IE303
Operations Research I

Week 11
Spring, 2025
Dr. Ozge Buyukdagli

THE TRANSPORTATION PROBLEM

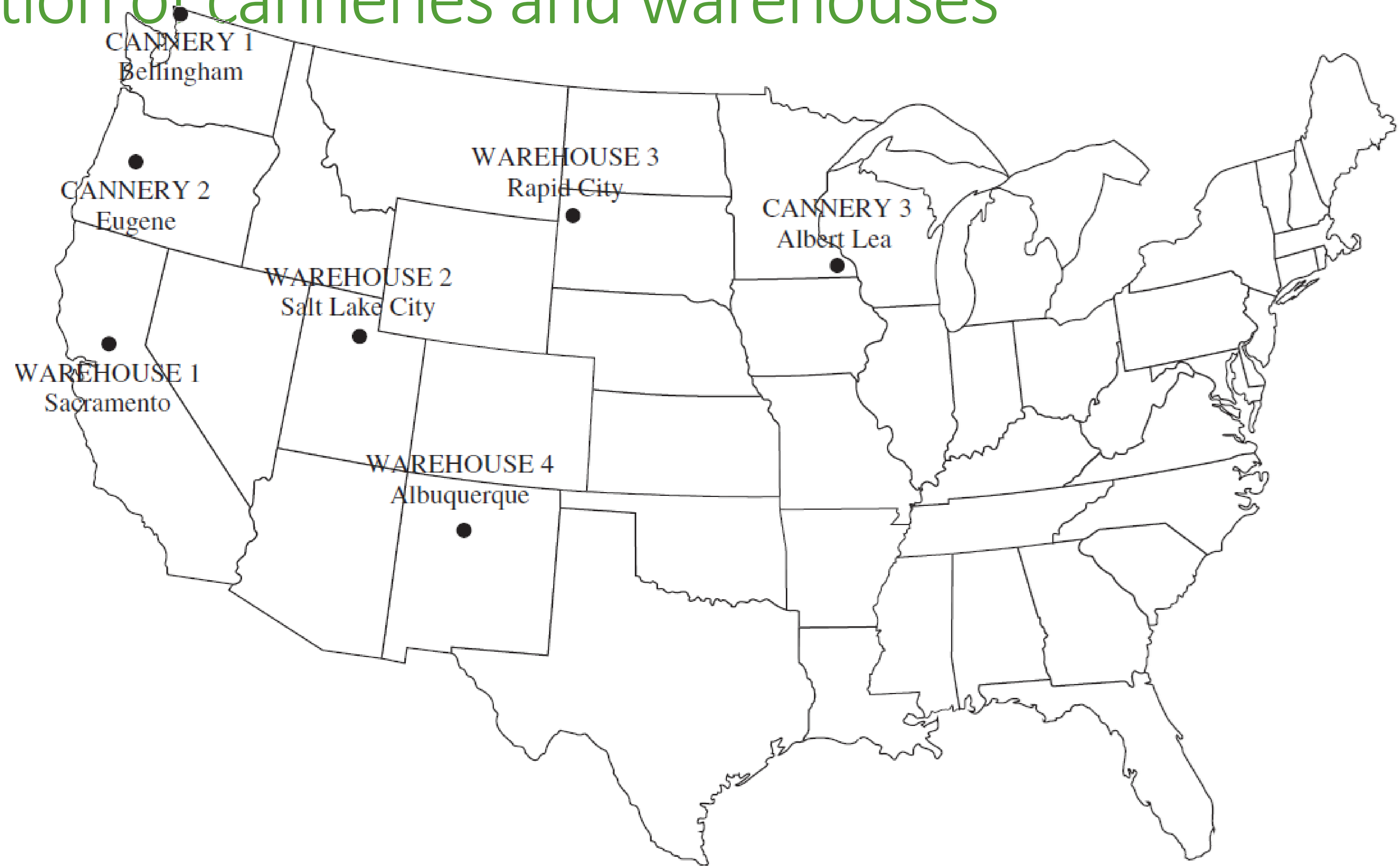
Prototype Example

One of the main products of the P & T COMPANY is canned peas.

The peas are prepared at three canneries (near **Bellingham**, Washington; **Eugene**, Oregon; and **Albert Lea**, Minnesota) and then shipped by truck to four distributing warehouses in the western United States (**Sacramento**, California; **Salt Lake City**, Utah; **Rapid City**, South Dakota; and **Albuquerque**, New Mexico),

The problem now is to determine which plan for assigning these shipments to the various cannery-warehouse combinations would *minimize the total shipping cost.*

Location of canneries and warehouses



≥ ≤

Shipping data for P & T Co.

