Unit II – Combinatorics
Part B

1 **Using Mathematical induction prove that** $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let $P(n) = \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

To Prove $P(1)$ is true

$$P(1) = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$$

Let us assume that $P(k)$ is true

$$P(k) = \frac{k(k+1)(2k+1)}{6}$$

Now to prove $P(k+1)$ is true.

i.e $P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$

$$P(k+1) = 1^2 + 2^2 + ... + k^2 + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$\therefore P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$$

Hence $P(k+1)$ is true.

2 **If *n* is a positive integer, then show that**
$$\mathbf{\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \forall n \geq 1.}$$

Solution:

Let $P(n) = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

We have to prove that $P(n)$ is true $\forall n \geq 1$.

Base Step : Put $n = 1$.

L.H.S

$$\therefore P(1) = \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{1.3} = \frac{1}{3}$$

R.H.S

$$\Rightarrow \frac{1}{2(1)+1} = \frac{1}{3}$$

$\Rightarrow$ LHS=RHS

$\therefore$ P(1) is true.

Inductive Step: Assume P(k) is true, k > 1.

$$\Rightarrow \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \text{ is true.}$$

To prove P(k+1) is true.

i.e to prove $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}=\frac{k+1}{2k+3}$ is true.

$$\text{L.H.S}=\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(\text{k}+1)}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

$\therefore P(k+1)$ is true.

Thus $P(k)$ is ture $\Rightarrow P(k+1)$ is true.

3 **Prove by mathematical induction that $6^{n+2}+7^{2n+1}$ is divisible by 43 for each positive integer *n*.**

Solution:

Let $P(1)$: Inductive step: for $n = 1$,

$6^{1+2}+7^{2+1}$=559, which is divisible by 43

So $P(1)$ is true.

Assume $P(k)$ is true.

$6^{k+2}+7^{2k+1}$ is divisible by 43.

i.e $6^{k+2}+7^{2k+1}=43m$ for some integer *m*.

To prove $P(k + 1)$ is true.

That is to prove $6^{(k+1)+2}+7^{2(k+1)+1}$ is divisible by 43.

Now

$$6^{k+3}+7^{2k+3}=6^{k+3}+7^{2k+1}.7^2$$

$$=6(6^{k+2}+7^{2k+1})+43.7^{2k+1}$$

$$=6.43m+43.7^{2k+1}$$

$$=43(6m+7^{2k+1})$$

This is divisible by 43.

So $P(k + 1)$ is true. By Mathematical Induction, $P(n)$ is true for all integer n.

4 **Using mathematical induction, show that for all positive integers *n*, $3^{2n+1}+2^{n+2}$ is divisible by 7.**

Solution:

Let $P(n)$: $3^{2n+1} + 2^{n+2}$ is divisible by 7.

Base Step:

To prove $P(1)$ is true,

$3^{2+1} + 2^{1+2} = 3^3 + 2^3 = 27 + 8 = 35 = (5)(7)$, which is divisible by 7.

$\therefore$ $P(1)$ is true.

Inductive Step:

Assume that $P(k)$ is true. That is $3^{2k+1} + 2^{k+2}$ is divisible by 7.

i.e., $3^{2k+1} + 2^{k+2} = 7m$ for some integer.

To prove $P(k+1)$ is true.

$$\begin{aligned}
\text{i.e., } 3^{2(k+1)+1} + 2^{(k+1)+2} &= 3^{2k+3} + 2^{k+3} \\
&= 3^{2k+1}(3^2) + 2^{k+2}(2) \\
&= 3^{2k+1}(9) + 2^{k+2}(2) \\
&= 2[3^{2k+1} + 2^{k+2}] + (7)\,3^{2k+1} \\
&= 2(7)(m) + (7)\,3^{2k+1}
\end{aligned}$$

which is divisible by 7.

Hence $P(k+1)$ is true whenever $P(k)$ is true.

By the principle of mathematical induction $P(n)$ is true for all positive integer n.

5 **Prove that Prove that $2^n < n!\ ,\ \forall n \geq 4$**

Solution : Let $P(n)$ be the proposition (or inequality)

$2^n < n!\ \ \forall n \geq 4$

We have to prove $P(n)$ is true $\forall n \geq 4$

Basis Step: Here $n_0 = 4$

$\therefore P(4)$ is $2^2 < 41 \Rightarrow$ $4 < 24$, which is true.

$\therefore P(4)$ is true.

Inductive Step: Assume $P(k)$ is true, $k > 1$.

$\Rightarrow$ $2^k < k!$ is true.

To prove $P(k+1)$ is true.

i.e. To prove $2^{k+1} < (k+1)\,!$ is true.

