Assignment - 2

M. Harish
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I affirm that I have neither given nor receive help or used any means which would make this assignment unfair.

— M. Harish

4.36

given an LTI system whose input

$$x(t) = [e^{-t} + e^{-3t}]u(t).$$

and output

$$y(t) = [2e^{-t} - 2e^{-4t}]u(t)$$

(a) frequency response of the system

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t) = x(t) * h(t)$$

As we know convolution in time domain is multiplication in frequency domain

$$Y(w) = X(w) \cdot H(w).$$

$$H(w) = \frac{Y(w)}{X(w)}$$

$$X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} \left(e^{-t} + e^{-3t}\right) u(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} \left(e^{-t} \cdot e^{-jwt} u(t) + e^{-3t} \cdot e^{-jwt} u(t)\right) dt$$

$$e^{-t} u(t) \xrightarrow{FT} \frac{1}{a+jw}$$

$$X(w) = \frac{1}{1+jw} + \frac{1}{3+jw}$$

$$= \frac{3+jw+1+jw}{(1+jw)(3+jw)} = \frac{4+2jw}{(1+jw)(3+jw)}$$

$$Y(w) = \int_{-\infty}^{\infty} y(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} \left(2e^{-t} u(t) - 2e^{-4t} u(t)\right) e^{-jwt}\, dt.$$

$$= \frac{2}{1+jw} - \frac{2}{4+jw}$$

$$= \frac{8 + \cancel{2jw} - 2 - \cancel{2jw}}{(1+jw)(4+jw)} = \frac{6}{(1+jw)(4+jw)}$$

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$$\Rightarrow H(\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{4+2j\omega}{\cancel{(1+j\omega)}(3+j\omega)} \times \frac{\cancel{(1+j\omega)}(4+j\omega)}{6}$$

$$= \frac{\cancel{2}(2+j\omega)(4+j\omega)}{{}_3\cancel{6}(3+j\omega)}$$

$$H(\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\cancel{6}^{\,3}}{\cancel{(1+j\omega)}(4+j\omega)} \times \frac{\cancel{(1+j\omega)}(3+j\omega)}{\cancel{2}(2+j\omega)}$$

$$\boxed{H(\omega) = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}}$$

(b) System's Impulse Response –

from the result of (a)

$$H(\omega) = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}$$

$$\frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)} = \frac{A}{4+j\omega} + \frac{B}{2+j\omega}$$

$$9+3j\omega = (2+j\omega)A + B(4+j\omega)$$

$$9+3j\omega = 2A + Aj\omega + 4B + Bj\omega.$$

$$9 + 3j\omega = 2A + 4B + j\omega(A + B)$$

Comparing we get

$$2A + 4B = 9 \rightarrow ①, \quad A + B = 3 \rightarrow ②$$

$$② \times 2 \Rightarrow \quad 2A + 2B = 6$$

$$(-) \quad 2A + 4B = 9$$

$$-2B = -3$$

$$\boxed{B = 3/2}$$

$$\Rightarrow \quad \boxed{A = 3/2}$$

$$\therefore \quad H(\omega) = \frac{3}{2}\left(\frac{1}{4 + j\omega}\right) + \frac{3}{2}\left(\frac{1}{2 + j\omega}\right)$$

$\Rightarrow$ we can observe that it is a linear combination

$$\therefore \quad \alpha x_1(t) + \beta x_2(t) \xleftrightarrow{FT} \alpha X_1(\omega) + \beta X_2(\omega)$$

$$\therefore \quad h(t) = \frac{3}{2}\left(e^{-2t}u(t)\right) + \frac{3}{2}\left(e^{-4t}u(t)\right)$$

$$\boxed{h(t) = \frac{3}{2}\left(e^{-2t} + e^{-4t}\right)u(t)}$$

(c) Differential equation relating input and output

Sol

$$H(\omega) = \frac{3(3+j\omega)}{(2+j\omega)(4+j\omega)}.$$

$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = \frac{9+3j\omega}{8+6j\omega+j^2\omega^2}$$

$$\Rightarrow Y(j\omega)\left[8+6j\omega+j^2\omega^2\right] = (9+3j\omega)X(j\omega)$$

$$8Y(\omega) + 6j\omega Y(\omega) + j^2\omega^2 Y(j\omega)$$

$$= 9X(j\omega) + 3j\omega X(\omega)$$

Applying Inverse F.T

$$\Rightarrow \boxed{8y(t) + 6\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} = 9x(t) + 3\cdot\frac{d}{dt}x(t)}$$

