Interpolating conformal algebra between the instant form and the front form of relativistic dynamics

Light Cone 2021: Contributed talk

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Motivation

- The instant form and the front form of relativistic dynamics introduced by Dirac in 1949 can be interpolated by introducing an interpolation angle parameter δ spanning between the instant form dynamics (IFD) at $\delta = 0$ and the front form dynamics, which is now known as the light-front dynamics (LFD) at $\delta = \frac{\pi}{4}$. **K.Hornboste, Phys. Rev. D 45, 3781 (1992)**.
- In **C.-R. Ji and C. Mitchell, Phys. Rev. D 64, 085013 (2001)**, they showed the Boost K^3 is dynamical in the region where $0 \leq \delta < \frac{\pi}{4}$ but becomes kinematic in the light-front limit ($\delta = \frac{\pi}{4}$).
- We extend the Poincaré algebra interpolation between instant and light-front time quantizations to the conformal algebra.
- Among the five more generators in the conformal algebra, only one generator known as the dilatation is kinematic for the entire region of the interpolation angle ($0 \leq \delta \leq \frac{\pi}{4}$).
- We find that one more generator from the Special Conformal Transformation (SCT) becomes kinematic in the light-front limit ($\delta = \frac{\pi}{4}$), i.e. the LFD.

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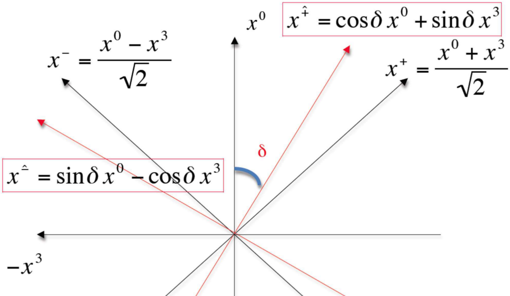


K. Hornbostel, PRD45, 3781 (1992) – RQFT
C.Ji and S.Rey, PRD53,5815(1996) – Chiral Anomaly
C.Ji and C. Mitchell, PRD64,085013 (2001) – Poincare Algebra
C.Ji and A. Suzuki, PRD87,065015 (2013) – Scattering Amps
C.Ji, Z. Li and A. Suzuki, PRD91, 065020 (2015) – EM Gauges
Z.Li, M. An and C.Ji, PRD92, 105014 (2015) – Spinors
C.Ji, Z.Li, B.Ma and A.Suzuki, in prepartion – Fermion Prop.

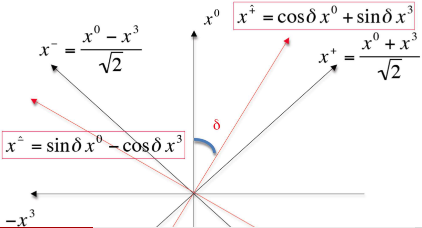
Method of Interpolation Angle

The interpolating space-time coordinates may be defined as a transformation from the ordinary space-time coordinates, $x^{\hat{\mu}} = \mathcal{R}^{\hat{\mu}}{}_{\nu} x^{\nu}$, i.e.

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{1}$$

in which the interpolation angle is allowed to run from 0 through 45°, $0 \le \delta \le \frac{\pi}{4}$.

Interpolation between Instant and Front Forms



Method of Interpolation Angle

In this interpolating basis, the metric becomes

$$g^{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}, \tag{2}$$

where $\mathbb{S} = \sin 2\delta$ and $\mathbb{C} = \cos 2\delta$.

The Poincaré matrix

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix} \tag{3}$$

transforms as well, so that

$$M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 \end{pmatrix} \tag{4}$$

where

$$\begin{aligned} E^{\hat{1}} &= J^2 \sin\delta + K^1 \cos\delta, \\ E^{\hat{2}} &= K^2 \cos\delta - J^1 \sin\delta, \\ F^{\hat{1}} &= K^1 \sin\delta - J^2 \cos\delta, \\ F^{\hat{2}} &= K^2 \sin\delta + J^1 \cos\delta. \end{aligned} \tag{5}$$

The Poincaré matrix

$$M_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\alpha}} M^{\hat{\alpha}\hat{\beta}} g_{\hat{\beta}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}, \tag{6}$$

where

$$\begin{aligned} \mathcal{K}^{\hat{1}} &= -K^1 \sin\delta - J^2 \cos\delta, \\ \mathcal{K}^{\hat{2}} &= J^1 \cos\delta - K^2 \sin\delta, \\ \mathcal{D}^{\hat{1}} &= -K^1 \cos\delta + J^2 \sin\delta, \\ \mathcal{D}^{\hat{2}} &= -J^1 \sin\delta - K^2 \cos\delta. \end{aligned} \tag{7}$$

