# Lattice QCD calculations of Transverse Momentum Dependent Parton Distribution Functions (TMDs)

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# Quarks & Gluons

**Hadrons are made of more fundamental particles, named "Quarks".**

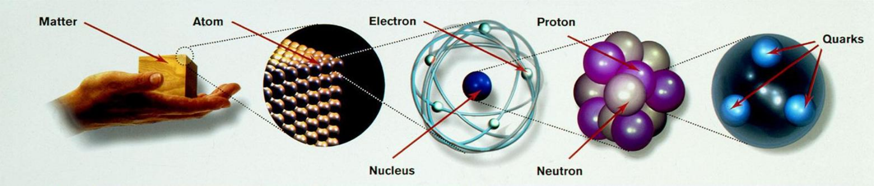



Figure 1: All known atoms are made of electrons and quarks (credit: CMS-CERN)

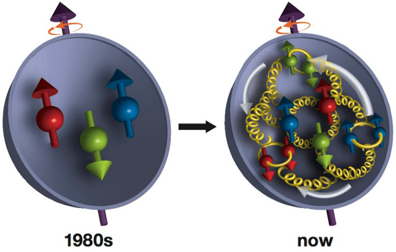



Figure 2: Internal structure of Hadrons [Quarks & Gluons] (credit: BNL)

# QCD (Quantum ChromoDynamics)

As a quantum field theory, the QCD Lagrange density is constructed from 2 types of particle fields:

- Spin-$\frac{1}{2}$ Dirac fields (quarks): $\boxed{\psi_f^i}$
  - color $i = 1, 2, 3 = N_c$
  - flavor $f = u, d, s, c, b, t$
- Massless spin-1 vector fields (gluons): $\boxed{A_\mu^a}$
  - color $a = 1, 2, \cdots, 8 = N_c^2 - 1$, with $SU(3)$ local color gauge symmetry

$$\boxed{\mathcal{L}_{QCD}(\psi_f, A_\mu) = -\frac{1}{4} F_{\mu\nu,a}^2[A] + \sum_f \bar{\psi}_f (iD_\mu[A]\gamma^\mu - m_f)\psi_f} \,. \tag{1}$$

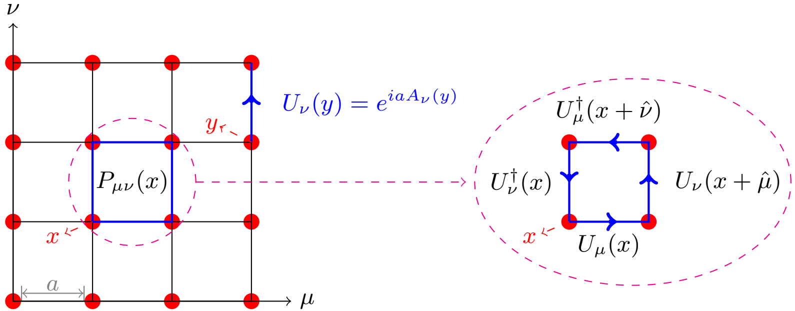
Here,

- the gluon field strength $\boxed{F_{\mu\nu,a}[A] = \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} - g f_{abc} A_{\mu,b} A_{\nu,c}}$
- the covariant derivative $\boxed{D_\mu[A] = \partial_\mu + igA_{\mu,a} t_a}$
- the generator $t_a$ and structure constant $f_{abc}$ define the $SU(3)$ color algebra: $\boxed{[t_a, t_b] = if_{abc} t_c}$
- $g$ is the strong coupling constant.

# Lattice QCD[1]

$\Lambda_4 = \{n_\mu = (n_1, n_2, n_3, n_4) | n_i \in a[0, 1, \cdots, L_i - 1]\}$

$$\boxed{S_{gauge} = \frac{2}{g^2} \sum_{x\in\Lambda_4} \sum_{\mu<\nu} (1 - Re\ Tr\ [P_{\mu\nu}(x)])} \tag{2}$$




where the elementary plaquette, $\boxed{P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)}$.

[1] K. G. Wilson, Confinement of Quarks, Phys. Rev. D10 (1974) 2445.

# Lattice QCD

$$\boxed{S_{gauge} = \frac{2}{g^2} \sum_{x \in \Lambda_4} \sum_{\mu<\nu} (1 - Re\ Tr\ [P_{\mu\nu}(x)]) = \frac{a^4}{2g^2} \sum_{x \in \Lambda_4} \sum_{\mu<\nu} tr\ [F_{\mu\nu}(x)^2] + \mathcal{O}(a^2)}.$$

Physical observables $\mathcal{O}$ are evaluated as an expectation value over the relevant degrees of freedom

$$\boxed{\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\ \mathcal{O}\ e^{-[S_{gauge} + \int dx \bar{\psi} \mathcal{M} \psi]}}{\int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] e^{-[S_{gauge} + \int dx \bar{\psi} \mathcal{M} \psi]}}}. \tag{3}$$

The quark fields $\psi$ & $\bar{\psi}$ are Grassmann variables:

$$\boxed{\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}[U]\ \widetilde{\mathcal{O}}\ det\mathcal{M}^2 e^{-[S_{gauge}]}}{\int \mathcal{D}[U] det\mathcal{M}^2 e^{-[S_{gauge}]}}}. \tag{4}$$

This integration results in the "contraction" of fermion–anti-fermion pairs in all possible ways (Wick's theorem), replacing them with quark propagators $\mathcal{M}^{-1}$.

