# LPP

October 2, 2023

# Linear Programming problem

## Mathematical Formulation

The procedure for mathematical formulation of a linear programming problem consists of the following problem:

**Step 1.** Study the given situation to find the key decisions to be made

**Step 2.** Identify the variable involved and designate them by symbols $x_j (j = 1, 2, \cdots, .)$

**Step 3.** State the feasible alternatives which generally are $x_j \leq 0$ for all $j$

**Step 4.** Identify the constraint in the problem and express them as linear inequalities or equations, LHS of which are linear functions of the decision variables.

**Step 5.** Identify the objective function and express it as a linear function of the decision variable.

# Example

A furniture dealer deals only two items viz., tables and chairs. He has to invest $Rs.10,000/-$ and a space to store at most 60 pieces. A table cost him $Rs.500/-$ and a chair $Rs.200/-$. He can sell all the items that he buys. He is getting a profit of $Rs.50$ per table and $Rs.15$ per chair. Formulate this problem as an LPP, so as to maximize the profit.

SOLUTION:

**Step 1.** Study the given situation to find the key decisions to be made

**Step 2.** Variables: Let $x_1$ and $x_2$ denote the number of tables and chairs respectively.

**Step 3.** State the feasible alternatives which generally are $x_1 \geq 0, x_2 \geq 0$ for all $j$

**Step 4.** Identify the constraint in the problem and express them as linear inequalities or equations.

The dealer has a space to store at most 60 pieces

$$x_1 + x_2 \leq 60$$

The cost of $x_1$ tables $= Rs.500x_1$
The cost of $x_2$ tables $= Rs.200x_2$
Total cost $= 500x_1 + 200x_2$ ,which cannot be more than 10000

$$500x_1 + 200x_2 \leq 10000$$
$$5x_1 + 2x_2 \leq 100$$

**Step 5.** Identify the objective function.
Profit on $x_1$ tables $= 50x_1$
Profit on $x_2$ chairs $= 15x_2$
Total profit $= 50x_1 + 15x_2$
Let $Z = 50x_1 + 15x_2$ , which is the objective function.
Since the total profit is to be maximized, we have to maximize $Z = 50x_1 + 15x_2$.

Thus, the mathematical formulation of the LPP is

$$\text{Maximize } Z = 50x_1 + 15x_2$$

Subject to the constraints:

$$x_1 + x_2 \leq 60$$
$$5x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

# Graphical Solutions

Linear programming involving two decision variable can be easily solved by using Graphical methods.

The major steps in the solution of LPP by graphical methods follows:

**Step 1.** Identify the problem :- The decision variable, the objective function and restrictions.

**Step 2.** Set up the mathematical formulation of the problem.

**Step 3.** Plot the graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is intersection of all regions represented by the constraints of the problem and is restricted to first quadrant only.

**Step 4.** The feasible region may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

**Step 5.** Find out the value of objective function at each corner points.

**Step 6.** Select the corner point and Optimize the value of the objective function. It give the optimum feasible solution.

# Example

Solve the given linear programming problems graphically:

$$\text{Minimize: } z = 20x + 10y$$

and the constraints are :

$$\begin{aligned} x + 2y &\leq 40, \\ 3x + y &\geq 30, \\ 4x + 3y &\geq 60, \\ x \geq 0, y &\geq 0 \quad \text{(non-negativity constraints)} \end{aligned}$$

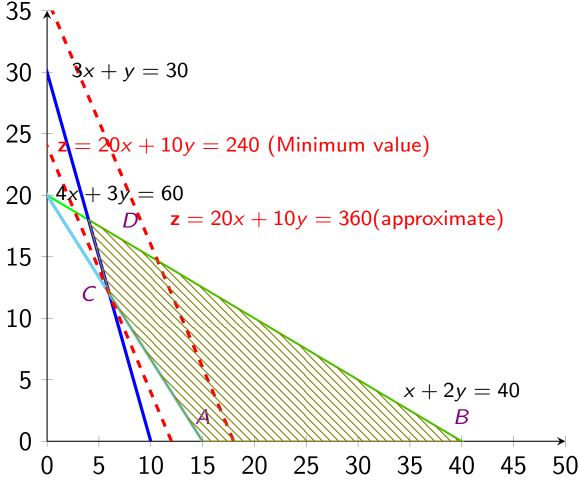
SOLUTION: **Step.1**: The appropriate model formulation of the given LPP

$$\text{Minimize: } z = 20x + 10y$$

and the constraints are :

$$\begin{aligned} x + 2y &\leq 40, \\ 3x + y &\geq 30, \\ 4x + 3y &\geq 60, \\ x \geq 0, y &\geq 0 \end{aligned}$$

**Step.2**: Now plot these points in the graph and find the feasible region.



**Step.3** The optimum value of the objective function occur at one of the extreme points of the feasible region.

That is, $A = (15, 0)$, $B = (40, 0)$, $C = (6, 12)$ and $D = (4, 18)$ be the extreme points.

**Step.4** Compute the $z$ values at the extreme points

| Extreme points | $(x_1, x_2)$ | $z = 20x + 10y$ |
|---|---|---|
| $A$ | $(15, 0)$ | 300 |
| $B$ | $(40, 0)$ | 800 |
| $C$ | $(6, 12)$ | 240 |
| $D$ | $(4, 18)$ | 260 |

**Step.5** The Optimum solution is that extreme point for which the objective function has least value. Thus Minimum $z = 20x + 10y$ is 240.

