CS7DS3 Applied Statistical Modelling
Assignment 1

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Course: MSc in Computer Science - Data Science

Q1

S_2 fails iff both components fails,

$$P(S_2\ fails) = P(x_1\ fails\ and\ x_2\ fails)$$

Given,

$$P(x_1\ fails) = P(x_2\ fails) = \theta$$

Since x_1 & x_2 failing are independent,

$$P(S_2\ fails) = P(x_1\ fails) * P(x_2\ fails)$$

$$\theta_1 = \theta * \theta$$

$$\theta_1 = \theta^2$$

Q2

A -> Total number of observed failures

$\theta = 0.08$; $n = 1000$

Each system follows a bernoulli trial, with the two outcomes; Component failure (success in trial) or Not component failure (failure in trial). The given n systems can thus be modeled with a binomial distribution with a probability of $\theta_1 = \theta^2 = 0.8^2$, and thus $\theta_1 = 0.0064$.

i

As stated in the **Probability Review II** notes under the section **Expectation and variance of random variables** for a binomial distribution,

Expectation,

$$\mathbb{E}[A] = n * \theta_1$$

$$\mathbb{E}[A] = 1000 * 0.0064$$

$$\mathbb{E}[A] = 6.4$$

Variance,

$$\mathbb{V}ar[A] = n * \theta_1 * (1 - \theta_1)$$

$$\mathbb{V}ar[A] = 1000 * 0.0064 * (1 - 0.0064)$$

$$\mathbb{V}ar[A] = 6.35904$$

ii

This was done via R code using the pbinom() function. The value $P(5 <= A <= 10)$ was computed as $P(A <= 10) - P(A < 5)$ which is the same as $P(A <= 10) - P(A <= 4)$ for our trials using `pbinom(10, 1000, 0.0064) - pbinom(4, 1000, 0.0064)`.

```
P(5 ⩽ A ⩽ 10) = 0.7050234
```

iii

This value was computed using qbinom() function in R. The value of k^* such that $P(A <= k^*) >= 0.95$ was computed as `qbinom(0.95, 1000, 0.0064)` and the result was checked to be strictly greater than the 95% percentile to satisfy $P(A <= k^*) > 0.95$ and the result was $k* = 11$ was strictly greater than the 95th percentile.

```
95th Percentile value for k = 11
P( A ⩽ 11 ) = 0.9697557
```

Q3

i

S_2 fails if either of the components fail (i.e) works iff both components work,

$$P(S_2\ works) = (1 - \theta) * (1 - \theta) = (1 - \theta)^2$$

$$P(S_2\ fails) = \theta_2 = 1 - (1 - \theta)^2$$

$$\theta_2 = 1 - (1 - \theta)^2$$

$$\theta_2 = 1 - (1 + \theta^2 - 2\theta)$$

$$\theta_2 = 2\theta - \theta^2$$

ii

S_3 fails if both of its parallel components fails, where each parallel component is composed of 2 serial subcomponents x_i,

$$P(S_3\,fails) = P(Parellel\ subcomponent\ 1\ fails\ and\ Parellel\ subcomponent\ 2\ fails)$$

$$\theta_3 = \theta_2 * \theta_2 = \theta_2^2$$

$$\theta_3 = [2\theta - \theta^2]^2$$

$$\theta_3 = 4\theta^2 + \theta^4 - 2\theta^2 * (2\theta)$$

$$\theta_3 = \theta^4 - 4\theta^3 + 4\theta^2$$

Q4

The experiment can be modeled with a binomial distribution (and likelihood) with probability θ, $n = 25$ experiments and $k = 3$ successes. Expert opinion suggest that there is a 5% to 10% chance each component fails, and that it is highly unusual for this to over 20%. This is the probability of failure of individual components (i.e) θ.

i

The specified prior is $Be(3, 30)$ with $a = 3; b = 30$. This is the number of expected successes and failures to be observed respectively.

Success here is the component failure and failure is the component non-failure.