# Numerical simulation for Lattice QCD

The vacuum expectation value of an observable in a Monte Carlo simulation approximation: (Sum over $U_n$ with probability $\propto e^{-S[U_n]}$)

$$\boxed{\langle O \rangle = \frac{\int \mathcal{D}[U] e^{-S_G[U]} O[U]}{\int \mathcal{D}[U] e^{-S_G[U]}}} \rightarrow \boxed{\langle O \rangle \approx \frac{1}{N} \sum_{U_n} O[U_n]}$$

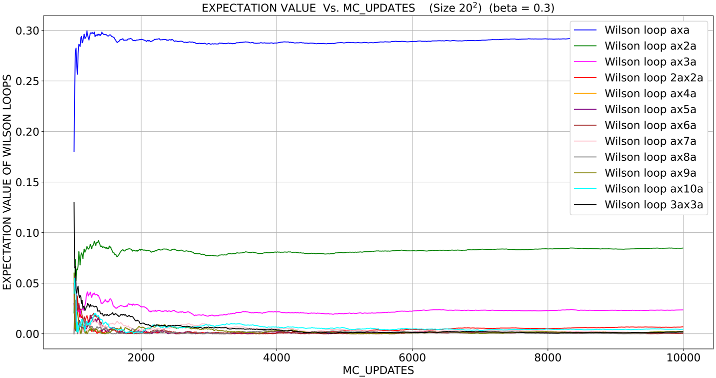
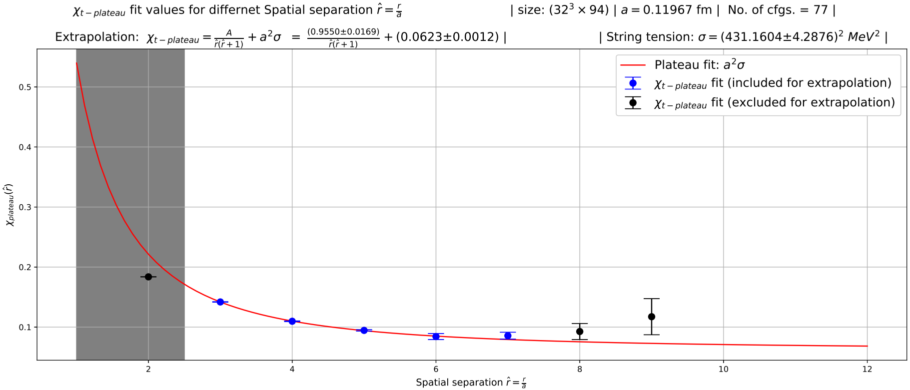



Figure 3: Markov chain - Monte Carlo simulations of $\mathbb{Z}_2$ lattice gauge theory $(1+1)$

# $SU(3)$ Wilson Loops & Quark String Tension

We calculated various sizes of Wilson loops $\langle W(r,t)\rangle$ and formed Creutz[2] ratios $\chi(r,t)$ to extract quark string tension $\sigma$

$$\boxed{\chi(r,t) = -\ln\left(\frac{\langle W(r,t)\rangle\langle W(r-1,t-1)\rangle}{\langle W(r,t-1)\langle W(r-1,t)\rangle}\right)} \rightarrow \boxed{\chi_{(\text{t-plateau})}(r) = \frac{A}{\hat{r}(\hat{r}+1)} + a^2\sigma}$$

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[2] Creutz, M., Phys. Rev. Lett. 45, 313.

## Proton Two Point Function

The proton mass can be determined from the two-point correlation function:

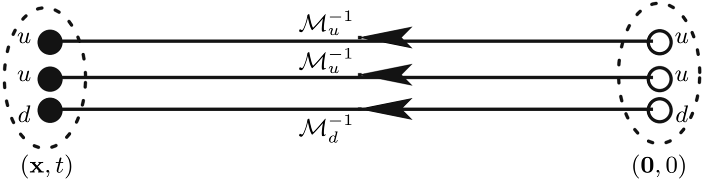
$$C_{\alpha\beta}(t,\mathbf{p}) = a^3 \sum_{\mathbf{x}} e^{-i\mathbf{p}.\mathbf{x}} C_{\alpha\beta}(t,\mathbf{x}) = a^3 \sum_{\mathbf{x}} e^{-i\mathbf{p}.\mathbf{x}} \langle 0 | \chi_\alpha(\mathbf{x},t)\widetilde{\chi}_\beta(\mathbf{0},0)|0\rangle \tag{5}$$

where

$$\chi_\alpha(\mathbf{x},t) = \epsilon^{ijk} u^i_\alpha(\mathbf{x},t) u^j_\gamma(\mathbf{x},t)[C^{-1}\gamma_5]_{\gamma\delta} d^k_\delta(\mathbf{x},t) \tag{6}$$

After integrating out the quark fields in the path integral formulation, this correlator is expressed in terms of products of the inverse of the Dirac operator

$$\begin{aligned} C_{\alpha\beta}(t,\mathbf{p}) = &- a^3 \sum_{\mathbf{x}} e^{-i\mathbf{p}.\mathbf{x}} \epsilon^{ijk}\epsilon^{i'j'k'}[C^{-1}\gamma_5]_{\alpha'\alpha''}[\gamma_5 C]_{\beta'\beta''} \\ &\times \left\langle [\mathcal{M}_d^{-1}]^{ki'}_{\alpha''\beta'} \left\{ [\mathcal{M}_u^{-1}]^{jj'}_{\alpha'\beta''}[\mathcal{M}_u^{-1}]^{ik'}_{\alpha\beta} - [\mathcal{M}_u^{-1}]^{ij'}_{\alpha\beta''}[\mathcal{M}_d^{-1}]^{ik'}_{\alpha'\beta} \right\} \right\rangle \end{aligned}$$



