DM Sem preparation

Unit - V

Topics to cover:
1) partial ordering
2) POSETS
3) Lattice as POSETS
4) Properties of Lattice
5) Lattice as algebraic systems
6) Sub Lattices
7) Direct product homomorphism
8) Some Special Lattices
9) Boolean Algebra
10) Sub boolean Algebra
11) Boolean homomorphism

Basic Definitions and 2 marks

1) Binary relation: If A×B are any two non empty sets, then a subset R of A×B is called Binary relation

2) partial order Relation:
A Relation R in a set P is called partial order relation or partial ordering in P iff R is Reflexive, antisymmetric, transitive

(1,2)(2,1) | (1,2)∈R (2,1)∉R unless a=b | (1,2) (2,3)∈R ~~E2~~ (1,3)∈R

3) POSET: A set is a POSET if it satisfies reflexive, antisymmetric and transitive also ~~[illegible]~~

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4) Totally ordered set (or) chain (or) simply ordered set

A POSET is said to be totally ordered set if any 2 elements are comparable

5) Upper Bound

Let $(P, \leq)$ be a POSET and A be any non empty subset of $P\,(A \subseteq P)$. An element $a \in P$ is an upper bound if $\boxed{a \geq x}$ $\boxed{\forall x \in A}$

6) Least upper Bound :

*) Least of all greatest

7) Lower Bound :

$a \leq x$ $\boxed{x \leq a \quad \forall x \in A}$

8) GLB : Greatest lower Bound

The greatest value of over all lower Bound

9) Lattice :

*) Is a POSET
*) Has GLB and LUB

10) Modular Lattice :

A lattice is said to be Modular lattice if $a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c \ \forall a, b, c \in L$

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1) Compliment:

Let $(L, \wedge, \vee)$ be a given Lattice. If it has both 0 element and 1 then it is Bounded Lattice, And if Each Lattice has compliment then are complimented Lattice

12) Boolean Algebra:

$(B, \wedge, \vee, 0, 1)$, also has compliment, if no compliment its not boolean algebra

Importants

2 Marks

1) Prove: every chain with three elements is always a complimented Lattice

if $(L, \wedge, \vee)$ is a chain, it means two elements are comparable wrt $0 \leq x \leq 1$

$0 \wedge x = 0$ or $x = x$

$1 \wedge x = x$, $1 \vee x = 1$

In both cases they do not have compliment!

2) Lattice homomorphism and Isomorphism:

homo: $f(a \vee b) = f(a) \oplus f(b)$

Iso: If Latice 1 - Lattice 2 are isomorphic they are one to one

3) $a \leq b$ and $c \leq d$ then $a * c \leq b * d$

$a \leq b = a * b = a$

$c * d = c * d = c$

T.P $(a * c) * (b * d) = (a * c)$

$= a * (c * b) * d$

$= a * c$

4) Prove lattice homomorphism order is preserving

$f: L_1 \to L_2$

$a \leq b$ then $GLB\{a,b\} = a$, $LUB\{a,b\} = b$

Now $f(a \wedge b) = f(a)$

$f(a) \wedge f(b) = f(a)$

$GLB\{f(a), f(b)\} = f(a)$, Therefore $f(a) \leq f(b)$

$a \leq b$ implies $f(a) \leq f(b)$ Therefore f is order preserving

5) Boolean Algebra 2^n, so no D_{12}

Part-B Sums

Important

1) Consider $D_{50} = \{1, 2, 5, 10, 25, 50\}$ is a poset then find the following:

a) Hasse

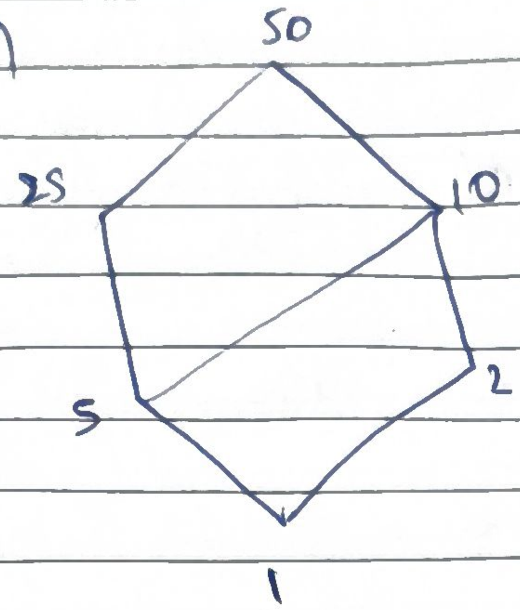
b) UB of 5 and 10, LUB of (5, 10)

c) LB of 5 and 10, GLB of (5, 10)

