The Poincaré Algebra Interpolation between Instant Form Dynamics (IFD) and Light Front Dynamics (LFD)

Dissertations External presentation

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ABSTRACT

The instant form and the front form of relativistic dynamics introduced by Dirac in 1949 can be interpolated by introducing an interpolation angle parameter δ spanning between the instant form dynamics (IFD) at $\delta = 0$ and the front form dynamics, which is now known as the light-front dynamics (LFD) at $\delta = \frac{\pi}{4}$. We present the Poincaré algebra interpolating between instant and light-front time quantizations. We show the Boost K^3 is dynamical in the region where $0 \leq \delta < \frac{\pi}{4}$ but becomes kinematic in the light-front limit ($\delta = \frac{\pi}{4}$). We show this will then be extended to Conformal algebra.

Commutations

The generators of the Poincaré group are

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}} \ , \tag{1}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right) \ , \tag{2}$$

Then the Poincaré algebra (commutation rules) can be derived as,
1) Commutation among P^{μ},

$$\begin{aligned}[][P^{\mu}, P^{\nu}] &= P^{\mu}P^{\nu} - P^{\nu}P^{\mu} = i^2(\partial^{\mu}\partial^{\nu} - \partial^{\nu}\partial^{\mu}) = 0 \ , \\ [P^{\mu}, P^{\nu}] &= 0\checkmark \ .\end{aligned} \tag{3}$$

2) Commutation among P^{ρ} and $L^{\mu\nu}$,

$$\begin{aligned}[][P^{\rho}, L^{\mu\nu}] &= P^{\rho}L^{\mu\nu} - L^{\mu\nu}P^{\rho} = -i^2(\partial^{\rho}\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right) - \left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right)\partial^{\rho}) \ , \\ &= -i^2\left(\partial^{\rho}x^{\mu}\partial^{\nu} + x^{\mu}\partial^{\rho}\partial^{\nu} - \partial^{\rho}x^{\nu}\partial^{\mu} - x^{\nu}\partial^{\rho}\partial^{\mu} - x^{\mu}\partial^{\nu}\partial^{\rho} + x^{\nu}\partial^{\mu}\partial^{\rho}\right) \ , \\ &= -i^2\left(\partial^{\rho}x^{\mu}\partial^{\nu} - \partial^{\rho}x^{\nu}\partial^{\mu}\right) = i\left(g^{\rho\mu}(-i\partial^{\nu}) - g^{\rho\nu}(-i\partial^{\mu})\right) \ , \\ \left[P^{\rho}, L^{\mu\hat{\nu}}\right] &= i\left(g^{\rho\mu}P^{\nu} - g^{\rho\nu}P^{\mu}\right)\checkmark \ .\end{aligned} \tag{4}$$

Commutations

3) Commutation among $L^{\mu\nu}$,

$$
\begin{aligned}
\left[L^{\alpha\beta}, L^{\rho\sigma}\right] &= L^{\alpha\beta} L^{\rho\sigma} - L^{\rho\sigma} L^{\alpha\beta} \ , \\
&= i^2 \left(\left(x^\alpha \partial^\beta - x^\beta \partial^\alpha\right)\left(x^\rho \partial^\sigma - x^\sigma \partial^\rho\right) - \left(x^\rho \partial^\sigma - x^\sigma \partial^\rho\right)\left(x^\alpha \partial^\beta - x^\beta \partial^\alpha\right)\right) \\
\left[L^{\alpha\beta}, L^{\rho\sigma}\right] &= -i\left(g^{\beta\sigma} L^{\alpha\rho} - g^{\beta\rho} L^{\alpha\sigma} + g^{\alpha\rho} L^{\beta\sigma} - g^{\alpha\sigma} L^{\beta\rho}\right) \checkmark \ .
\end{aligned}
$$

So, the Poincaré algebra are:

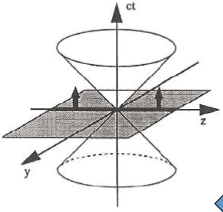
$$
\left[P^\mu, P^\nu\right] = 0 \ , \tag{5}
$$

$$
\left[P^\rho, L^{\mu\hat{\nu}}\right] = i\left(g^{\rho\mu} P^\nu - g^{\rho\nu} P^\mu\right) \ , \tag{6}
$$

$$
\left[L^{\alpha\beta}, L^{\rho\sigma}\right] = -i\left(g^{\beta\sigma} L^{\alpha\rho} - g^{\beta\rho} L^{\alpha\sigma} + g^{\alpha\rho} L^{\beta\sigma} - g^{\alpha\sigma} L^{\beta\rho}\right). \tag{7}
$$

LFD

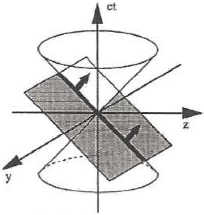
Dirac's Proposition



The instant form

1949

Can they be linked?



The front form

According to Dirac " ... the three-dimensional surface in space-time formed by a plane wave front advancing with the velocity of light. Such a surface will be called *front* for brevity". An example of a light-front is given by the equation $x^+ = x^0 + x^3 = 0$.

The variables $x^+ = \frac{x^0+x^3}{\sqrt{2}}$ and $x^- = \frac{x^0-x^3}{\sqrt{2}}$ are called light-front time and longitudinal space variables respectively. Transverse variable $x^\perp = (x^1, x^2)$. We denote the four-vector x^μ by

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, x^\perp, x^3)\ . \tag{8}$$

Scalar product

$$x.y = x^+ y^- + x^- y^+ - x^\perp . y^\perp . \tag{9}$$

The metric tensor is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\ , \tag{10}$$

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{11}$$

Let us denote the three generators of boosts by K^i and the three generators of rotations by J^i in equal-time dynamics. Define $E^1 = -K^1 + J^2$, $E^2 = -K^2 - J^1$, $F^1 = -K^1 - J^2$, and $F^2 = -K^2 + J^1$. The explicit expressions for the 6 generators K^3, E^1, E^2, J^3, F^1, and F^2 in the finite dimensional representation are

$$K^3 = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad E^1 = -i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{12}$$

$$E^2 = -i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J^3 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

$$F^1 = -i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad F^2 = -i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{14}$$

Note that K^3, E^1, E^2, and J^3 leave $x^+ = 0$ invariant and are kinematical generators while F^1 and F^2 do not and are dynamical generators.

From the Lagrangian density one may construct the stress tensor $T^{\mu\nu}$ and from the stress tensor one may construct a four-momentum P^{μ} and a generalized angular momentum $L^{\mu\nu}$.

$$P^{\mu} = \int dx^{-} d^{2}x^{\perp} \; T^{+\mu}, \tag{15}$$

$$L^{\mu\nu} = \int dx^{-} d^{2}x^{\perp} [x^{\nu} \; T^{+\mu} - x^{\mu} \; T^{+\nu}]. \tag{16}$$

Note that $L^{\mu\nu}$ is antisymmetric and hence has six independent components. Poincare algebra in terms of P^{μ} and $L^{\mu\nu}$ is

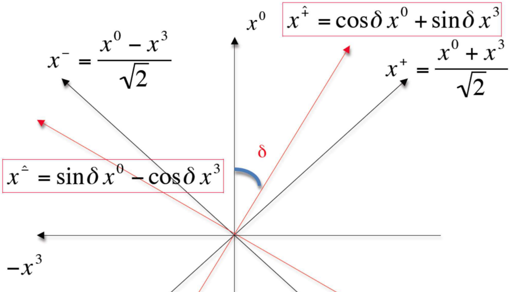
$$[P^{\mu}, P^{\nu}] = 0, \tag{17}$$

$$[P^{\mu}, L^{\rho\sigma}] = i[g^{\mu\rho} P^{\sigma} - g^{\mu\sigma} P^{\rho}], \tag{18}$$

$$[L^{\mu\nu}, L^{\rho\sigma}] = i[-g^{\mu\rho} L^{\nu\sigma} + g^{\mu\sigma} L^{\nu\rho} - g^{\nu\sigma} L^{\mu\rho} + g^{\nu\rho} L^{\mu\sigma}]. \tag{19}$$

In light-front dynamics P^{-} is the Hamiltonian and P^{+} and P^{i} $(i = 1, 2)$ are the momenta. $L^{-+} = K^{3}$ and $L^{+i} = E^{i}$ are the boosts. $L^{12} = J^{3}$ and $L^{-i} = F^{i}$ are the rotations.