Generators of Poincaré group

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}}, \tag{8}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right). \tag{9}$$

In the interpolating basis, the metric becomes

$$g^{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}, \tag{10}$$

The Poincaré algebra (Contra-variant form) in this interpolating basis is given by

$$\left[P^{\hat{\mu}}, P^{\hat{\nu}}\right] = 0, \tag{11a}$$

$$\left[P^{\hat{\rho}}, L^{\hat{\mu}\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}P^{\hat{\mu}}\right), \tag{11b}$$

$$\left[L^{\hat{\alpha}\hat{\beta}}, L^{\hat{\rho}\hat{\sigma}}\right] = -i\left(g^{\hat{\beta}\hat{\sigma}}L^{\hat{\alpha}\hat{\rho}} - g^{\hat{\beta}\hat{\rho}}L^{\hat{\alpha}\hat{\sigma}} + g^{\hat{\alpha}\hat{\rho}}L^{\hat{\beta}\hat{\sigma}} - g^{\hat{\alpha}\hat{\sigma}}L^{\hat{\beta}\hat{\rho}}\right). \tag{11c}$$

A comprehensive table of the 45 commutation relations among the co-variant components of the Poincare´ generators is presented below:

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P_-$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P_+$

Chueng-Ryong Ji and Chad Mitchell, Phys. Rev. **D 64**, 085013 (2001).
Chueng-Ryong Ji, Ziyue Li, and Alfredo Takashi Suzuki, Phys. Rev. **D 91**, 065020 (2015).

Kinematic and dynamic generators for different interpolation angles (Phys. Rev. **D 64**, 085013 (2001); Phys. Rev. **D 91**, 065020 (2015))

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P_-$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P_+$

- Among the ten Poincaré generators, the six generators $(\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P_1, P_2, P_{\hat{-}})$ are always kinematic in the sense that the $x^{\hat{+}} = 0$ plane is intact under the transformations generated by them. The operator $K^3 = M_{\hat{+}\hat{-}}$ is dynamical in the region where $0 \leq \delta < \pi/4$ but becomes kinematic in the light-front limit ($\delta = \pi/4$).
- To understand this, note that $[P^{\hat{+}}, K^{\hat{3}}] = i(\mathbb{S}P^{\hat{+}} - \mathbb{C}P^{\hat{-}}) \to iP^{\hat{+}}$ as $\delta \to \pi/4$. Similarly we have $[x^{\hat{+}}, L^{\hat{-}\hat{+}}] = i(\mathbb{S}x^{\hat{+}} - \mathbb{C}x^{\hat{-}}) \to ix^{\hat{+}}$ as $\delta \to \pi/4$. Therefore the instant defined by $x^{+} = 0$ becomes invariant under longitudinal boosts as we move to the light front.

Conformal Transformations

The Conformal transformation $x \longmapsto x'$ can be defined by, (1996 CFT Book by Francesco)

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta} = F(x) g_{\mu\nu} \tag{12}$$

Consider an infinitesimal translation,

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) \ . \tag{13}$$

The metric changes by,

$$\delta g_{\mu\nu} = \frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}} = \partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) \tag{14}$$

Conformality then requires,

$$\boxed{\partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) = F(x)\delta_{\mu\nu}} \quad \text{Conformal Killing Equation} \tag{15}$$

contraction with $\delta^{\mu\nu}$ yields

$$2\ \partial^{\mu}\epsilon_{\mu} = F(x)\ d \tag{16}$$

$$\implies \quad F(x) = \frac{2}{d}\partial_{\mu}\epsilon^{\mu} \tag{17}$$

Conformal Transformations

For $d \geq 3$, there are ONLY 4 calsses of solutions for $\epsilon_\mu(x)$

$$\begin{aligned} &\text{(Infinitesimal Translation)} \quad \epsilon^\mu(x) = a^\mu \quad \text{(constant)} && (18) \\ &\text{(Infinitesimal Rotation)} \quad \epsilon^\mu(x) = L^\mu_\nu x^\nu && (19) \\ &\text{(Infinitesimal Scaling)} \quad \epsilon^\mu(x) = \lambda x^\mu && (20) \\ &\text{(Infinitesimal SCT)} \quad \epsilon^\mu(x) = 2(b.x)x^\mu - x^2 b^\mu && (21) \end{aligned}$$