Now $2^{k+1} = 2^k \,.\, 2 < k!\ 2$

Since $k > 1$, $k + 1 > 2$.

$\therefore 2^{k+1} < k!.\,(k+1)$

$=> 2^{k+1} < (k+1)!$

$\therefore$ $P(k+1)$ is true.

Thus $P(k)$ is true = $P(k+1)$ is true.

Hence by first principle of induction $P(n)$ is true for $n \geq 4$.

$\Rightarrow 2^n < n!\ \ \forall n \geq 4$.

6

Prove by induction "every positive integer $n \geq 2$ is either a prime or can be written as a product of prime".

Solution: Let $P(n)$ denote the proposition "every integer $n \geq 2$ is either a prime or a product of primes". We have to prove $P(n)$ is true $\forall\ n \geq 2$.
Base Step: Put $n = 2$. $\therefore$ $P(2)$ is 2, which is a prime.
Thus $P(2)$ is true.
Inductive Step: Assume the proposition is true for all integers up to $k > 2$. i.e $P(3)$, $P(4)$ … $P(k)$ are true.
To prove $P(k+1)$ is true.
i. e to prove $(k+1)$ is either a prime or product of primes.
If $(k+1)$ is a prime, then we are through.
If $(k+1)$ is not a prime, it is a composite number and so it is a product of two positive integers x and y, where $1 < x, y < k+1$.
Since $x, y \leq k$, by induction hypothesis x and y are primes or product of primes.
$\therefore$ $k + 1 = x \,.\, y$ is a product of two or more primes.
$\therefore$ $P(k+1)$ is true.
Thus $P(3), P(4) \ldots P(k)$ are true $\Rightarrow P(k+1)$ is true.
Hence $P(n)$ is true for all $n \geq 2$.

7 **There are six men and five women in a room. Find the number of ways four persons can be drawn from the room.**

(1) They can be male or female
(2) Two must be men and two women
(3) They must all be of same sex

Solution
(i) Four persons can be drawn from 11 (6+5) persons is ${}^{11}C_4 = 330$ *ways*
(ii)Two men can be selected in ${}^{6}C_2$ ways and two women can be selected in ${}^{5}C_2$. Hence no. of ways of selecting 2 men and 2 women are ${}^{6}C_2 + {}^{5}C_2$ = 25 ways .
(iii) Number of ways of selecting four people and all of same sex is ${}^{6}C_4 + {}^{5}C_4$ = 20 ways .

8 **A Survey of 100 students with respect to their choice of the ice cream flavours Vanilla, Chocalate and strawberry shows that 50 students like Vanilla, 43 like chocolate , 28 like strawberry , 13 like Vanilla and chocolate, 11 like chocolate and strawberry , 12 like strawberry and Vanilla, and 5 like all of them. Find the number of students who like**

(i) Vanilla only
(ii) Chocolate only
(iii) Strawberry only
(iv) Chocolate but not Strawberry
(v) Chocolate and Strawberry but not Vanilla
(vi) Vanilla or Chocolate, but not Strawberry.
Also find the number students who do not like any of these flavours.

Given $|S \cap V \cap C| = 5$, $|S \cap V| = 12$, $|C \cap V| = 13$,$|C \cap S| = 11$
The Venn diagram shows the details

(i) Number of students who like Vanilla only = 30
(ii) Number of students who like Chocolate only = 24
(iii) Number of students who like Strawberry only = 10

(iv) Number of students who like Chocolate but not Strawberry = 24+8 = 32

(v) Number of students who like Chocolate and Strawberry but not Vanilla = 6

(vi) Number of students who like Vanilla or Chocolate but not Strawberry = 24 +8 + 30 = 62.

Number of students who do not like any of these flavours

= 100 – (50 + 16 + 24)

= 100 – 90 = 10

9 (a) In how many ways can the letters of the word MISSISSIPPI be arranged? (b) In how many of these arrangements, the P's are separated? (c) In how many arrangements, the I's are separated? (d) In how many arrangements, the P's are together?

Solution: (a) The word MISSISSIPPI contains 11 letters consisting of 4-I's, 4-S's, 2-P's, M.

$\therefore$ the number of arrangements $= \dfrac{11!}{4!\times 4!\times 2!} = \dfrac{3991680}{1152} = 34650$

(b) Since the P's are to be separated, first arrange the order 9 letters consisting of 4-I's, 4-S's and M. This can be done in $\dfrac{9!}{4!\times 4!}$ ways.

