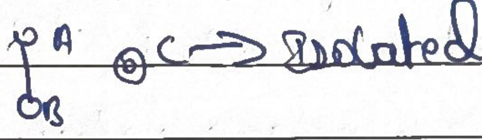
unit-3
Graph

Note: In case If you forget, My handwriting is worst If you don't understand it please don't ask 😊

2 marks and Basic definition

1) vertices → points in graph
   Edges → Lines connecting those
2) what is graph? A graph is G (V, E)
   V → vertices
   E → Edges
   are non empty sets
3) Isolated vertex:
   The vertex that has no edges connecting with others are called Isolated vertex



4) Directed graph or Digraph: Path is given for the edges and the ~~directed~~ directions are given
5) Self loop: The vertex has an edge that connects itself
6) Parallel Edge: When there are two or more edges there will be some parallel and distinct (not parallel) not perpendicular
7) Simple graph: It has only one edge
8) Multigraph: Consists of parallel edges

simple — undirected multigraph — Directed multigraph

9) Pseudographs: In which loops and parallel edges are allowed
10) weighted graph: A cost or weight is given to those edges
    Ex: heuristic in A* Algorithm

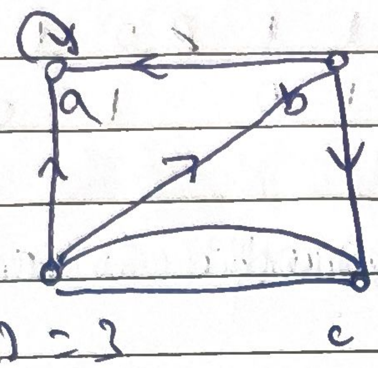
11) Degree of vertex: Number of edges

12) In degree : Number of edges connected to that vertex, v as terminal vertex the number of degree converse at v are called indegree

13) outdegree : The number of edges with v as initial vertex

14) $d^+(v)$

The vertex with 0 indegree



$deg^-(a) = 3$

$deg^+(a) = 1$ → Indegree

$deg^-(b) = 1$ → outdegree

$deg^+(b) = 2$

c is undirected

15) completed graph:
There is only one edge between two vertex

16) Note
The number of edges in K_n is nC_2 or $\frac{n(n-1)}{2}$

17) Cycle :
Starting and a final vertex are same $n \geq 3$

18) wheel W_n: Adding another vertex to cycle and connects to the rest of the vertex just like wheel's rim

19) Regular graph: If all vertex in the graph has the same degree they are called Regular graph

20) Bipartite graph: If graph is partitioned as V_1 and V_2 then every edge of G connects a vertex in V_1 and V_2

21) Sub graph: $H(V', E')$ are subgraph of $G(V, E)$

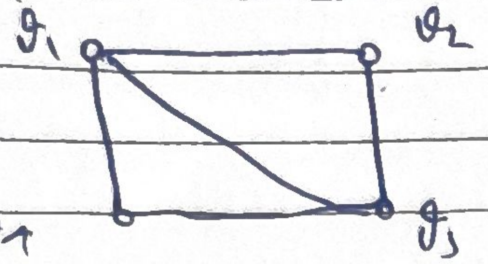
22) proper subgraph: If $V' = V$ then H is proper subgraph $V \geq V', E \geq E'$

23) Matrix Representation:

$n \times n$

$n \Rightarrow$ number of vertex

Ex



$$A_G = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$$

24) Isomorphic:

Two graphs are said to be isomorphic to each other when $G(V_1, E_1)$ and $G_2(V_2, E_2)$

$$\boxed{\begin{array}{c} V_1 = V_2 \\ E_1 = E_2 \end{array}}$$

25) Eulerian graph: (vertices can repeat)

*) Starting and end vertex are same just like cycle

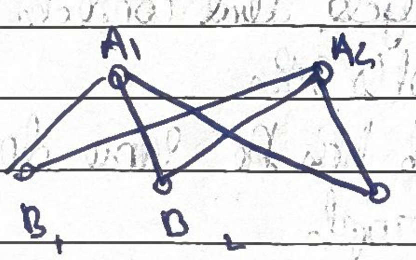
*) Eulerian path: A path of graph G is called Eulerian path if it includes each edge of G exactly once

