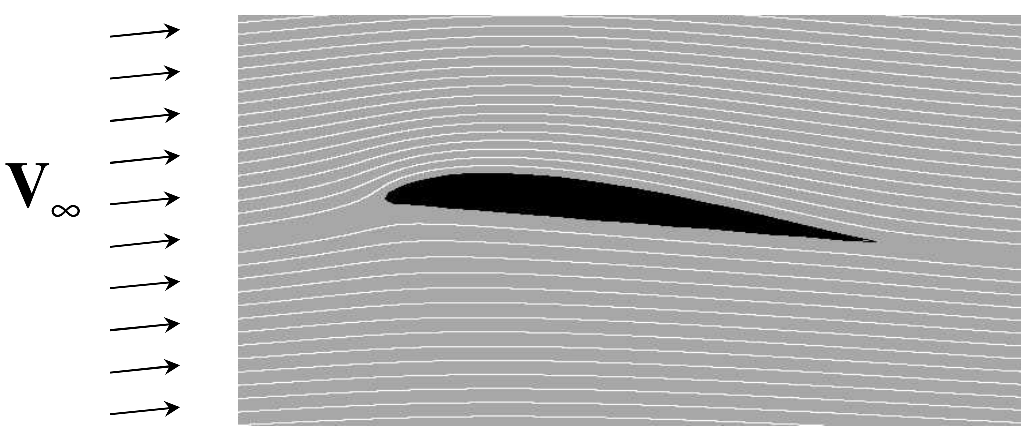
2. Two-dimensional airfoils at low M



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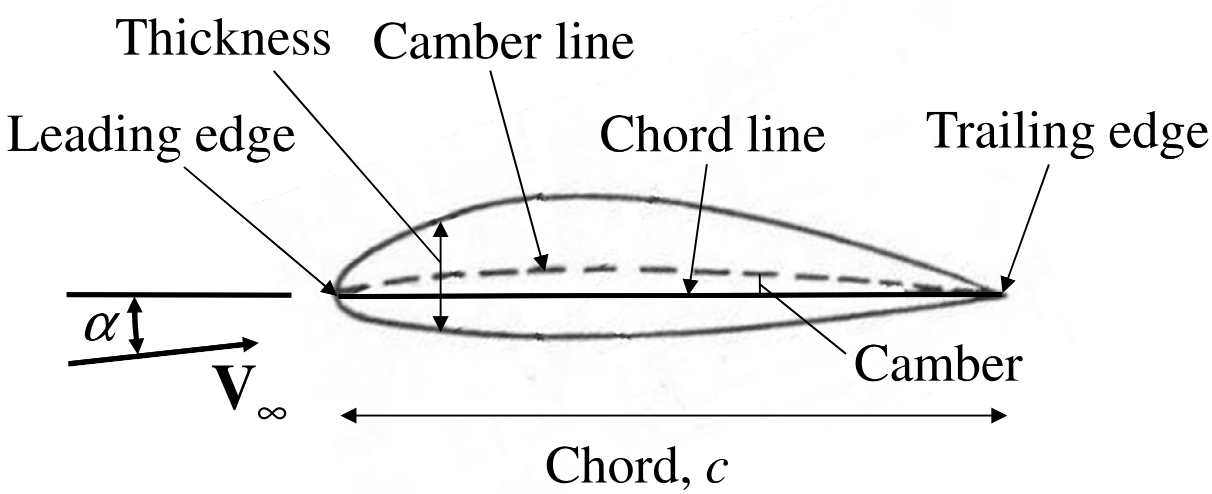
2.1 General characteristics of airfoils

Main aim of airfoil: provide lift with low drag over a range of operating conditions.

Nomenclature and notation:



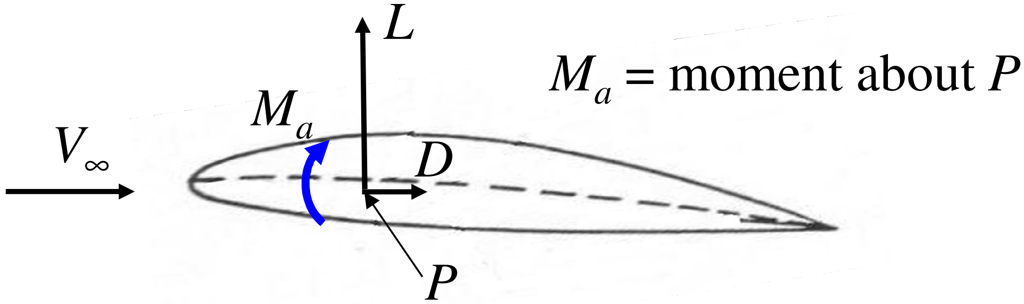
- chord line: joins leading and trailing edges.
- camber line: midway between surfaces.
- camber = distance from chord to camber line.
  Uncambered = symmetric.

α = angle of attack, c = chord
$\mathbf{V}_\infty$ = air velocity far from airfoil

Note: in general, $\alpha_{airfoil} \neq \alpha_{aircraft}$.

Aerodynamic coefficients

As for the aircraft as a whole, aerodynamic forces are split into lift and drag:



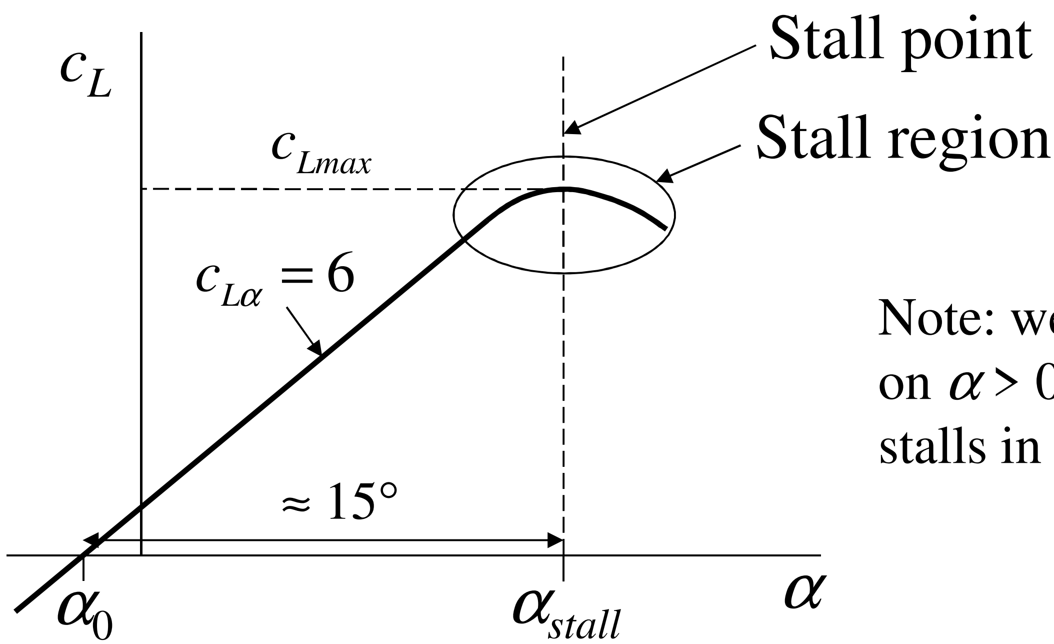
- a 2D airfoil is infinitely long $\Rightarrow$ forces are now *per unit length in the third dimension*
- M_a depends on choice of point P

Non-dimensional coefficients:

$$\boxed{\begin{gathered} L = \frac{1}{2}\rho c V_\infty^2 c_L \qquad D = \frac{1}{2}\rho c V_\infty^2 c_D \\ M_a = \frac{1}{2}\rho c^2 V_\infty^2 c_M \end{gathered}}$$

Lift coefficient versus α

Typically:



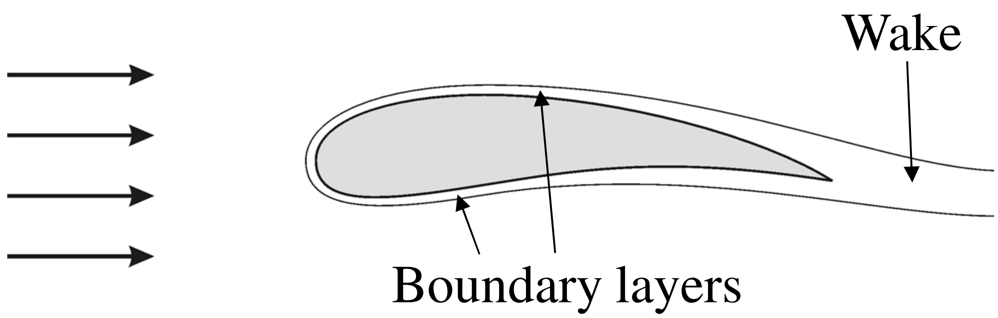
Note: we focus on $\alpha > 0$; also stalls in $\alpha < 0$.

