Introduction to Computer Science

Sample Solution 04

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Question 1

For arbitrary sets A, B and C, recall that $A \subseteq (B \cup C)$ is equivalent to $\forall x.\ x \in A \implies x \in B \vee x \in C$. Present the following statement in logical form:

$$(A \cap B) \subseteq (A \setminus C)$$

Answer 1

$\forall x.\ x \in (A \cap B) \implies x \in (A \setminus C)$	by definition of subset
$\forall x.\ (x \in A \vee x \in B) \implies x \in (A \setminus C)$	by definition of set union
$\forall x.\ (x \in A \vee x \in B) \implies (x \in A \wedge x \notin C)$	by definition of set difference

[1 mark]

Question 2

Let A, B and C be arbitrary sets. Prove the following:

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

Answer 2

Proof. Let x be an arbitrary element in the set $A \cup (B \cap C)$, we need to show that x is also an element of the set $(A \cup B) \cap (A \cup C)$. That $x \in A \cup (B \cap C)$ means that $x \in A$ **or** $x\ B \cap C$ by definition of set union. We proceed by case analysis:

Case I Let $x \in A$. It follows, from set union, that $x \in (A \cup B)$. Similarly, we have that $x \in (A \cup C)$ from set union. Thus, $x \in (A \cup B)$ **and** $x \in (A \cup C)$. Then, $x \in (A \cup B) \cap (A \cup C)$ follows from the definition of intersection.

Case II Let x $B \cap C$. Then, from set intersection, $x \in B$ **and** $x \in C$. Since $x \in B$ then $x \in A \cup B$ by definition of set union. Since also $x \in C$ then $x \in A \cup C$ by definition of set union. Thus, $x \in A \cup B$ **and** $x \in A \cup C$. That is, by definition of set intersection, $x \in (A \cup B) \cap (A \cup C)$

We have shown that $x \in (A \cup B) \cap (A \cup C)$ independently of whether $x \in A$ or x $B \cap C$. That is, $\forall x.\ x \in A \cup (B \cap C) \implies x \in (A \cup B) \cap (A \cup C)$, which is the same as $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ from definition of subset. **[1 mark]** $\square$

Question 3

Explain the meaning of the following statement:

$$\{n^2 \mid n \in \mathbb{N}\} \cap \{n^3 \mid n \in \mathbb{N}\} \neq \emptyset$$

Answer 3

The inequality expresses that the sets of squares and cubes of natural numbers are not disjoint (i.e., their intersection is not empty). The concrete meaning is:

> There exist numbers that simultaneously are a square and a cube of some natural numbers.

By the way, this statement is *true*. As a case, consider 64 which is the square of $8 \in \mathbb{N}$ and the cube of $4 \in \mathbb{N}$, i.e., $64 \in \{n^2 \mid n \in \mathbb{N}\} \cap \{n^3 \mid n \in \mathbb{N}\}$ **[1 mark]**

Question 4

For each of the following relations, specify if the relation is:

- Reflexive, irreflexive or neither.
- Symmetric, antisymmetric or neither.
- Transitive or not.

1. $I_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x = y\}$.
2. $L_{\mathbb{R}} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x < y\}$.

Answer 4

1. $I_{\mathbb{N}}$ is the *identity* relation on the set of naturals.

   This relation is *reflexive* since for all natural number $n \in \mathbb{N}$, we have that n is equal (identical) to itself $(n = n)$ implying that $(n, n) \in I_{\mathbb{N}}$.

   $I_{\mathbb{N}}$ is *symmetric* since for all $(n, m) \in I_{\mathbb{N}}$ we have that $n = m$ and then $m = n$ (by commutativity) implying that $(m, n) \in I_{\mathbb{N}}$.

   $I_{\mathbb{N}}$ is *transitive* since for all $n, m, p \in \mathbb{N}$ with $(n, m) \in I_{\mathbb{N}}$ and $(m, p) \in I_{\mathbb{N}}$, we have that $n = m$ and $m = p$. Then $n = p$ because equal numbers to the same number are equal to one another. This means that $(n, p) \in I_{\mathbb{N}}$.

   Observe that $I_{\mathbb{N}}$ is an *equivalence* relation because it is reflexive, symmetric and transitive.

   [1 mark]

2. $L_{\mathbb{R}}$ is the *less than* relation on the set of real numbers.

   $L_{\mathbb{R}}$ is *irreflexive* since no real number is less that itself. That is, there is no $x \in \mathbb{R}$ with $x < x$.

   $L_{\mathbb{R}}$ is *antisymmetric* since for all $x, y \in \mathbb{R}$ either $x < y$ **or** $y < x$ but not both. This means that $(x, y) \in L_{\mathbb{R}}$ **and** $(y, x) \in L_{\mathbb{R}}$ is always false, which makes the implication $((x, y) \in L_{\mathbb{R}} \wedge (y, x) \in L_{\mathbb{R}}) \implies x = y$ always true.

   $L_{\mathbb{R}}$ is *transitive* since for all $x, y, z \in \mathbb{R}$ with $(x, y) \in I_{\mathbb{R}}$ and $(y, z) \in L_{\mathbb{R}}$, we have that $x < y$ and $y < z$. It follows that $x < z$ and then $(y, z) \in L_{\mathbb{R}}$.