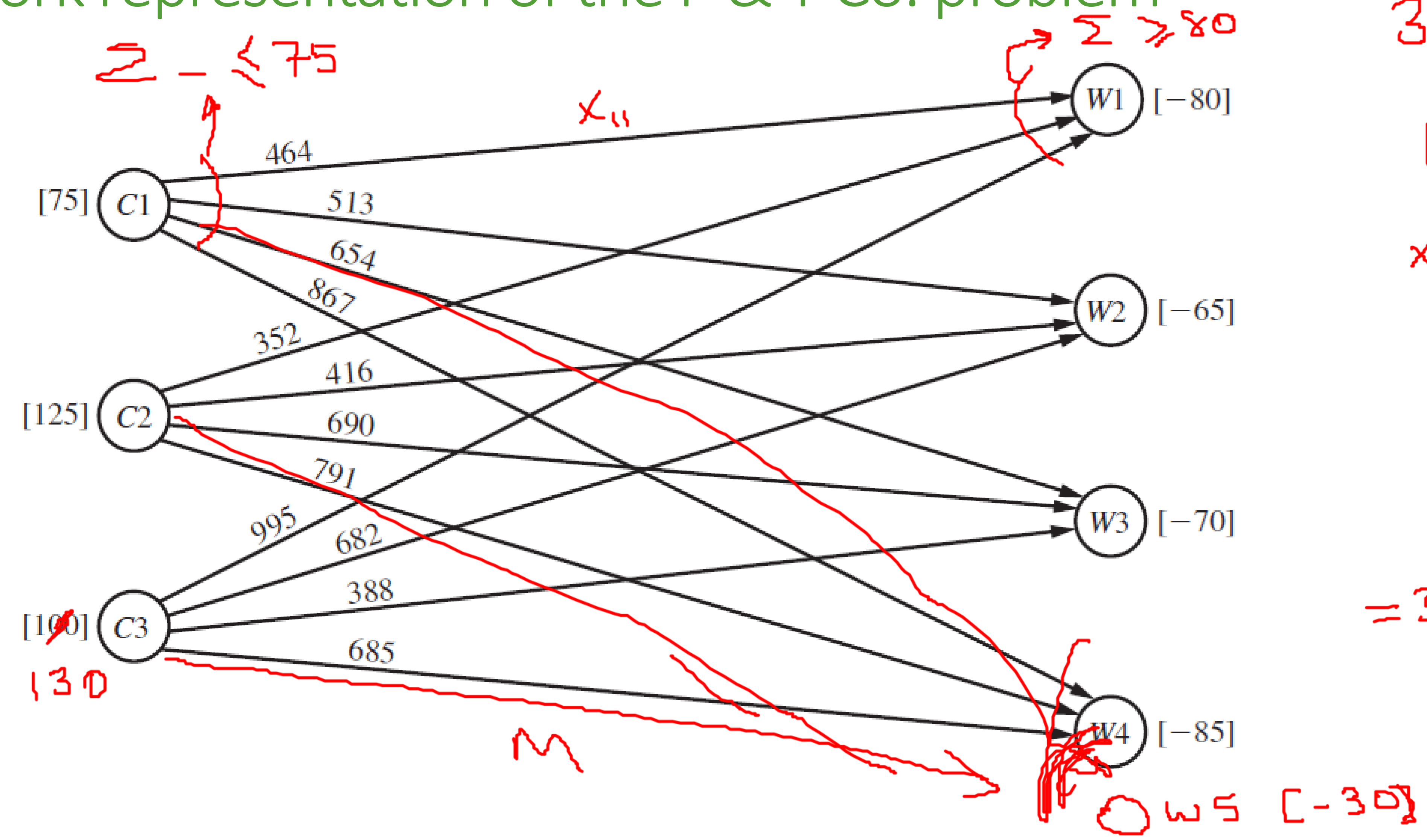
≠

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	~~300~~

330

300

Network representation of the P & T Co. problem



3x4

12 dv.

x_{ij}

= 30

LP Model

let x_{ij} $(i = 1, 2, 3;\ j = 1, 2, 3, 4)$ be the number of truckloads to be shipped from cannery i to warehouse j. Thus, the objective is to choose the values of these 12 decision variables (the x_{ij}) so as to

$$\text{Minimize} \quad Z = 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + 416x_{22} + 690x_{23} + 791x_{24} + 995x_{31} + 682x_{32} + 388x_{33} + 685x_{34},$$