50

Soln:

a)



b) UB of 5 : {5, 10, 25, 50}

UB of 10 : {10, 50, 25}

UB[5,10] = {10, 25, 50}

LUB[5,10] = {10}

c) LB of 5, 10 &

LB{5} = {1, 2}

LB{10} = {10, 5, 1, 2}

LB{5,10} = {5, 1, 2}

GLB{5,10} = {5}

2) If (A, R) is a POSET then (A, R^{-1}) is also a POSET, R^{-1} = {(b, a) | (a, b) ∈ R}

i) Reflexive : (a, a) ∈ R^{-1}

TP (a, a) ∈ R^{-1}

Since (A, R) is a POSET

(a, a) ∈ R

Thus (a, a) ∈ R^{-1}

ii) antisymmetric : TP (a, b) ∈ R^{-1} ∧ (a, b) ∈ R^{-1}

∀ a, b ∈ A

WKT (A, R) is a POSET Thus

(b, a) ∈ R ∧ (a, b) ∈ R

∴ (A, R) is POSET iff a = b

$(a,b) \in R^{-1} \wedge (b,a) \in R^{-1} \therefore a=b$
antisymmetric is true

iii) Transitive $(a,b) \wedge (b,c) \in R^{-1}$ $(a,b) \in R'$
if
$\forall a,b,c$ (A,R)
$(b,a) \wedge (a,b) \in R$
$(a,b) \wedge (b,c) \in R$
$\Rightarrow (a,c) \in R$
$\Rightarrow (c,a) \in R$
Thus (A,R) is a POSET
$(a,c) \in R^{-1}$
Transitive is True
Thus (A, R^{-1}) is a POSET

3) <u>Every chain is a distributive Lattice</u>

Soln Let: $(L, \leq)$ be a chain and let $a,b,c \in L$ then $a \leq b \leq c$ and
$a \geq b \geq c$
Claim) $a \leq b \leq c$
where

$\boxed{a \wedge (b \wedge c) = (a \wedge b) \vee (a \wedge c)}$
$a \wedge (b \wedge c)$ $a \wedge (b \vee c)$
$= a \wedge c$
LHS $= a$
RHS $= (a \wedge b) \vee (a \wedge c)$
$= a \vee a$
$= a$
LHS = RHS
distributive is true

Case 2:
$\boxed{a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)}$
LHS $= c$
RHS $= c$
LHS = RHS
$\therefore$ It will be proved

4) In every distributive lattice $a * b = a * c$ and $a \oplus b = a \oplus c$ implies $b = c$

Soln: Let's take $a \leq b \leq c$ ① ②

$a * b = a * c$ (GLB)

① $a = a$ LHS = RHS

② $a \oplus b = a \oplus c$ (LUB)

$b = c$

$b = c$ is proved //

5) Prove that every distributive lattice is modular but not conversely

Soln: A lattice $(L, \wedge, \vee)$ is said to be modular iff

$a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c$

$a \oplus (b * c) = (a \oplus b) * c$

$\boxed{\text{So if } a \leq c \text{ then it is modular}}$ converse is not true

6) Prove distributive Inequalities are true

Soln:

To prove: $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

i) To prove $a \vee (b \wedge c) \leq (a \vee b)$

wkt GLB $\{b, c\} = b \wedge c$

$\boxed{b \wedge c \leq b}$ $\boxed{b \wedge c \leq c}$

wkt LUB $\{a, b\}$

$\boxed{a \leq a \vee b}$ $\boxed{b \leq a \vee b}$

$b \wedge c \leq a \vee b$

Add a

$a \vee (b \wedge c) \leq a \vee b$

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Now To prove $a \vee (b \wedge c) \le a \vee c$

