Unit-1 Logic and Truth (Proofs)

Topics To cover :

- *) propositional Logic
- *) propositional equivalence
- *) predicates and quatifiers
- *) Rules of inference
- *) introduction to proofs
- *) proof methods and stateregs

Laws to use

Type 1: without using Truth Table

1) Commutative Law : $P \vee Q = Q \vee P$
   $P \wedge Q = Q \wedge P$
2) Associative Law : $P \vee (Q \vee R) = (P \vee Q) \vee R$
   $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$
3) Distributive Law : $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$
   $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
4) Demorgan : $\neg(P \vee Q) = \neg P \wedge \neg Q$
   $\neg(P \wedge Q) = \neg P \vee \neg Q$
5) Negation Law : $\neg P \vee P = T$
   $\neg P \wedge P = F$
6) Double Negation Law : $\neg(\neg P) = P$
7) Idempoten : $P \vee P = P$, $P \wedge P = P$
8) Absorption : $P \vee (P \wedge Q) = P$, $P \wedge (P \vee Q) = P$
9) Identity Law : $P \wedge T = P$, $P \vee F = P$
10) Dominant Law : $P \vee T = T$, $P \wedge F = F$
11) $P \to Q = \neg P \vee Q$
12) $P \leftrightarrow Q = (P \to Q) \wedge (Q \to R)$
13) Contrapositive $P \to Q = \neg Q \to \neg P$

$\neg \to$ negation
$\wedge$ Conjuction
$\vee$ Disjunction
$\to$ Conditional
$\leftrightarrow$ Biconditional

1) $\boxed{\neg P \wedge (\neg Q \wedge R)} \vee \boxed{(Q \wedge R) \vee (P \wedge R)} \Leftrightarrow R$

S.no	Statement	Reason
1)	$\neg P \wedge (\neg Q \wedge R)$	- (Taking one premise)
2)	$(\neg P \wedge \neg Q) \wedge R$	Asso. Law
3)	$\neg (P \vee Q) \wedge R$	DeMorgan
4)	$(Q \wedge R) \vee (P \wedge R)$	-
5)	$(Q \vee P) \wedge R$	Dist. Law
6)	$\neg (P \vee Q) \wedge R \vee Q \vee P \wedge R$	-
7)	$\neg (P \vee Q) \vee (P \vee Q) \wedge R \wedge R$	Commutative
8)	$T \wedge R \wedge R$	Neg Law
9)	R	Identity Law

Hence proved

2) $(P \vee Q) \wedge \neg (\neg P \wedge Q) \Leftrightarrow P$

S.no	Statement	Reason
1)	$P \vee Q$	-
2)	$\neg (\neg P \wedge Q)$	-
3)	$P \wedge \neg Q$	Double negation
4)	$(P \vee Q) \wedge (P \wedge \neg Q)$	

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S.No	Statement	Reason
1)	$\neg\neg(P \wedge Q)$	–
2)	$\neg\neg P \vee \neg Q$	Demorgans
3)	$P \vee \neg Q$	Double neg
4)	$P \vee Q$	–
5)	$(P \vee Q) \wedge (P \vee \neg Q)$	–
6)	$P \vee (Q \wedge \neg Q)$	Dist Law
7)	$P \vee F$	Negation Law
8)	P	Identity

∴ Proved

3) Proving Tautology

$((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R)) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R))$ is Tautology

S.No	Statement	Reason
1)	$((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R)) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R))$	–
2)	$((P \vee Q) \wedge \neg\neg P \wedge \neg(\neg Q \wedge \neg R) \vee \neg(P \vee Q) \vee \neg(P \vee R)$	Demorgan
3)	$((P \vee Q) \wedge \neg\neg P \wedge \neg(\neg Q \vee \neg R)) \vee \neg(P \vee Q) \wedge (P \vee R)$	Demorgans
4)	$(P \vee Q) \wedge \neg\neg(Q \wedge R) \vee \neg(P \vee Q) \wedge (P \vee R)$	Demorgans
5)	$(P \vee Q) \wedge P \wedge (Q \wedge R) \vee \neg(P \vee Q) \wedge (P \vee R)$	Double neg
6)	$(P \vee Q) \wedge (P \vee Q) \wedge (P \vee R) \vee \neg(P \vee Q) \wedge (P \vee R)$	Dist Law
7)	$(P \vee Q) \wedge (P \vee R) \vee \neg(P \vee Q) \wedge (P \wedge R)$	Idempotent Law
8)	T	Negation Law

4) $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Rightarrow \neg P \vee Q$

