1. Define: (a) Convergence of a sequence. (b) Divergence of a sequence to infinity. Prove using definitions that $\left\{ \dfrac{1}{n^p} \right\}$ converges if $p \geq 0$ and diverges to infinity if $p < 0$. (5)

Solution: (a) A sequence $\{x_n\}$ of real numbers is said to converge to a number L if for every $\epsilon > 0$, there exists a natural number N such that

$$|x_n - L| < \epsilon \text{ for all } n > N(\text{ or } n \geq N).$$

(1 Mark)

(b) A sequence $\{x_n\}$ of real numbers is said to diverge to $+\infty$ if for every positive number M, however large, there exists a natural number N such that

$$x_n > M \text{ for all } n \geq N.$$

(1 Mark)

We now consider the sequence $\left\{ \dfrac{1}{n^p} \right\}$.

Case 1: $p \geq 0$. If $p = 0$, then the sequnce becomes the constant sequence $\{1\}$ and so it converges to $L = 1$: For any $\epsilon > 0$, we have $|a_n - L| = |1 - 1| = 0 < \epsilon$ for all $n \geq N = 1$.

Let $p > 0$. In this case, we show that the sequence converges to 0. Let $\epsilon > 0$ be any positive number. Then we have to find a natural number N such that

$$\left| \frac{1}{n^p} - 0 \right| < \epsilon \quad n > N(\text{ or } n \geq N).$$

That is,

$$\frac{1}{n^p} < \epsilon \quad \forall n > N(\text{ or } n \geq N).$$

Now

$$\frac{1}{n^p} < \epsilon \Longleftrightarrow n > \left(\frac{1}{\epsilon} \right)^{1/p}.$$

Thus for $N = \left\lceil \left(\frac{1}{\epsilon}\right)^{1/p} \right\rceil$, we have

$$\left|\frac{1}{n^p} - 0\right| < \epsilon \text{ for all } n > N(\text{ or } n \geq N).$$

Hence the sequence converges. (2 Marks)

Case 2: $p < 0$. We prove that $\left\{\frac{1}{n^p}\right\}$ diverges to infinity. Let $q = -p$. Then $q > 0$ and $\left\{\frac{1}{n^p}\right\} = \{n^q\}$. Let M be any positive number. Then we must find a naural number N such that

$$\frac{1}{n^p} = n^q > M \text{ for all } n > N.$$

But

$$n^q > M \Longleftrightarrow n > M^{1/q}.$$

Thus $N = \lceil M^{1/q} \rceil$ is the required natural number. Hence the sequence diverges to ∞. (1 Mark)

2. Find $\lim_{n\to\infty}(n!)^{1/n^2}$

   Solution: (2)

   We know $(n!)^{1/n^2} \geq (1)^{1/n^2} = 1$. Also $(n!)^{1/n^2} \leq (n^n)^{1/n^2} = (n)^{1/n}$. From the above relations, we get

   $$1 \leq (n!)^{1/n^2} \leq (n)^{1/n}.$$

   Applying lim we get

   $$\lim_{n\to\infty} 1 \leq \lim_{n\to\infty}(n!)^{1/n^2} \leq \lim_{n\to\infty}(n)^{1/n} = 1.$$

   Therefore, by the sandwich theorem,

   $$\lim_{n\to\infty}(n!)^{1/n^2} = 1.$$

   (2 Marks)

OR

Let $y_n = (n!)^{1/n^2}$. Taking log on both sides, we get

$$\log(y_n) = \frac{\log n!}{n^2} = \frac{\log n + \log(n-1) + \cdots \log(1)}{n^2} \leq \frac{\log n}{n}.$$

Also

$$0 \leq \log(y_n) \leq \frac{\log n}{n}$$

and $\dfrac{\log n}{n} \longrightarrow 0$ (by L'Hôpital's rule).

Hence, by the sandwich theorem, $\log(y_n) \longrightarrow 0$.

