Exercises from 'Regression and SVD'

1. Show that for V_r defined in the SVD of phi, $V_r V_r^T$ is a projection matrix and that $tr(V_r V_r^T) = r$.

If $(V_r V_r^T)$ is a projection matrix, this implies it must be equal to its square:

$(V_r V_r^T)^2 = V_r V_r^T$

We know $\phi^T \phi = U_r \Lambda_r U_r^T => (U_r^T)^{-1} \Lambda_r^{-1} U_r^{-1} = (\phi^T \phi)^{-1}$

We also know $\phi = V_r \Lambda_r^{1/2} U_r^T$

We use this equality as well as $(\Lambda_r^{-1/2})^T = \Lambda_r^{-1/2}$ and $((U_r^T)^{-1})^T = (U_r)^{-1}$ to solve for V_r and V_r^T:

$=> V_r = \phi (U_r^T)^{-1} \Lambda_r^{-1/2}$

$=> V_r^T = \Lambda_r^{-1/2} (U_r)^{-1} \phi^T$

$$(V_r V_r^T)^2 = V_r V_r^T V_r V_r^T$$

Expand the inner $V_r^T V_r$:

$V_r V_r^T V_r V_r^T = V_r (\Lambda_r^{-1/2} (U_r)^{-1} \phi^T)(\phi (U_r^T)^{-1} \Lambda_r^{-1/2}) V_r^T$

$= V_r (\Lambda_r)^{-1/2} (U_r)^{-1} (\phi^T \phi) (U_r^T)^{-1} (\Lambda_r)^{-1/2} V_r^T$

Replace $\phi^T \phi$ with its equality from above:

$= V_r (\Lambda_r)^{-1/2} (U_r)^{-1} U_r \Lambda_r U_r^T (U_r^T)^{-1} (\Lambda_r)^{-1/2} V_r^T$

Cancel out terms identical to the identity matrix ($(U_r)^{-1} U_r$ and $U_r^T (U_r^T)^{-1}$):

$= V_r (\Lambda_r)^{-1/2} \Lambda_r (\Lambda_r)^{-1/2} V_r^T$

Because $\Lambda_r^{-1/2} \Lambda_r \Lambda_r^{-1/2} = I$:

$= V_r V_r^T$

thus shown $V_r V_r^T$ is a projection matrix.

Show the trace $= V_r V_r^T =$ r

$tr(V_r V_r^T) = tr(V_r^T V_r)$

$tr(V_r^T V_r) = tr(\Lambda_r^{-1/2} (U_r)^{-1} \phi^T \phi (U_r^T)^{-1} \Lambda_r^{-1/2})$

$= tr(\Lambda_r^{-1/2} (U_r)^{-1} U_r \Lambda_r U_r^T (U_r^T)^{-1} \Lambda_r^{-1/2})$

$= tr(\Lambda_r^{-1/2} \Lambda_r \Lambda_r^{-1/2})$

$= tr(I_{(r)}) = r$

2. Weighted least squares: Consider the following small variation of the regression problem. The observations are not homoscedastic, that is $var(t_n)$ changes with n. Mathematically, the linear regression model now writes as $t|X \sim ((\phi)w, (qD)^{-1})$ where D is a known precision matrix (typically diagonal in practice). Show that the normal equations in this case become $\phi^T D \phi(w) = \phi^T D_t$.

The multi-Gaussian probability density function becomes:

$N(t|x, \phi w, (qD)^{-1}) = 1/(2\pi)^{D/2} (qD)^{1/2} e^{(-1/2(t-\phi w)^T qD(t-\phi w))}$

$log(\ N(t|x, \phi w, (qD)^{-1}) = log(1/(2\pi)^{D/2}) + (1/2) log(qD) - (1/2)(t - \phi w)^T qD(t - \phi w))$

$= log(1/(2\pi)^{D/2}) + (1/2)log(qD) - (1/2)(q)(t^T Dt - t^T D\phi w - w^T \phi^T Dt + w^T \phi^T D\phi w)$

Differentiating with respect to w and equating to 0. (-1/2) and (q) get cancelled off.

$0 = -t^T D\phi - t^T D^T \phi + w^T(\phi^T D\phi + (\phi^T D\phi)^T)$

$2t^T D^T \phi = w^T(2\phi^T D\phi)$ because $D^T = D$ and $(\phi^T D\phi)^T = (\phi^T D^T (\phi^T)^T) = \phi^T D\phi$

