Available online at www.sciencedirect.com
ScienceDirect
Reliability Engineering and System Safety 93 (2008) 1584–1593
RELIABILITY ENGINEERING & SYSTEM SAFETY
www.elsevier.com/locate/ress

Matrix-based system reliability method and applications to bridge networks

Won-Hee Kang[a], Junho Song[a,*], Paolo Gardoni[b]

[a]*Department of Civil and Environmental Engineering, University of Illinois, Urbana-Champaign, IL 61801, USA*
[b]*Zachry Department of Civil Engineering, Texas A&M University, College Station, TX 77843, USA*

Received 7 February 2007; received in revised form 15 January 2008; accepted 9 February 2008
Available online 10 March 2008

Abstract

Using a matrix-based system reliability (MSR) method, one can estimate the probabilities of complex system events by simple matrix calculations. Unlike existing system reliability methods whose complexity depends highly on that of the system event, the MSR method describes any general system event in a simple matrix form and therefore provides a more convenient way of handling the system event and estimating its probability. Even in the case where one has incomplete information on the component probabilities and/or the statistical dependence thereof, the matrix-based framework enables us to estimate the narrowest bounds on the system failure probability by linear programming. This paper presents the MSR method and applies it to a transportation network consisting of bridge structures. The seismic failure probabilities of bridges are estimated by use of the predictive fragility curves developed by a Bayesian methodology based on experimental data and existing deterministic models of the seismic capacity and demand. Using the MSR method, the probability of disconnection between each city/county and a critical facility is estimated. The probability mass function of the number of failed bridges is computed as well. In order to quantify the relative importance of bridges, the MSR method is used to compute the conditional probabilities of bridge failures given that there is at least one city disconnected from the critical facility. The bounds on the probability of disconnection are also obtained for cases with incomplete information.

Keywords: Bayesian method; Bridge network; Complex system; Connectivity analysis; Fragility curve; Importance measure; Incomplete information; Reliability bounds; System reliability

1. Introduction

Many structures and lifelines are complex "systems" whose states are described as the Boolean functions of "component" events such as the occurrences of structural failure modes and the failures of constituent members or substructures. For reasonable decision-makings on structural designs, retrofits, repairs and damage mitigation strategies, it is essential to quantify the likelihood of such system events in an efficient manner. However, computing the probability of such a system event is often costly or infeasible due to the complexity of the system and/or the lack of complete information. For example, existing system reliability methods such as theoretical bounding formulas [1] and first-order system reliability method approximations [2] are applicable to only series and parallel systems with little flexibility in incorporating various types and levels of available information. It is possible to use theoretical bounding formulas for non-series or non-parallel systems by computing the probabilities of cut sets or link sets using the first-order system reliability method. However, this approach can be cumbersome and may provide inaccurate estimates. Song and Der Kiureghian [3] proposed a linear programming (LP) bounds method for computing bounds on the failure probability of any general systems with enhanced flexibility in incorporating available information. The method divides the sample space into the mutually exclusive events and describes the system failure probability and the available information by use of vectors and matrices. By solving an LP problem, one can obtain

*Corresponding author. Tel.: +1 217 244 9307; fax: +1 217 265 8040.
E-mail addresses: wkang3@uiuc.edu (W.-H. Kang), junho@uiuc.edu (J. Song), pgardoni@civil.tamu.edu (P. Gardoni).

0951-8320/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ress.2008.02.011

the narrowest possible bounds on the system failure probability.

Song and Kang [4] proposed a matrix-based system reliability (MSR) method that generalizes the LP bounds method to make use of the matrix-based representation of system events in the case of complete information as well. Unlike existing system reliability methods, this MSR method is uniformly applicable to general systems. The complexity of computations is not affected by that of the system event definition as the system reliability is computed by simple matrix calculations. Since the matrix formulation of a system can be obtained by algebraic manipulations of matrices representing component events or other system events, the MSR method provides a convenient way of identifying/handling the system events and estimating the probabilities thereof. In the case when only incomplete information is available, one can obtain the narrowest bounds on the system probability by solving an LP. This is equivalent to the LP bounds method. The MSR method has been further developed to compute the sensitivities of system reliabilities with respect to parameters in an efficient manner and to account for the statistical dependence between component events even in the case where it is not explicitly identified [5].

Bridge structures are important components of a transportation network that assure life safety, rescue and recovery efforts. However, they are vulnerable to seismic hazards and therefore often considered as the weakest links of a network. Thus, it is an essential task to estimate the seismic reliability of a transportation network based on the failure probabilities of bridge structures. This is a challenging task because the complexity of system failure events often makes it costly or impractical to identify and handle the system events using conventional cut sets or link sets. Hence, most of the previous studies on the risk of bridge networks used sampling-based system reliability methods [6–8]. The MSR method can estimate the probabilities of complex system events analytically and hence requires significantly less computation time than sampling-based methods. It also provides byproducts such as importance measures [9] and conditional probabilities that can quantify the relative contributions of components or subsystems to the likelihood of system failure events.

This paper presents the MSR method and applies it to a numerical example of transportation network for quantifying the reliability of a bridge network based on the seismic vulnerability of its constituent bridges. This study makes use of predictive fragility curves developed by a Bayesian methodology [10,11] for estimating the failure probabilities of the bridges. The probability of disconnection between each city/county and a critical facility is computed by the MSR method. We also estimate the probability mass function of the number of failed bridges and the conditional probabilities of component failures given the system failure. These conditional probabilities are used as importance measures that quantify the relative importance of the bridges with respect to disconnection events. The bounds on the probability of disconnection are also obtained for cases with incomplete information by solving an LP.

2. Matrix-based system reliability method

2.1. Matrix-based formulation of system reliability

Consider a system whose ith component has s_i distinct states, $i=1,\ldots,n$. The sample space can be divided into $m=\Pi_{i=1}^{n}s_i$ mutually exclusive and collectively exhaustive (MECE) events. We name these the "basic" MECE events and denote them by e_j, $j=1,\ldots,m$. One can describe any event by identifying the basic MECE events that belong to it. Therefore, a general system event can be represented by an "event" vector $\mathbf{c}$ whose jth element is 1 if e_j belongs to the system event and 0 otherwise. Let $p_j=P(e_j)$, $j=1,\ldots,m$, denote the probability of e_j. Due to the mutual exclusiveness of e_j's, the probability of a system event, E_{sys}, i.e., $P(E_{\text{sys}})$ is simply the sum of the probabilities of e_j's that belong to the system event. Therefore, the system failure probability is computed by a simple vector calculation

$$P(E_{\text{sys}}) = \sum_{j:e_j\subseteq E_{\text{sys}}} p_j = \mathbf{c}^{\mathrm{T}}\mathbf{p} \tag{1}$$

where $\mathbf{p}$ is the "probability" vector that contains p_j's.

