MODULE $3Bean$

EXTENDS $Integers$
CONSTANTS M, N, O
ASSUME $\land M \in 0 \,..\, 100$
$\quad\land N \in 0 \,..\, 100$
$\quad\land O \in 0 \,..\, 100$
$\quad\land (M + N + O) > 0$

```
--fair algorithm 3Bean{
  variable r = M, g = N, b = O ; {
   S: while ( TRUE ) {
      either
       { await (r > 1) ;
              r := r - 2 ;
             } ;
      or
      { await (g > 1) ;
               g := g - 2 ;
                } ;
      or
      { await (b > 1) ;
               b := b - 2 ;
                } ;
      or
    { await (r > 0 ∧ g > 0) ;
              r := r - 1 ;
              g := g - 1 ;
              b := b + 1 ;
               } ;
      or
    { await (r > 0 ∧ b > 0) ;
              r := r - 1 ;
              b := b - 1 ;
              g := g + 1 ;
               } ;
      or
    { await (b > 0 ∧ g > 0) ;
              b := b - 1 ;
              g := g - 1 ;
              r := r + 1 ;
               } ;
         }
     }
  }
```

BEGIN $TRANSLATION$ – the hash of the $PCal$ code: PCal-7d05fbd92b6d8292c628b5288105af4e

VARIABLES $r,\ g,\ b$

$vars \triangleq \langle r,\ g,\ b \rangle$

$$
\begin{aligned}
Init \triangleq\ & \text{Global variables} \\
& \land r = M \\
& \land g = N \\
& \land b = O
\end{aligned}
$$

$$
\begin{aligned}
Next \triangleq\ & \lor \land (r > 1) \\
& \quad \land r' = r - 2 \\
& \quad \land \text{UNCHANGED } \langle g,\ b \rangle \\
& \lor \land (g > 1) \\
& \quad \land g' = g - 2 \\
& \quad \land \text{UNCHANGED } \langle r,\ b \rangle \\
& \lor \land (b > 1) \\
& \quad \land b' = b - 2 \\
& \quad \land \text{UNCHANGED } \langle r,\ g \rangle \\
& \lor \land (r > 0 \land g > 0) \\
& \quad \land r' = r - 1 \\
& \quad \land g' = g - 1 \\
& \quad \land b' = b + 1 \\
& \lor \land (r > 0 \land b > 0) \\
& \quad \land r' = r - 1 \\
& \quad \land b' = b - 1 \\
& \quad \land g' = g + 1 \\
& \lor \land (b > 0 \land g > 0) \\
& \quad \land b' = b - 1 \\
& \quad \land g' = g - 1 \\
& \quad \land r' = r + 1
\end{aligned}
$$

$$
\begin{aligned}
Spec \triangleq\ & \land Init \land \Box[Next]_{vars} \\
& \land \mathrm{WF}_{vars}(Next)
\end{aligned}
$$

END *TRANSLATION* − the hash of the generated TLA code (remove to silence divergence warnings): TLA-1770dc3085475a4

$inv \triangleq (r + g + b) > 0 \lor (r + g + b) = 0$
$Termination \triangleq \Diamond(r + g + b < 2)$

1.1 Safety property :

Invariant function $\triangleq$ $(r + g + b) > 0 \lor (r + g + b) = 0$.

Invariant.$((r+g+b) > 0 \lor (r+g+b) = 0) =$ initial.$((M+N+O) > 0)$ & stable.$((r+g+b) > 0 \lor (r+g+b) = 0))$ holds.

Progress property :

Termination $\triangleq\ \diamond\ (r+g+b) < 2$. The program terminates once it reaches 1 bean or there are no beans left in the can as it satisfies none of the conditions.

1.2 Invariant $\triangleq\ (r+g+b)\ > 0 \vee\ (r+g+b)\ = 0.$

Invariant.$((r+g+b) > 0 \vee (r+g+b) = 0) =$ initial.$((M+N+O) > 0)$ & stable.$((r+g+b) > 0 \vee (r+g+b) = 0))$. The invariant holds because *initially*.$M+N+O > 0$ which is a subset of the $((r+g+b) > 0 \vee (r+g+b) = 0)$.

