Lattice QCD calculations of Sivers TMD

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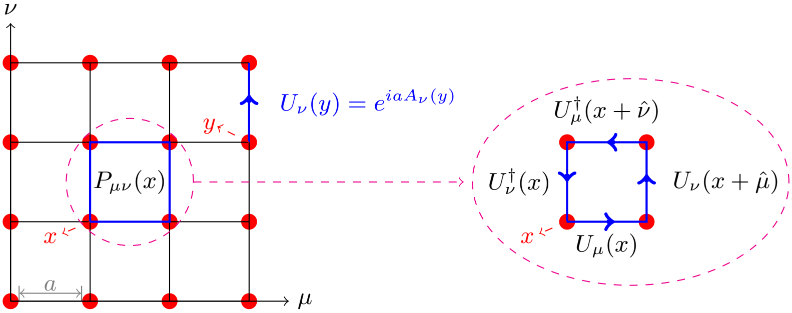
Lattice QCD[1]

$\Lambda_4 = \{n_\mu = (n_1, n_2, n_3, n_4) | n_i \in a[0, 1, \cdots, L_i - 1]\}$

$$\boxed{S_{gauge} = \frac{2}{g^2} \sum_{x \in \Lambda_4} \sum_{\mu < \nu} (1 - Re\ Tr\ [P_{\mu\nu}(x)])} \tag{1}$$



where the elementary plaquette, $\boxed{P_{\mu\nu}(x) = U_\mu(x) U_\nu(x + \hat{\mu}) U^\dagger_\mu(x + \hat{\nu}) U^\dagger_\nu(x)}$.

[1] K. G. Wilson, Confinement of Quarks, Phys. Rev. D10 (1974) 2445.

Lattice QCD

$$\boxed{S_{gauge} = \frac{2}{g^2}\sum_{x\in\Lambda_4}\sum_{\mu<\nu}(1 - Re\ Tr\ [P_{\mu\nu}(x)]) = \frac{a^4}{2g^2}\sum_{x\in\Lambda_4}\sum_{\mu<\nu} tr\ [F_{\mu\nu}(x)^2] + \mathcal{O}(a^2)}.$$

Physical observables $\mathcal{O}$ are evaluated as an expectation value over the relevant degrees of freedom

$$\boxed{\langle\mathcal{O}\rangle = \frac{\int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\ \mathcal{O}\ e^{-[S_{gauge}+\int dx\bar{\psi}\not{D}\psi]}}{\int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]e^{-[S_{gauge}+\int dx\bar{\psi}\not{D}\psi]}}}. \tag{2}$$

The quark fields ψ & $\bar{\psi}$ are Grassmann variables:

$$\boxed{\langle\mathcal{O}\rangle = \frac{\int \mathcal{D}[U]\ \widetilde{\mathcal{O}}\ det\not{D}^2 e^{-[S_{gauge}]}}{\int \mathcal{D}[U]det\not{D}^2 e^{-[S_{gauge}]}}}. \tag{3}$$

This integration results in the “contraction” of fermion–anti-fermion pairs in all possible ways (Wick’s theorem), replacing them with quark propagators $\not{D}^{-1}$.

Definition of TMDs

The starting point for our discussion of TMDs is the correlator of the general form:

$$\Phi^{[\Gamma]}(x, \boldsymbol{k}_\mathrm{T}; P, S; \ldots) \equiv \int dk^- \left(\int \frac{d^4 b}{(2\pi)^4}\, e^{ik\cdot b}\, \frac{\frac{1}{2}\,\langle P,S|\ \bar{q}(0)\,\Gamma\ \mathcal{U}[\mathcal{C}_b]\ q(b)\ |P,S\rangle}{\widetilde{\mathcal{S}}(-\boldsymbol{b}_\mathrm{T}^2;\ldots)} \right)$$

$$= \int \frac{d^2 \boldsymbol{b}_\mathrm{T}}{(2\pi)^2} \int \frac{d(b\cdot P)}{(2\pi)P^+}\ e^{ix(b\cdot P) - i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{k}_\mathrm{T}}\ \left.\frac{\frac{1}{2}\,\langle P,S|\ \bar{q}(0)\,\Gamma\ \mathcal{U}[\mathcal{C}_b]\ q(b)\ |P,S\rangle}{\widetilde{\mathcal{S}}(-\boldsymbol{b}_\mathrm{T}^2;\ldots)}\right|_{b^+=0} . \quad (4)$$