As an example, let us consider a three-component system in which each component has two distinct states, i.e., failure (E_i) and non-failure ($\bar{E}_i$). Fig. 1 illustrates its sample space divided into $m=2^3=8$ basic MECE events, i.e., $e_1,\ldots,e_8$. A general system event $E_{\text{sys}}=E_1E_2\cup E_3$ is the union of the basic MECE events e_1, e_2, e_3, e_4 and e_5 as shown by the shaded area in Fig. 1. Due to the mutual exclusivity of e_j's, the probability of the system event is the sum of the probabilities of e_j's, that is,

$$P(E_{\text{sys}}) = p_1+p_2+p_3+p_4+p_5 \tag{2}$$

Therefore, the vector $\mathbf{c}=[1\quad 1\quad 1\quad 1\quad 1\quad 0\quad 0\quad 0]^{\mathrm{T}}$ can represent the system event.

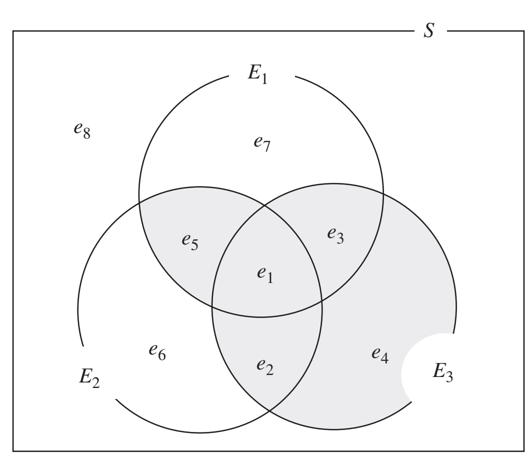

Fig. 1. Sample space for a three-component system.

The formulation in Eq. (1) can be generalized to compute the probabilities of multiple system events under multiple conditions of component failures by a single matrix multiplication

$$\mathbf{P}_{\text{sys}} = \mathbf{C}^{\text{T}}\mathbf{P} \tag{3a}$$

$$\mathbf{C} = [\,\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_{N_{sys}}\,] \tag{3b}$$

$$\mathbf{P} = [\,\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_{N_{\text{cond}}}\,] \tag{3c}$$

where $\mathbf{c}_i$, $i = 1,\ldots,N_{\text{sys}}$, is the event vector of the ith system event; $\mathbf{p}_j$, $i = 1,\ldots,N_{\text{cond}}$, denotes the probability vector for the jth condition; and $\mathbf{P}_{\text{sys}}$ is the matrix whose element at the ith row and the jth column is the probability of the ith system event under the jth condition. We name this as the MSR method [4].

The MSR method has the following merits over existing system reliability methods. First, the complexity of the system reliability computations is not affected by that of the system event definition because the reliability of a system event is calculated by a single matrix multiplication regardless of the system definition. Second, the matrix-based formulation allows us to identify/handle the system events conveniently and compute the corresponding probabilities efficiently. Third, if one has incomplete information on the component failure probabilities or their statistical dependence, the matrix-based framework enables us to obtain the narrowest possible bounds on any general system event. This is equivalent to the LP bounds method [3]. Fourth, one can calculate the conditional probabilities and various importance measures [9] using the MSR method without introducing further complexity. Fifth, the recent developments of matrix-based computer languages and software including MATLAB® and Octave rendered matrix calculations more efficient and easier to implement. Finally, the MSR method can be extended for evaluating various system reliability metrics such as average connectivity loss or service flow reduction [12] by describing the average or exceedance probability of such a metric in terms of **C** and **P**.

A potential drawback of the MSR method is that the sizes of vectors and matrices increase exponentially with the number of component events, which may require enormous capacity of computing memory for systems with a large number of components. However, we can overcome this drawback by transforming a large MSR problem into multiple smaller problems using the multi-scale approach [13] or evaluate the **p** vector elements only for the basic MECE events that are included in the system event.

In the following two subsections, we discuss how to identify the event vector **c** and calculate the probability vector **p** efficiently by use of MATLAB® language. While the examples in this paper deal with systems with bi-state components, the proposed methodology is applicable to systems with general multi-state components.

2.2. Identification of event vector c

One could directly identify the event vector **c** of a simple system event as demonstrated in the previous section. However, this approach may become infeasible or inefficient as the complexity of the system increases. An important merit of the MSR method is that one can construct the event vector of a system event by simple matrix manipulations of the event vectors of components or other subsystem events.

First, let $\mathbf{c}^E$ denote the event vector of a generic event E. The event vector of the complementary event of an event E is obtained by

$$\mathbf{c}^{\bar{E}} = \mathbf{1} - \mathbf{c}^E \tag{4}$$

where **1** denotes a vector of 1's. The intersection and the union of events $E_1, E_2, \ldots, E_n$ are, respectively,

$$\mathbf{c}^{E_1 \cdots E_n} = \mathbf{c}^{E_1} .\!* \mathbf{c}^{E_2} .\!* \cdots .\!* \mathbf{c}^{E_n} \tag{5a}$$

$$\mathbf{c}^{E_1 \cup \cdots \cup E_n} = \mathbf{1} - (\mathbf{1} - \mathbf{c}^{E_1}) .\!* (\mathbf{1} - \mathbf{c}^{E_2}) .\!* \cdots .\!* (\mathbf{1} - \mathbf{c}^{E_n}) \tag{5b}$$

where "$.*$" is the MATLAB® operator for element-by-element multiplication. Using a matrix-based language, one can perform the calculations in Eqs. (4) and (5) simply by single-line expressions with improved efficiency. We explain this with an example system event, $E_{\text{sys}} = E_1 E_2 \cup E_3$. The event vectors of the three component events are

$$\mathbf{c}^{E_1} = [1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0]^{\text{T}} \tag{6a}$$

$$\mathbf{c}^{E_2} = [1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0]^{\text{T}} \tag{6b}$$

$$\mathbf{c}^{E_3} = [1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^{\text{T}} \tag{6c}$$

These component event vectors can be generated easily by computers. Using the matrix manipulations in Eq. (5) subsequently, the event vector of E_{sys} is obtained as

$$\begin{aligned} \mathbf{c}^{E_{\text{sys}}} &= \mathbf{1} - (\mathbf{1} - \mathbf{c}^{E_1 E_2}) .\!* (\mathbf{1} - \mathbf{c}^{E_3}) \\ &= \mathbf{1} - (\mathbf{1} - \mathbf{c}^{E_1} .\!* \mathbf{c}^{E_2}) .\!* (\mathbf{1} - \mathbf{c}^{E_3}) \\ &= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]^{\text{T}} \end{aligned} \tag{7}$$

This is identical to the vector identified directly from Eqs. (1) and (2). It is evident that one can easily find the event vectors for any general system event from its Boolean descriptions.