$GLB\{b, c\} = b \wedge c$

$b \wedge c \le b, \quad b \wedge c \le c$

$LUB\{a, c\} = a \vee c$

$a \le a \vee c, \quad b \le a \vee c$

$b \wedge c \le a \vee c$

$\boxed{a \vee (b \wedge c) \le a \vee c}$

Same way we can prove the other case

7) Show that in distributive lattice and complimented lattice

$a \le b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \le a'$

1) $a \le b \Leftrightarrow a * b' = 0$

since $\boxed{a * b' = (a * b) * b' = 0}$

2) $a * b' = 0 \Leftrightarrow a' \oplus b = 1$

$a * b' = 0$

$a' \oplus b = (0)'$

$\boxed{a' \oplus b = 1}$

3) $a' \oplus b = 1 \Rightarrow b' \le a'$

$a' \oplus b = 1$

$= (a' \oplus b) * b' = 1 * b'$

$= (a' * b') \oplus (b * b') = b'$

$= a' * b' \oplus 0 = b'$

$= a' * b' = b'$

$\boxed{= b' \le a'}$

4) $b' \le a' \Rightarrow a \le b$

$b' \le a'$

$(b')' \le ((a)')'$

$\boxed{b \le a}$

8) Show that in any boolean algebra $(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$

$$LHS = (a+b'+0)(b+c'+0)(c+a'+0)$$
$$= (a+b'+cc')(b+c'+aa')(c+a'+bb')$$
$$= (a+b'+c)(a+b'+c')(b+c'+a)(b+c'+a')$$
$$(c+a'+b)(c+a'+b')$$
$$= (a'+b+cc')(b'+c+aa')(c'+a+bb')$$
$$= (a'+b+0)(b'+c+0)(c'+a+0)$$
$$= (a'+b)(b'+c)(c'+a)$$

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9) De Morgan

Prove i) $(a \vee b)' = a' \wedge b'$

To prove

$(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0$

$(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$

i) $\underbrace{(a \vee b)}_{b\ c} \wedge \underbrace{(\bar{a} \wedge \bar{b})}_{a} = 0$

$(\bar{a} \wedge \bar{b} \wedge a) \vee (\bar{a} \wedge \bar{b} \wedge b)$

$(a \wedge \bar{a} \wedge b) \vee (a \wedge \bar{b} \wedge b)$

$= 0$ proved

~~ii) $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$~~

~~$(\bar{a} \wedge \bar{b})$~~

ii) $\underbrace{(a \vee b)}_{a} \vee (\underbrace{\bar{a}}_{b} \wedge \underbrace{\bar{b}}_{c})$

$(a \vee b \vee \bar{a}) \wedge (a \vee b \vee \bar{b})$

$(1) \wedge (1)$

$= 1$

Do the same for $(a \wedge b)' = a' \vee b'$

i) $(a \wedge b) \wedge (\bar{a} \vee \bar{b}) = 0$

ii) $(a \wedge b) \vee (\bar{a} \vee \bar{b}) = 1$

$\boxed{\text{End of Importants give top priority to these questions}}$

Root of the lune

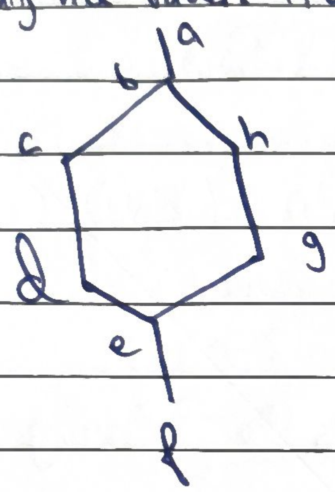
1) Prove $(N, \leq)$ is a lattice

$N = \{1, 2, 3 \ldots\}$

Lets say we have $N = \{1, 2, 3 \ldots\}$

$\boxed{\text{It has GLB and LUB so it must be a lattice}}$

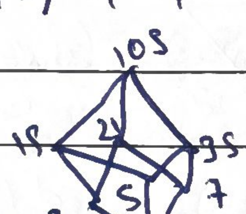
2) Finding the given Hasse diagram is a lattice



