Introduction to Computer Science

Sample Solution 02

Constructor University,
Bremen, Germany

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Question 1

For each of the following functions $f, g : \mathbb{N} \to \mathbb{R}$, determine the corresponding Landau set (Big-O notation):

1. $f(n) = 7n + 10n\log_2(n) - 2n^3 + 42$
2. $g(n) = 2\sqrt{n} + 3\log_2(n)$ n.b. $O(\log_2(n)) \subset O(n^{\frac{1}{2}})$

Answer 1

Recall that:

Theorem 1 (Landau Set Computation Rules).
There are three computation rules for Landau sets:

(i) *if* $k \neq 0$ *and* $f \in O(g)$, *then* $kf \in O(g)$.

(ii) *if* $f_1 \in O(g_1)$ *and* $f_2 \in O(g_2)$, *then* $(f_1 + f_2) \in O(\max\{g_1, g_2\})$.

(iii) *if* $f_1 \in O(g_1)$ *and* $f_2 \in O(g_2)$, *then* $(f_1 f_2) \in O(g_1 g_2)$.

From this Theorem 1, we elaborate the following:

1. $f(n) = 7n + 10n\log_2(n) - 2n^3 + 42$
   $f(n) = 7f_1(n) + 10f_1(n)f_2(n) - 2f_3(n) + f_4(n)$, where:
   - $f_1(n) = n$ with $f_1 \in O(n)$,
   - $f_2(n) = \log_2(n)$ with $f_2 \in O(\log_2(n))$,
   - $f_3(n) = n^3$ with $f_3 \in O(n^3)$, and
   - $f_4(n) = 42$ with $f_4 \in O(1)$.

Hence,

$7f_1(n) = 7n \in O(n)$ by *(i)*,
$10f_1(n) = 10n \in O(n)$ by *(i)*,
$10f_1(n)f_2(n) = 10n\log_2(n) \in O(n\log_2(n))$ by *(iii)*,
$2f_3(n) = 2n^3 \in O(n^3)$ by (i), and
$f(n) \in O(\max\{n, n\log_2 n, n^3, 1\}) = O(n^3)$ by *(ii)*.

Therefore, $f(n) \in O(n^3)$ **[1 mark]**

2. $g(n) = 2\sqrt{n} + 3\log_2(n)$
   $g(n) = 2g_1(n) + 3g_2(n)$ where:

   $g_1(n) = \sqrt{n} = n^{\frac{1}{2}} \in O(n^{\frac{1}{2}})$, and
   $g_2(n) = \log_2(n) \in O(\log_2(n))$,

   Hence,

   $2g_1(n) = 2\sqrt{n} \in O(n^{\frac{1}{2}})$ by *(i)*,
   $3g_2(n) = 3\log_2(n) \in O(\log_2(n))$ by *(i)*.
   Since $O(\log_2(n)) \subset O(n^{\frac{1}{2}})$ imply that $O(\max\{n^{\frac{1}{2}}, \log_2(n)\}) = O(n^{\frac{1}{2}})$ we have that:
   $g(n) \in O(\max\{n^{\frac{1}{2}}, \log_2(n)\}) = O(n^{\frac{1}{2}})$ by *(ii)*.

   Therefore, $f(n) \in O(n^{\frac{1}{2}})$ **[1 mark]**

Question 2

Consider a set of cards where every card has a number $n \in \{0, 1, 2, 3\}$ on one side and a suit (clubs $\clubsuit$, diamonds $\diamondsuit$, hearts $\heartsuit$ or spades $\spadesuit$) on the other side. Assume that all the cards are placed on a table in such a way that only one of the two sides (suit or number) is visible. A magician makes the following claim,

> *All cards that have a heart suit $\heartsuit$ on one side, always have number 2 on the other side.*

In order to verify whether the magician's claim is true, you need to turn over some cards. Assume that this must be done with the minimum possible number of turned over cards.

Which cards would you turn over to verify that a heart suit $\heartsuit$ on one side *implies* a number 2 on the other side? Justify!

Answer 2

First, let us consider the truth table corresponding to the implication: $\heartsuit \rightarrow 2$, namely "$\heartsuit$ on one side *implies* a number 2 on the other side"

	$\heartsuit$	2	$\heartsuit \rightarrow 2$
(i)	True	True	True
(ii)	**True**	**False**	**False**
(iii)	False	True	True
(iv)	False	False	True

As it can be seen from the truth table, the only way in which the implication is falsified occurs in row (ii) when it is *true* that one side has a heart suit $\heartsuit$, but it is *false* that the other side has a number 2. Thus, we need to verify that this situation never arises. There are two cases:

1. The visible side of a card shows a suit. If the suit is **not** the heart suit $\heartsuit$, then we will be in row (iii) or (iv) of the truth table, where the implication is *true*, so there is no need to turn such cards. Otherwise, if the suit is the heart suit $\heartsuit$, then we will be in row (i) or (ii) of the truth table, so we need to turn around the card to verify whether there is a 2 or not on the other side. **[1 mark]**
2. The visible side of a card shows a number. If the number is 2, then we will be in row (i) or (iii) of the truth table, where the implication is *true*, so there is no need to turn around such cards. Otherwise, if the number is **not** 2, then we will be in row (ii) or (iv) of the truth table, so we need to turn around the card to verify whether there is a $\heartsuit$ or not on the other side. **[1 mark]**

Consequently, we must turn around those cards that show a heart suit $\heartsuit$, and those that show a number different from 2.

