GEODESICS OF 2-TORUS

Saturday, October 31, 2020 6:06 PM

GEODESICS

GEODESICS are defined as the shortest path between two well defined points on a curved-space, generally a smooth manifold. On a curved space they are obtained by minimizing the equation of length using VARIATION technique. In variation method we define the infinitesimal change in path/function/curve is essentially equal to zero, we obtain then the equation which satisfies local extremum of length.

It is possible that several different curves between two points minimize the distance. Example on a sphere between two diametrically opposite points, on a cylinder between two points diametrically opposite to each other.

Geodesics equations gives us the local extremum of the length. Geodesics are not actually the "Shortest Curves".
GEODESICS --- Locally shortest curves.

GEODESICS ON 2-TORUS.

Generalized co-ordinates for the manifold of 2-torus in R^3 are theta, phi. Torus in configuration space is $\mathbf{S^1 \times S^1}$ **in** $\mathbf{R^4}$. Torus is formed by revolving a circle of radius 'b' around a center in a circle of radius 'a'. Now defining $\frac{\boldsymbol{a}}{\boldsymbol{b}}$ **=n** essentially means Torii having same "n" are nothing but enlargement of size(scaling) and the properties doesn't change. As what matters is value of n , without loss of generality we use b =1.
Choosing b =1 doesn't affect the shape of geodesic but only when a particle is travelling on Geodesic curve the speed of it depends on "b". It can be easily verified when geodesics compared with a particle motion in shortest time. Choosing b=1 gives us the "**unit-speed Geodesics**".

Thus any general point will be represented as *x=(n+cos(phi))cos(theta) , y = (n+cos(phi))sin(theta) and z = sin(phi)*.

Thus $dS^2 = (n+\cos(phi))^2 dtheta^2 + b^2 dphi^2$ -----1 ==> Metric tensor(g) = $\begin{matrix} (n+cos(phi))^2 & 0 \\ 0 & 1 \end{matrix}$

Using the geodesic equations , one can obtain the geodesic equation curves and the Christoffel symbols is easy to solve if we pre assume some derivatives from g-matrix. Only $\frac{\partial g_{theta\ theta}}{\partial\ phi}$ term survives so the geodesic equations will be(calc 1)

$$\boxed{\begin{matrix} \frac{d^2\ theta}{dS^2} - \frac{2\sin(phi)}{n+\cos(phi)} \frac{d\ theta}{dS} \frac{d\ phi}{dS} = 0 \quad \text{---------2} \\ \frac{d^2\ phi}{dS^2} + (n+\cos(phi))\sin(phi) \frac{d\ theta}{dS} \frac{d\ theta}{dS} = 0 \quad \text{-------3} \end{matrix}}$$

The equations representing the Geodesic are obtained now but they are not in form to analytically get the curves and relation between theta and phi explicitly.
For our convenience we can change the system of equations into a four dimensional space variables relates as

$$\frac{d\ theta}{dS} = U \ \ and \ \ \frac{d\ phi}{dS} = V$$

$$\frac{d\ U}{dS} - \frac{2\sin(phi)}{n+\cos(phi)} UV = 0 \ and \ \frac{d\ V}{dS} + (n+\cos(phi))\sin(phi)\ U^2 = 0$$

The third expression is in form of a linear differential equation and using (4) in (3) expression gives (calculation 2-last page)

$$\boxed{\begin{matrix} \frac{d\ theta}{dS} (n+\cos(phi))^2 = constant = c \quad \text{------- 4} \\ \frac{d\ phi}{dS} = \sqrt{1 - \frac{c^2}{(n+cos(phi))^2}} \quad \text{--------- 5} \end{matrix}}$$

dividing (4) and (5) gives us the equation we use to get the curves i.e.,

$$\boxed{\frac{d\ theta}{d\ phi} = \frac{c}{(n+\cos(phi))\sqrt{(n+\cos(phi))^2 - c^2}} \quad \text{--------6}}$$

For a 2-torus n>1 always and in the denominator when n+cos(phi) = c implies $\frac{d\ phi}{dS}$ = 0 reprensting the geodesic curve is constant in phi variation at that instant .

Rearranging (6) gives $\frac{1}{(n+\cos(\ phi))^4} (\frac{d\ phi}{d\ theta})^2 + \frac{1}{(n+\cos(\ phi))^2} = \frac{1}{c^2}$ --------- 7

It can be seen as countour lines of F(phi, $\frac{d\ phi}{d\ theta}$) = $\frac{1}{(n+\cos(\ phi))^4} (\frac{d\ phi}{d\ theta})^2 + \frac{1}{(n+\cos(\ phi))^2}$ represents the variation of $\frac{d\ phi}{d\ theta}$ with respect to phi i.e, equation 7