# Definitions:

1. Decision variables
2. Objective function
3. Constraints
4. Non-negative constraints
5. Feasible solutions
6. Infeasible solutions
7. Corner points

# Example

$$\text{Maximize:} z = 4x_1 + 3x_2$$

and the constraints are :

$$\begin{aligned} 2x_1 + x_2 &\leq 1000 \\ x_1 + x_2 &\leq 800 \\ x_1 \leq 400 &\text{ and } x_2 \leq 700 \\ x_1 \geq 0 &\text{ and } x_2 \geq 0 \quad \text{(non-negativity constraints)} \end{aligned}$$

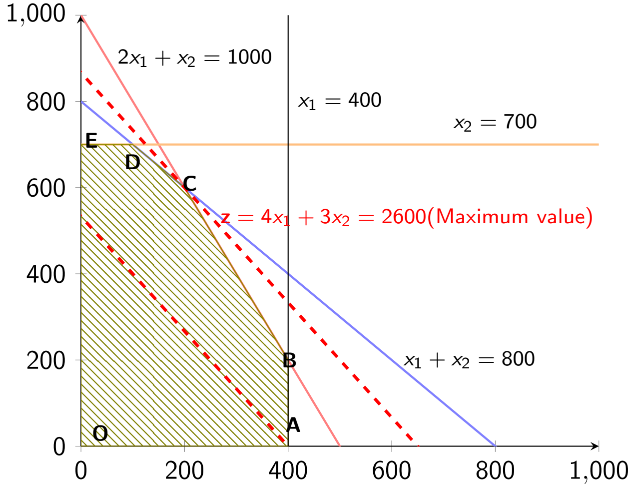
SOLUTION:

**Step.1**: The appropriate model formulation of the given LPP.

**Step.2**: Now plot these points in the graph and find the feasible region. As each point has the coordinates 0f the type $(x_1, x_2)$; any point satisfying the condition $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant.

Now the inequality are graphed taking the equations, $2x_1 + x_2 \leq 1000$ graphed as $2x_1 + x_2 = 1000$, and also $x_1 + x_2 \leq 800$ as $x_1 + x_2 = 800$

Further the constraints are are plotted on the graph which represents the area between the lines $x_1 = 400$ and $x_2 = 700$




Now all the constraints are graphed. The area bounded by all these constraints are called feasible region, as shown by the shaded region.

**Step.3** The optimum value of the objective function occur at one of the extreme points of the feasible region.

That is, $O = (0, 0), A = (400, 0), B = (400, 200), C = (200, 600), D = (100, 700)$ and $E = (0, 700)$

**Step.4** Compute the $z$ values at the extreme points

| Extreme points | $(x_1, x_2)$ | $z = 4x_1 + 3x_2$ |
|---|---|---|
| $O$ | $(0, 0)$ | 0 |
| $A$ | $(400, 0)$ | 1600 |
| $B$ | $(400, 200)$ | 2200 |
| $C$ | $(200, 600)$ | 2600 |
| $D$ | $(100, 700)$ | 2500 |
| $E$ | $(0, 700)$ | 2100 |

**Step.5** The Optimum solution is that extreme point for which the objective function has least value. Thus Maximum $z = 4x_1 + 3x_2$ is 2600.

# Simplex Method

A standard method of maximizing a linear function of several variables under several constraints on other linear functions.
Simplex method is an approach to solving linear programming models by hand using slack variables, tableau's, and pivot variables as a means to finding the optimal solution of an optimization problem. Simplex tableau is used to perform row operations on the linear programming model as well as for checking optimality.

## Definition (Basic Solution)

Given a system of $m$ simultaneous linear equations in $n$ unknowns ($m < n$)

$$\mathbf{Ax} = b, \quad \mathbf{x}^T \in \mathbb{R}^n$$

where $\mathbf{A}$ be an $m \times n$ Matrix of rank $m$. Let $\mathbf{B}$ be an $m \times n$ sub-matrix, formed by $m$ linearly independent column of $\mathbf{A}$. Then a solution obtained by setting $n - m$ variables not associated with the column of $\mathbf{B}$, equals to zero, and solving the resulting system, is called **Basic solution** to the given system of equations

## Definition (Feasible Solution)

Any solution to a general LPP which is also satisfies the non-negative restrictions of the problem, is called a feasible solution to the General LPP.

## Definition (Degenerate solution)

A basic solution to the system is called degenerate if one or more of the basic variable vanish

## Definition (Basic feasible solution)

A feasible solution to the LPP, which is also a basic solution to the problem is called basic feasible solution to the LPP

## Definition (Improved feasible solution)

Let $\mathbf{X}_B$ and $\hat{\mathbf{X}}_B$ be two feasible solution to the LPP. Then $\hat{\mathbf{X}}_B$ is said to be the improved feasible solution, as compared to $\mathbf{X}_B$, if

$$\hat{\mathbf{C}}_B\hat{\mathbf{X}}_B \geq \mathbf{C}_B\mathbf{X}_B$$

where $\mathbf{C}_B$ is constituted of cost component corresponding $\hat{\mathbf{X}}_B$

## Definition (Optimum basic feasible solution)

A basic feasible solution $\mathbf{X}_B$ to the LPP.:

$$\text{Maximize } z = c\mathbf{x} \text{ subject to certain constraint } \mathbf{A}\mathbf{x} = b, \mathbf{x} \geq 0$$

is called the optimum basic feasible solution if, $z_0 = \mathbf{C}_B\mathbf{X}_B \geq z^*$, where $z^*$ is the value of objective function for any feasible solution.