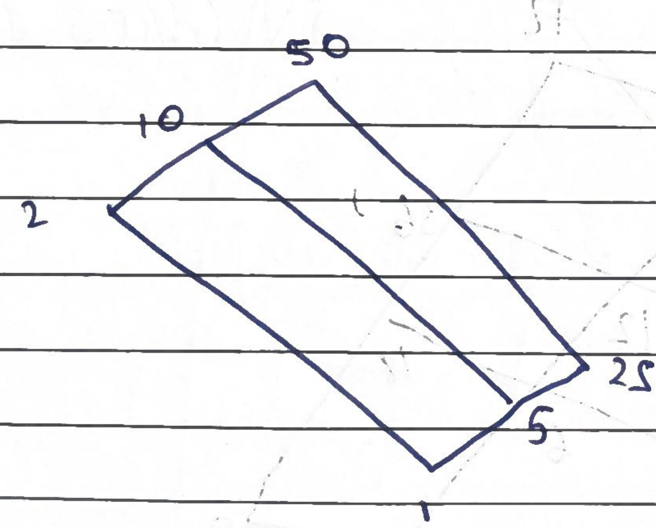
is a lattice it has GLB and LUB

3) Drawing Hasse diagram based on number of elements

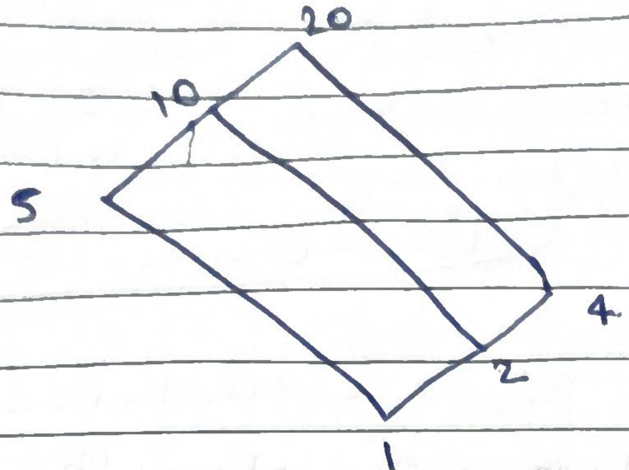
$D_{105} = \{1, 3, 5, 7, 15, 21, 35, 105\}$ = 8 (cube)



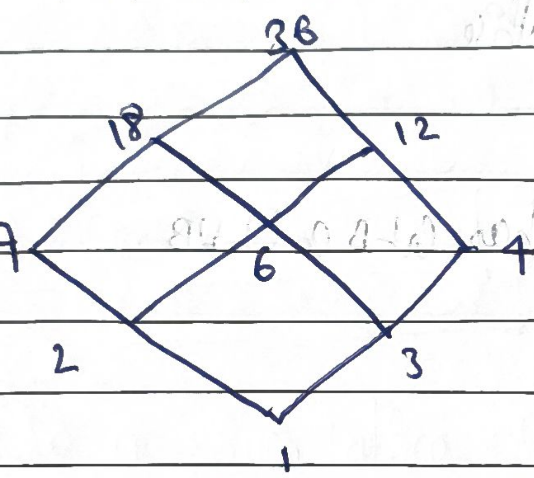
D_{50} $\{1, 2, 5, 10, 25, 50\}$ = 6 (window)



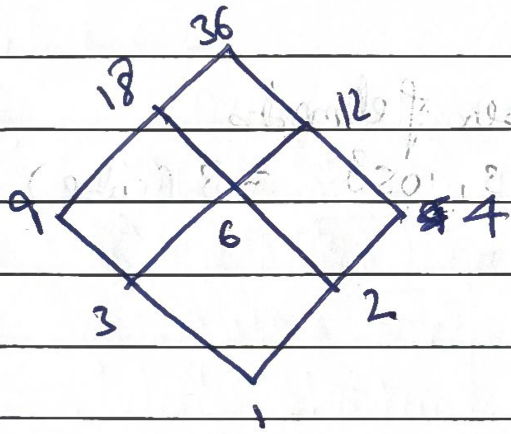
$D_{20} = \{1, 2, 4, 5, 10, 20\} = 6$ (window)