$+ M x_{15}$

$+ M x_{25}$

$+ M x_{35}$

subject to the constraints

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &= 75 \\
x_{21} + x_{22} + x_{23} + x_{24} &= 125 \\
x_{31} + x_{32} + x_{33} + x_{34} &= 100 \\
x_{11} + x_{21} + x_{31} &= 80 \\
x_{12} + x_{22} + x_{32} &= 65 \\
x_{13} + x_{23} + x_{33} &= 70 \\
x_{14} + x_{24} + x_{34} &= 85
\end{aligned}$$

350

$= 125$

and

$$x_{ij} \geq 0 \qquad (i = 1, 2, 3;\ j = 1, 2, 3, 4).$$

cont

Constraint coefficients for P & T Co.

Coefficient of:

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} x_{11} & x_{12} & x_{13} & x_{14} & x_{21} & x_{22} & x_{23} & x_{24} & x_{31} & x_{32} & x_{33} & x_{34} \end{array} \\
\mathbf{A} = & \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 \\
1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 & & & & 1
\end{array}\right]
\begin{array}{l}
\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Cannery constraints} \\
\left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{Warehouse constraints}
\end{array}
\end{array}
$$

it is the special structure in the pattern of these coefficients that distinguishes this problem as a transportation problem, not its context.

The Transportation Problem Model

the general transportation problem is concerned with distributing any commodity from any group of supply centers, called **sources**,

to any group of receiving centers, called **destinations**, in such a way as to *minimize the total distribution cost*.

TABLE 9.4 Terminology for the transportation problem

Prototype Example	General Problem
Truckloads of canned peas	Units of a commodity
Three canneries	*m* sources
Four warehouses	*n* destinations
Output from cannery *i*	Supply s_i from source *i*
Allocation to warehouse *j*	Demand d_j at destination *j*
Shipping cost per truckload from cannery *i* to warehouse *j*	Cost c_{ij} per unit distributed from source *i* to destination *j*

The Transportation Problem Model

The requirements assumption: Each source has a fixed *supply* of units, where this entire supply must be distributed to the destinations. (We let s_i denote the number of units being supplied by source i, for $i = 1, 2, \ldots, m$.) Similarly, each destination has a fixed *demand* for units, where this entire demand must be received from the sources. (We let d_j denote the number of units being received by destination j, for $j = 1, 2, \ldots, n$.)

The feasible solutions property: A transportation problem will have feasible solutions if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

Fortunately, these sums are equal for the P & T Co.

The Transportation Problem Model

In some real problems, the supplies actually represent *maximum amounts* (rather than fixed amounts) to be distributed.

it is possible to **reformulate** the problem so that they then fit this model by introducing a **dummy destination or a dummy source** to take up the slack between the actual amounts and maximum amounts being distributed.

The Transportation Problem Model

The cost assumption: The cost of distributing units from any particular source to any particular destination is *directly proportional* to the number of units distributed. Therefore, this cost is just the *unit cost* of distribution *times* the *number of units distributed.* (We let c_{ij} denote this unit cost for source i and destination j.)

The only data needed for a transportation problem model are the supplies, demands, and unit costs.

These are the *parameters of the model.*

Parameter table for the transportation problem

		Cost per Unit Distributed				
		Destination				
		1	2	...	n	Supply
Source	1	c_{11}	c_{12}	...	c_{1n}	s_1
	2	c_{21}	c_{22}	...	c_{2n}	s_2
	⋮	...	...	...	...	⋮
	m	c_{m1}	c_{m2}	...	c_{mn}	s_m
Demand		d_1	d_2	...	d_n	

Since a transportation problem can be formulated simply by either filling out a parameter table or drawing its network representation, it is not necessary to write out a formal mathematical model for the problem.

Model of the general transportation problem

$$\text{Minimize} \qquad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i \qquad \text{for } i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j.$$

Integer solutions property: For transportation problems where every s_i and d_j have an integer value, all the basic variables (allocations) in *every* basic feasible (BF) solution (including an optimal one) also have *integer* values.

An Example with a Dummy Destination

The NORTHERN AIRPLANE COMPANY builds commercial airplanes for various airline companies around the world.

The last stage in the production process is to produce the jet engines and then to install them (a very fast operation) in the completed airplane frame.

The company has been working under some contracts to deliver a considerable number of airplanes in the near future, and the production of the jet engines for these planes must now be scheduled for the next four months.