In each of these arrangements of 9 letters, there are 10 gaps in which the 2-P's can be arranged in $\dfrac{P(10,2)}{2!}$ ways. $= \dfrac{10\times 9}{2!} = 45 \text{ways}$.

$\therefore$ the total number of ways of arranging all the 11 letters which the P's are separated is $= \dfrac{9!}{4!\times 4!}\times 45 = \dfrac{16329600}{24\times 24} = 28,350$.

(c) Since the I's are to be separated from one another, first arrange the other 7 letters consisting of 4-I's, 2-P's and M. This can be done in $\dfrac{7!}{4!\times 2!}$ ways.

In each of these arrangements of 7 letters, there are 8 gaps in which the 4-I's can be arranged in $\dfrac{P(8,4)}{4!}$ ways. $= \dfrac{8\times 7\times 6\times 5}{4!} = 70 \text{ways}$.

$\therefore$ the total number of ways of arranging all the 11 letters which the I's are separated is $= \dfrac{7!}{4!\times 2!}\times 70 = 7350$ ways.

(d) Since the P's are to be together, treat them as one unit. The remaining 9 letters consisting of 4-I's, 4-S's and M as 9 units. Thus we have 10units, which can be arranged in $\dfrac{10!}{4!\times 4!}$ ways.

Since the P's are identical, by interchanging them we don't get any new arrangement.

Hence total number of arrangements in which the P's are together $\dfrac{10!}{4!\times 4!} = 6300$ ways.

10 The password for a computer system consists of eight distinct alphabetic characters. Find the number of passwords possible that

(a) end in the string MATH

(b) begin with the string CREAM

(c) contain the word COMPUTER as a substring

Solution: There are 26 English alphabets. Password consists of 8 different alphabets.

(a) The password should end with MATH.
The other four places must be filled with the remaining 22 alphabets choosing 4 at a time. This can be done in P(22, 4) ways.
$\therefore$ the total number of passwords = P(22, 4) = 22 × 21 × 20 × 19 = 175560.

(b) The password should begin with the string CREAM.
So, the 3places must be filled up with 3 letters from the remaining 21 letters in P(21, 3) ways. $\therefore$ number of passwords = P(21, 3) = 21 × 20 × 19 = 7980

(c) The word COMPUTER contains 8 letters and so it is itself the password.
$\therefore$ the number of ways of forming the password is 1.

11 How many bit strings of length 10 contains

(A) exactly four 1's
(B) at most four 1's
(C)at least four 1's
(D) an equal number of 0's and 1's

Solution:

A bit string of length 10 can be considered to have 10 positions. These 10 positions should be filled with four 1's and six 0's.

No. of required bit strings = $\frac{10!}{4!6!} = 210$

2.The 10 positions should be filled with (i) no 1's and ten 0's (ii) one 1's and nine 0's (iii) two 1's and eight 0's (iv) three 1's and seven 0's (v) four 1's and six 0's.Therefore Required no. of bit strings = $\frac{10!}{0!10!} + \frac{10!}{1!9!} + \frac{10!}{2!8!} + \frac{10!}{3!7!} + \frac{10!}{4!6!} = 386 ways$

3 The ten position are to be filled up with (i) four 1's and six 0's (or) (ii) five 1'and five 0's (or) six 1's and four 0's etc……ten 1's and zero 0's. Therefore no. of bit strings =
$\frac{10!}{4!6!} + \frac{10!}{5!5!} + \frac{10!}{6!4!} + \frac{10!}{3!7!} + \frac{10!}{8!2!} + \frac{10!}{9!11!} + \frac{10!}{10!0!} = 848\, ways$

4. The ten positions are to be filled up with five 1's and five 0's . Therefore no. of bit strings

$$\frac{10!}{5!5!} = 252\, ways$$

12 From a committee consisting of 6 men and 7 women, in how many ways can we select a committee of (i) 3 men and 4 women. (ii) 4 members which has atleast one women. (iii) 4 persons that has atmost one man. (iv) 4 persons of both sexes.

Solution:

(i) Three men can be selected from 6 men in $6C_3$ ways, 4 women can be selected from 7 women in $7C_4$ ways.

