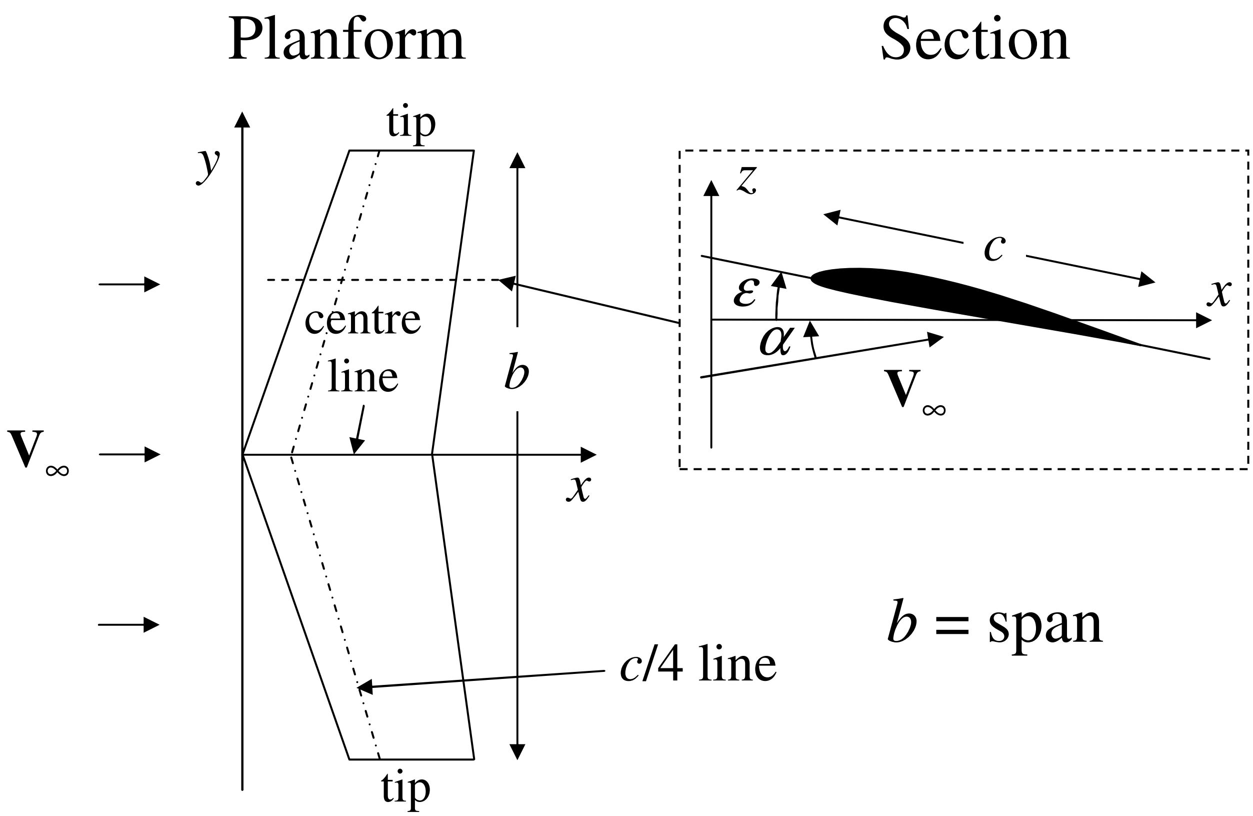
3. Finite wings at low Mach number

Contents

3.1 Wing geometry

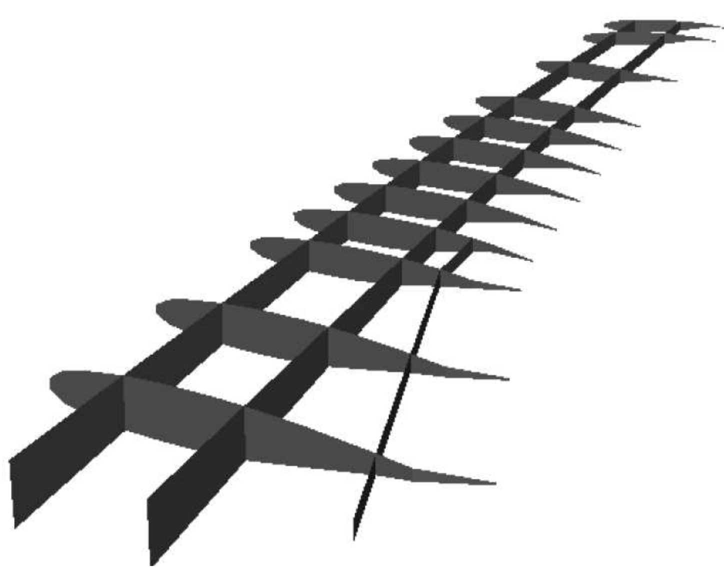


b = span

- wing symmetry plane ⇒ $y = 0$
- wing design specifies span, b, and:
  - airfoil shape as a function of y
  - chord $c(y)$
  - twist $\varepsilon(y)$
  - sweepback $x_{c/4}(y)$, dihedral $z_{c/4}(y)$
- for simplicity: straight longitudinal flight

Wing geometry (contd)