The NORTHERN AIRPLANE COMPANY

$(X_1 - D_1) \cdot sc.$

$X_1 = 15$
$X_2 = 20$
$X_3 = 30$
$X_4 = 10$

To meet the contracted dates for delivery, the company must supply engines for installation in the quantities indicated in the second column.

Because of the variations in production costs, it may well be worthwhile to produce some of the engines a month or more before they are scheduled for installation, and this possibility is being considered.

The drawback is that such engines must be stored until the scheduled installation at a storage cost of $15,000 per month

$X_1 \cdot 1.08 + X_2 \cdot 1.11$

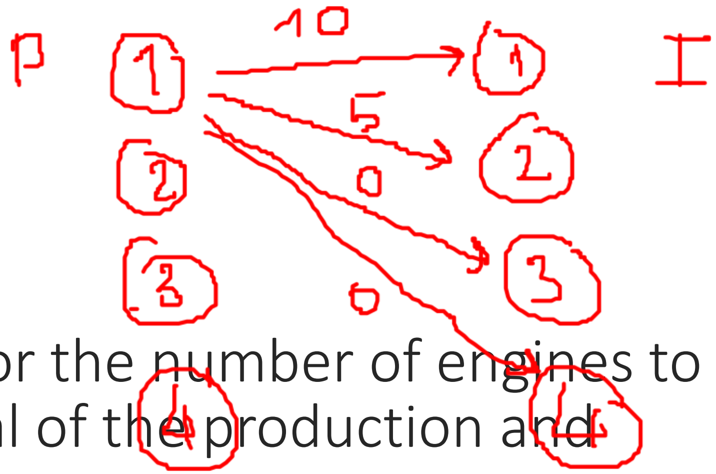
TABLE 9.7 Production scheduling data for Northern Airplane Co.

Month	Scheduled Installations	Maximum Production	Unit Cost* of Production	Unit Cost* of Storage
1	10	25	1.08	0.015
2	15	35	1.11	0.015
3	25	30	1.10	0.015
4	20	10	1.13	

*Cost is expressed in millions of dollars.

P_i S I_1

Formulation.



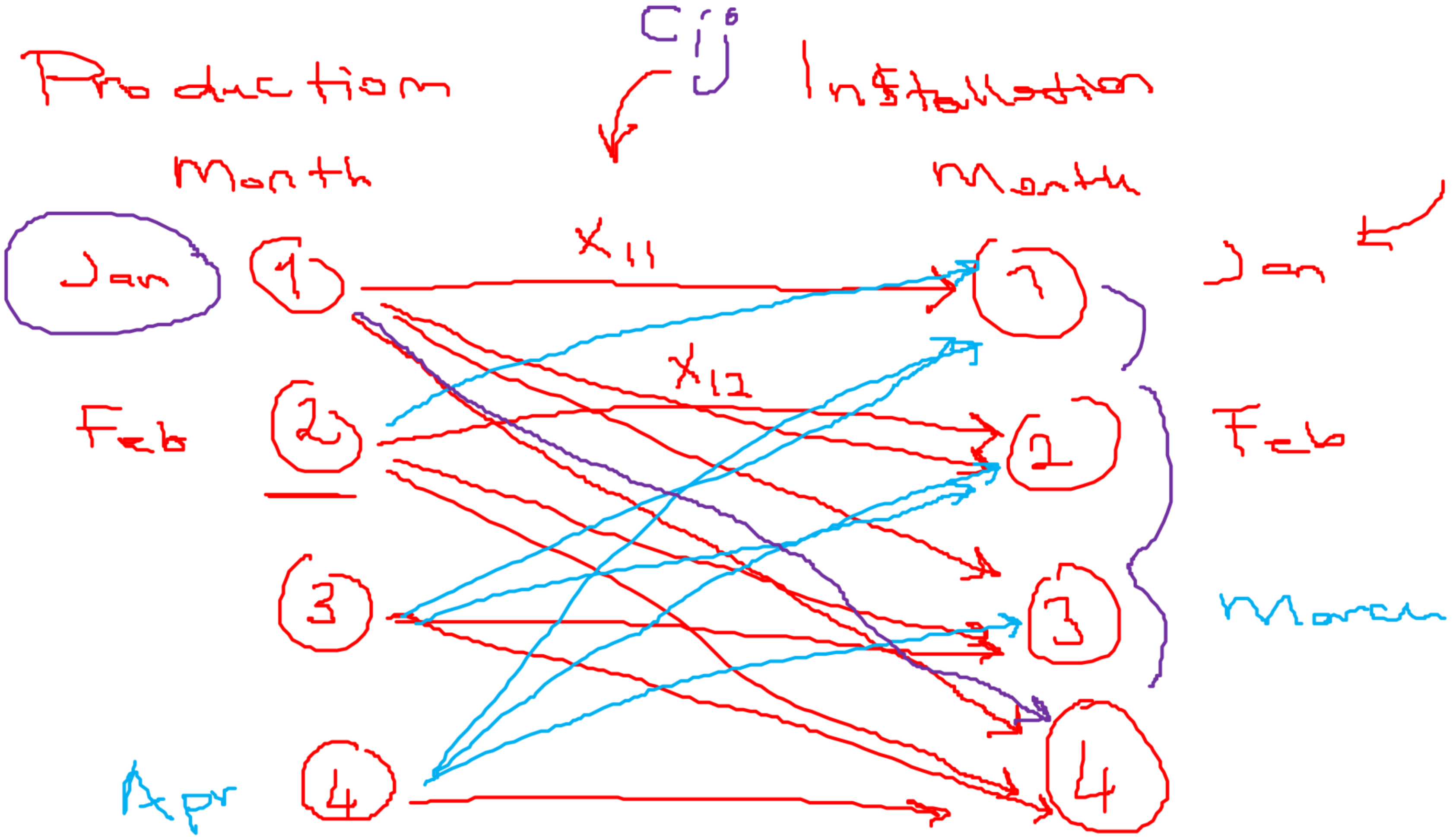
The production manager wants a schedule developed for the number of engines to be produced in each of the four months so that the total of the production and storage costs will be minimized.