1) $(\neg(P \wedge Q))$ —

~~2) $\neg P \wedge \neg Q$~~

2) $\neg P \vee \neg Q$ Demors

3) $(\neg P \vee (\neg P \vee Q))$

9) $((\neg P \vee \neg P) \vee Q)$ Asso

10) $\neg P \vee Q$ Idempotent

11) $\dfrac{\neg P \vee \neg Q}{P} \to \dfrac{\neg P \vee Q}{Q}$

12) $\neg P \vee Q$

13) $\neg(\neg P \vee \neg Q) \vee \neg P \vee Q$ Double neg

14) $(P \wedge Q) \vee \boxed{(\underset{b}{\neg P} \vee \underset{c}{Q})}$

Q^{a}

15) $((P \wedge Q) \vee \neg P) \wedge ((P \wedge Q) \vee Q)$ Dis Law

16) $P \wedge Q \vee \neg P \wedge P \wedge Q \vee Q$

17) $P \vee \neg P \wedge Q \wedge P \wedge Q \vee Q$

18) $Q \wedge (P \vee Q)$

19) $(Q \wedge P) \vee (Q \wedge Q)$

20) $Q \wedge P \vee Q$

21) $\neg P \vee Q$

$\therefore$ Hence proved

~~15) $(\neg P \vee Q \vee P)$~~

15) $(\neg P \vee Q) \vee P) \wedge ((P \vee Q) \vee Q)$

16) $P \vee \neg P \vee Q \wedge \neg P \vee Q \vee Q$

17) $Q \wedge \neg P \vee Q$

18) $Q \wedge Q \vee \neg P$

$= Q \vee \neg P$

$\Rightarrow \neg P \vee Q$ $\therefore$ Hence proved

Converse, contrapositive and inverse

let $P \to Q$ be the conditional proposition
$q \to p$ is its converse
$\neg q \to \neg p$ is called contrapositive
$\neg p \to \neg q$ is called inverse

1) Obtain the converse, corresponding and inversion of
"Team India win whenever dhoni is captain"

Sol:

q : India wins
p : dhoni is captain

Converse	: $q \to p$
Converse	: $q - p$
contrapositive	: $\neg q \to \neg p$
Inverse	: $\neg p \to \neg q$

Elementary product (DNF)

A product of statement variable and negation
Ex : $P \wedge \neg P$, $Q \wedge \neg Q$

Elementary sum (CNF)

Sum of statement variables or negation is called elementary sum
$P \vee \neg P$, $Q \vee \neg Q$

DNF:

[elementary product] $\vee$ [elementary product]

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2) CNF

[Elementary sum] ∩ [Elementary sum]

PDNF: (minterms) v (minterms) . . v (minterms)
PCNF: (Maxterms) ∧ (Maxterms) . . . . . .
Example: To obtain DNF

$P \rightarrow (P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

Statement	Reason
$P \rightarrow (P \rightarrow Q) \wedge (\neg(\neg Q \vee \neg P))$	—
$\neg P \vee (P \rightarrow Q) \wedge \neg\neg Q \wedge \neg\neg P$	$P \rightarrow Q = \neg P \vee Q$
$(\neg P \vee \neg P \vee Q) \wedge (Q \wedge P)$	
$(\neg P \vee Q) \wedge (Q \wedge P)$	
$\neg P \vee Q \wedge Q \wedge P$	Idempotent
$(\neg P \vee Q) \wedge P$	
$(P \wedge \neg P) \vee (P \wedge Q)$	Distributive
Thus DNF is obtained	

3) CNF: $P \Rightarrow (P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

Statement — Reason

$P \rightarrow (P \rightarrow Q) \wedge \neg(\neg Q \wedge \neg P)$

$\neg P \vee (P \rightarrow Q) \wedge \neg\neg Q \wedge \neg\neg P$ — $P \rightarrow Q \Rightarrow \neg P \vee Q$

$\neg P \vee (\neg P \vee Q) \wedge Q \vee P$ — Double neg

$(\neg P \vee Q) \wedge (Q \vee P)$

$(\neg P \vee Q) \wedge (Q \vee P)$ — Associative

$(\neg P \vee Q) \wedge P$

$(\neg P) \vee (Q \wedge (Q \vee P))$

$(\neg P) \vee ((Q \wedge Q) \vee Q \wedge P)$

$(\neg P) \wedge (Q \vee (Q \wedge R))$

$(\neg P \vee (Q \wedge Q)) \vee (\neg P \vee (Q \vee P)))$

$(\neg P \vee Q) \wedge (\neg P \vee (Q \vee P)))$ — CNF is obtained

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4) To obtain PCNF and PDNF
$(\neg P \to R) \wedge (Q \leftrightarrow P)$