- some derived parameters:
  - wing area S = projection on x-y plane
  - mean chord:
    $$\overline{c} = b^{-1}\int_{-b/2}^{b/2} c\left(y\right)dy$$
  - aspect ratio:
    $$A = b^2 / S \approx b / \overline{c}$$
- moveable elements, e.g. flaps, included in airfoil geometry
- note: fuselage not included, wing geometry extrapolated to centre line
- example:



Aerodynamic coefficients

Much as for aircraft:

$$L = \frac{1}{2}\rho S V_\infty^2 C_L \qquad D = \frac{1}{2}\rho S V_\infty^2 C_D$$

$$M_a = \frac{1}{2}\rho S \bar{c} V_\infty^2 C_M$$

Dependence of coefficients on:

$$\left.\begin{array}{l}\text{- angle of attack } \alpha \\ \text{- wing geometry}\end{array}\right\} \text{main parameters}$$

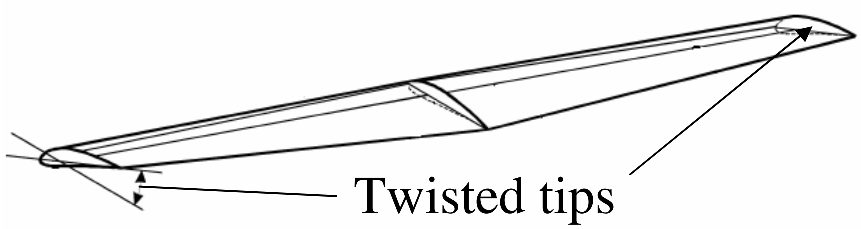
$$\left.\begin{array}{l}\text{- Reynolds number} \\ \text{- Mach number}\end{array}\right\} \text{neglected}$$

- stall if α too large
- unstalled $\Rightarrow$ lift and moment linear in α
- wing has aerodynamic centre (on centre line)

Some effects of wing geometry

Twist

• tips usually given negative twist to produce "washout", discouraging tip stall



Tip stall is bad because: reduced effectiveness of ailerons and less warning of impending stall.

Sweepback

• alleviates compressibility effects

• however, encourages tip stall

• $x_{a.c.}$ of wing increased

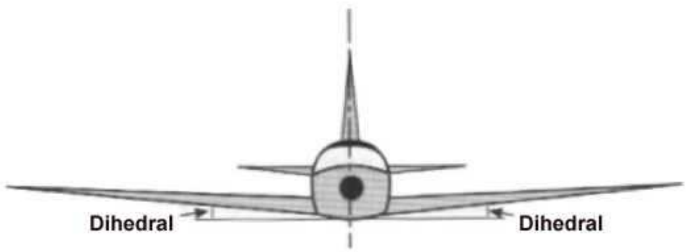
• promotes lateral stability

Normally used only for compressibility reasons (most modern airliners employ sweepback).

Some effects of wing geometry (contd)

Dihedral

• positive dihedral promotes lateral stability



Chord and airfoil shape distributions

• most wings are tapered, i.e. chord decreases towards tips

• airfoil shape varies continuously along wing

Overall, the choice of wing geometry is crucial in aircraft performance, but is far from simple.

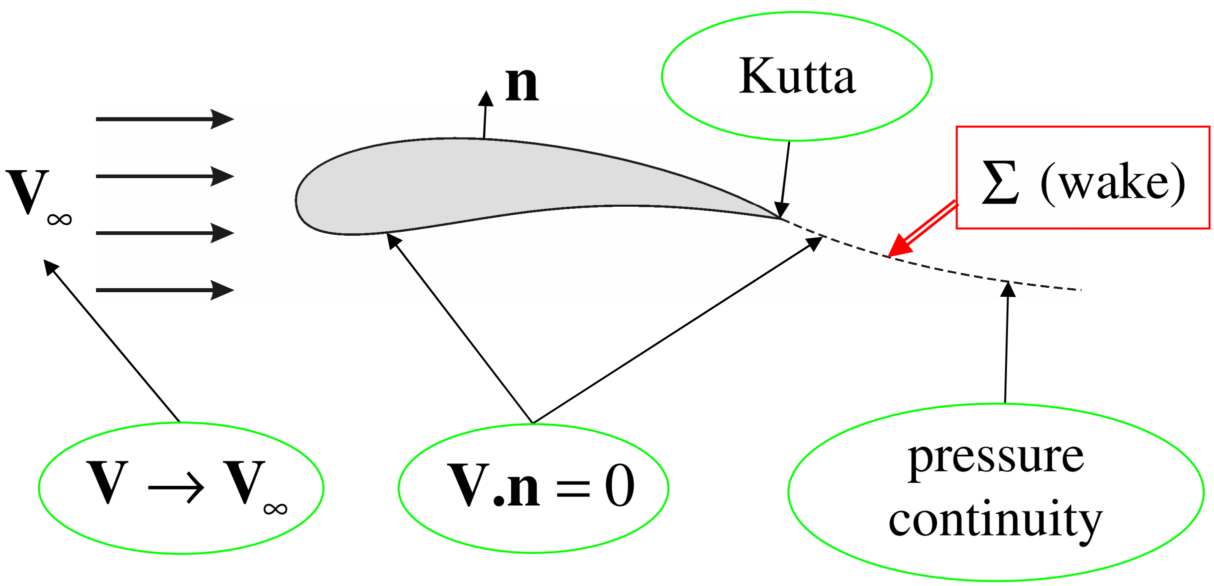
3.2 Lifting-surface theory

Reminder of inviscid boundary-value problem to solve for velocity field $\mathbf{V}$:

$$\nabla \times \mathbf{V} = 0 \qquad \text{irrotationality}$$

$$\nabla . \mathbf{V} = 0 \qquad \text{incompressibility}$$

with boundary conditions:



Σ = discontinuity in $\mathbf{V}_t$ (trailing vortex sheet). Absent in 2D, but important in 3D.