One way: let *x_j* be the number of jet engines to be produced in month *j,*

for j = 1, 2, 3, 4.

- This does *not fit the transportation problem type.*

On the other hand, by adopting a different viewpoint, we can instead formulate the problem as a transportation problem that requires *much less effort to solve.*



x_{ij} = # of engi prod in month i and inst. in month j

X impossible

j < i ?

m

when prod is after install.

$$c_{ij} = \begin{cases} p_i \cdot x_{ij} & i = j \\ (j-i) \cdot 5 \cdot x_{ij} + p_i x_{ij} & i < j \end{cases}$$

Formulation

Because the units being distributed are jet engines, each of which is to be scheduled for production in a particular month and then installed in a particular (perhaps different) month

$$\text{Source } i = \text{production of jet engines in month } i \ (i = 1, 2, 3, 4)$$

$$\text{Destination } j = \text{installation of jet engines in month } j \ (j = 1, 2, 3, 4)$$

$$x_{ij} = \text{number of engines produced in month } i \text{ for installation in month } j$$

$$c_{ij} = \text{cost associated with each unit of } x_{ij}$$

$$= \begin{cases} \text{cost per unit for production and any storage} & \text{if } i \leq j \\ ? & \text{if } i > j \end{cases}$$

$$s_i = ?$$

$$d_j = \text{number of scheduled installations in month } j.$$

Incomplete Parameter Table

+30

■ **TABLE 9.8** Incomplete parameter table for Northern Airplane Co.

		Cost per Unit Distributed				
		Destination				
		1	2	3	4	Supply
Source	1	1.080	1.095	1.110	1.125	? 25
	2	? M	1.110	1.125	1.140	? 35
	3	? M	? M	1.100	1.115	? 30
	4	? M	? M	? M	1.130	? 10
Demand		10	15	25	20	

0.615 0.615 5 100 70

I want these assignments to be impossible

Not obvious, not fixed..

Formulation

it is necessary to assign some fixed number to every entry in the table, including those in the supply column, to have a transportation problem.

The only change from the standard model for the transportation problem is that these constraints are in the form of inequalities instead of equalities.

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 25,$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 35,$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 30,$$
$$x_{41} + x_{42} + x_{43} + x_{44} \leq 10.$$

To convert these inequalities to equations in order to fit the transportation problem model, we use the familiar device of *slack variables:*

are allocations to a single **dummy destination** that represent the *unused production capacity in the respective months.*

Complete Parameter Table

S = D

Because the demand for the dummy destination is the total unused capacity, this demand is:

(25 + 35 + 30 + 10) - (10 + 15 + 25 + 20) = 30.

With this demand included, the sum of the supplies now equals the sum of the demands.

TABLE 9.9 Complete parameter table for Northern Airplane Co.