- strongly resembles $C_L(\alpha)$ for whole aircraft (not a coincidence, but avoid making equation: *aircraft = airfoil*)
- stall causes loss of lift (and increased drag): determines c_{Lmax}

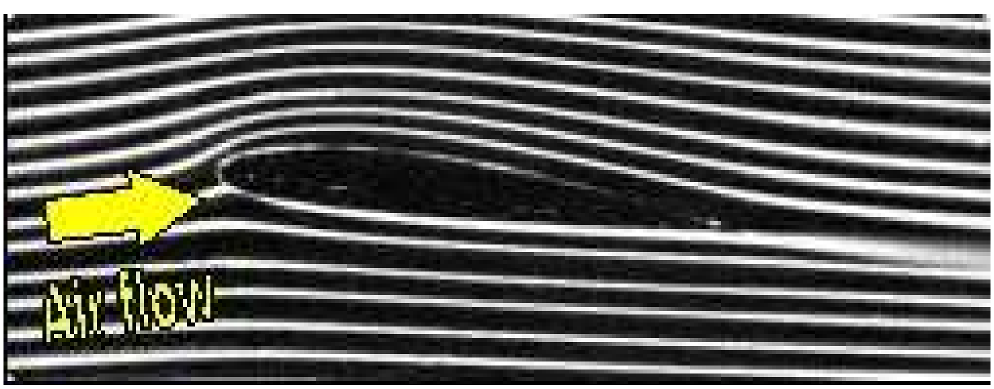
Avoiding stall is central to airfoil design.

Flow structure

Reynolds number, $Re = V_\infty c / \nu \sim 10^7$, is very large $\Rightarrow$ thin boundary layers and wake:



- here, layer thickness exaggerated for clarity
- stall arises from separation of boundary layer over upper surface:



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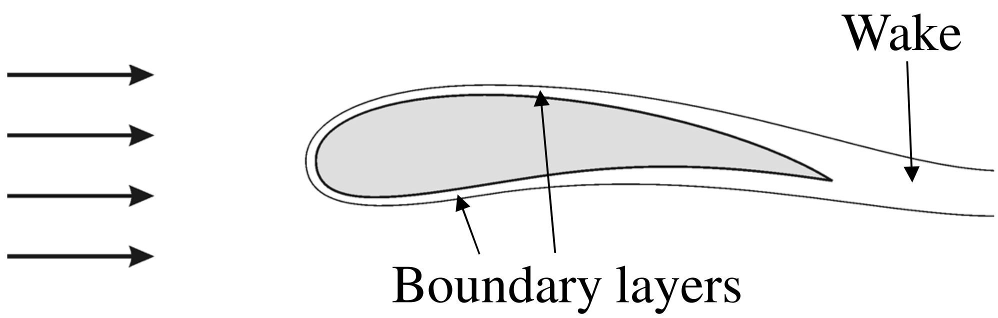

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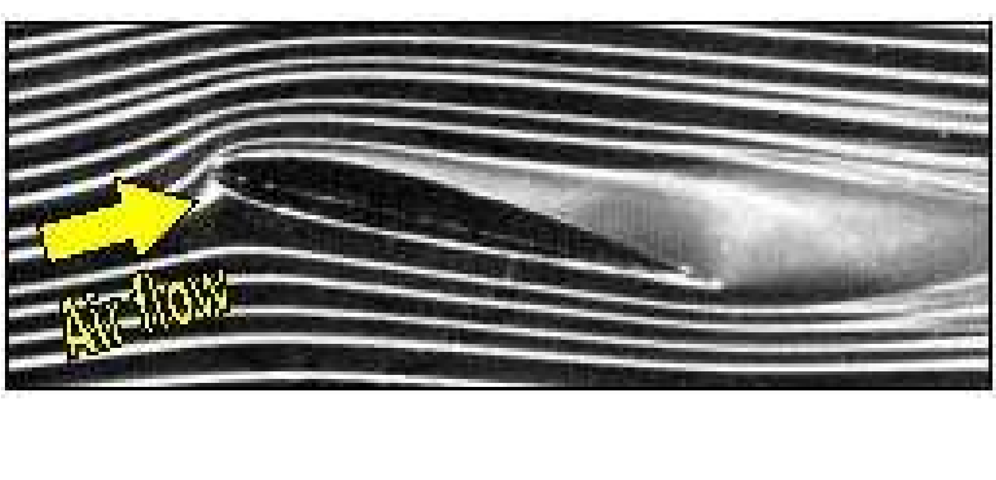

Stalled: note thicker wake $\Rightarrow$ increased drag

Flow structure

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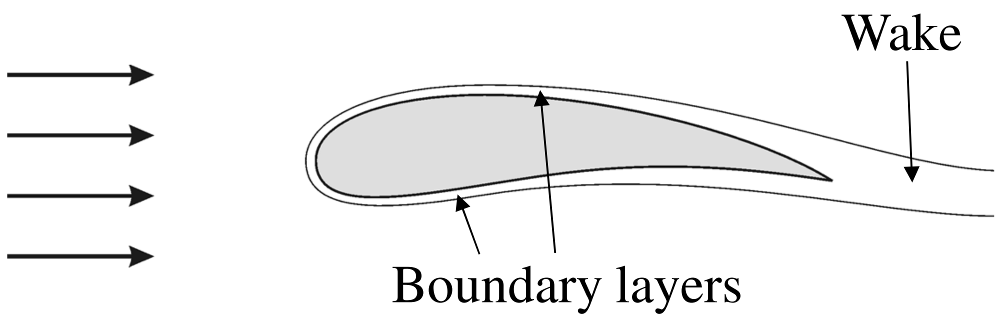
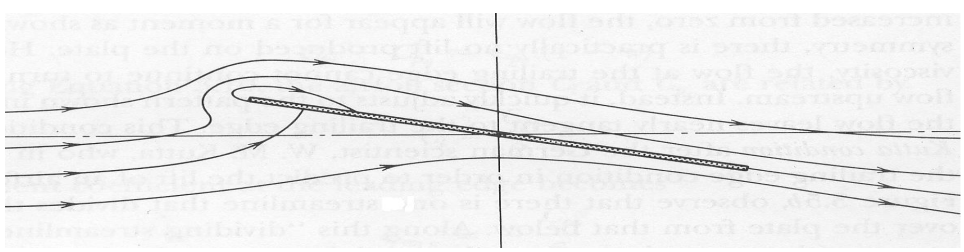


- here, layer thickness exaggerated for clarity
- stall arises from separation of boundary layer over upper surface:

Discouraging stall

☞ airfoil thin with small angle of attack
☞ sharp trailing edge: allows top and bottom boundary layers to join smoothly and form narrow wake

So let's try a flat plate:



As α increases from zero:

- small α: leading-edge separation "bubble" (reattachment of boundary layer)