- The beta prior is conjugate to the binomial distribution, and leads to a beta posterior (As given in the **Bayesian Inference: beta-binomial model** notes under section **Conjugacy**).
- The mean of the beta distribution is,

  $$mean = \frac{a}{a+b} = \frac{3}{3+30}$$

  $$mean \approx 0.09091$$

  This suggests that the expected failure is around 9.1%, which aligns well with the expert opinion that the failure rate is expected to be 5% - 10%.
- The variance is ,

  $$Var = \frac{ab}{(a+b)^2(a+b+1)}$$

  $$Var = \frac{3(30)}{(3+30)^2(3+30+1)} = \frac{90}{(33)^2(34)}$$

$$Var \approx 0.00243$$

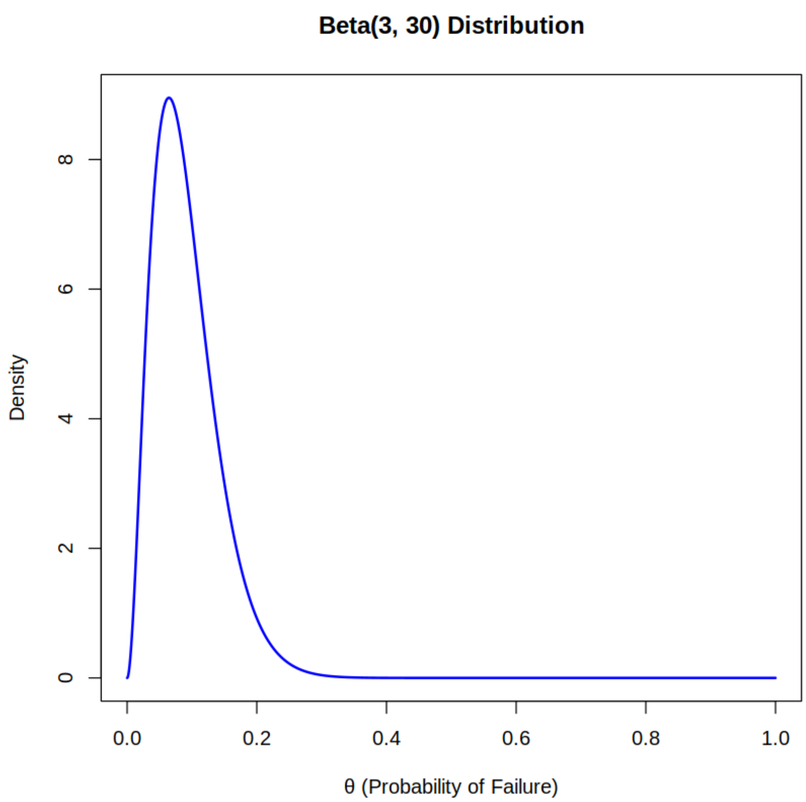
This tells us that the values of θ are concentrated around the mean and not too spread out.

- The mode is,

$$mode = \frac{a-1}{a+b-2} = \frac{3-1}{3+30-2}$$
$$mode \approx 0.06452$$

This tells us that the most likely value of θ is 6.45% which is consistent with the expert opinion.

- The distribution $Be(3, 30)$ is highly skewed towards lower values of θ (since $a << b$). The plot was created by generating the distribution using `dbeta(x, 3, 30)` with x ranging from 0 to 1 and plotting the results using `plot()`.



- The probabilty of the value being over 20% (stated to be highly unlikely) under this prior is,

$$P(\theta >= 0.20) \approx 0.03169$$

The value was computed using pbeta() as `1 - pbeta(0.20, 3, 30)`. This gives approximately a 3.2% chance for θ values over 20% and is thus consistent with expert opinion.