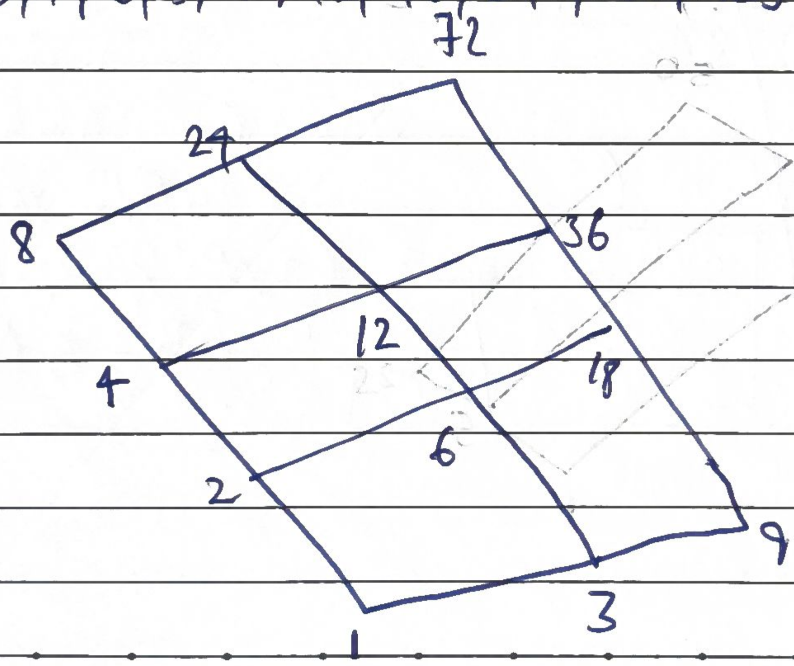
$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



(wrong)



$D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\} = 12$ elements



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4) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

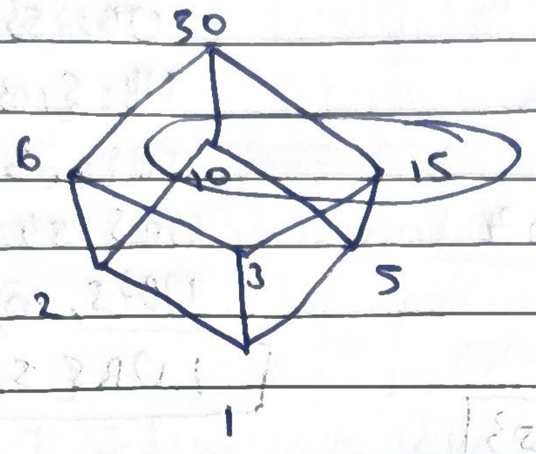
Find a) LB of 10 and 15 and GLB

b) UB of 10 and 15 and LUB

c) Hasse

Soln:

c) Hasse diagram



a) LB of $10 = \{10, 5, 3, 2, 1\}$

LB of $15 = \{15, \cancel{10}, 5, 3, 2, 1\}$

~~LB = {10, 5, 3, 2, 1}~~ LB = $\{5, 3, 2, 1\}$

$\boxed{\text{GLB}\{10, 15\} = 5}$

b) UB of 10 $\{10\} = \{10, \cancel{\ }, 30\}$

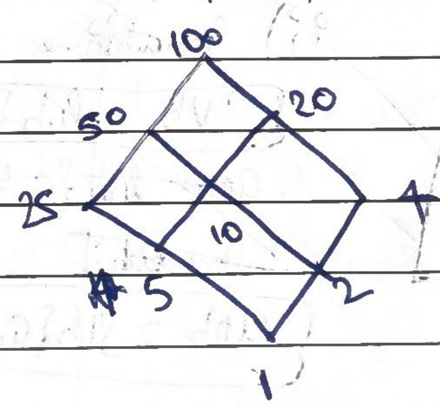
$\{15\} = \{15, 30\}$

$= \{30\}$ common

$\boxed{\text{LUB}\{10, 15\} = \{30\}}$

5) Let D_{100} be division of 100

$D_{100} : \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$



i) GLB {10, 20}

LB {10} = {10, 5, 2, 1}

LB {20} = {20, 4, 2, 1, 10}

LB {10, 20} = {10, 5, 2, 4, 1}

$\boxed{\text{GLB } \{10, 20\} = \{10\}}$

ii) LUB {10, 20}

UB {10} = {10, 50, 20, 100}

UB {20} = {20, 100}

UB {10, 20} = {20, 100}

$\boxed{\text{LUB } \{10, 20\} = \{20\}}$

iii) GLB {5, 10, 20, 25}

LB {5} = {5, 1}

LB {10} = {10, 5, 2, 1}

LB {20} = {20, 4, 2, 1, 10}

LB {25} = {5, 1}

LB = {5, 1}

$\boxed{\text{GLB } \{5, 10, 20, 25\} = \{5\}}$

iv) LUB = {5, 10, 20, 25}

UB {5} = {5, 25, 10, 20, 100}

UB = {10} = {10, 50, 20, 100}

UB {20} = {20, 100}

UB {25} = {50, 100}

UB {5, 10, 20, 25} = {100}

$\boxed{\text{LUB } \{5, 10, 20, 25\} = \{100\}}$

6) Let $(L, \wedge, \vee)$ be a given Lattice. Then $\wedge$ and $\vee$ Satisfies the following Conditions for all $a, b, c \in L$