The generators of conformal transformations are:

$$\begin{aligned} &\text{(translation)} \quad P^\mu = -i\partial^\mu \ , \\ &\text{(dilation)} \quad D = -ix_\mu \partial^\mu \ , \\ &\text{(rotation)} \quad L^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \ , \\ &\text{(SCT)} \quad \mathfrak{K}^\mu = -i\left(2x^\mu x_\nu \partial^\nu - x^2 \partial^\mu\right) \ . \end{aligned}$$

Finite conformal transformations

$$
\begin{aligned}
\text{(translation)} \quad & x'^{\mu} = x^{\mu} + a^{\mu} \\
\text{(dilatation)} \quad & x'^{\mu} = \alpha x^{\mu} \\
\text{(rotation)} \quad & x'^{\mu} = M^{\mu}_{\nu} x^{\nu} \\
\text{(SCT)} \quad & x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2}
\end{aligned}
$$

Let us also note that for finite Special Conformal Transformations, we can re-write the expression as follows

$$
\begin{aligned}
x'^{\mu} &= \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2} = \frac{x^2}{|x - bx^2|^2}(x^{\mu} - b^{\mu} x^2), \\
\implies \frac{1}{x'^{\mu}} &= \frac{x^{\mu} - b^{\mu} x^2}{x^2}, \\
\implies \frac{x'^{\mu}}{x'^2} &= \frac{x^{\mu}}{x^2} - b^{\mu}
\end{aligned}
$$

SCT

From $\boxed{\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu}$, we see that the SCT can be understood as an inversion of x^μ, followed by a translation b^μ, and followed again by an inversion.

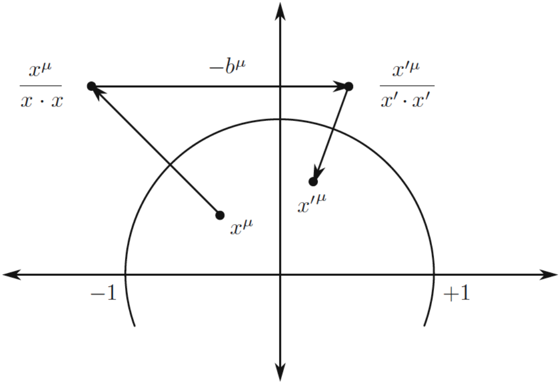


Figure: Illustration of a finite SCT

The generators of conformal transformations:

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}} \tag{22}$$

$$\text{(dilation)} \quad D = -ix_{\hat{\mu}}\partial^{\hat{\mu}} \tag{23}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right) \tag{24}$$

$$\text{(SCT)} \quad \mathfrak{K}^{\hat{\mu}} = -i\left(2x^{\hat{\mu}}x_{\hat{\nu}}\partial^{\hat{\nu}} - x^2\partial^{\hat{\mu}}\right) \tag{25}$$

Therefore the full Conformal algebra is given by

$$\begin{aligned}
&\left[P^{\hat{\mu}}, P^{\hat{\nu}}\right] = 0,\\
&\left[\mathfrak{K}^{\hat{\mu}}, \mathfrak{K}^{\hat{\nu}}\right] = 0,\\
&\left[D, P^{\hat{\mu}}\right] = iP^{\hat{\mu}},\\
&\left[D, \mathfrak{K}^{\hat{\mu}}\right] = -i\mathfrak{K}^{\hat{\mu}},\\
&\left[P^{\hat{\rho}}, L^{\mu\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}P^{\hat{\mu}}\right),\\
&\left[\mathfrak{K}^{\hat{\rho}}, L^{\hat{\mu}\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}\mathfrak{K}^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}\mathfrak{K}^{\hat{\mu}}\right),\\
&\left[L^{\hat{\alpha}\hat{\beta}}, L^{\hat{\rho}\hat{\sigma}}\right] = -i\left(g^{\hat{\beta}\hat{\sigma}}L^{\hat{\alpha}\hat{\rho}} - g^{\hat{\beta}\hat{\rho}}L^{\hat{\alpha}\hat{\sigma}} + g^{\hat{\alpha}\hat{\rho}}L^{\hat{\beta}\hat{\sigma}} - g^{\hat{\alpha}\hat{\sigma}}L^{\hat{\beta}\hat{\rho}}\right),\\
&\left[\mathfrak{K}^{\hat{\mu}}, P^{\hat{\nu}}\right] = 2i\left(g^{\hat{\mu}\hat{\nu}}D - L^{\hat{\mu}\hat{\nu}}\right),\\
&\left[D, L^{\hat{\mu}\hat{\nu}}\right] = 0.
\end{aligned}$$