In the case when the system event has not been identified as a Boolean description due to the complexity or a large number of cut sets or link sets, one can develop a computer algorithm to construct the event vector from the vectors of components or other system events. When only a subset of cut sets or link sets is identified, an MSR analysis employing the event vector based on the subset provides lower and upper bounds on the system failure probability.

2.3. Computation of probability vector p

Let us consider a system whose component failure probabilities are all available and the component events are statistically independent. In this case, the probability vector has as many elements as the basic MECE events while each element is the product of probabilities of components and their complementary events. If each element is computed one by one, this can be an enormous numerical task. In order to overcome this, we propose a matrix-based procedure to construct the probability vectors in an efficient manner.

The probability vector for a system with component 1 only is

$$\mathbf{p}_{[1]} = \begin{bmatrix} P_1 \\ \bar{P}_1 \end{bmatrix} \tag{8}$$

where P_1 and $\bar{P}_1 = 1 - P_1$, respectively, denote the probability of the component event 1 and its complementary event. The probability vector for a system with two and three components can be constructed by sequential matrix calculations as follows:

$$\mathbf{p}_{[2]} = \begin{bmatrix} \mathbf{p}_{[1]} \cdot P_2 \\ \mathbf{p}_{[1]} \cdot \bar{P}_2 \end{bmatrix} = \begin{bmatrix} P_1P_2 \\ \bar{P}_1P_2 \\ P_1\bar{P}_2 \\ \bar{P}_1\bar{P}_2 \end{bmatrix} \tag{9a}$$

$$\mathbf{p}_{[3]} = \begin{bmatrix} \mathbf{p}_{[2]} \cdot P_3 \\ \mathbf{p}_{[2]} \cdot \bar{P}_3 \end{bmatrix} = \begin{bmatrix} P_1P_2P_3 \\ \bar{P}_1P_2P_3 \\ P_1\bar{P}_2P_3 \\ \bar{P}_1\bar{P}_2P_3 \\ P_1P_2\bar{P}_3 \\ \bar{P}_1P_2\bar{P}_3 \\ P_1\bar{P}_2\bar{P}_3 \\ \bar{P}_1\bar{P}_2\bar{P}_3 \end{bmatrix} \tag{9b}$$

Thus, we can construct the probability vector for a system with n components by sequentially performing the matrix calculation

$$\mathbf{p}_{[i+1]} = \begin{bmatrix} \mathbf{p}_{[i]} \cdot P_{i+1} \\ \mathbf{p}_{[i]} \cdot \bar{P}_{i+1} \end{bmatrix} \tag{10}$$

for $i = 1,2,\ldots,(n-1)$.

Fig. 2 compares the CPU times required to construct the probability vectors for systems with 2–20 components by element-wise computations and by the matrix calculations in Eq. (10). For the test, MATLAB® on a computer with Dual Pentium Processors (2.80 GHz each) and 1 GB of RAM was used. The matrix-based approach constructs the probability vectors with much more efficiency. For example, the CPU times for a system with 20 components were 1219.0 s (element-wise) and 0.0629 s (Eq. (10)), respectively.

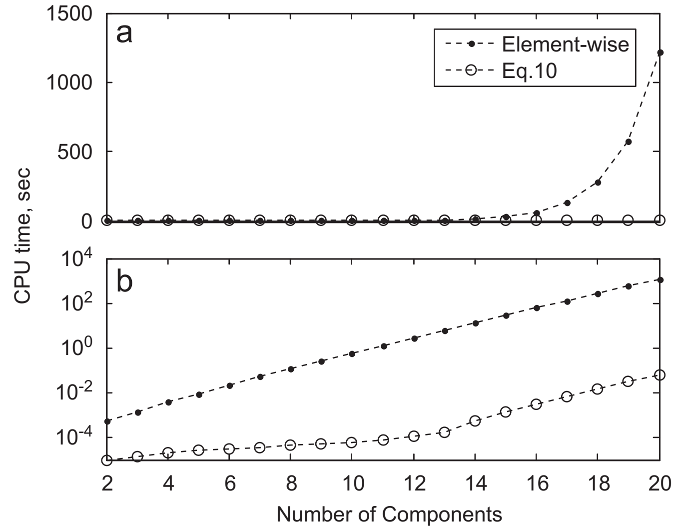

Fig. 2. CPU times to construct probability vectors: (a) linear scale and (b) log scale.

2.4. Statistical dependence between components

When component events have statistical dependence, it may be a daunting task to construct the **p** vector because the basic MECE events cannot be computed simply by products of component probabilities. However, in many structural system reliability problems, we can achieve conditional independence between component events given outcomes of a few random variables representing the "environmental dependence" or "common source effects". For example, when a system event of a bridge network is considered, the component (bridge) failures may be considered conditionally independent of the failures of other bridges given a seismic intensity, especially when common source effects by other randomness are negligible.

Let $\mathbf{X}$ and $f_{\mathbf{X}}(\mathbf{x})$ denote a vector of random variables causing the statistical dependence between components, termed as common source random variables (CSRVs) and its joint probability distribution function (PDF). Then, by the total probability theorem, the system failure probability can be computed as

$$P(E_{\text{sys}}) = \int_{\mathbf{x}} P(E_{\text{sys}}|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{11}$$

where $P(E_{\text{sys}}|\mathbf{x})$ denotes the conditional failure probability of the system given outcome $\mathbf{X} = \mathbf{x}$. Using the MSR formulation in Eq. (1), one can compute the system failure probability as

$$P(E_{\text{sys}}) = \int_{\mathbf{x}} \mathbf{c}^{\mathrm{T}} \mathbf{p}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{12}$$

where $\mathbf{p}(\mathbf{x})$ is the vector of the conditional probabilities of the basic MECE events given $\mathbf{X} = \mathbf{x}$. Due to the conditional independence of the components given $\mathbf{X} = \mathbf{x}$, one can construct $\mathbf{p}(\mathbf{x})$ efficiently by the sequential matrix manipulations in Eqs. (8) and (10). When the CSRVs are not explicitly identified, one can represent them by random variables using the Dunnett–Sobel correlation coefficient matrix [5].