So, by the continuous function theorem, we have $(n!)^{1/n^2} = y_n = e^{\log y_n} \longrightarrow e^0 = 1$. (2 Marks)

3. Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges. (3)

   Solution:

   Let $\{a_{m_k}\}$ and $\{a_{n_k}\}$ be subsequences of $\{a_n\}$ such that $a_{m_k} \to L_1$ and $a_{n_k} \to L_2$, where $L_1 \neq L_2$. We must show that $\{a_n\}$ diverges.

   Suppose, for contradicition, that $\{a_n\}$ converges, say, to L. Let $\epsilon = |L_1 - L_2| > 0$. Then corresponding to $\epsilon/4 > 0$, there is a natural number N_0 such that $|a_n - L| < \epsilon/4$ for all $n \geq N_0$.

   Wlog., suppose $|L - L_1| \leq |L - L_2|$; i.e., L is closer to L_1.

   Now $a_{m_k} \to L_1$ and $a_{n_k} \to L_2$. Thus, corresponding to $\epsilon/4 > 0$, we can find integers K_1 and K_2 such that

   $$|a_{m_k} - L_1| < \epsilon/4 \text{ for all } k \geq K_1$$

   and

   $$|a_{n_k} - L_2| < \epsilon/4 \text{ for all } k \geq K_2.$$

Let $N = \max(N_0, K_1, K_2)$. Then this implies the following:

$$|a_n - L| < \epsilon/4 \text{ for all } n \geq N$$

and

$$|a_{n_k} - L_2| < \epsilon/4 \text{ for all } k \geq N.$$

That is, the former inquality is satisfied by all a_n's with $n \geq N$ and the latter inequality is satisfied by infinitely many a_{n_k}'s as the subsequence $\{a_{n_k}\}$ has infinitely many terms by the definition of a subsequence.

But $|L - L_2| \geq \epsilon/2$ by our assumption on L. Thus we also have that none of $a_N, a_{N+1}, a_{N+2}, \ldots$ satisfies the second inequality since $n_k \geq k \geq N$. This is a contradiction. Hence we conclude that the sequence diverges. (3 Marks)

4. Present series of nonzero terms with sum (a) 0 and (b) π^2. (2)

Solution:

(a) We have

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

Hence the series

$$-2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n} + \cdots$$

converges and has sum 0. (1 Mark)

(b) We have

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

Hence the series

$$\pi^2 - 2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n} + \cdots$$

converges and has sum π^2. (1 Mark)

OR

We have $\sum \frac{1}{n^2} = \pi^2/6$. Hence

$$\sum \frac{6}{n^2} = \pi^2.$$

(1 Mark)

5. Prove or disprove: (3)

   (a) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ converges;

   (b) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converges.

   Solution: (a) We have

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1.$$

Thus, by limit comparison theorem $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ both converge or both diverge. But the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So, $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ also diverges. (2 Marks)

(b) Since $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0$. So, $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ diverges by the nth term test for divergence. (1 Mark)

6. Consider the series $\sum a_n$, where $a_n = \begin{cases} n/2^n & \text{if } n \text{ is prime;} \\ 1/2^n & \text{otherwise.} \end{cases}$
   Does it converge? Give reasons. (2)

   Solution: The series converges. We can prove this by using the root test. The root test is applicable since the terms are positive. Here

   $$1/2 \leq a_n^{1/n} \leq n^{1/n}/2 \quad \text{for all} \quad n.$$

   That is the sequence $a_n^{1/n}$ is sandwiched between the sequences $\{1/2\}$ and $\{n^{1/n}/2\}$ and both of these converge to $1/2$.

   So, $a_n^{1/n} \longrightarrow 1/2$ by the sandwich theorem.

   Since $1/2 < 1$, the series converges by the root test. (2 Marks)

7. Prove that the alternating p-series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if $p > 0$ and diverges if $p \leq 0$. (3)

   Solution: *Case 1:* $p > 0$. Then (1) $\dfrac{1}{n^p} > 0$; (2) $\dfrac{1}{n^p} > \dfrac{1}{(n+1)^p}$ since $n^p < (n+1)^p$; and (3) $\dfrac{1}{n^p} \longrightarrow 0$. Hence the series converges by the alternating series test. (2 Marks)

   Case 2: $p \leq 0$. Then $\dfrac{(-1)^n}{n^p} \not\to 0$. Hence the series diverges by the nth term test for divergence. (1 Mark)

8. Consider a power series $\sum a_n x^n$. Prove: (a) If the power series converges for $x = c \neq 0$, then it converges absolutely for $|x| < |c|$. (b) If the power series diverges for $x = d$, then it diverges for $|x| > |d|$. (4)

   Solution: (a) Suppose the series $\displaystyle\sum_{n=0}^{\infty} a_n c^n$ converges.