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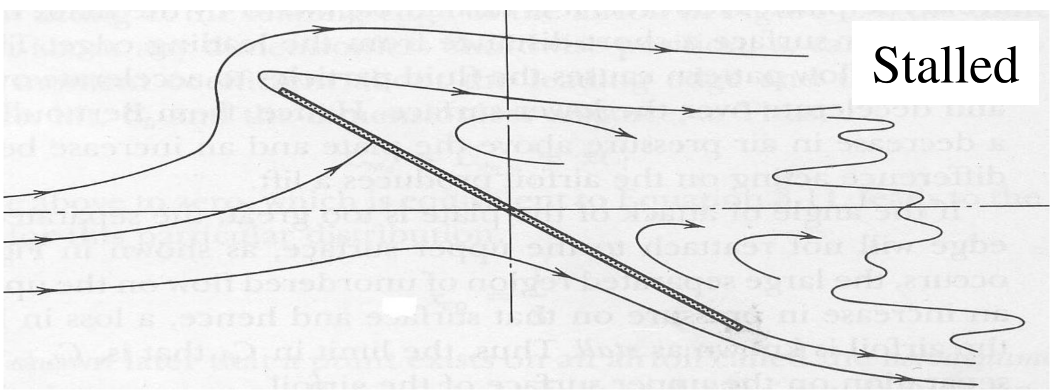


As α increases from zero:

- small α: leading-edge separation "bubble" (reattachment of boundary layer)
- larger α: no longer reattaches - stalls at $\alpha \approx 8°$ with c_{Lmax} only 0.8

Discouraging stall (contd)

☞ rounded leading edge discourages separation at that edge ⇒ increases c_{Lmax}

☞ camber further increases c_{Lmax}

⇒ typical airfoil shape. Detailed design fixes distributions of camber and thickness for a given application. In particular:

- radius of leading edge,
- locations and values of maximum camber and thickness.

• some airfoils, e.g. stabilizers, must work at either positive or negative lift ⇒ camber may be inappropriate.

• here we have focused on stalling and maximum lift. Other factors, e.g. drag, compressibility effects (Ma) and structural considerations affect design.

LE versus TE stall

Stall can occur at either leading or trailing edge:

Leading-edge separation

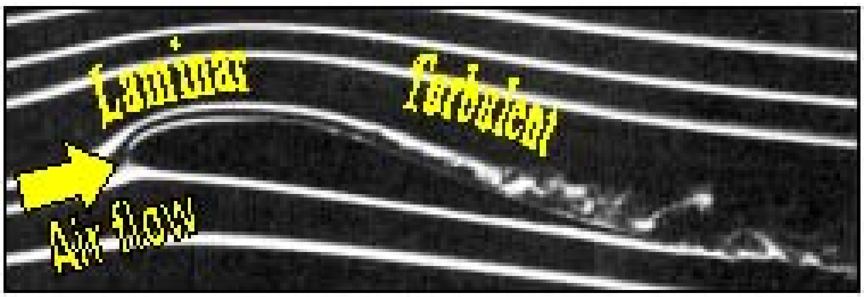
- thin airfoils with sharp leading edges (archetype = flat plate),
- stall may happen abruptly via bursting of separation bubble. Usually bad because:
  - harder to control aircraft,
  - pilot has less warning of stall.

Trailing-edge separation

- thicker airfoils with rounded leading edges (and encouraged by increasing *Re*),
- usually gradual: separation point moves progressively upstream from trailing edge as α increases.

Boundary-layer turbulence

Boundary layer is usually turbulent over rear part of upper airfoil surface:



- aerodynamic force on surface = pressure + viscous skin friction
- turbulent skin friction considerably higher than laminar $\Rightarrow$ increased drag $\Rightarrow$ delaying transition can reduce drag
- but, turbulent boundary layers are more resistant to separation than laminar ones $\Rightarrow$ turbulence can have a beneficial effect!
- lower-surface boundary layer can also be turbulent near trailing edge

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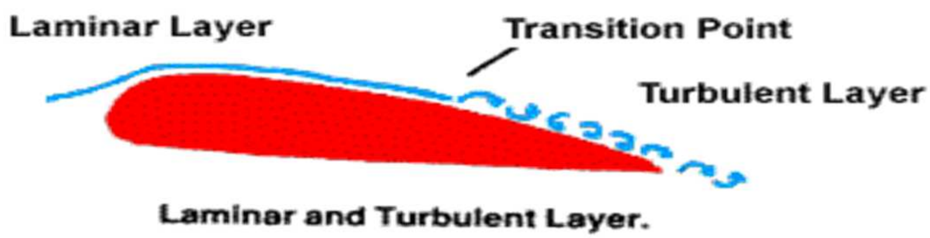

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Boundary-layer turbulence (contd)

Transition location determined by surface pressure distribution, Reynolds number and surface roughness.

Reynolds number and roughness effects:

☞ increasing Reynolds number causes transition point to move upstream

☞ above a certain threshold size, roughness can provoke “premature” transition

☞ roughness can also have a direct effect on skin friction and hence increase drag

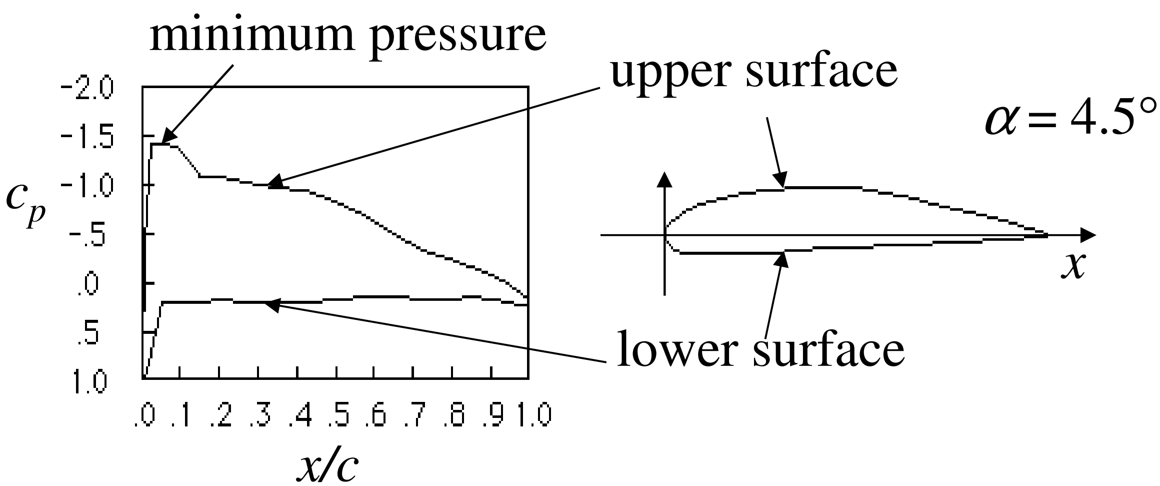
Effects of surface pressure distribution...

2.2 Surface pressure distribution

Usually expressed in nondimensional form:

$$c_p = \frac{p - p_\infty}{\rho V_\infty^2 / 2}$$

Typically:



- lift comes (mainly) from pressure difference between lower and upper surfaces (area between curves)
- maximum pressure at stagnation point near leading edge
- minimum pressure on upper surface

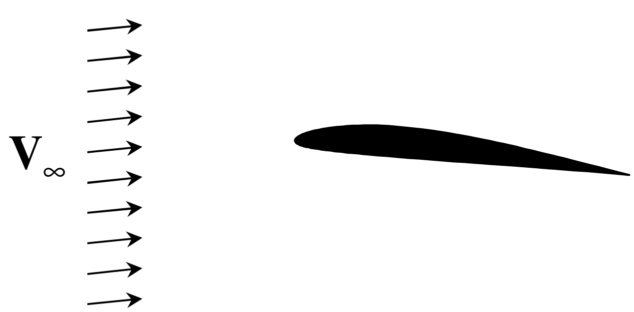
Consequences of Bernoulli

Assuming unseparated: pressure $p(x)$ imposed from outside boundary layer, where viscosity is negligible. Bernoulli applied to external flow:

$$p + \frac{1}{2}\rho V_{ext}^2 = constant \Rightarrow c_p = 1 - \left(\frac{V_{ext}}{V_\infty}\right)^2$$

where V_{ext} = velocity just outside layer.