Since the system event definition, represented by the event vector $\mathbf{c}$, is not affected by the outcome of the CSRVs, the system failure probability can be alternatively estimated as

$$P(E_{\text{sys}}) = \mathbf{c}^{\mathrm{T}} \int_{\mathbf{x}} \mathbf{p}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \mathbf{c}^{\mathrm{T}} \tilde{\mathbf{p}} \tag{13}$$

It is noteworthy that even with the consideration of statistical dependence, we need to perform the matrix multiplication only once. The "predictive" probability vector $\tilde{\mathbf{p}}$ can be obtained by discretization over the space of $\mathbf{x}$ and numerical integration. This approach is efficient when $\mathbf{X}$ has a small number of random variables. In the case when the system has a relatively large number of CSRVs, the system probability in Eq. (12) can be estimated by an efficient probability integration algorithm.

2.5. *Incomplete information*

Sometimes it is impossible to construct the probability vector completely if some component probabilities are missing or only their bounds are known. If the conditional independence is not achievable for a system with statistically dependent components, one may have only low-order joint probabilities. Even in this case, the matrix-based system formulation enables us to obtain the narrowest possible bounds on the probability of a system event by solving the LP problem:

$$\text{minimize(maximize)}\, \mathbf{c}^{\mathrm{T}}\mathbf{p} \tag{14a}$$

$$\text{subject to } \mathbf{A}_1\mathbf{p} = \mathbf{b}_1 \tag{14b}$$

$$\mathbf{A}_2\mathbf{p} \geqslant \mathbf{b}_2 \tag{14c}$$

$$\mathbf{A}_3\mathbf{p} \leqslant \mathbf{b}_3 \tag{14d}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ denote the matrices whose rows are the event vectors for which exact probabilities or bounds are available, and $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ are the vectors of available probabilities or bounds. This "LP bounds method" [3] has been successfully applied to structures, lifelines and structural systems under stochastic excitations [3,14,15]. For use in a large-scale problem, a multi-scale approach has also been proposed [13].

2.6. *Conditional probability and importance measures*

In order to measure the relative importance of components or cut sets, many importance measures (IM) have been introduced and used in the system engineering community. For example, Fussell–Vesely IM [16] of the ith component event E_i is defined as

$$\mathrm{FV}_i = \frac{P(\cup_{k:C_k \supseteq E_i} C_k)}{P(E_{\text{sys}})} \tag{15}$$

where C_k denotes the kth cut set of the system event. Song and Der Kiureghian [9] reviewed several IMs and proposed methods to compute them by the LP bounds method. We propose to use the conditional probability of the component event given the system failure as an IM of the component as well. By the definition of the conditional probability, this IM is computed as

$$\mathrm{CIM}_i = P(E_i | E_{\text{sys}}) = \frac{P(E_i E_{\text{sys}})}{P(E_{\text{sys}})} \tag{16}$$

As seen in Eqs. (15) and (16), most IMs are defined as the ratio of the probability of a new system event E'_{sys} to that of the system event of interest E_{sys}. Therefore in the MSR formulation, an IM is computed by

$$\mathrm{IM} = P(E'_{\text{sys}})/P(E_{\text{sys}}) = (\mathbf{c}'^{\mathrm{T}}\mathbf{p})/(\mathbf{c}^{\mathrm{T}}\mathbf{p}) \tag{17}$$

where $\mathbf{c}'$ is the event vector of E'_{sys}. Note that once the MSR method is performed for the system event of interest, the only significant tasks required for calculating various IMs is to find the event vector for the new system event E'_{sys} and another matrix multiplication. The probability vector does not have to be re-constructed. The event vector of the new system event is easily obtained by the matrix manipulations introduced in Eqs. (4) and (5).

3. Example: connectivity of bridge network

As an application, we consider a traffic network that connects eight cities by highways with 12 bridges. Fig. 3 shows the cities and the bridges in the network by circles and squares, respectively, along with their identification numbers. For simplicity, we assume that there are no other routes between cities than the highways shown in Fig. 3. It is also assumed that the bridges are the only components of the highway system whose seismic damages may cause paths to be disconnected. Suppose City 1 has a major hospital that should be accessible from the other cities in case of emergency. The concurrent failures of some bridges may isolate a city from the hospital for a prolonged period after an earthquake event. For decision-makings on the retrofits of bridges or general mitigation strategies, it is essential to estimate the probability of such disconnection events based on the fragility estimates of bridge structures and a seismic hazard model. However, the events of disconnections are so complex that it is difficult to identify all the cut sets or link sets, and to compute the probability of disconnections analytically. This example demonstrates the merits of the proposed MSR method in identifying/handling various complex system events and estimating the probabilities thereof.

For the bridge failure probabilities given a seismic intensity, we make use of predictive fragility estimates based on multi-variate probabilistic capacity and demand models developed by a Bayesian framework [10,11]. These models properly account for both aleatory and epistemic uncertainties, and correct the conservatism inherent in the deterministic models. We consider two bridge configurations,

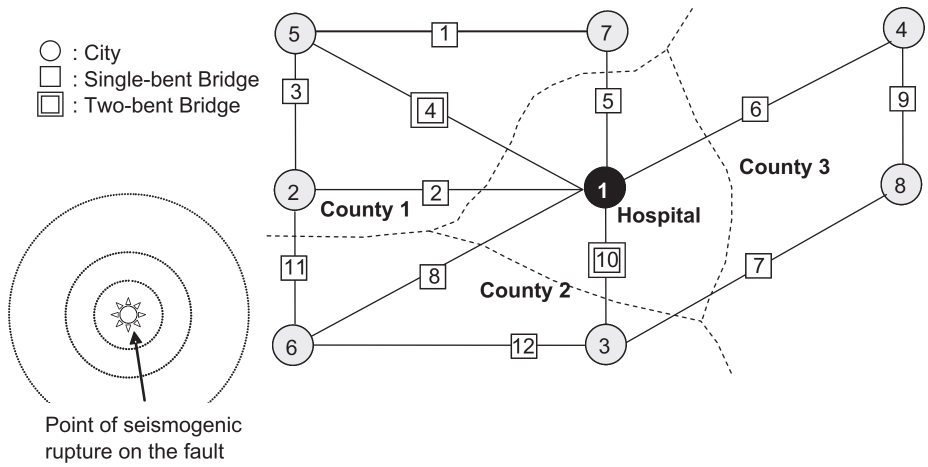

Fig. 3. Example bridge network.