Full conformal algebra

A comprehensive table of the **105 commutation** relations among the co-variant components of the Conformal generators is presented below:

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$	$\mathfrak{K}_{\hat{+}}$	$\mathfrak{K}_{\hat{1}}$	$\mathfrak{K}_{\hat{2}}$	$\mathfrak{K}_{\hat{-}}$	D
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$-2i(\mathbb{S}D - K^{\hat{3}})$	$-iP_{\hat{+}}$
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0	$2i\mathcal{D}^{\hat{1}}$	$2iD$	$-2iJ^{\hat{3}}$	$2i\mathcal{K}^{\hat{1}}$	$-iP_{\hat{1}}$
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0	$2i\mathcal{D}^{\hat{2}}$	$2iJ^{\hat{3}}$	$2iD$	$2i\mathcal{K}^{\hat{2}}$	$-iP_{\hat{2}}$
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	0	0	$-i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	0
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{+}}$	0	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0	0	$i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{1}}$	0	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{-}}$	0	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{-}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-2i(\mathbb{S}D + K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$2i\mathbb{C}D$	$-iP_{\hat{-}}$
$\mathfrak{K}_{\hat{+}}$	$2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	$2i(\mathbb{S}D + K^{\hat{3}})$	0	0	0	0	$i\mathfrak{K}_{\hat{+}}$
$\mathfrak{K}_{\hat{1}}$	$2i\mathcal{D}^{\hat{1}}$	$-2iD$	$-2iJ^{\hat{3}}$	0	$i\mathfrak{K}_{\hat{+}}$	0	$-i\mathfrak{K}_{\hat{2}}$	$i\mathfrak{K}_{\hat{-}}$	0	$2i\mathcal{K}^{\hat{1}}$	0	0	0	0	$i\mathfrak{K}_{\hat{1}}$
$\mathfrak{K}_{\hat{2}}$	$2i\mathcal{K}^{\hat{2}}$	$2iJ^{\hat{3}}$	$-2iD$	0	0	$i\mathfrak{K}_{\hat{+}}$	$i\mathfrak{K}_{\hat{1}}$	0	$i\mathfrak{K}_{\hat{-}}$	$2i\mathcal{K}^{\hat{2}}$	0	0	0	0	$i\mathfrak{K}_{\hat{2}}$
$\mathfrak{K}_{\hat{-}}$	$2i(\mathbb{S}D - K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	$-2i\mathbb{C}D$	0	0	0	0	$i\mathfrak{K}_{\hat{-}}$
D	$iP_{\hat{+}}$	$iP_{\hat{1}}$	$iP_{\hat{2}}$	0	0	0	0	0	0	$iP_{\hat{-}}$	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{-}}$	0