The above correlator can be decomposed into TMDs

$$\Phi^{[\gamma^+]}(x, \boldsymbol{k}_\mathrm{T}; P, S, \ldots) = f_1 - \left[\frac{\epsilon_{ij}\, \boldsymbol{k}_i\, \boldsymbol{S}_j}{m_N}\ f_{1T}^{\perp}\right]_{\mathrm{odd}} \quad (5)$$

Parametrization in position space

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S| \;\bar{q}(0)\, \Gamma\, \mathcal{U}[0, \eta v, \eta v + b, b]\; q(b)\; |P, S\rangle \tag{6}$$

For the Γ-structures at the leading twist, the correlator can be written in the form[2]

$$\frac{1}{2P^+} \widetilde{\Phi}^{[\gamma^+]}_{\text{unsubtr.}} = \widetilde{A}_{2B} + i m_N \epsilon_{ij} \boldsymbol{b}_i \boldsymbol{S}_j\, \widetilde{A}_{12B} \tag{7}$$

(Decompositions analogous to work by A. Metz et al.[3] in momentum space)

[2] Musch, B. U., Hägler, P., Engelhardt, M., Negele, J. W., & Schäfer, A.Phys. Rev. **D 85**(2012), 094510
[3] Phys. Lett. **B618** (2005) 90-96.

Link geometry

Staple-shaped gauge connection:

$$\mathcal{U}[\mathcal{C}_b^{(\eta v)}] = \mathcal{U}[0, \eta v, \eta v + b, b], \tag{8}$$

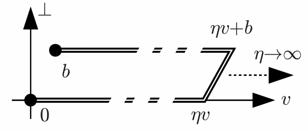


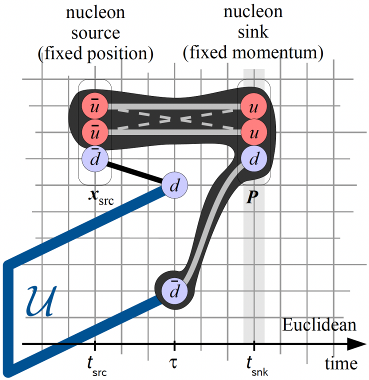
Figure 1: The four vectors v and P give the direction of the staple and the momentum, while b defines the separation between the quark operators. *(arXiv:1111.4249v2 [hep-lat])*

The Lorentz-invariant quantity characterizing the direction of v is the Collins-Soper type parameter

$$\boxed{\hat{\zeta} \equiv \zeta/2m_N = \frac{v \cdot P}{\sqrt{|v^2|}\sqrt{P^2}}.} \tag{9}$$

The light-like direction $v = n$ can be approached in the limit $\zeta \to \infty$.

Lattice Setup [4]



- Evaluate directly $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S| \;\bar{q}(0)\,\Gamma\, \mathcal{U}[0, \eta v, \eta v + b, b]\; q(b)\; |P, S\rangle$
- Euclidean time: Place the entire operator at one time slice, i.e., b, ηv purely spatial.
- Extrapolate $\eta \longrightarrow \infty$, $\hat{\zeta} \longrightarrow \infty$ numerically.