# Proton Two Point Function

On inserting a complete set of states between the interpolating operators[3]

$$C_{\alpha\beta}(t,\mathbf{p}) = a^3 \sum_{\mathbf{x}} e^{-i\mathbf{p}.\mathbf{x}} \langle 0| \chi_\alpha(\mathbf{x},t)\widetilde{\chi}_\beta(\mathbf{0},0)|0\rangle \tag{7}$$

$$= a^3 \sum_{n,\sigma} \frac{e^{-E_n(\mathbf{p})t}}{2E_n(\mathbf{p})} \langle 0| \chi_\alpha |n;\mathbf{p},\sigma\rangle \langle n;\mathbf{p},\sigma| \widetilde{\chi}_\beta |0\rangle \tag{8}$$

$$= a^3 Z(\mathbf{p}) \sum_{\sigma} u_\alpha(n=0,\mathbf{p},\sigma)\widetilde{u}_\beta(n=0,\mathbf{p},\sigma)\frac{e^{-E_n(\mathbf{p})t}}{2E_n(\mathbf{p})} + \cdots \tag{9}$$

Tracing this correlator against a given Dirac structure, often chosen to be $\Gamma^+ = \frac{1}{2}(1+\gamma_4)$, leads to

$$\boxed{C_{\Gamma^+}(t,\mathbf{p})} = \Gamma^+_{\beta\alpha} C_{\alpha\beta}(t,\mathbf{p}) \xrightarrow{t\to\infty} \boxed{Ce^{-E_n(\mathbf{p})t}} \tag{10}$$

where $C$ is a time-independent constant.

[3] Refer: Boussarie, Renaud, et al. "TMD Handbook." arXiv:2304.03302 [hep-ph].

# Proton Two Point Function

For $\mathbf{p} = 0$ we find $C_{\Gamma+}(t, \mathbf{0}) \xrightarrow{t \to \infty} Ce^{-m_H t}$

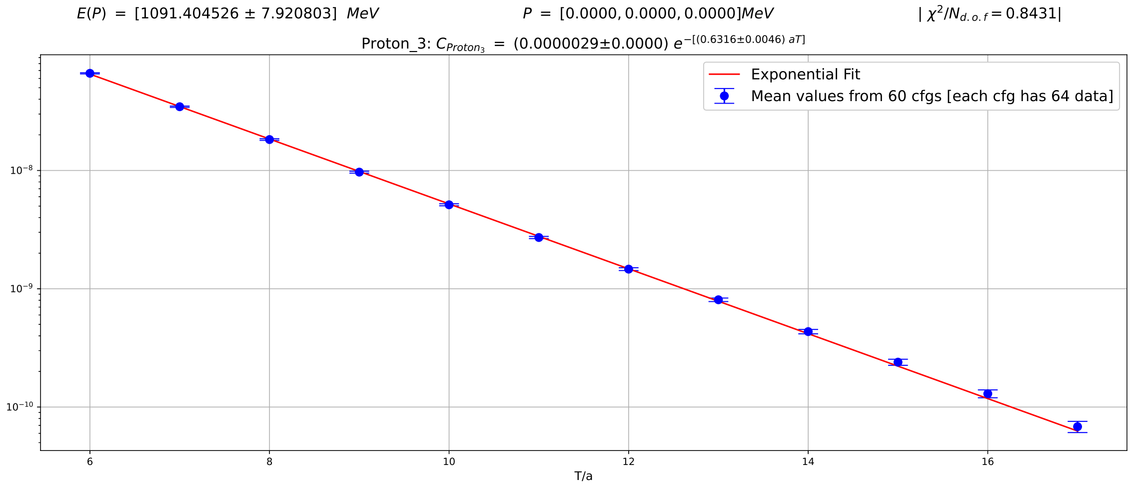



Figure 4: Proton two-point function $C_{\Gamma+}(t, \mathbf{0})$ from lattice size:$(32^3, 94)$ with $a = 0.11967$ fm

# Proton Two Point Function

Given the discrete time series $C_{\Gamma^+}(t, \mathbf{p})$ from the Monte Carlo sampling, the proton mass dispersion relation can be extracted from fits to the time dependence of $C_{\Gamma^+}(t, \mathbf{p}) \xrightarrow{t\to\infty} Ce^{-E_n(\mathbf{p})t}$:

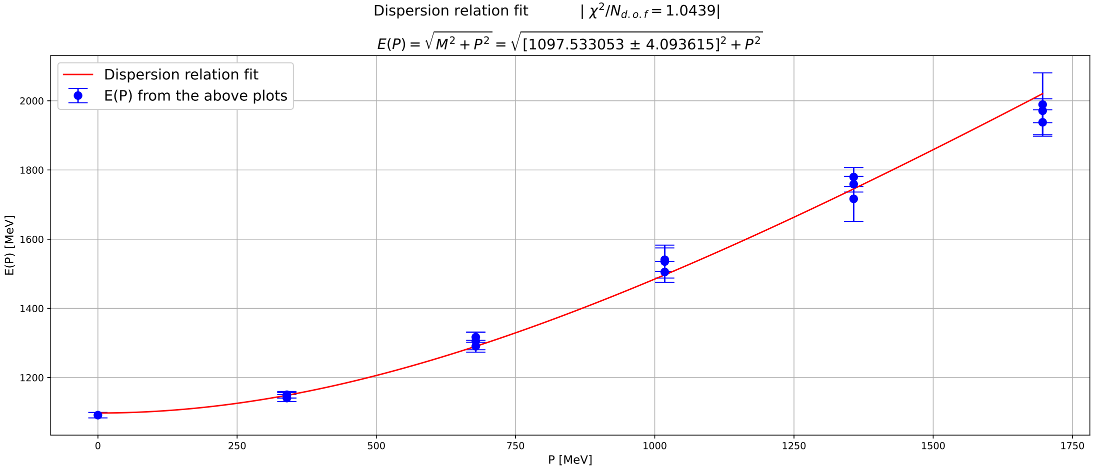
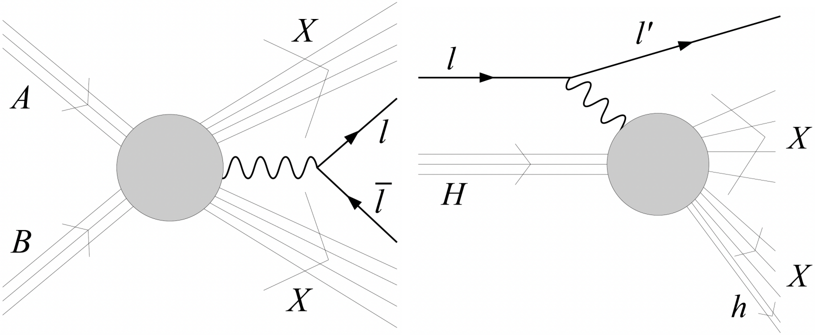



Figure 5: Proton dispersion relation from lattice size:$(32^3, 94)$ with $a = 0.11967$ fm

# Introduction: TMDs

The **intrinsic motion of quarks and gluons** inside the proton or neutron, specifically **with respect to the transverse momentum**, can be described in terms of Transverse Momentum Dependent Parton Distribution Functions (TMDs)




(a) The Drell-Yan process.

(b) The SIDIS process.

Figure 6: Two examples of processes sensitive to TMD PDFs. We draw the leading contributions, in which a single electroweak gauge boson (wiggled lines) is exchanged.

## Introduction: TMDs

In the SIDIS cross section

$$\frac{d\sigma}{d^3P_h d^3P_{l'}} \propto L_{\mu\nu}W^{\mu\nu} \tag{11}$$

$$\implies \quad W^{\mu\nu}(P,q,P_h) = \int \frac{d^4l}{(2\pi)^4} e^{iq\cdot l} \sum_X \langle N(P,S)|\, J^\mu(-b)\, |Xh(P_h,S_h)\rangle \, \langle Xh(P_h,S_h)|\, J^\nu(0)\, |N(P,S)\rangle$$

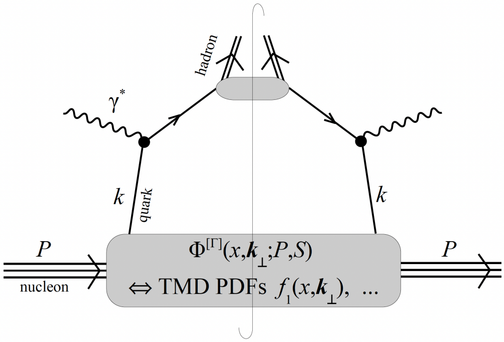



Figure 7: Simplifed factorized tree level diagram of the hadron tensor in SIDIS. *arXiv:0907.2381*

# Definition of TMDs

- Consider a frame where the nucleon has large momentum in $z$-direction, i.e., $P^+ \gg m_N$, $\boldsymbol{P}_{\mathrm{T}} = 0$. In light cone coordinates, the components $\boxed{\mathrm{k}^+ \colon \boldsymbol{k}_{\mathrm{T}} \colon k^- \sim P^+/m_N \colon 1 \colon m_N/P^+}$, under boosts along the $z$-axis.

The starting point for our discussion of TMDs are the correlator of the general form

$$\Phi^{[\Gamma]}(k, P, S; \ldots) \equiv \int \frac{d^4 b}{(2\pi)^4} e^{ik\cdot b} \frac{\overbrace{\frac{1}{2} \langle P, S| \ \bar{q}(0)\, \Gamma\, \mathcal{U}[\mathcal{C}_b] \ q(b) \ |P, S\rangle}^{\equiv \widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S; \ldots)}}{\widetilde{\mathcal{S}}(b^2; \ldots)} \tag{12}$$

- The gauge link $\mathcal{U}[\mathcal{C}_b]$ brings divergences; so we divide it by soft factor $\tilde{\mathcal{S}}$

