Group 8 - GLM Exercises

November 10, 2015

1. A density that belongs in the one-parameter exponential family has the following (canonical) representation:

$p(z) = ...$

where θ is called the natural parameter, and c and h are functions whose exact form depends on the particular density. For many members of this family, q > 0, is a further unknown parameter, often called a precision parameter.

1. The following densities belong to the exponential family. Identify θ, q and an appropriate $c(\theta)$, for each of them (take into account that for some z might be a transformation of t):

Normal

[ADD ME]

Bernoulli

$t\tilde{}Bern(n,p) = p^z(1-p)^{1-z}$

$exp\left\{ log(p^z) + log((1-p)^{1-z}) \right\}$

$exp\left\{ zlog(p) + (1-z)log(1-p) \right\}$

$exp\left\{ zlog(p) + log(1-p) - zlog(1-p) \right\}$

$exp\left\{ zlog(\frac{p}{1-p}) + log(1-p) \right\}$

Result:

$\theta = log(\frac{p}{1-p})$

$q = 1$

$c(\theta) = -log(1-p)$

Binomial

$t\tilde{}Bin(n,p) = \binom{n}{k}p^k(1-p)^{n-k}$

$exp\left\{ log(\binom{n}{k}) + log(p^k) + log((1-p)^{n-k}) \right\}$

$$exp\left\{log(\binom{n}{k}) + klog(p) + (n-k)log(1-p)\right\}$$

$$exp\left\{log(\binom{n}{k}) + klog(\frac{p}{1-p}) + nlog(1-p)\right\}$$

$$exp\left\{n\left[\frac{k}{n}log(\frac{p}{1-p}) + log(1-p)\right] + log(\binom{n}{k})\right\}$$

Result:

$\theta = log(\frac{p}{1-p})$

$q = n$

$c(\theta) = log(\binom{n}{k})$

Poisson

[ADD ME]

2. Identify the canonical link function for each of the models given above.

I JUST LOOKED THIS UP I'M NOT SURE IT'S CORRECT.

Normal

μ

Bernoulli

$logit(\mu)$

Binomial

$logit(\mu)$

Poisson

$log(\mu)$

3. Consider now the generalised linear model:

$t_n|x_n -> NdEF(\theta(x_n, w), q\gamma_n)$ with

$\theta(x_n, w) = (c')^{-1}(g^{-1}(\phi(x_n)^T w)) =: f(\phi(x_n)^T w)$

2. R^2 and deviance

1. Show that for any linear regression model:

$-2logp(t|X, w_{MLE}, q_{MLE}) = Nloge^T e + const$

where "const" does not depend on M or X.

We know that:

$-2logp(t|X, w, q) = -Nlogq + q(t - \phi w)^T (t - \phi w) + const$

Maximizing with respect to w gives us:

$w_{mle} = (\phi^T \phi)^{-1} \phi^T t$

We plug this w_{mle} back into the log likelihood to get the likelihood function given w_{mle}:

$-2logp(t|X, w_{mle}, q) = -Nlog(q) + qe^T e$

Now, we can take the derivative wrt to q to solve for q_{mle} which gives us:

$q_{mle} = \left(\frac{1}{N} e^T e \right)^{-1}$

We use this to solve for the deviance of the maximum likelihood function:

$-2logp(t|X, w_{mle}, q_{mle}) = -Nlog(q_{mle}) + q_{mle} e^T e$

$-Nlog(q_{mle}) + q_{mle} e^T e = -Nlog\left((\frac{1}{N} e^T e)^{-1} \right) + \left(\frac{1}{N} e^T e \right)^{-1} e^T e$

$= Nlog\left((\frac{1}{N} e^T e) \right) + N$

$= N[log(e^T e) - log(N)] + N$

$= Nlog(e^T e) - N(log(N) + 1)$

$= Nlog(e^T e) + const$