Statement | Reason

$(\neg P \to R) \wedge (Q \leftrightarrow P)$

$P \wedge R \wedge (Q \to P) \wedge (P \to Q)$

$(P \wedge R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$ — condition and biconditional laws

Now PCNF should have all variable in each then lets Add F to Add Rem

$\cancel{CPV}$

$((P \wedge R) \vee R) \wedge ((\neg Q \vee P) \vee F) \wedge ((\neg P \vee Q) \vee F))$

$[(P \wedge R \vee (\neg Q \wedge Q))] \wedge [((\neg Q \vee P) \vee (\neg R \wedge R))] \wedge$
$[((\neg P \vee Q) \vee (\neg R \wedge R))]$

$[(P \wedge R \vee Q) \wedge (P \wedge R \vee \neg Q)] \wedge [(\neg Q \vee P \vee \neg R) \wedge (\neg Q \vee P \vee R)] \wedge [(\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R)]$

$\Leftrightarrow$ PCNF is obtained

Maxterms of PQR are

$(P \vee Q \vee \bar{R})$

$\cancel{(\bar{P} \vee Q \vee \bar{R})}$, $\cancel{(P \vee \bar{Q} \vee R)}$, $\cancel{(P \vee Q \vee \bar{R})}$, $(P \vee \bar{Q} \vee R)$

$(P \vee \bar{Q} \vee \bar{R})$ $\cancel{(\bar{P} \vee Q \vee \bar{R})}$ $(\bar{P} \vee \bar{Q} \vee R)$ $\cancel{(P \vee Q \vee R)}$

$\neg\neg S$ = all maxterms without PCNF

$= (\bar{P} \vee \bar{Q} \vee R) \wedge (\bar{P} \vee \bar{Q} \vee \bar{R}) \wedge (P \vee Q \vee \neg R)$

Apply demorgan

$= (P \wedge Q \wedge R) \vee (\bar{P} \wedge \bar{Q} \wedge R) \vee (P \wedge Q \wedge \bar{R})$

PDNF is obtained.

Theory of Inference

It is used to check logical validity of given premises or simply we are proving!

Direct method:

we apply rules and we get conclusion

Indirect method:

we will take assumed value and prove it false thus the solution will be true

Formula

1) $P, P \to Q \Rightarrow Q$
2) $\neg Q, P \to Q \Rightarrow \neg P$
3) $\neg P, P \vee Q \Rightarrow Q$
4) $P \to Q, Q \to R \Rightarrow P \to R$
5) $P, Q \Rightarrow P \wedge Q$
6) $P \wedge Q \Rightarrow P, Q$
7) $P, Q \Rightarrow P \vee Q$
8) $P \wedge \neg Q \Leftrightarrow \neg(P \to Q)$

1) Direct method : Prove

$(P \to Q) \wedge (R \to S), (Q \to T) \wedge (S \to V), \neg (T \wedge V)$
and $(P \to R) \Rightarrow \neg P$

S.No	Statement	Formula
1)	$(P \to Q) \wedge (R \to S)$	Rule P{0}
2)	$(P \to Q)$	Rule T{1} $P \wedge Q \equiv P, Q$
3)	$(R \to S)$	Rule T{1} $P \wedge Q \Rightarrow P, Q$
4)	$(Q \to T) \wedge (S \to V)$	Rule {P}
5)	$(Q \to T)$	Rule T{4} $P \wedge Q \Rightarrow P, Q$
6)	$(S \to V)$	Rule T
7)	$(P \to Q), (Q \to T)$ $\boxed{\equiv (P \to T)}$	Rule T{2, 5}
8)	$(R \to S), (S \to V) \boxed{\Rightarrow (R \to V)}$	Rule T{3, 6}
9)	$\neg(T \wedge V)$	Rule P
10)	$\neg T$	Rule T{9} $\neg(T \wedge V) \equiv \neg T, V$
11)	V	
12)	$\neg T, P \to T \Rightarrow \neg P$	Rule T{7, 10}
13)	$V, P \to V \Rightarrow P$	Rule T{4, 1}
14)	$P, P \to T \Rightarrow T$	
15)	$\neg P, T \Rightarrow \neg P$	
16)	$\neg P$	

$\therefore$ obtained

Relty analls:

Direct method

$(P \to Q) \wedge (R - S), (Q \to T) \wedge (S \to V), \neg(T \wedge V)$ and $(P \to R) \Rightarrow \neg P$