26) Hamiltonian: (vertices can not except for start and end)

*) Same like Eulerian except for initial and final vertex

27) Incident: If $\boxed{V_1}$ is a vertex of some edge $\boxed{E_1}$ that is said to be $\boxed{\text{Incident}}$ with $\boxed{V_1}$

28) Complete bipartite example:



29) To find number of edges base on degree

$2e = v(d)$ $\quad\quad$ (d) - degrees of freedom

$= 10 \times 6$

$\boxed{e = 30}$

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30) Adjacency of Matrix : A Matrix represented with $n \times n$ number of vertices with Number of edge connecting them as values inside

31) Hamiltonian path: a path in graph that visits every vertex exactly once unlike hamiltonian cycle which start and end with same vertex

Part B Importants and Sums

Hand shaking Theorem

Theorem 1) If the vertices of an undirected graph are each of odd degree k, show that the number of edges is a multiple of k

Sol:

Since the number of vertices of odd degree is even, Let it be $2n$. Let the number of edges be ne

then by Hand shaking Theorem

$$\boxed{\Sigma \deg(v) = 2ne}$$

$$\underset{(\deg \text{ is odd})}{\Sigma \deg(v)} + \underset{\deg \text{ is even}}{\Sigma \deg(v)} = 2ne$$

Since all vertices are of odd degree 'k' and no vertices are of even degree

$$\underset{\text{even}}{\Sigma \deg(v)} = 0$$

Hence

$$\sum_{i=1}^{2n} \deg(v) = 2ne$$

$$\sum_{i=1}^{2n} k = 2ne$$

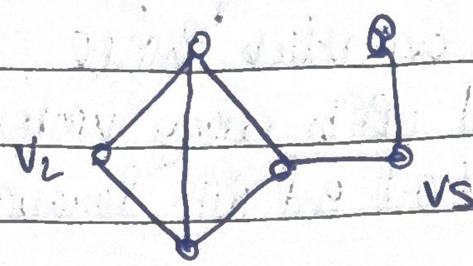
$$2nk = 2ne$$

$$\boxed{ne = \cancel{2n}\, nk}$$

∴ They are multiples of k

Theorem 2) The number of odd degree vertices are even

Diagram



$\boxed{\text{odd deg vertices} = 4}$

Soln: Let $G = \{V, E\}$ V => vertices
E => Edges

Let $v_1, v_2, v_3 \dots v_k$ be the vertices of odd degree
$v_1', v_2', v_3' \dots v_k'$ be the vertices of even degree

From diagram

$$\boxed{\sum d(v_j)' = d(v_2) + d(v_5) = 4 \text{ (even)}}$$

$\therefore \sum_{j=1}^{m} d(v_j)'$ is even let it be $2l$

$$\boxed{\sum_{j=1}^{m} d(v_j)' = 2l}$$

Now to prove number of odd degree vertices is even

$$\sum_{i=1}^{n} d(v_i) = 2e \quad \text{--- (3)}$$

(2) and (3)

$$\sum_{i=1}^{\alpha} d(v_i) + \sum_{i=1}^{k} d(v_i)' = 2e$$

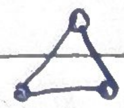
$$+ 2l = 2e$$

$$= 2e - 2l$$

$$\boxed{\sum_{i=1}^{k} d(v_i) = 2(e - l)}$$ $\therefore$ Multiple of 2 is always even

Theorem: Show that the degree of vertex of a simple graph G on n vertices can not exceed n-1.

