More on conformal symmetry

for Dr. Ji's group meeting

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October 28, 2022

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Conformal Transformations

Let us consider a flat space in d dimensions and transformations thereof which locally preserve the angle between any two lines. A conformal transformation is a smooth, invertible map $x \to x'$ such that

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x), \tag{1.1}$$

where the positive function $\Lambda(x)$ is called the scale factor.
Note furthermore, for flat spaces the scale factor $\Lambda(x) = 1$ corresponds to the Poincaré group consisting of translations and rotations, respectively Lorentz transformations.
Let us next consider the infinitesimal coordinate transformations which up to first order in a small parameter $\epsilon(x) << 1$ read

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2). \tag{1.2}$$

Conformal Transformations

For $d \geq 3$, there are ONLY 4 calsses of solutions for $\epsilon_\mu(x)$ of $x'_\mu = x_\mu + \epsilon_\mu(x) + \mathcal{O}(\epsilon^2)$.

$$\text{(Infinitesimal Translation)} \quad \epsilon^\mu(x) = a^\mu \quad \text{(constant)} \tag{1.3}$$

$$\text{(Infinitesimal Rotation)} \quad \epsilon^\mu(x) = L^\mu_\nu x^\nu \tag{1.4}$$

$$\text{(Infinitesimal Scaling)} \quad \epsilon^\mu(x) = \lambda x^\mu \tag{1.5}$$

$$\text{(Infinitesimal SCT)} \quad \epsilon^\mu(x) = 2(b.x)x^\mu - x^2 b^\mu \tag{1.6}$$

The Finite conformal transformations are:

$$\begin{aligned}
\text{(translation)} \quad & x'^\mu = x^\mu + a^\mu \\
\text{(rotation)} \quad & x'^\mu = L^\mu_\nu x^\nu \\
\text{(dilatation)} \quad & x'^\mu = \alpha x^\mu \\
\text{(SCT)} \quad & x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}
\end{aligned}$$

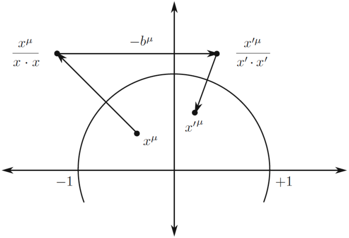
SCT

Inversions is given by

$$\boxed{x^\mu} \longrightarrow \boxed{x'^\mu = \frac{x^\mu}{x^2}} \tag{1.7}$$

The SCTs can be understood as an inversion of x^μ, followed by a translation b^μ, and followed again by an inversion.

$$\boxed{x^\mu} \longrightarrow \boxed{x'^\mu = \frac{x^\mu}{x^2}} \longrightarrow \boxed{x''^\mu = \frac{x^\mu}{x^2} - b^\mu} \longrightarrow \boxed{x'''^\mu = \frac{\frac{x^\mu}{x^2} - b^\mu}{(\frac{x^\mu}{x^2} - b^\mu)^2} = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}}$$



SCT

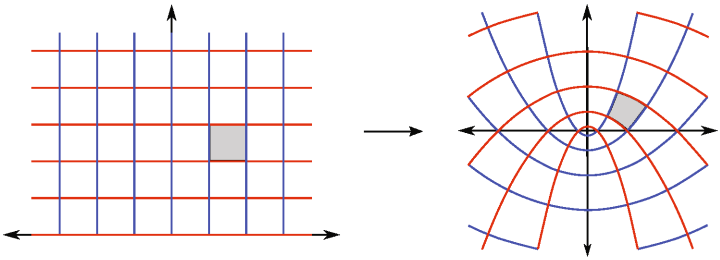


Figure: Conformal transformation in two dimensions

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}\eta_{\mu\nu} = \frac{1}{(1-2(b\cdot x)+b^2x^2)^2}\eta_{\alpha\beta} \tag{1.8}$$

So, $\boxed{\Lambda = \frac{1}{(1-2(b\cdot x)+b^2x^2)^2}}$.

The generators of conformal transformations:

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}} \tag{2.1}$$

$$\text{(dilation)} \quad D = -ix_{\hat{\mu}}\partial^{\hat{\mu}} \tag{2.2}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right) \tag{2.3}$$

$$\text{(SCT)} \quad \mathfrak{K}^{\hat{\mu}} = -i\left(2x^{\hat{\mu}}x_{\hat{\nu}}\partial^{\hat{\nu}} - x^2\partial^{\hat{\mu}}\right) \tag{2.4}$$

Therefore the full Conformal algebra is given by

$$\begin{aligned}
&\left[P^{\hat{\mu}}, P^{\hat{\nu}}\right] = 0, \\
&\left[\mathfrak{K}^{\hat{\mu}}, \mathfrak{K}^{\hat{\nu}}\right] = 0, \\
&\left[D, P^{\hat{\mu}}\right] = iP^{\hat{\mu}}, \\
&\left[D, \mathfrak{K}^{\hat{\mu}}\right] = -i\mathfrak{K}^{\hat{\mu}}, \\
&\left[P^{\hat{\rho}}, L^{\mu\hat{\hat{\nu}}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}P^{\hat{\mu}}\right), \\
&\left[\mathfrak{K}^{\hat{\rho}}, L^{\hat{\mu}\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}\mathfrak{K}^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}\mathfrak{K}^{\hat{\mu}}\right), \\
&\left[L^{\hat{\alpha}\hat{\beta}}, L^{\hat{\rho}\hat{\sigma}}\right] = -i\left(g^{\hat{\beta}\hat{\sigma}}L^{\hat{\alpha}\hat{\rho}} - g^{\hat{\beta}\hat{\rho}}L^{\hat{\alpha}\hat{\sigma}} + g^{\hat{\alpha}\hat{\rho}}L^{\hat{\beta}\hat{\sigma}} - g^{\hat{\alpha}\hat{\sigma}}L^{\hat{\beta}\hat{\rho}}\right), \\
&\left[\mathfrak{K}^{\hat{\mu}}, P^{\hat{\nu}}\right] = 2i\left(g^{\hat{\mu}\hat{\nu}}D - L^{\hat{\mu}\hat{\nu}}\right), \\
&\left[D, L^{\hat{\mu}\hat{\nu}}\right] = 0.
\end{aligned}$$

Full conformal algebra

A comprehensive table of the **105 commutation** relations among the co-variant components of the Conformal generators is presented below:

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$	$\mathfrak{K}_{\hat{+}}$	$\mathfrak{K}_{\hat{1}}$	$\mathfrak{K}_{\hat{2}}$	$\mathfrak{K}_{\hat{-}}$	D
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$-2i(\mathbb{S}D - K^{\hat{3}})$	$-iP_{\hat{+}}$
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0	$2i\mathcal{D}^{\hat{1}}$	$2iD$	$-2iJ^{\hat{3}}$	$2i\mathcal{K}^{\hat{1}}$	$-iP_{\hat{1}}$
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0	$2i\mathcal{D}^{\hat{2}}$	$2iJ^{\hat{3}}$	$2iD$	$2i\mathcal{K}^{\hat{2}}$	$-iP_{\hat{2}}$
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	0	0	$-i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	0
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{+}}$	0	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0	0	$i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{1}}$	0	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{-}}$	0	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{-}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-2i(\mathbb{S}D + K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$2i\mathbb{C}D$	$-iP_{\hat{-}}$
$\mathfrak{K}_{\hat{+}}$	$2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	$2i(\mathbb{S}D + K^{\hat{3}})$	0	0	0	0	$i\mathfrak{K}_{\hat{+}}$
$\mathfrak{K}_{\hat{1}}$	$2i\mathcal{D}^{\hat{1}}$	$-2iD$	$-2iJ^{\hat{3}}$	0	$i\mathfrak{K}_{\hat{+}}$	0	$-i\mathfrak{K}_{\hat{2}}$	$i\mathfrak{K}_{\hat{-}}$	0	$2i\mathcal{K}^{\hat{1}}$	0	0	0	0	$i\mathfrak{K}_{\hat{1}}$
$\mathfrak{K}_{\hat{2}}$	$2i\mathcal{K}^{\hat{2}}$	$2iJ^{\hat{3}}$	$-2iD$	0	0	$i\mathfrak{K}_{\hat{+}}$	$i\mathfrak{K}_{\hat{1}}$	0	$i\mathfrak{K}_{\hat{-}}$	$2i\mathcal{K}^{\hat{2}}$	0	0	0	0	$i\mathfrak{K}_{\hat{2}}$
$\mathfrak{K}_{\hat{-}}$	$2i(\mathbb{S}D - K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	$-2i\mathbb{C}D$	0	0	0	0	$i\mathfrak{K}_{\hat{-}}$
D	$iP_{\hat{+}}$	$iP_{\hat{1}}$	$iP_{\hat{2}}$	0	0	0	0	0	0	$iP_{\hat{-}}$	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{-}}$	0