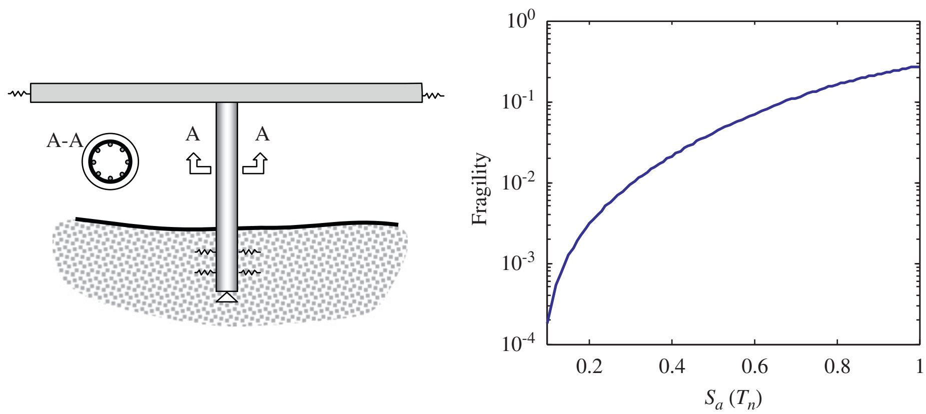

Fig. 4. Example single-bent overpass bridge (not to scale) (left) and corresponding fragility function (right) [11].

single-bent (Fig. 4) and two-bent overpasses (Fig. 5). In the example network, Bridges 4 and 10 have two-bent overpasses while the other bridges have one-bents. The bridges are designed by Mackie and Stojadinovic [17] according to Caltrans' Bridge Design Specification and Seismic Design Criteria [18]. Details on the design parameters for the overpass bridges are defined in Gardoni et al. [11]. In particular, the means of the natural periods of the single-bent and the two-bent overpasses are 0.8 and 1.01 s, respectively. Figs. 4 and 5 also show the corresponding fragility estimates for given spectral acceleration, S_a, where the fragility is defined as the conditional probability of attaining or exceeding specified shear and deformation performance levels for a given value of S_a. It is noted that the shear and deformation fragilities used in the paper are conditional probabilities of failure of a bridge. In this case, a bridge subject to an earthquake is in either of the following two damage states: failure and not failure. However, other fragilities that correspond to intermediate performance levels (e.g., immediate service level, and repairable damage) could also be used. In this case a bridge subject to an earthquake can be in multiple damage states, for example, no or insignificant damage, moderate damage, heavy damage and complete failure. Each performance level can be used to delimit two damage states and the fragilities can be used to compute the probability of being in each damage state. The probabilities of being in each damage state can then be used with the proposed MSR method to assess the probability of being in a network-level damage state that can be expressed, for example, in terms of number of lanes open between network nodes instead of simply connectivity.

We consider earthquakes that may occur at a point of seismogenic rupture on the nearby fault shown in Fig. 3. The earthquake magnitude M is assumed to follow the truncated exponential distribution [19] whose PDF $f_M(m)$ is given as

$$f_M(m) = \begin{cases} \dfrac{\beta \exp[-\beta(m-m_0)]}{1-\exp[-\beta(m_\mathrm{p}-m_0)]} & \text{for } m_0 \leqslant m \leqslant m_\mathrm{p} \\ 0 & \text{elsewhere} \end{cases} \tag{18}$$

where β is a parameter that determines the shape of the distribution, and m_0 and m_p are minimum and maximum

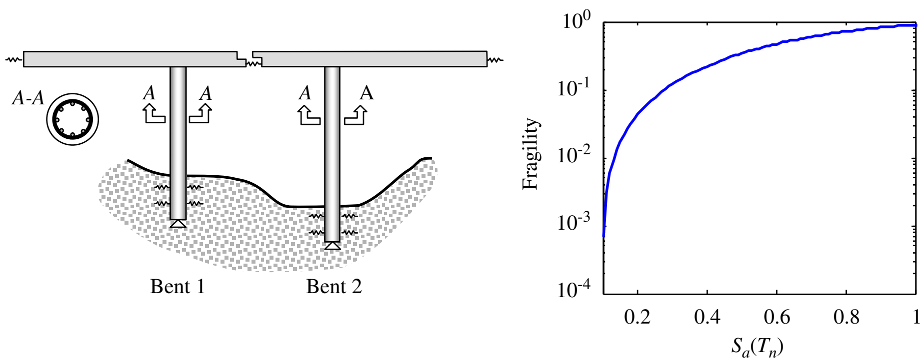

Fig. 5. Example two-bent overpass bridge (not to scale) (left) and corresponding fragility function (right) [11].

thresholds of the considered magnitudes. In this example, we use $\beta = 0.76$, $m_0 = 6.0$ and $m_{\mathrm{p}} = 8.5$.

The spectral accelerations S_{a} at the bridge sites are estimated in two steps. First we propagate the horizontal component of the peak ground acceleration A_{H}, from the epicenter of the earthquake to the location of the bridge. Then we convert A_{H} into the corresponding S_{a} based on the natural period of the bridge. The following attenuation relationship developed by Campbell [20] is used for propagating A_{H}:

$$\begin{aligned}\ln(A_{\mathrm{H}}) = & -3.512 + 0.904M \\ & -1.328 \ln\left\{\sqrt{r^2 + [0.149 \exp(0.647M)]^2}\right\} \\ & + [1.125 - 0.112 \ln r - 0.0957M]F \\ & + [0.440 - 0.171 \ln r]S_{\mathrm{SR}} + [0.405 - 0.222 \ln r]S_{HR}\end{aligned} \tag{19}$$

where F represents the fault type, here assumed to be 0 for strike-slip type faulting; S_{SR} and S_{HR} define the local site conditions, here assumed to be alluvium or firm soil ($S_{\mathrm{SR}} = S_{\mathrm{HR}} = 0$); and r is the distance between the site of interest (bridge) and the epicenter. The distances of Bridges 1–12 in the network are 12.0, 9.4, 7.5, 10.1, 14.6, 19.2, 18.0, 8.3, 23.9, 13.0, 3.3 and 9.6 km, respectively. The value of S_{a} for each bridge is obtained by multiplying A_{H} at each bridge site by 1.1 for the single-bent overpass, and by 0.75 for the two-bent overpass considering their expected natural periods. These multiplication factors are obtained from the response spectrum in Chopra [21].

Example 1. Disconnection between cities and hospital.

We first estimate the probability that each city is disconnected from the hospital by using the MSR method. If a path has at least one failed bridge, the path is not available. When all the paths connecting a city and the hospital are unavailable, the city is disconnected from the hospital. Therefore, the disconnection event is represented by a link set system event. Taking advantage of the matrix-based framework, we identify a single system event vector directly from matrix manipulations instead of identifying and handling numerous link sets.