• pressure from Bernoulli

Velocity-potential formulation

Irrotationality and incompressibility $\Rightarrow$

$$\mathbf{V} = \nabla \phi \qquad \text{velocity potential}$$

$$\nabla^2 \phi = 0 \qquad \text{Laplace in 3D}$$

and velocity boundary conditions rewritten in terms of ϕ.

Boundary-value problem to solve for:

☞ velocity potential ϕ

☞ unknown surface Σ

Σ unknown $\Rightarrow$ difficult nonlinear problem

- ϕ discontinuous across Σ
- 3D Laplacian:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

- note: stream function unavailable in 3D

Lifting-surface approximation

Velocity decomposed as:

$$\mathbf{V} = \mathbf{V}_\infty + \underbrace{\mathbf{V}_w}_{\textit{Effect of wing}}$$

Wing and trailing vortex sheet represented by:

$z = z_+(x, y)$ upper surface of wing

$z = z_-(x, y)$ lower surface of wing

$z = z_\Sigma(x, y)$ vortex sheet

Approximation supposes:

☞ z_+, z_- and z_Σ small compared to chord
☞ small angle of attack

• like thin-airfoil approximation in 2D

• wing is close to being a flat plate at zero incidence $\Rightarrow V_w << V_\infty$

Lifting-surface approximation (contd)

• approximate form of Bernoulli:

$$p - p_\infty = \frac{1}{2}\rho\left(V_\infty^2 - V^2\right) \approx -\rho V_\infty V_{wx}$$

Boundary conditions on wing surface

$$\mathbf{V.n} = 0 \quad \Rightarrow \quad V_z = V_x \frac{\partial z_\pm}{\partial x} + V_y \frac{\partial z_\pm}{\partial y} \quad z = z_\pm$$

approximated as:

$$V_{wz} = V_\infty\left(\frac{\partial z_\pm}{\partial x} - \alpha\right) \quad z = 0\pm,\ \ x_L(y) < x < x_T(y)$$

$x = x_L(y),\ x_T(y)$: leading and trailing edges

Boundary conditions on vortex sheet

$\mathbf{V.n} = 0$ and pressure continuity $\Rightarrow$

$$\left.\begin{array}{c} V_{wz}(z=0+) = V_{wz}(z=0-) = V_\infty\left(\dfrac{\partial z_\Sigma}{\partial x} - \alpha\right) \\ V_{wx}(z=0+) = V_{wx}(z=0-) \end{array}\right\} x > x_T$$

Lifting-surface boundary-value problem

$$\nabla^2 \phi_w = 0$$

$$\frac{\partial \phi_w}{\partial z} = V_\infty \left(\frac{\partial z_\pm}{\partial x} - \alpha \right) \quad z = 0\pm,\ x_L(y) < x < x_T(y)$$

$$\left. \begin{aligned} \phi_w(z=0+) - \phi_w(z=0-) &= \Gamma(y) \\ \frac{\partial \phi_w}{\partial z}(z=0+) &= \frac{\partial \phi_w}{\partial z}(z=0-) \end{aligned} \right\} x > x_T(y)$$