# Definition of TMDs

Integrating the correlator over the suppressed momentum component $k^-$ yields

$$\Phi^{[\Gamma]}(x, \boldsymbol{k}_{\mathrm{T}}; P, S; \ldots) \equiv \int dk^- \Phi^{[\Gamma]}(k, P, S; \ldots)$$
$$= \int \frac{d^2 \boldsymbol{b}_{\mathrm{T}}}{(2\pi)^2} \int \frac{d(b \cdot P)}{(2\pi) P^+} \; e^{ix(b\cdot P) - i\boldsymbol{b}_{\mathrm{T}} \cdot \boldsymbol{k}_{\mathrm{T}}} \; \left. \frac{\frac{1}{2} \langle P,S| \; \bar{q}(0)\, \Gamma \; \mathcal{U}[\mathcal{C}_b] \; q(b) \; |P,S\rangle}{\widetilde{\mathcal{S}}(-\boldsymbol{b}_{\mathrm{T}}^2; \ldots)} \right|_{b^+=0} . \quad (13)$$

The above correlator can be decomposed into TMDs.

$$\Phi^{[\gamma^+]}(x, \boldsymbol{k}_{\mathrm{T}}; P, S, \ldots) = f_1 - \left[ \frac{\epsilon_{ij}\, \boldsymbol{k}_i\, \boldsymbol{S}_j}{m_N} \; f_{1T}^{\perp} \right]_{\mathrm{odd}} , \quad (14)$$

$$\Phi^{[\gamma^+ \gamma^5]}(x, \boldsymbol{k}_{\mathrm{T}}; P, S, \ldots) = \Lambda\, g_1 + \frac{\boldsymbol{k}_{\mathrm{T}} \cdot \boldsymbol{S}_{\mathrm{T}}}{m_N} \; g_{1T} , \quad (15)$$

$$\Phi^{[i\sigma^{i+} \gamma^5]}(x, \boldsymbol{k}_{\mathrm{T}}; P, S, \ldots) = \boldsymbol{S}_i \; h_1 + \frac{(2\boldsymbol{k}_i \boldsymbol{k}_j - \boldsymbol{k}_{\mathrm{T}}^2 \delta_{ij}) \boldsymbol{S}_j}{2m_N^2} \; h_{1T}^{\perp}$$
$$+ \frac{\Lambda \boldsymbol{k}_i}{m_N} h_{1L}^{\perp} + \left[ \frac{\epsilon_{ij} \boldsymbol{k}_j}{m_N} h_1^{\perp} \right]_{\mathrm{odd}} . \quad (16)$$

# Parametrization of the correlator in position space

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S| \ \bar{q}(0)\, \Gamma\, \mathcal{U}[0, \eta v, \eta v + b, b] \ q(b) \ |P, S\rangle \tag{17}$$

For the $\Gamma$-structures at leading twist, the correlator can be written in the form

$$\frac{1}{2P^+} \widetilde{\Phi}^{[\gamma^+]}_{\text{unsubtr.}} = \widetilde{A}_{2B} + i m_N \epsilon_{ij} \boldsymbol{b}_i \boldsymbol{S}_j \ \widetilde{A}_{12B} \tag{18}$$

$$\frac{1}{2P^+} \widetilde{\Phi}^{[\gamma^+ \gamma^5]}_{\text{unsubtr.}} = -\Lambda \ \widetilde{A}_{6B} + i \left\{ (b\cdot P)\Lambda - m_N (\boldsymbol{b}_{\mathrm{T}} \cdot \boldsymbol{S}_{\mathrm{T}}) \right\} \ \widetilde{A}_{7B} \tag{19}$$

$$\begin{aligned} \frac{1}{2P^+} \widetilde{\Phi}^{[i\sigma^{i+} \gamma^5]}_{\text{unsubtr.}} &= i m_N \epsilon_{ij} \boldsymbol{b}_j \ \widetilde{A}_{4B} - \boldsymbol{S}_i \ \widetilde{A}_{9B} - i m_N \Lambda \boldsymbol{b}_i \ \widetilde{A}_{10B} \\ &\quad + m_N \left\{ (b\cdot P)\Lambda - m_N (\boldsymbol{b}_{\mathrm{T}} \cdot \boldsymbol{S}_{\mathrm{T}}) \right\} \boldsymbol{b}_i \ \widetilde{A}_{11B} \end{aligned} \tag{20}$$

(Decompositions analogous to work by Metz et al. Phys. Lett. **B618** (2005) 90-96. in momentum space)

# Strategy

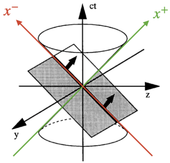



Figure 8: Light-front coordinates

- The separation $b$ of the quark field operators has a transverse component, $\boxed{b = nb^- + b_\perp}$. So, this separation is space-like

$$\Phi^{[\Gamma]}(k, P, S; \ldots) \equiv \int \frac{d^4 b}{(2\pi)^4} e^{ik\cdot b} \frac{\overbrace{\frac{1}{2} \langle P, S| \; \bar{q}(0)\, \Gamma\, \mathcal{U}[\mathcal{C}_b]\; q(b) \; |P, S\rangle}^{\equiv \widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S; \ldots)}}{\widetilde{\mathcal{S}}(b^2; \ldots)} \tag{21}$$

- We parametrized this correlator in terms of Lorentz-invariant amplitudes
- We choose the Lorentz frame in which this nonlocal operator is defined at one single time
- The computation of the nonlocal matrix element can be cast in terms of a Euclidean path integral and performed employing the standard methods of lattice QCD

# Link geometry

The gauge link employed in this work reads

$$\mathcal{U}[\mathcal{C}_b^{(\eta v)}] = \mathcal{U}[0, \eta v, \eta v + b, b], \tag{22}$$

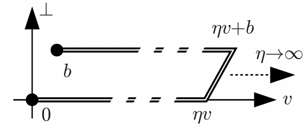



Figure 9: Staple-shaped gauge connection. The four-vectors $v$ and $P$ give the direction of the staple and the momentum, while b defines the separation between the quark operators. *(arXiv:1111.4249v2 [hep-lat])*

The Lorentz-invariant quantity characterizing the direction of $v$ is the Collins-Soper type parameter

$$\boxed{\hat{\zeta} \equiv \zeta/2m_N = \frac{v \cdot P}{\sqrt{|v^2|}\sqrt{P^2}}.} \tag{23}$$

The light-like direction $v = n$ can be approached in the limit $\zeta \to \infty$.

