Week 1

Q:1 Verify using graphing calc.

Q:2

(a) a general polynomial of n^{th} degree,

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

break into even odd parts

$$p(x) = \sum_{j\,(even)} a_j x^j + \sum_{k\,(odd)} a_k x^k$$

qed

(b)
$$f(x) = \frac{f(x)}{2} + \frac{f(x)}{2} + 0$$

$$= \frac{f(x)}{2} + \frac{f(x)}{2} + \frac{f(-x)}{2} - \frac{f(-x)}{2}$$

$$= \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

this is even part (easy to verify!)

(c) $\frac{1}{x+a} = \frac{\frac{1}{x+a} + \frac{1}{-x+a}}{2} + \frac{\frac{1}{x+a} - \frac{1}{-x+a}}{2}$

$$= \frac{\frac{\cancel{2}a}{a^2-x^2}}{\cancel{2}} + \frac{\frac{-\cancel{2}x}{a^2-x^2}}{\cancel{2}}$$

$$= \frac{a}{a^2-x^2} - \frac{x}{a^2-x^2}$$

even part

odd part

Q:4

(a) $\frac{dA}{dr} = ?$ $A(r) = \pi r^2$

$$\frac{d(\pi r^2)}{dr} = 2\pi r$$

Can you visualise this?

(b) Likewise!

Q:5

$$f(x) = \begin{cases} cx^2 + 4x + 1 & ; \ x \geq 1 \\ ax + b & ; \ x < 1 \end{cases}$$

continuity $\downarrow$

$$c + 5 = a + b$$

smoothness $\downarrow$

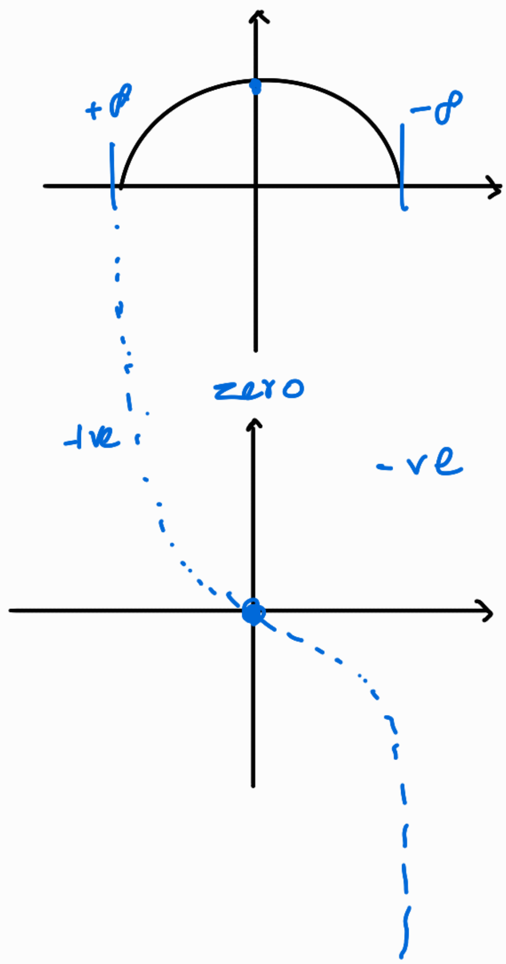
$$2c + 4 = a$$

$$\boxed{a = 4 + 2c}$$

get, b

Q:6

e.g a) Semicircle;



$x^2 + y^2 = 1$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

Note:
First Draw
and then
verify with
analytic
procedure.

Q:7 Trivial $\rightarrow$ asking for points
satisfying $\frac{dy}{dx} = 0$

Q:8 $$\frac{dy}{dx} = \frac{1}{n}x^{\frac{1}{n}-1} = \frac{1}{n}x^{\frac{1-n}{n}}$$

Q:9) $y = uv$

$$y' = u'v + v'u$$

$$y'' = u''v + u'v' + v'u' + v''u$$

$$= u''v + 2u'v' + v''u$$

Note
(Similarity to
a binomal
expansion)

$$y''' = u'''v + u''v' + 2u''v' + 2u'v'' + v''u' + v'''u$$

$$= u'''v + 3u''v' + 3u'v'' + v'''u$$

$y^{(p+q)}$ of $y = x^p(1+x)^q$

$$\Rightarrow y^{(p+q)} = \sum_{k=0}^{p+q} \binom{p+q}{k} {x^p}^{(k)} {(1+x)^q}^{(p+q-k)}$$

$$= \binom{p+q}{0} {(1+x)^q}^{(p+q)} x^p + \binom{p+q}{1} {x^p}' {(1+x)^q}^{(p+q-1)}$$

$\downarrow$

This will not survive as highest power of binomial is x^q and there is a greater than q derivative that will turn it zero

convince yourself that the only surviving term here is:

$$\binom{p+q}{p} {x^p}^{(p)} {(1-x)^q}^{(q)}$$

$$p+q-p = q$$

let $a = x^p \Rightarrow a^{(p)} = p!\, x^0 = p!$

$$\Rightarrow y^{(p+q)} = \frac{(p+q)!}{\cancel{p!}\,\cancel{q!}}\,\cancel{p!}\,\cancel{q!} = (p+q)!$$