Kutta condition at $z = 0,\ x = x_T(y)$

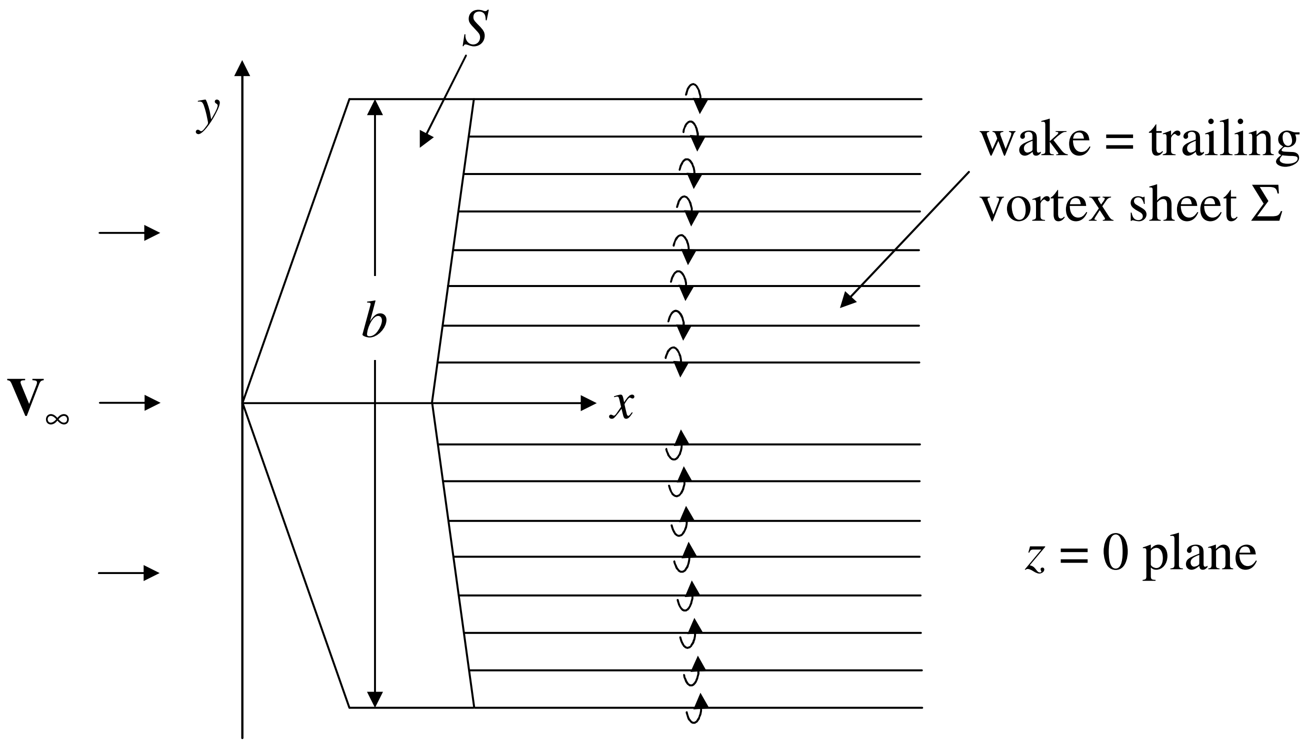
$\phi_w \to 0$ at upstream infinity

- linear problem to solve for ϕ_w (and $\Gamma(y)$)
- once ϕ_w known:

$$\mathbf{V}_w = \nabla \phi_w \qquad\qquad p - p_\infty = -\rho V_\infty V_{wx}$$

- boundary conditions now at $z = 0 \Rightarrow$ removes difficulty of unknown Σ

Trailing vortices



- $\Gamma(y)$ = wing circulation defined using any curve C for which $y = constant$
- according to boundary conditions, only V_y is discontinuous across Σ:

$$V_y(z=0+) - V_y(z=0-) = \frac{d\Gamma}{dy}$$

- trailing vortex sheet is limit of infinitely many streamwise line vortices of circulation $d\Gamma$: vortices reflect varying wing circulation.

Elementary solutions of Laplace in 3D

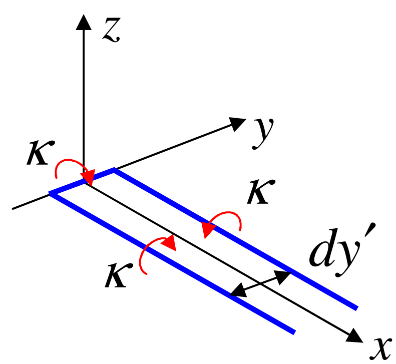
Point source of volume flux q at $\mathbf{x}'$:

$$\phi(\mathbf{x}) = qG_s(\mathbf{x}-\mathbf{x}') \qquad G_s(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}$$

Elementary horseshoe vortex of circulation κ and infinitesimal span dy' at $\mathbf{x}'$:

$$\phi(\mathbf{x}) = \kappa G_v(\mathbf{x}-\mathbf{x}')\,dy'$$

$$G_v(\mathbf{x}) = \frac{z}{4\pi\left(y^2+z^2\right)}\left\{1+\frac{x}{|\mathbf{x}|}\right\}$$



- dy' infinitesimal $\Rightarrow G_v$ in the limit $dy' \to 0$, $\kappa \to \infty$ (similar to dipole construction)
- $\mathbf{x} = (x, y, z)$, $\mathbf{x}' = (x', y', z')$
- care needed when using line vortex segments

Superposition of elementary solutions

Look for solution of lifting-surface problem:

$$\phi_w = \phi^{(v)} + \phi^{(s)}$$

where

$$\phi^{(v)}(\mathbf{x}) = \iint_S \gamma(x', y') G_v(\mathbf{x} - \mathbf{x}')\, dx'dy'$$

$$\phi^{(s)}(\mathbf{x}) = \iint_S \sigma(x', y') G_s(\mathbf{x} - \mathbf{x}')\, dx'dy'$$

Interpretation: superposition of point sources and horseshoe vortices distributed over wing.

Already a solution of Laplace, as well as boundary conditions on Σ and at infinity. There remain the wing and Kutta conditions.

Determination of γ and σ

Calculation of V_{wz} on wing surface $S \Rightarrow$

$$V_{wz}(z = 0\pm) = \frac{\partial \phi_w}{\partial z}(z = 0\pm) =$$
$$\frac{\partial}{\partial y}\iint_S \gamma(x', y') G(x - x', y - y') dx'dy' \pm \frac{1}{2}\sigma(x, y)$$

where

$$G(x, y) = -\frac{1}{4\pi y}\left\{1 + \frac{|\mathbf{x}|}{x}\right\}$$

Applying wing boundary conditions:

$$\sigma = V_\infty \frac{\partial \tau}{\partial x}$$

$$\boxed{\begin{aligned}&\frac{\partial}{\partial y}\iint_S \gamma(x', y') G(x - x', y - y') dx'dy' = \\ &\qquad V_\infty\left(\frac{\partial \zeta}{\partial x} - \alpha - \varepsilon(y)\right)\end{aligned}} \qquad (*)$$

$$\tau(x, y) = z_+ - z_- \qquad \text{thickness}$$

$$\zeta(x, y) = \frac{1}{2}(z_+ + z_-) - z_L + \varepsilon(x - x_L) \qquad \text{camber}$$