• $V_{ext}\uparrow \Rightarrow p\downarrow$ and vice versa:



inviscid theory

• maximum pressure c_p = 1 at stagnation point
• minimum pressure $\Leftrightarrow$ maximum V_{ext}

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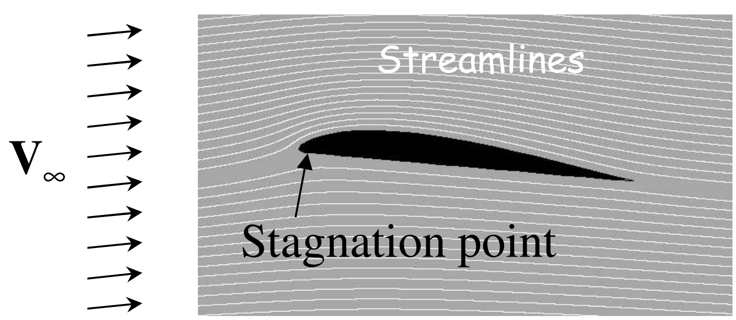

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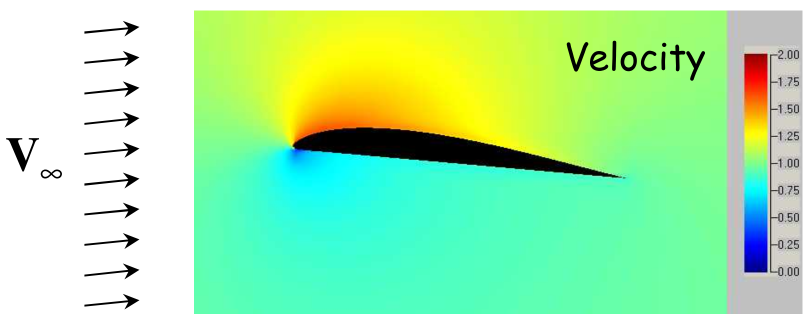

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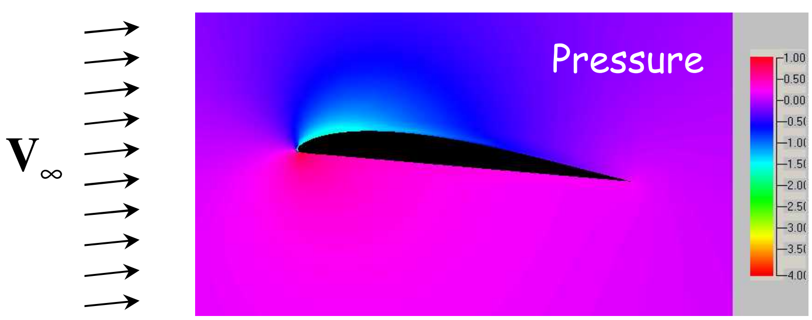


inviscid theory

- maximum pressure c_p = 1 at stagnation point
- minimum pressure $\Leftrightarrow$ maximum V_{ext}

Effects of surface pressure gradient

"Adverse" gradient = tries to slow the fluid and thicken the boundary layer.
"Favorable" gradient = tries to accelerate the fluid and thin the boundary layer.

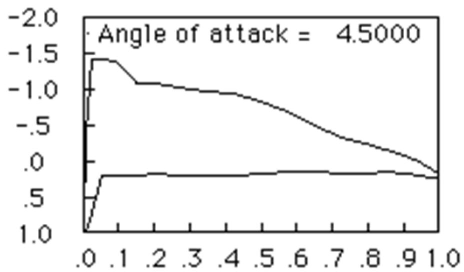
> An adverse pressure gradient encourages both separation and transition.

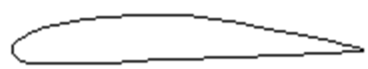
- airfoils have an adverse gradient over rear of upper surface, hence tendency for separation and turbulence there
- low-drag airfoils designed to delay transition by reducing the region of adverse gradient
- but, surface roughness can increase drag both directly and via premature transition

Example of roughness: insects on low-drag designed wings.

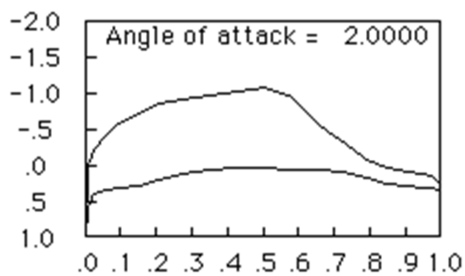
Examples of airfoils

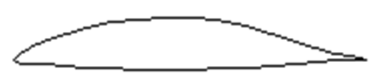
Conventional cambered airfoil (given before):





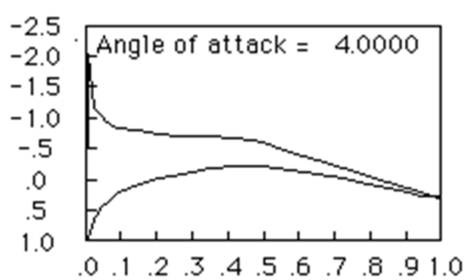
Low-drag design:

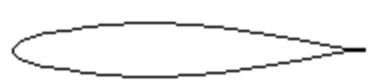




Minimum pressure further back

Symmetric airfoil:





Note sharp peak in pressure at nose

Examples of airfoils (contd)

- apparently minor changes in geometry can $\Rightarrow$ significant differences in pressure distribution and hence characteristics
- airfoil design mainly consists of tailoring the pressure distribution for a given application and range of operating conditions (including different α)

2.3 Aerodynamic coefficients

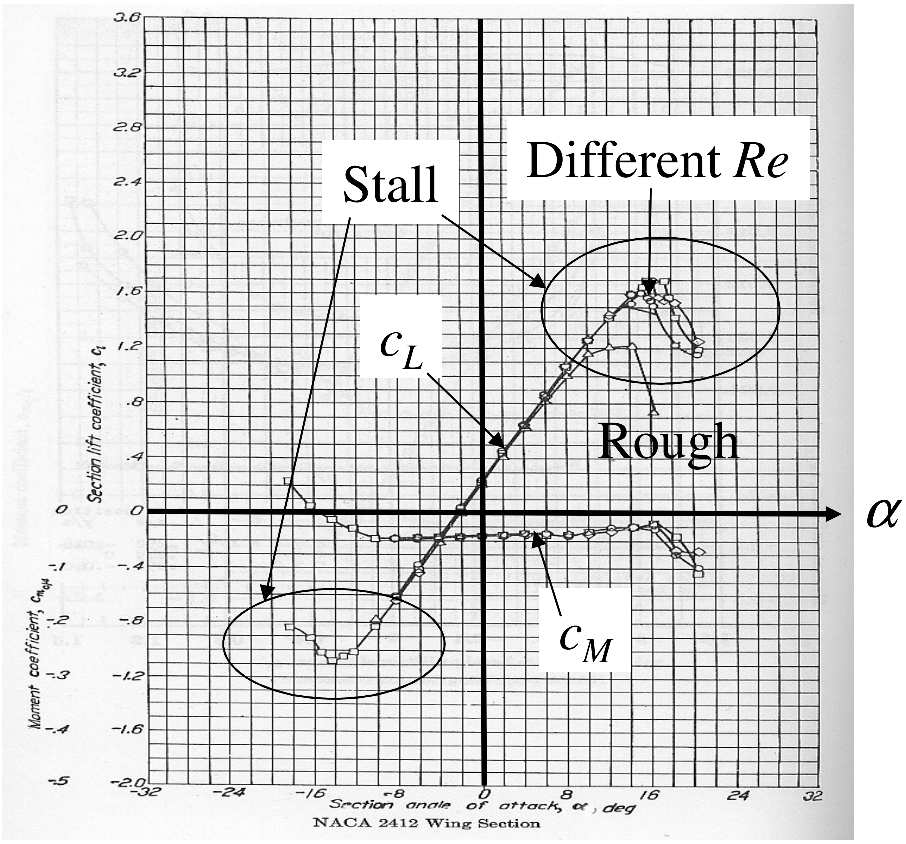
Most important airfoil characteristics. Depend on:

☞ airfoil shape (and roughness)
☞ angle of attack α
☞ Mach number (compressibility effects, not treated in this chapter)
☞ Reynolds number

Coefficients determined using:

- experiments (c.f. literature, e.g. Abbott and Von Doenhoff). Final judge and jury.
- numerical methods (differing degrees of accuracy and sophistication). Design.
- analytical methods (e.g. thin airfoil theory). Understanding and preliminary design.