Kinematic and dynamic generators of the Conformal group[1,2]

The generators of conformal transformations:

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}} \tag{2.5}$$

$$\text{(dilation)} \quad D = -ix_{\hat{\mu}}\partial^{\hat{\mu}} \tag{2.6}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right) \tag{2.7}$$

$$\text{(SCT)} \quad \mathfrak{K}^{\hat{\mu}} = -i\left(2x^{\hat{\mu}}x_{\hat{\nu}}\partial^{\hat{\nu}} - x^2\partial^{\hat{\mu}}\right) \tag{2.8}$$

Since $\left[\mathfrak{K}^{\hat{+}}, x^{\hat{+}}\right] = -i\left(2x^{\hat{+}}x^{\hat{+}} - (x^{\hat{\alpha}}.x_{\hat{\alpha}})\mathbb{C}\right) \to \; -i(x^0.x^0 + \vec{x}.\vec{x})$ as $\delta \to 0$, and $\left[\mathfrak{K}^{\hat{+}}, x^{\hat{+}}\right] = -i\left(2x^{\hat{+}}x^{\hat{+}} - (x^{\hat{\alpha}}.x_{\hat{\alpha}})\mathbb{C}\right) \to \; -i(2x^{+}.x^{+})$ as $\delta \to \pi/4$, the conformal generator (LF time component) $\mathfrak{K}_{-}$ is Kinematic in LFD, but Dynamic in IFD. And $[D, x^{\hat{+}}] = -ix^{\hat{+}}$, so D is always Kinematic in both IFD and LFD.

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2,\ \mathcal{K}^{\hat{2}} = J^1,\ J^3, P^1, P^2, P^3,\ D$	$\mathcal{D}^{\hat{1}} = -K^1,\ \mathcal{D}^{\hat{2}} = -K^2,\ K^3, P^0,\ \mathfrak{K}_0,\ \mathfrak{K}_1,\ \mathfrak{K}_2,\ \mathfrak{K}_3$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}},\ D$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}},\ \mathfrak{K}_{\hat{+}},\ \mathfrak{K}_{\hat{1}},\ \mathfrak{K}_{\hat{2}},\ \mathfrak{K}_{\hat{-}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1,\ \mathcal{K}^{\hat{2}} = -E^2,\ J^3, K^3, P^1, P^2, P_{-},\ D,\ \mathfrak{K}_{-}$	$\mathcal{D}^{\hat{1}} = -F^1,\ \mathcal{D}^{\hat{2}} = -F^2,\ P_{+},\ \mathfrak{K}_{+},\ \mathfrak{K}_1,\ \mathfrak{K}_2$

[1] Chueng-Ryong Ji and Chad Mitchell, Phys. Rev. **D 64**, 085013 (2001).

[2] Chueng-Ryong Ji, Ziyue Li, and Alfredo Takashi Suzuki, Phys. Rev. **D 91**, 065020 (2015).

Boost as Rotations in 4D

The 4D Angular momentum tensor is given by:

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}. \tag{3.1}$$

$$\left[L^{\alpha\beta}, L^{\rho\sigma}\right] = -i\left(g^{\beta\sigma}L^{\alpha\rho} - g^{\beta\rho}L^{\alpha\sigma} + g^{\alpha\rho}L^{\beta\sigma} - g^{\alpha\sigma}L^{\beta\rho}\right) \tag{3.2}$$

There are $\frac{n(n-1)}{2}$ number of planes in n dimension:

Dimension	# of planes
1D	0
2D	1
3D	3
4D	6
5D	10
6D	15

(3.3)

Conformal algebra in simpler form

In order to make the conformal commutation rules into a simpler form, we define the following generators:

$$J_{a,b} = \begin{pmatrix} 0 & D & \frac{-\mathfrak{K}_0}{\sqrt{2}} & \frac{-\mathfrak{K}_1}{\sqrt{2}} & \frac{-\mathfrak{K}_2}{\sqrt{2}} & \frac{-\mathfrak{K}_3}{\sqrt{2}} \\ -D & 0 & \frac{P_0}{\sqrt{2}} & \frac{P_1}{\sqrt{2}} & \frac{P_2}{\sqrt{2}} & \frac{P_3}{\sqrt{2}} \\ \frac{\mathfrak{K}_0}{\sqrt{2}} & \frac{-P_0}{\sqrt{2}} & 0 & -K^1 & -K^2 & -K^3 \\ \frac{\mathfrak{K}_1}{\sqrt{2}} & \frac{-P_1}{\sqrt{2}} & K^1 & 0 & J^3 & -J^2 \\ \frac{\mathfrak{K}_2}{\sqrt{2}} & \frac{-P_2}{\sqrt{2}} & K^2 & -J^3 & 0 & J^1 \\ \frac{\mathfrak{K}_3}{\sqrt{2}} & \frac{-P_3}{\sqrt{2}} & K^3 & J^2 & -J^1 & 0 \end{pmatrix} \tag{4.1}$$

where $J_{a,b} = -J_{b,a}$ and $a, b \in \{-2, -1, 0, 1, 2, 3\}$. These new generators obey the $SO(4+1, 1)$ commutation relations:

$$[J_{ab}, J_{cd}] = -i\left(g_{bd}J_{ac} - g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc}\right) \tag{4.2}$$

where,

$$g_{a,b} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.3}$$

The algebra Eq.(7.4) with the above $g_{a,b}$ is equivalent to the conformal algebra.

Interpolating conformal algebra in simpler form

For interpolating this algebra Eq.(7.4) between IFD and LFD, let us define a interpolation (transformation) matrix (6×6) which is given by,

$$(\mathcal{R}^{b}_{\hat{a}})_{6\times 6} = (\mathcal{R}^{b}_{\hat{a}})^{T}_{6\times 6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\delta & 0 & 0 & \sin\delta \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \tag{4.4}$$

Then in interpolation form $J_{\hat{a},\hat{b}}$ becomes $J_{\hat{a},\hat{b}} = \mathcal{R}^{c}_{\hat{a}} J_{c,d} \mathcal{R}^{d}_{\hat{b}}$, that is

$$J_{\hat{a},\hat{b}} = \begin{pmatrix} 0 & D & -\frac{\mathfrak{K}_{\hat{+}}}{\sqrt{2}} & -\frac{\mathfrak{K}_{1}}{\sqrt{2}} & -\frac{\mathfrak{K}_{2}}{\sqrt{2}} & -\frac{\mathfrak{K}_{\hat{-}}}{\sqrt{2}} \\ -D & 0 & \frac{P_{\hat{+}}}{\sqrt{2}} & \frac{P_{1}}{\sqrt{2}} & \frac{P_{2}}{\sqrt{2}} & \frac{P_{\hat{-}}}{\sqrt{2}} \\ \frac{\mathfrak{K}_{\hat{+}}}{\sqrt{2}} & -\frac{P_{\hat{+}}}{\sqrt{2}} & 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^{3} \\ \frac{\mathfrak{K}_{1}}{\sqrt{2}} & -\frac{P_{1}}{\sqrt{2}} & -\mathcal{D}^{\hat{1}} & 0 & J^{3} & -\mathcal{K}^{\hat{1}} \\ \frac{\mathfrak{K}_{2}}{\sqrt{2}} & -\frac{P_{2}}{\sqrt{2}} & -\mathcal{D}^{\hat{2}} & -J^{3} & 0 & -\mathcal{K}^{\hat{2}} \\ \frac{\mathfrak{K}_{\hat{-}}}{\sqrt{2}} & -\frac{P_{\hat{-}}}{\sqrt{2}} & -K^{3} & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix} \tag{4.5}$$