		Cost per Unit Distributed					
		Destination					
		1	**2**	**3**	**4**	**5(*D*)**	**Supply**
Source	1	1.080	1.095	1.110	1.125	0	25
	2	*M*	1.110	1.125	1.140	0	35
	3	*M*	*M*	1.100	1.115	0	30
	4	*M*	*M*	*M*	1.130	0	10
Demand		10	15	25	20	30	

THE ASSIGNMENT PROBLEM

The **assignment problem** is a special type of linear programming problem where **assignees** are being assigned to perform **tasks**.

To fit the definition of an assignment problem, these kinds of applications need to be formulated in a way that satisfies the following assumptions.

1. The number of assignees and the number of tasks are the same. (This number is denoted by n.)
2. Each assignee is to be assigned to exactly *one* task.
3. Each task is to be performed by exactly *one* assignee.
4. There is a cost c_{ij} associated with assignee i ($i = 1, 2, \ldots, n$) performing task j ($j = 1, 2, \ldots, n$).
5. The objective is to determine how all n assignments should be made to minimize the total cost.

Any problem satisfying all these assumptions can be solved extremely efficiently by algorithms designed specifically for assignment problems.

The JOB SHOP COMPANY

Prototype Example

The JOB SHOP COMPANY has purchased three new machines of different types.

There are four available locations in the shop where a machine could be installed.

Some of these locations are more desirable than others for particular machines because of their proximity to work centers that will have a heavy work flow to and from these machines.

Therefore, the objective is to assign the new machines to the available locations to minimize the total cost of materials handling.

The JOB SHOP COMPANY

The estimated cost in dollars per hour of materials handling involving each of the machines is given in Table.

Location 2 is not considered suitable for machine 2, so no cost is given for this case.

min

$\sum c_{ij} \cdot x_{ij}$

s.t

$\sum_{j} x_{ij} = 1 \quad \forall i$

$\sum_{i} x_{ij} \leq 1 \quad \forall j$ → location

$x_{ij} = \begin{cases} 1 \\ 0 \end{cases}$

TABLE 9.24 Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	2	3	4
Machine	1	13	16	12	11
	2	15	—	13	20
	3	5	7	10	6

c_{ij}

Final Cost Table

To formulate this problem as an assignment problem, we must introduce a *dummy machine for the extra location.*

Also, an extremely large cost M should be attached to the assignment of machine 2 to location 2 to prevent this assignment in the optimal solution.

TABLE 9.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	**2**	**3**	**4**
	1	13	16	12	11
Assignee	2	15	*M*	13	20
(Machine)	3	5	7	10	6
	4(*D*)	0	0	0	0

The Assignment Problem Model

The mathematical model for the assignment problem uses the following decision variables:

$$x_{ij} = \begin{cases} 1 & \text{if assignee } i \text{ performs task } j, \\ 0 & \text{if not,} \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Thus, each x_{ij} is a *binary variable*

Binary variables are important in OR for representing *yes/no decisions.*

In this case, the yes/no decision is:

Should assignee *i perform task j?*

We will see more on these, in Integer Programming Chapter.

The Assignment Problem Model

By letting Z denote the total cost, the assignment problem model is

$$\text{Minimize} \qquad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

each assignee is to perform exactly one task

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1, 2, \ldots, n,$$

each task to be performed by exactly one assignee.

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j$$
$$(x_{ij} \text{ binary}, \qquad \text{for all } i \text{ and } j).$$

Now compare this model (without the binary restriction) with the transportation problem model

In class exercise

9.3-4.* The coach of an age group swim team needs to assign swimmers to a 200-yard medley relay team to send to the Junior Olympics. Since most of his best swimmers are very fast in more than one stroke, it is not clear which swimmer should be assigned to each of the four strokes. The five fastest swimmers and the best times (in seconds) they have achieved in each of the strokes (for 50 yards) are

Stroke	Carl	Chris	David	Tony	Ken
Backstroke	37.7	32.9	33.8	37.0	35.4
Breaststroke	43.4	33.1	42.2	34.7	41.8
Butterfly	33.3	28.5	38.9	30.4	33.6
Freestyle	29.2	26.4	29.6	28.5	31.1

The coach wishes to determine how to assign four swimmers to the four different strokes to minimize the sum of the corresponding best times.

Conclusion

The general simplex method is a powerful algorithm that can solve surprisingly large versions of any of these problems.

However, some of these problem types have such simple formulations that they can be solved much more efficiently by *streamlined algorithms that* exploit their *special structure.*

These streamlined algorithms can cut down tremendously on the computer time required for large problems, and they sometimes make it computationally feasible to solve huge problems.

transportation simplex method.