4.44

given relation between output & input of causal LTI system.

$$\frac{d}{dt}y(t) + 10y(t) = \int_{-\infty}^{\infty} x(\tau)\cdot z(t-\tau)\,d\tau - x(t)$$

$$z(t) = e^{-t}u(t) + 3\delta(t).$$

$$\Rightarrow \int_{-\infty}^{\infty} x(\tau) z(t-\tau)\, d\tau = x(t) * z(t)$$

$$\frac{d}{dt} y(t) + 10 y(t) = \int_{-\infty}^{\infty} x(t) * z(t)\, d\tau - x(t)$$

$$= x(t) * z(t) - x(t)$$

Applying F.T

$$j\omega Y(\omega) + 10 Y(\omega) = X(\omega) \cdot Z(\omega) - X(\omega)$$

$$Y(\omega)(10 + j\omega) = X(\omega)[Z(\omega) - 1]$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{Z(\omega) - 1}{10 + j\omega} = H(\omega)$$

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(e^{-t} u(t) + 3\delta(t)\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-t} u(t) \cdot e^{-j\omega t}\, dt + 3\int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt.$$

$\downarrow$ FT $\qquad\qquad \delta(t) \xrightarrow{FT} 1$

$$= \frac{1}{1 + j\omega} + 3$$

$$Z(\omega) = \frac{4 + 3j\omega}{1 + j\omega}$$

$$\Rightarrow H(\omega) = \frac{\frac{4+3j\omega}{1+j\omega} - 1}{10+j\omega}$$

$$= \frac{4+\cancel{3}j\omega - 1 - \cancel{j\omega}}{(1+j\omega)(10+j\omega)} = \frac{3+2j\omega}{(1+j\omega)(10+j\omega)}$$

Frequency Response $\boxed{H(\omega) = \frac{3+2j\omega}{(1+j\omega)(10+j\omega)}}$

(b) Impulse Response

using partial fraction

$$\frac{3+2j\omega}{(1+j\omega)(10+j\omega)} = \frac{A}{1+j\omega} + \frac{B}{10+j\omega}$$

$$3+2j\omega = A(10+j\omega) + B(1+j\omega)$$

$$= 10A + Aj\omega + B + Bj\omega$$

$$= 10A + B + j\omega(A+B)$$

Comparing LHS & RHS.

$$10A + B = 3 \rightarrow (1) \qquad A + B = 2 \rightarrow (2)$$

① $\Rightarrow$ $10A + B = 3$

②$\times 10$ $\Rightarrow$ $10A + 10B = 20$

$-9B = -17$

$B = \frac{17}{9}$, $A = \frac{1}{9}$

$\therefore$

$$H(\omega) = \frac{1}{9}\left(\frac{1}{1+j\omega}\right) + \frac{17}{9}\left(\frac{1}{10+j\omega}\right)$$

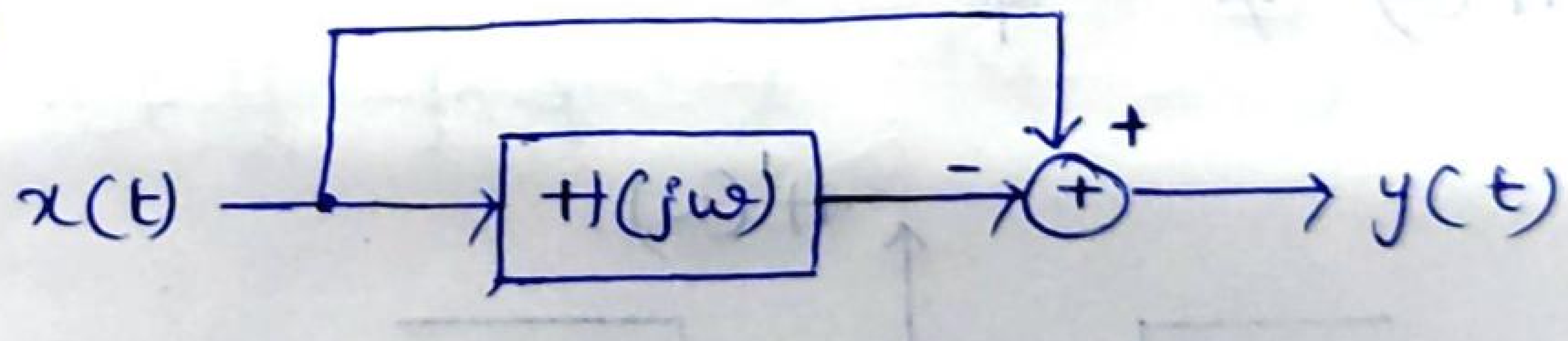
from inverse F.T

$$h(t) = \frac{1}{9}\left[e^{-t}u(t)\right] + \frac{17}{9}\left[e^{-10t}u(t)\right].$$