[4]Figure Credits: Dr. Engelhardt (NMSU)

TMDs in Fourier space and invariant amplitudes

$\widetilde{A}_i(b^2, b\cdot P, (b\cdot P)R(\hat{\zeta}^2)/m_N^2, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)$

- x-integrated TMDs in Fourier space $\Longleftrightarrow$ $\widetilde{A}_{iB}$ evaluated at $\boxed{\mathrm{b}\cdot P = 0}$

$$\tilde{f}_1^{[1](0)}(\boldsymbol{b}_\mathrm{T}^2;\hat{\zeta},\ldots,\eta v\cdot P) = 2\,\widetilde{A}_{2B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)/\widetilde{\mathcal{S}}(b^2;\ldots)\,,$$

$$\tilde{f}_{1T}^{\perp[1](1)}(\boldsymbol{b}_\mathrm{T}^2;\hat{\zeta},\ldots,\eta v\cdot P) = -2\,\widetilde{A}_{12B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)/\widetilde{\mathcal{S}}(b^2;\ldots)\,,$$

Generalized Sivers shifts from amplitudes

All other renormalization and soft factor-related dependencies cancel out in the ratio.

- $\langle \boldsymbol{k}_y \rangle^{\text{Sivers}} = \langle \boldsymbol{k}_y \rangle_{TU}$ is T-odd, it describes a feature of the transverse momentum distribution of (unpolarized) quarks in a transversely polarized proton.

$$\begin{aligned} \langle \boldsymbol{k}_y \rangle_{TU}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \eta v\cdot P) \;&\equiv\; m_N \frac{\tilde{f}_{1T}^{\perp[1](1)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)}{\tilde{f}_{1}^{[1](0)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)} \\ &= -m_N \frac{\widetilde{A}_{12B}(-\boldsymbol{b}_{\mathrm{T}}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)}{\widetilde{A}_{2B}(-\boldsymbol{b}_{\mathrm{T}}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)} \\ &\xrightarrow{\boldsymbol{b}_{\mathrm{T}}^2=0} \left. \frac{\int dx \int d^2\boldsymbol{k}_{\mathrm{T}}\, \boldsymbol{k}_y \;\; \Phi^{[\gamma^+]}(x, \boldsymbol{k}_{\mathrm{T}}, P, S; \ldots)}{\int dx \int d^2\boldsymbol{k}_{\mathrm{T}} \;\; \Phi^{[\gamma^+]}(x, \boldsymbol{k}_{\mathrm{T}}, P, S; \ldots)} \right|_{\boldsymbol{S}_{\mathrm{T}} = (1,0)} \end{aligned} \tag{10}$$

Numerical Results

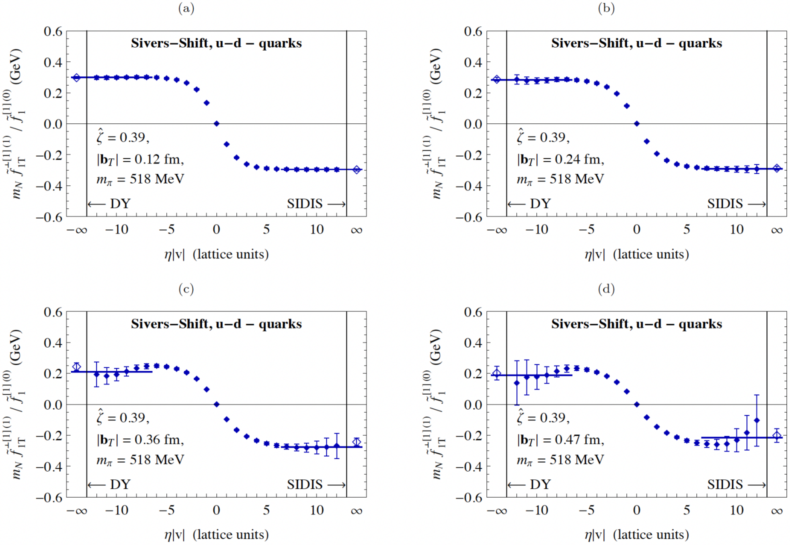


Figure 2: Extraction of the generalized Sivers shift on the lattice with $m_\pi = 518$MeV *(arXiv:1111.4249v2 [hep-lat])*