$$t^T D^T \phi = w^T \phi^T D\phi$$

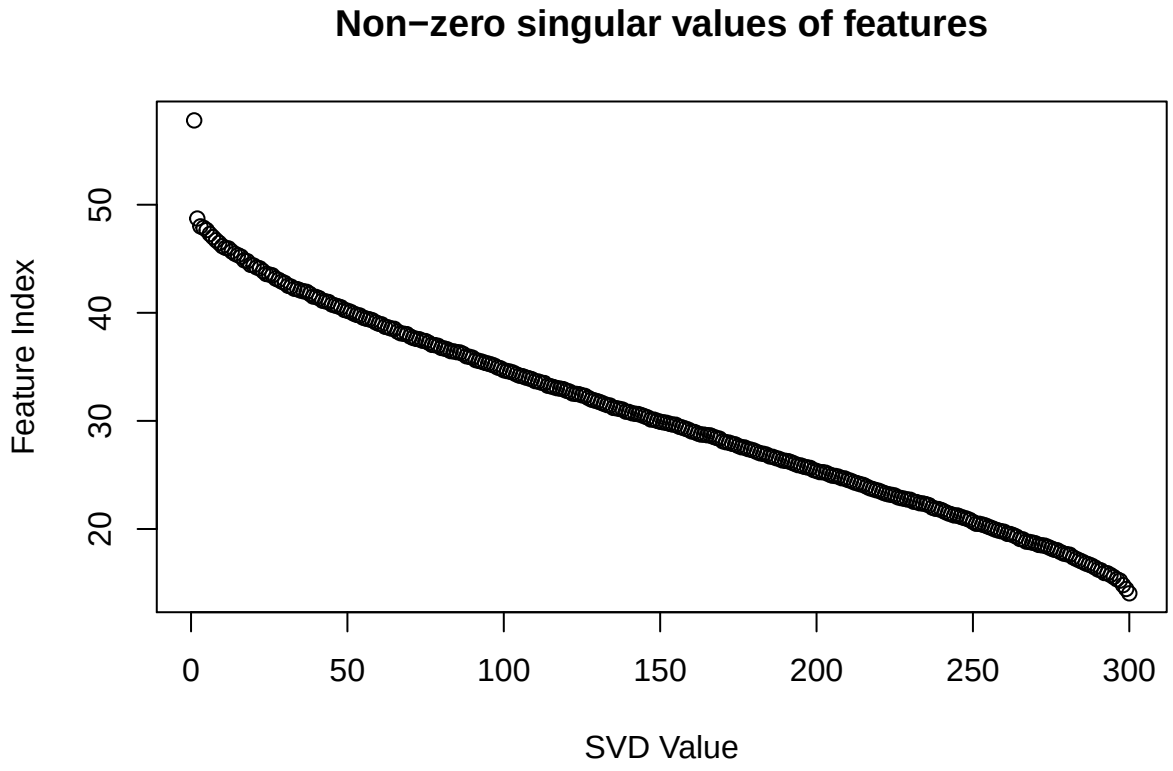
Taking the transpose on both sides:

$$\phi^T D(t^T)^T = \phi^T D^T (\phi^T)^T (w^T)^T$$

$$\phi^T Dt = \phi^T D\phi w$$

3. Extract the first 300 rows and 1001 columns from the synthetic_regression.txt dataset; hence you have 300 replica- tions and 1000 input variables.

3.1 Plot the non-zero singular values of the input matrix.

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## numeric(0)
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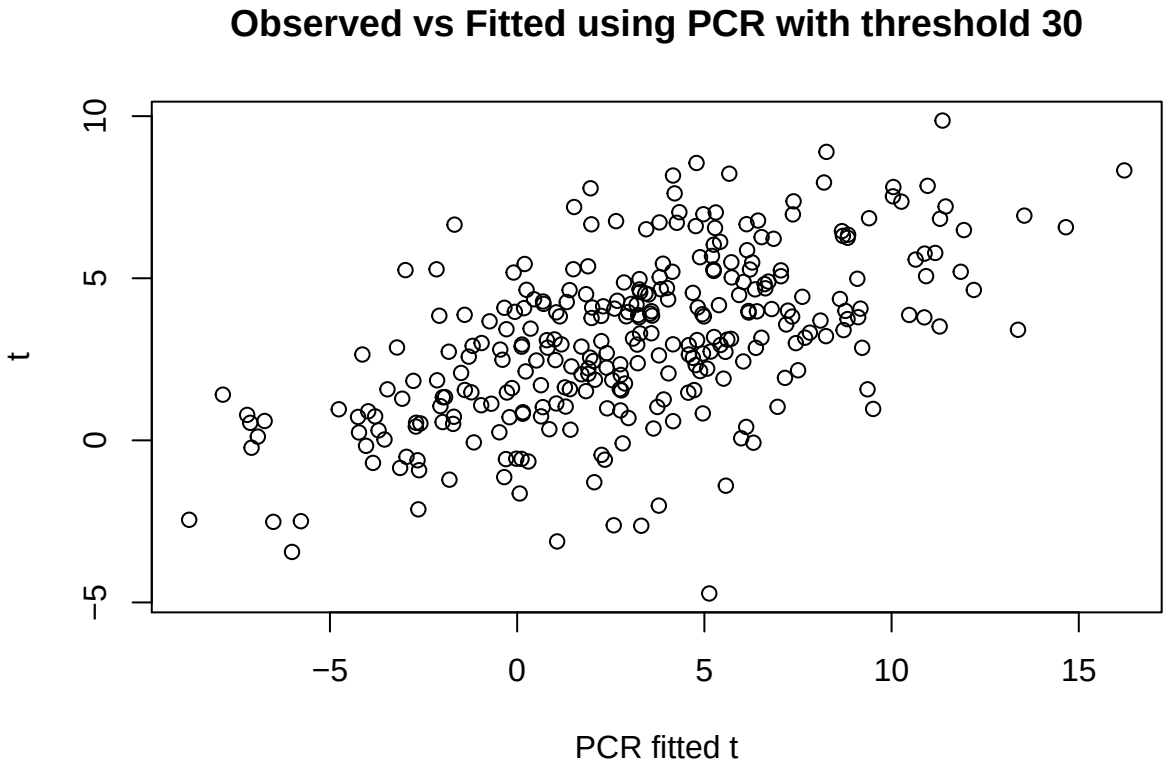
3.2 Determine r, the rank of the input matrix.

The rank of the input matrix is 300.

3.3 Do a principal component regression based on the largest 30 eigenvalues of the input matrix and by including an intercept term (hence 31 features in total). Plot fitted vs observed values according to this model. Compare the fit of this model to the one you obtained in a previous

exercise using the first (in order of appearance) 30 input variables and the intercept. Is it better or worse? Explain in at most 3 lines your findings and explanation.

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## numeric(0)
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The fit of the PCA model is slightly worse than the fit of the linear model.

The reason is the linear model minimizes the error with respect to the observed data plus the features whereas the PCA minimizes the error with respect to the features themselves. So the linear model is a better fit of the observed data.