By product rule the committee of 3 men and 4 women can be selected in $6C_3 \times 7C_4 = 700$

(ii)For the committee of atleast one women we have the following possibilities
(a) 1 women and 3 men
(b) 2 women and 2 men
(c) 3 women and 1 men
(d) 4 women and 0 men
Therefore, the selection can be done in

$$7C_1 \times 6C_3 + 7C_2 \times 6C_2 + 7C_3 \times 6C_1 + 7C_4 \times 6C_0 \ ways$$
$$= 7 \times 20 + 21 \times 15 + 35 \times 6 + 35 \times 1$$
$$= 140 + 315 + 210 + 35 = 700 \ ways$$

(iii) For the committee of almost one men we have the following possibilities
(a) 1 men and 3 women
(b) 0 men and 4 women
Therefore, the selection can be done in

$$6C_1 \times 7C_3 + 6C_0 \times 7C_4 \ ways$$
$$= 6 \times 35 + 1 \times 35$$
$$= 245 \ ways$$

(iv) For the committee of both sexes, we have the following possibilities
(a) 1 men and 3 women
(b) 2 men and 2 women
(b) 3 men and 1 women
Therefore, the selection can be done in

$$6C_1 \times 7C_3 + 6C_2 \times 7C_2 + 6C_3 \times 7C_1 \ ways$$
$$= 6 \times 35 + 15 \times 21 + 20 \times 7$$
$$= 210 + 315 + 140 \ ways$$
$$= 665 \ ways$$

13 Find the number of integers between 1 and 100 that are not divisible by any of the integers 2, 3, 5 or 7.

$$\therefore \ |A| = \left\lfloor \frac{100}{2} \right\rfloor = 50, \quad |B| = \left\lfloor \frac{100}{3} \right\rfloor = 33$$
$$|C| = \left\lfloor \frac{100}{5} \right\rfloor = 20, \quad |D| = \left\lfloor \frac{100}{7} \right\rfloor = 14$$
$$|A \cap B| = \left\lfloor \frac{100}{LCM(2,3)} \right\rfloor = \left\lfloor \frac{100}{2 \times 3} \right\rfloor = 16$$
$$|A \cap C| = \left\lfloor \frac{100}{LCM(2,5)} \right\rfloor = \left\lfloor \frac{100}{2 \times 5} \right\rfloor = 10$$
$$|A \cap D| = \left\lfloor \frac{100}{LCM(2,7)} \right\rfloor = \left\lfloor \frac{100}{2 \times 7} \right\rfloor = 7$$
$$|B \cap C| = \left\lfloor \frac{100}{LCM(3,5)} \right\rfloor = \left\lfloor \frac{100}{3 \times 5} \right\rfloor = 6$$
$$|B \cap D| = \left\lfloor \frac{100}{LCM(3,7)} \right\rfloor = \left\lfloor \frac{100}{3 \times 7} \right\rfloor = 4$$
$$|C \cap D| = \left\lfloor \frac{100}{LCM(5,7)} \right\rfloor = \left\lfloor \frac{100}{5 \times 7} \right\rfloor = 2$$
$$|A \cap B \cap C| = \left\lfloor \frac{100}{LCM(2,3,5)} \right\rfloor = \left\lfloor \frac{100}{2 \times 3 \times 5} \right\rfloor = 3$$

$$|A\cap B\cap D| = \left\lfloor \frac{100}{LCM(2,3,7)} \right\rfloor = \left\lfloor \frac{100}{2\times 3\times 7} \right\rfloor = 2$$

$$|A\cap C\cap D| = \left\lfloor \frac{100}{LCM(2,5,7)} \right\rfloor = \left\lfloor \frac{100}{2\times 5\times 7} \right\rfloor = 1$$

$$|B\cap C\cap D| = \left\lfloor \frac{100}{LCM(3,5,7)} \right\rfloor = \left\lfloor \frac{100}{3\times 5\times 7} \right\rfloor = 0$$

$$|A\cap B\cap C\cap D| = \left\lfloor \frac{100}{LCM(2,3,5,7)} \right\rfloor = \left\lfloor \frac{100}{2\times 3\times 5\times 7} \right\rfloor = 0$$

$$\begin{aligned}|A\cup B\cup C\cup D| = {} & |A|+|B|+|C|+|D|-|A\cap B|-|A\cap C|-|A\cap D|-|B\cap C| \\ & -|B\cap D|+|C\cap D|+|A\cap B\cap C|+|A\cap B\cap D|+|A\cap C\cap D| \\ & +|B\cap C\cap D|-|A\cap B\cap C\cap D| \\ = {} & 78\end{aligned}$$

Therefore not divisible by any of the integers 2, 3, 5 and 7=100 – 78 = 22.

14 **Determine the number of positive integers n, 1$\leq n \leq 1000$, that are not divisible by 2, 3 or 5 but are divisible by 7**.