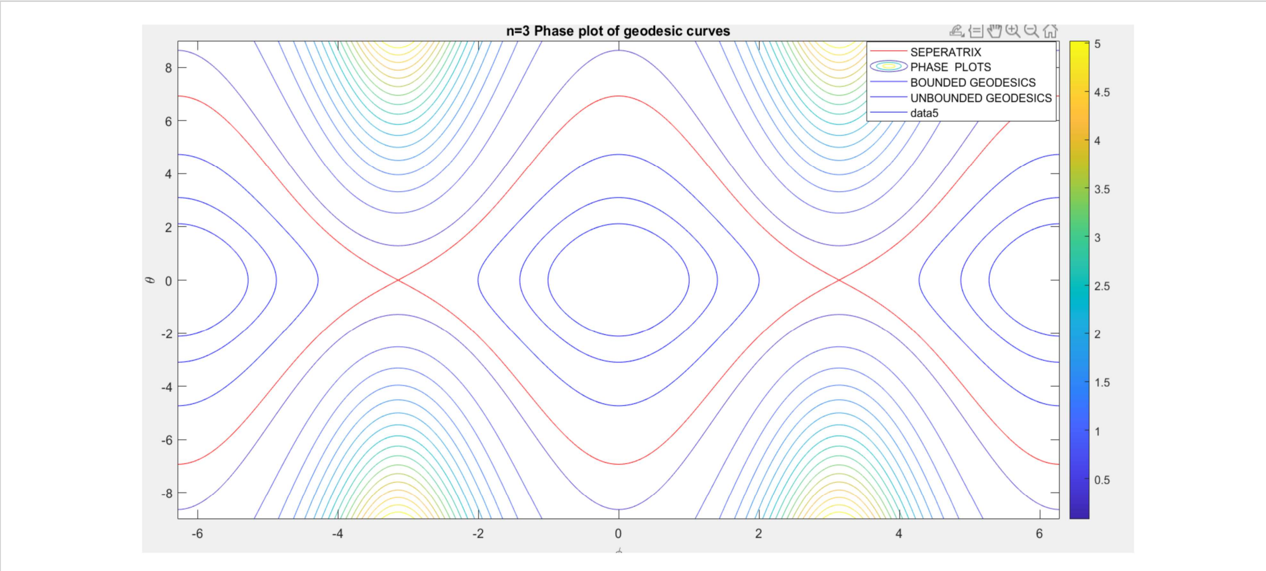


Figure 1:

plot between phi, $\frac{d\,phi}{d\,theta}$

The curves are like phase plots. They actually have the information that the nature of geodesic i.e., bounded or unbounded.

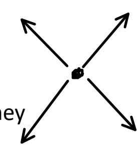
1. The closed curves (in plot) represents the bounded geodesic curves.
2. The curves which are not closed(in plot) represents the unbounded geodesic.
3. Red line is the separatrix, which is also a bridge between Bounded and Unbounded Geodesics.

<u>Information from Phase Plot</u>

Bounded geodesics:
By the name suggests, the curve is restricted and the curve doesn't contain all the points.
Unbounded geodesics:
By the name suggests, the curve is not restricted and the curve contain all the points.

The plot above gives the difference between the bounded and unbounded geodesics, and the truth is there exists a separatrix between them. The fact is that separatrix doesn't touch phi =pi or odd multiples of pi, like Hyperbolic point in phase plot. It essentially means that the curve tend to approach phi=pi or odd multiples of pi.

Hyperbolic point looks like evolving near a point and they don't touch the point at center.



The separatrix has two sets to discuss, 1. at phi=pi and 2. phi ≠pi...... In (7) substitute phi = π and c = n-1 results $\frac{d\,phi}{d\,theta} = 0$.
So c = n-1 corresponds to separatrix.

Now one can extend this ideology of bounded and unbounded curves for nature and curvature of different type of geodesics. As the value of "c "at separatrix is n-1 , we can discuss the **different type of geodesics as of c < n-1 and c = n-1 and c > n-1** ("c" has physical significance). As of now we have not discussed the limitations of "c", from (6) in denominator

$$(n+\cos(phi))^2 - c^2 \geq 0 \quad \text{implies} \quad c \in [0,n+1]$$

:: The value "c" also provides the relative information about angle made with inner and outer equator. The value of $\frac{d\,phi}{d\,theta}$ at even multiples of π is always higher than odd multiples of π ,which means for a fixed change in theta the increase in phi is more at even multiples of π(outer equator) than odd multiples of pi. As small variation in phi resuts in linear variation of z-height,increase in height is more at outer eqautor.

<u>The angle made at outer equator is always more than inner equator.</u>

<u>Different types of Geodesics Possible on 2-Torus</u>

<u>If c = 0</u>

$$\frac{d\ theta}{d\ phi} = \frac{c}{(n+\cos(phi))\sqrt{(n+\cos(phi))^2-c^2}} \quad \text{and c = 0} \quad => \frac{d\ theta}{d\ phi} = 0$$

1. The Geodesic curve doesn't have any variation in theta as phi changes.
2. The curves are then the **<u>meridians</u>** .
3. The corresponding phase plot(figure 1) is undetermined as
   c = 0 ....F(phi, $\frac{d\ phi}{d\ theta}$) = $\frac{1}{c^2}$ diverges.