Determination of γ and σ (contd)

- source strength σ given explicitly in terms of wing thickness distribution
- γ requires solution of integral equation (*) with Kutta condition: $\gamma = 0$ at trailing edge
- γ depends on:
    - angle of attack
    - wing planform S
    - twist and camber distributions
- integral in (*) should be interpreted as a principal value

Discretised version of (*) $\Rightarrow$ wing divided into panels, on each of which is placed a horseshoe vortex $\Rightarrow$ numerical panel method.

Circulation and lift

- Interpretation of γ as jump in V_x across S:

$$V_x(z=0+) - V_x(z=0-) = \gamma(x,y)$$

- Wing circulation at spanwise location y:

$$\Gamma(y) = \int_{x_L(y)}^{x_T(y)} \gamma(x,y)\,dx$$

- Pressure difference:

$$p(z=0-) - p(z=0+) = \rho V_\infty \gamma$$

- Lift associated with spanwise element dy:

$$dL = \rho V_\infty \Gamma(y)\,dy$$

generalisation of 2D result relating lift and circulation

- Total lift:

$$\boxed{L = \rho V_\infty \int_{-b/2}^{b/2} \Gamma(y)\,dy}$$

- Moment (about y-axis):

$$M_a = -\rho V_\infty \iint_S x\gamma(x,y)\,dx\,dy$$

Circulation and lift (contd)

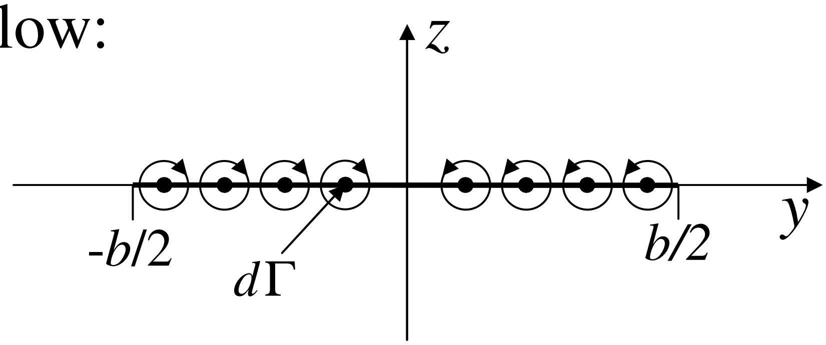
- Linearity of problem for $\gamma \Rightarrow$

  $$C_L = C_{L\alpha}\left(\alpha - \alpha_0^{wing}\right) \quad C_M = C_{M\alpha}\left(\alpha - \alpha_{M0}^{wing}\right)$$

  where $C_{L\alpha}$ and $C_{M\alpha}$ depend only on wing planform.

Far wake

At downstream distances x large compared with span, the flow becomes independent of x $\Rightarrow$ 2D flow:



Velocity field:

$$V_x = V_\infty, \quad \underbrace{V_y = V_y\left(y, z\right), \quad V_z = V_z\left(y, z\right)}_{\textit{Due to trailing vortices}}$$

Kinetic energy (per unit downstream length) in frame of reference of air:

$$\frac{1}{2}\rho\iint\left(V_y^2 + V_z^2\right)dydz =$$

$$\frac{\rho}{4\pi}\int_{-b/2}^{b/2}\int_{-b/2}^{b/2}\Gamma\left(y\right)\frac{d\Gamma}{dy'}\frac{dy'dy}{y-y'}$$

Induced drag

In frame of reference of air, work done by drag force provides kinetic energy of wake $\Rightarrow$

$$D_i = \frac{\rho}{4\pi}\int_{-b/2}^{b/2}\int_{-b/2}^{b/2}\Gamma(y)\frac{d\Gamma}{dy'}\frac{dy'dy}{y-y'}$$

In 3D, there is drag according to inviscid theory. Drag is due to trailing vortices.

- referred to as "induced" drag to distinguish it from "parasite" drag, whose existence is due to viscous effects (the latter exists in 2D)
- induced drag is an inevitable accompaniment of lift.

Minimisation of induced drag

Writing

$$\Gamma(y) = V_\infty b \sum_{n=1}^{\infty} B_n \sin n\theta$$

where $y = \frac{1}{2} b \cos\theta \ (0 < \theta < \pi)$,

$$C_{D_i} = \frac{1}{4}\pi A \sum_{n=1}^{\infty} n B_n^2$$

$$C_L = \frac{2}{S V_\infty} \int_{-b/2}^{b/2} \Gamma dy = \frac{1}{2}\pi A B_1$$

Minimum C_{D_i} for given $C_L \Rightarrow$

$$\Gamma(y) = V_\infty b B_1 \sin\theta = \Gamma_0 \left(1 - \left(\frac{2y}{b} \right)^2 \right)^{1/2}$$

Minimum induced drag $\Rightarrow$ “elliptic” loading.

$$C_{D_i} = \frac{C_L^2}{\pi A} \qquad\qquad A = \frac{b^2}{S}$$

Drag modelling

For general loading:

$$C_{D_i} = \frac{C_L^2}{e\pi A}$$

$e < 1$: Oswald's efficiency factor.