For example, consider the event that City 5 is disconnected from City 1. There are six distinct paths between the cities: $\{5\to1\}$, $\{5\to2\to1\}$, $\{5\to7\to1\}$, $\{5\to2\to6\to1\}$, $\{5\to2\to6\to3\to1\}$ and $\{5\to2\to6\to3\to8\to4\to1\}$. We first construct an event vector representing the failure of bridges $\mathbf{c}^{E_i}$, $i = 1,2,\ldots,12$. From these component event vectors, the event vector of the disconnection of each path is obtained by Eq. (5b). For example, the path $\{5\to2\to1\}$ involves Bridges 2 and 3, and therefore its event vector is

$$\mathbf{c}^{\{5\to2\to1\}} = 1 - (1 - \mathbf{c}^{E_2}).*(1 - \mathbf{c}^{E_3}) \tag{20}$$

Then, we construct the event vector of the disconnection from the event vectors of all the paths using Eq. (5a):

$$\begin{aligned}\mathbf{c} = & \mathbf{c}^{\{5\to1\}}.*\mathbf{c}^{\{5\to2\to1\}}.*\mathbf{c}^{\{5\to7\to1\}}.*\mathbf{c}^{\{5\to2\to6\to1\}} \\ & .*\mathbf{c}^{\{5\to2\to6\to3\to1\}}.*\mathbf{c}^{\{5\to2\to6\to3\to8\to4\to1\}}\end{aligned} \tag{21}$$

All these processes are performed by a matrix-based computer code.

In this example, we assume the only source of statistical dependence between the failures of bridges is the uncertain earthquake magnitude M. While there may exist dependencies due to commonalities in the experienced deteriorating conditions, maintenance, etc., conditional independence of bridge failures given a seismic intensity is usually a reasonable approximation. However, it is noted that the MSR method can account for the statistical dependence between the component events in general even in the case where the statistical dependence is not explicitly identified [5]. For a given magnitude $M = m$, we estimate the spectral acceleration S_a at each bridge site and find the corresponding failure probability from the fragility function (Fig. 4 or 5). Taking advantage of the conditional independence, we construct the conditional probability vector for a given earthquake magnitude $M{=}m$, denoted by $\mathbf{p}(m)$, using the matrix-based procedure in Eq. (10) and the bridge fragilities. Then, the conditional probability of the disconnection event given an earthquake magnitude is computed by

$$P(E_{\mathrm{sys}}|M = m) = \mathbf{c}^{\mathrm{T}}\mathbf{p}(m) \tag{22}$$

Note that the event vector is not affected by the magnitude and hence obtained only once. Fig. 6 shows $P(E_{sys}|M = m)$ for the eight cities considered. A decision-maker may be interested in the probability that at least one city is disconnected from the hospital. This event is the union of the disconnection events of the cities. Therefore, the event vector for this new system event is easily obtained by Eq. (5b) using the event vectors already identified for the cities. The conditional probabilities of this event are shown in Fig. 6.

Using Eq. (13), we compute the probabilities of disconnections for an earthquake with uncertain magnitude. Fig. 7 shows the computed probabilities along with the probability of at least one disconnection. These disconnection probabilities are influenced by three important factors: distances between the bridges and the epicenter, the network configuration and the fragilities of the bridges. This example demonstrates that the system reliability analysis by the MSR method systematically accounts for all these factors in estimating the likelihood of various events. First, it is seen that Cities 4, 7 and 8 have the highest probabilities of disconnection. These cities have only two outgoing paths while the other cities have three. This network configuration makes the three cities have lower probabilities of exit and thus increases the probabilities of disconnection. It is also noteworthy that, among these three cities, City 7 shows the highest failure probability. This is because the paths coming out of City 7 are closer to the seismic source than those of the other two cities. Because of the seismic attenuation, the bridges on the closer paths are subjected to stronger ground motions. Finally, the actual fragility of a bridge is important in determining the disconnection probability since the latter is a direct function of the bridge fragilities. Increasing the overall safety of the bridges in a network increases the network safety, reducing the disconnection probability. Conversely, the deterioration of the bridges in a network over time leads to more vulnerable bridges and thus to a more vulnerable network. The network topology and the distance of each bridge to the epicenter define how each bridge fragility defines the network fragility.

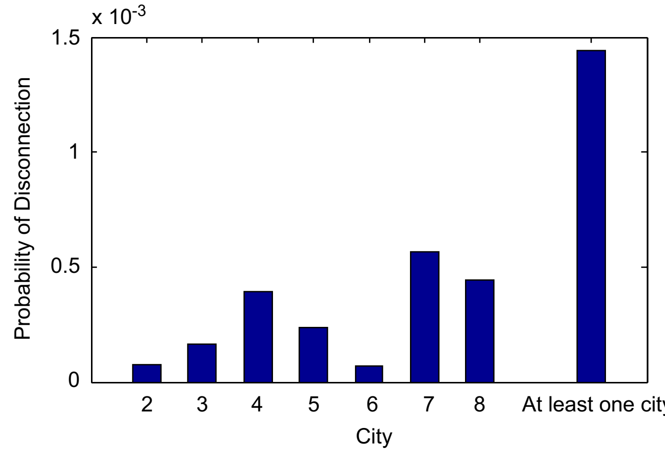

Fig. 7. Probability of disconnection between each city and hospital.

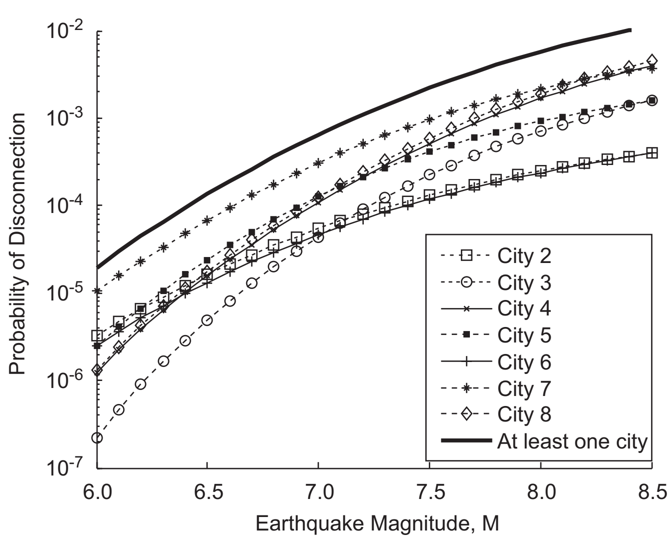

Fig. 6. Conditional probability of disconnection between each city and hospital for given earthquake magnitude.

Example 2. Disconnection between county and hospital.

