Calculus and Analytical Geometry - 2

Week 1: Prelude

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Note: All these questions are based on your previous understandings of basic geometry, algebra, and analysis.

Questions

Q1) Sketch, using translation and change of scale:

a) $y = 1 + |x + 2|$

b) $y = \frac{2}{(x-1)^2}$

Q2)

a) Show that every polynomial is the sum of an even and an odd function.

b) Generalize part (a) to an arbitrary function $f(x)$ by writing:

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

Verify this equation, and then show that the two functions on the right are respectively even and odd.

c) How would you write $\frac{1}{x+a}$ as the sum of an even and an odd function?

Q3)

a) Graph the function f that consists of straight-line segments joining the points $(-1, -1), (1, 2), (3, -1)$, and $(5, 2)$. Such a function is called piecewise linear.

b) Extend the graph of f periodically. What is its period?

c) Graph the function $g(x) = 3f\left(\frac{x}{2} - 1\right) - 3$

Q4) Use the difference quotient definition of derivative:

a) Calculate the rate of change of the area of a disk with respect to its radius. (Your answer should be the circumference of the disk.)

b) Calculate the rate of change of the volume of a ball with respect to the radius. (Your answer should be the surface area of the ball.)

Q5) Find the values of the constants a, b, and c for which the following function is differentiable. (Give a and b in terms of c.)

$$f(x) = \begin{cases} cx^2 + 4x + 1, & x \geq 1; \\ ax + b, & x < 1. \end{cases}$$

Q6) In Figure 1, graph the derivative of the following functions directly below the graph of the function. It is very helpful to know that the derivative of an odd function is even and the derivative of an even function is odd.

Q7) Find the points (x, y) of the graph $y = x^3 + x^2 - x + 2$ at which the slope of the tangent line is horizontal.

Q8) Find $\frac{dy}{dx}$ for $y = x^{1/n}$.

Q9) Let $y = u(x)v(x)$.

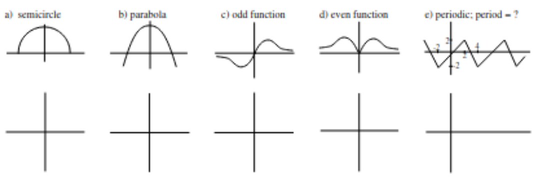

Figure 1:

a) Find y', y'', and y'''.

b) The general formula for $y^{(n)}$, the n-th derivative, is called Leibniz' formula:

$$y^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

Use this to check your answers in part (a), and use it to calculate $y^{(p+q)}$, if $y = x^p(1+x)^q$.

Q10) The half-life λ of a radioactive substance decaying according to the law $y = y_0 e^{-kt}$ is defined to be the time it takes for the amount to decrease to $\frac{1}{2}$ of the initial amount y_0.

a) Express the half-life λ in terms of k.

b) Show that if at time t_1 the amount is y_1, then at time $t_1 + \lambda$ it will be $\frac{y_1}{2}$, no matter what t_1 is.

Q11) Find the Taylor expansion of $f(x) = \ln(1+x)$ up to the fourth order.
Q12) Solve for x (Hint: put $u = e^x$, solve first for u):

a) $\frac{e^x + e^{-x}}{e^x - e^{-x}} = y$

b) $y = e^x + e^{-x}$

Q13)

a) Show that a chord of the unit circle with angle θ has length $\sqrt{2 - 2\cos\theta}$. Deduce from the half-angle formula

$$\sin(\theta/2) = \sqrt{\frac{1 - \cos\theta}{2}}$$

that the length of the chord is $2\sin(\theta/2)$.

b) Calculate the perimeter of an equilateral n-gon with vertices at a distance 1 from the center. Show that as $n \to \infty$, the perimeter tends to 2π, the circumference of the unit circle.

Q14) Evaluate the following limits:

a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(1 + \frac{2i}{n}\right)$

b) $\lim_{x\to\infty} \frac{e^x}{e^x - 1}$

c) $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

Q15) Let $f(x)$ be a continuous function satisfying the conditions of the Mean Value Theorem on the interval $[a, b]$. Prove the theorem by showing that there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Find such a point c for $f(x) = x^2$ on the interval $[1, 4]$.