Soln



$n=3$
deg of $v=2$

Given G is a simple graph since
*It has no self loop
*It has no multiple edges
Let v_1 be any vertex of G
Since G is simple graph so v_1 can't be
Adjacent to remaining $n-1$ vertices of G
$\therefore$ The vertex v_1 may have maximum degree $n-1$
Similarly remaining vertices also have maximum
degree $n-1$
Thus the degree of vertices of a simple graph
can not exceed $\boxed{n-1}$

Theorem 4: Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

Sol: To prove WKT,

$$\sum d(v_i) = 2e$$

Lets put n in Σ

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\deg(v_1) + \deg(v_2) \ldots \deg(v_n) = 2e$$

WKT
deg(v) simple graph cannot exceed $(n-1)$

$$(n-1)+(n-1)\ldots(n-1) = 2e$$

$$n(n-1) = 2e$$

$$\boxed{e = \frac{n(n-1)}{2}} \quad \therefore \text{Proved.}$$

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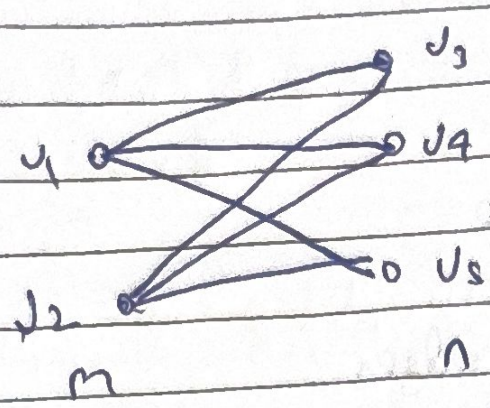
Theorem 5: Show that G is a bipartite simple graph with n vertices and e edges then $e \leq \frac{n^2}{4}$

Diagram



$\therefore$ complete bipartite graph

No of V = no of edges
$= M \times M$
M and n

No of edges $= M \times n$

Lets say there exist a graph of vertices n_1 and n_2
Then the maximum number of vertices could be $n_1 \times n_2$

$$\boxed{e \leq n_1 \times n_2}$$

WKT

$$\frac{n_1+n_2}{2} \geq \sqrt{n_1 n_2} \quad [AM \geq GM]$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

S.B on bs

$n_1+n_2=n$

$$\left(\frac{n_1+n_2}{2}\right)^2 \geq n_1 n_2$$

$$\frac{n^2}{4} \geq n_1 n_2 \qquad n_1 \cdot n_2 = e$$

$$\boxed{\left(\frac{n^2}{4}\right) \geq e}$$

$\therefore$ proved

Theorem 6: Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges

Sln : Let G be graph and has more than $\frac{(n-1)(n-2)}{2}$ edges

i.e

$$|E(G)| > \frac{(n-1)(n-2)}{2}$$

To prove G is connected by contradiction

G is not connected

G must have atleast two components let it be G_1 and G_2

$V_1 \rightarrow$ vertices of G_1

$V_2 \rightarrow$ vertices of G_2 $\quad$ n total no of vertices

$|V_1| = m$

$|V_2| = n-m$

Also i) $1 \leq m \leq n-1$

ii) There are no edges joining V_1 and V_2

iii) $|V_2| = n-m \geq 1$

Now

$$|E(G)| = |E(G_1 \cup G_2)| = |E(G_1)| + |E(G_2)|$$

$$= \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$

$$= \frac{1}{2}[m^2 - m + n(n-m-1) - m(n-m-1)]$$

$$= \frac{1}{2}[m^2 - m + n^2 - nm - n - mn + m^2 + m]$$

Adding and Subtracting $2n-2$

$$= \frac{1}{2}[m^2 - m + n(n-1) - nm - mn + m^2 + m + (2n-2) - (2n-2)]$$

$$= \frac{1}{2}[n(n-1) + 2m^2 - 2nm + 2n - 2 - 2(n-1)]$$

$= \frac{1}{2}[(n-1)(n-2) + 2(m^2-1) - 2n(m-1)]$

$|E(G)| = \frac{1}{2}[(n-1)(n-2) + (m-1)(m-n+1)] \le \frac{1}{2}(n-1)(n-2)$

$\therefore$ It is contradiction

$\therefore$ proved

Theorem 7: A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges

Soln: Let G be simple graph of n vertices k components

For a simple graph max no of vertices are $\frac{n(n-1)}{2}$

Let $n_1, n_2, n_3 \ldots n_q$ be number of vertices in each of k components of G