   Observe that $L_{\mathbb{R}}$ is *asymmetric* because is *antysymmetric* and *irreflexive*. In other words, for all $(x, y) \in L_{\mathbb{R}}$ we have that $x < y$, so it cannot be the case that $y < x$ and, hence, $(y, x) \notin L_{\mathbb{R}}$

   [1 mark]

Question 5

Consider the set $\mathcal{P}$ of all persons of a certain family and the set $\mathcal{F} \subseteq \mathcal{P}$ of all female persons in this family. Suppose you are given also the relations,

- $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$where $(x, y) \in \mathcal{S}$ iff x is a **son** of y,
- $\mathcal{D} \subseteq \mathcal{P} \times \mathcal{P}$ where $(x, y) \in \mathcal{D}$ iff x is a **daughter** of y and
- $\mathcal{I}_{\mathcal{F}} \subseteq \mathcal{F} \times \mathcal{F}$ is the **identity relation over females**, i.e., for all $x, y \in \mathcal{F}$ we have $(x, y) \in \mathcal{I}_{\mathcal{F}}$ if $x = y$. For instance, if *teresa* and *lulu* are females, and *martin* is male then $(\mathit{lulu}, \mathit{lulu}) \in \mathcal{I}_{\mathcal{F}}$, but $(\mathit{teresa}, \mathit{lulu}) \notin \mathcal{I}_{\mathcal{F}}$ and $(\mathit{martin}, \mathit{martin}) \notin \mathcal{I}_{\mathcal{F}}$.

From these relations, new relations can be defined by means of the operations of intersection $\cap$, union $\cup$, difference $\setminus$, composition $\circ$ and converse $(\cdot)^\mathsf{T}$. Recall that the converse of a given relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ is the relation $\mathcal{R}^\mathsf{T} \subseteq \mathcal{Y} \times \mathcal{X}$ resulting from switching the order of the elements. For instance, if $(\mathit{teresa}, \mathit{lulu}) \in \mathcal{D}$ then $(\mathit{lulu}, \mathit{teresa}) \in \mathcal{D}^\mathsf{T}$.

1. Explain in words what the following relation stands for:
   $(\mathrm{S} \cup \mathrm{D})^\mathsf{T} \circ \mathcal{I}_\mathcal{F}$
2. Define (construct) the relation:
   $\mathcal{Z} \subseteq \mathcal{P} \times \mathcal{P}$ where $(x, y) \in \mathcal{Z}$ iff x is a **sister** of y.

Answer 5

1. The relation is **mother**.

   This relation starts from all *females*, i.e., $\mathcal{I}_\mathcal{F}$ composed with (followed by) their correspondent children. Note that the children of a person are obtained from the converse of the relation *son* **or** (union) *daughter*.

   [1 mark]
2. $\mathcal{Z} = ((\mathrm{S} \cup \mathrm{D})^\mathsf{T} \circ \mathrm{D}) \setminus \mathcal{I}_\mathcal{F}$

   We start from the relation *daughter* to reach the parents of a daughter. This is composed with (followed by) the children of the parent. Finally, we remove the *identity over females* since a person cannot be a sister of herself. For example, if $(\mathit{teresa}, \mathit{lulu}) \in \mathcal{D}$ and $(\mathit{luis}, \mathit{lulu}) \in \mathcal{S}$ then $(\mathit{teresa}, \mathit{luis}) \in (\mathrm{S} \cup \mathrm{D})^\mathsf{T} \circ \mathrm{D}$ and $(\mathit{teresa}, \mathit{teresa}) \in (\mathrm{S} \cup \mathrm{D})^\mathsf{T} \circ \mathrm{D}$. Moreover, $(\mathit{teresa}, \mathit{luis}) \in ((\mathrm{S} \cup \mathrm{D})^\mathsf{T} \circ \mathrm{D}) \setminus \mathcal{I}_\mathcal{F}$ but $(\mathit{teresa}, \mathit{teresa}) \notin ((\mathrm{S} \cup \mathrm{D})^\mathsf{T} \circ \mathrm{D}) \setminus \mathcal{I}_\mathcal{F}$. In other words, $(\mathit{teresa}, \mathit{luis}) \in \mathcal{Z}$ but $(\mathit{teresa}, \mathit{teresa}) \notin \mathcal{Z}$.

   [1 mark]

Question 6

Inspired by set intersection, define a Haskell function `intersect` that given two lists, returns a list with the elements common to both lists without repetition. Define the function with the most general possible type.

For example:

```
intersect [2,4,8,16] [1,2,3,4,5,6]
⟹ [2,4]
```

```
intersect [42,27,32] [1,2,3,4,5,6]
⟹ []

intersect ‘‘Hello’’ Haskell’’
⟹ ‘‘Hel’’

intersect [True, True, True] [False, True, False, True]
⟹ [True]
```

Answer 6

A recursive solution:

```
intersect :: Eq a => [a] -> [a] -> [a]
intersect [] _ = []
intersect (x:xs) ys
      | elem x ys = x : intersect xs (filter (/= x) ys)
      | otherwise = intersect xs ys
```

[3 marks]