# Fundamental properties of the solution

**Theorem (Reduction of a feasible solution to a Basic feasible solution)**

*If an LPP has a feasible solution, then it also has basic feasible solution.*

**Proof.**

Let the L.P.P be to determine $\mathbf{x}$ so as to

$$\text{Maximize} \quad \mathbf{z=c\ x}, \mathbf{c}, \mathbf{x}^T \in \mathbb{R}^n$$

Subject to constraints:

$$\mathbf{A\ x} = \mathbf{b}, \ \mathbf{x} \geq 0$$

where $A$ is an $m \times n$ real matrix and $\mathbf{b}, \mathbf{c}$ are $m \times 1$ and $1 \times n$ real matrices respectively. Let $\rho(A) = m$.

- Existence of feasible solution $\Longleftarrow \rho(A, b) = \rho(A)$ and $m < n$.
- $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a feasible solution with $x_j \geq 0$ for all $j$.
- $\mathbf{x}$ has $p$ positive components and remaining $n - p$ components are zero.

□

Let us so relabel our components that the positive components are the first components and assume that the columns of $A$ have been relabelled accordingly. Then

$$a_1x_1 + a_2x_2 + \cdots + a_px_p = b$$

Where $a_1, a_2, \ldots a_p$ are the first $p$ columns of $A$.

Two cases now do arise:

(i) The vectors $a_1, a_2, \ldots, a_p$ forms a linearly independent set. Then $p \leq m$.
If $p = m$, the given solution is a non-degenerate basic feasible solution, $x_1, x_2, \ldots, x_p$ as the basic variables.
If $p < m$ then the set $\{a_1, a_2, a_p\}$ can be extended to $\{a_1, a_2, \ldots, a_p, a_{p+1}, \ldots, a_n\}$ form a basis for the columns of $A$.
Then we, have

$$a_1x_1 + a_2x_3 + \cdots + a_mx_m = b$$

Where $x_j = 0$ for $j = p+1, p+2, \ldots, m$
Thus we have, in the case, a degenerate basic feasible solution with $m - p$ of the basics variables zero.

## Proof cts....

(ii) The set $\{a_1, a_2, \ldots, a_p\}$ is linearly dependent. Obviously $p > m$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be a set of constants (not all zero) such that

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_p a_p = 0$$

Suppose that for any index $r, \alpha_r \neq 0$. Then $a_r = -\sum_{j \neq r}^{p} \frac{\alpha_j}{\alpha_r} a_j$

$$\sum_{j \neq r}^{p} a_j x_j + \left( -\sum_{j \neq r}^{p} \frac{\alpha_j}{\alpha_r} a_j \right) x_r = b$$

Or

$$\sum_{j \neq r}^{p} \left( x_j - x_r \frac{\alpha_j}{\alpha_r} \right) a_j = b$$

Thus we have a solution with not more than $p - 1$ non-zero components. $\square$

## Proof cts....

(ii) [cts...] To ensure that these are positive, we shall choose $a_r$ in such a way that

$$x_j - x_r \frac{\alpha_j}{\alpha_r} \geq 0 \text{ for all } j \neq r$$

This requires that either $\alpha_j = 0$ or

$$\frac{x_j}{\alpha_j} \geq \frac{x_r}{\alpha_r}, \text{ if } \alpha_j > 0 \text{ and } \frac{x_j}{\alpha_j} \leq \frac{x_r}{\alpha_r}, \text{ if } \alpha_j < 0$$

Thus, if we select $a_r$ such that

$$\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j}, \alpha_j > 0 \right\}$$

Then for each of the $p-1$ variables $x_j - x_r \frac{\alpha_j}{\alpha_r}$ is non negative, and so we have a feasible solution with not more than $p-1$ non zero components. □

# Proof continue ...

1. Consider now this new feasible solution with not more than $p-1$ non-zero components.
2. If the corresponding set of $p-1$ columns of $A$ is linearly independent. Case (i) applies and we have arrived at a basic feasible solution.
3. If this set is again linearly dependent, we may repeat the process to arrive at a feasible solution with not more than $p-2$ non zero components.
4. The argument can be repeated. Ultimately, we get a feasible solution with associated set of column vectors of $A$ linearly independent.
5. The discussion of case (i) then applies and we do get a basic feasible solution.

This completes the proof.

## Theorem (Replacement of a basic vector)

*Let an LPP have a basic feasible solution. If we drop one of the basic vectors and introduce a non-basic vector in the basic set, then the new solution obtained is also a basic feasible solution.*

## Theorem (Improved Basic feasible solution)

*Let $\mathbf{X}_B$ be a feasible solution to the LPP:*

$$\textit{Maximize } z = c\mathbf{x} \;\; \textit{subject to constraint} \;\; \mathbf{A}\mathbf{x} = b, \mathbf{x} \geq 0$$

*Let $\hat{\mathbf{X}}_B$ be another feasible solution obtained by admiting a non-basis column vector $a_j$ in the basis, for which the net evaluation $z_j - c_j$ is negative. Then $\hat{\mathbf{X}}_B$ is an improved basic feasible solution to the problem, that is*

$$\hat{\mathbf{C}}_B \hat{\mathbf{X}}_B > \mathbf{C}_B \mathbf{X}_B$$

## Theorem (Unbounded solution)

*Let there exist a basic feasible solution to a given LPP. If for at-least one $j$, for which $y_{ij} \leq 0 (i = 1, 2, \cdots, m)$, and $z_j - c_j$ is negative, then there does not exist any optimum solution to the LPP.*

## Theorem (Condition for optimality)

*A sufficient condition for a basic feasible solution to an LPP to be an optimum(maximum) is that $z_j - c_j \geq 0$ for all $j$ for which the column vector $a_j \in A$ is not in the basis $B$*