i) Idempotent Law = $a \vee a = a$ and $a \wedge a = a$

ii) Commutative : $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$

iii) Associative Law : $(a \vee b) \vee c = a \vee (b \vee c)$ and ~~$a \wedge (b \wedge c)$ $(a \wedge (b \wedge c))$~~ $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

iv) Absorption Law : $a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

Soln:

i) Idempotent Law : $a \vee a = a$

$a \wedge a = a$

$$\boxed{\begin{aligned} a \vee a &= \text{LUB}\{a, a\} = a \\ a \wedge a &= \text{GLB}\{a, a\} = a \end{aligned}}$$

ii) Commutative

$\boxed{a \vee b = \text{lub}\{a, b\} = \text{lub}\{b, a\} = b \vee a}$

~~$a \wedge b = \text{lub}\{a, b\} = \text{lub}$~~

~~$a \wedge b = \text{lub}\{$~~

$\boxed{a \wedge b = \text{glb}\{a, b\} = \text{glb}\{b, a\} = b \wedge a}$

~~iii) Associative Law~~

~~$a \vee (b \vee c) = (a \vee b) \vee c$~~

~~TP: $a \vee (b \vee c) = (a \vee b) \vee c$~~

~~$a \vee (b \vee c) = d$~~

~~$a \vee (b \vee a)$~~

~~$(a \vee b) \vee c = e$~~

~~Thus $a \vee (b \vee c) = LUB\{a, b \vee c\}$~~

~~st $a \le d$~~

~~$b \vee c \le d$~~

~~Thus $(a \vee b) \vee c = Lub\{a \vee b, c\}$~~

~~$a \vee b$~~

iii) ~~Absorption Law~~: Associative Law

$a \vee (b \vee c) = (a \vee b) \vee c$

TP $a \vee (b \vee c) = (a \vee b) \vee c$

$a \vee (b \vee c) = d$

$(a \vee b) \vee c = e$

~~$b \vee c = \ldots$~~

$a \vee (b \vee c) = Lub\{a, b \vee c\}$

$\boxed{\begin{array}{l} a \le d \\ b \vee c \le d \end{array}}$ — 1

$b \vee c = Lub\{b, c\}$

$\left.\begin{array}{l} b \le b \vee c \\ c \le b \vee c \end{array}\right\}$ 4

From 1 & 4

$a \le d$

$b \le d$

$c \le d$

$d \ge c$

$a \vee b \le d$

$c \le d$

From 2 e is $LUB\{a \vee b, c\}$

d is ub $\{a, b \vee c\}$

$\boxed{\begin{array}{l} d = e \\ a \vee (b \vee c) = \\ (a \vee b) \vee c \\ \text{Similarly} \\ a \wedge (b \wedge c) \\ = (a \wedge b) \wedge c \\ \text{is true} \end{array}}$

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iv) Absorption Law :

$a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$

$a \wedge b$ is glb $\{a, b\}$ $\qquad$ $a \vee b$ is lub $\{a, b\}$

$\left.\begin{array}{l} a \leq a \wedge b \\ b \leq a \wedge b \end{array}\right\}$ ① $\qquad$ $\left.\begin{array}{l} a \leq a \vee b \\ b \leq a \vee b \end{array}\right\}$ 2

$a \leq a$

$a \leq a$ ②

$a \vee (a \vee b) \leq a$ ④

$\boxed{a \vee (a \wedge b) \leq a}$ ②

$a \leq a \vee (a \vee b)$ ④

from ④ ⑤

$\boxed{a \leq a \vee (a \wedge b)}$

$\boxed{a = a \vee (a \vee b)}$

$\boxed{a = a \vee (a \wedge b)}$

∴ Proved

7) Lattice Isotope property is true

Soln

Let $(L, \wedge, \vee)$ be given lattice for any $a, b, c \in L$

If $b \leq c$ Then $\boxed{\begin{array}{l} a \wedge b \leq a \wedge c \\ a \vee b \leq a \vee c \end{array}}$ $\begin{array}{l} GLB\{b, c\} = b \wedge c = b \\ LUB\{b, c\} = b \vee c = c \end{array}$