Solution:

Let A, B, C and D denote respectively the number of integers between1-1000, that are not divisible by 2,3 5 and 7 respectively. Now

$$|D| = \left[\frac{1000}{7}\right] = [142.8] = 142$$

$$|\mathrm{A}\cap \mathrm{B}\cap \mathrm{C}\cap \mathrm{D}| = \left[\frac{1000}{2\times 3\times 5\times 7}\right] = 7$$

The number between 1-1000 that are divisible by 7 but not divisible by 2,3,5 and 7 $= |D| - |\mathrm{A}\cap\mathrm{B}\cap\mathrm{C}\cap\mathrm{D}| = 138$

15 **Prove that in any group of six people there must be at least three mutual friends or three mutual enemies.**

Proof:

Let the six people be A, B, C, D, E and F. Fix A. The remaining five people can accommodate into two groups namely (1) Friends of A and (2) Enemies of A

Now by generalized Pigeonhole principle, at least one of the group must contain $\left(\frac{5-1}{2}\right)+1=3$ people.

Let the friend of A contain 3 people. Let it be B, C, D.

Case (1): If any two of these three people (B, C, D) are friends, then these two together with A form three mutual friends.

Case (2): If no two of these three people are friends, then these three people (B, C, D) are mutual enemies.

In either case, we get the required conclusion.

If the group of enemies of A contains three people, by the above similar argument, we get the required conclusion.

16 A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French and Russian, how many students have taken a course in all three languages?

Solution: Let S-Spanish, F-French, R-Russian.

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S\cap F| = 103$, $|S\cap R| = 23$, $|F\cap R| = 14$, $|S\cup F\cup R| = 2092$.

$|S\cup F\cup R| = |S| + |F| + |R| - |S\cap F| - |S\cap R| - |F\cap R| + |S\cap F\cap R|$

$= 1232 + 879 + 114 - 103 - 23 - 14 + 2092$

$\therefore |S\cap F\cap R| = 7$

17 Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, $n \geq 2$, $a_0 = 2, a_1 = 3.$

Solution: Given $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 2$, $a_1 = 3$.

$\Rightarrow a_n - 6a_{n-1} + 9a_{n-2} = 0$

Since $n - (n - 2) = 2$, it is of order 2.

The characteristic equation is

$r^2 - 6r + 9 = 0 \Rightarrow (r - 3)^2 = 0 \Rightarrow r = 3, 3$

The roots are real and equal.

$\therefore$ The general solution is $a_n = (A + Bn)\, 3^n$.

We shall now find the values of A, B using $a_0 = 2$, $a_1 = 3$.

Put $n = 0$, $\therefore\ a_0 = A$ $\Rightarrow A = 2$

Put $n = 1$, $\therefore\ a_0 = (A + B)\, 3 \Rightarrow 3(2 + B) = 3$

$\Rightarrow B = -1$

$\therefore$ The general solution is $a_n = (2 - n)\, 3^n$, $n \geq 0$.

18 Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$ where $n \geq 2$, $a_0 = 1$, $a_1 = 2$.

Solution: Given

$a_n = 2(a_{n-1} - a_{n-2})$

$= a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic equation is given by

$\lambda^2 - 2\lambda + 2 = 0$

$$\therefore \lambda = \frac{2 \pm \sqrt{4-4(2)}}{2} = \frac{2 \pm i2}{2} = 1 \pm i$$

$$\therefore \lambda = 1+i \ \ ,1-i$$

$$\therefore \text{Solution is } a_n = A(1+i)^n + B(1-i)^n$$

Where A and B are arbitrary constants

Now, we have

$$z = x + iy$$
$$= r[\cos\theta + i\sin\theta]$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

By Demoivre's theorem we have,

$$(1+i)^n = [\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)]^n$$

$$=[\sqrt{2}]^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$$\text{and } (1-i)^n = [\sqrt{2}]^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

Now,

$$a_n = A[[\sqrt{2}]^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)] + B[[\sqrt{2}]^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)]$$

$$=[\sqrt{2}]^n\left((A+B)\cos\frac{n\pi}{4} + i(A-B)\sin\frac{n\pi}{4}\right)$$

$$\therefore a_n = [\sqrt{2}]^n\left(C_1\cos\frac{n\pi}{4} + C_2\sin\frac{n\pi}{4}\right)] \qquad (1)$$

Is the required solution. Let $C_1 = A + B, \quad C_2 = i(A - B)$

Since $a_0 = 1, a_1 = 2$

$$(1) \Rightarrow a_0 = (\sqrt{2})[C_1\cos 0 + C_2\sin 0] = 0$$
$$\Rightarrow 1 = C_1$$

$$a_1 = [\sqrt{2}]^1\left(C_1\cos\frac{\pi}{4} + C_2\sin\frac{\pi}{4}\right)$$

$$2 = \sqrt{2}\left(C_1\frac{1}{\sqrt{2}} + C_2\sin\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow 2 = C_1 + C_2$$
$$\Rightarrow C_2 = 1$$

