Simplified Matrix-Based Approach to Connectivity-Focused Network Reliability Analysis

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Introduction

Kang et al. [2] introduce the matrix-based approach to system reliability analysis and demonstrate this approach on bridge networks. This approach has been applied to in various contexts [].

Alkaff et al. [1] develop the

Problem: Obtain the connectivity between hospital and town or county centers through a bridge network with failure probability matrix P.

Analytical Solutions Using Markov Chain

The computation $(I-P)^{-1}$ as a means of finding the probability of connectivity between nodes in a network is rooted in the field of matrix analysis and graph theory, particularly in the context of reliability and Markov chains. To understand why this method works, we can refer to the concepts of reliability in stochastic networks and the properties of the inverse matrix in relation to connectivity.

Key Concepts

1. **Stochastic Adjacency Matrix**:
   For a network represented by an adjacency matrix A, where A_{ij} indicates the presence (1) or absence (0) of an edge, we adapt this to a reliability matrix R where R_{ij} indicates the probability of the edge (i, j) being operational.

2. **Failure Probability Matrix P**:
   If P_{ij} represents the probability of edge (i, j) failing, then $1 - P_{ij}$ represents the probability of edge (i, j) being operational.

3. **Effective Adjacency Matrix**:
   The matrix $I - P$ is an adjusted form of the identity matrix I, where P represents the failure probabilities of the edges. This matrix modification incorporates the failure probabilities into the network's connectivity.

The Inverse Matrix in Network Connectivity

$$P_{\text{connectivity}}(i, j) = \left[(I - P)^{-1}\right]_{ij} \tag{1}$$

This formula computes the probability of paths existing between nodes, considering all possible paths and the correlation structure in the failure probabilities.

To understand why $(I-P)^{-1}$ provides the probability of connectivity, we need to delve into the properties of the inverse matrix and its interpretation in terms of walks or paths in the graph.

1. **Series Expansion and Path Counting**:
   Consider the Neumann series expansion for $(I-P)^{-1}$:
   $$(I-P)^{-1} = I + P + P^2 + P^3 + \cdots$$
   This series expansion sums up all possible powers of P, which correspond to walks of different lengths in the graph.
   Derivation: use one-step dynamic programming.

2. **Summing All Paths**:
   Each power P^k represents the probability of traveling from one node to another via a path of length k while considering the failure probabilities of the edges. By summing up these probabilities (for all k), $(I-P)^{-1}$ essentially captures the overall probability of connectivity between any two nodes, considering all possible paths.

3. **Graph Theory Interpretation**:
   In graph theory, this approach is related to the concept of computing the reachability or connectivity matrix, which indicates the presence of a path (and its reliability) between nodes in the network.

Mathematical Justification

The use of the inverse matrix to calculate connectivity probabilities can be justified by considering the stochastic interpretation of matrix P and its impact on the network's structure:

- **Stochastic Matrix**:
  If P is considered as a transition probability matrix in a Markov chain, $I-P$ can be interpreted as the fundamental matrix of the Markov process. The fundamental matrix captures the expected number of steps (or transitions) required to reach a certain state (or node).

- **Reliability Theory**:
  In reliability theory, the matrix $(I-P)^{-1}$ can be seen as a way to compute the expected reliability of the network by incorporating the failure probabilities in the path calculations. Each entry $(I-P)^{-1}_{ij}$ represents the total reliability of reaching node j from node i.

Summary

The inverse matrix $(I-P)^{-1}$ effectively aggregates the probabilities of all possible paths between nodes, taking into account the failure probabilities of the edges. This approach is grounded in both reliability theory and the properties of stochastic matrices, providing a comprehensive measure of network connectivity. The series expansion and the fundamental matrix interpretation provide the mathematical underpinning for why this method accurately computes connectivity probabilities.

Correlation Between Link Failure Probabilities

1. Mean-filed approximation The effective failure probability matrix P'_{ij}

In a bridge network with spatially correlated link failures, the mean-field approximation (MFA) simplifies the computation of effective failure probabilities by averaging the influence of correlations across the network. The objective is to derive an effective failure probability matrix P' that incorporates the impact of spatial correlations.

Let $P = \{P_{ij}\}$ denote the matrix of independent failure probabilities, where P_{ij} represents the probability of failure for the link between nodes i and j. Let $\Sigma = \{\Sigma_{(ij)(kl)}\}$ represent the pairwise correlation matrix, where $\Sigma_{(ij)(kl)}$ quantifies the correlation between the failures of links (i,j) and (k,l). Under the mean-field approximation, the effective failure probability for link (i,j) is influenced by the failure probabilities of all other links in the network, weighted by their correlations, which can be expressed as:

$$P'_{ij} = P_{ij} + \sum_{(k,l)\neq(i,j)} \Sigma_{(ij)(kl)} \cdot \langle P_{kl} \rangle, \tag{2}$$

where $\langle P_{kl} \rangle$ is the mean-field term representing the average failure probability over all links. To estimate $\langle P_{kl} \rangle$, the network-wide average failure probability, adjusted for correlations, is used:

$$\langle P_{kl} \rangle = \frac{\sum_{(k,l)} P_{kl} \left(1 + \sum_{(i,j)\neq(k,l)} \Sigma_{(kl)(ij)}\right)}{|\mathcal{E}|}, \tag{3}$$

where $|\mathcal{E}|$ is the total number of links in the network.

Since the effective failure probabilities depend on each other, the mean-field approximation requires iterative computation. Starting with an initial estimate $P'_{ij} = P_{ij}$, the values of P'_{ij} are updated iteratively as:

$$P'_{ij} = P_{ij} + \sum_{(k,l)\neq(i,j)} \Sigma_{(ij)(kl)} \cdot \langle P'_{kl} \rangle. \tag{4}$$

Convergence is achieved when the difference between successive iterations of P' is sufficiently small:

$$\|P' - P'_{\text{prev}}\| < \epsilon, \tag{5}$$

where ϵ is a predefined threshold. Once convergence is reached, the effective failure probability matrix P' is constructed. From P', the effective operational probability matrix $C' = \{C'_{ij}\}$ is computed as:

$$C'_{ij} = 1 - P'_{ij}. \tag{6}$$

In spatially correlated networks, the correlation matrix Σ often reflects the decay of correlations with distance. A common choice for $\Sigma_{(ij)(kl)}$ is:

$$\Sigma_{(ij)(kl)} = \exp\left(-\frac{\text{dist}((i,j),(k,l))}{\xi}\right), \tag{7}$$

where $\text{dist}((i,j),(k,l))$ represents the spatial distance between links (i,j) and (k,l), and ξ is a correlation decay parameter.

The mean-field approximation thus provides a practical way to incorporate spatial correlations into failure probability calculations that can strike a balance between computational efficiency and accuracy for large-scale network analyses.

Compare with Monte Carlo simulation for approximate failure probability matrix: 1. Generate correlated random variables for link failures using Σ and a copula model; 2. Compute the empirical failure probability of link (i,j) from the Monte Carlo samples.

Finally, compare with MCS in obtaining the connection probability.

References

[1] Abdullah Alkaff, Mochamad Nur Qomarudin, and Yusuf Bilfaqih. Network reliability analysis: Matrix-exponential approach. *Reliability Engineering & System Safety*, 204:107192, 2020.

[2] Won-Hee Kang, Junho Song, and Paolo Gardoni. Matrix-based system reliability method and applications to bridge networks. *Reliability Engineering & System Safety*, 93(11):1584–1593, 2008.