# TMDs in Fourier space and $x$-integrations (Mellin moments)

$$\tilde{f}(x, \boldsymbol{b}_\mathrm{T}^2; \ldots) \equiv \int d^2\boldsymbol{k}_\mathrm{T}\, e^{i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{k}_\mathrm{T}}\; f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots) \tag{24}$$

$$\tilde{f}^{(n)}(x, \boldsymbol{b}_\mathrm{T}^2 \ldots) \equiv n! \left(-\frac{2}{m_N^2}\partial_{\boldsymbol{b}_\mathrm{T}^2}\right)^n \; \tilde{f}(x, \boldsymbol{b}_\mathrm{T}^2; \ldots) \tag{25}$$

In the limit $|\boldsymbol{b}_\mathrm{T}| \to 0$, one recovers conventional $\boldsymbol{k}_\mathrm{T}$-moments of TMDs:

$$\tilde{f}^{(n)}(x, 0; \ldots) = \int d^2\boldsymbol{k}_\mathrm{T} \left(\frac{\boldsymbol{k}_\mathrm{T}^2}{2m_N^2}\right)^n f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots) \equiv f^{(n)}(x)\ . \tag{26}$$

$\boldsymbol{k}_\mathrm{T}$-moments like $f_1^{(0)}(x)$ and $f_{1T}^{\perp(1)}(x)$ are ill-defined without further regularization, we therefore do not attempt to extrapolate to $\boldsymbol{b}_\mathrm{T} = 0$, but rather state our results at finite $|\boldsymbol{b}_\mathrm{T}|$.
**In our studies so far, we only considered the first $x$-moments (accessible at $b \cdot P = 0$), rather than scanning range of $b \cdot P$**

$$f^{[1]}(\boldsymbol{k}_\mathrm{T}^2; \ldots) \equiv \int_{-1}^{1} dx\; f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots)\ . \tag{27}$$

where, $x = \frac{k^+}{P^+}$

# TMDs in Fourier space and invariant amplitudes

$\widetilde{A}_i(b^2, b\cdot P, (b\cdot P)R(\hat{\zeta}^2)/m_N^2, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)$

Certain $x$-integrated TMDs in Fourier space directly correspond to the amplitudes $\widetilde{A}_{iB}$ evaluated at $b\cdot P = 0$ :

$$\tilde{f}_1^{[1](0)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P) = 2\,\widetilde{A}_{2B}(-\boldsymbol{b}_{\mathrm{T}}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)/\widetilde{\mathcal{S}}(b^2; \ldots)\,,$$
$$\tilde{f}_{1T}^{\perp[1](1)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P) = -2\,\widetilde{A}_{12B}(-\boldsymbol{b}_{\mathrm{T}}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)/\widetilde{\mathcal{S}}(b^2; \ldots)\,,$$

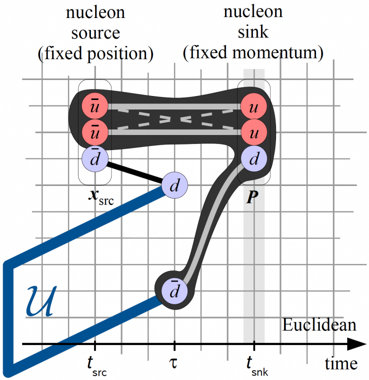
# Generalized Sivers shifts from amplitudes

All other renormalization and soft factor related dependences cancel out in the ratio.

- $\langle \boldsymbol{k}_y \rangle^{\text{Sivers}} = \langle \boldsymbol{k}_y \rangle_{TU}$ is T-odd, it describes a feature of the transverse momentum distribution of (unpolarized) quarks in a transversely polarized proton.

$$\begin{aligned}
\langle \boldsymbol{k}_y \rangle_{TU}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \eta v \cdot P) \;&\equiv\; m_N \frac{\tilde{f}_{1T}^{\perp [1](1)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v \cdot P)}{\tilde{f}_1^{[1](0)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v \cdot P)} \\
&= -m_N \frac{\widetilde{A}_{12B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N \hat{\zeta})^2, \eta v \cdot P)}{\widetilde{A}_{2B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N \hat{\zeta})^2, \eta v \cdot P)} \\
&\xrightarrow{\boldsymbol{b}_\mathrm{T}^2 = 0} \left. \frac{\int dx \int d^2 \boldsymbol{k}_\mathrm{T}\, \boldsymbol{k}_y \; \Phi^{[\gamma^+]}(x, \boldsymbol{k}_\mathrm{T}, P, S; \ldots)}{\int dx \int d^2 \boldsymbol{k}_\mathrm{T} \quad \Phi^{[\gamma^+]}(x, \boldsymbol{k}_\mathrm{T}, P, S; \ldots)} \right|_{\boldsymbol{S}_\mathrm{T} = (1,0)}
\end{aligned} \tag{28}$$

# Lattice Setup [4]




- Evaluate directly $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2}\ \langle P, S|\ \bar{q}(0)\,\Gamma\, \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b)\ |P, S\rangle$
- Euclidean time: Place entire operator at one time slice, i.e., $b$, $\eta v$ purely spatial.
- Extrapolate $\eta \longrightarrow \infty$, $\hat{\zeta} \longrightarrow \infty$ numerically.