   Then $a_n c^n \to 0$. Therefore, corresponding to $\epsilon = 1$, there is an integer N such that, for $n \geq N$,

$$|a_n c^n| < 1 \quad \text{or} \quad |a_n| < \frac{1}{|c|^n}.$$

Now take any x such that $|x| < |c|$ and consider

$$|a_0| + |a_1 x| + \ldots + |a_{N-1}x^{N-1}| + |a_N x^N| + |a_{N+1}x^{N+1}| + \ldots.$$

There are only a finite number of terms prior to $|a_N x^N|$ and so their sum is finite.

Starting from $|a_N x^N|$, the sum of the terms is less than

$$\left|\frac{x}{c}\right|^N + \left|\frac{x}{c}\right|^{N+1} + \left|\frac{x}{c}\right|^{N+2} + \ldots.$$

But the above series is a geometric series with common ratio less than 1 since $|x| < |c|$.

So, it converges. Thus it follows that the given power series converges absolutely for $|x| < |c|$. (3 Marks)

(b) To prove this part of the theorem, we can use the first half, namely (a).

Suppose, for contradiction, that the power series diverges at $x = d$ and converges at a value x_0 with $|x_0| > |d|$.

Then by taking $c = x_0$, we can conclude by the first half of the theorem that the power series converges at $x = d$, which is a contradiction.

Thus, it follows that if the power series diverges for $x = d$, then it diverges for all x with $|x| > |d|$. (1 Mark)

9. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^{10}}$. Where does it converge absolutely? Where does it converge conditionally? (3)

Solution: Here $\displaystyle\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1)^{10}}{n^{10}} = \left(1 + \frac{1}{n}\right)^{10} \longrightarrow 1.$

Thus the radius of convergence of the series is $R = 1$. Hence the power series converges absolutely for $|x| < 1$. (1 Mark)

For $x = \pm 1$, the power series is either a p-series or an alternating p-series with $p = 10$. Hence the power series converges absolutely for $x = \pm 1$. (1 Mark)

Hence the interval of convergence of the power series is $[-1, 1]$ and for every x in this interval the power series converges absolutely. (1 Mark)

10. Find the Maclaurin series of $f(x) = \sqrt{1+x}$. In particular, find its general term. (3)

Solution: Here

$$\begin{aligned}
f(x) &= \sqrt{1+x} = (1+x)^{1/2} \\
f'(x) &= 1/2(1+x)^{-1/2} \\
f''(x) &= (1/2)(-1/2)(1+x)^{-3/2} \\
f'''(x) &= (1/2)(-1/2)(-3/2)(1+x)^{-5/2} \\
&\vdots \\
f^{(n)}(x) &= (1/2)(-1/2)(-3/2)\ldots(1/2-(n-1))(1+x)^{1/2-n} \\
&\vdots
\end{aligned}$$

So,

$$f(0) = 1, f'(0) = 1/2, f''(0) = (1/2)(-1/2), f'''(x) = (1/2)(-1/2)(-3/2), \ldots$$

$$f^{(n)}(0) = (1/2)(-1/2)(-3/2)\ldots(1/2-(n-1)), \ldots$$

Hence the Maclaurin series of $f(x) = \sqrt{1+x}$ is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

$$= 1 + \frac{1/2}{1!}x + \frac{(1/2)(-1/2)}{2!}x^2 + \ldots + \frac{(1/2)(-1/2)(-3/2)\ldots(1/2-(n-1))}{n!}x^n + \ldots.$$

(1 + 2 Marks (2 Marks is for the general term))