Conformal algebra in IFD

	P_0	P_1	P_2	K^3	$-K^1$	$-K^2$	J^3	$-J^2$	J^1	P_3	$\mathfrak{K}_0$	$\mathfrak{K}_1$	$\mathfrak{K}_2$	$\mathfrak{K}_3$	D
P_0	0	0	0	iP_3	iP_1	iP_2	0	0	0	0	$-2iD$	$2iK^1$	$-2iJ^1$	$-2iK^3$	$-iP_0$
P_1	0	0	0	0	iP_0	0	$-iP_2$	iP_3	0	0	$-2iK^1$	$2iD$	$-2iJ^3$	$-2iJ^2$	$-iP_1$
P_2	0	0	0	0	0	iP_0	iP_1	0	iP_3	0	$-2iK^2$	$2iJ^3$	$2iD$	$2iJ^1$	$-iP_2$
K^3	$-iP_3$	0	0	0	iJ^2	$-iJ^1$	0	iK^1	iK^2	$-iP_0$	$-i\mathfrak{K}_3$	0	0	$-i\mathfrak{K}_0$	0
$-K^1$	$-iP_1$	$-iP_0$	0	$-iJ^2$	0	$-iJ^3$	iK^2	$-iK^3$	0	0	$-i\mathfrak{K}_1$	$-i\mathfrak{K}_0$	0	0	0
$-K^2$	$-iP_2$	0	$-iP_0$	iJ^1	iJ^3	0	$-iK^1$	0	$-iK^3$	0	$-i\mathfrak{K}_2$	0	$-i\mathfrak{K}_0$	0	0
J^3	0	iP_2	$-iP_1$	0	$-iK^2$	iK^1	0	iJ^1	iJ^2	0	0	$i\mathfrak{K}_2$	$-i\mathfrak{K}_1$	0	0
$-J^2$	0	$-iP_3$	0	$-iK^1$	iK^3	0	$-iJ^1$	0	iJ^3	iP_1	0	$-i\mathfrak{K}_3$	0	$i\mathfrak{K}_1$	0
J^1	0	0	$-iP_3$	$-iK^2$	0	iK^3	$-iJ^2$	$-iJ^3$	0	iP_2	0	0	$-i\mathfrak{K}_3$	$i\mathfrak{K}_2$	0
P_3	0	0	0	iP_0	$i0$	$i0$	0	$-iP_1$	$-iP_2$	0	$-2iK^3$	$2iJ^2$	$-2iJ^1$	$2iD$	$-iP_3$
$\mathfrak{K}_0$	$2iD$	$2iK^1$	$2iK^2$	$i\mathfrak{K}_3$	$i\mathfrak{K}_1$	$i\mathfrak{K}_2$	0	0	0	$2iK^3$	0	0	0	0	$i\mathfrak{K}_0$
$\mathfrak{K}_1$	$-2iK^1$	$-2iD$	$-2iJ^3$	0	$i\mathfrak{K}_0$	0	$-i\mathfrak{K}_2$	$i\mathfrak{K}_3$	0	$-2iJ^2$	0	0	0	0	$i\mathfrak{K}_1$
$\mathfrak{K}_2$	$2iJ^1$	$2iJ^3$	$-2iD$	0	0	$i\mathfrak{K}_0$	$i\mathfrak{K}_1$	0	$i\mathfrak{K}_3$	$2iJ^1$	0	0	0	0	$i\mathfrak{K}_2$
$\mathfrak{K}_3$	$2iK^3$	$2iJ^2$	$-2iJ^1$	$i\mathfrak{K}_0$	0	0	0	$-i\mathfrak{K}_1$	$-i\mathfrak{K}_2$	$-2iD$	0	0	0	0	$i\mathfrak{K}_3$
D	iP_0	iP_1	iP_2	0	0	0	0	0	0	iP_3	$-i\mathfrak{K}_0$	$-i\mathfrak{K}_1$	$-i\mathfrak{K}_2$	$-i\mathfrak{K}_3$	0

Conformal algebra in LFD

	P_+	P_1	P_2	K^3	$-F^1$	$-F^2$	J^3	$-E^1$	$-E^2$	P_-	$\mathfrak{K}_+$	$\mathfrak{K}_1$	$\mathfrak{K}_2$	$\mathfrak{K}_-$	D
P_+	0	0	0	$-iP_+$	0	0	0	iP_1	iP_2	0	0	$2iF^1$	$2iE^2$	$-2i(D-K^3)$	$-iP_+$
P_1	0	0	0	0	iP_+	0	$-iP_2$	iP_-	0	0	$-2iF^1$	$2iD$	$-2iJ^3$	$-2iE^1$	$-iP_1$
P_2	0	0	0	0	0	iP_+	iP_1	0	iP_-	0	$-2iF^2$	$2iJ^3$	$2iD$	$-2iE^2$	$-iP_2$
K^3	iP_+	0	0	0	$-iF^1$	$-iF^2$	0	iE^1	iE^2	$-iP_-$	$i\mathfrak{K}_+$	0	0	$-i\mathfrak{K}_-$	0
$-F^1$	0	$-iP_+$	0	iF^1	0	0	iF^2	$-iK^3$	$-iJ^3$	$-iP_1$	0	$-i\mathfrak{K}_+$	0	$-i\mathfrak{K}_1$	0
$-F^2$	0	0	$-iP_+$	iF^2	0	0	$-iF^1$	iJ^3	$-iK^3$	$-iP_2$	0	0	$-i\mathfrak{K}_+$	$-i\mathfrak{K}_2$	0
J^3	0	iP_2	$-iP_1$	0	$-iF^2$	iF^1	0	$-iE^2$	iE^1	0	0	$i\mathfrak{K}_2$	$-i\mathfrak{K}_1$	0	0
$-E^1$	$-iP_1$	$-iP_-$	0	$-iE^1$	iK^3	$-iJ^3$	iE^2	0	0	0	$-i\mathfrak{K}_1$	$-i\mathfrak{K}_-$	0	0	0
$-E^2$	$-iP_2$	0	$-iP_-$	$-iE^2$	iJ^3	iK^3	$-iE^1$	0	0	0	$-i\mathfrak{K}_2$	0	$-i\mathfrak{K}_-$	0	0
P_-	0	0	0	iP_-	iP_1	iP_2	0	0	0	0	$-2i(D+K^3)$	$2iE^1$	$2iE^2$	0	$-iP_-$
$\mathfrak{K}_+$	0	$2iF^1$	$2iF^2$	$-i\mathfrak{K}_+$	0	0	0	$i\mathfrak{K}_1$	$i\mathfrak{K}_2$	$2i(D+K^3)$	0	0	0	0	$i\mathfrak{K}_+$
$\mathfrak{K}_1$	$-2iF^1$	$-2iD$	$-2iJ^3$	0	$i\mathfrak{K}_+$	0	$-i\mathfrak{K}_2$	$i\mathfrak{K}_-$	0	$-2iE^1$	0	0	0	0	$i\mathfrak{K}_1$
$\mathfrak{K}_2$	$-2iE^2$	$2iJ^3$	$-2iD$	0	0	$i\mathfrak{K}_+$	$i\mathfrak{K}_1$	0	$i\mathfrak{K}_-$	$-2iE^2$	0	0	0	0	$i\mathfrak{K}_2$
$\mathfrak{K}_-$	$2i(D-K^3)$	$2iE^1$	$2iE^2$	$i\mathfrak{K}_-$	$i\mathfrak{K}_1$	$i\mathfrak{K}_2$	0	0	0	0	0	0	0	0	$i\mathfrak{K}_-$
D	iP_+	iP_1	iP_2	0	0	0	0	0	0	iP_-	$-i\mathfrak{K}_+$	$-i\mathfrak{K}_1$	$-i\mathfrak{K}_2$	$-i\mathfrak{K}_-$	0