$\therefore a_n = [\sqrt{2}]^n \left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4} \right)$

19 Solve the recurrence

$$S(n) - 2S(n-1) - 3S(n-2) = 0,\ n \ge 2 \text{ with } S(0) = 3 \text{ and } S(1) = 1.$$

20 Find the general solution of the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 4^n,\ n \ge 2.$

$\text{Given } a_n - 5a_{n-1} + 6a_{n-2} = 4^n,\ n \ge 2 \ ----(1)$

The corresponding homogeneous recurrence relation is

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

Since $n-(n-2) = 2$, it is of order 2.

$\therefore$ the characteristic equation is $r^2 - 5r + 6 = 0$

$\Rightarrow (r-2)(r-3) = 0 \Rightarrow r = 2,\ 3$

$a_n^{(h)} = A.2^n + B.3^n$

Given $f(n) = 4^n$, 4 is not a root of the characteristic equation

$\therefore$ the particular solution is $a_n^{(p)} = C.4^n$

Substituting in the given equation (1), we get

$C.4^n - 5C.4^{n-1} + 6C.4^{n-2} = 4^n$

$4^{n-2}C[16 - 20 + 6] = 4^n$

Dividing by 4^{n-2}

$\Rightarrow 2C = 16$

$C = 8$

$\therefore a_n^{(p)} = 8 \times 4^n$

Hence the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$= A.2^n + B.3^n + \left(8 \times 4^n\right)$$

21 Solve the recurrence $a_n - 3a_{n-1} = 2\text{n},\ a_1 = 3$

Solution:

$a_n - 3a_{n-1} = 2n,\ a_1 = 3$

$\therefore$ the homogenous recurrence relation is $a_n - 3a_{n-1} = 0$

Since $n-(n-1) = 1$, the order is 1

The characteristic equation is $r - 3 = 0 \Rightarrow r = 3$

Hence $a_n^{(h)} = C.3^n$

Given $f(n) = 2n$, which is a polynomial of degree 1.

Hence Particular solution is $a_n = A_0 + A_1 n$

$\therefore A_0 + A_1 n - 3\left(A_0 + A_1(n-1)\right) = 2n$

$-2nA_1 = 2 \Rightarrow A_1 = -1$

$3A_1 - 2A_0 = 0$

$\Rightarrow 2A_0 = 3A_1$

$\Rightarrow 2A_0 = 3(-1)$

$\Rightarrow A_0 = \dfrac{-3}{2}$

$a_n^{(p)} = \dfrac{-3}{2} - n = \dfrac{-1}{2}(3+2n)$

$\therefore$ the general solution of the given recurrence relation is

$a_n = a_n^{(h)} + a_n^{(p)}$

$\Rightarrow a_n = C.3^n - \dfrac{1}{2}(3+2n) ----(1)$

Given: $a_1 = 3$

Putting $n = 1$ in (1)

$a_1 = C.3 - \dfrac{1}{2}(3+2)$

$\Rightarrow 3 = 3C - \dfrac{5}{2} \Rightarrow C = \dfrac{11}{6}$

$\therefore$ the general solution of the given recurrence relation is

$a_n = \dfrac{11}{6}.3^n - \dfrac{1}{2}(3+2n)$

22 Solve the recurrence relation for the Fibonacci sequence.

Solution:

The sequence of Fibonacci numbers satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ .....(1) and satisfies the initial conditions $f_1 = 1, f_2 = 1$.

(1) $\Rightarrow f_n - f_{n-1} - f_{n-2} = 0$ ...(2)

Let $f_n = r^n$ be a solution of the given equation.

The characteristic equation is $r^2 - r - 1 = 0$

$r = \dfrac{1 \pm \sqrt{1+4}}{2}$

Let $r_1 = \dfrac{1+\sqrt{5}}{2}, r_2 = \dfrac{1-\sqrt{5}}{2}$

$\therefore$ By theorem

$$f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n \text{ ...(3)}$$

$$f_1 = 1 \Rightarrow f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$(1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 2$ ...(4)

$$f_2=1\Rightarrow f_2=\alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^2+\alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^2=1$$

$$=\alpha_1\frac{(1+\sqrt{5})^2}{4}+\alpha_2\frac{(1-\sqrt{5})^2}{4}=1$$

$$=(1+\sqrt{5})^2\alpha_1+(1-\sqrt{5})^2\alpha_2=4 \qquad ...(5)$$

$$(4)\times(1-\sqrt{5})\Rightarrow$$

$$(1-\sqrt{5})\quad(1+\sqrt{5})\alpha_1+(1-\sqrt{5})^2\alpha_2=2\quad(1-\sqrt{5})\qquad ...(6)$$

