PHYS-520 Quantum Computing (Spring 2024)
Final Project
Complexity, and AdS/CFT

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Abstract

I present a comprehensive overview of the AdS/CFT (Anti de-Sitter/Conformal Field Theories) correspondence, highlighting the non-preservation of "complexity" between AdS space (bulk) and conformal boundary. Furthermore, I provide a heuristic overview of the quantum circuit complexity of states and the relationship between certain conceptual puzzles in quantum gravity and knowing whether certain states and operations had exponential quantum circuit complexity[1].

1 Introduction

In general, quantum circuit complexity measures or quantifies the amount of resources that we need to implement an operation. Every unitary can be exactly implemented using finitely many two local gates; in particular, we need at most 4^n gates for n-qubit circuit because we need 4^n complex numbers to specify n-qubit quantum state. The quantum notion of circuit complexity also makes some astonishing appearances in actual physical systems, such as in AdS/CFT correspondence. This has opened up new avenues for exploring the fundamental nature of the universe and has the potential to deepen our understanding of quantum mechanics and gravity.

The AdS/CFT (Anti de-Sitter/Conformal Field Theories) correspondence was originally proposed by Juan M. Maldacena [2] in 1997, which is a conjectured duality between a theory of quantum gravity and anti-sitter space (a quantum mechanical CFT). The conjecture is that these two theories are connected by a dictionary, which maps states in one theory to states in the other theory and operators in one theory to operators of the other, so this dictionary is a sort of duality between these two theories. In 2014, Leonard Susskind discovered a paradox related to the growth of wormholes [3]. The paradox suggests that certain gravitational systems do not equilibrate quickly. For instance, the volume of a wormhole on the AdS side takes an exponential amount of time to equilibrate. However, on the CFT side, the expectation values of observables equilibrate rapidly. Therefore, Susskind raised the question, *"What is the corresponding quantity on the quantum mechanical CFT to the wormhole volume on the AdS side?"*

In the upcoming section, I briefly introduce AdS/CFT and why the wormhole volume is postulated to correspond to the quantum circuit complexity. Finally, I mention a few recent advances in this field.

2 AdS/CFT

The AdS/CFT correspondence [2] postulates a duality between quantum gravity in anti-de-Sitter (AdS) spacetime and Conformal Field Theories (CFTs).

Anti-de-Sitter (AdS) is a hyper spacetime with a negative cosmological constant in $d + 1$ dimensions. We can conformally map hyperbolic space to a finite disc in the plane to visualize this. An AdS spacetime can be visualized as a solid, infinitely long cylinder where time runs along its infinite length. Every horizontal section of the cylinder is a hyperbolic space representing an equal time slice. The interior of the cylinder is called the bulk, and its boundary is the boundary of AdS spacetime. In the AdS/CFT correspondence, we explore the physics of a quantum gravitational theory on AdS spacetime.

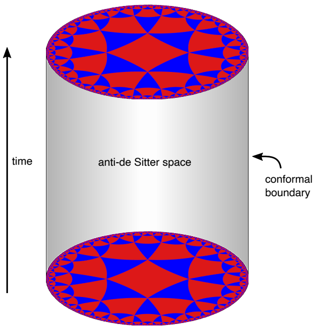

Figure 1: Three-dimensional anti-de Sitter space is like a stack of hyperbolic disks, each one representing the state of the universe at a given time. The resulting spacetime looks like a solid cylinder [4].

Conversely, Conformal Field Theory (CFT) is a quantum field theory with powerful symmetry $SO(2,d)$ called conformal invariance. The action of CFT is invariant under conformal transformation; from Noether's theorem, we can find all the generators of all conformal transformations, and the commutation relation among all the conformal generators obeys $SO(2,d)$ algebra, which is the same isometry group of Anti-de-Sitter (AdS) space. In the AdS/CFT correspondence, the consequence of the $SO(2,d)$ isometry of AdS_{d+1} is the conformal invariance of the dual field theory [5].

We can discretize the space and time on which the CFT is defined and idealize it as a quantum circuit acting on a finite number of qubits:

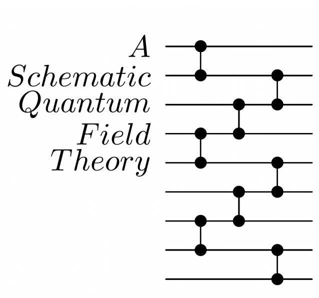

Figure 2: A schematic quantum CFT [1]

It is important to note that CFT interactions occur at all times and between every pair of neighboring points. Therefore, the above diagram is just a schematic diagram.

The match in symmetries of both AdS and CFT gives rise to the AdS/CFT correspondence. This correspondence can be thought of as a dictionary, where each operator in the CFT that generates objects and states must correspond to a quantum field in the AdS theory; for example, the graviton (quantum particle for gravity [not proven to exist yet]) field in AdS space is associated with the stress-energy tensor of the dual CFT.

After establishing a dictionary that matches the objects and states on both the Anti-de Sitter space (AdS) and conformal field theory (CFT) sides, the AdS/CFT duality asserts that the same result can be obtained by performing the calculation on either side of the correspondence. One more important thing to note is that this AdS/CFT correspondence is still conjectured and has not yet been proven for the general case.