Numerical Results

Results: Sivers shift

Dependence of SIDIS limit on $|b_T|$

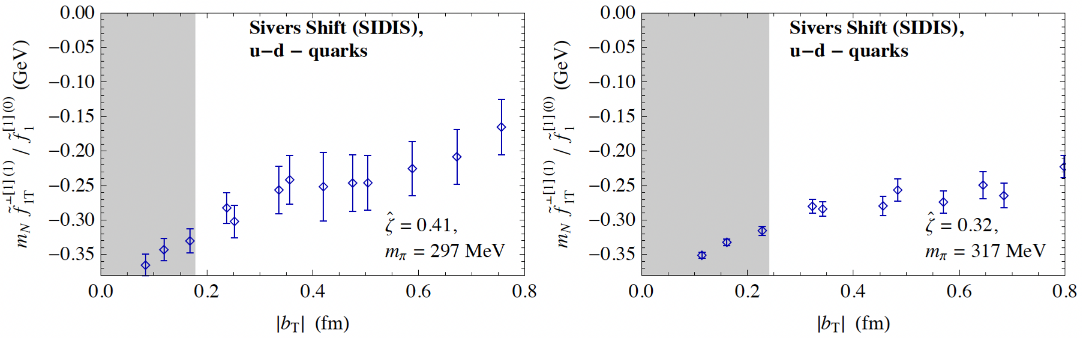


Figure 3: Generalized Sivers shift as a function of the quark separation $|\boldsymbol{b}_T|$ for the SIDIS case ($|\eta v| = \infty$).
arXiv:2301.06118 [hep-lat]

Numerical Results

Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$

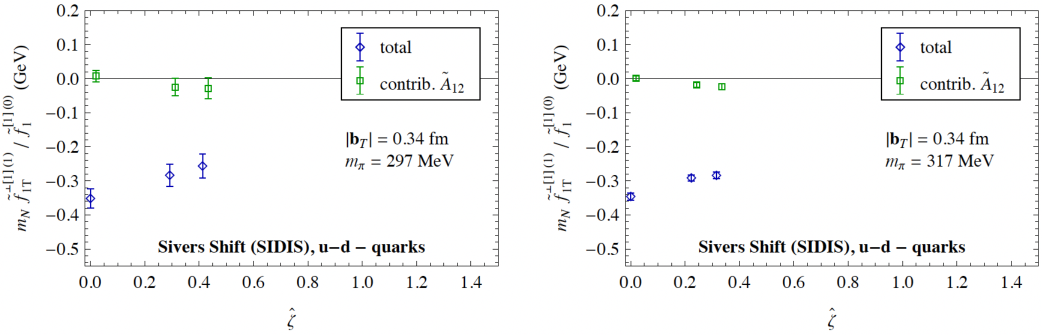


Figure 4: we show the $\hat{\zeta}$-dependence of the generalized Sivers shift, depicting both the full result and the result obtained with just $\tilde{A}_{12}$ in the numerator. *arXiv:2301.06118 [hep-lat]*

Few More Numerical Results

- M. Engelhardt, *et al.*, PoS **LATTICE2022**, 103 (2023), [arXiv:2301.06118 [hep-lat]].
- B. Yoon, M. Engelhardt, R. Gupta, T. Bhattacharya, J. R. Green, B. U. Musch, J. W. Negele, A. V. Pochinsky, A. Schäfer and S. N. Syritsyn, Phys. Rev. D **96**, no.9, 094508 (2017), [arXiv:1706.03406 [hep-lat]].
- M. Engelhardt, B. Musch, T. Bhattacharya, J. R. Green, R. Gupta, P. Hägler, S. Krieg, J. Negele, A. Pochinsky and A. Schäfer, *et al.*, EPJ Web Conf. **112**, 01008 (2016)
- M. Engelhardt, B. Musch, T. Bhattacharya, J. R. Green, R. Gupta, P. Haegler, J. Negele, A. Pochinsky, A. Schafer and S. Syritsyn, *et al.*, PoS **QCDEV2015**, 018 (2015)
- M. Engelhardt, B. Musch, T. Bhattacharya, R. Gupta, P. Hägler, S. Krieg, J. Negele, A. Pochinsky, S. Syritsyn and B. Yoon, PoS **LATTICE2015**, 117 (2016)