Kinematic and dynamic generators of the Conformal group

The generators of conformal transformations:

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}} \tag{26}$$

$$\text{(dilation)} \quad D = -ix_{\hat{\mu}}\partial^{\hat{\mu}} \tag{27}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right) \tag{28}$$

$$\text{(SCT)} \quad \mathfrak{K}^{\hat{\mu}} = -i\left(2x^{\hat{\mu}}x_{\hat{\nu}}\partial^{\hat{\nu}} - x^2\partial^{\hat{\mu}}\right) \tag{29}$$

Since $\left[\mathfrak{K}^{\hat{+}}, x^{\hat{+}}\right] = -i\left(2x^{\hat{+}}x^{\hat{+}} - (x^{\hat{\alpha}}.x_{\hat{\alpha}})\mathbb{C}\right) \rightarrow \; -i(x^0.x^0 + \vec{x}.\vec{x})$ as $\delta \rightarrow 0$, and $\left[\mathfrak{K}^{\hat{+}}, x^{\hat{+}}\right] = -i\left(2x^{\hat{+}}x^{\hat{+}} - (x^{\hat{\alpha}}.x_{\hat{\alpha}})\mathbb{C}\right) \rightarrow \; -i(2x^+.x^+)$ as $\delta \rightarrow \pi/4$, the conformal generator (LF time component) $\mathfrak{K}_-$ is Kinematic in LFD, but Dynamic in IFD. And $[D, x^{\hat{+}}] = -ix^{\hat{+}}$, so D is always Kinematic in both IFD and LFD.

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2,\ \mathcal{K}^{\hat{2}} = J^1,\ J^3, P^1, P^2, P^3,\ D$	$\mathcal{D}^{\hat{1}} = -K^1,\ \mathcal{D}^{\hat{2}} = -K^2,\ K^3, P^0,\ \mathfrak{K}_0,\ \mathfrak{K}_1,\ \mathfrak{K}_2,\ \mathfrak{K}_3$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}},\ D$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}},\ \mathfrak{K}_{\hat{+}},\ \mathfrak{K}_{\hat{1}},\ \mathfrak{K}_{\hat{2}},\ \mathfrak{K}_{\hat{-}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1,\ \mathcal{K}^{\hat{2}} = -E^2,\ J^3, K^3, P^1, P^2, P_-,\ D,\ \mathfrak{K}_-$	$\mathcal{D}^{\hat{1}} = -F^1,\ \mathcal{D}^{\hat{2}} = -F^2,\ P_+,\ \mathfrak{K}_+,\ \mathfrak{K}_1,\ \mathfrak{K}_2$