3 Wormhole-volume & complexity

Susskind [3] pointed out that the map between bulk and boundary doesn't seem to be complexity-preserving. Simple states on one side (like the volume of a wormhole) can map onto complex states on the CFT side. Let's consider two separate regions in the bulk connected by a wormhole from quantum gravity (quantum general relativity). The time evolution will make the volume of this wormhole grow linearly for an exponentially long time. In the far future, the volume will reach equilibrium when it exhausts all possible finite Hilbert space. Suppose we wait a doubly exponentially long time. In that case, the system will undergo recurrence, so the volume will dip but return to equilibrium (but this won't happen until a very, very long time).

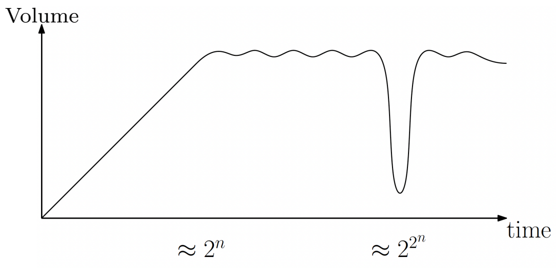

Figure 3: Volume of the wormhole in the AdS side [6, 7]

When we look at the CFT dual, we will get some initial state in some spatial slice, which is Thermofield Double State $|\psi_0\rangle$ is maximally entangled between two identical Hilbert spaces:

$$|\psi_0\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in(0,1)^n} |y\rangle \otimes |y\rangle \tag{3.1}$$

and the time evolution is generated by the Hamiltonian (time evolution on the AdS side will make the volume of the wormhole longer and longer) in CFT dual

$$H = \sum_{i<j} h_{ij} \tag{3.2}$$

where h_{ij} is some form of Hamiltonian. After time t (on one side, we time evolve forward, and on the other side, we evolve backward)

$$|\psi_t\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in(0,1)^n} e^{-iHt} |y\rangle \otimes e^{iHt} |y\rangle \tag{3.3}$$

$$|\psi_t\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in(0,1)^n} |y\rangle \otimes U^t |y\rangle \tag{3.4}$$

where, $U = (e^{-iHt})^T e^{iHt}$. If we apply $UU\cdots U|0\rangle^{\otimes n}$, we'll get something that very quickly "thermalizes." But on the AdS side, the wormhole undergoes lengthening instead of rapid thermalization.

Then Susskind [3] raised the question: "What function of this state in CFT $f(|\psi_t\rangle)$ captures the property of the wormhole volume in the AdS side?". The proposal was the complexity of the quantum circuit $\mathcal{C}(|\psi_t\rangle)$ of the state $|\psi_t\rangle$

$$\implies f(|\psi_t\rangle) = \mathcal{C}(|\psi_t\rangle) \tag{3.5}$$

Because one can generate the CFT state $|\psi_t\rangle$ from n simple initial state (Bell pairs) by simply applying $U = (e^{-iHt})^T e^{iHt}$ repeatedly t times, and also upper bound 2^n times the polynomial in n because circuit complexity of every n qubit state is upper bounded,

$$\mathcal{C}(|\psi_t\rangle) \leq min\{t, 2^n\} n^{\mathcal{O}(1)} \tag{3.6}$$

To summarize, one can expect the state complexity to behave as shown below. One can also

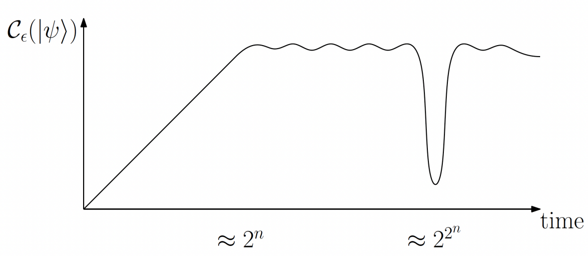

Figure 4: quantum circuit complexity $\mathcal{C}(|\psi_t\rangle)$ of the state $|\psi_t\rangle$ [1, 6]

show that near $t \sim 2^{2^n}$, we'll reach a t for which U^t is very close to the identity, in which case the state has low complexity so there is a dip at $t \sim 2^{2^n}$ [1, 6].

If we multiply U^{-1} instead of U, the wormhole shrinks at the same speed it previously grew. This is what happens to the state complexity; please refer [1]. Also, if we apply some other unitary $\tilde{U} \not\approx U$ before we switch to U^{-1} from U, The prediction from quantum gravity is that the wormhole would continue growing instead of shrinking, which aligns with the state complexity [8].

3.1 Some Recent Advances

- Haferkamp et al. [9] have proven a conjecture proposed by Brown and Susskind for random quantum circuits: from their finding, a local random circuit's quantum complexity grows linearly in the number of gates until reaching a value exponential in the system size.
- Lower-bounding quantum complexity is a long-standing open problem in quantum information theory. The core difficulty is that the gates performed early in a circuit may partially cancel with gates performed later. Please refer to Aaronson's discussion on bower bounds in [1].

3.2 Conclusion

The concept that circuit complexity is equivalent to the volume of wormholes is still considered to be a conjecture that has not yet been proven. This complexity dictionary may even limit what one can learn about wormhole interiors from CFT [10]. But for now, circuit complexity has established itself as a valuable tool in studying the AdS/CFT correspondence.

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