Interpolating conformal algebra in simpler form

Then the simplified conformal algebra in interpolation is:

$$\left[J_{\hat{a}\hat{b}}, J_{\hat{c}\hat{d}}\right] = -i\left(g_{\hat{b}\hat{d}}J_{\hat{a}\hat{c}} - g_{\hat{b}\hat{c}}J_{\hat{a}\hat{d}} + g_{\hat{a}\hat{c}}J_{\hat{b}\hat{d}} - g_{\hat{a}\hat{d}}J_{\hat{b}\hat{c}}\right) \tag{4.6}$$

where,

$$g_{\hat{a},\hat{b}} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix} \tag{4.7}$$

The algebra Eq.(4.6) with the above interpolating 6×6 metric will reproduce the explicit commutation relations in interpolation between IFD and LFD mentioned in previous Table. With the IFD limit $\mathbb{C} \longrightarrow 1$ & $\mathbb{S} \longrightarrow 0$ and LFD limit $\mathbb{C} \longrightarrow 0$ & $\mathbb{S} \longrightarrow 1$, the algebra Eq.(4.6) will reproduce the explicit commutation relations.

Isomorphism between conformal algebra and Dirac matrices

The defining property for the gamma matrices to generate a Clifford algebra is the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I_4, \tag{5.1}$$

Covariant gamma matrices are defined by

$$\gamma_\mu = \eta_{\mu\nu}\gamma^\nu = \left\{\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3\right\}. \tag{5.2}$$

Further basis $\sigma_{\mu\nu}$ elements of the Clifford algebra are given by

$$\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu] = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{5.3}$$

$$\sigma_{\mu\nu} = \frac{1}{4}\left(\gamma_\mu\gamma_\nu - 2g_{\mu\nu} + \gamma_\mu\gamma_\nu\right) \tag{5.4}$$

$$\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - g_{\mu\nu}) \tag{5.5}$$

where it satisfy,

$$\sigma_{\mu\nu} = -\,\sigma_{\nu\mu}; \tag{5.6}$$

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = (-g_{\nu\sigma}\sigma_{\mu\rho} + g_{\nu\rho}\sigma_{\mu\sigma} - g_{\mu\rho}\sigma_{\nu\sigma} + g_{\mu\sigma}\sigma_{\nu\rho}). \tag{5.7}$$

Let's go to $d = 6$; we have $\sigma_{a,b}$ where $a, b \in \{-2, -1, 0, 1, 2, 3\}$.
Then $\gamma_{\tilde{a}} = (\gamma_5, \gamma_0, \gamma_1, \gamma_2, \gamma_3)$. Where, $\tilde{a}, \tilde{b} \in \{-1, 0, 1, 2, 3\}$.
Let's define

$$J_{\tilde{a},\tilde{b}} \equiv i\sigma_{\tilde{a},\tilde{b}};\ J_{-2,\tilde{a}} = -J_{\tilde{a},-2} = \frac{i}{2}\gamma_{\tilde{a}} \tag{5.8}$$

then $[\sigma_{ab}, \sigma_{cd}] = (-g_{bd}\sigma_{ac} + g_{bc}\sigma_{ad} - g_{ac}\sigma_{bd} + g_{ad}\sigma_{bc})$ will be (since, $\sigma_{a,b} = -iJ_{a,b}$)

$$(-1)\,[J_{ab}, J_{cd}] =(-i)\,(-g_{bd}J_{ac} + g_{bc}J_{ad} - g_{ac}J_{bd} + g_{ad}J_{bc}) \tag{5.9}$$

$$[J_{ab}, J_{cd}] = -\,i\,(g_{bd}J_{ac} - g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc}) \tag{5.10}$$

$$J_{a,b} = \frac{i}{2}\begin{pmatrix} 0 & \gamma_5 & \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_5 & 0 & \gamma_5\gamma_0 & \gamma_5\gamma_1 & \gamma_5\gamma_2 & \gamma_5\gamma_3 \\ -\gamma_0 & \gamma_0\gamma_5 & 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3 \\ -\gamma_1 & \gamma_1\gamma_5 & \gamma_1\gamma_0 & 0 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ -\gamma_2 & \gamma_2\gamma_5 & \gamma_2\gamma_0 & \gamma_2\gamma_1 & 0 & \gamma_2\gamma_3 \\ -\gamma_3 & \gamma_3\gamma_5 & \gamma_3\gamma_0 & \gamma_3\gamma_1 & \gamma_3\gamma_2 & 0 \end{pmatrix}_{6\times 6} \tag{5.11}$$

Isomorphism

$$J_{a,b} = \frac{i}{2}\begin{pmatrix} 0 & \gamma_5 & \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_5 & 0 & \gamma_5\gamma_0 & \gamma_5\gamma_1 & \gamma_5\gamma_2 & \gamma_5\gamma_3 \\ -\gamma_0 & \gamma_0\gamma_5 & 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3 \\ -\gamma_1 & \gamma_1\gamma_5 & \gamma_1\gamma_0 & 0 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ -\gamma_2 & \gamma_2\gamma_5 & \gamma_2\gamma_0 & \gamma_2\gamma_1 & 0 & \gamma_2\gamma_3 \\ -\gamma_3 & \gamma_3\gamma_5 & \gamma_3\gamma_0 & \gamma_3\gamma_1 & \gamma_3\gamma_2 & 0 \end{pmatrix}_{6\times 6} \tag{5.12}$$

is isomorphism to $SO(4+1,1)$ group

$$J_{a,b} = \begin{pmatrix} 0 & D & \frac{1}{2}(P_0 - \mathfrak{K}_0) & \frac{1}{2}(P_1 - \mathfrak{K}_1) & \frac{1}{2}(P_2 - \mathfrak{K}_2) & \frac{1}{2}(P_3 - \mathfrak{K}_3) \\ -D & 0 & \frac{1}{2}(P_0 + \mathfrak{K}_0) & \frac{1}{2}(P_1 + \mathfrak{K}_1) & \frac{1}{2}(P_2 + \mathfrak{K}_2) & \frac{1}{2}(P_3 + \mathfrak{K}_3) \\ -\frac{1}{2}(P_0 - \mathfrak{K}_0) & -\frac{1}{2}(P_0 + \mathfrak{K}_0) & 0 & -K^1 & -K^2 & -K^3 \\ -\frac{1}{2}(P_1 - \mathfrak{K}_1) & -\frac{1}{2}(P_1 + \mathfrak{K}_1) & K^1 & 0 & J^3 & -J^2 \\ -\frac{1}{2}(P_2 - \mathfrak{K}_2) & -\frac{1}{2}(P_2 + \mathfrak{K}_2) & K^2 & -J^3 & 0 & J^1 \\ -\frac{1}{2}(P_3 - \mathfrak{K}_3) & -\frac{1}{2}(P_3 + \mathfrak{K}_3) & K^3 & J^2 & -J^1 & 0 \end{pmatrix}_{6\times 6} \tag{5.13}$$