Consider Counties 1, 2 and 3 that consist of Cities (2,5,7), (3,6) and (4,8), respectively. Suppose the officials of a county want to know the probability that their county will be disconnected from the hospital. An important merit of the MSR method is that we can construct the event vector of a new system event easily from other component/system events. Moreover, since there is no change in component probabilities, we do not need to construct the probability vector again. In this example, the event of a county's disconnection is the intersection of the events of the disconnections of all the constituent cities. Therefore, the event vector of a county's disconnection is conveniently obtained by Eq. (5a). For example, the event vector for County 1 is $\mathbf{c}^{\text{County1}} = \mathbf{c}^{\text{City2}}.{*}\mathbf{c}^{\text{City5}}.{*}\mathbf{c}^{\text{City7}}$. Using the conditional probability vector $\mathbf{p}(m)$ in the previous example and the new event vectors, the conditional probabilities of county disconnections are calculated (Fig. 8). The probability of a county's disconnection for an earthquake with uncertain intensity is easily estimated by use of the county's event vector and the predictive probability vector $\tilde{\mathbf{p}}$, which was estimated in the previous example. The disconnection probabilities of Counties 1–3 are $5.98 \times 10^{(-6)}$, $1.09 \times 10^{(-5)}$ and $2.64 \times 10^{(-4)}$ respectively.

Example 3. Number of failed bridges.

The event that a certain number of bridges will fail is also a complex system event. In the case when the components have the same probabilities and they are statistically independent, one can compute the probability of such event using the binomial distribution. MSR can compute such probabilities conveniently even in the case of non-homogeneous component probabilities and/or dependent components. For example, the event that exactly one

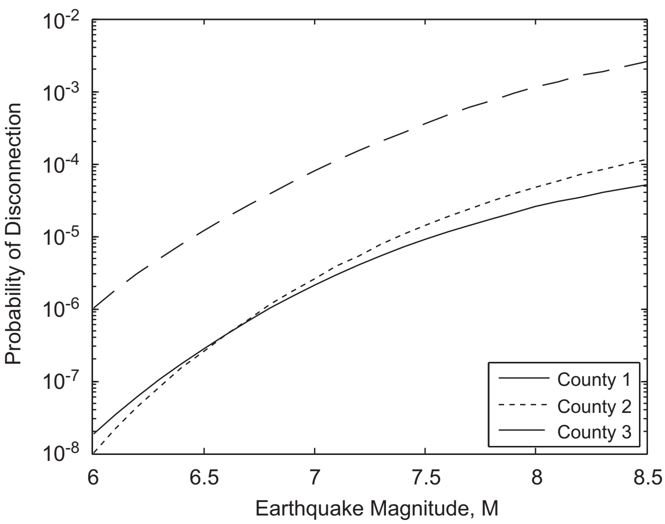

Fig. 8. Conditional probability of disconnection between each county and hospital for given earthquake magnitude.

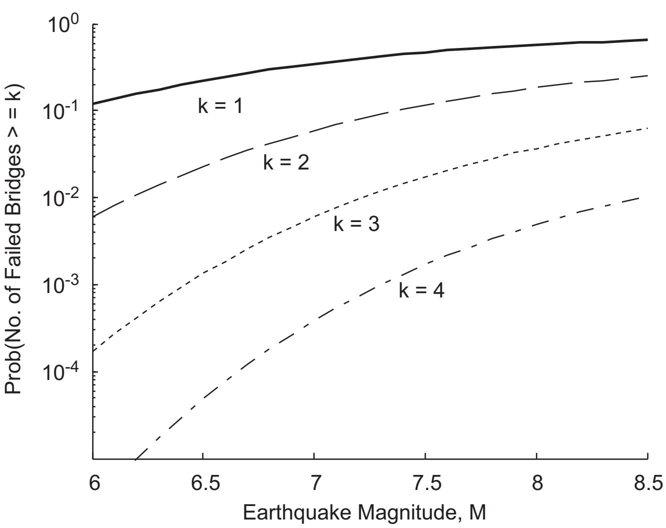

Fig. 9. Probability that at least k ($k = 1,\ldots,4$) bridges fail for given earthquake magnitude.

bridge will fail is defined as

$$E_{\text{onebridgefails}} = E_1\bar{E}_2\bar{E}_3\cdots\bar{E}_{12} \cup \bar{E}_1E_2\bar{E}_3\cdots\bar{E}_{12} \cup \cdots \cup \bar{E}_1\bar{E}_2\bar{E}_3\cdots\bar{E}_{11}E_{12} \tag{23}$$

The corresponding event vector is obtained by matrix manipulations in Eqs. (4) and (5). Since the conditional and predictive probability vectors in the previous examples are reused, there is no additional cost on computation. Fig. 9 shows the probabilities that the number of failed bridges is at least $k = 1,\ldots,4$.

Example 4. Disconnection between City 5 and hospital in case of no information on Bridge 12.

Suppose we try to estimate the probability of disconnection between City 5 and hospital but the fragility of Bridge 12 is not available due to insufficient information or data.

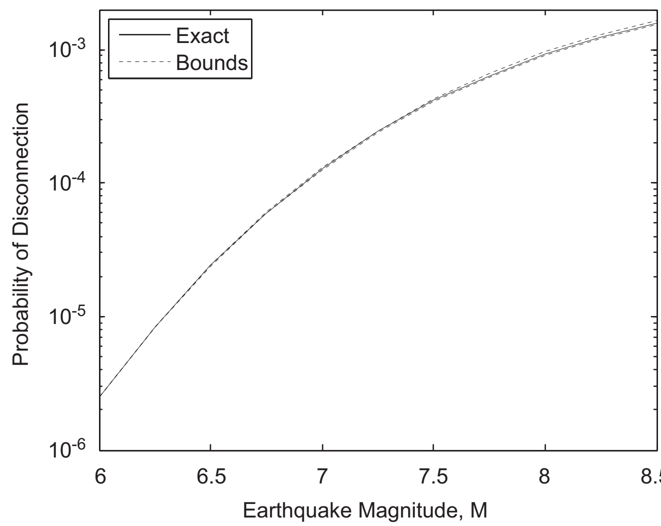

Fig. 10. Conditional probability of disconnection between City 5 and hospital for given earthquake magnitude.

The MSR framework still allows us to obtain the narrowest bounds on the system probability by use of the LP bounds method in Eq. (14). For the vector **c** in the LP problem, we use the event vector previously identified for City 5. Instead of constructing the vector **p** completely, we use it as an unknown decision variable for the LP. We compute the elements of the vector for which we have information and use them as equality constraints in Eq. (14b). If bounds on the probabilities are available, we use them as inequality constraints in Eqs. (14c) and (14d). The axioms of probability should be added as constraints as well [3]. By solving the LP, we find the two **p** vectors that minimize or maximize the system probability $\mathbf{c}^T\mathbf{p}$. Fig. 10 compares the LP bounds with the probabilities evaluated based on the complete information. Note that we can still estimate narrow bounds on the system probabilities in spite of the incomplete information on a bridge without introducing an arbitrary assumption.

Example 5. Importance measures of bridges.