To prove : stable.$((r+g+b) > 0 \vee (r+g+b) = 0))$: $((r+g+b) > 0 \vee (r+g+b) = 0))next((r+g+b) > 0 \vee (r+g+b) = 0))$.

Action 1: $((r+g+b) > 0 \vee (r+g+b) = 0))\ r > 1 \rightarrow r := r-2((r+g+b) > 0 \vee (r+g+b) = 0))$
Applying assignment axiom with guard:
$((r+g+b) > 0 \vee (r+g+b) = 0))\ \wedge r > 1 = ((r-2+g+b > 0) \vee (r-2+g+b = 0)$
$((r+g+b) > 1 = ((r+g+b > 2) \vee (r+g+b = 2)$
$(r+g+b) > 1\ implies((r+g+b > 2) \vee (r+g+b = 2). Therefore\ after\ the\ action\ is\ performed,\ it\ remains\ within\ the(r+g+b > 1)state,\ satisfying\ stable\ condition$

Action 2 :
$((r+g+b) > 0 \vee (r+g+b) = 0))g > 1 \rightarrow g := g-2((r+g+b) > 0 \vee (r+g+b) = 0))$
$((r+g+b) > 0 \vee (r+g+b) = 0)) \wedge g > 1 = ((r+g-2+b > 0) \vee (r+g-2+b = 0)$
$(r+g+b) > 1\ implies(r+g+b > 2 \vee (r+g+b = 2)$

Action 3 :
$((r+g+b) > 0 \vee (r+g+b) = 0))b > 1 \rightarrow b := b-2((r+g+b) > 0 \vee (r+g+b) = 0))$
$((r+g+b) > 0 \vee (r+g+b) = 0)) \wedge b > 1 = ((r+g+b-2 > 0) \vee (r+g+b-2 = 0)$
$(r+g+b) > 1\ implies(r+g+b > 2 \vee (r+g+b = 2)$

Action 4 :
$((r+g+b) > 0 \vee (r+g+b) = 0))(r > 0 \wedge g > 0) \rightarrow r := r-1; g := g-1; b := b+1((r+g+b) > 0 \vee (r+g+b) = 0))$
Since $r > 0$ & $g > 0$, this condition $r+g+b > 0$ is true:
$r+g+b > 0 \wedge r > 0 \wedge g > 0 =\ (r-1+g-1+b+1)\ > 0$
$r+g+b > 1$ implies $(r+g+b)\ > 1$

Action 5: $((r+g+b) > 0 \vee (r+g+b) = 0))\ (b > 0 \wedge g > 0)\ \rightarrow b := b-1;g := g-1;r := r+1((r+g+b) > 0 \vee (r+g+b) = 0))$
$(r+g+b)\ > 0 \wedge b > 0 \wedge g > 0 = ((r+1+g-1+b-1) > 0$
$(r+g+b) > 1\ implies(r+g+b) > 1$

Action 6 :
$((r+g+b) > 0 \vee (r+g+b) = 0))\ (b > 0 \wedge r > 0)\ \rightarrow b := b-1;r := r-1;g := g+1((r+g+b) > 0 \vee (r+g+b) = 0))$
$((r+g+b) > 0 \vee (r+g+b) = 0))\ \wedge r > 0 \wedge b > 0 =\ (r-1+g+1+b-1)\ > 0\ (r+g+b)\ > 1$ implies $(r+g+b)\ > 1$

Test cases:

1.2.1) $r = 1$,$g = 1$,$b = 1$:

The invariant is satisfied at this point as initially $r+g+b > 0$ & *stable*$((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is satisfied.

Beans picked $\rightarrow$ 1 red & 1 green.

Action performed $\rightarrow r := r-1;g := g-1;b := b+1 \rightarrow$ Therefore, 2 beans are thrown out and 1 blue bean is added.

Leftover beans $\rightarrow b = 2 \rightarrow$ $stable((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is satisfied as $r+g+b > 0$.

Beans picked $\rightarrow$ 2 blue beans $\rightarrow$ Both of the beans are thrown out, $r+g+b$ reduces by 2 and lower bound is 0.

Leftover beans $\rightarrow 0 \rightarrow$ $stable((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is satisfied as $r+g+b = 0$.

Similarly, invariant holds for all of the other combinations as $r+g+b > 0$ or $r+g+b = 0$.