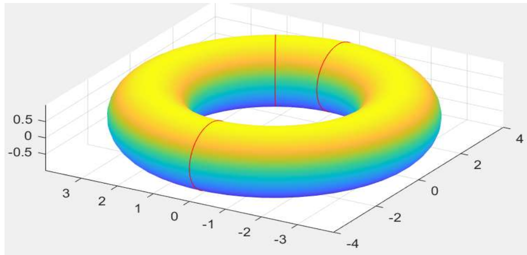
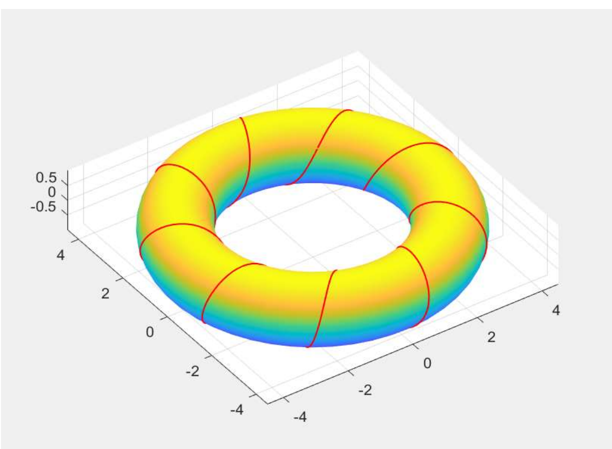


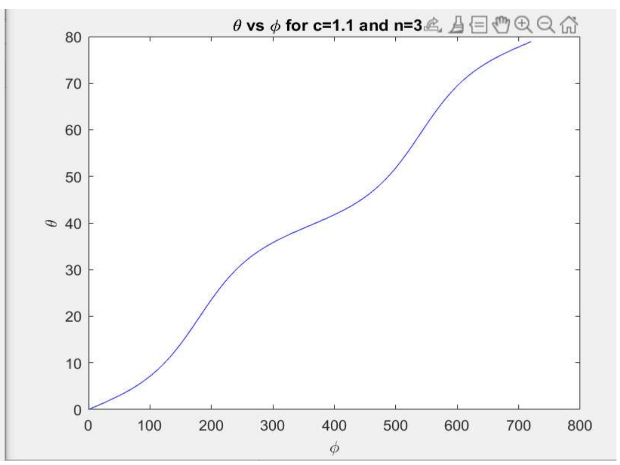
Figure 2:
The curves are represented in red, n = 3.5. The geodesic curves are circles of fixed theta.

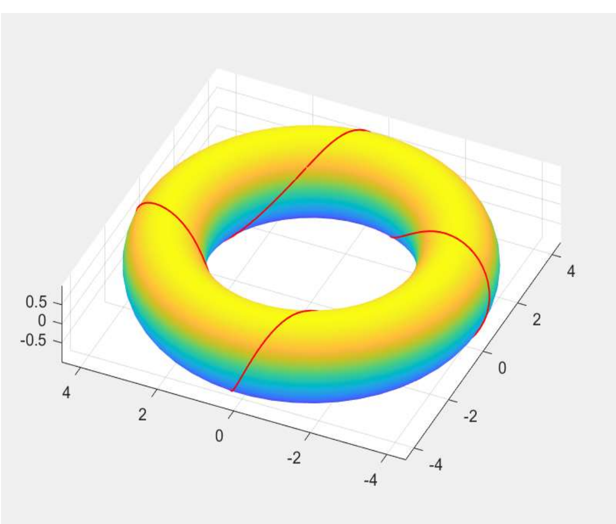
<u>If 0 < c < n-1</u>

$$\frac{d\ theta}{d\ phi} = \frac{c}{(n+\cos(phi))\sqrt{(n+\cos(phi))^2-c^2}}$$

1. According to phase plot (figure 1) , c < n-1 represents the curves above separatrix and corresponds to **unbounded geodesics**.
2. Unbounded geodesics have one special property that, they **<u>pass through all the points on torus</u>**. Because from phase plot one can easily see that the curve has every phi in its domain i.e., all range of phi.
3. Here the angle made at outer equator is high than inner equator which can be seen from figure 3 and 4 clearly.