- the Northwest Corner Rule
- Vogel's approximation

Hungarian algorithm
(for assignment)

INNOVATIVE USES OF BINARY VARIABLES IN MODEL FORMULATION

So far we have seen the *basic decisions, yes-or-no type.*

There are some other interesting ways to use them.

Lets see…

Either-Or Constraints

Consider the important case where a choice can be made between two constraints, so that *only one (either one) must hold.*

For example, there may be a choice as to which of two resources to use for a certain purpose, so that it is necessary for only one of the two resource availability constraints to hold mathematically.

$$\begin{array}{ll} \text{Either} & 3x_1 + 2x_2 \leq 18 \\ \text{or} & x_1 + 4x_2 \leq 16, \end{array}$$

at least one of these two inequalities must hold but not necessarily both.

How would you formulate this problem?

Either-Or Constraints

Either
$$3x_1 + 2x_2 \le 18$$
$$x_1 + 4x_2 \le 16 + M$$

or
$$3x_1 + 2x_2 \le 18 + M$$
$$x_1 + 4x_2 \le 16.$$

The key is that adding *M to the right-hand side of such constraints has the effect of* eliminating them, because they would be satisfied automatically by any solutions that satisfy the other constraints of the problem.

We may introduce auxiliary variables

$$3x_1 + 2x_2 \le 18 + My$$
$$x_1 + 4x_2 \le 16 + M(1 - y).$$

y must be either 0 or 1

M is an extremely large positive number

1. Should $x_1 + 4x_2 \le 16$ be selected as the constraint that must hold?
2. Should $3x_1 + 2x_2 \le 18$ be selected as the constraint that must hold?

K out of N Constraints Must Hold

Consider the case where the overall model includes a set of *N possible constraints such that* only some *K of these constraints must hold. (K<N)*

This case is a direct generalization of the preceding case, *K=1 and N=2.*

Denote the N possible constraints by

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &\le d_1 \\
f_2(x_1, x_2, \ldots, x_n) &\le d_2 \\
&\vdots \\
f_N(x_1, x_2, \ldots, x_n) &\le d_N.
\end{aligned}
$$

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &\le d_1 + My_1 \\
f_2(x_1, x_2, \ldots, x_n) &\le d_2 + My_2 \\
&\vdots \\
f_N(x_1, x_2, \ldots, x_n) &\le d_N + My_N \\
\sum_{i=1}^{N} y_i &= N - K,
\end{aligned}
$$

and

$$y_i \text{ is binary,} \qquad \text{for } i = 1, 2, \ldots, N,$$

Functions with *N Possible Values*

Consider the situation where a given function is required to take on any one of N given values. Denote this requirement by

$$f(x_1, x_2, \ldots, x_n) = d_1 \qquad \text{or} \qquad d_2, \ldots, \qquad \text{or} \qquad d_N.$$

The equivalent IP formulation of this requirement is the following:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{N} d_i y_i$$

$$\sum_{i=1}^{N} y_i = 1$$

and

$$y_i \text{ is binary,} \qquad \text{for } i = 1, 2, \ldots, N.$$

In this case, there are *N yes/no questions being asked, should d_i be the value chosen (i=1,2,...,N)? Because the y_i respectively represent these yes/no decisions, the second constraint makes them* ***mutually exclusive alternatives****.*

Functions with *N Possible Values*

To illustrate how this case can arise, reconsider the Wyndor Glass Co. problem

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$\begin{aligned} x_1 \quad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0$$

Eighteen hours of production time per week in Plant 3 currently is unused and available for the two new products *or for certain future products that will be ready for production soon.*

In order to leave any remaining capacity in usable blocks for these future products, management wants to impose the restriction that the production time used by the two current products be 6 *or 12 or 18 hours per week.*

$$3x_1 + 2x_2 = 6 \quad \text{or} \quad 12 \quad \text{or} \quad 18.$$

$$3x_1 + 2x_2 = 6y_1 + 12y_2 + 18y_3$$
$$y_1 + y_2 + y_3 = 1$$

and

$$y_1, y_2, y_3 \text{ are binary.}$$

This replacement yields a very tractable **MIP formulation.**

The Fixed-Charge Problem

It is quite common to incur a fixed charge or setup cost when undertaking an activity.

For example, such a charge occurs when a production run to produce a batch of a particular product is undertaken and the required production facilities must be set up to initiate the run.

In such cases, the total cost of the activity is the sum of a variable cost and the setup cost.

