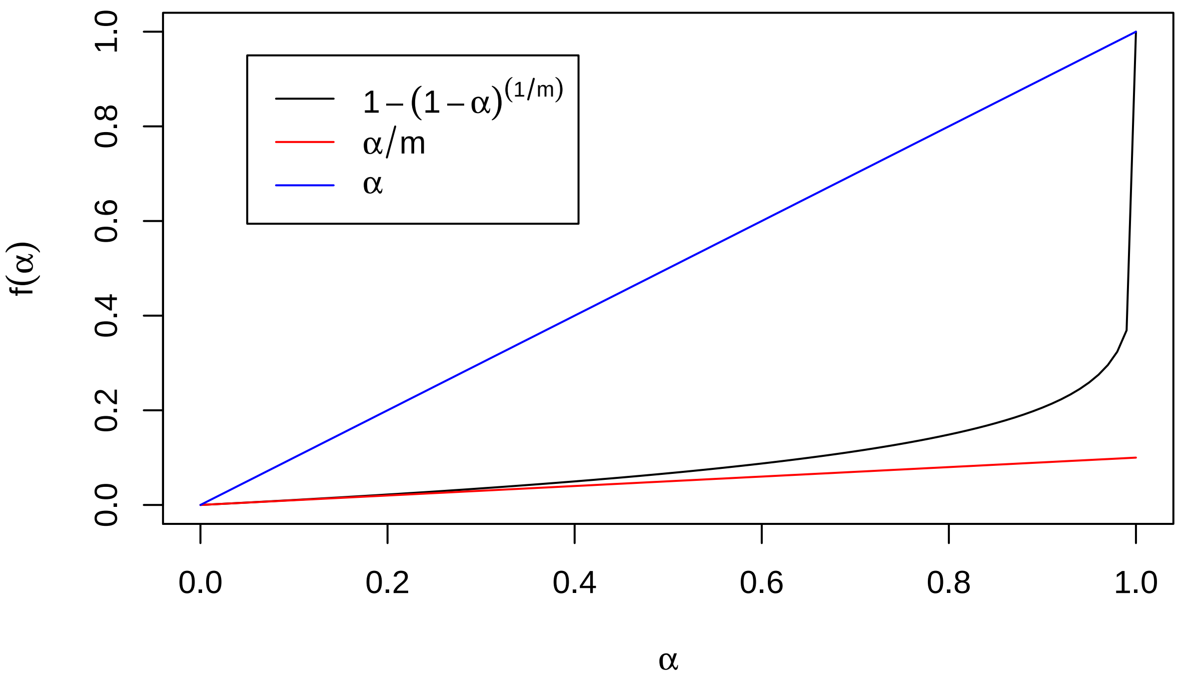
$p_r = 1 - (1-\alpha)^{\frac{1}{m}}$

2.4 For $\alpha \in [0, 1]$, check that $f_2 \leq f_1 \leq f_3$.

$f_1(\alpha) = 1 - (1-\alpha)^{\frac{1}{m}}$

$f_2(\alpha) = \frac{\alpha}{m}$

$f_3(\alpha) = \alpha$



2.5 (optional)

We can say $f_1 \leq f_3$ from the following:

$1 - (1-\alpha)^{\frac{1}{m}} \leq \alpha$

$1 - (1-\alpha)^{\frac{1}{m}} - \alpha \leq 0$

We know the expression $(1-\alpha)^{\frac{1}{m}}$ when $m = 1$ because $m \in [1, \inf]$ and $1 - \alpha \in [0, 1]$, and a fraction raised to a fraction is a greater fraction, increasing with the denominator of the exponent.

So the minimum value of $1 - (1-\alpha)^{\frac{1}{m}} - \alpha$ is given by $m = 1$ and

$1 - 1 + \alpha - \alpha = 0 \leq 0$, and $f_1 \leq f_3$ Q.E.D.

We can say $f_2 \leq f_1$ from the following:

$\frac{\alpha}{m} \leq 1 - (1-\alpha)^{\frac{1}{m}}$

$0 \leq m - m(1-\alpha)^{1/m} - \alpha$

The expression $m - m(1-\alpha)^{1/m}$ will always be less than or greater than alpha because as described above, a fraction raised a fraction is a greater fraction so the difference in these two terms m and $m - m(1-\alpha)^{1/m}$ is always greater than α and the difference increases with m. The minimum of m is 1 so for further proof, we can take the minimum to be:

$0 \leq 1 - (1-\alpha) - \alpha = 0$ Q.E.D.