S.NO	Statement	Formula
1)	$(P \to Q) \wedge (R \to S)$	Rule P $\boxed{\text{Additional premise}}$
2)	$(P \to Q)$	Rule T {1}
3)	$(R \to S)$	Rule T {1}
4)	$(Q \to T) \wedge (S \to V)$	Rule P
5)	$(Q \to T)$	Rule T {4}
6)	$(S \to V)$	Rule T {4}
7)	$(P \to Q), (Q \to T)$ $\boxed{= (P \to T)}$	Rule T {5, 2}
8)	$\boxed{R \to V}$	Rule T {6, 3}
9)	$\neg(T \wedge V)$	Rule P
10)	$P \to R$	Rule P
11)	$\neg T$	Rule T {9}
12)	V	Rule T {9}
13)	$(P \to R), (R \to V)$ $\boxed{= P \to V}$	Rule T {8, 10}
14)	$\neg T, P \to T \Rightarrow \neg P$	Rule T {11, 7}
15)	$V, P \to V \Rightarrow P$	Rule T {13, 12}
16)	$P, P \to T \Rightarrow T$	Rule T {15, 7}
17)	$T \wedge \neg P \Rightarrow \neg P$	Rule T {16, 14}

$\neg P$ is obtained.

Indirect method

$P \to Q, Q \to R, \overline{P \vee R} \Rightarrow R$

S.No	Statement	Reason
1)	$\neg R$	Rule P Additional premise
2)	$P \vee R$	Rule P
3)	$P \to Q$	Rule P
4)	$Q \to R$	Rule P
5)	$(P \to Q)(Q \to R) = \boxed{P \to R}$	Rule T {3, 4}
6)	$\neg R, P \vee R \Rightarrow \neg P$	Rule T {2, 1}
7)	$\neg P, P \to Q \Rightarrow \neg Q$	Rule T {6, 3}
8)	$\neg Q, Q \to R \Rightarrow \neg R$	Rule T {7, 5}
9)	$R, \neg R$	Rule T {1, 8}
10)	$R \wedge \neg R = F$	Rule T {9} = F

$\therefore$ They are True, Additional is false

Consistency and Inconsistency of premises

Consistent $\Rightarrow$ Tautology

Inconsistent $\Rightarrow$ Contradiction

Note: Mostly do not use any other laws than rules of inferences for these problems, mainly for conditionals

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Now we'll see of converting the Sentence to Statements

Ex 1) Rajkumar misses many classes through illness, he fails
2) If he fails he is uneducated
3) If he reads he is not uneducated
4) Raj misses mas class due to illness and read lot of Books

E: Raj skipped classes
S: He fails
A: He is uneducated
B: He reads Books

1) $E \rightarrow S$
2) $S \rightarrow A$
3) $B \rightarrow \neg A$
4) $E \wedge B$

use them and check for consistency or inconsistency

Quantifiers

*) They are used to quantify the nature of variable
*) There are two types
*) for all $(\forall x)$
*) for some $(\exists x)$

Rule :

[Rule US] : $(\forall x)\ P(x) \Rightarrow P(y)$
[Rule ES] : $(\exists x)\ P(x) \Rightarrow P(y)$
[Rule UG] : $P(y) \Rightarrow (\forall x)\ P(x)$
[Rule EG] : $P(y) \Rightarrow (\exists x)\ P(x)$
$\neg [(\exists x)(P(x))] \Rightarrow (\forall x)(\neg P(x))$

1) Test validity "All Integer are rational, some Integers are powers of 2"
"Therefore some of Integers are Rational and powers of 2"

$I(x)$: Int
$R(x)$: Rational
$P(x)$: Powers of 2

→ Universal
∧ Existential

Given :

1) $\forall x \, [I(x) \rightarrow R(x)]$
2) $\exists x \, [I(x) \wedge P(x)]$

Conclu

3) $\exists x \, [R(x) \wedge P(x)]$

use Rule P, T, US, UG, ES, EG for sure you will obtain the conclusion

Direct proof sums

1) Prove sum of two odd integers are even

Soln Lets assum the two odd as

$m = 2k+1$
$n = 2l+1$

$m+n = 2k+1+2l+2$
$= 2(k+l+1)$

∴ It's multiple of 2 ∴ they are even

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2) Prove $\sqrt{2}$ is irrational
soln

$\sqrt{2} = \frac{p}{q} \forall p, q \in \mathbb{Z}, q \neq 0$

p & q have no common divisor
Sq on bs

$2 = \frac{p^2}{q^2}$

$2q^2 = p^2 \rightarrow ①$

p is even

$p = 2m$

Sub 2 in ①

$2q^2 = (2m)^2$

$2q^2 = 4m^2$

$q^2 = 2m^2$

$q = 2m$

q is an even

Thus $\sqrt{2} = \frac{2m}{2k}$

They are irrational since they have common factor

NOTE P->Q have F when P: T Q: F
P<->Q have F when both are not same