1.2.2) $r = 2$,$g = 2$,$b = 2$:

The invariant is satisfied at this point as initially $r+g+b > 0$ & $stable((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is satisfied.

Beans picked $\rightarrow$ 2 red beans.

Action performed $\rightarrow r := r - 2 \rightarrow$ Both of these beans are thrown out.

Leftover beans $\rightarrow g = 2,\ b = 2 \rightarrow$ $stable((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is satisfied as $r+g+b > 0$.

Beans picked $\rightarrow$ 2 green beans.

Leftover beans $\rightarrow b = 2 \rightarrow$ $stable((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is satisfied as $r+g+b > 0$.

Beans picked $\rightarrow$ 2 blue beans.

Action performed $\rightarrow b := b - 2$.

Leftover beans $\rightarrow 0 \rightarrow$ $stable((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is still satisfied as $r+g+b = 0$.

Another state *transition[for the same initial values]*:

Beans picked $\rightarrow$ 1 red & 1 green bean.

Action performed $\rightarrow r := r - 1; g := g - 1; b := b + 1 \rightarrow$ these beans are thrown out & 1 blue bean is added.

Leftover beans $\rightarrow r = 1,\ g = 1,\ b = 3 \rightarrow$ $stable((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is satisfied as $r+g+b > 0$.

Beans picked $\rightarrow$ 1 green bean & 1 blue bean.

Action performed $\rightarrow b := b - 1; g := g - 1; r := r + 1 \rightarrow$ Both of these beans are thrown out & 1 red bean is added.

Leftover beans $\rightarrow r = 2,\ g = 0,\ b = 2 \rightarrow$ $stable((r+g+b) > 0 \vee (r+g+b) \triangleq 0)$ is satisfied as $r+g+b > 0$.

Beans picked $\rightarrow$ 1 red bean & 1 blue bean.

Action performed $\rightarrow b := b - 1; r := r - 1; g := g + 1 \rightarrow$ these 2 beans are thrown out & 1 green bean is added.

Leftover beans $= r = 1,\ g = 1,\ b = 1$.

From the previous test case where initially $r = 1,\ g = 1,\ b = 1$ was already observed to be stable, in conclusion this test case is also stable and therefore invariant.

1.3 Fixed point $\triangleq$ $(r+g+b \leq 1)$

$FP = ((r > 1) \wedge (b > 1) \wedge (g > 1) \wedge (r > 0 \wedge g > 0) \wedge (r > 0 \wedge b > 0) \wedge (b > 0 \wedge g > 0))$.
$FP = \ (r \leq 1 \wedge b \leq 1 \wedge g \leq 1 \wedge (r = 0 \vee g = 0) \wedge (r = 0 \vee b = 0) \wedge (g = 0 \vee b = 0)$.
$FP = r + g + b \leq 1$.

The number of beans cannot be below 0 [*i.e.* it is bounded by 0], so the fixed points are $\Rightarrow r + g + b = 0$ & $r + g + b = 1$

1.4 Termination:

Termination $\triangleq \ \Diamond \ (r + g + b < 2)$.

Variant function $\triangleq \ r + g + b$ [It is a decreasing function]. For every action, $r + g + b$ reduces by 1 bean or 2 beans and it is bounded by 0.

To prove : $\{P\}$a$\{Q\}$ where Pre-condition is $r + g + b$ & post-condition is $\{(r + g + b) - 1\} \vee \{(r + g + b) - 2\}$.

Action 1:
$(r + g + b) \ r > 1 \rightarrow r := r - 2((r + g + b) - 1) \vee \ (r + g + b) - 2))$
$(r + g + b) \ r > 1 \rightarrow r := r - 2 = r - 2 + g + b$
$(r + g + b) \ r > 1 \rightarrow r := r - 2 = \ (r + g + b) - 2$

Action 2:
$(r + g + b) \ g > 1 \rightarrow g := g - 2((r + g + b) - 1) \vee \ (r + g + b) - 2))$
$(r + g + b) \ g > 1 \rightarrow g := g - 2 = \ (r + g + b) - 2$

Action 3:
$(r + g + b) \ b > 1 \rightarrow b := b - 2((r + g + b) - 1) \vee \ (r + g + b) - 2))$
$(r + g + b) \ b > 1 \rightarrow b := b - 2 = \ (r + g + b) - 2$