## Corollary

*A necessary and sufficient condition for which a basic feasible solution ta an LPP to be an optimum(maximum) $z_j - c_j \geq 0$ for all $j$ for which $a_j \notin B$*

## Theorem

*any convex combination of $k$ different optimal solutions to an LPP is again an optimum solution to the problem*

# Computational procedure

The two fundamental condition on which the simplex method is based are

- **Condition of feasibility:** It consumes that if the initial solution is basic feasible the only feasible solution occur during computing.
- **Condition for Optimality:** It guarantees that only better solution will be encountered

The computation of simplex method will requires the construction of simplex table. The initial simplex table is constructed by writing out the coefficients and the constraints of LPP in a systematic tabular manner

| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $y_1$ | $y_2$ | $\cdots$ | $y_n$ |
|---|---|---|---|---|---|---|
| $\mathbf{C}_{B1}$ | $\mathbf{y}_{B1}$ | $\mathbf{x}_{B1}$ | $y_{11}$ | $y_{12}$ | $\cdots$ | $y_{1n}$ |
| $\mathbf{C}_{B2}$ | $\mathbf{y}_{B2}$ | $\mathbf{x}_{B2}$ | $y_{21}$ | $y_{22}$ | $\cdots$ | $y_{2n}$ |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| $\mathbf{C}_{Bm}$ | $\mathbf{y}_{Bm}$ | $\mathbf{x}_{Bm}$ | $y_{m1}$ | $y_{m2}$ | $\cdots$ | $y_{mn}$ |
| | | $z_0$ | $z_1 - c_1$ | $z_2 - c_2$ | $\cdots$ | $z_n - c_n$ |

The optimal of the general LPP is obtained in the following steps:

**Step.1.** Select an initial basic feasible solution to initiate the algorithm.

**Step.2.** Check the objective function to see whether there is some non basic variables that would improve the objective function if brought in the basis. If such available exists go to next step otherwise stop.

**Step.3.** Determine how large the variable found in the **step.2** can be made until one of the basic variable in the current solution become zero. Eliminate the latter variable and let the next trial solution contain the newly found variable instead

**Step.4.** Check for optimality the current solution.

**Step.5.** Continue the iterations until either an optimal solution is attained or there is an indication than an unbounded solution exists.

# Simplex algorithm

The steps for computation of the optimum solution as follows:

**Step.1.** Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we can change it to maximizing problem as

$$\text{Minimum}(z) = -\text{Maximum}(z)$$

**Step.2.** Check weather all $b_i$ are non-negative, if any of the $b_i$ are negative then multiply the corresponding in-equation of the constraints by $(-1)$ so that all the $b_i$ are non-negative

**Step.3.** Convert all the inequality of the constraints into equations slack and/surplus variables in the constraints. Put the cost of the variable equals zero.

**Step.4.** Obtain the initial basic feasible solution of the form $\mathbf{X}_B = \mathbf{B}^{-1}b$ and put it in the first column of the simplex table.

**Step.5.** Compute the net evaluations $z_j - c_j (j = 1, 2, \cdots, n)$ by using the relation $z_j - c_j = \mathbf{C}_B y_j - c_j$
Examine the sign of $z_j - c_j$

- If all $z_j - c_j \geq 0$ the the initial basic feasible solution $\mathbf{x}_B$ is the optimum basic feasible solution.
- If at-least one $z_j - c_j < 0$, proceed on to next step

**Step.6.** If there are more than one negative $z_j - c_j$, then choose the most negative of them.Let it be $z_r - c_r$ for some $j = r$

- If all the $\mathbf{y}_{ir} \leq 0 (1 = 1, 2, \cdots, m)$, then there is an unbounded solution to the given problem.
- If at-least one $\mathbf{y}_{ir} > 0$, then the corresponding $\mathbf{y}_r$ enters the basis $\mathbf{y}_B$

**Step.7.** Compute the ratios $\left\{ \frac{\mathbf{x}_{Bi}}{\mathbf{y}_{ir}}, \mathbf{y}_{ir} > 0, i = 1, 2, \cdots, m \right\}$ and choose the minimum of them. Let the minimum of these ratio be $\frac{\mathbf{x}_{Bk}}{\mathbf{y}_{kr}}$. Then $\mathbf{y}_k$ will level the basis $\mathbf{y}_B$. The common element $\mathbf{y}_{kr}$, which is in the $k^{th}$ row and $r^{th}$ column is known as the leading element or the pivotal element of the table.

**Step.8.** Convert the leading element to unity by dividing its row by the leading element itself and all other elements in its column to zeroes by making use of the relation:

$$\begin{aligned} \hat{y}_{ij} &= y_{ij} - \frac{y_{kj}}{y_{kr}} y_{ir} \\ \hat{y}_{kj} &= \frac{y_{kj}}{y_{kr}} \end{aligned}$$

**Step.9.** Got to **Step.5** and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

# Example

$$\text{Maximize} z = 4x_1 + 10x_2$$

subject to the constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 50 \\ 2x_1 + 5x_2 &\leq 100 \\ 2x_1 + 3x_2 &\leq 90 \\ x_1 \geq 0, \text{ and } x_2 &\geq 0 \end{aligned}$$

SOLUTION:

By introducing slack variables $s_1 \geq 0, s_2 \geq 0$ and $s_3 \geq 0$ respectively, the constraints of LPP are converted to system of equations.