$$\boxed{h(t) = \frac{1}{9}\left(e^{-t}u(t) + 17e^{-10t}u(t)\right)}$$

6.33

given



$$y(t) = x(t) - \left(x(t) * h(t)\right)$$

$$\Updownarrow \text{FT}$$

$$Y(\omega) = X(\omega) - \left(X(\omega) \cdot H(\omega)\right)$$

$$= X(\omega)\left(1 - H(\omega)\right)$$

$$\frac{Y(\omega)}{X(\omega)} = 1 - H(\omega) \Rightarrow H'(\omega) = 1 - H(\omega)$$

(a) given $H(\omega)$ is a low pass filter with cutoff frequency ω_{lp}

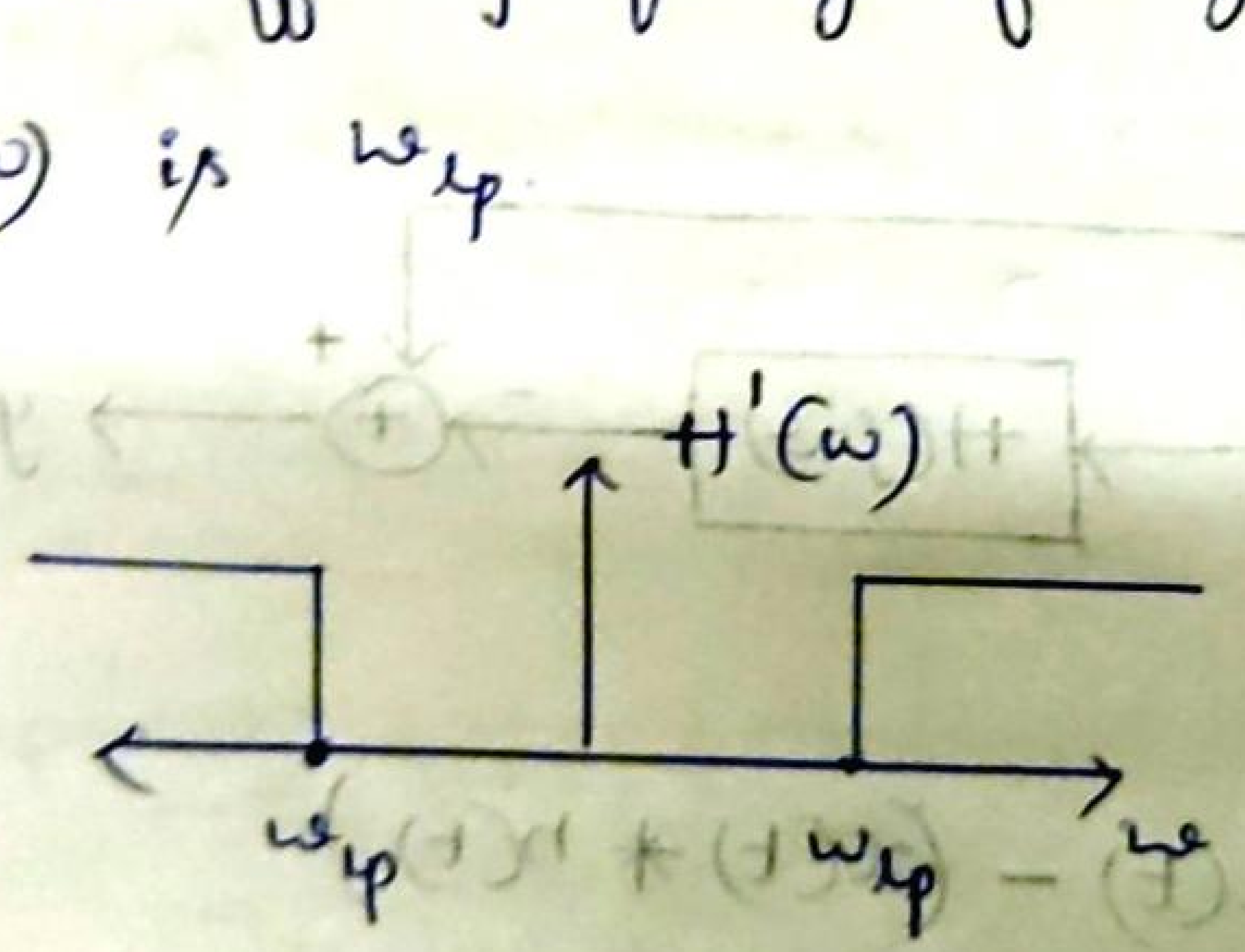
ie

$$H(\omega) = \begin{cases} 1 & |\omega| < \omega_{lp} \\ 0 & |\omega| > \omega_{lp} \end{cases}$$

$$\underbrace{H'(\omega) = \begin{cases} 0 & |\omega| < \omega_{lp} \\ 1 & |\omega| > \omega_{lp} \end{cases}}$$

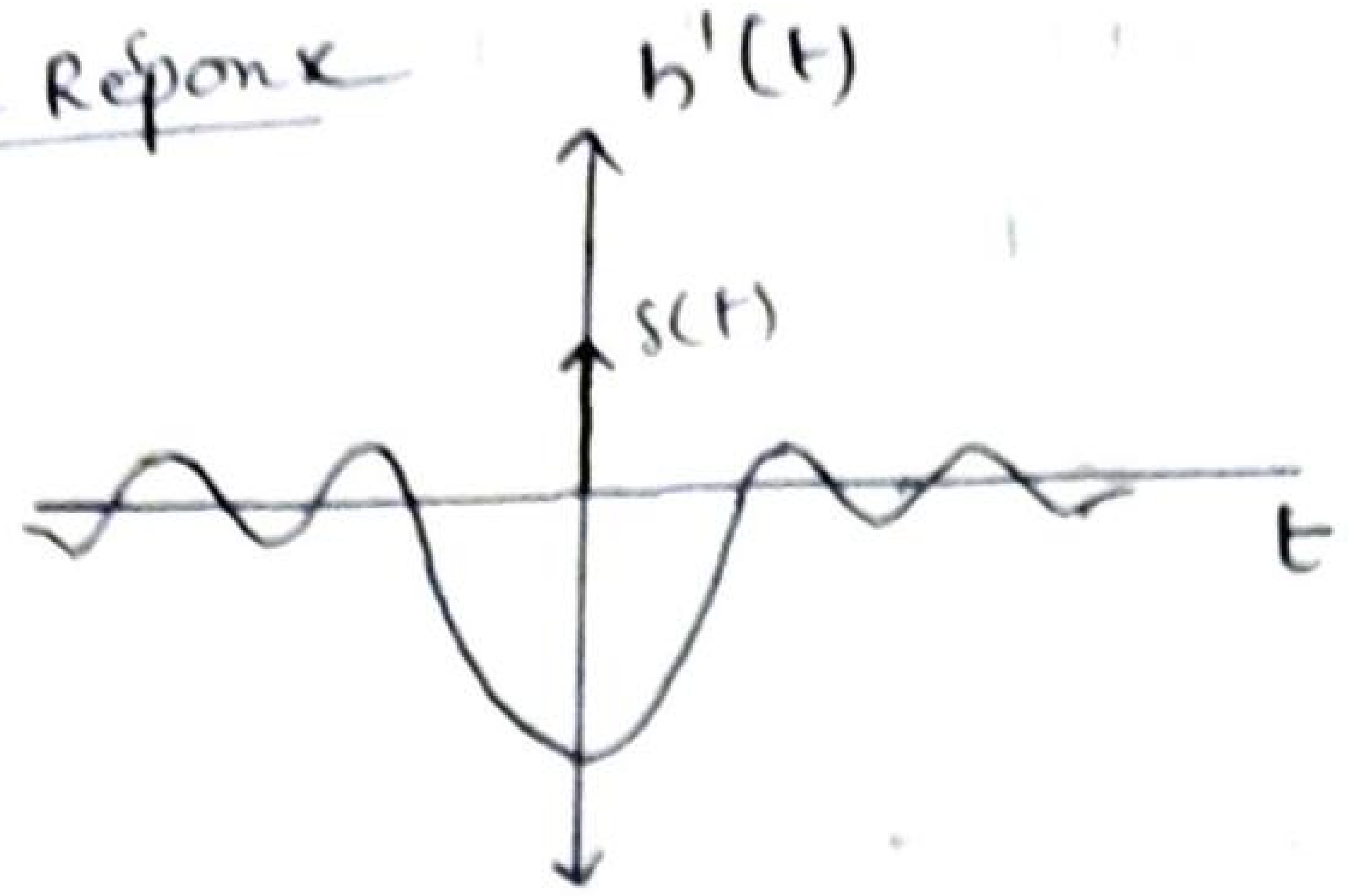
↳ system corresponds to an ideal highpass filter.