$$(6)-(5)\Rightarrow\alpha_1(1+\sqrt{5})[1-\sqrt{5}-1-\sqrt{5}]=2-2\sqrt{5}-4$$

$$\alpha_1(1+\sqrt{5})[-2\sqrt{5}]=-2-2\sqrt{5}$$

$$\alpha_1(1+\sqrt{5})[-2\sqrt{5}]=-2(1+\sqrt{5})$$

$$\alpha_1=\frac{1}{\sqrt{5}}$$

$$4)\Rightarrow(1+\sqrt{5})\frac{1}{\sqrt{5}}+(1-\sqrt{5})\alpha_2=2$$

$$\frac{1}{\sqrt{5}}+1+(1-\sqrt{5})\alpha_2=2$$

$$(1-\sqrt{5})\alpha_2=2-\frac{1}{\sqrt{5}}-1$$

$$=1-\frac{1}{\sqrt{5}}$$

$$(1-\sqrt{5})\alpha_2=\frac{\sqrt{5}-1}{\sqrt{5}}$$

$$\alpha_2=\frac{-1}{\sqrt{5}}$$

$$(3)\Rightarrow f_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

23 Use generating function to solve the recurrence relation $S(n+1)-2S(n)=4^n$, with $S(0)=1$ and $n\geq 0$.

Solution: Given $\boldsymbol{S(n+1)-2S(n)=4^n}$, The recurrence relation can be written as $a_{n+1}-2a_n-4^n=0, n\geq 0 \quad (1)$

Multiply (1) by x^n and summing from n = 1 to ∞

$$\sum_{n=0}^{\infty} a_{n+1}x^n - 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 4^n x^n = 0$$

$$= \frac{1}{x}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 4^n x^n = 0$$

$$= \frac{1}{x}\left[G(x) - a(0) - 2G(x)\right] - \frac{1}{1-4x} = 0$$

$$= \frac{1}{x}(G(x)-1) - 2G(x) = \frac{1}{1-4x}$$

$$G(x)\left(\frac{1}{x}-2\right) = \frac{1}{1-4x} + \frac{1}{x} = \frac{x+1-4x}{x(1-4x)} = \frac{1-3x}{x(1-4x)}$$

$$G(x) = \frac{1-3x}{(1-4x)} \times \frac{1}{1-2x} = \frac{1-3x}{(1-2x)((1-4x)}$$

$$\frac{1-3x}{(1-2x)((1-4x)} = \frac{A}{1-2x} + \frac{B}{1-4x}$$

By solving we get A=1/2 and B=1/2

$$\therefore G(x) = \frac{\frac{1}{2}}{1-2x} + \frac{\frac{1}{2}}{1-4x}$$

$$= \frac{1}{2}\left[1-2x\right]^{-1} + \frac{1}{2}\left[1-4x\right]^{-1}$$

$$= \frac{1}{2}\left[1+2x+(2x)^2+\ldots\right] + \frac{1}{2}\left[1+4x+(4x)^2+\ldots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} 2^n x^n + \frac{1}{2}\sum_{n=0}^{\infty} 4^n x^n$$

$a_n =$ coefficient of $x^n in\, G(x)$

$$a_n = \frac{2^n}{2} + \frac{4^n}{2} = 2^{n-1} + 2(4)^{n-1}$$

24 Using generating function solve $a_n - 3a_{n-1} = 2,\ \forall\ n \geq 1, a_0 = 2$.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $\{a_n\}$.

Given $\qquad a_n - 3a_{n-1} = 2$

Multiplying by x^n, $\quad a_n x^n - 3a_{n-1}x^n = 2x^n$

$$\Rightarrow a_n x^n - 3x a_{n-1}x^{n-1} = 2x^n$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = 2\sum_{n=1}^{\infty} x^n$$

$$\Rightarrow a_0 + \sum_{n=1}^{\infty} a_n x^n - 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = a_0 + 2(x + x^2 + x^3 + \ldots)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - 3xG(x) = 2 + 2x(1 + x + x^2 + \ldots)$$

