# IE303
# Operations Research I

Week 9

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# Integer Programming

In many practical problems, the decision variables actually make sense only if they have integer values.

- to assign people, machines, and vehicles to activities in integer quantities.

If requiring integer values is the only way in which a problem deviates from a linear programming formulation,

- then it is an ***integer programming*** *(IP)* ***problem.***

If only *some of the variables are required to have integer values (so the divisibility* assumption holds for the rest),

- this model is referred to as ***mixed integer programming (MIP).***

# Integer Programming

There have been numerous applications of integer programming that involve a direct extension of linear programming where the divisibility assumption must be dropped.

An important area: problems involving a number of interrelated "yes-or-no decisions."

Should we undertake a particular fixed project?

Should we make a particular fixed investment?

Should we locate a facility in a particular site?

$$x_j = \begin{cases} 1 & \text{if decision } j \text{ is yes} \\ 0 & \text{if decision } j \text{ is no.} \end{cases}$$

IP problems that contain only binary variables sometimes are called **binary integer programming (BIP) problems**

# IP - PROTOTYPE EXAMPLE

The CALIFORNIA MANUFACTURING COMPANY is considering expansion by building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities.

It also is considering building at most one new warehouse, but the choice of location is restricted to a city where a new factory is being built.

| Decision Number | Yes-or-No Question | Decision Variable | Net Present Value | Capital Required |
|---|---|---|---|---|
| 1 | Build factory in Los Angeles? | $x_1$ | \$9 million | \$6 million |
| 2 | Build factory in San Francisco? | $x_2$ | \$5 million | \$3 million |
| 3 | Build warehouse in Los Angeles? | $x_3$ | \$6 million | \$5 million |
| 4 | Build warehouse in San Francisco? | $x_4$ | \$4 million | \$2 million |
| | | | Capital available: | \$10 million |

The objective is to find the feasible combination of alternatives that maximizes the total net present value.

# The BIP Model

$$x_j = \begin{cases} 1 & \text{if decision } j \text{ is yes,} \\ 0 & \text{if decision } j \text{ is no,} \end{cases} \qquad (j = 1, 2, 3, 4).$$

Let

$Z$ = total net present value of these decisions.

$$Z = 9x_1 + 5x_2 + 6x_3 + 4x_4.$$

amount of capital expended on the four facilities cannot exceed $10 million.

$$6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10.$$

the last two decisions represent *mutually exclusive alternatives*

$$x_3 + x_4 \le 1$$

decisions 3 and 4 are *contingent on decisions* 1 and 2

$x_3 = 0$ if $x_1 = 0$ $\qquad$ $x_4 = 0$ if $x_2 = 0$

$$x_3 \le x_1$$

$$x_4 \le x_2$$

# The complete BIP model

$$\text{Maximize} \quad Z = 9x_1 + 5x_2 + 6x_3 + 4x_4,$$

subject to

$$\begin{aligned}
6x_1 + 3x_2 + 5x_3 + 2x_4 &\le 10 \\
x_3 + x_4 &\le 1 \\
-x_1 + x_3 &\le 0 \\
-x_2 + x_4 &\le 0 \\
x_j &\le 1 \\
x_j &\ge 0
\end{aligned}$$

and

$$x_j \text{ is integer,} \quad \text{for } j = 1, 2, 3, 4.$$

$$x_j \text{ is binary,} \quad \text{for } j = 1, 2, 3, 4.$$

# BIP Applications

Investment Analysis

Site Selection

Designing a Production and Distribution Network

Dispatching Shipments

Scheduling Interrelated Activities

Airline Applications

*And many others with yes/no questions*

# Covering All Characteristics

SOUTHWESTERN AIRWAYS needs to assign its crews to cover all its upcoming flights.

| | Feasible Sequence of Flights | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Flight** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** |
| 1. San Francisco to Los Angeles | 1 | | | 1 | | | 1 | | | 1 | | |
| 2. San Francisco to Denver | | 1 | | | 1 | | | 1 | | | 1 | |
| 3. San Francisco to Seattle | | | 1 | | | 1 | | | 1 | | | 1 |
| 4. Los Angeles to Chicago | | | | 2 | | | 2 | | 3 | 2 | | 3 |
| 5. Los Angeles to San Francisco | 2 | | | | | 3 | | | | 5 | 5 | |
| 6. Chicago to Denver | | | | 3 | 3 | | | | 4 | | | |
| 7. Chicago to Seattle | | | | | | | 3 | 3 | | 3 | 3 | 4 |
| 8. Denver to San Francisco | | 2 | | 4 | 4 | | | | 5 | | | |
| 9. Denver to Chicago | | | | | 2 | | | 2 | | | 2 | |
| 10. Seattle to San Francisco | | | 2 | | | | 4 | 4 | | | | 5 |
| 11. Seattle to Los Angeles | | | | | | 2 | | | 2 | 4 | 4 | 2 |
| Cost, $1,000's | 2 | 3 | 4 | 6 | 7 | 5 | 7 | 8 | 9 | 9 | 8 | 9 |

(The numbers in each column indicate the order of the flights.)

Exactly three of the sequences need to be chosen (one per crew) in such a way that every flight is covered.

# Formulation with Binary Variables

The objective is to minimize the total cost of the three crew assignments that cover all the flights.

Should sequence $j$ be assigned to a crew? $\quad (j = 1, 2, \ldots, 12)$

Therefore, we use 12 binary variables to represent these respective decisions:

$$x_j = \begin{cases} 1 & \text{if sequence } j \text{ is assigned to a crew} \\ 0 & \text{otherwise.} \end{cases}$$

Consider: Seattle to Los Angeles (LA)]. Five sequences (namely, sequences 6, 9, 10, 11, and 12) include this flight. Therefore

$$x_6 + x_9 + x_{10} + x_{11} + x_{12} \geq 1.$$

# The complete BIP model

$$\text{Minimize} \quad Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12},$$

subject to

$$
\begin{aligned}
x_1 + x_4 + x_7 + x_{10} &\geq 1 && \text{(SF to LA)}\\
x_2 + x_5 + x_8 + x_{11} &\geq 1 && \text{(SF to Denver)}\\
x_3 + x_6 + x_9 + x_{12} &\geq 1 && \text{(SF to Seattle)}\\
x_4 + x_7 + x_9 + x_{10} + x_{12} &\geq 1 && \text{(LA to Chicago)}\\
x_1 + x_6 + x_{10} + x_{11} &\geq 1 && \text{(LA to SF)}\\
x_4 + x_5 + x_9 &\geq 1 && \text{(Chicago to Denver)}\\
x_7 + x_8 + x_{10} + x_{11} + x_{12} &\geq 1 && \text{(Chicago to Seattle)}\\
x_2 + x_4 + x_5 + x_9 &\geq 1 && \text{(Denver to SF)}\\
x_5 + x_8 + x_{11} &\geq 1 && \text{(Denver to Chicago)}\\
x_3 + x_7 + x_8 + x_{12} &\geq 1 && \text{(Seattle to SF)}\\
x_6 + x_9 + x_{10} + x_{11} + x_{12} &\geq 1 && \text{(Seattle to LA)}\\
\sum_{j=1}^{12} x_j &= 3 && \text{(assign three crews)}
\end{aligned}
$$

and

$$x_j \text{ is binary}, \qquad \text{for } j = 1, 2, \ldots, 12.$$

# Installing security cameras

To promote campus security, the Campus safety department is in process of installing security cameras at selected locations. The department wants to install the minimum number of cameras that watches each street on campus. Each camera has 180 degrees vision capacity. The figure gives the map of campus