Suppose we intend to improve the post-hazard connectivity of the whole region by retrofitting selected bridges in the network. With limited budget, we may want to identify the bridges whose upgrade would enhance the connectivity in the most efficient manner. Here we define the important bridges in terms of the likelihood that there is at least one disconnected city in the region. Using the MSR method, we compute the importance measure *CIM* in Eq. (16) of each bridge for a given earthquake magnitude. By total probability theorem, we can evaluate *CIM* for an unknown magnitude as well. Fig. 11 shows the CIMs of the 12 bridges. Since CIMs quantify the contributions of the likelihood of the individual bridge failures to that of the system failure event of interest, retrofits or upgrades on the bridges with high CIMs are expected to reduce the system failure probability in the most efficient manner. Bridges 1 and 5 are identified as the most important ones.

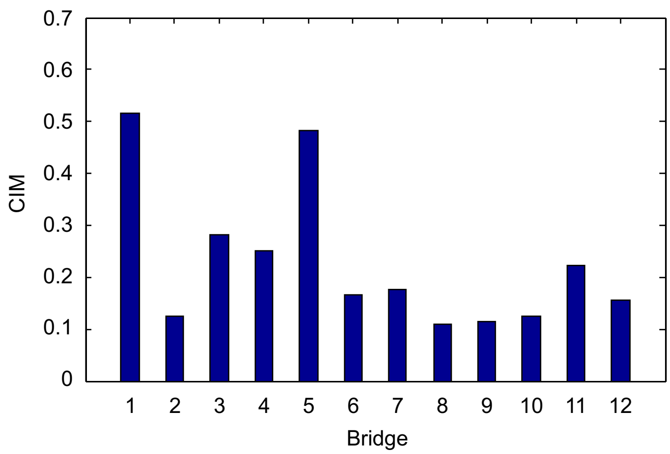

Fig. 11. CIM of bridges with respect to disconnection of at least one city.

It is noteworthy that these are on the paths connecting City 7, the most vulnerable city according to the results in Fig. 7. The MSR method allows us to compute the various importance measures based on different definitions of importance without additional cost in computation.

4. Summary and conclusions

The matrix-based system reliability (MSR) method can estimate the probabilities of complex system events by simple matrix calculations. Using matrix-based computing languages and software, one can efficiently identify the matrices that represent the system event and the component risks. When there exists significant dependence between component failure events, the MSR method can estimate the system failure probability by use of conditional statistical independence of components given outcomes of uncertain common sources. When one has incomplete information on the component failures or their dependence, the matrix-based framework still helps obtain the narrowest bounds on the system failure probability for given information. The MSR can also estimate various importance measures and conditional probabilities that help quantify the relative contribution of components to the system events of interest. The merits of the MSR method are demonstrated by its applications to a bridge network for estimating the probabilities of various system events such as the disconnection of cities and counties from a hospital. The probability distribution of the number of failed bridges and conditional probabilities are also obtained by the proposed method. The example also demonstrates the ability of the MSR method to estimate narrow bounds on the system failure probability in spite of the lack of complete information.

Acknowledgment

This work was supported by the Mid-America Earthquake Center under a grant from the Earthquake Engineering Research Center's program of the National Science Foundation (Award no. EEC-97010785).

References

[1] Ditlevsen O. Narrow reliability bounds for structural systems. J Struct Mech 1979;7(4):453–72.
[2] Hohenbichler M, Rackwitz R. First-order concepts in system reliability. Struct Saf 1983;1(3):177–88.
[3] Song J, Der Kiureghian A. Bounds on system reliability by linear programming. J Eng Mech ASCE 2003;129(6):627–36.
[4] Song J, Kang W-H. Matrix-based system reliability method and applications to structural systems. In: The 18th ASCE Engineering Mechanics Division conference, Blacksburg, VA, USA, 2007.
[5] Song J, Kang W-H. System reliability and sensitivity under statistical dependence by matrix-based system reliability method. Struct Saf, to appear.
[6] Guikema S, Gardoni P. Reliability estimation for networks of reinforced concrete bridges. J Infrastruct Syst ASCE, to appear.
[7] Liu M, Frangopol DM. Time dependent bridge network reliability: novel approach. J Struct Eng ASCE 2005;131(2):329–37.
[8] Liu M, Frangopol DM. Balancing connectivity of deteriorating bridge networks and long-term maintenance cost through optimization. J Bridge Eng ASCE 2005;10(4):468–81.
[9] Song J, Der Kiureghian A. Component importance measures by linear programming bounds on system reliability. In: The ninth international conference on structural safety and reliability (ICOSSAR9), Rome, Italy, 2005.
[10] Gardoni P, Der Kiureghian A, Mosalam KM. Probabilistic capacity models and fragility estimates for reinforced concrete columns based on experimental observations. J Eng Mech ASCE 2002;128(10): 1024–38.
[11] Gardoni P, Mosalam KM, Der Kiureghian A. Probabilistic seismic demand models and fragility estimates for RC bridges. J Earthquake Eng 2003;7:79–106.
[12] Dueñas-Osorio L, Craig JI, Goodno BJ. Tolerance of interdependent infrastructures to natural hazards and intentional disruptions. Mid-America Earthquake Center report; 2006.
[13] Der Kiureghian A, Song J. Multi-scale reliability analysis and updating of complex systems by use of linear programming. J Reliab Eng Syst Saf 2008;93(2):288–97.
[14] Song J, Der Kiureghian A. Bounds on system reliability by linear programming and applications to electrical substations. In: The ninth international conference on applications of statistics and probability in civil engineering (ICASP9), San Francisco, CA, USA, 2003.
[15] Song J, Der Kiureghian A. Joint first-passage probability and reliability of systems under stochastic excitation. J Eng Mech ASCE 2006;132(1):65–77.
[16] Fussell BJ. How to hand-calculate system reliability characteristics. IEEE Trans Reliab 1973;24(3).
[17] Mackie K, Stojadinovic B. Probabilistic seismic demand model for California highway bridges. J Bridge Eng ASCE 2001;6(6):468–81.
[18] Caltrans. Seismic design criteria. California Department of Transportation, 1.1 Edition. 1999.
[19] Cosentino P, Ficarra V, Luzio D. Truncated exponential frequency–magnitude relationship in earthquake statistics. Bull Seismol Soc Am 1977;67(6):1615–23.
[20] Campbell K. Empirical near-source attenuation relationship for horizontal and vertical components of peak ground acceleration, peak ground velocity, and pseudo-absolute acceleration response spectral. Seismol Res Lett 1997;68(1):154–97.
[21] Chopra AK. Dynamics of structures: theory and applications to earthquake engineering. Upper Saddle River, NJ: Prentice-Hall; 1995. p. 220.