$$G(x) - 3xG(x) \quad = 2 + 2x(1-x)^{-1}$$

$$G(x)[1-3x] \quad = 2+\frac{2x}{1-x}=\frac{2}{1-x}$$

$$G(x)=\frac{2}{(1-x)(1-3x)}$$

Split $\frac{2}{(1-x)(1-3x)}$ by partial fraction

Let $\frac{2}{(1-x)(1-3x)}=\frac{A}{(1-x)}+\frac{B}{(1-3x)}$

$$\Rightarrow 2=A(1-3x)+B(1-x)$$

When $x=1$, $\quad 2=A(1-3)\Rightarrow A=-1$

When $x=\frac{1}{3}$, $\quad 2=B\left(1-\frac{1}{3}\right)\Rightarrow B=3$

$$\therefore G(x)=\frac{2}{(1-x)(1-3x)}=\frac{-1}{(1-x)}+\frac{3}{(1-3x)}$$

$$=-(1-x)^{-1}+3(1-3x)^{-1}$$

$$\Rightarrow a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots=-(1+x+x^2+\ldots+x^n+\ldots)$$
$$+3(1+3x+3^2x^2+\ldots+3^nx^n+\ldots)$$

$\therefore a_n=-1+3.3^n,\quad n\geq 0$

$a_n=-1+3^{n+1},\quad n\geq 0$

25 Use the method of generating function to solve

$$\boldsymbol{a_{n+1}-8a_n+16a_{n-1}=4^n,\ n\geq 1, a_0=1, a_1=8}$$

Let $G(x)=\sum_{n=0}^{\infty}a_nx^n$ be the generating function of the sequence $\{a_n\}$.

Given $\quad a_{n+1}-8a_n+16a_{n-1}=4^n$

Multiplying by x^n, $\quad a_{n+1}x^n-8a_nx^n+16a_{n-1}x^n=4^nx^n$

$$\Rightarrow \sum_{n=1}^{\infty}a_{n+1}x^n-8\sum_{n=1}^{\infty}a_nx^n+16\sum_{n=1}^{\infty}a_{n-1}x^n=\sum_{n=1}^{\infty}4^nx^n$$

$$\Rightarrow \frac{1}{x}\sum_{n=1}^{\infty}a_{n+1}x^{n+1}-8\sum_{n=1}^{\infty}a_nx^n+16x\sum_{n=1}^{\infty}a_{n-1}x^{n-1}=\sum_{n=1}^{\infty}(4x)^n \quad ----(1)$$

But $\sum_{n=1}^{\infty}a_{n+1}x^{n+1}=a_2x^2+a_3x^3+a_4x^4+\ldots$

$$=a_0+a_1x+a_2x^2+a_3x^3+\ldots-a_0-a_1x$$
$$=G(x)-1-8x$$

$$\sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots - a_0$$

$$= G(x) - 1$$

$$\therefore (1) \Rightarrow \frac{1}{x}\left[G(x) - 1 - 8x\right] - 8\left[G(x) - 1\right] + 16xG(x) = 1 + (4x) + (4x)^2 + \ldots - 1$$

$$G(x)\left[\frac{1}{x} - 8 + 16x\right] - \frac{1}{x} - 8 + 8 = \frac{1}{1-4x} - 1$$

$$G(x)\left[\frac{1 - 8x + 16x^2}{x}\right] - \frac{1}{x} = \frac{1-(1-4x)}{1-4x}$$

$$G(x)\left[\frac{(1-4x)^2}{x}\right] = \frac{4x}{1-4x} + \frac{1}{x}$$

$$G(x)\left[\frac{(1-4x)^2}{x}\right] = \frac{4x^2 + 1 - 4x}{x(1-4x)}$$

$$G(x) = \frac{1 - 4x + 4x^2}{(1-4x)^3}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \left(1 - 4x + 4x^2\right)\left(1-4x\right)^{-3}$$

$$= \left(1 - 4x + 4x^2\right)\frac{1}{1.2}\left(\begin{array}{r} 1.2 + 2.3(4x) + 3.4(4x)^2 + \ldots + (n-1)n(4x)^{n-2} + n(n+1)(4x)^{n-1} \\ + (n+1)(n+2)(4x)^n + \ldots \end{array}\right)$$

$$\therefore a_n = \textit{coefficient of } x^n$$

$$\Rightarrow a_n = \frac{1}{2}\left[(n+1)(n+2)4^n - 4n(n+1)4^{n-1} + 4(n-1)n4^{n-2}\right]$$

$$= \frac{1}{2}\left[4^n(n+1)(n+2-n) + n(n-1)4^{n-1}\right]$$

$$= \frac{1}{2}\left[4^n(n+1)(2) + (n^2 - n)4^{n-1}\right]$$

$$= \frac{4^{n-1}}{2}\left[8(n+1) + (n^2 - n)\right]$$

$$a_n = \frac{4^{n-1}}{2}\left[n^2 + 7n + 8\right]$$