TP $a \wedge b \leq a \wedge c$

Then prove $(a \wedge b) \wedge (a \wedge c) = a \wedge b$

$(a \wedge b \wedge a \wedge c)$

$(a \wedge a \wedge b \wedge c)$ $\qquad$ $\boxed{\begin{array}{l} \text{note: } a \wedge a = a \\ b \wedge c = b \end{array}}$

$= (a \wedge b)$

∴ proved

TP $a \vee b \leq a \vee c$

$(a \vee b) \vee (a \vee c) = a \vee c$

$(a \vee b \vee a \vee c)$

$\Rightarrow a \vee c$ $\qquad$ $\boxed{\begin{array}{l} a \vee a = a \\ a \vee c = c \end{array}}$

8) Let $(L, \leq)$ be a lattice for any $a, b \in L$

$a \leq b \iff a \vee b = b \iff a \wedge b = a$

i) $a \leq b \iff a \vee b = b$

To prove $a \vee b \leq b$

Given $a \leq b$

wkt $a \leq a$ $b \leq b$

$a \vee b \leq b$

$b \leq a \vee b$

$a \vee b = \text{lub}\{a, b\}$

$a \leq a \vee b$

$b \leq a \vee b$

$\boxed{a \vee b = b}$

ii) $a \vee b = b \iff a \wedge b = a$

LHS $a \wedge b$

Sub b from last eq

$= a \wedge (a \vee b)$

$= a$ (Absorption Law)

It is True

iii) $a \wedge b = a$ to prove $a \leq b$

$a \wedge b = a$

$a \wedge b = \text{GLB}\{a, b\} = a$

$\boxed{\therefore a \leq b}$

9) In any distributive Lattice $(L, \wedge, \vee)$ prove that $(a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Soln: a dist Law

$\Rightarrow (a \vee b) \wedge \boxed{(b \vee c)\ (c \vee a)}$

$= \underbrace{(a \vee b)}_{a} \wedge (\underbrace{b}_{b}\ \underbrace{c \vee (a \wedge b)}_{c}\ )$ a

$= ((a \vee b) \wedge c) \vee ((a \vee b) \wedge (a \wedge b))$ dist Law

b c

$\div$

$\Rightarrow ((c\wedge a)\vee(c\wedge(b)))\vee((b\wedge a)\wedge a)\vee((b\wedge a)\wedge b)$

$= ((c\wedge a)\vee((c\wedge b)\vee(b\wedge a\wedge a)\vee(b\wedge a\wedge b)$

$= (a\wedge b)\vee(b\wedge c)\vee(c\wedge a)$

$=$ RHS

proved

$\boxed{\text{5th chapter over}}$

Remaining Missed sum

1) Every finite lattice is bounded

proof: Let $(L, \wedge, \vee)$ be a given lattice

$\therefore L$ is a lattice both GLB and LUB exist

Let a be GLB of L and

b be LUB of L

Then $x\in L$ $a \leq x \leq b$

GLB$\{a,x\} = a\wedge x = a$

LUB$\{a,x\} = a\vee x = x$

and GLB$\{b,x\} = x$

LUB$\{b,x\} = b$

$\therefore$ Any finite lattice is Bounded

2) If $(L,\wedge,\vee)$ is a lattice $a\leq c \Leftrightarrow a\vee(b\wedge c) \leq (a\vee b)\wedge c$

Soln $a \leq c$

$a\vee c = c$

By dist inequality

$a\vee(b\wedge c) \leq (a\vee b)\wedge(a\vee c)$

$\Rightarrow a\vee(b\wedge c) \leq (a\vee b)\wedge c$

$\therefore a\leq c \Rightarrow a\vee(b\wedge c) \leq (a\vee b)\wedge c$

Conversely

$a \vee (b \wedge c) \leq (a \vee b) \wedge c$

By the defn

$a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$

$\Rightarrow a \leq c$

$\therefore \; a \vee (b \wedge c) \leq (a \wedge b) \wedge c \Rightarrow a \leq c$

From Both we get

$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$