After the 6 dimensional rotation,

$$J'_{a,b} = \begin{pmatrix} 0 & D & \frac{-\mathfrak{K}_0}{\sqrt{2}} & \frac{-\mathfrak{K}_1}{\sqrt{2}} & \frac{-\mathfrak{K}_2}{\sqrt{2}} & \frac{-\mathfrak{K}_3}{\sqrt{2}} \\ -D & 0 & \frac{P_0}{\sqrt{2}} & \frac{P_1}{\sqrt{2}} & \frac{P_2}{\sqrt{2}} & \frac{P_3}{\sqrt{2}} \\ \frac{\mathfrak{K}_0}{\sqrt{2}} & \frac{-P_0}{\sqrt{2}} & 0 & -K^1 & -K^2 & -K^3 \\ \frac{\mathfrak{K}_1}{\sqrt{2}} & \frac{-P_1}{\sqrt{2}} & K^1 & 0 & J^3 & -J^2 \\ \frac{\mathfrak{K}_2}{\sqrt{2}} & \frac{-P_2}{\sqrt{2}} & K^2 & -J^3 & 0 & J^1 \\ \frac{\mathfrak{K}_3}{\sqrt{2}} & \frac{-P_3}{\sqrt{2}} & K^3 & J^2 & -J^1 & 0 \end{pmatrix}_{6\times 6} \tag{5.14}$$

$$\boxed{J'_{a,b} = \frac{i}{2} \begin{pmatrix} 0 & \gamma_5 & \frac{(1-\gamma_5)\gamma_0}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_1}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_2}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_3}{\sqrt{2}} \\ -\gamma_5 & 0 & \frac{(1+\gamma_5)\gamma_0}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_1}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_2}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_3}{\sqrt{2}} \\ \frac{-(1-\gamma_5)\gamma_0}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_0}{\sqrt{2}} & 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_1}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_1}{\sqrt{2}} & \gamma_1\gamma_0 & 0 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_2}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_2}{\sqrt{2}} & \gamma_2\gamma_0 & \gamma_2\gamma_1 & 0 & \gamma_2\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_3}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_3}{\sqrt{2}} & \gamma_3\gamma_0 & \gamma_3\gamma_1 & \gamma_3\gamma_2 & 0 \end{pmatrix}_{6\times 6}} \tag{5.15}$$

So fifteen matrices of the γ's and their products can be grouped as the fifteen components of a skew angular momentum tensor $J_{a,b}$ in six dimensions[3].

[3] The inhomogeneous lorentz group and the conformal group, W. A. Hepner

4×4 representation

The representation of conformal group in terms of 4×4 gamma matrices are the following;

$$P_\mu = \frac{i}{2}(1+\gamma_5)\gamma_\mu; \tag{5.16}$$

$$\mathfrak{K}_\mu = \frac{-i}{2}(1-\gamma_5)\gamma_\mu; \tag{5.17}$$

$$K^1 = \frac{i}{2}\gamma_1\gamma_0; \quad K^2 = \frac{i}{2}\gamma_2\gamma_0; \quad K^3 = \frac{i}{2}\gamma_3\gamma_0; \tag{5.18}$$

$$J^1 = \frac{i}{2}\gamma_2\gamma_3; \quad J^2 = \frac{i}{2}\gamma_3\gamma_1; \quad J^3 = \frac{i}{2}\gamma_1\gamma_2; \tag{5.19}$$

$$D = \frac{i}{2}\gamma_5. \tag{5.20}$$

4×4 representation

Explicitly,

$$J^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad J^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad J^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$K^1 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}; \quad K^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad K^3 = \frac{1}{2}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix};$$

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}; \quad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\mathfrak{K}_0 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathfrak{K}_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathfrak{K}_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathfrak{K}_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \frac{i}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using these explicit 4×4 matrix representations of generators in the instant form and from the definition of light front generators, one can obtain the 4×4 matrix representations in LFD.

Conformal Group as a Rotation Group in $D = 6$

Let's start with:

$$J_{a,b} = \begin{pmatrix} 0 & D & \frac{-\mathfrak{K}_0}{\sqrt{2}} & \frac{-\mathfrak{K}_1}{\sqrt{2}} & \frac{-\mathfrak{K}_2}{\sqrt{2}} & \frac{-\mathfrak{K}_3}{\sqrt{2}} \\ -D & 0 & \frac{P_0}{\sqrt{2}} & \frac{P_1}{\sqrt{2}} & \frac{P_2}{\sqrt{2}} & \frac{P_3}{\sqrt{2}} \\ \frac{\mathfrak{K}_0}{\sqrt{2}} & \frac{-P_0}{\sqrt{2}} & 0 & -K^1 & -K^2 & -K^3 \\ \frac{\mathfrak{K}_1}{\sqrt{2}} & \frac{-P_1}{\sqrt{2}} & K^1 & 0 & J^3 & -J^2 \\ \frac{\mathfrak{K}_2}{\sqrt{2}} & \frac{-P_2}{\sqrt{2}} & K^2 & -J^3 & 0 & J^1 \\ \frac{\mathfrak{K}_3}{\sqrt{2}} & \frac{-P_3}{\sqrt{2}} & K^3 & J^2 & -J^1 & 0 \end{pmatrix}_{6\times 6} \tag{6.1}$$

where $J_{a,b} = -J_{b,a}$ and $a, b \in \{-2, -1, 0, 1, 2, 3\}$. These new generators obey the $SO(4+1, 1)$ commutation relations:

$$[J_{ab}, J_{cd}] = -i\left(g_{bd}J_{ac} - g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc}\right) \tag{6.2}$$

where,

$$g_{a,b} = g^{a,b} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{6\times 6} \quad ; \qquad {g^a}_b = {g_a}^b = \delta_{a,b} \tag{6.3}$$

Ansatz:

In 6-dimensional space, there are 15 planes ($\frac{n(n-1)}{2}$ planes in n dimension); rotation on each plane corresponds to each conformal transformations. We can also write

$$J_{ab} = i(x_a \partial_b - x_b \partial_a) \tag{6.4}$$

where $a, b \in \{-2, -1, 0, 1, 2, 3\}$. And it obeys,

$$[J_{ab}, J_{cd}] = -i\left(g_{bd} J_{ac} - g_{bc} J_{ad} + g_{ac} J_{bd} - g_{ad} J_{bc}\right) \tag{6.5}$$

Let's say that A^a is 6-vector; suppose A^a and B^a transform under 6d rotation:

$$A^{a\prime} = R^a{}_b A^b, \qquad B^{a\prime} = R^a{}_c B^c. \tag{6.6}$$

Then the inner products $A'.B'$ and $A.B$ can be written as

$$A'_b B'^b = (g_{ab} R^a_c R^b_d) A^c B^d, \tag{6.7}$$

$$A_b B^b = g_{cd} A^c B^d. \tag{6.8}$$

In order for $A'.B' = A.B$ to hold for any A and B, the coefficients of $A^c B^d$ should be the same term by term:

$$g_{ab} R^a_c R^b_d = g_{cd}. \tag{6.9}$$

Ansatz:

Let's start by looking at a 6d rotation transformation which is (has to be) infinitesimally close to the identity:

$$R^a{}_b = g^a{}_b + \omega^a{}_b \ , \tag{6.10}$$

where ω^a_b is a set of small (real) numbers. Inserting this to the defining condition ((6.9)), we get

$$\begin{aligned} g^{cd} &= R^c_b R^{bd} \ , \quad (6.11) \\ &= (g^c_b + \omega^c_b)(g^{bd} + \omega^{bd}) \ , \\ &= g^{cd} + \omega^{cd} + \omega^{dc} + \mathcal{O}(\omega^2). \quad (6.12) \end{aligned}$$

Keeping terms to the first order in ω, we then obtain

$$\omega^{ab} = -\omega^{ba} \ . \tag{6.13}$$

Namely, ω^{ab} is anti-symmetric (which is true when the indices are both subscript or both superscript; in fact, ω^a_b is not anti-symmetric under $a \longleftrightarrow b$).