Question 3

Suppose Lulu, a Constructor University student, satisfies the following conditions:

§1. If Lulu studies, then she receives good marks.

§2. If Lulu does not study, then she enjoys university life.

§3. If Lulu does not receive good marks, then she does not enjoy university life.

By means of a propositional logic argument show that from the above premises, it follows that: "Lulu receives good marks".

Answer 3

Proof. By the principle of excluding the middle (*tertium non datur*), we know that any proposition or its negation is true. So, it must be the case that *Lulu studies* OR *Lulu does not study.* Let us proceed by case analysis:

- Case I: –*Lulu studies*–. Then, it follows from §1 that –*Lulu receives good marks*– by implication elimination (*modus ponens*). **[1 mark]**
- Case II: *Lulu does not study.* Then, it follows from §2, by implication elimination, that –*Lulu enjoys university life*–(1) . By way of contradiction, let us now assume that –*Lulu does not receive good marks*–(2). In that case, it follows from §3 that –*Lulu does not enjoy university life*–(3). Clearly, propositions (1) and (3) are in contradiction! This means that assumption (2) is *false*, (*reductio ad absurdum*), so its negation must be true, namely that –*Lulu receives good marks*–. **[1 mark]**

Since in both Case I and Case II, we have reached the same conclusion, so by case analysis (proof by cases), it follows inescapably that –*Lulu receives good marks*–. $\square$

Alternative solution The same propositional logic argument can be derived from a truth table. For this, let propositional variables S, G and E stand, respectively, for *Lulu studies*, *Lulu receives good marks* and *Lulu enjoys university life.*

	S	G	E	$S \to G$	$\neg S \to E$	$\neg G \to \neg E$
(i)	True	**True**	True	**True**	**True**	**True**
(ii)	True	**True**	False	**True**	**True**	**True**
(iii)	True	False	True	False	True	False
(iv)	True	False	False	False	True	True
(v)	False	**True**	True	**True**	**True**	**True**
(vi)	False	True	False	True	False	True
(vii)	False	False	True	True	True	False
(viii)	False	False	False	True	False	True

As it can be seen from the truth table, our three premises $S \to G$, $\neg S \to E$ and $\neg G \to \neg E$ are *True* simultaneously only in rows (i), (ii) and (v). In these same rows (i), (ii) and (v), we have that propositional variable G is *True*. Therefore, if the three premises are *True*, then G (*Lulu receives good marks*) holds.

Truth table **[1 mark]** and argument **[1 mark]**.

Question 4

Inside a container, there are balloons of various colors and of three sizes, namely small, medium or large. The following premise holds *for all* balloons,

> *If a balloon is not of yellow color, then this balloon is neither of medium size.*

Which of the following sentences is a logical consequence of this premise, Justify!

(a) If there is a medium size balloon then there is also a yellow balloon
(b) All balloons are either small or large
(c) There are no balloons of yellow color
(d) There are some medium size balloons which are not yellow colored
(e) If there is a yellow balloon then this balloon cannot be of large size
(f) All balloons are yellow colored and all of them are of medium size

[1 mark]

Answer 4

For answering this question, we take the *contrapositive* of the premise. That is, *for all* balloons,

> *If a balloon is of medium size, then this balloon is of yellow color.*

This means that all balloons of medium size are yellow. Notice, that this does not say that a yellow balloon must be of medium size. A yellow balloon can be of any size, but a medium size balloon can only be yellow.

Consequently, (a) is *True*. Namely, *if there is a medium size balloon then there is also a yellow balloon.* We do not know if there is a medium size balloon, but if there is one, we know that the same balloon is yellow, so there will be a yellow balloon. **[1 mark]**

Question 5

Define a Haskell function: `prefix :: Eq a => a -> [a] -> [a]` such that for any arbitrary element `e :: a` and list `l ::[a]`, the function `prefix e l` produces the least prefix of list of `l` that includes the element `e`.

In other words, this function returns the (sub)-list of `l` obtained by removing all the elements placed after the first occurrence of `e` in `l`. If `e` is not in `l` then the resulting list must be the original list `l`.

For example:

```
prefix 4 [1,2,3,4,5,6]
⟹ [1,2,3,4]

prefix 3 [42,99,3,4,3,5,3,6,3,7]
⟹ [42,99,3]

prefix 'a' ''Hello Haskell''
⟹ ''Hello Ha''

prefix 10 [99,42,24]
⟹ [99,42,24]
```

Answer 5

This solution was discussed during the lectures.

```
prefix :: Eq a => a -> [a] -> [a]
prefix _ []     = []
prefix e (x:xs) | e == x    = [e]
                | otherwise = x : (prefix e xs)
```

Base case (empty list) **[1 mark]**

Recursive case **[2 marks]**