$\therefore n_1 + n_2 \ldots n_k = |V(G)|$

$$\boxed{\text{i.e } \sum_{i=1}^{k} n_i = n} \quad \text{— (1)}$$

Now $\sum_{i=1}^{k} (n_i - 1) = (n_1 - 1)(n_2 - 1) + \ldots (N_k - 1)$

$= (n_1 + n_2 + \ldots + n_k) - (1 + 1 \ldots k)$ k times

$$\boxed{\sum_{i=1}^{k} (n_i - 1) = n - k}$$

Sq on BS

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = [n-k]^2$$

$$\boxed{\left[[n_1-1]+[n_2-1]+\dots[n_k-1]\right]^2 = n^2-2nk+k^2} \quad \text{—(E)}$$

we have $(n_1-1)^2+(n_2-1)^2+\dots(n_k-1)^2 \le (n-1)^2+(n_2-1)^2 + (n_k-1)^2+2(n_1-1)(n_2-2) + \dots 2((n_{k-1})-1)(n_k-1)$
$= n^2-2nk+k^2$

$$(n_1^2-2n_1+1)+(n_2^2-2n_2+1)\dots(n_k^2-2n_k+1) \le n^2-2nk+k^2$$

$$[n_1^2+n_2^2\dots+n_k^2]-2[n_1+n_2+n_k]+[1+1\dots+1] \le n^2-2nk+k^2$$

$$\sum_{i=1}^{k} n_i^2 -2n+k \le n^2-2nk+k^2$$

$$\boxed{\sum_{i=1}^{k} n_i^2 \le n^2-2nk+k^2+2n-k}$$

WKT $\frac{n(n-1)}{2}$ $\frac{n_i(n_i-1)}{2}$

$$a = \frac{n_1(n_1-1)}{2} + \dots + \frac{n_k(n_k-1)}{2}$$

$$= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k}(n_i^2-n_i)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right]$$

$= \frac{1}{2}[n^2 - 2nk + k^2 + 2n - k] - n]$

$= \frac{1}{2}[(n-k)^2 + (n-k)]$

$= \frac{1}{2}(n-k)^2 + (n-k)$

$\therefore$ proved.

Theorem 8 : Define a self complimentary graph. Show that if G is a self complementary simple graph with n vertices then $n = 0$ or $1 \pmod 4$

or

Every simple self complimentary graph has $4k$ or $4k+1$ vertices

Soln

$G \cong \bar{G}$

WKT $G \cup \bar{G} = K_n$

$|E(G \cup \bar{G})| = |E(K_n)|$

$|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$ $\quad G \cong \bar{G}$ $\quad \therefore |E(G)| = |E(\bar{G})|$

$2|E(G)| = \frac{n(n-1)}{2}$

$E(G) = \frac{n(n-1)}{4}$

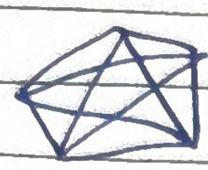
$\boxed{n = 4k \text{ or } n-1 = 4k}$

$\therefore$ True

→ stupid sum just leave it

Theorem 7: A connected graph is Euler graph if and only if each of its vertices is of even degree
(or)
State and prove necessary and sufficient conditions of a graph to be an Euler graph

Soln: Let G be connected also Euler graph
To prove all of G is even
Let C be Eulerian Circuit of G



Since v is a internal vertex then

$d(v) = 2 \left\{ \begin{array}{l} \text{number of times} \\ \text{v occurs through the} \\ \text{Euler circuit} \end{array} \right. = \text{even}$

Since the Euler circuit C start and ends with a
$\therefore d(a) = 2+2 \times \left\{ \begin{array}{l} \text{number of times a occurs} \\ \text{inside C} \end{array} \right.$
$= \text{even}$

∴ G has all the vertices of even degree
Conversely assume that each of the vertex of G has an even degree
To prove G is Euler