Interpolation between Instant and Front Forms



K. Hornbostel, PRD45, 3781 (1992) – RQFT
C.Ji and S.Rey, PRD53,5815(1996) – Chiral Anomaly
C.Ji and C. Mitchell, PRD64,085013 (2001) – Poincare Algebra
C.Ji and A. Suzuki, PRD87,065015 (2013) – Scattering Amps
C.Ji, Z. Li and A. Suzuki, PRD91, 065020 (2015) – EM Gauges
Z.Li, M. An and C.Ji, PRD92, 105014 (2015) – Spinors
C.Ji, Z.Li, B.Ma and A.Suzuki, in prepartion – Fermion Prop.

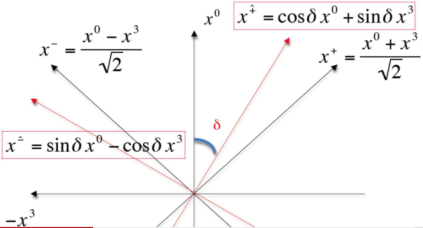
Method of Interpolation Angle

The interpolating space-time coordinates may be defined as a transformation from the ordinary space-time coordinates, $x^{\hat{\mu}} = \mathcal{R}^{\hat{\mu}}{}_{\nu} x^{\nu}$, i.e.

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{20}$$

in which the interpolation angle is allowed to run from 0 through 45°, $0 \le \delta \le \frac{\pi}{4}$.

Interpolation between Instant and Front Forms



Method of Interpolation Angle

In this interpolating basis, the metric becomes

$$g^{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}, \tag{21}$$

where $\mathbb{S} = \sin 2\delta$ and $\mathbb{C} = \cos 2\delta$.

The Poincaré matrix

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix} \tag{22}$$

transforms as well, so that

$$M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 \end{pmatrix} \tag{23}$$

where

$$\begin{aligned} E^{\hat{1}} &= J^2 \sin\delta + K^1 \cos\delta, \\ E^{\hat{2}} &= K^2 \cos\delta - J^1 \sin\delta, \\ F^{\hat{1}} &= K^1 \sin\delta - J^2 \cos\delta, \\ F^{\hat{2}} &= K^2 \sin\delta + J^1 \cos\delta. \end{aligned} \tag{24}$$

The Poincaré matrix

$$M_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\alpha}} M^{\hat{\alpha}\hat{\beta}} g_{\hat{\beta}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}, \tag{25}$$

where

$$\begin{aligned} \mathcal{K}^{\hat{1}} &= -K^1 \sin\delta - J^2 \cos\delta, \\ \mathcal{K}^{\hat{2}} &= J^1 \cos\delta - K^2 \sin\delta, \\ \mathcal{D}^{\hat{1}} &= -K^1 \cos\delta + J^2 \sin\delta, \\ \mathcal{D}^{\hat{2}} &= -J^1 \sin\delta - K^2 \cos\delta. \end{aligned} \tag{26}$$

Generators of Poincaré group

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}}, \tag{27}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right). \tag{28}$$

In the interpolating basis, the metric becomes

$$g^{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}, \tag{29}$$

The Poincaré algebra (Contra-variant form) in this interpolating basis is given by

$$\left[P^{\hat{\mu}}, P^{\hat{\nu}}\right] = 0, \tag{30a}$$

$$\left[P^{\hat{\rho}}, L^{\hat{\mu}\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}P^{\hat{\mu}}\right), \tag{30b}$$

$$\left[L^{\hat{\alpha}\hat{\beta}}, L^{\hat{\rho}\hat{\sigma}}\right] = -i\left(g^{\hat{\beta}\hat{\sigma}}L^{\hat{\alpha}\hat{\rho}} - g^{\hat{\beta}\hat{\rho}}L^{\hat{\alpha}\hat{\sigma}} + g^{\hat{\alpha}\hat{\rho}}L^{\hat{\beta}\hat{\sigma}} - g^{\hat{\alpha}\hat{\sigma}}L^{\hat{\beta}\hat{\rho}}\right). \tag{30c}$$

A comprehensive table of the 45 commutation relations among the co-variant components of the Poincare´ generators is presented below:

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P_-$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P_+$

Chueng-Ryong Ji and Chad Mitchell, Phys. Rev. **D 64**, 085013 (2001).
Chueng-Ryong Ji, Ziyue Li, and Alfredo Takashi Suzuki, Phys. Rev. **D 91**, 065020 (2015).