The chosen prior is thus a good match and is consistent with the expert opinion. Thus it can be used as the prior to model the parameter distribution.

ii

Under the Beta-Binomial model, for the prior $Be(a, b)$ the posterior can be constructed (with $k = \Sigma x_i$) as $Be(a_n, b_n)$ where $a_n = k + a$; $b_n = n - k + b$ (As stated in the **Bayesian Inference: beta-binomial model** notes under section **Beta-binomial model**).

$$a_n = k + a = 3 + 3$$

$$a_n = 6$$

$$b_n = n - k + b = 25 - 3 + 30$$

$$b_n = 52$$

Thus the posterior distribution is $\theta|x \sim Be(6, 52)$ - The mean of the beta distribution is,

$$mean = \frac{a}{a+b} = \frac{6}{6+52}$$

$$mean \approx 0.1035$$

This suggests that the expected failure is around 10.35%, which is a bit higher due to observed failures but aligns well with the expert opinion.

- The variance is ,

$$Var = \frac{ab}{(a+b)^2(a+b+1)}$$

$$Var = \frac{6(52)}{(6+52)^2(6+52+1)} = \frac{312}{(58)^2(59)}$$

$$Var \approx 0.00157$$

  The extimate is now more confident and the variance has decreased.

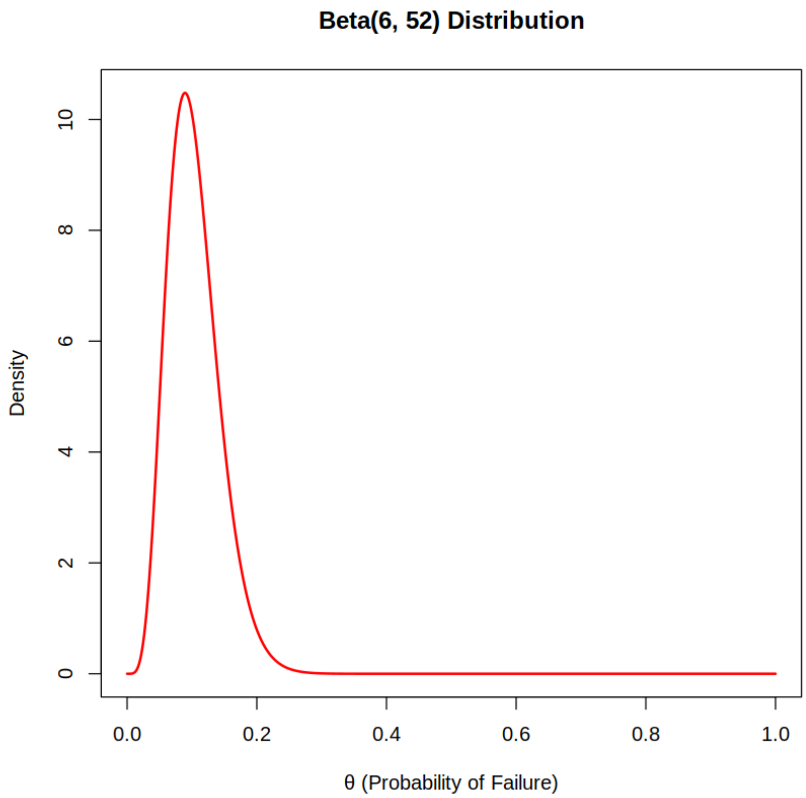
- The mode is,

$$mode = \frac{a-1}{a+b-2} = \frac{6-1}{6+52-2}$$

$$mode \approx 0.0893$$

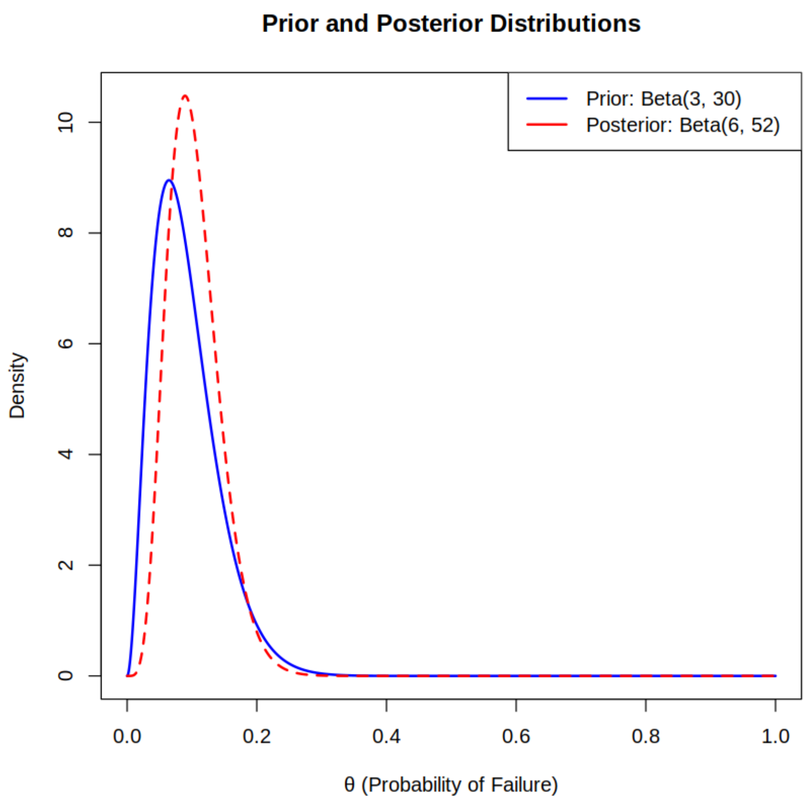
  This tells us that the most likely value of θ is 8.93% which has also increased due to the observed failures.

- The distribution $Be(6, 52)$ is also highly skewed towards lower values of θ (since $a << b$). The plot was created by generating the distribution using `dbeta(x, 6, 52)` with x ranging from 0 to 1 and plotting the results using `plot()`.



iii

- The prior mean was 0.09091(~9%) and mode was 0.06452(~6.45%), both of which which increased a bit in the posterior to 0.1035(~10.4%) and 0.0893(~8.93%).
- The variance has decreased from 0.00243(~0.24%) to 0.00157(~0.15%), which shows an increase in our certainty.
- While both the distributions are skewed towards lower values, the posterior is narrower. The plot was created by generating the distribution using `dbeta(x, 3, 30)` & `dbeta(x, 6, 52)` with x ranging from 0 to 1 and plotting the results using `plot()`.



- The prior was strong, but not too strong as the observed data was still able to influence the posterior in a non-insignificant way.
- Since the data consisted only of 25 observations, a bit less the number of pseudo-observations in the prior (33), the selected prior is slightly more informative than the data alone in the construction of the prior but not by a huge margin as seen by the data's ability to influence the posterior