In principle, e depends on angle of attack. However, for a given wing, a constant value is found to yield reasonable results.

A model including parasite drag:

$$C_D = C_{D0} + \frac{C_L^2}{e\pi A}$$

where C_{D0} is zero-lift drag coefficient and e is modified by viscous effects.

This type of model can also be used for an entire aircraft. The parameters C_{D0} and e are usually determined from engineering rules of thumb or experimentally.

Drag modelling (contd)

- for aircraft, drag includes contributions, e.g. fuselage, other than wings: these modify C_{D0} and e
- if Mach and Reynolds number effects are allowed for, C_{D0} and e depend on M and Re
- typically, for an aircraft:
  - $e \sim 0.6\text{-}0.8$
  - $C_{D0} \sim 0.02\text{-}0.05$

3.3 Lifting-line theory

Assumptions, over and above those of lifting surface theory:

- ☞ large aspect ratio, A, i.e. span much greater than chord
- ☞ no sweep

Lifting-surface integral equation:

$$\frac{\partial}{\partial y}\iint_S \gamma(x', y')\,G(x-x', y-y')\,dx'dy' = V_\infty\left(\frac{\partial \zeta}{\partial x} - \alpha - \varepsilon(y)\right)$$

In lifting-line theory $(x-x')^2 << (y-y')^2$ over most of wing $\Rightarrow$

$$G(x-x', y-y') = \frac{1}{4\pi(y'-y)}\left\{1 + \frac{|\mathbf{x}-\mathbf{x}'|}{x-x'}\right\}$$

$$\sim \frac{1}{4\pi(y'-y)} + \frac{\operatorname{sgn}(y'-y)}{4\pi(x-x')}$$

Lifting-line theory (contd)

$$\Rightarrow \quad \frac{\partial}{\partial y}\iint_S \gamma(x',y')\,G(x-x',y-y')\,dx'dy' \sim$$

$$\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{d\Gamma}{dy'}\frac{dy'}{y'-y}+\frac{1}{2\pi}\int_{x_L(y)}^{x_T(y)}\frac{\gamma(x',y)}{x'-x}dx'$$

Hence:

$$\boxed{\frac{1}{2\pi}\int_{x_L}^{x_T}\frac{\gamma(x',y)}{x-x'}dx' = V_\infty\left(\alpha_e(y)-\frac{\partial\zeta}{\partial x}\right)} \qquad (**)$$

where

$$\alpha_e(y)=\alpha+\underbrace{\varepsilon(y)}_{twist}+\underbrace{\alpha_i(y)}_{induced}$$

$$\alpha_i(y)=\frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{d\Gamma}{dy'}\frac{dy'}{y'-y}$$

(**) = 2D, thin-airfoil theory with angle of attack $\alpha_e \Rightarrow \ \gamma(x,y)$

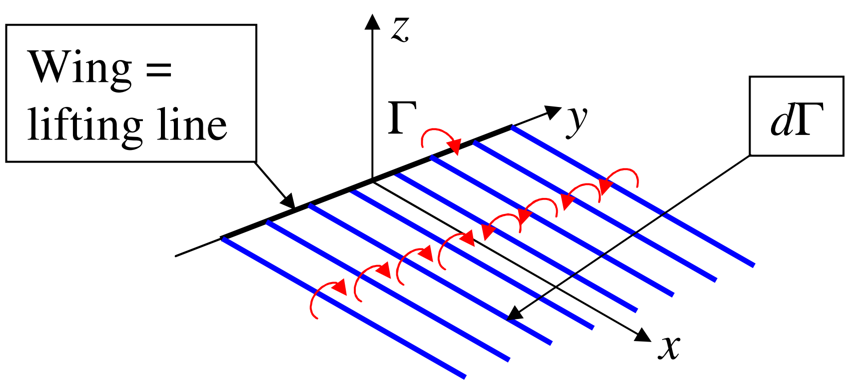
- wing = succession of 2D airfoil sections

Interpretation of α_e and α_i

$$\alpha_e(y) = \alpha + \underbrace{\varepsilon(y)}_{twist} + \underbrace{\alpha_i(y)}_{induced}$$

$$\alpha_i(y) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{d\Gamma}{dy'}\frac{dy'}{y'-y}$$

- α_e = effective angle of attack seen by airfoil
- $\alpha + \varepsilon$ = geometric angle of attack
- $\alpha_i < 0$: angle of attack *decreased* by downwash due to trailing vortices:



Trailing vortex $d\Gamma \Rightarrow$ upwash velocity on lifting line:

$$\frac{d\Gamma}{4\pi(y'-y)}$$

sum over vortices and divide by $V_\infty \Rightarrow \alpha_i < 0$

Lifting-line integral equation

From 2D, thin-airfoil theory with angle of attack α_e:

$$\Gamma = \pi c V_\infty \left(\alpha_e - \alpha_0\right)$$

$\alpha_0(y)$ = known zero-lift angle of 2D airfoil

Hence, integral equation for $\Gamma(y)$:

$$\boxed{\Gamma = \pi c V_\infty \left(\alpha + \varepsilon - \alpha_0 + \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{d\Gamma}{dy'} \frac{dy'}{y' - y} \right)}$$

or equivalently

$$\sum_{n=1}^{\infty} \left(1 + \frac{n\pi c}{2b \sin\theta} \right) B_n \sin n\theta = \frac{\pi c}{b} \left(\alpha + \varepsilon - \alpha_0 \right)$$

for the coefficients B_n. Numerical solution needed in general.