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$$

The modified function is to maximize

$$z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution is now represented as in the simplex table

The modified function is to maximize

$$z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution is now represented as in the simplex table

| | | | | | |
|---|---|---|---|---|---|
| $c_j$ | 4 | 10 | 0 | 0 | 0 |

The modified function is to maximize

$$z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution is now represented as in the simplex table

| | | $c_j$ | 4 | 10 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1[y_1]$ | $x_2[y_2]$ | $s_1[y_3]$ | $s_2[y_4]$ | $s_3[y_5]$ |

The modified function is to maximize

$$z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution is now represented as in the simplex table

| | | $c_j$ | 4 | 10 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1[y_1]$ | $x_2[y_2]$ | $s_1[y_3]$ | $s_2[y_4]$ | $s_3[y_5]$ |
| 0 | $s_1$ | 50 | 2 | 1 | 1 | 0 | 0 |
| 0 | $s_2$ | 100 | 2 | 5 | 0 | 1 | 0 |
| 0 | $s_3$ | 90 | 2 | 3 | 0 | 0 | 1 |

The modified function is to maximize

$$z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution is now represented as in the simplex table

| | | $c_j$ | 4 | 10 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1[y_1]$ | $x_2[y_2]$ | $s_1[y_3]$ | $s_2[y_4]$ | $s_3[y_5]$ |
| 0 | $s_1$ | 50 | 2 | 1 | 1 | 0 | 0 |
| 0 | $s_2$ | 100 | 2 | 5 | 0 | 1 | 0 |
| 0 | $s_3$ | 90 | 2 | 3 | 0 | 0 | 1 |
| | $z_j$ | | 0 | 0 | 0 | 0 | 0 |

The modified function is to maximize

$$z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution is now represented as in the simplex table

| | | $c_j$ | 4 | 10 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1[y_1]$ | $x_2[y_2]$ | $s_1[y_3]$ | $s_2[y_4]$ | $s_3[y_5]$ |
| 0 | $s_1$ | 50 | 2 | 1 | 1 | 0 | 0 |
| 0 | $s_2$ | 100 | 2 | 5 | 0 | 1 | 0 |
| 0 | $s_3$ | 90 | 2 | 3 | 0 | 0 | 1 |
| | $z_j$ | | 0 | 0 | 0 | 0 | 0 |
| | $z_j - c_j$ | | -4 | -10 | 0 | 0 | 0 |

The modified function is to maximize

$$z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution is now represented as in the simplex table

| | | $c_j$ | 4 | 10 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1[y_1]$ | $x_2[y_2]$ | $s_1[y_3]$ | $s_2[y_4]$ | $s_3[y_5]$ | $\theta$ |
| 0 | $s_1$ | 50 | 2 | 1 | 1 | 0 | 0 | $50/1$ |
| $\leftarrow \boxed{0}$ | $\boxed{s_2} \leftarrow$ | 100 | 2 | $\boxed{5}$ | 0 | 1 | 0 | $\boxed{100/5} \leftarrow$ |
| 0 | $s_3$ | 90 | 2 | 3 | 0 | 0 | 1 | $90/3$ |
| | $z_j$ | | 0 | 0 | 0 | 0 | 0 | |
| | $z_j - c_j$ | | -4 | $\boxed{-10} \uparrow$ | 0 | 0 | 0 | |

where $c_j =$ coefficient of variables in $z$, $z_j = C_B \times y_j$,
$\theta = \left\{ \frac{\mathbf{x}_{Bi}}{\mathbf{y}_{ir}}, \mathbf{y}_{ir} > 0, (i = 1, 2, 3) \right\}$ and choose the minimum of them.

Convert the leading element into unity and all other elements in the column to zero,

$$\hat{y}_{ij} = y_{2j} - \frac{y_{2j}}{y_{22}} y_{2r}$$
$$\hat{y}_{kj} = \frac{y_{kj}}{y_{kr}}$$

$$\hat{y}_{21} = \frac{y_{21}}{y_{22}} = 2/5; y_{20} = \frac{y_{20}}{y_{22}} = 100/5 = 20; \text{and so on to the same row}$$
$$\hat{y}_{10} = y_{10} - \frac{y_{20}}{y_{22}} y_{12} = 50 - \frac{100}{5} \times 1 = 30$$
$$\hat{y}_{30} = y_{30} - \frac{y_{20}}{y_{22}} y_{32} = 90 - \frac{100}{5} \times 3 = 30$$
$$\hat{y}_{31} = y_{31} - \frac{y_{21}}{y_{22}} y_{31} = 2 - 2/5 \times 3 = 4/5$$
$$\hat{y}_{11} = y - 11 - \frac{y_{21}}{y_{22}} y_{12} = 2 - 2/5 \times 1 = 8/5$$
$$\hat{y}14 = y_{14} - \frac{y_{21}}{y_{22}} y_{12} = 0 - 1/5 \times 1 = -1/5$$

and so on.

Using computation the iterated simplex table is

| | | $c_j$ | 4 | 10 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ |
| 0 | $s_1$ | 30 | 8/5 | 0 | 1 | -1/5 | 0 |
| $\boxed{10}$ | $\boxed{x_2}$ | 20 | 2/5 | $\boxed{1}$ | 0 | 1/5 | 0 |
| 0 | $s_3$ | 30 | 4/5 | 0 | 0 | -3/5 | 1 |
| | $z_j$ | | 4 | 10 | 0 | 2 | 0 |
| | $z_j - c_j$ | $z(= 200)$ | 0 | 0 | 0 | 2 | 0 |

In the above simplex table yields a new basic feasible solution with increased value of $z$. Moreover no futher improvement in the value of $z$ is possible since all $z_j - c_j \geq 0$.