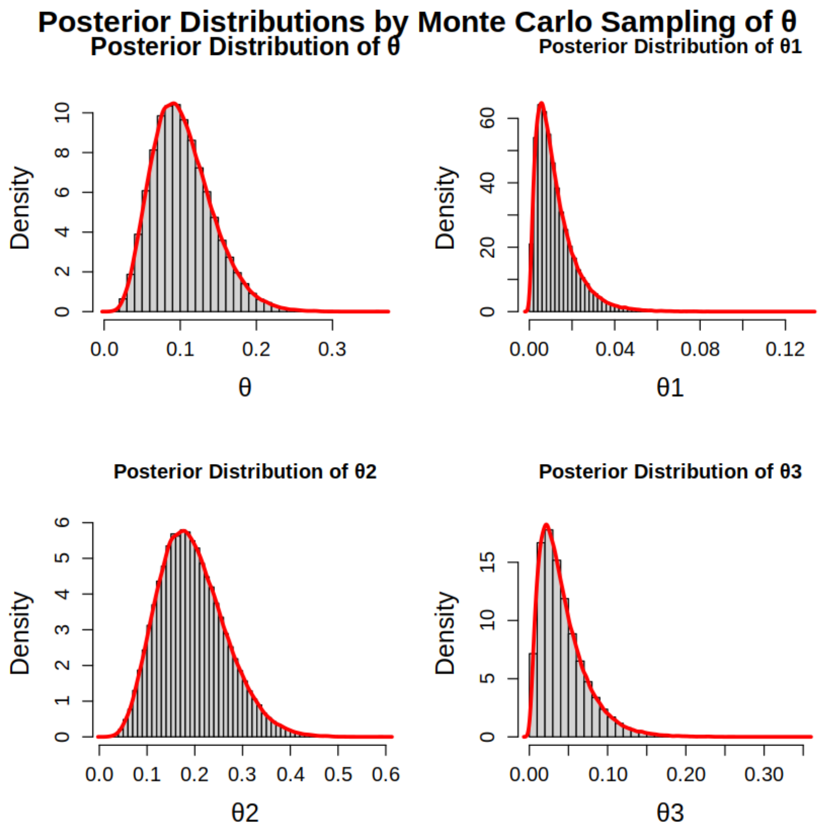
iv

The posterior distributions $P(\theta_1|x,\theta)$, $P(\theta_2|x,\theta)$, $P(\theta_3|x,\theta)$ are constructed by Monte-Carlo sampling of θ from the posterior distribution $P(\theta|x)$ and transformation of the sampled values in terms of θ_1, θ_2, and θ_3 - 100000 values from the posterior of θ_1 are sampled using `rbeta(1e5, 6, 52)`. All the plots in this section are made using `hist(freq=False)` to plot the densities histogram.

- The sampled values are transformed into θ_1 as `mc_post_theta_1 <- mc_post_theta ^ 2` since $\theta_1 = \theta^2$
- The sampled values are transformed into θ_2 as `mc_post_theta_2 <- (2`

`* mc_post_theta) - (mc_post_theta ^ 2)` since $\theta_2 = 2\theta - \theta^2$

- The sampled values are transformed into θ_3 as `mc_post_theta_3 <- (mc_post_theta ^ 4) - (4 * mc_post_theta ^ 3) + (4 * mc_post_theta^2)` since $\theta_3 = \theta^4 - 4\theta^3 + 4\theta^2$



- The mean, variance and the 95% interval for the posteriors of $\theta_1, \theta_2, \theta_3$ are computed from the derived values by using `mean()`, `var()`, `quantile(c(0.025, 0.975))`

```
Posterior for θ:
Mean: 0.1034516
Variance: 0.001584095
95% Interval: 0.03957243 0.1940732

Posterior for θ1:
Mean: 0.0122863
Variance: 9.221551e-05
95% Interval: 0.001565977 0.03766439

Posterior for θ2:
Mean: 0.1946168
Variance: 0.004945259
95% Interval: 0.07757889 0.3504819

Posterior for θ3:
Mean: 0.04282091
Variance: 0.000951169
95% Interval: 0.006018484 0.1228376
```

- The posterior of θ_1 is highly right-skewed distribution with a mean much smaller than that of θ ($0.0123 < 0.104$), the variance is also smaller ($9.17307e - 05 < 0.0016$) so the values are highly concentrated around the mean.
- The posterior of θ_2 stretches the distribution distribution resulting in a increase of mean θ ($0.195 > 0.104$), the variance is also higher ($0.0049 > 0.0016$) as the values are more spread out.
- The transformations of θ_3 is is a combination of the transformations from θ_1 & θ_2 resulting in a right-skewed posterior distribution like that of θ_1 but more stretched out than θ_1 similar to θ_2 and a decrease of mean from that of θ ($0.0.0429 < 0.104$), the variance is lower ($0.0009 < 0.0016$) as the values are more concentrated but is higher than that of θ_1.