- once $\Gamma(y)$ is known, all other quantities can be determined, e.g. $\alpha_i(y)$, lift, induced drag.

Elliptic loading

With elliptic loading:

$$\alpha + \varepsilon - \alpha_0 = B_1 \left(\frac{b}{\pi c} \sin\theta + \frac{1}{2} \right)$$

Elliptic loading for all α requires:

$$c(y) = c_0 \sin\theta = c_0 \left(1 - \left(\frac{2y}{b} \right)^2 \right)^{1/2}$$ elliptic planform

$$\varepsilon(y) - \alpha_0(y) = constant$$

Lift coefficient of such a wing:

$$C_L = C_{L\alpha}\left(\alpha - \alpha_0^{wing}\right)$$

$$C_{L\alpha} = \frac{2\pi}{1 + 2/A} \qquad \alpha_0^{wing} = \alpha_0 - \varepsilon$$

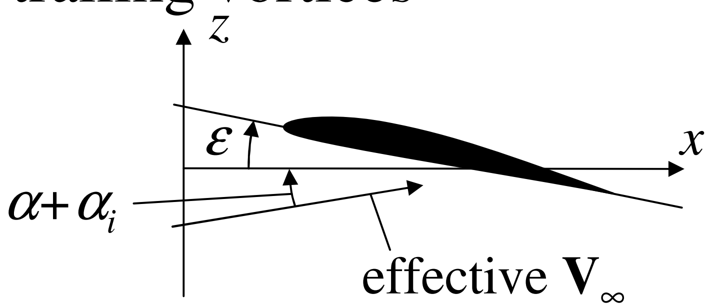
- finite span reduces $C_{L\alpha}$ by factor $\left(1 + 2/A\right)^{-1}$
- α_0^{wing} has intuitively expected value

Elliptic loading (contd)

Induced angle of attack:

$$\alpha_i = -\frac{C_L}{\pi A}$$

- α_i constant $\Rightarrow$ direction of incident flow $\mathbf{V}_\infty$ seen by wing modified by induced velocity of trailing vortices



- positive lift $\Rightarrow$ $\alpha_i < 0 \Rightarrow$ reduced angle of attack, hence reduced $C_{L\alpha}$
- wing produces lift perpendicular to modified direction of incidence, but this is not generally perpendicular to the real $\mathbf{V}_\infty \Rightarrow$ alternative interpretation of induced drag.

General loading: some empirical relations

The above results are only strictly valid for elliptic loading. For general loading, the lifting-line integral equation should be solved numerically.

However, the qualitative characteristics of the elliptic case are found to still hold and the main quantitative result:

$$C_{L\alpha} \approx \frac{2\pi}{1+2/A}$$

provides a reasonable estimate of the effects of finite span. A somewhat better approximation at low aspect ratios is:

$$C_{L\alpha} \approx \frac{2\pi A}{2+\sqrt{4+A^2}}$$

Helmbold equation

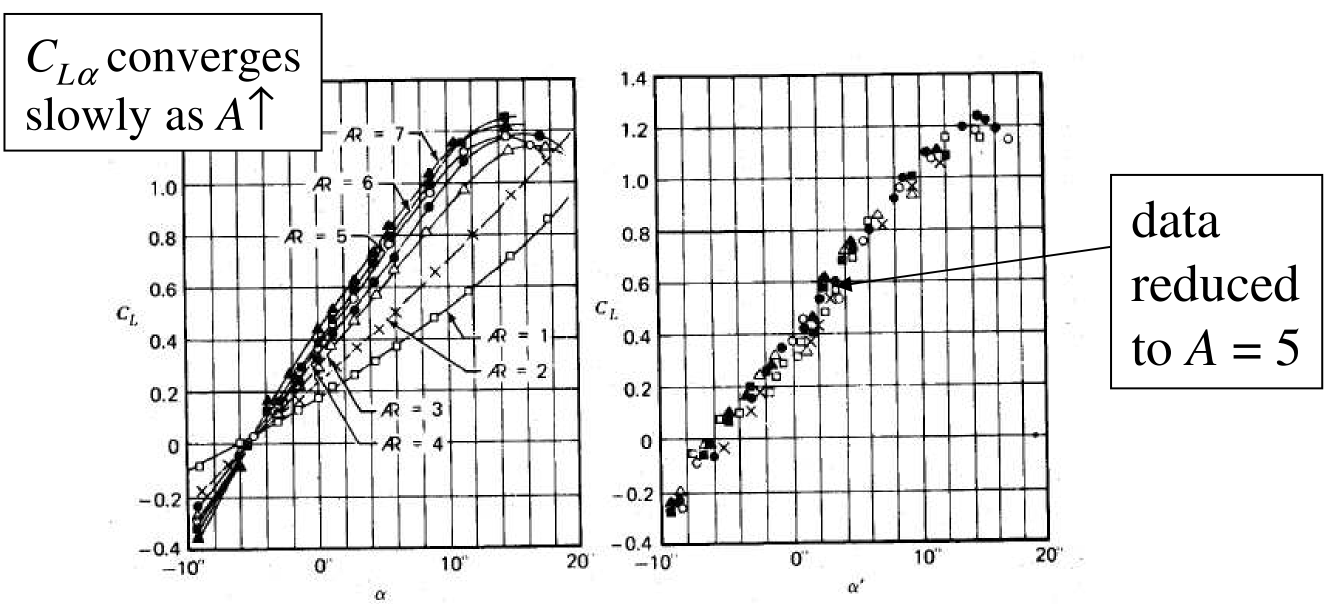
3.4 Summary of finite-span effects

Main effects:

☞ vortex-sheet wake reflects spanwise variations of wing circulation

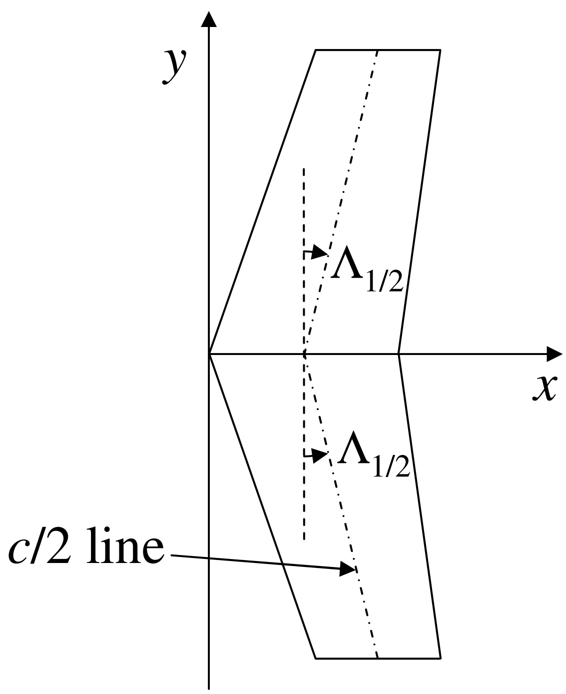
☞ induced drag (inviscid) accompanies lift and reflects energy needed to create trailing vortex sheet

☞ reduced lift slope due to velocity induced by trailing vortices, which decreases the angle of attack seen by the wing



Effect of sweep on $C_{L\alpha}$

Like finite span, sweep (not allowed for by lifting-line theory) tends to decrease $C_{L\alpha}$.



Linearly tapered wing with half-chord sweepback angle $\Lambda_{1/2}$.

Other measures of sweep, e.g. $\Lambda_{1/4}$ or Λ_{LE}, often used.

Empirical relation:

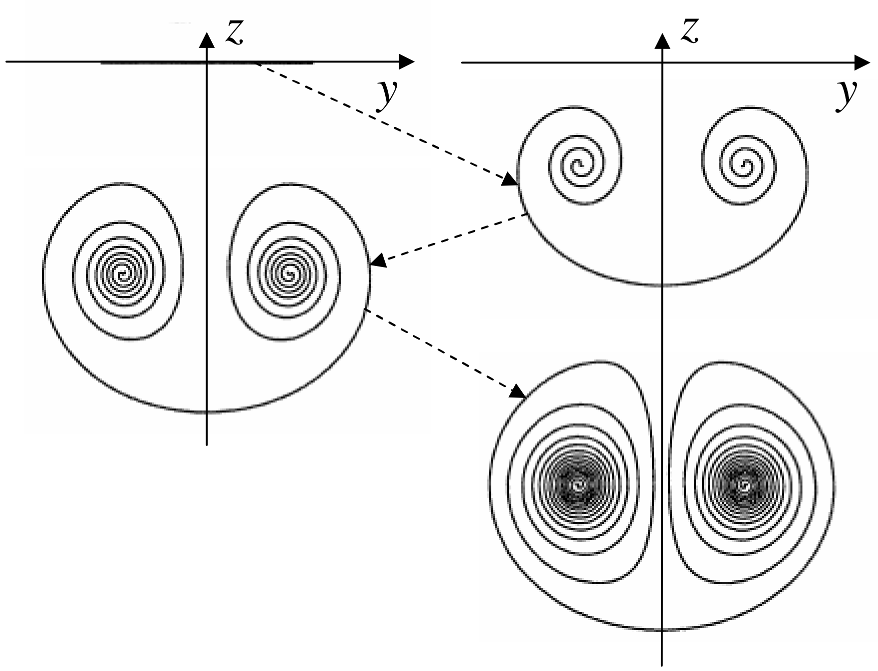
$$C_{L\alpha} \approx \frac{2\pi A}{2 + \sqrt{4 + \left(A / \cos \Lambda_{1/2} \right)^2}}$$

$\Rightarrow$

$$C_{L\alpha} \to 2\pi \cos \Lambda_{1/2} \quad \text{as } A \to \infty$$

3.5 Trailing-vortex dynamics

Trailing vortex sheet evolves with downstream distance due to its self-induced velocity (not allowed for in lifting-surface theory).



2D, unsteady calculation

- sheet rolls up and moves downwards
- end result is a pair of counter-rotating vortices
- often visible as contrails

Trailing-vortex dynamics (contd)