IFD

The following table summarizes the commutation relations (contra-variant form) between the Poincare generators explicitly in Instant Form Dynamics (IFD) (when interpolation angle, $\delta = 0$),

	P^0	P^1	P^2	$-K^3$	K^1	K^2	J^3	J^2	$-J^1$	P^3
P^0	0	0	0	iP_3	iP^1	iP^2	0	0	0	0
P^1	0	0	0	0	iP_0	0	$-iP^2$	$-iP_3$	0	0
P^2	0	0	0	0	0	iP_0	iP^1	0	$-iP_3$	0
$-K^3$	$-iP_3$	0	0	0	iJ^2	$-iJ^1$	0	iK^1	iK^2	iP_0
K^1	$-iP^1$	$-iP_0$	0	$-iJ^2$	0	$-iJ^3$	$-iK^2$	iK^3	0	0
K^2	$-iP^2$	0	$-iP_0$	iJ^1	iJ^3	0	iK^1	0	iK^3	0
J^3	0	iP^2	$-iP^1$	0	iK^2	$-iK^1$	0	$-iJ^1$	$-iJ^2$	0
J^2	0	iP_3	0	$-iK^1$	$-iK^3$	0	iJ^1	0	iJ^3	iP^1
$-J^1$	0	0	$+iP_3$	$-iK^2$	0	$-iK^3$	iJ^2	$-iJ^3$	0	iP^2
P^3	0	0	0	$-iP_0$	0	0	0	$-iP^1$	$-iP^2$	0

LFD

The following table summarizes the commutation relations (contra-variant form) between the Poincare generators explicitly in Light-Front Dynamics (LFD) (when interpolation angle, $\delta = \frac{\pi}{4}$),

	P^+	P^1	P^2	K^3	E^1	E^2	J^3	F^1	F^2	P^-
P^+	0	0	0	iP_-	0	0	0	iP^1	iP^2	0
P^1	0	0	0	0	iP_-	0	$-iP^2$	iP_+	0	
P^2	0	0	0	0	0	iP_-	iP^1	0	iP_+	0
K^3	$-iP_-$	0	0	0	$-iE^1$	$-iE^2$	0	iF^1	iF^2	iP_+
E^1	0	$-iP_-$	0	iE^1	0	0	$-iE^2$	$-iK^3$	$-iJ^3$	$-iP^1$
E^2	0	0	$-iP_-$	iE^2	0	0	iE^1	iJ^3	$-iK^3$	$-iP^2$
J^3	0	iP^2	$-iP^1$	0	iE^2	$-iE^1$	0	iF^2	$-iF^1$	0
F^1	$-iP^1$	$-iP_+$	0	$-iF^1$	iK^3	$-iJ^3$	$-iF^2$	0	0	0
F^2	$-iP^2$	0	$-iP_+$	$-iF^2$	iJ^3	iK^3	iF^1	0	0	0
P^-	0	0	0	$-iP_+$	iP^1	iP^2	0	0	0	0

Kinematic and dynamic generators for different interpolation angles (Phys. Rev. **D 64**, 085013 (2001); Phys. Rev. **D 91**, 065020 (2015))

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P_-$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P_+$

- Among the ten Poincaré generators, the six generators $(\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P_1, P_2, P_{\hat{-}})$ are always kinematic in the sense that the $x^{\hat{+}} = 0$ plane is intact under the transformations generated by them. The operator $K^3 = M_{\hat{+}\hat{-}}$ is dynamical in the region where $0 \leq \delta < \pi/4$ but becomes kinematic in the light-front limit ($\delta = \pi/4$).
- To understand this, note that $[P^{\hat{+}}, K^{\hat{3}}] = i(\mathbb{S}P^{\hat{+}} - \mathbb{C}P^{\hat{-}}) \to iP^{\hat{+}}$ as $\delta \to \pi/4$. Similarly we have $[x^{\hat{+}}, L^{\hat{-}\hat{+}}] = i(\mathbb{S}x^{\hat{+}} - \mathbb{C}x^{\hat{-}}) \to ix^{\hat{+}}$ as $\delta \to \pi/4$. Therefore the instant defined by $x^+ = 0$ becomes invariant under longitudinal boosts as we move to the light front.