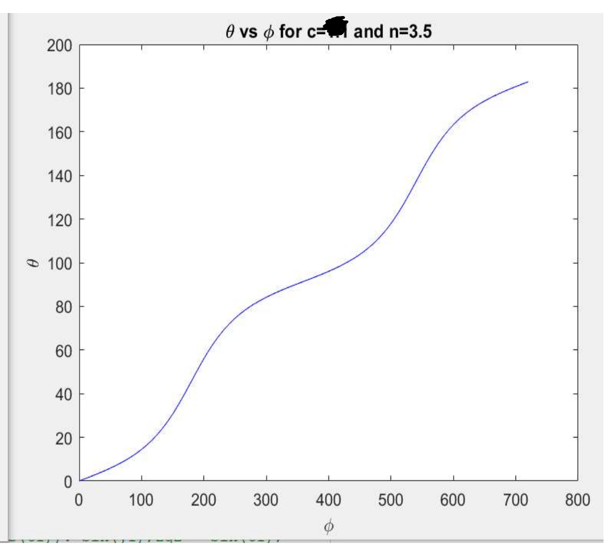


Figure 3
n=3.5, c = 1.1

figure 4
n=3.5, c = 1.8

The right side plot is phi between 0 and 720 degrees/2*pi. We can observe that 'theta' in plot of figure(3) is near 80 degrees and theta in plot of figure(4) is near 180 degrees. So number of times the curve touches inner equator is more in first curve than in second curve.

The cycles will increase as c decreases.

4. From figure 3 and 4 ,the change of value "c" from c = 1.1 to 1.8 results in variation of angle made by geodesic at inner equator(phi = pi). As c tends to n-1 the **geodesic approaches tangentially to inner equator**. This is straight forward from phase plot i.e, as the c tends to n-1 $\frac{d\,phi}{d\,theta}$ approches to zero at phi =pi or odd multiples of pi, which means phi will not have change at inner equator.

If c = n-1 :: The separatrix

$$\frac{d\,theta}{d\,phi} = \frac{c}{(n+\cos(phi))\sqrt{(n+\cos(phi))^2-c^2}}$$

As we see for c = n-1 the phase plot will be separatrix, so some special happens here. The separatrix consists of essentially two cases

1. Initially if phi = pi from (5)--- $\frac{d\,phi}{dS} = \sqrt{1-\frac{c^2}{(n+cos(phi))^2}}$

at phi = pi ; $\frac{d\,phi}{dS}$ = 0.(explain 1) The phase plot of separatrix then is point at phi = pi, which means the geodesic curve is inner equator .

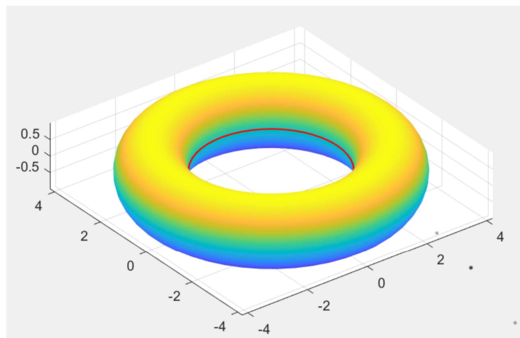


Figure 5
n = 3.2 , c =2.2

2. Initially if phi $\neq$ pi from (5)--- $\frac{d\,phi}{dS} = \sqrt{1-\frac{c^2}{(n+cos(phi))^2}}$, the sepratrix just approches to phi = pi and donot touch the point. Which gives the information that the geodesic curve approches tangentially to inner equator (i.e, phi = pi).

:: This property of extending tangentially to inner equator(case of c=n-1) can be observed from 0 < c < n-1 case, we discussed that curve at inner equator approaching more slantly as c increases(comparing figure (3) and (4)) . Also this can be said from phase plot, as c increases $\frac{d\,phi}{dtheta}$ decreases (for every phi,as one curve lies entirely below the other in phase plot) resulting in less change in phi i.e less change in height z, for a particular change in theta.