Systems cutoff frequency of high pass filter $H'(\omega)$ is ω_{lp}.



$$h'(t) = \delta(t) - h(t) = \delta(t) - \frac{\sin(\omega_{lp} t)}{\pi t}.$$

Impulse Response



(b) given $H(\omega)$ is a ideal ~~low~~ high pass filter with cutoff frequency ω_{hp}.

$$H(\omega) = \begin{cases} 0 & |\omega| < \omega_{hp} \\ 1 & |\omega| > \omega_{hp} \end{cases}$$

overall system frequency response

$$H'(\omega) = \begin{cases} 1 & |\omega| < \omega_{hp} \\ 0 & |\omega| > \omega_{hp} \end{cases}$$

we can see that overall system's frequency response corresponds to an ideal lowpass filter with cutoff frequency ω_{hp}.

7.24.

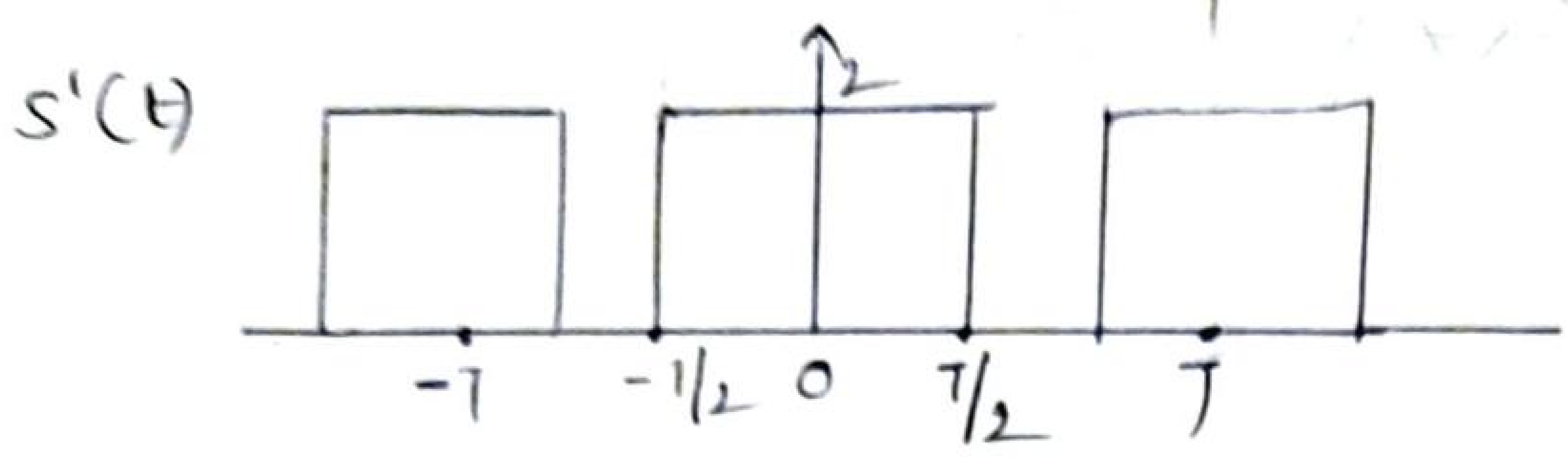
given $x(t)$ an input signal multiplied by a periodic square wave $s(t)$.