Ansatz

Thus ω^{ab} has 15 independent parameters:

$$\omega^{ab} = \begin{pmatrix} 0 & \omega^{-2-1} & \omega^{-20} & \omega^{-21} & \omega^{-22} & \omega^{-23} \\ -\omega^{-2-1} & 0 & \omega^{-10} & \omega^{-11} & \omega^{-12} & \omega^{-13} \\ -\omega^{-20} & -\omega^{-10} & 0 & \omega^{01} & \omega^{02} & \omega^{03} \\ -\omega^{-21} & -\omega^{-11} & -\omega^{01} & 0 & \omega^{12} & \omega^{13} \\ -\omega^{-22} & -\omega^{-12} & -\omega^{02} & -\omega^{12} & 0 & \omega^{23} \\ -\omega^{-23} & -\omega^{-13} & -\omega^{03} & -\omega^{13} & -\omega^{23} & 0 \end{pmatrix}_{(6\times6)} . \tag{6.14}$$

This can be conveniently parameterized using 15 anti-symmetric matrices as

$$\begin{aligned} \omega^{ab} = -i\Big[&\omega^{-2-1}(J_{-2-1})^{ab} + \omega^{-20}(J_{-20})^{ab} + \omega^{-21}(J_{-21})^{ab} + \omega^{-22}(J_{-22})^{ab} \\ &+ \omega^{-23}(J_{-23})_{ab} + \omega^{-10}(J_{-10})^{ab} + \omega^{-11}(J_{-11})^{ab} + \omega^{-12}(J_{-12})^{ab} \\ &+ \omega^{-13}(J_{-13})^{ab} + \omega^{01}(J_{01})^{ab} + \omega^{02}(J_{02})^{ab} + \omega^{03}(J_{03})^{ab} \\ &+ \omega^{23}(J_{23})^{ab} + \omega^{13}(J_{13})^{ab} + \omega^{12}(J_{12})^{ab}\Big] , \end{aligned} \tag{6.15}$$

$$= -i\sum_{c<d} \omega^{cd}(J_{cd})^{ab} , \tag{6.16}$$

Ansatz:

with

$$(J_{-2-1})^{ab} = \begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{-20})^{ab} = \begin{pmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{-21})^{ab} = \begin{pmatrix} 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ;$$

$$(J_{-22})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{-23})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{-10})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ;$$

$$(J_{-11})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{-12})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{-13})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \end{pmatrix} ;$$

$$(J_{01})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{02})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{03})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{pmatrix} ;$$

$$(J_{12})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad (J_{13})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix} ; \quad (J_{23})^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix} .$$

Note that for a given pair of c and d, $(J_{cd})^{ab}$ is a 6×6 matrix, while ω^{cd} is a real number. The elements $(J_{cd})^{ab}$ can be written in a concise form as follows: first, we note that in the upper right half of each matrix (i.e. for $a < b$), the element with $(a, b) = (c, d)$ is 1 and all else are zero, which can be written as $g^a_c g^b_d$. For the lower half, all we have to do is to flip a and b and add a minus sign. Combining the two halves, we get

$$(J_{cd})^{ab} = g^a_c g^b_d - g^b_c g^a_d. \tag{6.17}$$

This is defined only for $c < d$ so far. For $c > d$, we will use this same expression ((6.17)) as the definition; then, $(J_{cd})^{ab}$ is anti-symmetric with respect to $(c \longleftrightarrow d)$:

$$(J_{cd})^{ab} = -(J_{dc})^{ab}, \tag{6.18}$$

which also means $(J_{cd})^{ab} = 0$ if $c = d$. Together with $\omega^{cd} = -\omega^{dc}$, ((6.16)) becomes

$$\omega^{ab} = -i \sum_{c<d} \omega^{cd} (J_{cd})^{ab} = -i \sum_{c>d} \omega^{cd} (J_{cd})^{ab} = -\frac{i}{2} \omega^{cd} (J_{cd})^{ab} \ , \tag{6.19}$$

where in the last expression, sum over all values of c and d is implied.

The infinitesimal transformation (6d rotation) ((6.10)) can then be written a

$$R^a_b = g^a_b - \frac{i}{2}\omega^{cd}(J_{cd})^a_b \ , \tag{6.20}$$

or in matrix form,

$$R = I - \frac{i}{2}\omega^{cd}J_{cd} \ . \tag{6.21}$$

The generator representation $(J_{cd})^a_b$ can be obtained by

$$(J_{cd})^a_b = g^{af}(J_{cd})_{fb} \tag{6.22}$$

The representation matrices of conformal generators are defined by taking the first index to be superscript and the second subscript:

$$\frac{-\mathfrak{K}_\mu}{\sqrt{2}} \equiv (J_{-2\mu})^a_b \ ; \quad \frac{P_\mu}{\sqrt{2}} \equiv (J_{-1\mu})^a_b \ ; \quad D \equiv (J_{-2-1})^a_b \ ; \tag{6.23}$$

$$-K_i \equiv (J_{i0})^a_b \ ; \quad J_i \equiv (J_{\epsilon_{ijk}jk})^a_b \ . \tag{6.24}$$

6 × 6 representation

Explicitly,

$$P_0=\sqrt{2}\begin{pmatrix}0&0&-i&0&0&0\\0&0&0&0&0&0\\0&-i&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\quad P_1=\sqrt{2}\begin{pmatrix}0&0&0&-i&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&i&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\quad P_2=\sqrt{2}\begin{pmatrix}0&0&0&0&-i&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&i&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\tag{6.25}$$

$$P_3=\sqrt{2}\begin{pmatrix}0&0&0&0&0&-i\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&i&0&0&0&0\end{pmatrix};\quad \mathfrak{K}_0=\sqrt{2}\begin{pmatrix}0&0&0&0&0&0\\0&0&i&0&0&0\\i&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\quad \mathfrak{K}_1=\sqrt{2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&i&0&0\\0&0&0&0&0&0\\-i&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\tag{6.26}$$

$$\mathfrak{K}_2=\sqrt{2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&i&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\-i&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\quad \mathfrak{K}_3=\sqrt{2}\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&i\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\-i&0&0&0&0&0\end{pmatrix};\quad D=\begin{pmatrix}i&0&0&0&0&0\\0&-i&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\tag{6.27}$$

$$J^1=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-i\\0&0&0&0&i&0\end{pmatrix};\quad J^2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&i\\0&0&0&0&0&0\\0&0&0&-i&0&0\end{pmatrix};\quad J^3=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-i&0\\0&0&0&i&0&0\\0&0&0&0&0&0\end{pmatrix};\tag{6.28}$$

$$K^1=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-i&0&0\\0&0&-i&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\quad K^2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-i&0\\0&0&0&0&0&0\\0&0&-i&0&0&0\\0&0&0&0&0&0\end{pmatrix};\quad K^3=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-i\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-i&0&0&0\end{pmatrix}.\tag{6.29}$$

More Ansatzes

So the infinitesimal transformation (6d rotation) can then be written a

$$R^a_b = g^a_b - \frac{i}{2}\omega^{cd}(J_{cd})^a_b \ , \tag{6.30}$$

And the representation matrices of conformal generators are defined by taking the first index to be superscript and the second subscript:

$$\begin{aligned} \frac{-\mathfrak{K}_\mu}{\sqrt{2}} \equiv (J_{-2\mu})^a_b \ ; \quad \frac{P_\mu}{\sqrt{2}} \equiv (J_{-1\mu})^a_b \ ; \quad D \equiv (J_{-2-1})^a_b \ ; \\ -K_i \equiv (J_{i0})^a_b \ ; \quad J_i \equiv (J_{\epsilon_{ijk}jk})^a_b \ . \end{aligned} \tag{6.31}$$