Extension to include the dependence on $x = \frac{k^+}{P^+}$

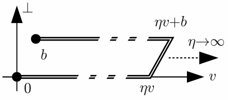
$$\frac{1}{2} \langle P,S| \; \bar{q}(0) \; \gamma^+ \; \mathcal{U}[\mathcal{C}_b] \; q(b) \; |P,S\rangle = 2P^+ \left(\widetilde{A}_{2B} + i m_N \epsilon_{ij} \boldsymbol{b}_i \boldsymbol{S}_j \; \widetilde{A}_{12B} \right) \tag{11}$$

$$\Longrightarrow \; \langle \boldsymbol{k}_y \rangle_{TU}(\boldsymbol{b}_{\mathrm{T}}^2, x, \hat{\zeta}, \eta v\cdot P) \; \equiv \; m_N \frac{\tilde{f}_{1T}^{\perp(1)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)}{\tilde{f}_1^{(0)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)} \tag{12}$$

$$= -m_N \frac{\int d(b\cdot P) e^{ix(b\cdot P)} \widetilde{A}_{12B}(b^2, b\cdot P, (b\cdot P) R(\hat{\zeta}^2)/m_N^2, -1/(m_N \hat{\zeta})^2, \eta v\cdot P)}{\int d(b\cdot P) e^{ix(b\cdot P)} \widetilde{A}_{2B}(b^2, b\cdot P, (b\cdot P) R(\hat{\zeta}^2)/m_N^2, -1/(m_N \hat{\zeta})^2, \eta v\cdot P)} \tag{13}$$

- The range of accessible $b \cdot P$ is limited:

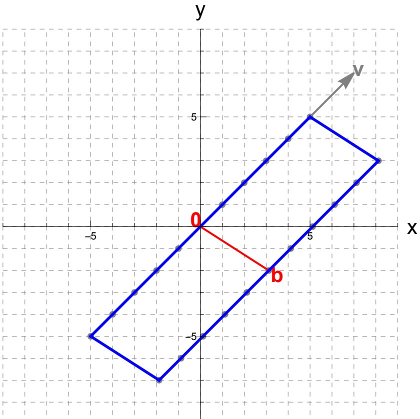
$$\boxed{\frac{v\cdot b}{v\cdot P} = b\cdot P \frac{1 - \sqrt{1 + \hat{\zeta}^{-2}}}{m_N^2}}$$



Extension to include the dependence on $x = \frac{k^+}{P^+}$

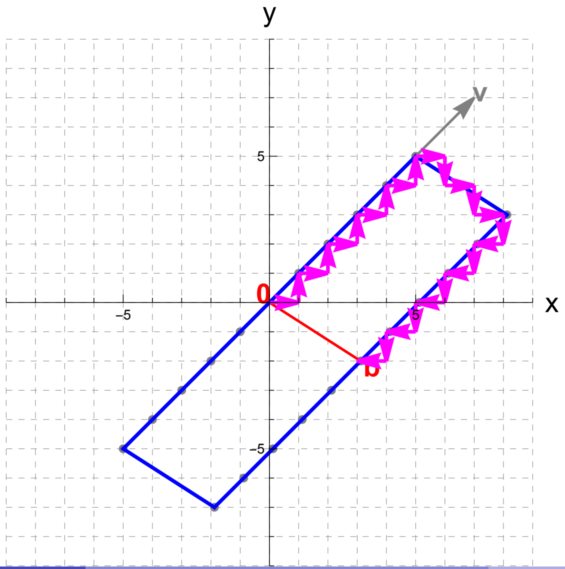
Example Parameters of Lattice QCD calculations for $\boxed{\frac{v\cdot b}{v\cdot P} = b\cdot P \frac{1-\sqrt{1+\hat{\zeta}^{-2}}}{m_N^2}}$