Hence the maximum feasible solution is $x_1 = 0, x_2 = 20$ with Maximum $z = 200$

# Example

$$\text{Maximize} \quad z = 107x_1 + x_2 + 2x_3$$

subject to the constraints:

$$\begin{aligned} 14x_1 + x_2 - 6x_3 + 3x_4 &= 7 \\ 16x_1 + x_2 - 6x_3 &\leq 5 \\ 3x_1 - x_2 - x_3 &\leq 0 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

SOLUTION:

By introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$ respectively, the set of constraints of given LPP are converted as

$$\begin{bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}$$

The initial basic feasible solution is now represented as in the simplex table.

| | | $c_j$ | 107 | 1 | 2 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $\theta$ |
| 0 | $x_4$ | 7/3 | 14/3 | 1/3 | -2 | 1 | 0 | 0 | 1/2 |
| 0 | $s_1$ | 5 | 16 | 1 | -6 | 0 | 0 | 0 | 5/16 |
| $\boxed{0} \leftarrow$ | $\boxed{s_2} \leftarrow$ | 0 | $\boxed{3}$ | -1 | -1 | 0 | 0 | 1 | $\boxed{0} \leftarrow$ |
| | $z_j$ | | 0 | 0 | 0 | 0 | 0 | | |
| | $z_j - c_j$ | | $\boxed{-107} \uparrow$ | -1 | -2 | 0 | 0 | 0 | |

The Final iteration. **Unbounded solution**

| | | $c_j$ | 107 | 1 | 2 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ |
| 0 | $x_4$ | 7/3 | 0 | 17/9 | -4/9 | 1 | 0 | -14/9 |
| 0 | $s_1$ | 5 | 0 | 19/3 | -2/3 | 0 | 1 | -16/3 |
| $\boxed{107}$ | $\boxed{x_1}$ | 0 | 1 | -1/3 | -1/3 | 0 | 0 | 1/3 |
| | $z_j$ | | 107 | -107/3 | -107/3 | 0 | 0 | 107/3 |
| | $z_j - c_j$ | $z = 0$ | 0 | -110/3 | $\boxed{-113/3}$ $\uparrow$ | 0 | 0 | 107/3 |

We can see that $z_2 - c_2$ and $z_3 - c_3$ both are negative, but the most negative one is $z_3 - c_3$. Now $x_3$ will not enter to the basis $\mathbf{y}_B$ since $x_{i3}$ are non-positive. Thus we can say that the solution is unbounded.

# Use of Artificial Variable

- If original constraint is an equation or is of the form type ($\geq$) we may no longer have a ready starting basic feasible solution.
- In order to obtain an initial basic feasible solution, we first put the given L.P.P into its standard form and then a non-negative variable is added to the left side of each equation have the much needed starting basic variables.
- The added variable is called an artificial variable and plays the same role as a slack variable in providing initial basic feasible solution.
- Artificial variables have no physical meaning from the point of the original problem, the method will be valid only we are able to force these variables to be out or at zero level when the optimum solution attained.

# TWO-PHASE METHOD

In the first phase of this method. the sum of the artificial variables is minimize. subject to the given constraints (known as auxiliary L.P.P.) to get a basic feasible solution to the original L.P.P. Second phase then optimizes the original objective function starting With the basic feasible solution obtained at the end of Phase 1.

The iterative procedure of the algorithm may he summarise as below

**step 1.** Write the given L.P.P. into. its standard form and check whether there exists starting basic feasible solution.

1. If there is a ready starting basic feasible solution, go to in Phase 2 .
2. If there does not exist a ready starting basic feasible solution., go on to the next step

**PHASE I**

**Step 2.** Add the artificial variable to the left side of the each equation that lacks the needed starting basic variables. Construct an auxiliary objective function aimed at minimizing the sum of all artificial variables. Thus, the new objective is to
Minimize $\quad z = A_1 + A_2 + \ldots + A_n$
Maximize $\quad z^* = -A_1 - A_2 - \ldots - A_n$
where $A, (i = 1.2, \ldots, m)$ are the non-negative artificial variables.

**Step 3**. Apply simplex algorithm to the specially constructed L.P.P. The following three cases may arise at the least interaction:

1. $\max z^* \leq 0$ and at least one artificial variable is present in the basis with positive value. In such a case, the original L. P.P. does not possess any feasible solution.
2. $\max z^* = 0$ and at least one artificial variable is present in the basis at zero value. In such a case, the original L.P.P. possess the feasible solution. In order to get basis feasible solution we may proceed directly to Phase 2 of else eliminate the artificial variable and then proceed to Phase 2.
3. $\max z^* = 0$ and no artificial variable is present in the basis. In such a case, a basic feasible solution to the original L.P.P. has been found. Go to Phage: 2 .

**PHASE 2**

**Step 4.** Consider the optimum BASIC feasible solution of Phase.1 as a starting basic feasible solution for the original L.P.P. Assign actual coefficients to the variables in the objective function and a value zero to the artificial variables that appear at zero value in the final simplex table of Phase 1.

Apply usual simplex method to the modified simplex table to get optimum solution.