Q:10)

$$y = y_0 e^{-kt}$$

$$y(t') = \frac{1}{2} y(t)$$

$$e^{-kt'} = \frac{1}{2} e^{-kt} \quad ; \quad 2 = e^{-kt+kt'}$$

$$2 = e^{k(t'-t)}$$ half-life

$$\ln 2 = kT$$

$$T = \frac{\ln 2}{k}$$

Q:11

Trivial

easily verifiable

Q:12

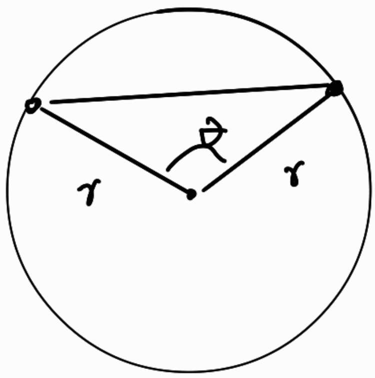
$$\frac{u+\frac{1}{u}}{u-\frac{1}{u}} = y \qquad \frac{u^2+1}{u^2-1} = y$$

$$\frac{\frac{u^2+1}{\cancel{u}}}{\frac{u^2-1}{\cancel{u}}} = y$$

$$u^2+1 = yu^2 - y$$

solve like a quadratic.

Q:13



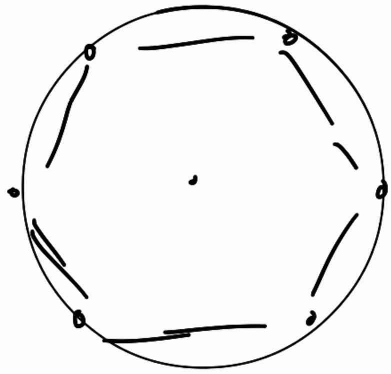
(a)

$r = 1$

Hint: cosine law

verification trivial

(b) as $n \to \infty$ $\theta \to 0$



circumference

$$n \times \sin\frac{\theta}{2}$$

$$n \times \theta = 2\pi$$

$$n = \frac{2\pi}{\theta}$$

$$\lim_{\theta \to 0} 2\pi \frac{\sin \theta/2}{\theta}$$

$$= 2\pi$$

Q:14

a) $$\frac{2}{n}\Bigg(\underbrace{\left(1+\frac{2}{n}\right)+\left(1+\frac{4}{n}\right)+\cdots\left(1+\frac{2\cancel{n}}{\cancel{n}}\right)}_{n \text{ times } 1 \,=\, n}\Bigg)$$

$$\frac{2}{n}\left(n+\frac{2}{n}\left(1+\underbrace{2+\cdots-n}_{n\text{ - natural sum } = \frac{n(n+1)}{2}}\right)\right)$$

$$\frac{2}{n}\left(n+\frac{\cancel{2}}{\cancel{n}}\frac{\cancel{n}(n+1)}{\cancel{2}}\right)$$

$$\frac{2}{n}(2n+1)$$

$$n\to\infty \qquad 4+\frac{2}{n} \qquad = 4$$

(b) DIY Ans : $\underline{1}$

(c) DIY Ans : $-1/6$

Note: You may use L'Hôpital Rule. → Remember this is applicable for otherwise $\frac{\infty}{\infty}$ or $\frac{0}{0}$ forms.

Mean-Value Theorem,

Statement;

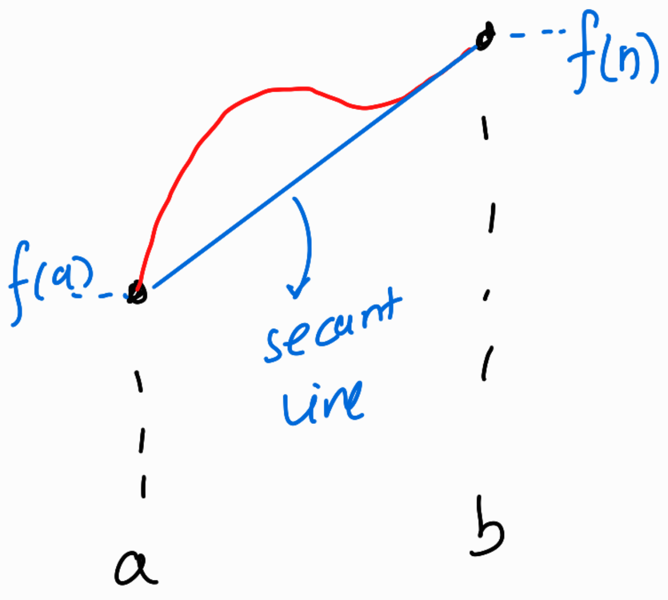
"If f is a function that is continiuous on $[a,b]$ and differentiable on (a,b), $\exists$ some $c \in (a,b)$ where;

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

Proof;

$A \to (a, f(a))$

$B \to (b, f(b))$



eqn of secant line;

$$y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

Define;

$$F(x) = f(x) - y$$

+ve where $f(x)$ is above the line and vice-versa.

$$F(a) = F(b)$$

by Rolle's Theorem

Prove seperately

$$\exists \; F'(c) = 0$$

$$\Rightarrow 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

// qed

Note : a more visual aspect of this theorem exists. worth digging.