2.6: Conclude that, under the independence assumptions, the level needed for each test can be determined and it is smaller than the approach that makes no correction for multiple testing.

In 2.3 it was shown the level needed for each test is $p_r = 1 - (1-\alpha)^{\frac{1}{m}}$ in order to achieve a level of α for multiple tests. The level that makes no correction for multiple testing is assuming $m = 1$:

No correction: $p_{uncorrected} = 1 - (1 - \alpha)$

Corrected: $p_{corrected} = 1 - (1-\alpha)^{\frac{1}{m}}$

We want to show that $p_{corrected} \leq p_{uncorrected}$

$1 - (1-\alpha)^{\frac{1}{m}} \leq 1 - (1 - \alpha)$

$-(1-\alpha)^{\frac{1}{m}} \leq -(1-\alpha)$ Multiplying both sides by -1, we reverse the sign.

$(1-\alpha)^{\frac{1}{m}} \geq (1-\alpha)$

Since $m \in [1, \inf]$ and $\alpha \in [0,1]$, this comparison is always true.

A conservative but robust test: Bonferroni

3. Upper-bounding the probability of at least one rejection.

Show that $\alpha \leq P_{C-H}[\cup_{i=1}^{m}\{p_i(Y_i) < \alpha\}] \leq m\alpha$

Using Boole's inequality we can say that:

$P_{C-H}[\cup_{i=1}^{m}\{p_i(Y_i) < \alpha\}] \leq \sum_{i=1}^{m} P(p_i(Y_i) < \alpha)$

From section 1 of the project we know that $P(P(p_i(Y_i) < \alpha) = \alpha$. Therefore, $\sum_{i=1}^{m} P(p_i(Y_i) < \alpha) = m\alpha$.

Hence, we have proved the upper bound $P_{C-H}[\cup_{i=1}^{m}\{p_i(Y_i) < \alpha\}] \leq m\alpha$.

The lower bound of α turns into an equality at $P_{C-H}[\cup_{i=1}^{m}\{p_i(Y_i) < \frac{\alpha}{m}\}]$, since, solving for p_{size},

$P_{C-H}[\cup_{i=1}^{m}\{p_i(Y_i) < p_{size}\}] \leq \sum_{i=1}^{m} P(p_i(Y_i) < p_{size}) = \alpha$

$mp_{size} = \alpha$

$p_{size} = \frac{\alpha}{m}$

This means that using a threshold probability of $\frac{\alpha}{m}$ for rejecting each individual test ensures that the overall type I error equals α.

Now, if $\alpha = P_{C-H}[\cup_{i=1}^{m}\{p_i(Y_i) < \frac{\alpha}{m}\}]$, for any $p_{size} \geqslant \frac{\alpha}{m}$ such as $p_{size} = \alpha$, $\alpha \leq P_{C-H}[\cup_{i=1}^{m}\{p_i(Y_i) < p_{size}\}]$, proving the lower bound, which will obsiusly turn into an equality when we only test once ($m = 1$). Alternatively, it is easy to see that for any set of events A_i where $i \in [1, m]$, $P(A_1) \leqslant P(\cup_{i=1}^{m} A_i)$. In this case, event i is $P(p_i(Y_i) < \alpha)$.

Ordered p-values, family-wise error rate and a new MT correction

4. Show that under the complete null the probability of at least one false rejection is α

If we reject test i when $p_{(i)} < \frac{i\alpha}{m}$, as long as y_i follow a uniform distribution between 0 and 1, we can also state that we reject test i when $y_i < \frac{i\alpha}{m}$. Thus, the probability of at least one false rejection is:

$P_H[\cup_{i=1}^{m}\{y_i < \frac{i\alpha}{m}\}]$

The probability of none of the tests being rejected is the complement of at least one rejection.

$P_H[\cup_{i=1}^{m}\{y_i < \frac{i\alpha}{m}\}] = 1 - P_H[\cap_{i=1}^{m}\{y_i > \frac{i\alpha}{m}\}]$

And we know that $P_H[\cap_{i=1}^{m}\{y_i > \frac{i\alpha}{m}\}] = 1 - \alpha$. Therefore,

$P_H[\cup_{i=1}^{m}\{y_i < \frac{i\alpha}{m}\}] = 1 - (1-\alpha)^m = 1 - (1-\alpha) = \alpha$