Properties of lift coefficient $c_L(\alpha)$



- NACA-2412
- $c_{M_{c/4}}$
- different vertical scales

- virtually independent of *Re* (here ranging from 3.1×10^6 to 8.9×10^6)
- unstalled $c_L(\alpha)$ nearly linear: $c_L = c_{L\alpha}(\alpha - \alpha_0)$

  $c_{L\alpha} = 0.11$ per degree varies little between airfoils

  α_0 ($= -2°$ here) depends on airfoil (zero for uncambered airfoils)

Properties of lift coefficient (contd)

- stall angle $\Rightarrow c_{L\max}$: may be sensitive to roughness. Present smooth stall angle $\approx 16°$ and $c_{L\max} \approx 1.6$ are typical.
- in practice, α usually lies between α_0 (zero lift angle) and stall angle, a range of 18° in present smooth-surface case.

Properties of moment coefficient $c_M(\alpha)$

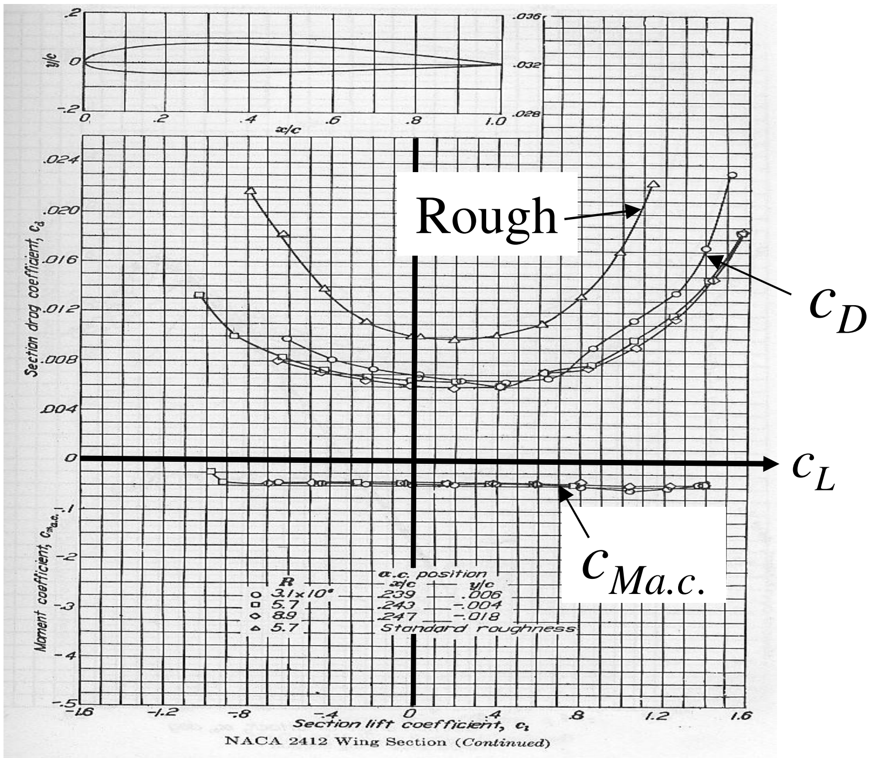
- in figure, moment about *quarter-chord point*
- note different scale than for c_L
- like $c_L(\alpha)$, virtually independent of *Re*
- very nearly constant over unstalled range of α (slight linear increase)

Properties of moment coefficient (contd)

Aerodynamic centre = point on chord line about which the unstalled moment is independent of α.

- analogous to neutral point for whole aircraft
- can be thought of as point of application of aerodynamic forces: unstalled $c_M = c_{Ma.c.}$ no longer depends on α
- aerodynamic centre is close to the quarter-chord point
- unstalled $c_{Ma.c.}$ zero for symmetric airfoils ($\alpha = 0 \Rightarrow L = M_a = 0$ by symmetry)

Properties of drag coefficient $c_D(\alpha)$



- plotted as function of c_L over unstalled range
- note different vertical scales
- more significant *Re* effects on drag
- roughness effects even when unstalled
- quadratic approximation:

$$c_D = c_{D0} + kc_L^2$$

2.4 Inviscid theory

Conditions for inviscid theory:

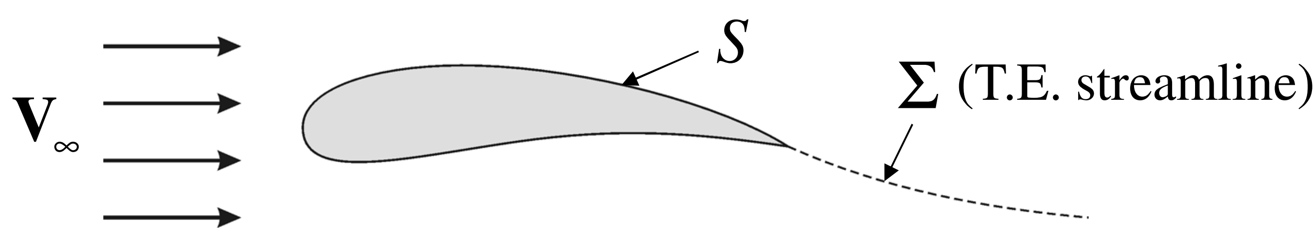
- large Reynolds number
- no separation of boundary layers

☞ viscous term dropped ⇒

$$\frac{D\mathbf{V}}{Dt} = -\frac{1}{\rho}\nabla p \qquad \nabla\boldsymbol{.}\mathbf{V} = 0 \qquad \text{Equations of motion}$$

where second equation (incompressibility) supposes $Ma << 1$, as throughout this chapter.

☞ boundary layer and wake infinitely thin:



Note: flow assumed *steady* from here on.

Irrotationality

Vorticity defined by:

$$\boldsymbol{\omega} = \nabla \times \mathbf{V}$$

evolves following a fluid particle according to:

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\boldsymbol{.}\nabla\mathbf{V}$$

$\Rightarrow$ irrotationality ($\boldsymbol{\omega} = 0$) of a particle conserved

At upstream infinity, $\mathbf{V} = \mathbf{V}_\infty$ is irrotational.
Following any particle in from infinity $\Rightarrow$

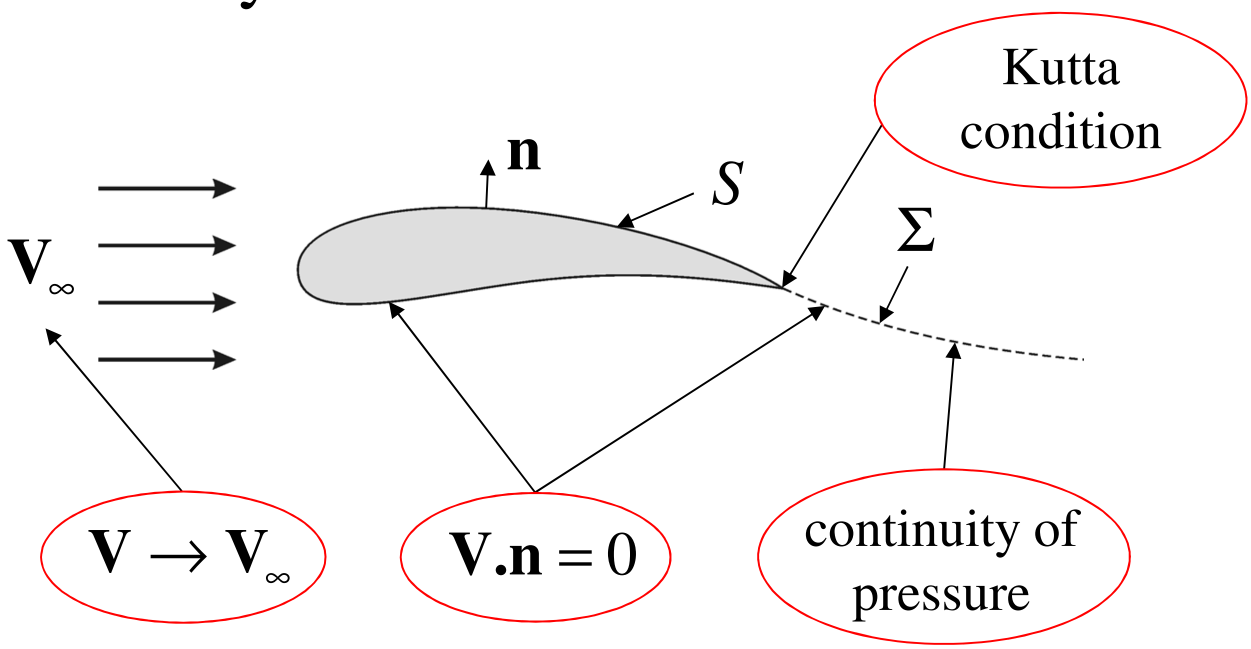
$$\boxed{\text{Flow is irrotational everywhere.}}$$