The ω^{cd} is the infinitesimal angel in which 6d planes are being rotated;

$$\omega^{cd} = 2\begin{pmatrix} 0 & \alpha & -\sqrt{2}b^0 & -\sqrt{2}b^1 & -\sqrt{2}b^2 & -\sqrt{2}b^3 \\ -\alpha & 0 & \sqrt{2}a^0 & \sqrt{2}a^1 & \sqrt{2}a^2 & \sqrt{2}a^3 \\ \sqrt{2}b^0 & -\sqrt{2}a^0 & 0 & -\xi^1 & -\xi^2 & -\xi^3 \\ \sqrt{2}b^1 & -\sqrt{2}a^1 & \xi^1 & 0 & \theta^3 & -\theta^2 \\ \sqrt{2}b^2 & -\sqrt{2}a^2 & \xi^2 & -\theta^3 & 0 & \theta^1 \\ \sqrt{2}b^3 & -\sqrt{2}a^3 & \xi^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix}_{6\times 6} \tag{6.32}$$

Correspondence between 6D De-Sitter space and 4D space-time

From

$$J_{a,b} = \begin{pmatrix} 0 & D & \frac{-\mathfrak{K}_0}{\sqrt{2}} & \frac{-\mathfrak{K}_1}{\sqrt{2}} & \frac{-\mathfrak{K}_2}{\sqrt{2}} & \frac{-\mathfrak{K}_3}{\sqrt{2}} \\ -D & 0 & \frac{P_0}{\sqrt{2}} & \frac{P_1}{\sqrt{2}} & \frac{P_2}{\sqrt{2}} & \frac{P_3}{\sqrt{2}} \\ \frac{\mathfrak{K}_0}{\sqrt{2}} & \frac{-P_0}{\sqrt{2}} & 0 & -K^1 & -K^2 & -K^3 \\ \frac{\mathfrak{K}_1}{\sqrt{2}} & \frac{-P_1}{\sqrt{2}} & K^1 & 0 & J^3 & -J^2 \\ \frac{\mathfrak{K}_2}{\sqrt{2}} & \frac{-P_2}{\sqrt{2}} & K^2 & -J^3 & 0 & J^1 \\ \frac{\mathfrak{K}_3}{\sqrt{2}} & \frac{-P_3}{\sqrt{2}} & K^3 & J^2 & -J^1 & 0 \end{pmatrix}_{6\times 6} \tag{7.1}$$

One can write;

$$\text{(Translation)} \quad \sqrt{2}\, J_{-1,\mu} = P_\mu = -i\partial_\mu \tag{7.2a}$$

$$\text{(Rotation)} \quad J_{\mu,\nu} = L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \tag{7.2b}$$

$$\text{(Dilation)} \quad J_{-2,-1} = D = -ix_\mu\partial^\mu \tag{7.2c}$$

$$\text{(SCT)} \quad \sqrt{2}\, J_{\mu,-2} = \mathfrak{K}_\mu = -i\left(2x_\mu(x\cdot\partial) - (x\cdot x)\partial_\mu\right) \tag{7.2d}$$

Correspondence between 6D De-Sitter space and 4D space-time

As we know, in 6-dimensional space, there are 15 planes ($\frac{n(n-1)}{2}$ planes in n dimension); rotation on each plane corresponds to each conformal transformations. We can also write

$$J_{ab} = i(\tilde{x}_a \tilde{\partial}_b - \tilde{x}_b \tilde{\partial}_a) \tag{7.3}$$

where $a, b \in \{-2, -1, 0, 1, 2, 3\}$. And it obeys,

$$[J_{ab}, J_{cd}] = -i\left(g_{bd} J_{ac} - g_{bc} J_{ad} + g_{ac} J_{bd} - g_{ad} J_{bc}\right) \tag{7.4}$$

From 7.3, one can also write in 6-dimensional space;

$$\text{(Translation)} \quad \sqrt{2}\ J_{-1,\mu} = P_\mu = i\sqrt{2}(\tilde{x}_{-1}\tilde{\partial}_\mu - \tilde{x}_\mu \tilde{\partial}_{-1}) \tag{7.5a}$$

$$\text{(Rotation)} \quad J_{\mu,\nu} = L_{\mu\nu} = i(\tilde{x}_\mu \tilde{\partial}_\nu - \tilde{x}_\nu \tilde{\partial}_\mu) \tag{7.5b}$$

$$\text{(Dilation)} \quad J_{-2,-1} = D = i(\tilde{x}_{-2}\tilde{\partial}_{-1} - \tilde{x}_{-1}\tilde{\partial}_{-2}) \tag{7.5c}$$

$$\text{(SCT)} \quad \sqrt{2}\ J_{\mu,-2} = \mathfrak{K}_\mu = i\sqrt{2}(\tilde{x}_\mu \tilde{\partial}_{-2} - \tilde{x}_{-2}\tilde{\partial}_\mu) \tag{7.5d}$$

Correspondence between 6D De-Sitter space and 4D space-time

On comparing 7.2 and 7.5,

$$P_\mu = -i\partial_\mu \longleftrightarrow i\sqrt{2}(\tilde{x}_{-1}\tilde{\partial}_\mu - \tilde{x}_\mu\tilde{\partial}_{-1}) \tag{7.6a}$$

$$\mathfrak{K}_\mu = -i\left(2x_\mu(x\cdot\partial) - (x\cdot x)\partial_\mu\right) \longleftrightarrow i\sqrt{2}(\tilde{x}_\mu\tilde{\partial}_{-2} - \tilde{x}_{-2}\tilde{\partial}_\mu) \tag{7.6b}$$

$$D = -ix_\mu\partial^\mu \longleftrightarrow i(\tilde{x}_{-2}\tilde{\partial}_{-1} - \tilde{x}_{-1}\tilde{\partial}_{-2}) \tag{7.6c}$$

$$L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \longleftrightarrow i(\tilde{x}_\mu\tilde{\partial}_\nu - \tilde{x}_\nu\tilde{\partial}_\mu) \tag{7.6d}$$

by comparing the coefficients, we can find the correspondence between 6 dimensional de sitter space and the usual 4-dimensional space, which is

$$\tilde{x}_{-1} = \frac{-\lambda}{\sqrt{2}}; \qquad \tilde{\partial}_{-1} = 0; \tag{7.7}$$

$$\tilde{x}_{-2} = \frac{-\lambda}{\sqrt{2}}(x^\mu.x_\mu); \qquad \tilde{\partial}_{-2} = \frac{-\sqrt{2}}{\lambda}(x^\mu.\partial_\mu); \tag{7.8}$$

$$\tilde{x}_\mu = \lambda x_\mu; \qquad \tilde{\partial}_\mu = \frac{1}{\lambda}\partial_\mu; \tag{7.9}$$

$$\tag{7.10}$$

6D De-Sitter space

Now, let's consider the 6d dot product,

$$g_{ab}\tilde{x}^a\tilde{x}^b = \tilde{x}^{-2}\tilde{x}_{-2} + \tilde{x}^{-1}\tilde{x}_{-1} + \tilde{x}^{\mu}\tilde{x}_{\mu} \tag{7.11}$$

$$\tilde{x}_a.\tilde{x}^a = -2\tilde{x}_{-2}\tilde{x}_{-1} + \tilde{x}^{\mu}\tilde{x}_{\mu} \qquad (\text{since, } \tilde{x}_{-1} = -\tilde{x}^{-2} \ \& \ \tilde{x}_{-2} = -\tilde{x}^{-1}) \tag{7.12}$$

$$\tilde{x}_a.\tilde{x}^a = -2\frac{-\lambda}{\sqrt{2}}(x^{\mu}.x_{\mu})\frac{-\lambda}{\sqrt{2}} + \lambda^2 x^{\mu}x_{\mu} = -\lambda^2 x^{\mu}x_{\mu} + \lambda^2 x^{\mu}x_{\mu} \tag{7.13}$$

$$\tilde{x}_a.\tilde{x}^a = 0 \tag{7.14}$$

we may recover coordinates, x_μ, near the origin and coordinates $\frac{x_\mu}{x^2}$, near infinity, on Minkowski space by forming the ratios

$$x_\mu = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_\mu}{\tilde{x}_{-1}} \tag{7.15}$$

$$\frac{x_\mu}{x^2} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_\mu}{\tilde{x}_{-1}}\frac{2\tilde{x}_{-1}\tilde{x}_{-1}}{\tilde{x}_\mu\tilde{x}^\mu} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_\mu}{\tilde{x}_{-1}}\frac{2\tilde{x}_{-1}\tilde{x}_{-1}}{2\tilde{x}_{-2}\tilde{x}_{-1}} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_\mu}{\tilde{x}_{-2}} \tag{7.16}$$

Also, $x^2 = \frac{\tilde{x}_{-2}}{\tilde{x}_{-1}}$.