# Example

Maximize $z = 5x_1 - 4x_2 + 3x_3$ subject to the constraints:

$$\begin{aligned} 2x_1 + x_2 - 6x_3 &= 20 \\ 6x_1 + 5x_2 + 10x_3 &\leq 76 \\ 8x_1 - 3x_2 + 6x_3 &\leq 50 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

SOLUTION:

We convert the LPP into standard form by using slack variables $s_1 \geq 0$ and $s_2 \geq 0$ and artificial variable $A_1$

Maximize $z = 5x_1 - 4x_2 + 3x_3$ subject to the constraints:

$$\begin{aligned} 2x_1 + x_2 - 6x_3 + A_1 &= 20 \\ 6x_1 + 5x_2 + 10x_3 + s_1 &= 76 \\ 8x_1 - 3x_2 + 6x_3 + s_2 &= 50 \\ x_1, x_2, x_3, s_1, s_2, A_1 &\geq 0 \end{aligned}$$

An initial basic feasible solution $x_1 = x_2 = x_3 = 0,\ A_1 = 20, s_1 = 76, s_2 = 50$

Phase: I

Auxiliary LPP

$$\text{Maximize} \quad z = -A_1 \quad \text{subject to the constraints:}$$

$$\begin{aligned} 2x_1 + x_2 - 6x_3 + A_1 &= 20 \\ 6x_1 + 5x_2 + 10x_3 + s_1 &= 76 \\ 8x_1 - 3x_2 + 6x_3 + s_2 &= 50 \end{aligned}$$

writing in the Matrix form $AX = B$

$$\begin{bmatrix} 2 & 1 & -6 & 0 & 0 & 1 \\ 6 & 5 & 10 & 1 & 0 & 0 \\ 8 & -3 & 6 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ A_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 76 \\ 50 \end{bmatrix}$$

| | | $c_j$ | 0 | 0 | 0 | 0 | 0 | -1 | min ratio |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_3$ | $s_1$ | $s_2$ | $A_1$ | $X_B/x_i, x_i > 0$ |
| -1 | $A_1$ | 20 | 2 | 1 | -6 | 0 | 0 | 1 | 10 |
| 0 | $s_1$ | 76 | 6 | 5 | 10 | 1 | 0 | 0 | 76/6 |
| $\mathbf{0} \leftarrow$ | $\mathbf{s_2} \leftarrow$ | 50 | $\mathbf{8}$ | -3 | 6 | 0 | 1 | 0 | $\mathbf{50/8} \leftarrow$ |
| | $z_j - c_j$ | | $\mathbf{-2} \uparrow$ | -1 | 6 | 0 | 0 | 0 | |

$z_j - c_j = C_B X_j - C_j$

| | | $c_j$ | 0 | 0 | 0 | 0 | 0 | -1 | min ratio |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_3$ | $s_1$ | $s_2$ | $A_1$ | $X_B/x_i, x_i > 0$ |
| -1 | $\mathbf{A_1} \leftarrow$ | 15/2 | 0 | **7/4** | -15/4 | 0 | -1/4 | 1 | **4.28** $\leftarrow$ |
| 0 | $s_1$ | 77/2 | 0 | 22/4 | 11/2 | 1 | -3/4 | 0 | 5.31 |
| 0 | $x_1$ | 25/4 | 1 | -3/8 | 3/4 | 0 | 1/8 | 0 | Negative |
| | $z_j - c_j$ | | 0 | $-\mathbf{7/4} \uparrow$ | 15/4 | 0 | 1/4 | 0 | |

| | | $c_j$ | 0 | 0 | 0 | 0 | 0 | -1 | min ratio |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_3$ | $s_1$ | $s_2$ | $A_1$ | $X_B/x_i, x_i > 0$ |
| 0 | $x_2$ | 30/7 | 0 | 1 | -30/7 | 0 | -1/7 | 4/7 | |
| 0 | $s_1$ | 52/7 | 0 | 0 | 256/7 | 1 | 2/7 | -29/7 | |
| 0 | $x_1$ | 55/7 | 1 | 0 | -6/7 | 0 | 1/14 | 3/14 | |
| | $z_j - c_j$ | | 0 | 0 | 0 | 0 | 0 | 1 | |

Thus we get the optimal value,

$$\max z^* = 0$$

$\max z^* = 0$ and no artificial variable is present in the basis. In such a case, a basic feasible solution to the original L.P.P. has been found. Go to Phage: 2.

**Phase:II**

Consider the final simplex table of Phase: I, considered the actual cost associated with the original variables. Delete the artificial variable column $A_1$ from the table as it is been eliminated from phase:I

| | | $c_j$ | 5 | -4 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_3$ | $s_1$ | $s_2$ |
| -4 | $x_2$ | 30/7 | 0 | 1 | -30/7 | 0 | -1/7 |
| 0 | $s_1$ | 52/7 | 0 | 0 | 256/7 | 1 | 2/7 |
| 5 | $x_1$ | 55/7 | 1 | 0 | -6/7 | 0 | 1/14 |
| | $z_j - c_j$ | | 0 | 0 | 69/7 | 0 | 13/14 |

$z_j - c_j = C_B X_j - C_j$
Since all $z_j - c_j \geq 0$, an optimal feasible solution has been reached. Hence, an optimum feasible solution to the given LPP is

$$x_1 = 55/7 \quad x_2 = 30/7 \quad x_3 = 0$$

$$\text{Max } z = (-4 \times 30/7 + 5 \times 55/7 + 0) = 155/7$$

# Big M:- Method

- If an LP has any $\geq$ or $=$ constraints, a starting BFS may not be readily avialable.
- In order to use the simplex method, a BFS is needed. To remedy the predicament, artificial variables are created.

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**Solution Strategy**

1. In the optimal solution, all artificial variables must be set equal to zero.
2. To accomplish this, in a Min LP, a term $MA_i$ is added to the objective function for each artificial variable $A_i$. For a Max LP, the term $-MA_i$ is added to the objective function for each $A_i$.
3. $M$ represents some very large number.