Timeperiod of $s(t)$ is T

$x(t)$ is a bandlimited signal $|X(j\omega)| = 0$ for $|\omega| > \omega_m$

we can express $s(t) = s'(t) - 1$

where $s'(t)$ is a rectangular wave of amplitude 2



$$s(t) = s'(t) - 1$$

Applying F.T

$$S(\omega) = S'(\omega) - 2\pi\delta(\omega) \qquad 1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$= \sum_{K=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T} \cdot K\right) \frac{2\pi}{T} \cdot \frac{4\sin(\Delta\omega)}{\omega} - 2\pi\delta(j\omega)$$

$$= \left(\sum_{K=-\infty}^{\infty} \frac{4\sin\left(\Delta \times \frac{2\pi}{T} K\right)}{\cancel{\frac{2\pi}{T}} K} \cdot \cancel{\frac{2\pi}{T}} \right) - 2\pi X(j\omega)$$

$$= \sum_{K=-\infty}^{\infty} \frac{4\sin\left(\frac{2\pi K\Delta}{T}\right)}{K} X\left(j\left(\omega - \frac{2\pi K}{T}\right)\right) - 2\pi X(j\omega)$$

(a) given $\Delta = T/3$.

$\omega_s = \frac{2\pi}{T}$ from above eq.

Condition for no aliasing

$$\omega_s > 2\omega_m$$

$$\Rightarrow \frac{2\pi}{T} > 2\omega_m$$

$$\frac{\pi}{T} > \omega_m$$

$$T < \frac{\pi}{\omega_m} \qquad \Rightarrow \omega_m < \frac{\pi}{T}$$

Since replicas are spaced $2\pi/T$ apart.
So to avoid aliasing ω_m should be less than π/T.

$$\therefore \; T_{max} = \frac{\pi}{\omega_m}.$$

(b) $\Delta = T/4$

~~we can~~

Since replices are spaced $\frac{4\pi}{T}$ the Condition for not aliasing.

$$\omega_s > 2\omega_m.$$

$$\frac{4\pi}{T} > 2\omega_m$$

$$\frac{2\pi}{T} > \omega_m$$

$$\Rightarrow \omega_m < \frac{2\pi}{T}$$

ω_m should be less than $2\pi/T$ for not aliasing.

$$T_{max} = \frac{2\pi}{\omega_m}.$$

(7.37)

given

1) $x(t)$ is band limited ; $X(j\omega) = 0 \quad |\omega| > W$.

2) $p(t)$ is non uniformly spaced periodic pulse train

3) $f(t)$ is a periodic waveform with period $T = 2\pi/\omega$ since $f(t)$ multiplies an impulses train, only its values $f(0) = a$ & $f(\Delta) = b$ at $t=0$ & $t=\Delta$ respectively are significant.

4) $H_1(j\omega)$ is a 90° phase shifter.

$$H_1(j\omega) = \begin{cases} j, & \omega > 0 \\ -j, & \omega < 0 \end{cases}$$

5) $H_2(j\omega)$ is an ideal lowpass filter ;

that is

$$H_2(j\omega) = \begin{cases} K & 0 < \omega < W \\ K^* & -W < \omega < 0 \\ 0 & |\omega| > W \end{cases}$$

(a)

$p(t)$, $y_1(t)$ $y_2(t)$ & $y_3(t)$ fourier transforms (FT).

$p(t)$

we may write $p(t)$ as

$$p(t) = p_1(t) + p_2(t-\Delta) \rightarrow (1)$$

where
$$p_1(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2\pi k/W).$$

$$\cancel{P(j\omega) = (1+e^{-j\omega\Delta})\, P_1(j\omega)}.$$

$$P_1(j\omega) = W \sum_{k=-\infty}^{\infty} \delta(\omega - kW)$$

$$p_2(t-\Delta) \xleftrightarrow{FT} P_1(j\omega)\cdot e^{-j\omega\Delta}$$

$$P_2(j\omega) = e^{-j\omega\Delta}\left[W \sum_{k=-\infty}^{\infty} \delta(\omega - kW\right].$$

from (1) applying F-T

$$P(j\omega) = W \sum_{k=-\infty}^{\infty} \delta(\omega - kW) + e^{-j\omega\Delta}\left[W \sum_{k=-\infty}^{\infty} \delta(\omega - kW)\right]$$

$$\boxed{P(j\omega) = (1 + e^{-j\omega\Delta})\, W \sum_{k=-\infty}^{\infty} \delta(\omega - kW)}$$