$\Rightarrow$ equations of motion become:

$$\nabla \times \mathbf{V} = 0 \qquad \nabla\boldsymbol{.}\mathbf{V} = 0$$

$$p = p_\infty + \frac{1}{2}\rho\left(V_\infty^2 - V^2\right) \qquad \text{(B)}$$

Boundary conditions



- Kutta condition: finite velocity at trailing edge
- in 3D, the tangential component of $\mathbf{V}$ can have a jump discontinuity at Σ (trailing vortex sheet: see next chapter), but...

Specialising to 2D for the remainder of this chapter, (B) and boundary conditions on $\Sigma \Rightarrow$

No trailing vortex sheet in 2D case $\Rightarrow$ Σ and its boundary conditions dropped.

Boundary-value problem

Boundary-value problem to determine **V**:

$$\nabla \times \mathbf{V} = 0 \qquad \text{irrotationality}$$

$$\nabla . \mathbf{V} = 0 \qquad \text{incompressibility}$$

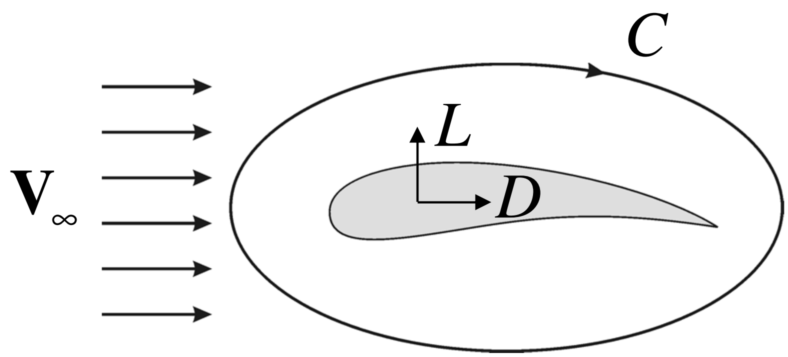
$$\mathbf{V.n} = 0 \text{ on airfoil surface } S$$

Kutta condition at trailing edge

$$\mathbf{V} \rightarrow \mathbf{V}_\infty \text{ at infinity}$$

- once velocity field is known, pressure follows from Bernoulli (B)
- the above problem for **V** is linear (thanks to irrotationality) $\Rightarrow$ much easier than might have been expected

Lift, drag and circulation



C = any closed curve going once around airfoil in indicated sense

Circulation: $$\Gamma = \oint_C \mathbf{V}.d\mathbf{x}$$

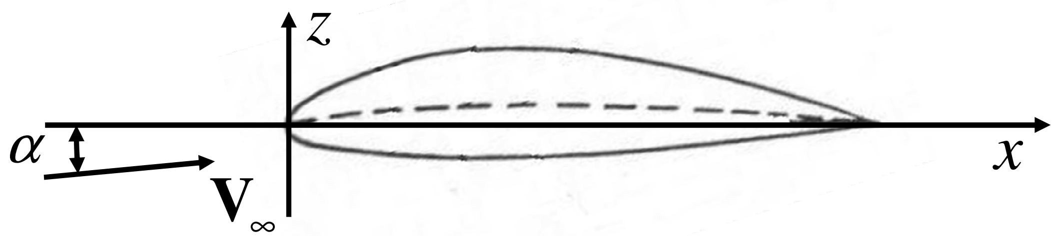
- irrotationality and Stokes theorem $\Rightarrow$ Γ is independent of choice of C

Lift and drag determined by a momentum audit (over a large circle; see MF1 course):

$$L = \rho \Gamma V_\infty \qquad\qquad D = 0$$

- no drag according to 2D inviscid theory
- lift determined by circulation, in turn fixed by the Kutta condition

Stream function



Incompressibility condition:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z} = 0$$

$\Rightarrow$ stream function $\psi(x, z)$:

$$V_x = \frac{\partial \psi}{\partial z} \qquad V_z = -\frac{\partial \psi}{\partial x}$$

Vorticity:

$$\boldsymbol{\omega} = (0, \omega, 0) \qquad \omega = \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Irrotationality $\Rightarrow$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \qquad \text{Laplace equation}$$

Stream function (contd)

Streamlines:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial z} dz = V_x dz - V_z dx = 0$$

$\Rightarrow$ curves of constant ψ.

Reformulation of boundary-value problem:

$$\boxed{\begin{gathered} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \\ \psi = K \text{ on airfoil surface } S \\ |\nabla \psi| \text{ finite at trailing edge} \\ \psi \sim V_\infty \left(z \cos\alpha - x \sin\alpha \right) \text{ at infinity} \end{gathered}}$$

- simpler because only a single unknown: ψ
- K = arbitrary constant (can be chosen at will)
- solution of Laplace via standard methods

Velocity potential

Irrotationality:

$$\omega = \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} = 0$$

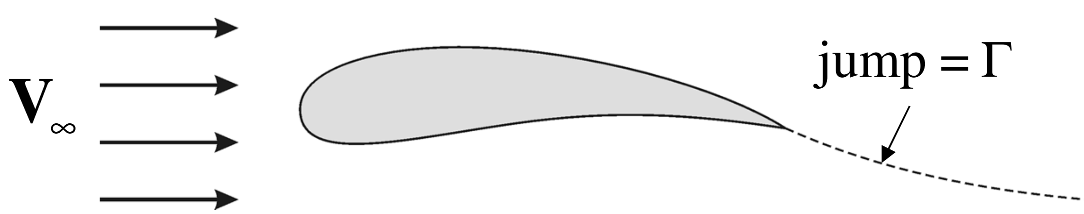
$\Rightarrow$ velocity potential $\phi(x, z)$:

$$V_x = \frac{\partial \phi}{\partial x} \qquad V_z = \frac{\partial \phi}{\partial z}$$

Incompressibility $\Rightarrow$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{Laplace again}$$

- ϕ discontinuous:



- unlike stream function, velocity potential useable in 3D: $\mathbf{V} = \nabla \phi$

Velocity potential (contd)

• formulation of boundary-value problem for velocity potential left as an exercise...

Laplace equation arises in both stream-function and velocity-potential formulations. Methods for solving Laplace include:

- conformal mapping
- thin-airfoil theory (approximate)
- panel methods (numerical)

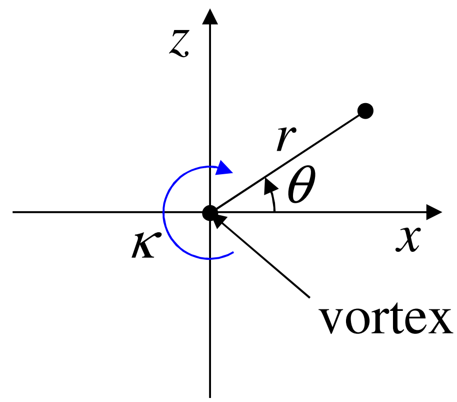
Analytical solutions for certain geometries, but numerics needed in general.