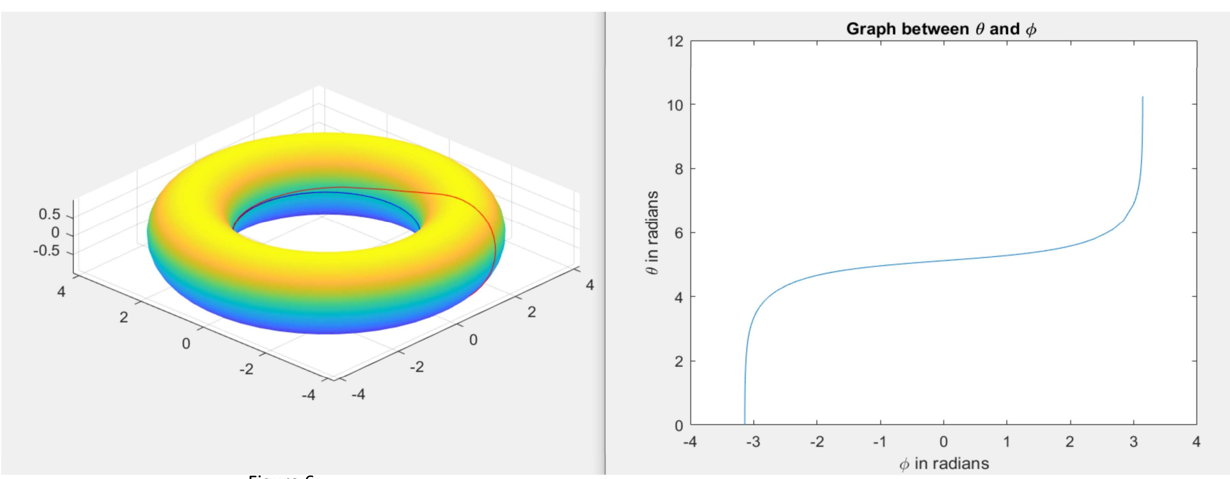


Figure 6:
1.The left side is the geodesic curve on torus with n=3.2 and c = 2.2.
2.The geodesic curve approaches tangentially to inner equator.
3.Right side plot represents that there is no change in phi when phi approaches pi.
Analogously the curve approaches tangentially at phi = -pi also due to symmetry.

The separatrix condition is the bridge between bounded and unbounded geodesics of torus. Now bounded geodesics by the name implies that the family of geodesics on torus doesn't cover entire points/entire torus. Which says the restriction and it is obvious as we get closed curves in phase plot for phi(limited/restricted range).

Bounded geodesics implies c > n-1 and c ϵ [0 , n+1]

$$\frac{d\ theta}{d\ phi} = \frac{c}{(n+\cos(phi))\sqrt{(n+\cos(phi))^2-c^2}}$$

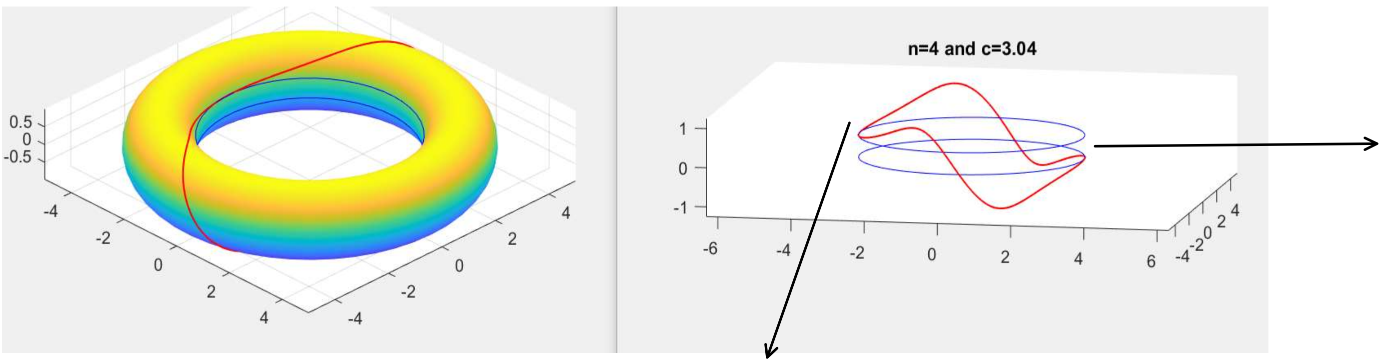
$(n + \cos(phi))^2 > c^2$ implies $\cos(phi) > c - n$ ---------- 8

<u>If n-1 < c < n+1</u>

$$\frac{d\ theta}{d\ phi} = \frac{c}{(n+\cos(phi))\sqrt{(n+\cos(phi))^2-c^2}}$$

from (8) cos(phi) > c - n and c lies between (n-1 , n+1) . Let cos(phi_not) = c - n implies cos(phi) > cos(phi_not). So phi lies between -phi_not and phi_not

<u>The geodesic curve oscillates between -phi not and phi not.</u>



Point 1

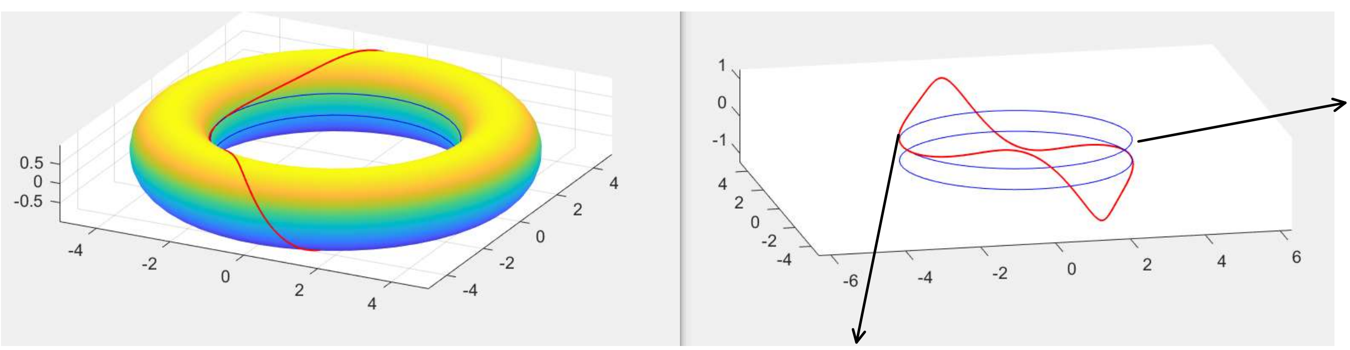
point 2

Figure 7:
n= 4 and c = 3.04 for both figures.
Both figures above are the same i.e., one with torus and other without torus plot for better visualizing of plot. We can clearly see that the Geodesic curve touches/ evolves tangentially to the phi =±phi_not . So this is a bounded geodesic. The phase plot can be justified for figure (7)..i.e.,