$y_1(t)$:- given from figure

$$y_1(t) = x(t) \cdot p(t) \cdot f(t)$$

let us consider

$$s(t) = p(t) \cdot f(t)$$

$$= p_1(t) f(t) + p_2(t-\Delta) f(t).$$

$$s(t) = p_1(t) \cdot a + p_2(t-\Delta) \cdot b$$

$$= a p_1(t) + b p_2(t-\Delta)$$

$f(t) = a, \ t=0$

$f(t) = b, \ t=\Delta$

$$S(j\omega) = a \cdot P_1(j\omega) + b P_2(j\omega)$$

$$= a\left(W \sum_{k=-\infty}^{\infty} \delta(\omega - kW)\right) + b\left[e^{-j\omega\Delta} W \sum_{k=-\infty}^{\infty} \delta(\omega - kW)\right]$$

$$S(j\omega) = W \sum_{k=-\infty}^{\infty} \left(a + b e^{-j\omega\Delta}\right) \delta(\omega - kW)$$

$$y_1(t) = x(t) \cdot s(t)$$

$$Y_1(j\omega) = \frac{1}{2\pi}\left(X(\omega) * S(\omega) \right)$$

$$= \frac{1}{2\pi} W \sum_{k=-\infty}^{\infty} (a+be^{-j\omega\Delta})\, X(j(\omega-kW))$$

given $Y_1(j\omega)$ for $0<\omega<W$.

$$Y_1(j\omega) = \frac{W}{2\pi}\left[a+be^{0} X(j\omega) + a+be^{-j\omega\Delta} X(j(\omega-kW))\right]$$

$$\boxed{Y_1(j\omega) = \frac{W}{2\pi}\left[(a+b)X(j\omega) + (a+be^{-j\omega\Delta})X(j(\omega-kW)) \right]}$$

$\underline{Y_2(t)}$:-

$$y_2(t) = y_1(t) * H_1(t).$$

$$Y_2(j\omega) = Y_1(j\omega) \cdot H_1(j\omega)$$

for $0<\omega<W$.

$$Y_2(j\omega) = \frac{Wj}{2\pi}\left[(a+b)X(j\omega) + (a+be^{-j\omega\Delta}) X(j(\omega-kW)) \right]$$

for $\omega>0$ $H(j\omega)=j$.

$$\boxed{Y_2(j\omega) = \frac{Wj}{2\pi}\left[(a+b)X(j\omega) + (a+be^{-j\omega\Delta})X(j(\omega-kW))\right]}$$

$\underline{y_3(t)}$

given $y_3(t) = x_p(t)$

$$y_3(t) = x(t) \cdot p(t)$$

$$Y_3(j\omega) = \left(X(j\omega) * P(j\omega)\right) \times \frac{1}{2\pi}$$

$$= \frac{1}{2\pi} \times W\left(1+e^{-jw\Delta}\right) \sum_{k=-\infty}^{\infty} \delta(w-kW)\, X(jw)$$

$$= \frac{W}{2\pi}\left(1+e^{-jw\Delta}\right) \sum_{k=-\infty}^{\infty} X(j(w-kW)).$$

given for $0 < w < W$

$$Y_3(jw) = \frac{W}{2\pi}\left(1+e^{-j\Delta(0)}\right) X(jw) + \frac{W}{2\pi}\left(1+e^{-jW}\right) X(w-W)$$

$$\boxed{Y_3(jw) = \frac{W}{2\pi}\left[2X(jw) + \left(1+e^{-j\Delta W}\right) X(w-W)\right]}$$

(b) given a, b, K ?

$z(t) = x(t)$ for any band limited $x(t)$ and any Δ such that $0 < \Delta < \pi/W$.

$\Rightarrow$ let $y_4(t) = y_2(t) + y_3(t)$.

$z(t) = y_4(t) * h_2(t)$.

Applying fourier Transform

$$z(j\omega) = Y_4(j\omega) \cdot H_2(j\omega)$$

$$= \left(Y_2(j\omega) + Y_3(j\omega) \right) \cdot H_2(j\omega)$$

$z(j\omega)$ $0 < \omega < W$

$$\therefore \quad z(j\omega) = K\left[Y_2(j\omega) + Y_3(j\omega) \right] \rightarrow 1$$

$$X(j\omega) = K\left[\frac{W}{2\pi}\left[j(a+b)X(j\omega) + \left(a + be^{-j\omega\Delta}\right) X(j(\omega - K\omega)) \right] \right.$$

$$\left. + \frac{W}{2\pi}\left[2X(j\omega) + \left(1 + e^{-j\omega\Delta}\right) X(j(\omega - W)) \right] \right]$$

$$X(j\omega) = \left[\frac{KW}{2\pi}\left(2 + aj + bj\right) X(j\omega) \right] +$$

$$\left[\frac{KW}{2\pi}\left(1 + e^{-j\omega\Delta} + aj + bje^{-j\Delta W}\right) X(j(\omega - W)) \right]$$