Q5

i

The probability of failure of a S_3 system is given by the posterior $P(\theta_3|x)$. The 1000 S_3 systems can be modeled as a binomial distribution with $n = 1000$ and probability $P(\theta_3|x)$. - Monte Carlo samples (100000) are drawn from this distribution using the `rbinom()` function each sample simulating the failures from $n = 1000$ trials with the probabilities taken from the θ_3 estimated from Monte Carlo samples of θ with each sample from the binomial distribution using one of the probabilities from the θ_3 distribution. - The simulated failures are used to find the expected payments using `ifelse()` conditions for all the samples. - The Expected price can be computed as the average price over the samples,

$$E[Price] = \frac{1}{N}\sum_{i=1}^{N} Price_i$$

and is computed as `sum(mc_post_payments) / length((mc_post_payments))` and the result expected payments is **924.81** (approx).

ii

- If the non-informative prior $\theta \sim Be(1, 1)$ was used under the same conditions, the posterior distribution is then given by $Be(4, 23)$.
- θ_3 is sampled from this new posterior, and `rbinom()` is used to simulate the trials with the sampled probabilities.
- The Expected price is computed the same way as in Q5(i) and the result expected payments is **656.76** (approx).
- The expected price decreases significantly when using a non-informative prior.
- Thus the results are sensitive to the expert prior used
- Using a non-informative prior significantly increases the chances of failure and consequently reduces the expected price.

Q6

Given there are n systems with k failures and the probability of each S_1 system failing is given by θ_1. The observed data follow a Binomial distribution $y = y_1, y_2, ..., y_n \sim Binom(n, \theta_1)$.

i

The lieklihood function is,

$$L(\theta_1|y) = P(y|\theta_1) = \binom{n}{k}\theta_1^k(1-\theta_1)^{n-k}$$

In terms of θ (since $\theta_1 = \theta^2$),

$$L(\theta|y) = P(y|\theta) = \binom{n}{k}(\theta^2)^k(1-\theta^2)^{n-k}$$

$$L(\theta|y) = \binom{n}{k}\theta^{2k}(1-\theta^2)^{n-k}$$

ii

The likelihood function is,

$$L(\theta|y) = \binom{n}{k}\theta^{2k}(1-\theta^2)^{n-k}$$

Taking the log likelihood,

$$l = log(L) = log\ \binom{n}{k} + log\ \theta^{2k} + log\ (1-\theta^2)^{n-k}$$

The term $log\ \binom{n}{k}$ is a constant with respect to θ,

$$l = 2k\ log(\theta) + (n-k)\ log(1-\theta^2) + c$$

Taking the derivative of l with respect to θ,

$$\frac{dl}{d\theta} = \frac{2k}{\theta} + \frac{n-k}{1-\theta^2}(-2\theta)$$

Setting $\frac{dl}{d\theta} = 0$,

$$\frac{2k}{\theta} + \frac{n-k}{1-\theta^2}(-2\theta) = 0$$

$$\frac{2k}{\theta} = \frac{n-k}{1-\theta^2}(2\theta)$$

$$2k - 2k\theta^2 = 2n\theta^2 - 2k\theta^2$$

On simplifying,

$$n\theta^2 = k$$

$$\theta^2 = \frac{k}{n}$$

Thus,

$$\hat{\theta} = \sqrt{\frac{k}{n}}$$

where; $\hat{\theta}$ is the Maximum Likelihood estimate for θ.

iii

- The system failure is given by θ_1 which is a non-linear transformation of $\theta(\theta_1 = \theta^2)$, and the likelihood (in terms of θ) is given by,

$$L(\theta|y) = \binom{n}{k}\theta^{2k}(1-\theta^2)^{n-k}$$

- The likelihood in terms of θ_1 is binomial and the parameter can be modeled with a beta distribution by the beta-binomial conjugacy.
- But the likelihood in terms of θ is non-linear in θ and using a beta prior $(Be(a,b))$ for θ results in the posterior distribution,

$$P(\theta|y,a,b) \propto \theta^{2k}(1-\theta^2)\theta^{a-1}\theta^{b-1}$$

- The posterior is not in the form of a beta distribution and thus the conjugacy is lost.
- Thus a closed-form solution is not possible, and although Monte-Carlo methods can still be used to estimate the posterior, it is relatively difficult to estimate θ using a Bayesian approach.