Rotation in 6D De-Sitter space

Now for example let's consider a special conformal transformation in x^0 direction. We have

$$SCT(0) = \exp(ib^0\mathfrak{K}_0) \tag{7.17}$$

$$= \exp\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}b^0 & 0 & 0 & 0 \\ -\sqrt{2}b^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.18}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}b^0 & 0 & 0 & 0 \\ -\sqrt{2}b^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2(b^0)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.19}$$

$$SCT(0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ (b^0)^2 & 1 & -\sqrt{2}b^0 & 0 & 0 & 0 \\ -\sqrt{2}b^0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.20}$$

Rotation in 6D De-Sitter space

then,

$$\begin{pmatrix} \tilde{x}'_{-2} \\ \tilde{x}'_{-1} \\ \tilde{x}'_0 \\ \tilde{x}'_1 \\ \tilde{x}'_2 \\ \tilde{x}'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ (b^0)^2 & 1 & -\sqrt{2}b^0 & 0 & 0 & 0 \\ -\sqrt{2}b^0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_{-2} \\ \tilde{x}_{-1} \\ \tilde{x}_0 \\ \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \tag{7.21}$$

$$\begin{pmatrix} \tilde{x}'_{-2} \\ \tilde{x}'_{-1} \\ \tilde{x}'_0 \\ \tilde{x}'_1 \\ \tilde{x}'_2 \\ \tilde{x}'_3 \end{pmatrix} = \begin{pmatrix} \tilde{x}_{-2} \\ (b^0)^2\tilde{x}_{-2} + \tilde{x}_{-1} - \sqrt{2}b^0\tilde{x}_0 \\ -\sqrt{2}b^0\tilde{x}_{-2} + \tilde{x}_0 \\ \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \tag{7.22}$$

$$\tag{7.23}$$

Rotation in 6D De-Sitter space

The coordinates near the origin:

$$x'_0 = \frac{-1}{\sqrt{2}} \frac{\tilde{x}'_0}{\tilde{x}'_{-1}} \tag{7.24}$$

$$= \frac{-1}{\sqrt{2}} \frac{-\sqrt{2}b^0\tilde{x}_{-2} + \tilde{x}_0}{(b^0)^2\tilde{x}_{-2} + \tilde{x}_{-1} - \sqrt{2}b^0\tilde{x}_0} = \frac{-1}{\sqrt{2}} \frac{-\sqrt{2}b^0\tilde{x}_{-2} + \tilde{x}_0}{(b^0)^2\tilde{x}_{-2} + \tilde{x}_{-1} - \sqrt{2}b^0\tilde{x}_0} \tag{7.25}$$

$$= \frac{-1}{\sqrt{2}} \frac{-\sqrt{2}b^0\frac{\tilde{x}_{-2}}{\tilde{x}_{-1}} + \frac{\tilde{x}_0}{\tilde{x}_{-1}}}{(b^0)^2\frac{\tilde{x}_{-2}}{\tilde{x}_{-1}} + 1 - \sqrt{2}b^0\frac{\tilde{x}_0}{\tilde{x}_{-1}}} = -\frac{1}{\sqrt{2}} \frac{-\sqrt{2}b^0x^2 + \left(-\sqrt{2}x_0\right)}{(b^0)^2x^2 + 1 - \sqrt{2}b^0\left(-\sqrt{2}x_0\right)} \tag{7.26}$$

$$= \frac{b^0x^2 + x_0}{(b^0)^2x^2 + 1 + 2b^0x_0} \tag{7.27}$$

$$x'_0 = \frac{x_0 + b_0x^2}{1 + 2b^0x_0 + (b^0)^2x^2}\ . \tag{7.28}$$

Rotation in 6D De-Sitter space

For the inverse coordinate,

$$y'_0 = \frac{\tilde{x}'_0}{\tilde{x}'^0.\tilde{x}'_0} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_0}{\tilde{x}_{-2}} = -\frac{1}{\sqrt{2}}\frac{-\sqrt{2}b^0\tilde{x}_{-2} + \tilde{x}_0}{\tilde{x}_{-2}} = b^0 - \frac{\tilde{x}_0}{\sqrt{2}\tilde{x}_{-2}} = b^0 + y_0 \tag{7.29}$$

$$y'_0 = y_0 + b_0 \tag{7.30}$$

we have a simple translation at infinity.

Constraints of Conformal Symmetry

We would like to understand how the conformal symmetry group acts on quantum states and fields. In the case of the Poincaré group, it is often convenient to choose fields that are eigenvectors of the momentum operator P_μ. In the context of the conformal symmetry group, P_μ no longer plays as privileged a role. P_μ does not commute with K_μ nor with D.

In the case of conformal symmetry, dilatation D largely replaces the privileged role of P^t. The commutation relations $[D, P_\mu] = iP_\mu$ and $[D, K_\mu] = -iK_\mu$ are suggestively close to the commutation relations for the raising and lower operators of the harmonic oscillator with the identifications $H \sim D$, $P_\mu \sim a^\dagger$ and $K_\mu \sim a$. Recall that for the harmonic oscillator, the raising and lower operators commute to give $[a, a^\dagger] = 1$ and the Hamiltonian can be written as a combination of these raising and lower operators: $H = a^\dagger a + E_0$, where E_0 is a constant (the ground state energy). A short computation leads to the conclusion $[H, a] = -a$ and $[H, a^\dagger] = a^\dagger$. If there is a lowest weight state $|0\rangle$, such that $a|0\rangle = 0$, then $H|0\rangle = E_0|0\rangle$. Moreover, the relation $H(a^\dagger)^n|0\rangle = (E_0 + n)(a^\dagger)^n|0\rangle$ follows from the commutation relations of H with $a^\dagger$.

We can play a very similar game with the conformal group. We declare a lowest weight state – or *primary* state – to be an eigenvector of the dilatation operator and also annihilated by special conformal transformations

$$D|\phi_I\rangle = i\Delta|\phi_I\rangle \ , \tag{3.1}$$

$$K_\mu|\phi_I\rangle = 0 \ . \tag{3.2}$$

Constraints of Conformal Symmetry

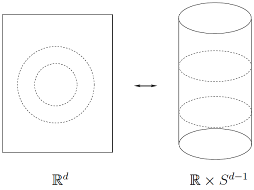
Actually, it turns out that the dilatation operator D can be though of as the Hamiltonian of another theory. Consider $\mathbb{R}^d$ in spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega_{d-1} = r^2 \left[\frac{dr^2}{r^2} + d\Omega_{d-1}\right]$$

Now let $t = \log r$ so that

$$\frac{dr^2}{r^2} + d\Omega_{d-1} = dt^2 + d\Omega_{d-1}$$

which is the metric on $\mathbb{R} \times S^{d-1}$. Now, if we are considering a CFT on $\mathbb{R}^d$, the theory should be invariant under rescaling of the metric! so that studying a CFT on $\mathbb{R}^d$ should be equivalent to study the theory on $\mathbb{R} \times S^{d-1}$:



A very interesting feature of this map is that it takes circles of constant radius in $\mathbb{R}^d$ to constant t slices on $\mathbb{R} \times S^{d-1}$. As a consequence, the dilatation operator on $\mathbb{R}^d$, which maps circles onto circles with different radius, corresponds to time translations on $\mathbb{R} \times S^{d-1}$, so it behaves as a Hamiltonian!