Comparing LHS & RHS

$$\frac{KW}{2\pi}\left(2+a_j+b_j\right)=1$$

$$\boxed{K=\frac{2\pi}{W(2+a_j+b_j)}}$$

& $$\frac{KW}{2\pi}\left(1+e^{-jw\Delta}+a_j+b_j e^{-j\Delta w}\right)=0.$$

$$1+e^{-jw\Delta}+a_j+b_j e^{-j\Delta w}=0.$$

$$\Rightarrow 1+\cos(w\Delta)-j\sin(w\Delta)+a_j+b_j\left[\cos(w\Delta)-j\sin(w\Delta)\right)=0.$$

$$\Rightarrow 1+\cos(w\Delta)+b\sin(w\Delta)+j\left[a+b\cos(w\Delta)-\sin(w\Delta)\right]=0$$

If $x+iy=0$ then $x=0,\ y=0$ for $x\in R$.

$$1+\cos(w\Delta)+b\sin(w\Delta)=0 \rightarrow (a)$$

$$a+b\cos(w\Delta)-\sin(w\Delta)=0 \rightarrow (b).$$

$$b\sin(w\Delta)=-\ 1+\cos(w\Delta)$$

$$\boxed{b=-\frac{(1+\cos(w\Delta))}{\cancel{b}\sin(w\Delta)}}$$

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Substituting $\cancel{X}$ b in (a).

$$\Rightarrow a + \frac{-(1+\cos(\omega\Delta))}{\sin(\omega\Delta)} \cdot \cos(\omega\Delta) - \sin(\omega\Delta) = 0.$$

$$a - \cot(\omega\Delta)(1+\cos(\omega\Delta)) - \sin(\omega\Delta) = 0$$

$$a = \sin(\omega\Delta) + \frac{1+\cos(\omega\Delta)}{\tan(\omega\Delta)}.$$

$$\therefore \boxed{a = \sin(\omega\Delta) + \frac{1+\cos(\omega\Delta)}{\tan(\omega\Delta)}.}$$

7.39

given $x(t) = \cos\left(\frac{\omega_s}{2} t + \phi\right)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

$T = 2\pi/\omega_s$

(a) find $g(t)$?

given $x(t) = \cos(\phi)\cos\left(\frac{\omega_s}{2} t\right) + g(t)$

$$\Rightarrow \cos\left(\frac{\omega_s}{2} t + \phi\right) = \cos(\phi)\cos\left(\frac{\omega_s}{2} t\right) + g(t)$$

$$\Rightarrow \cos\left(\frac{\omega_s}{2} t\right)\cos(\phi) - \sin\left(\frac{\omega_s}{2} t\right)\sin(\phi)$$

$$= \cos(\phi)\cos\left(\frac{\omega_s}{2} t\right) + g(t)$$

$$\therefore \boxed{g(t) = -\sin\left(\frac{\omega_s}{2} t\right)\sin(\phi)}$$

(b) to show ~~that~~

$g(nT) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$

$$g(nT) = -\sin\left(\frac{\cancel{2}\pi}{\cancel{2T}} n \cancel{T}\right)\sin(\phi)$$

$$= -\sin(n\pi)\cdot\sin(\phi)$$

$\hookrightarrow = 0$ for $n \in \mathbb{Z}$

$$g(nT) = 0 \cdot -\sin(\phi) = 0$$

(c)

$$x_p(t) = \sum_{\eta=-\infty}^{+\infty} x(\eta T)\, \delta(t-\eta T).$$

$$= \sum_{\eta=-\infty}^{\infty} \delta(t-\eta T)\left(\cos\phi \cdot \cos\left(\frac{\omega_s}{2}\eta t\right) + \underbrace{g(\eta T)}_{= 0 \rightarrow (b)} \right)$$

$$= \sum_{\eta=-\infty}^{\infty} \delta(t-\eta T)\, \cos\phi \cos\left(\frac{\omega_s}{2}\eta T\right)$$

$$\boxed{x_p(t) = \sum_{\eta=-\infty}^{\infty} \delta(t-\eta T) \cos\phi \cos\left(\frac{\omega_s}{2}\eta T\right)}$$

When $x_p(t)$ signal is passed through a lowpass filter, $\omega_c = \omega_s/2$ we can reconstruct in effect of performing band limited interpolation.

This results in the signal

$$y(t) = \cos\left(\frac{\omega_s}{2} t\right) \cos(\phi).$$