Action 4:
$(r+g+b) \ (r > 0 \wedge g > 0) \ \rightarrow r := r-1; g := g-1; b := b+1((r+g+b)-1) \vee \ (r+g+b)-2))$
$(r + g + b) \ (r > 0 \wedge g > 0) \ \rightarrow r := r - 1; g := g - 1; b := b + 1 = r - 1 + g - 1 + b + 1$
$(r + g + b) \ (r > 0 \wedge g > 0) \ \rightarrow r := r - 1; g := g - 1; b := b + 1 = \ (r + g + b) - 1$

Action 5:
$(r+g+b) \ (b > 0 \wedge g > 0) \ \rightarrow b := b-1; g := g-1; r := r+1((r+g+b)-1) \vee \ (r+g+b)-2))$
$(r + g + b) \ (b > 0 \wedge g > 0) \ \rightarrow b := b - 1; g := g - 1; r := r + 1 = r + 1 + g - 1 + b - 1$
$(r + g + b) \ (b > 0 \wedge g > 0) \ \rightarrow b := b - 1; g := g - 1; r := r + 1 = \ (r + g + b) - 1$

Action 6:
$(r+g+b) \ (b > 0 \wedge r > 0) \ \rightarrow b := b-1; r := r-1; g := g+1((r+g+b)-1) \vee \ (r+g+b)-2))$
$(r + g + b) \ (b > 0 \wedge r > 0) \ \rightarrow b := b - 1; r := r - 1; g := g + 1 = r - 1 + g + 1 + b - 1$
$(r + g + b) \ (b > 0 \wedge r > 0) \ \rightarrow b := b - 1; r := r - 1; g := g + 1 = \ (r + g + b) - 1$

Testcases:

1.4.1) $r = 1, \ g = 1, \ b = 1$:

Beans picked $\rightarrow$ 1 red & 1 green.

Action performed $\rightarrow r := r - 1; g := g - 1; b := b + 1 \rightarrow$ Therefore, 2 beans are thrown out and 1 blue bean is added. $r + g + b$ reduces by 1.

Leftover beans $\rightarrow b = 2$.

Beans picked $\rightarrow$ 2 blue beans $\rightarrow$ Both of the beans are thrown out, $r + g + b$ reduces by 2 and lower bound is 0.

Similarly for the other combinations, the metric function reduces by 1 bean or 2 beans. And the program terminates when the number of beans is 0 or 1.

1.4.2) $r = 2,\ g = 1,\ b = 1$:

Beans picked $\rightarrow$ 2 red beans.

Action performed $\rightarrow r := r - 2 \rightarrow$ Both the beans are thrown out. $r + g + b$ reduces by 2.

Leftover beans $\rightarrow g = 1,\ b = 1$.

Beans picked $\rightarrow$ 1 green & 1 blue bean $\rightarrow b := b - 1; g := g - 1; r := r + 1 \rightarrow$ 1 red bean is added & these two beans are thrown out. $r + g + b$ reduces by 1. And the program terminates with 1 red bean in the can.

Another state *transition*(*for the same initial values*):

Beans picked $\rightarrow$ 1 red bean & 1 green bean $\rightarrow r := r - 1; g := g - 1; b := b + 1 \rightarrow$ 1 blue bean is added & these two beans are thrown out. $r + g + b$ reduces by 1.

Leftover beans $\rightarrow r = 1,\ b = 2$.

Beans picked $\rightarrow$ 1 red bean & 1 blue bean $\rightarrow b := b - 1; r := r - 1; g := g + 1 \rightarrow$ 1 green bean is added & these two are thrown out. $r + g + b$ reduces by 1.

Leftover beans $\rightarrow g = 1,\ b = 1$.

Beans picked $\rightarrow$ Last two beans $\rightarrow b := b - 1; g := g - 1; r := r + 1 \rightarrow$ 1 red bean is added & these two are thrown out. $r + g + b$ reduces by 1. The program terminates at 1 bean.

* Modification History
* Last modified Sun *Oct* 18 19:44:55 *EDT* 2020 by *harita*
* Created *Fri Oct* 16 15:48:17 *EDT* 2020 by *harita*