Point 1 ------ $\frac{d\ phi}{d\ theta} = 0$ and phi is negative represents left hand side of phase plot.

Point 2 ------ $\frac{d\ phi}{d\ theta} = 0$ and phi is positive represents right hand side of phase plot.

The figure is very difficult to visualize, unless seen with 3-D rotating in matlab.



Point 1

point 2

Figure 8:
n= 3.5 and c = 3.5 for both figures.
Both figures above are the same i.e., one with torus and other without torus plot for better visualizing of plot. We can clearly see that the Geodesic curve touches/ evolves tangentially to the phi =±phi_not . So this is a bounded geodesic. The phase plot can be justified from figure (7)..i.e.,

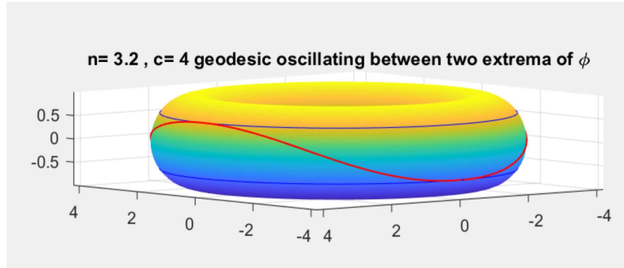
Point 1 ------ $\frac{d\ phi}{d\ theta} = 0$ and phi is negative represents left hand side of phase plot.

Point 2 ------ $\frac{d\ phi}{d\ theta} = 0$ and phi is positive represents right hand side of phase plot.

The figure is very difficult to visualize, unless seen with 3-D rotating in MATLAB.

As the blue circles(the extrema) are moving towards outer equator (from figure (7) and (8))and can also be seen from phase plot as we move towards higher "c" , the curve approaches towards origin i.e., phi = 0 => Outer equator.

Figure 9:
clearly we can observe the oscillations pattern here.

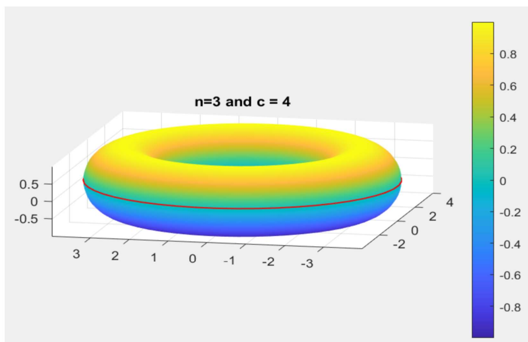


If c = n+1

$$\frac{d\ phi}{dS} = \sqrt{1 - \frac{c^2}{(n+cos(phi))^2}} \quad \text{------ 5}$$

When c = n+1 the only possible value of phi are 0 ,2 pi,4 pi.... Which are analogus to outer equator. So the only possible curve is the outer equator. And the phase plot corresponding to this are point at origin or even multiples of pi. This is also a Global minimum of

$$F(phi, \frac{d\ phi}{d\ theta}) = \frac{1}{(n+\cos(\ phi))^4} (\frac{d\ phi}{d\ theta})^2 + \frac{1}{(n+\cos(\ phi))^2}$$