Constraints of Conformal Symmetry

Just as the harmonic oscillator has excited states that are formed by acting with $a^\dagger$ on the ground state, conformal primary states have *descendant* states which are constructed by acting with derivatives $P_\mu = -i\partial_\mu$ on the conformal primary state. Acting with P_μ n times increases the conformal weight $\Delta \to \Delta + n$. Acting with K_μ decreases the weight.

Most of the conformal field theory literature is phrased in terms of operators and correlation functions rather than states. We thus replace these conformal primary states with operators at the origin acting on the vacuum that create these states. A conformal primary operator $\phi_I(x)$ is one such that

$$\phi_I(0)|0\rangle = |\phi_I\rangle \ . \tag{3.4}$$

Part of the definition of the vacuum is that it is conformally invariant; it is annihilated by all of the generators of the conformal group. We could have chosen any point in space-time to insert the operator as all points are related via the conformal group. However, our choice of generators, for example $D = -ix^\mu\partial_\mu$, make the origin a simpler choice.

The action of the group on the operator is then given in terms of commutation relations:

$$[D, \phi_I(0)] = i\Delta\phi_I(0) \ , \tag{3.5}$$
$$[M_{\mu\nu}, \phi_I(0)] = (M_{\mu\nu})_I{}^J\phi_J(0) \ , \tag{3.6}$$
$$[K_\mu, \phi_I(0)] = 0 \ . \tag{3.7}$$

Constraints of Conformal Symmetry

To recover the action of D, $M_{\mu\nu}$ and K_μ on $\phi_I(x)$ away from the origin, we use the fact that $\phi_I(x) = e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x}$ and the commutator algebra of the conformal group. For instance

$$\begin{aligned}
[D, \phi_I(x)] &= De^{iP\cdot x}\phi_I(0)e^{-iP\cdot x} - e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x}D \\
&= e^{iPx}(e^{-iP\cdot x}De^{iP\cdot x}\phi_I(0) - \phi_I(0)e^{-iP\cdot x}De^{iP\cdot x})e^{-iP\cdot x} \\
&= e^{iP\cdot x}[\hat{D}, \phi_I(0)]e^{-iP\cdot x} \ , \qquad (3.8)
\end{aligned}$$

where we have defined $\hat{D} = e^{-iP\cdot x}De^{iP\cdot x}$. We then compute $\hat{D}$ explicitly,

$$\begin{aligned}
\hat{D} &= \left(1 - ix\cdot P - \frac{(x\cdot P)^2}{2} + \ldots\right) D \left(1 + ix\cdot P - \frac{(x\cdot P)^2}{2} + \ldots\right) \\
&= D - ix^\mu[P_\mu, D] - \frac{1}{2}x^\mu x^\nu[P_\mu, [P_\nu, D]] + \ldots \qquad (3.9)
\end{aligned}$$

and from the commutator algebra conclude that $[P_\mu, [P_\mu, D]]$ and all higher order terms vanish. In short $\hat{D} = D - x^\mu P_\mu$ and

$$[D, \phi_I(x)] = i(\Delta + x^\mu\partial_\mu)\phi_I(x) \ . \qquad (3.10)$$

A similar simplification occurs for the other elements of the conformal group.

Constraints of Conformal Symmetry

The complete list of the action of conformal generators on operators is

$$\begin{aligned}
[P_\mu, \phi_\alpha(x)] &= i\partial_\mu \phi_\alpha(x) \\
[D, \phi_\alpha(x)] &= i\left(\Delta + x^\mu \partial_\mu\right)\phi_\alpha(x) \\
[L_{\mu\nu}, \phi_\alpha(x)] &= -i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)\phi_\alpha(x) + i(S_{\mu\nu})_{\alpha\beta}\Phi_\beta(x) \\
[K_\mu, \phi_\alpha(x)] &= 2ix_\mu \Delta \phi_\alpha(x) + i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right)\phi_\alpha(x) + 2ix^\rho (S_{\rho\mu})_{\alpha\beta}\Phi_\beta(x)
\end{aligned}$$

A. O. Barut - Unification of the external conformal symmetry group and the internal conformal dynamical group

If X_μ and Π_μ are conjugate variables,

$$[\Pi_\mu, X_\nu] = ig_{\mu\nu}, \quad g = (+,-,-,-), \tag{2.3}$$

we may write the conformal generators as follows:

$$L_{\mu\nu} = X_\mu \Pi_\nu - X_\nu \Pi_\mu,$$

$$P_\mu = L_{\mu 6} + L_{\mu 4} = \Pi_\mu,$$

$$K_\mu = L_{\mu 6} - L_{\mu 4} = 2X_\mu (X_\nu \Pi^\nu + iH) - (X_\nu X^\nu)\Pi_\mu,$$

$$D = \quad L_{64} = X_\nu \Pi^\nu + iH. \tag{2.4}$$

Here H is a number called the homogeneity (for reasons given below) and is related to the Casimir invariant of the algebra by

$$Q = \tfrac{1}{2} L_{ab} L^{ab} = H^2 - 4H. \tag{2.5}$$

A. O. Barut - Unification of the external conformal symmetry group and the internal conformal dynamical group

Acting as differential operators on a function space of X, Π_μ would be represented by $i\partial/(\partial X^\mu)$. Direct verification shows that the generators L_{ab} obey the $O(4,2)$ commutation relation

$$[L_{ab}, L_{cd}] = -i(g_{ac}L_{bd} + g_{bd}L_{ac} - g_{ad}L_{bc} - g_{bc}L_{ad}), \qquad (2.6)$$

where g is the diagonal metric

$$g = \begin{pmatrix} + & - & - & - & - & + \\ 0=5 & 1 & 2 & 3 & 4 & 6 \end{pmatrix}. \qquad (2.7)$$

A. O. Barut - Unification of the external conformal symmetry group and the internal conformal dynamical group

The specific commutators are

$$[L_{\mu\nu}, L_{\lambda\rho}] = -i(g_{\mu\lambda}L_{\nu\rho} + g_{\nu\rho}L_{\mu\lambda} - g_{\mu\rho}L_{\nu\lambda} - g_{\nu\lambda}L_{\mu\rho}),$$

$$[P_\mu, K_\nu] = 2i(g_{\mu\nu}D - L_{\mu\nu}),$$

$$[L_{\mu\nu}, P_\lambda] = -i(g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu), \quad [D, P_\lambda] = -iP_\lambda,$$

$$[L_{\mu\nu}, K_\lambda] = -i(g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu), \quad [D, K_\lambda] = iK_\lambda. \qquad (2.8)$$

All other commutators vanish.

A. O. Barut - Unification of the external conformal symmetry group and the internal conformal dynamical group

The special conformal transformation (2.1) may be linearized by introducing K as a scale parameter along with the following new coordinates

$$\begin{aligned} &Y^\mu \equiv KX^\mu, \\ &Y^4 - Y^6 \equiv K, \\ &Y^4 + Y^6 \equiv \Lambda \equiv KX_\mu X^\mu, \end{aligned} \tag{2.9}$$

having the property

$$Y \circ Y \equiv Y_a Y^a = Y_\mu Y^\mu - (Y^4 - Y^6)(Y^4 + Y^6) = Y_\mu Y^\mu - K\Lambda = 0. \tag{2.10}$$

The generators (2.4) may then be written in the simple form

$$L_{ab} = Y_a Q_b - Y_b Q_a, \tag{2.11}$$

Future works and Reviews

- A. O. Barut and G. L. Bornzin (Unification of the external conformal symmetry group and the internal conformal dynamical group).
- Abdus Salam's paper on CFT Representation [Ann. Phys. N.Y.J 53,174 (1969) - Finite-Component Field Representations of the Conformal Group].