Suppose G is not Euler
Then G has no Euler circuit
Since each of the vertex are even degree
∴ G contains circuit

Let C_1 be a circuit of maximum length
Since by assuming C_1 is not euler
$G - E(C_1)$ has some other components G'
here all the vertices of G has even degree
and all the vertices of C_1 has even degree
Since G is a connected graph
$\therefore$ There is a vertex common to C_1 & C_2 let it be v
Now g and C_1 and C_2 using the common vertex v
we get a new circuit $C_1 \cup C_2$ in G
Here $E(C_1 \cup C_2) > E(C_1)$

G is Euler graph

Left over Theorems:

1) Prove that at least two vertices has at least two vertices of same deg

$$M + m = \frac{n(n-1)}{2}$$

$$2m = \frac{n(n-1)}{2}$$

$$\boxed{n(n-1) = 4m}$$

Dont care this ↑

Problems:

1) A undirected graph G has 16 edges and all vertices of degree 2. Find of edge

$d(v_1) + d(v_2) \ldots d(v_n) = 2e$ [Handshaking theorem]

$2[1+1+ \cdots 1] = 2e$

$\cancel{2}[n] = \cancel{2}e$

$n = \cancel{2}e$

$n = e$

$\boxed{n = 16}$

2) The number of vertices = 10, vertices degree = 3 find e

$3[10] = 2e$

$30 = 2e$

$\boxed{e = 15}$

3) Does there a graph exist with 13 vertices with deg 3

$\Sigma\, d(v_i) = 2e$

$3[13] = 2e$

$39 = 2e$

$e = \frac{39}{2}$ ≠ divisible

∴ graph dose not exist

4) Can you draw a graph of 5 vertices with deg of seque 1, 2, 3, 4, 5

$\Sigma\, d(v_i) = 2e$

$= 15 \neq$ even

Can not draw!

5) If 5 vertices are there with 1, 3, 7, 2, 3 degree

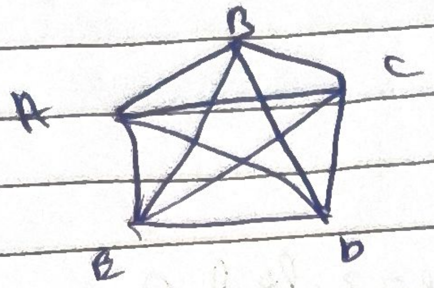
Soln

deg sq = [illegible] ≠ even

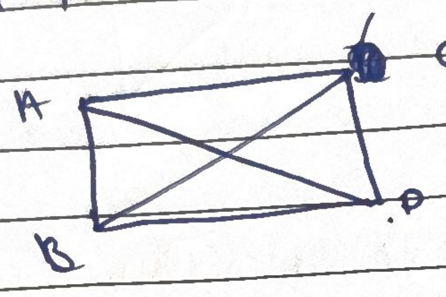
can not draw

6) Subgraph from the complete graph of 5 vertices for complete graph with vertices

