Assignment 3

Question 1

lemma 2 : If n is an odd number, then n^2 is an odd integer

n is odd → $n = 2k+1$, for some integer $k \in \mathbb{Z}$

$n^2 = (2k+1)^2 = 4k^2 + 4k + 2$

Simplified → $2k^2 + 2k + 1$ (because the factor does not change anything)

$2(k^2+k)+1$

no matter if k^2+k is odd or even the $\times 2$ turns it even as proved from previous example (Lemma 1)

thus ~~proved~~ an even number +1 gives odd hence proved

OR

Let $m = k^2+k$ → $2m+1$: same form as n thus n^2 is odd

Question 2

lemma 3 : if n is an integer, then n^2+n+6 is even

if n is odd:

$n \to 2k+1$

$(2k+1)^2 + 2k+1+6$

$= 4k^2 + 4k + 1 + 2k + 1 + 6$

$= 4k^2 + 6k + 8 \to 4k^2 + 6k + 8$

$= 2(2k^2+3k+4) \to 2m$: $m = 2k^2+3k+4$

same as $n = 2k$ from definition 1

thus even

if n is even

$n \to 2k$

$(2k)^2 + (2k) + 6$

$4k^2 + 2k + 6 \to 2(2k^2+k+3) \to 2m$

thus even from definition 1

Question 3

Lemma 4: Suppose n is an integer. If n^2 is odd, then n is odd

contrapositive

$$(P \to Q) \equiv (\neg Q \to \neg P)$$

$$n = 2k$$

$$(n^2) = 4k^2$$

Since $4k^2$ is even, n^2 cannot be odd

therefore if n^2 is odd, then n must be odd

Question 4

Lemma 5: For any natural number $n \in \mathbb{N}$

$$\sum_{i=0}^{n} i + \sum_{i=0}^{n+1} i = (n+1)^2$$

Proof by induction on n

- base case:

$n = 0$

$$\sum_{i=0}^{0} i = 0$$

$$\sum_{0}^{0+1} i = 1 \qquad 0+1 = 1 \qquad \text{Thus equality holds}$$

$$(0+1)^2 = 1$$

- Induction step: $\forall x \in \mathbb{N} . P(x) \leftrightarrow (P(0) \wedge \forall y \in \mathbb{N} . P(y) \to P(succ(y)))$

induction hypothesis:

Assume for an arbitrary $y \in \mathbb{N}$ with $y \geq 0$

$$\sum_{i=0}^{y} i + \sum_{i=0}^{y+1} i = (y+1)^2$$

To show $y+1$ holds:

$$\sum_{i=0}^{y+1} i + \sum_{i=0}^{y+2} i = (y+2)^2$$

(Sucessor)

LHS: $\sum_{i=0}^{y+1} i = (y+1)^2 - \sum_{i=0}^{y} i + \sum_{i=0}^{y+2} i$

$$\sum_{i=0}^{y+2} i = \sum_{i=0}^{y} i + \sum_{0}^{2} i$$

$$(y+1)^2 + (1+2)$$

$$= (y+1)^2 + 3$$

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LHS : $(y+1)^2 + 3 \longrightarrow y^2 + 2y + 4$

RHS : $(y+2)^2 \longrightarrow y^2 + 2y + 4$

HENCE RHS = LHS

Thus equation holds

Question 5

proof by induction on n:

recursive $0 = 3$
recursive n = recursive$(n-1) + 2n$

formula $n = 3 + n(n+1)$

- Base case $n = 0$:

  recursive $0 \longrightarrow 3$

  formula $0 \longrightarrow 3 + 0(0+1) = 3$

Thus equality holds

- Induction step : $\forall x \in \mathbb{N} . P(x) \longleftrightarrow (P(0) \wedge \forall y \in \mathbb{N} . P(y) \rightarrow P(\mathrm{succ}(y)))$

  Induction hypothesis:

  Assume for an arbitrary $y \in \mathbb{N}$ with $y \geq 0$

  let $y = 2$
  recursive y = recursive$(y-1) + 2y$

  recursive 2 = recursive$(1) + 2(2)$

  $\hookrightarrow$ recursive(1) = recursive$(0) + 2(1)$

  $\hookrightarrow$ recursive$(0) = 3$

  recursive$(1) = 3 + 2 = 5$
  recursive$(2) = 5 + 4 = 9$

  formula $y = 3 + y(y+1)$

  formula $2 = 3 + 2(2+1) = 3 + 6 = 9$

  Thus recursive n = formula n

CONTINUATION

To show $y+1$ holds:

recursion $(y+1)$ = formula $(y+1)$

recursive $(y+1)$ = recursive (y) + $2(y+1)$

by hypothesis:

recursive $(y) = 3 + y(y+1)$: Equation from formula

thus recursive $(y+1) = y(y+1) + 3 + 2y + 2$

$= y^2 + 3y + 5$

formula $(y+1) = 3 + (y+1)(y+2)$

$= 3 + y^2 + 2y + y + 2$

$= y^2 + 3y + 5$

Thus recursive $(y+1)$ = formula $(y+1)$

Hence equation holds