Group 8 - GLM Exercises

November 10, 2015

1. A density that belongs in the one-parameter exponential family has the following (canonical) representation:

$p(z) = ...$

where θ is called the natural parameter, and c and h are functions whose exact form depends on the particular density. For many members of this family, q $>$ 0, is a further unknown parameter, often called a precision parameter.

1. The following densities belong to the exponential family. Identify θ, q and an appropriate $c(\theta)$, for each of them (take into account that for some z might be a transformation of t):

Normal

[ADD ME]

Bernoulli

$t\tilde{}Bern(n,p) = p^z(1-p)^{1-z}$

$= exp\left\{ log(p^z) + log((1-p)^{1-z}) \right\}$

$= exp\left\{ zlog(p) + (1-z)log(1-p) \right\}$

$= exp\left\{ zlog(p) + log(1-p) - zlog(1-p) \right\}$

$= exp\left\{ zlog(\frac{p}{1-p}) + log(1-p) \right\}$

Result:

$\theta = log(\frac{p}{1-p})$

$q = 1$

$c(\theta) = -log(1-p)$

Binomial

$t\tilde{}Bin(n,p) = \binom{n}{k}p^k(1-p)^{n-k}$

$= exp\left\{ log(\binom{n}{k}) + log(p^k) + log((1-p)^{n-k}) \right\}$

$$= exp\left\{ log(\binom{n}{k}) + klog(p) + (n-k)log(1-p) \right\}$$

$$= exp\left\{ log(\binom{n}{k}) + klog(\frac{p}{1-p}) + nlog(1-p) \right\}$$

$$= exp\left\{ n\left[\frac{k}{n}log(\frac{p}{1-p}) + log(1-p) \right] + log(\binom{n}{k}) \right\}$$

Result:

$\theta = log(\frac{p}{1-p})$

$q = n$

$c(\theta) = log(\binom{n}{k})$

Poisson

$t\tilde{}Poisson(\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$

$$= exp\left\{ log(\lambda^k e^{-\lambda}) - log(k!) \right\}$$

$$= exp\left\{ k * log(\lambda) - \lambda log(e) - log(k!) \right\}$$

$$= exp\left\{ k * log(\lambda) - \lambda - log(k!) \right\}$$

Result:

$\theta = log(\lambda)$

$q = 1$

$c(\theta) = -\lambda$

2. Identify the canonical link function for each of the models given above.

I JUST LOOKED THIS UP I'M NOT SURE IT'S CORRECT.

Normal

μ

Bernoulli

$logit(\mu)$

Binomial

$logit(\mu)$

Poisson

$log(\mu)$

3. Consider now the generalised linear model:

$t_n|x_n -> NdEF(\theta(x_n, w), q\gamma_n)$ with

$\theta(x_n, w) = (c')^{-1}(g^{-1}(\phi(x_n)^T w)) =: f(\phi(x_n)^T w)$

2. R^2 and deviance

1. Show that for any linear regression model:

$-2logp(t|X, w_{MLE}, q_{MLE}) = Nloge^T e + const$

where "const" does not depend on M or X.

We know that:

$-2logp(t|X, w, q) = -Nlogq + q(t - \phi w)^T(t - \phi w) + const$

Maximizing with respect to w gives us:

$w_{mle} = (\phi^T \phi)^{-1} \phi^T t$

We plug this w_{mle} back into the log likelihood to get the likelihood function given w_{mle}:

$-2logp(t|X, w_{mle}, q) = -Nlog(q) + qe^T e$

Now, we can take the derivative wrt to q to solve for q_{mle} which gives us:

$q_{mle} = \left(\frac{1}{N} e^T e\right)^{-1}$

We use this to solve for the deviance of the maximum likelihood function:

$-2logp(t|X, w_{mle}, q_{mle}) = -Nlog(q_{mle}) + q_{mle} e^T e$

$-Nlog(q_{mle}) + q_{mle} e^T e = -Nlog\left((\frac{1}{N} e^T e)^{-1}\right) + \left(\frac{1}{N} e^T e\right)^{-1} e^T e$

$= Nlog\left((\frac{1}{N} e^T e)\right) + N$

$= N[log(e^T e) - log(N)] + N$

$= Nlog(e^T e) - N(log(N) + 1)$

$= Nlog(e^T e) + const$

2. Show that in the null model,

$w_{0,MLE} = \bar{t}$

[ADD ME]

3. The null model is nested within the saturated model, and it corresponds to the special case where w1 = · · · = wM = 0. Let D0 be the deviance of the null model and D1 be that of the saturated model. Show that:

$D_0 - D_1 = -Nlog(1 - R^2)$

where R^2 is the coefficient R^2 for the saturated model.

We know from **part 1** that:

$-2logp(t|X, w_{mle}, q_{mle}) = Nlog(e^T e) + const$

Which also what we equate as D_M for model M, so:

$D_0 - D_1 = Nlog(e_0^T e_0) - Nlog(e_1^T e_1)$

$= N\left[log(\frac{e_0^T e_0}{e_1^T e_1})\right]$

We can use the property that $e = t - \hat{t}$

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