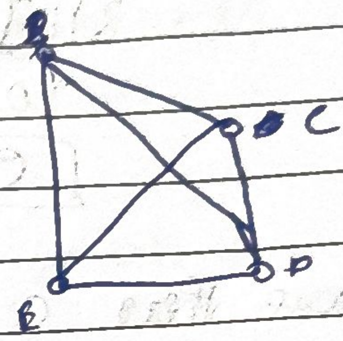
i) with 5



ii) with 4 without E



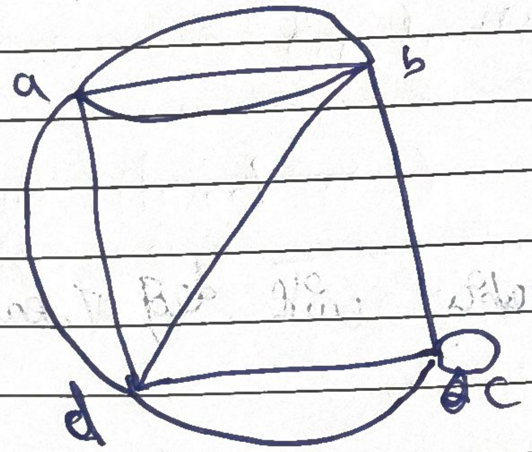
iii) with 4 without A



So draw like that

7) Adjacency matrix of simple graph or any graph

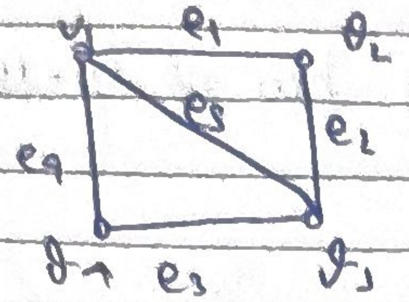
Soln



4x4

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 3 & 0 & 2 \\ B & 3 & 0 & 1 & 1 \\ C & 0 & 1 & 1 & 2 \\ D & 2 & 1 & 2 & 0 \end{array}$$

8) Incidence Matrix



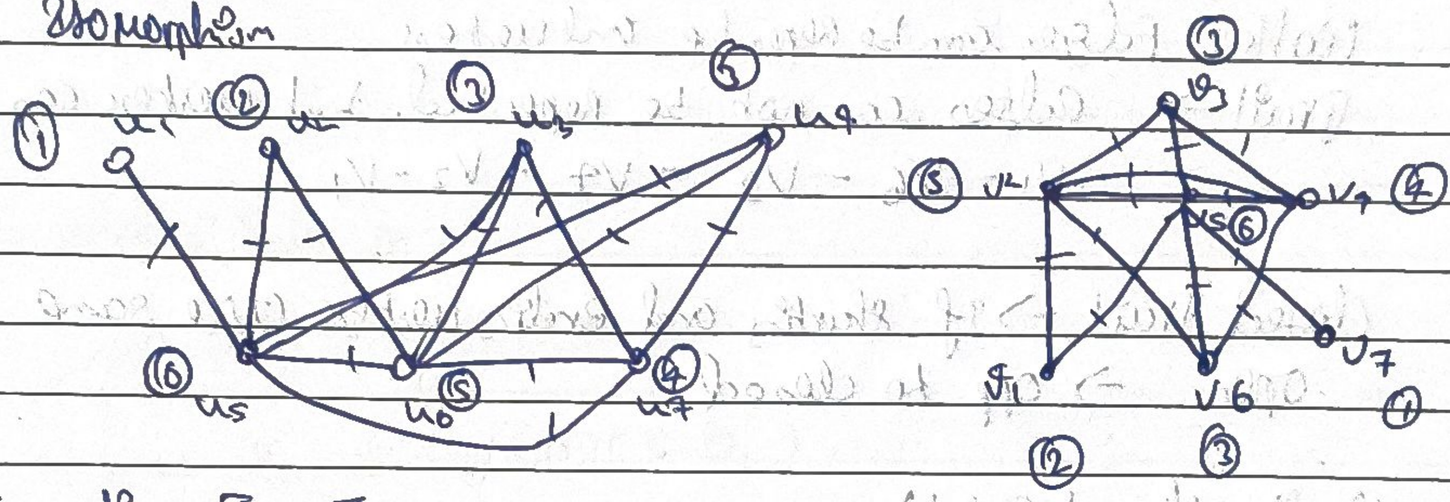
$$
\begin{array}{c|ccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 \\
\hline
v_1 & 1 & 0 & 0 & 1 & 1 \\
v_2 & 1 & 1 & 0 & 0 & 0 \\
v_3 & 0 & 1 & 1 & 0 & 1 \\
v_4 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

9) Graph Isomorphism

It should satisfy

1) Number of vertices are same
2) Number of edges are same
3) Deg Sequence are same
4) Adjacency are same

Check Isomorphism



1) N of vertices - 7 = 7
2) N of edge = 12
3) deg Seq = Same
4) Adjacency : vertices with same deg of freedom on both are adjacent