$\boldsymbol{b}/a$	$\eta v/a$	$P \cdot aL/(2\pi)$
$(3.12, -2, 0)$	$\pm n' \cdot (1,1,0)$	$(-1,0,0)$



Extension to include the dependence on $x = \frac{k^+}{P^+}$

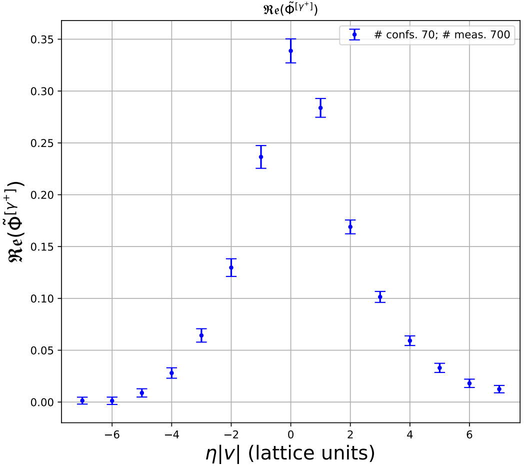
On the lattice:

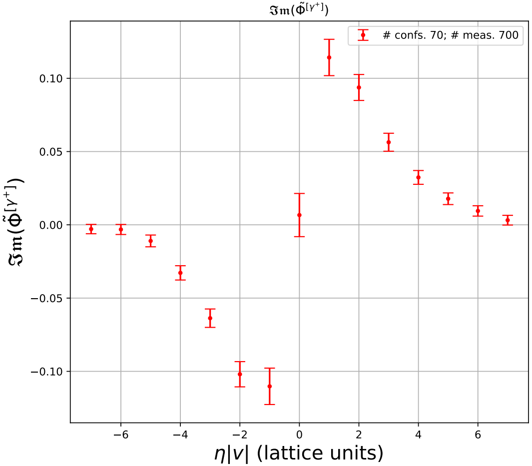


Extension to include the dependence on $x = \frac{k^+}{P^+}$

Lattice calculation from off-axis staple link: (low statistics)

$b^\mu = (0, 3.12, -2, 0)a; \quad v^\mu = (0, 1, 1, 0)a; \quad \hat{\zeta} = 0.225; \quad b \cdot P = 0.613 \neq 0; \quad P^1 = = -339.3$ MeV





Extension to include the dependence on $x = \frac{k^+}{P^+}$

Extract b_L-even component of imaginary part of γ^+ correlator

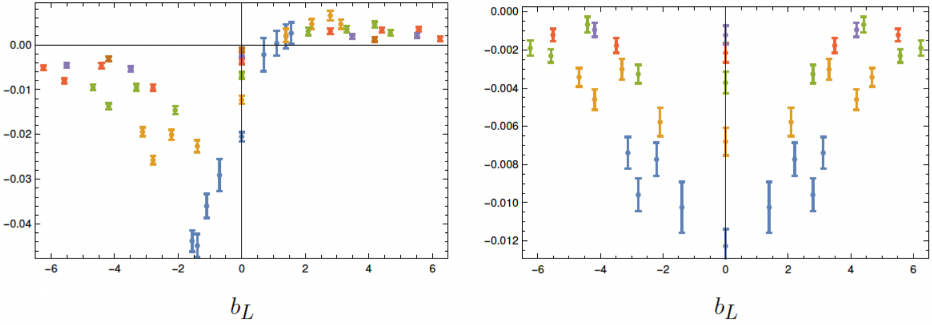


Figure 5: [5] $\left[\frac{1}{2}\langle P,S|\ \bar{q}(0)\ \gamma^+\ \mathcal{U}[\mathcal{C}_b]\ q(b)\ |P,S\rangle = 2P^+\left(\widetilde{A}_{2B} + im_N\epsilon_{ij}\boldsymbol{b}_i\boldsymbol{S}_j\ \widetilde{A}_{12B}\right)\right]$