Conformal Transformations

The Conformal transformation $x \longmapsto x'$ can be defined by,

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta} = F(x) g_{\mu\nu} \tag{31}$$

Consider an infinitesimal translation,

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) \ . \tag{32}$$

The metric changes by,

$$\delta g_{\mu\nu} = \frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}} = \partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) \tag{33}$$

Conformality then requires,

$$\boxed{\partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) = F(x)\delta_{\mu\nu}} \quad \text{Conformal Killing Equation} \tag{34}$$

contraction with $\delta^{\mu\nu}$ yields

$$2\ \partial^{\mu}\epsilon_{\mu} = F(x)\ d \tag{35}$$

$$\implies \quad F(x) = \frac{2}{d}\partial_{\mu}\epsilon^{\mu} \tag{36}$$

Conformal Transformations

For $d \geq 3$, there are ONLY 4 calsses of solutions for $\epsilon_\mu(x)$

$$\text{(Infinitesimal Translation)} \quad \epsilon^\mu(x) = a^\mu \quad \text{(constant)} \tag{37}$$

$$\text{(Infinitesimal Rotation)} \quad \epsilon^\mu(x) = L^\mu_\nu x^\nu \tag{38}$$

$$\text{(Infinitesimal Scaling)} \quad \epsilon^\mu(x) = \lambda x^\mu \tag{39}$$

$$\text{(Infinitesimal SCT)} \quad \epsilon^\mu(x) = 2(b.x)x^\mu - x^2 b^\mu \tag{40}$$

The generators of conformal transformations are:

$$\begin{aligned}
\text{(translation)} \quad & P^\mu = -i\partial^\mu \ , \\
\text{(dilation)} \quad & D = -ix_\mu\partial^\mu \ , \\
\text{(rotation)} \quad & L^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) \ , \\
\text{(SCT)} \quad & \mathfrak{K}^\mu = -i\left(2x^\mu x_\nu\partial^\nu - x^2\partial^\mu\right) \ .
\end{aligned}$$

Conformal algebra

the full Conformal algebra is given by

$$
\begin{aligned}
&[P^\mu, P^\nu] = 0,\\
&[\mathfrak{K}^\mu, \mathfrak{K}^\nu] = 0,\\
&[D, P^\mu] = iP^\mu,\\
&[D, \mathfrak{K}^\mu] = -i\mathfrak{K}^\mu,\\
&\left[P^\rho, L^{\mu\hat{\nu}}\right] = i\left(g^{\rho\mu}P^\nu - g^{\rho\nu}P^\mu\right),\\
&[\mathfrak{K}^\rho, L^{\mu\nu}] = i\left(g^{\rho\mu}\mathfrak{K}^\nu - g^{\rho\nu}\mathfrak{K}^\mu\right),\\
&\left[L^{\alpha\beta}, L^{\rho\sigma}\right] = -i\left(g^{\beta\sigma}L^{\alpha\rho} - g^{\beta\rho}L^{\alpha\sigma} + g^{\alpha\rho}L^{\beta\sigma} - g^{\alpha\sigma}L^{\beta\rho}\right),\\
&[\mathfrak{K}^\mu, P^\nu] = 2i\left(g^{\mu\nu}D - L^{\mu\nu}\right),\\
&[D, L^{\mu\nu}] = 0.
\end{aligned}
$$

Conclusion & Future Scope

We presented the Poincaré algebra in Interpolation form. We showed the Boost K^3 is dynamical in the region where $0 \leq \delta < \frac{\pi}{4}$ but becomes kinematic in the light-front limit $(\delta = \frac{\pi}{4})$.
Then, we formally developed the Conformal algebra and showed that the set of conformal transformations manifestly forms a group, and it has the Poincaré group as a subgroup. Our future work is to extend the Interpolation method to Conformal algebra.

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