[4]Figure Credits: Dr. Engelhardt (NMSU)

# Numerical Results

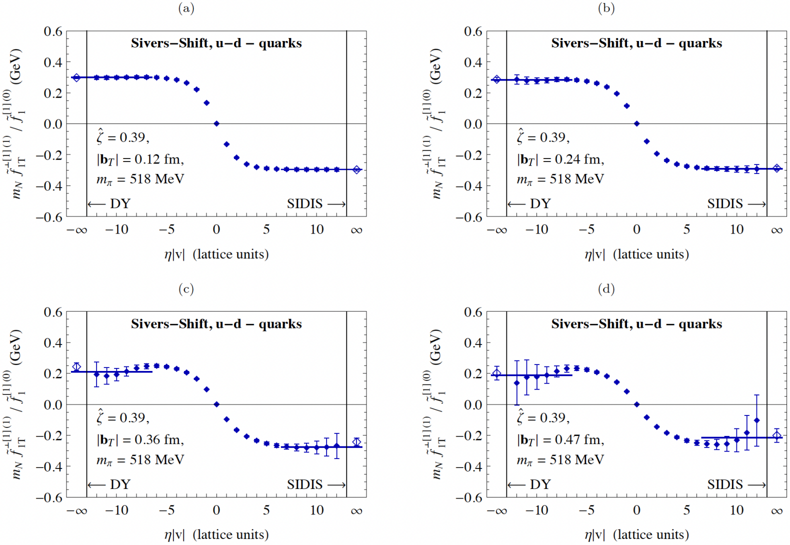



Figure 10: Extraction of the generalized Sivers shift on the lattice with $m_\pi = 518\text{MeV}$ *(arXiv:1111.4249v2 [hep-lat])*

# Numerical Results

## Results: Sivers shift

Dependence of SIDIS limit on $|b_T|$

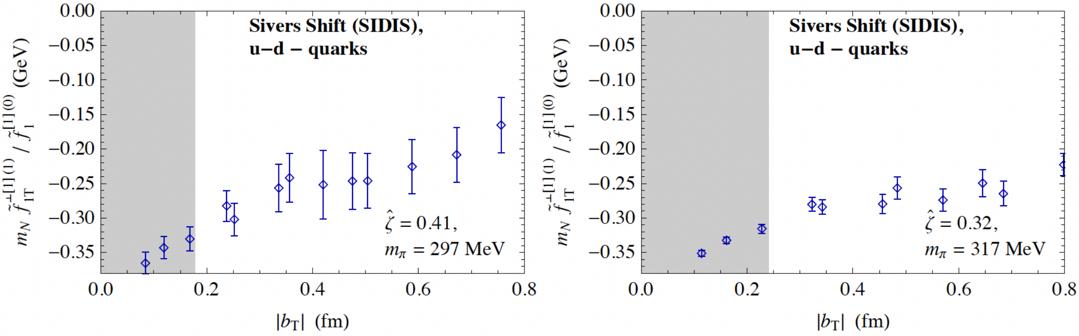



Figure 11: Generalized Sivers shift as a function of the quark separation $|\boldsymbol{b}_\mathrm{T}|$ for the SIDIS case ($|\eta v| = \infty$). *arXiv:2301.06118 [hep-lat]*

# Numerical Results

## Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$

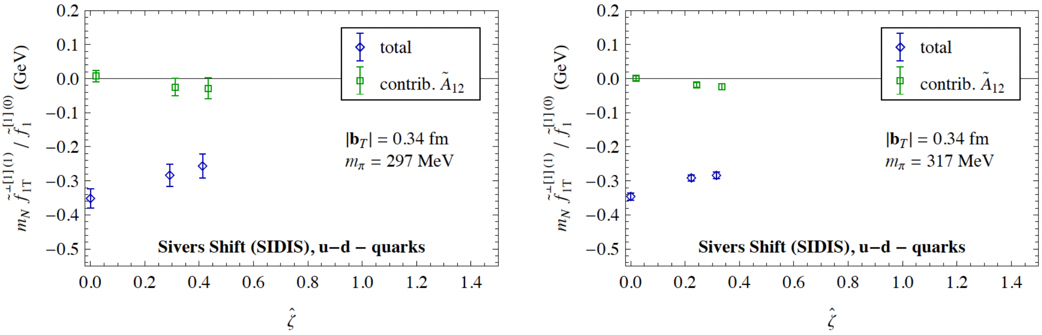



Figure 12: we show the $\hat{\zeta}$-dependence of the generalized Sivers shift, depicting both the full result and the result obtained with just $\tilde{A}_{12}$ in the numerator. *arXiv:2301.06118 [hep-lat]*