Elementary solutions of Laplace

Line vortex of circulation κ

$$\psi = \frac{\kappa}{2\pi}\log r \qquad \phi = -\frac{\kappa}{2\pi}\theta$$

$$\Rightarrow \quad V_r = 0 \qquad V_\theta = -\frac{\kappa}{2\pi r}$$



- Laplace satisfied apart from singularity (vortex)
- streamlines = circles centred on vortex
- velocity is infinite at vortex
- corresponds to Dirac distribution of vorticity at vortex location, i.e. on *y*-axis

Airfoil at large distances compared to chord:

$$\psi \sim \underbrace{V_\infty\left(z\cos\alpha - x\sin\alpha\right)}_{\textit{Uniform flow } \mathbf{V}_\infty} + \underbrace{\frac{\Gamma}{2\pi}\log r}_{\textit{Effect of airfoil}} + \ldots$$

$\Rightarrow$ at large *r*, airfoil looks like a line vortex.

Elementary solutions (contd)

Line source of strength q

$$\psi = \frac{q}{2\pi}\theta \qquad \phi = \frac{q}{2\pi}\log r$$

$$\Rightarrow \quad V_r = \frac{q}{2\pi r} \qquad V_\theta = 0$$

- corresponds to line source of fluid on y-axis
- streamlines = radially out from source
- velocity is infinite at source
- q = volume flux of source per unit length

Linearity of Laplace equation $\Rightarrow$ can construct more general solutions by superposition of elementary ones.

- other elementary solutions, e.g. multipoles

Pros and cons of inviscid theory

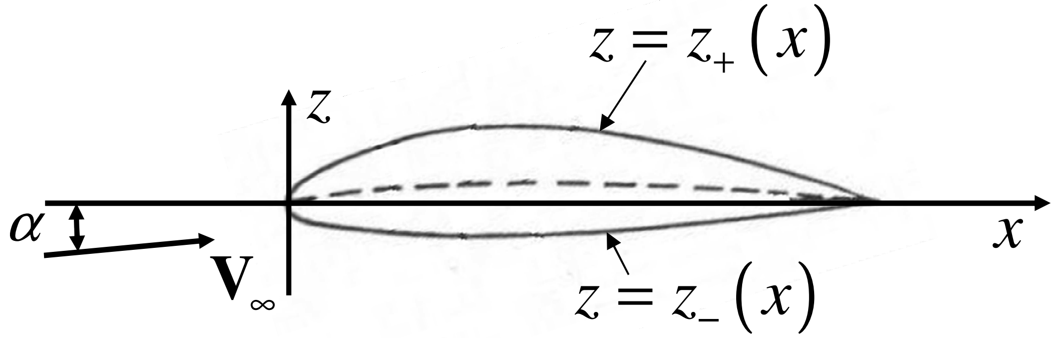
Inviscid theory:

- approximation which has nothing directly to say about drag or separation
- simple and low cost determination of pressure distribution, lift, moment and location of aerodynamic centre

Based on results of inviscid calculation:

- pressure distribution allows diagnosis of e.g. separation
- a subsequent boundary-layer calculation can be performed, yielding information on e.g. skin friction and transition
- a second inviscid calculation may be carried out, allowing for effects of the boundary layer on the external flow

2.5 Thin-airfoil theory



$$\zeta(x) = \frac{1}{2}(z_+ + z_-) \quad \text{camber}$$

$$\tau(x) = z_+ - z_- \quad \text{thickness}$$

Writing

$$\psi = \underbrace{V_\infty (z\cos\alpha - x\sin\alpha)}_{\textit{Uniform flow } \mathbf{V}_\infty} + \underbrace{\psi_a}_{\textit{Effect of airfoil}}$$

boundary condition on airfoil surface is

$$\psi_a = V_\infty \left(x\sin\alpha - z_\pm(x)\cos\alpha\right) + K \qquad z = z_\pm(x)$$

Thin-airfoil theory approximations:

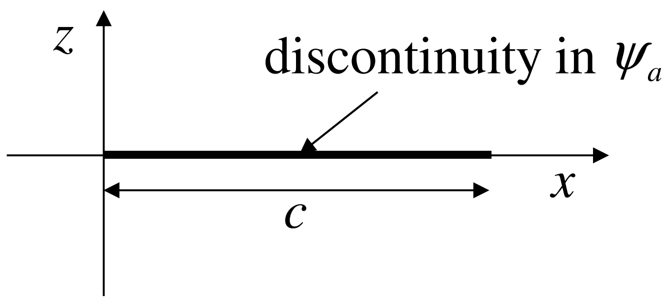
☞ thickness and camber small compared with chord $\Rightarrow$ above boundary conditions are applied at $z = 0$ instead of $z = z_\pm(x)$

☞ small $\alpha \Rightarrow \sin\alpha \approx \alpha, \cos\alpha \approx 1$

Thin-airfoil approximation

Surface boundary condition becomes:

$$\psi_a = V_\infty\left(\alpha x - z_\pm(x)\right) + K \qquad z = 0\pm, \quad 0 < x < c$$



Airfoil replaced by line discontinuity

Thin-airfoil boundary-value problem for ψ_a:

$$\frac{\partial^2 \psi_a}{\partial x^2} + \frac{\partial^2 \psi_a}{\partial z^2} = 0$$

$$\psi_a(x, 0\pm) = V_\infty\left(\alpha x - z_\pm(x)\right) + K \quad 0 < x < c$$

Kutta condition

$$\psi_a = o(r) \quad \text{as } r \to \infty$$

Line vortex/source components

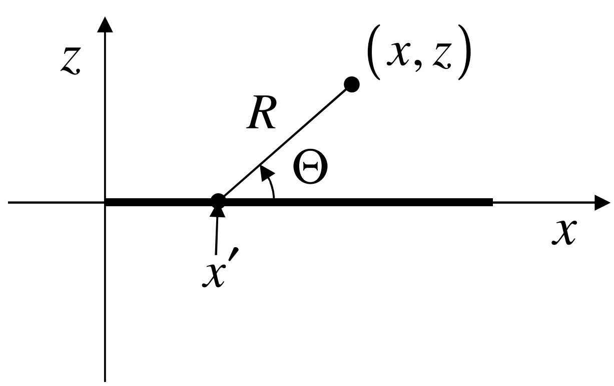
Look for solution:

$$\psi_a = \psi^{(v)} + \psi^{(s)}$$

where

$$\psi^{(v)}(x,z) = \frac{1}{2\pi}\int_0^c \gamma(x')\log R(x,z;x')\,dx'$$

$$\psi^{(s)}(x,z) = \frac{1}{2\pi}\int_0^c \sigma(x')\,\Theta(x,z;x')\,dx'$$

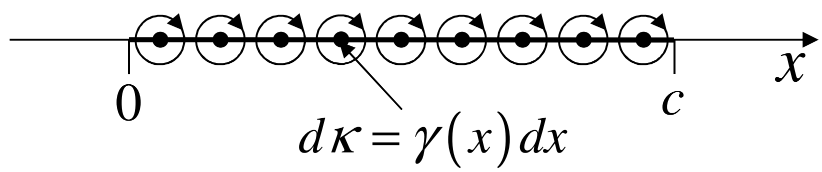


$$-\pi < \Theta \le \pi$$

Interpretation: superposition of elementary line vortices ($\psi^{(v)}$) and line sources ($\psi^{(s)}$).

Laplace and condition at ∞ already satisfied.

Vortex component $\psi^{(v)}$



$\psi^{(v)}$ = sum of many elementary vortices, becomes a surface distribution of vorticity (a vortex sheet) in the limit.

$\gamma(x)$ = strength of vortex sheet.

From definition of $\psi^{(v)}$, surface velocities induced by sheet:

$$V_x^{(v)}(z=0+) = -V_x^{(v)}(z=0-) = \frac{1}{2}\gamma$$

$$V_z^{(v)}(z=0+) = V_z^{(v)}(z=0-)$$

A vortex sheet of strength $\gamma \Rightarrow$ tangential velocity jumps discontinuously by γ.

Source component $\psi^{(s)}$

Likewise, $\psi^{(s)}$= continuous limit of the sum of many line sources, inducing surface velocities:

$$V_x^{(s)}(z=0+) = V_x^{(s)}(z=0-)$$

$$V_z^{(s)}(z=0+) = -V_z^{(s)}(z=0-) = \frac{1}{2}\sigma(x)$$

$\Rightarrow$ normal velocity jumps discontinuously by σ.