MATLAB codes for Different Geodesics(all cases included here)

```
%/ k and c are variables
k=;    % shape of Torus(ratio of a to b)
c=;  %Nature of geodesic

%Generating Torus
theta_f= linspace(0,2*pi,50);
phi_f = linspace(0,2*pi,50);
[T,P] = meshgrid(theta_f,phi_f);
%Plotting Torus
xf = (k+cos(P)).*cos(T);yf = (k+cos(P)).*sin(T);zf = sin(P);
figure
surf(xf,yf,zf)
axis equal
shading interp

if c == 0
   %
   theta1=0;
   % /can be given any value
  z = sin(P).*cos(T-T);x = (k+cos(P))*cos(theta1);y =(k+cos(P))*sin(theta1);
  z1 = sin(P).*cos(T-T);x1 = (k+cos(P))*cos(theta1+(pi/6));y1 =(k+cos(P))*sin(theta1+(pi/6));
  z2 = sin(P).*cos(T-T);x2 = (k+cos(P))*cos(theta1+(pi));y2 =(k+cos(P))*sin(theta1+(pi));
  hold on
  plot3(x,y,z,'r',x1,y1,z1,'r',x2,y2,z2,'r')
  axis equal


   elseif (0 < c)&& (c < (k-1))
   %/
   theta1=0;
   %/
   phi1=0;phi2=360*1;
```

```
%ODE45 helps in solving differential equations using RUNGA-KETTA method
dydt = @(t,y) c/((k+cos(t))*sqrt(((k+cos(t))^2 -c^2)));
[t,y] = ode45(dydt , [phi1*pi/180,phi2*pi/180],theta1);
   p = y(end);
  a = floor(2*pi/p);
  hold on
  for i=1:a+1
   xt1 = (k+cos(t)).*cos(y+(i-1)*p);yt1 = (k+cos(t)).*sin(y+(i-1)*p);zt1 = sin(t);
    plot3(xt1,yt1,zt1,'r','linewidth',1)
  end
%plotting only upto phi = 720 degrees becuase it is a repetitive plot
figure
plot(t*180/pi,y*180/pi,'b',(t*180/pi)+360,(p+y)*180/pi,'b')
xlabel('\phi')
ylabel('\theta')
title('\theta vs \phi')


elseif c == k-1
   %/
   theta1=0;
   Phi1 = 2;
   %/
   if phi1 == pi
      z =sin(P-P); x = (k-1)*cos(P);y =(k-1)*sin(P);
     hold on
     plot3(x,y,z,'r','linewidth',1.1)
      axis equal
   else
   phi1=-pi+1e-3;phi2=pi-1e-3;

dydt = @(t,y) c/((k+cos(t))*sqrt(((k+cos(t))^2 -c^2)));
[t,y] = ode45(dydt,[phi1,phi2],theta1);
xg = (k+cos(t)).*cos(y);yg = (k+cos(t)).*sin(y);zg = sin(t);
z1 =sin(P-P); x1 = (k-1)*cos(P);y1 =(k-1)*sin(P);
hold on
plot3(xg,yg,zg,'r',x1,y1,z1,'b')
axis equal
figure
plot(t,y,'b')
grid on
xlabel('\phi')
ylabel('\theta')
title('\theta vs \phi')
   end

elseif ((k-1)<c)&&(c<(k+1))
   phi11 = -acosd(c-k)*pi/180+(1e-6);
   phi21 = acosd(c-k)*pi/180-(1e-6);
   theta1 = 0;

   dydt = @(t,y) c/((k+cos(t))*sqrt(((k+cos(t))^2 -c^2)));
   [t,y] = ode45(dydt , [phi11,phi21],theta1);
   xg = (k+cos(t)).*cos(y);yg = (k+cos(t)).*sin(y);zg = sin(t);
   p = y(end)
   a = floor(2*pi/p)

hold on
for i=1:a+1
   xt1 = (k+cos(t)).*cos(y+(i-1)*p);yt1 = (k+cos(t)).*sin(y+(i-1)*p);zt1 = sin(((-1)^(i-1))*t);
    plot3(xt1,yt1,zt1,'r','linewidth',1)
end
hold on

z2 =sin(acosd(c-k)*pi/180)*cos(T-T);z3 =-sin(acosd(c-k)*pi/180)*cos(T-T); x2 =c*cos(P);y2 =c*sin(P);
plot3(x2,y2,z2,'b',x2,y2,z3,'b')
axis equal

figure
for i=1:a+1
hold on
plot(((-1)^(i-1))*t*180/pi,(y+p*(i-1))*180/pi,'b')
grid on
xlabel('\phi')
ylabel('\theta')
title('\theta vs \phi. Oscillation pattern i.e., phi is bounded')
end
elseif c == k+1
   z =0*cos(P-P); x = c*cos(P);y =c*sin(P);
     hold on
     plot3(x,y,z,'r','linewidth',1)
     axis equal
end
```

MATLAB code for phase plot

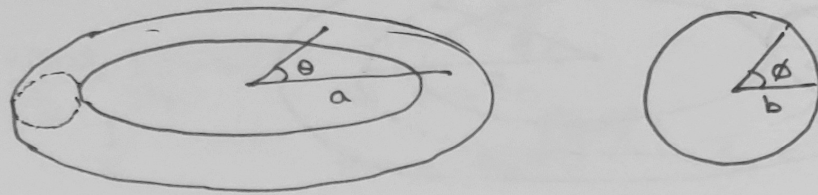
```
%/
n=;
%/
k=2;
phi = linspace(-k*pi,k*pi,500);
y = linspace(-9,9,500);
[X,Y] = meshgrid(phi,y);

F = ((Y.*Y)+((n+cos(X)).*(n+cos(X)))))./((n+cos(X)).*(n+cos(X)).*(n+cos(X)).*(n+cos(X)));
v1 =[0.25,0.25];
v2 =[0.1,0.1];
v3 =[0.08,0.08];
v4 =[0.15,0.15];

figure
contour(X,Y,F,v1,'r')
xlabel('\phi')
ylabel('d\phi/d\theta')
title('Phase plot of geodesic curves')
colorbar
grid on

hold on
contour(X,Y,F,17)
contour(X,Y,F,v2,'b')
contour(X,Y,F,v3,'b')
contour(X,Y,F,v4,'b')
```