# Few More Numerical Results


- M. Engelhardt, *et al.*, PoS **LATTICE2022**, 103 (2023), [arXiv:2301.06118 [hep-lat]].
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## My PhD work: Extension to include the dependence on $x = \frac{k^+}{P^+}$

$$\frac{1}{2} \langle P, S| \ \bar{q}(0) \ \gamma^+ \ \mathcal{U}[\mathcal{C}_b] \ q(b) \ |P, S\rangle = 2P^+ \left( \widetilde{A}_{2B} + i m_N \epsilon_{ij} \boldsymbol{b}_i \boldsymbol{S}_j \ \widetilde{A}_{12B} \right) \tag{29}$$

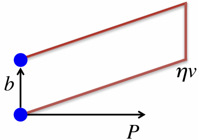
$$\Longrightarrow \ \langle \boldsymbol{k}_y \rangle_{TU} (\boldsymbol{b}_\mathrm{T}^2, x, \hat{\zeta}, \eta v{\cdot}P) \ \equiv \ m_N \frac{\tilde{f}_{1T}^{\perp(1)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v{\cdot}P)}{\tilde{f}_1^{(0)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v{\cdot}P)} \tag{30}$$

$$= -m_N \frac{\int d(b \cdot P) e^{ix(b\cdot P)} \widetilde{A}_{12B}(b^2, b{\cdot}P, (b{\cdot}P)R(\hat{\zeta}^2)/m_N^2, -1/(m_N\hat{\zeta})^2, \eta v{\cdot}P)}{\int d(b \cdot P) e^{ix(b\cdot P)} \widetilde{A}_{2B}(b^2, b{\cdot}P, (b{\cdot}P)R(\hat{\zeta}^2)/m_N^2, -1/(m_N\hat{\zeta})^2, \eta v{\cdot}P)} \tag{31}$$

- The range of accessible $b \cdot P$ is limited:

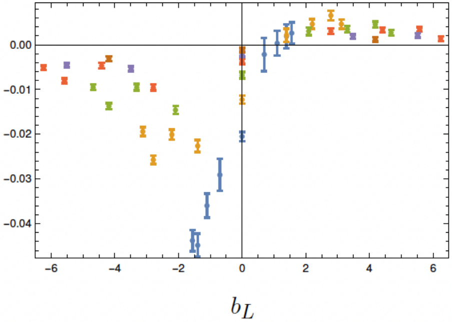
$$\boxed{\frac{v{\cdot}b}{v{\cdot}P} = b{\cdot}P \frac{R(\hat{\zeta}^2)}{m_N^2} ,} \qquad \because (b^+ = 0 \text{ and } v_T = \mathbf{P}_T = 0) \tag{32}$$

where $R(\hat{\zeta}^2) \equiv 1 - \sqrt{1 + \hat{\zeta}^{-2}} = \frac{m_N^2}{v\cdot P} \frac{v^+}{P^+}$.



# My PhD work: Extension to include the dependence on $x = \frac{k^+}{P^+}$

**Extract $b_L$-even component of imaginary part of $\gamma^+$ correlator**



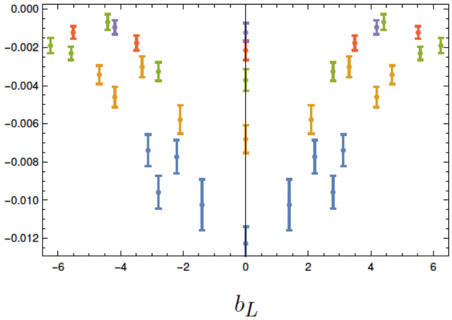




Figure 13: [5] $\left[\frac{1}{2}\langle P,S|\ \bar{q}(0)\ \gamma^+\ \mathcal{U}[\mathcal{C}_b]\ q(b)\ |P,S\rangle = 2P^+\left(\widetilde{A}_{2B} + im_N\epsilon_{ij}\boldsymbol{b}_i\boldsymbol{S}_j\ \widetilde{A}_{12B}\right)\right]$

[5] PDFLattice 2019: M. Engelhardt

# My PhD work: Extension to include the dependence on $x = \frac{k^+}{P^+}$

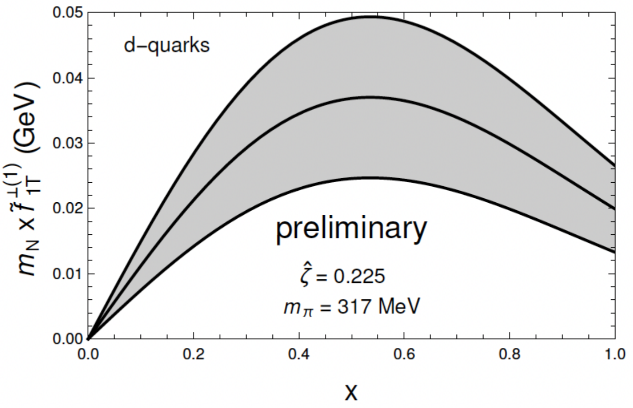



Figure 14: Nucleon SIDIS $d$-quark generalized Sivers shift as a function of momentum fraction $x$, multiplied by $x$ [6,7]

[6] "TMD Handbook." arXiv:2304.03302 [hep-ph].

[7] M. Engelhardt, J. R. Green, S. Krieg, S. Meinel, J. Negele, A. Pochinsky et al., to be published

# Conclusions

- It is feasible to obtain the $x$-dependence of TMD ratios: **Sivers shift**
- Inspite of constraints $\frac{v\cdot b}{v\cdot P} = b\cdot P\,\frac{R(\hat{\zeta}^2)}{m_N^2}$, it is possible to improve the analysis