Calculating the airfoil circulation using a curve C running along the surface:

$$\Gamma = \int_0^c \left(V_x(x,0+) - V_x(x,0-)\right) dx = \int_0^c \gamma(x)\, dx$$

i.e. total circulation is sum of the elementary circulations $d\kappa = \gamma(x)\, dx$.

Equations for $\gamma(x)$ and $\sigma(x)$

Applying the surface boundary conditions to $\psi_a = \psi^{(v)} + \psi^{(s)}$:

$$\frac{1}{2\pi}\int_0^c \gamma(x')\log|x-x'|\,dx' \pm \frac{1}{2}\int_x^c \sigma(x')\,dx' = V_\infty\left(\alpha x - z_\pm(x)\right) + K$$

$\Rightarrow$

$$\frac{1}{2\pi}\int_0^c \gamma(x')\log|x-x'|\,dx' = V_\infty\left(\alpha x - \zeta(x)\right) + K$$

$$\int_x^c \sigma(x')\,dx' = -V_\infty \tau(x)$$

whose x-derivatives $\Rightarrow$

$$\boxed{\begin{gathered}\frac{1}{2\pi}\int_0^c \frac{\gamma(x')\,dx'}{x-x'} = V_\infty\left(\alpha - \frac{d\zeta}{dx}\right)\\ \sigma(x) = V_\infty \frac{d\tau}{dx}\end{gathered}} \qquad (*)$$

as equations for $\gamma(x)$ and $\sigma(x)$.

Equations for $\gamma(x)$ and $\sigma(x)$ (contd)

Camber, incidence and thickness decouple:

$$\left.\begin{array}{l}\text{camber}\\ \text{incidence}\end{array}\right\} \Rightarrow \gamma(x) \Rightarrow \psi^{(v)}$$

$$\text{thickness} \quad \Rightarrow \sigma(x) \Rightarrow \psi^{(s)}$$

Camber, incidence and thickness can be treated separately, then summed.

- integral in (*) should be interpreted as a principal value
- equation for $\sigma(x)$ is explicit, whereas that for $\gamma(x)$ needs to be solved...

Solution for $\gamma(x)$

Change variable to:

$$x = \frac{1}{2}c\left(1 - \cos\eta\right) \qquad 0 \le \eta \le \pi$$

and look for solution of (*) of the form:

$$\gamma = 2V_\infty \left\{ A_0 \cot\frac{1}{2}\eta + \sum_{n=1}^{\infty} A_n \sin n\eta \right\} \qquad (**)$$

$\Rightarrow$

$$\boxed{\begin{aligned} A_0 &= \alpha - \frac{1}{\pi}\int_0^\pi f(\eta)\,d\eta \\ A_n &= \frac{2}{\pi}\int_0^\pi f(\eta)\cos n\eta\,d\eta \qquad n \ge 1 \end{aligned}}$$

where

$$f(\eta) = \frac{d\zeta}{dx}$$

- Kutta condition implicit in (**) ($\gamma(\eta = \pi) = 0$)
- thin-airfoil solution now complete.

Surface pressure

Bernoulli $\Rightarrow$

$$c_p = \frac{p - p_\infty}{\rho V_\infty^2 / 2} = 1 - \left(\frac{V}{V_\infty}\right)^2 = -2\frac{\mathbf{V}_\infty . \mathbf{V}_a}{V_\infty^2} - \left(\frac{V_a}{V_\infty}\right)^2$$

where

$$\mathbf{V} = \mathbf{V}_\infty + \mathbf{V}_a$$

Order of magnitude analysis of thin-airfoil solution $\Rightarrow$ $V_a / V_\infty << 1 \Rightarrow$

$$c_p \approx -2\frac{\mathbf{V}_\infty . \mathbf{V}_a}{V_\infty^2} \approx -2\frac{V_{ax}}{V_\infty}$$

$$V_{ax} = \frac{\partial \psi_a}{\partial z} = \frac{\partial \psi^{(v)}}{\partial z} + \frac{\partial \psi^{(s)}}{\partial z}$$

$\Rightarrow$ surface-pressure coefficient:

$$c_p = -\frac{1}{V_\infty}\left\{ \pm\gamma + \frac{1}{\pi}\int_0^c \frac{\sigma(x')\,dx'}{x - x'} \right\} \quad z = 0 \pm$$

Aerodynamic coefficients

Pressure difference:

$$p\left(x,0-\right)-p\left(x,0+\right)=\rho V_{\infty}\gamma$$

Integrating over $x \Rightarrow$

$$L=\rho V_{\infty}\int_{0}^{c}\gamma dx=\rho V_{\infty}\Gamma \quad \text{(usual relation)}$$

$$M_{0}=-\rho V_{\infty}\int_{0}^{c}x\gamma dx \qquad \text{(LE moment)}$$

Using Fourier series for γ and integrating:

$$c_{L}=\frac{L}{\rho c V_{\infty}^{2}/2}=2\pi\left(A_{0}+\frac{1}{2}A_{1}\right)$$

$$c_{M0}=\frac{M_{0}}{\rho c^{2}V_{\infty}^{2}/2}=\frac{1}{2}\pi\left(\frac{1}{2}A_{2}-A_{0}-A_{1}\right)$$

$$\Rightarrow \quad c_{Mc/4}=\frac{M_{0}+cL/4}{\rho c^{2}V_{\infty}^{2}/2}=\frac{1}{4}\pi\left(A_{2}-A_{1}\right)$$

$\Rightarrow$ aerodynamic centre at quarter chord.

Principal results

Lift

$$\boxed{c_L = 2\pi\left(\alpha - \alpha_0\right)}$$

$$\alpha_0 = \frac{1}{\pi}\int_0^\pi f\left(\eta\right)\left(1 - \cos\eta\right)d\eta$$

Aerodynamic centre and moment

$$\boxed{x_{a.c.} = \frac{1}{4}c}$$

$$c_{Ma.c.} = \frac{1}{2}\int_0^\pi f\left(\eta\right)\left(\cos 2\eta - \cos\eta\right)d\eta$$

Pressure distribution

$$c_p = \underbrace{\mp 2\left(A_0 \cot\frac{1}{2}\eta + \sum_{n=1}^{\infty} A_n \sin n\eta\right)}_{Camber/incidence} + \underbrace{\frac{1}{\pi}\int_0^c \frac{d\tau}{dx'}\frac{dx'}{x' - x}}_{Thickness}$$

Principal results (contd)

Only c_p depends on the thickness distribution. In particular, lift and moment determined entirely by the camber distribution and (in the case of lift) angle of attack.

- linear dependence of c_L on α
- $c_{L\alpha} = 2\pi \equiv 0.11$ per degree
- aerodynamic centre at quarter chord
- pressure infinite at leading edge: this is an artifact of thin-airfoil theory, but reflects the pressure peak near the leading edge

Pros and cons of thin-airfoil theory

- approximations (and hence limitations) in addition to those noted earlier for inviscid theory
- explicit analytic expressions for aerodynamic coefficients
- general results, e.g. $c_L(\alpha)$ linear
- $c_{L\alpha} = 2\pi$ and $x_{a.c.} = c/4$ close to reality
- pressure distribution not very well represented (e.g. infinite singularity)

Thin-airfoil theory gives a first approximation of airfoil characteristics. A full inviscid calculation is the next step up in sophistication.

2.6 Unsteadiness

Unsteadiness of flow around airfoil can arise from time dependence of:

- airfoil geometry or orientation (e.g. pilot uses controls)
- upstream flow (e.g. airfoil acceleration, atmospheric "turbulence")

Time dependence of airfoil circulation (defined using a curve *C* which runs along the airfoil surface) implies vortex shedding, e.g.:

Vortex shedding maintains constant overall circulation (a result of Kelvin's theorem).

Varying circulation $\Rightarrow$ trailing-edge vortex sheet. Sheet tends to roll up $\Rightarrow$ vortex.