Note: While simulating the figures, every time closed curves are not possible. Here i had plotted/simulated only up to certain range of theta and phi, some figures may be closed(although rarely) and some are not closed. If we want to get closed curves we have to extend the range of theta and phi and also the range is not able to determine exactly always. Also MATLAB takes several steps and much time to plot if large range is given. As we are interested in only the nature of geodesic it is not mandatory to get closed curves.



$$\ddot{q}_l + \frac{1}{2}\Gamma^{l}_{ij}\frac{dq_i}{ds}\frac{dq_j}{ds} = 0 \qquad q_1 \equiv \theta,\ q_2 \equiv \phi$$

$$\Gamma^{l}_{ij} = m^{lk}\left[\frac{\partial m_{ki}}{\partial q_j} + \frac{\partial m_{kj}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right]$$

Where $M = \begin{pmatrix} (n+\cos(\phi))^2 & 0 \\ 0 & 1 \end{pmatrix}$ assumed $b = 1$, $a/b = n$ for generality.

$$\ddot{q}_1 + \frac{1}{2}\Gamma^{\theta}_{ij}\frac{dq_i}{ds}\frac{dq_j}{ds} = 0$$

$$\Gamma^{\theta}_{\theta\theta} = m^{\theta\theta}\left[0 + 0 - 0\right] + 0(\quad) = 0$$

$$\Gamma^{\theta}_{\theta\phi} = m^{\theta\theta}\left[0 + \frac{\partial m_{\theta\theta}}{\partial\phi} - 0\right] + 0(\quad)$$

$$= \frac{1}{(n+\cos\phi)^2} \times -2a\sin\phi\,(n+\cos\phi) = \frac{-2a\sin\phi}{(n+\cos\phi)}$$

$$\Gamma^{\theta}_{\phi\theta} = m^{\theta\theta}\left[\frac{\partial m_{\theta\theta}}{\partial\phi} + 0 - 0\right] + 0(\quad)$$

$$= \frac{-2a\sin\phi}{n+\cos\phi}$$

$$\Gamma^{\theta}_{\phi\phi} = m^{\theta\theta}\left[0 + 0 - 0\right] + 0(\quad) = 0$$

$$\boxed{\ddot{\theta} - \frac{2\sin\phi}{n+\cos\phi}\dot{\theta}\dot{\phi} = 0} \qquad \dot{\theta} = d\theta/ds \quad \dot{\phi} = \frac{d\phi}{ds}$$

Similarly

$$\ddot{q}_2 + \frac{1}{2}\Gamma^{\phi}_{ij}\frac{dq_i}{ds}\frac{dq_j}{ds} = 0$$

Similarly

$$\ddot{q_2} + \frac{1}{2}\Gamma^{\phi}_{ij}\frac{dq_i}{ds}\frac{dq_j}{ds} = 0$$

$$\Gamma^{\phi}_{\theta\theta} = 0(\quad) + 1(-2\sin\phi(n+\cos\phi))$$

$$= 2\sin\phi(n+\cos\phi)$$

$$\Gamma^{\phi}_{\theta\phi} = 0(\quad) + 1(0+0-0)$$

$$\Gamma^{\phi}_{\phi\theta} = 0(\quad) + 1(0+0-0)$$

$$\Gamma^{\phi}_{\phi\phi} = 0(\quad) + 1(0+0-0)$$

$$\boxed{\ddot{\phi} + \sin\phi(n+\cos\phi)\left(\frac{d\theta}{ds}\right)^2 = 0}$$

Calculation - 2

$$\frac{d^2\theta}{ds^2} - \frac{2\sin(\phi)}{(n+\cos\phi)}\frac{d\theta}{ds}\frac{d\phi}{ds} = 0$$

$$(d\theta/ds)^2 = U \Rightarrow \frac{dU}{ds} = 2\frac{d\theta}{ds} \times \frac{d^2\theta}{ds^2}$$

$$\frac{1}{2} \times \frac{dU}{ds} - \frac{2\sin(\phi)}{n+\cos\phi} U \frac{d\phi}{ds} = 0$$

$$\frac{dU}{U} = \frac{4\sin\phi}{n+\cos\phi}d\phi \longrightarrow \ln(U) = -4\ln(n+\cos\phi) + c_1$$

$$\Rightarrow U(n+\cos\phi)^4 = c_1$$

$$\Rightarrow \boxed{\frac{d\theta}{ds} \times (n+\cos\phi)^2 = c} \quad \text{— (4)}$$