[5] PDFLattice 2019: M. Engelhardt

Extension to include the dependence on $x = \frac{k^+}{P^+}$

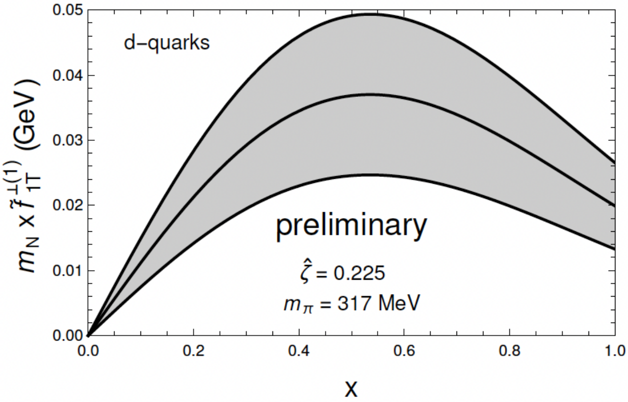


Figure 6: Nucleon SIDIS d-quark generalized Sivers shift as a function of momentum fraction x, multiplied by x [6,7]

[6] "TMD Handbook." arXiv:2304.03302 [hep-ph].

[7] M. Engelhardt, J. R. Green, S. Krieg, S. Meinel, J. Negele, A. Pochinsky et al., to be published

Conclusions

- It is feasible to obtain the x-dependence of TMD ratios: **Sivers shift**
- In spite of constraints $\frac{v \cdot b}{v \cdot P} = b \cdot P \frac{R(\hat{\zeta}^2)}{m_N^2}$, it is possible to improve the analysis

Backup

TMDs in Fourier space and x-integrations (Mellin moments)

$$\tilde{f}(x, \boldsymbol{b}_{\mathrm{T}}^2; \ldots) \equiv \int d^2\boldsymbol{k}_{\mathrm{T}}\, e^{i\boldsymbol{b}_{\mathrm{T}}\cdot\boldsymbol{k}_{\mathrm{T}}}\; f(x, \boldsymbol{k}_{\mathrm{T}}^2; \ldots) \tag{14}$$

$$\tilde{f}^{(n)}(x, \boldsymbol{b}_{\mathrm{T}}^2 \ldots) \equiv n! \left(-\frac{2}{m_N^2}\partial_{\boldsymbol{b}_{\mathrm{T}}^2}\right)^n \; \tilde{f}(x, \boldsymbol{b}_{\mathrm{T}}^2; \ldots) \tag{15}$$

In the limit $|\boldsymbol{b}_{\mathrm{T}}| \to 0$, one recovers conventional $\boldsymbol{k}_{\mathrm{T}}$-moments of TMDs:

$$\tilde{f}^{(n)}(x, 0; \ldots) = \int d^2\boldsymbol{k}_{\mathrm{T}} \left(\frac{\boldsymbol{k}_{\mathrm{T}}^2}{2m_N^2}\right)^n f(x, \boldsymbol{k}_{\mathrm{T}}^2; \ldots) \equiv f^{(n)}(x)\ . \tag{16}$$

In our studies so far, we only considered the first x-moments (accessible at $b \cdot P = 0$) rather than scanning range of $b \cdot P$

$$f^{[1]}(\boldsymbol{k}_{\mathrm{T}}^2; \ldots) \equiv \int_{-1}^{1} dx\; f(x, \boldsymbol{k}_{\mathrm{T}}^2; \ldots)\ . \tag{17}$$

where, $x = \frac{k^+}{P^+}$