If this is the case, the *total cost of the activity*

$$f_j(x_j) = \begin{cases} k_j + c_j x_j & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0, \end{cases}$$

where x_j denotes the level of activity j ($x_j \geq 0$), k_j denotes the setup cost, and c_j denotes the cost for each incremental unit.

The Fixed-Charge Problem

To formulate the overall model, suppose that there are *n activities, each with the preceding* cost structure

$$\text{Minimize} \qquad Z = f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n),$$

$$Z = \sum_{j=1}^{n} (c_j x_j + k_j y_j),$$

where

$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0. \end{cases}$$

HOW CAN WE CONNECT xs and ys?

$$x_j \leq M y_j \qquad \text{for } j = 1, 2, \ldots, n$$

the *y_j can be viewed as contingent decisions*

SOME FORMULATION EXAMPLES

We now present a series of examples that illustrate a variety of formulation techniques with binary variables.

For the sake of clarity, these examples have been kept very small.

Making Choices When the Decision Variables Are Continuous

The Research and Development Division of the GOOD PRODUCTS COMPANY has developed three possible new products.

- **Restriction 1:** From the three possible new products, *at most two should be* chosen to be produced. Each of these products can be produced in either of two plants.
- **Restriction 2:** Just one of the two plants should be chosen to be the sole producer of the new products.

GOOD PRODUCTS COMPANY Data

	Production Time Used for Each Unit Produced			Production Time Available per Week
	Product 1	Product 2	Product 3	
Plant 1	3 hours	4 hours	2 hours	30 hours
Plant 2	4 hours	6 hours	2 hours	40 hours
Unit profit	5	7	3	(thousands of dollars)
Sales potential	7	5	9	(units per week)

The objective is to choose the products, the plant, and the production rates of the chosen products so as to maximize total profit.

The Model

Maximize $\quad Z = 5x_1 + 7x_2 + 3x_3,$

subject to

$$
\begin{aligned}
3x_1 + 4x_2 + 2x_3 &\le 30 \\
4x_1 + 6x_2 + 2x_3 &\le 40 \\
x_1 \qquad\qquad\quad &\le 7 \\
x_2 \qquad\quad &\le 5 \\
x_3 &\le 9
\end{aligned}
$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

restriction 1

The number of strictly positive decision variables (x_1, x_2, x_3) must be ≤ 2.

restriction 2

$$
\begin{aligned}
&\text{Either} \qquad 3x_1 + 4x_2 + 2x_3 \le 30 \\
&\text{or} \qquad\quad\; 4x_1 + 6x_2 + 2x_3 \le 40
\end{aligned}
$$

Formulation with Auxiliary Binary Variables

To deal with requirement 1

$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \text{ can hold (can produce product } j) \\ 0 & \text{if } x_j = 0 \text{ must hold (cannot produce product } j), \end{cases}$$

for $j = 1, 2, 3.$

$$\begin{aligned} &x_1 \le My_1 \\ &x_2 \le My_2 \\ &x_3 \le My_3 \\ &y_1 + y_2 + y_3 \le 2 \\ &y_j \text{ is binary,} \qquad \text{for } j = 1, 2, 3. \end{aligned}$$

Formulation with Auxiliary Binary Variables

To deal with requirement 2

$$y_4 = \begin{cases} 1 & \text{if } 4x_1 + 6x_2 + 2x_3 \leq 40 \text{ must hold (choose Plant 2)} \\ 0 & \text{if } 3x_1 + 4x_2 + 2x_3 \leq 30 \text{ must hold (choose Plant 1).} \end{cases}$$

$$3x_1 + 4x_2 + 2x_3 \leq 30 + My_4$$

$$4x_1 + 6x_2 + 2x_3 \leq 40 + M(1 - y_4)$$

y_4 is binary.

The complete model

$$\text{Maximize} \quad Z = 5x_1 + 7x_2 + 3x_3,$$

subject to

$$\begin{aligned} x_1 &\le 7 \\ x_2 &\le 5 \\ x_3 &\le 9 \\ x_1 - My_1 &\le 0 \\ x_2 - My_2 &\le 0 \\ x_3 - My_3 &\le 0 \\ y_1 + y_2 + y_3 &\le 2 \\ 3x_1 + 4x_2 + 2x_3 - My_4 &\le 30 \\ 4x_1 + 6x_2 + 2x_3 + My_4 &\le 40 + M \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0$$

$$y_j \text{ is binary}, \qquad \text{for } j = 1, 2, 3, 4.$$

This now is an MIP model,

with 3 variables *not required to be integer and*

4 binary variables