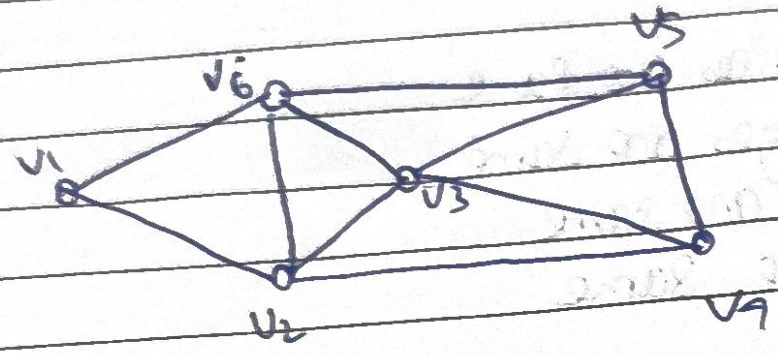
∴ They are Isomorphic

Also we can check by Matrix too

If one has 3 cycle and the other has no cycle they are not isomorphic

Path Reachility connecte

10) walk: A walk of a graph G is defined as a finite alternating sequence of vertices and edges of the form $v_0, e_1, v_1, e_2, v_2, e_3$ ... such that begining and ending with vertex such that each edge e_i is incident with v_{i-1} & v_i



$V_1 - V_6 - V_3 - V_4$

$V_1 - V_6 - V_2 - V_3 - V_6 - V_5$

Walk: Edges can be Repeate and vertex

Trail -> Edges can not be Repeated but vertex can

$V_1 - V_6 - V_3 - V_4 - V_2 - V_1$

closed Trail -> If starting and ending vertex are same

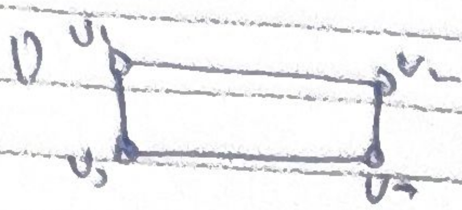
open -> opp to closed

Circuit (Closed Trial)

Cycle : Start — — Start with out Repetition edges and vertex except start

11) Complimented Graph

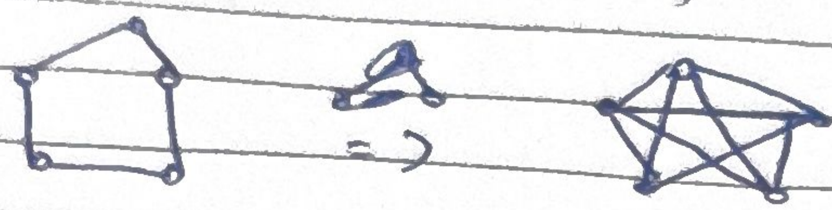
1)



V(G) = V(G)' is complemented

(connect whichever not connected)

2)

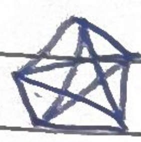


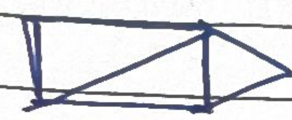
12) Self Compliment: A simple graph G is said to be self compliment if G is isomorphic to $\bar{G}$ ie $G \cong \bar{G}$

13) Eulerian and Hamiltonian circuit

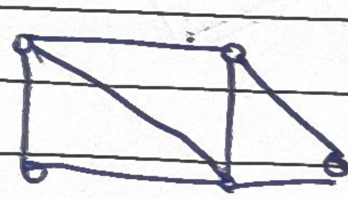
↓ edges only once

↓ vertex exactly once except start and end





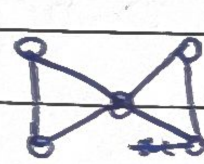
14) Check whether Euler or not



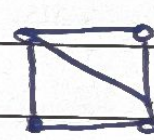
(not Euler)

15) Give example

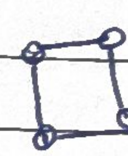
i) Eulerian not Hamiltonian



ii) Hamil Not Euler



iii) Both



iv neither