# Big M:- Method

1. Modify the constraints so that the RHS of each constraint is non-negative. Identify each constraint that is now an $=$ or $\geq$ constraint.
2. Convert each inequality constraint to standard form (add a slack variable for $\leq$ constraints, add an excess variable for $\geq$ constraints.
3. For each $\geq$ constraints or $=$ constraint, add artificial variables. Add sign restriction $A_i \geq 0$.
4. Let $M$ denote a very large positive number. Add (for each artificial variable) $MA_i$ to Min problem objective functions or $-MA_i$ Max problem objective functions.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Remembering $M$ represents a very large number, solve the transformed problem by the simplex.

# Example

$$\text{Maximize} \quad z = 6x_1 + 4x_2 \quad \text{subject to the constraints:}$$

$$2x_1 + x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

**SOLUTION:**

Introducing slack variables $s_1, s_2$, surplus variable $s_3$ and artificial variable $A_1$ in the constraints of LPP,

$$\text{Maximize} \quad z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 \quad \text{subject to the constraints:}$$

$$2x_1 + x_2 + s_1 = 30$$
$$3x_1 + 2x_2 + s_2 = 24$$
$$x_1 + x_2 - s_3 = 3$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

We don't have the initial feasible solution. So we introduce artificial variable $A_1$ in the third constraint. Then the initial feasible solution is $s_1 = 30, s_2 = 24$ and $A_1 = 3$

Now the iterative simplex table

| | | $c_j$ | 6 | 4 | 0 | 0 | 0 | -M |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $A_1$ |
| 0 | $s_1$ | 30 | 2 | 3 | 1 | 0 | 0 | 0 |
| 0 | $s_2$ | 24 | 3 | 2 | 0 | 1 | 0 | 0 |
| -M | $\mathbf{A_1} \leftarrow$ | 3 | $\boxed{1}$ | 1 | 0 | 0 | -1 | 1 |
| | $z_j$ | -3M | -M | -M | 0 | 0 | M | 0 |
| | $z_j - c_j$ | | $\mathbf{-M-6} \uparrow$ | -M-4 | 0 | 0 | M | 0 |

*Initial iteration*: introduce $x_1$ and drop $A_1$

| | | $c_j$ | 6 | 4 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ |
| 0 | $s_1$ | 24 | 0 | 1 | 1 | 0 | 2 |
| 0 | $\mathbf{s_2} \leftarrow$ | 15 | 0 | -1 | 0 | 1 | $\boxed{3}$ |
| 6 | $x_1$ | 3 | 1 | 1 | 0 | 0 | -1 |
| | $z_j$ | 18 | 6 | 6 | 0 | 0 | -6 |
| | $z_j - c_j$ | | 0 | 2 | 0 | 0 | $-\mathbf{6} \uparrow$ |

Introduce $s_3$ and drop $s_2$

Optimal solution:

| | | $c_j$ | 6 | 4 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ |
| 0 | $s_1$ | 14 | 0 | 5/3 | 1 | -2/3 | 0 |
| 0 | $s_3$ | 5 | 0 | -1/3 | 0 | 1/3 | 1 |
| 6 | $x_1$ | 8 | 1 | 2/3 | 0 | 1/3 | 0 |
| | $z_j$ | 48 | 6 | 4 | 0 | 2 | 0 |
| | $z_j - c_j$ | | 0 | 0 | 0 | 2 | 0 |

Since all $z_j - c_j$, an optimal solution has been reached. Thus it is at $x_1 = 8, x_2 = 0,\ z = 48$

# Alternate solution

It is evident from the net evaluation of the optimum table that the net evaluation to the non-feasible variable $x_2$ is 0. This is an indication that the current solution is not unique and alternate solution exists. Thus we bring $x_2$ into the basis instead of $s_1$ or $s_3$, thus

| | | $c_j$ | 6 | 4 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ |
| 4 | $x_2$ | 42/5 | 0 | 1 | 3/5 | -2/5 | 0 |
| 0 | $s_3$ | 39/5 | 0 | 0 | 1/5 | 1/5 | 1 |
| 6 | $x_1$ | 12/5 | 1 | 0 | -2/5 | 3/5 | 0 |
| | $z_j$ | 48 | 6 | 4 | 0 | 2 | 0 |
| | $z_j - c_j$ | | 0 | 0 | 0 | 2 | 0 |

Thus we observe that the optimum value is same. hence the feasible solutions are $x_1 = [8, 0]$ and $x_2 = [12/5, 42/5]$

# Example

Minimize $z = 5x_1 - 6x_2 - 7x_3$ subject to the constraints:

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &\geq 15 \\ 5x_1 - 6x_2 + 10x_3 &\geq 0 \\ x_1 + x_2 + x_3 &= 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

SOLUTION:

**Constraint 1**: It has a sign "$\geq$" (greater than or equal) so the surplus variable $s_1$ will be subtracted and the artificial variable $A_1$ will be added.
**Constraint 2**: It has a negative or null independent term. All coefficients will be multiplied by -1 and their sign will be changed from "$\geq$" In this way it corresponds to add the slack variable $s_2$.
**Constraint 3**: It has an "$=$" sign (equal) so the artificial variable $A_2$ will be added.

Minimize $z = 5x_1 - 6x_2 - 7x_3 + 0s_1 + 0s_2 + MA_1 + MA_2$ subject to the constraints:

$$x_1 + 5x_2 - 3x_3 - s_1 + A_1 = 15$$
$$-5x_1 + 6x_2 - 10x_3 + s_2 = 0$$
$$x_1 + x_2 + x_3 + A_2 = 5$$
$$x_1, x_2, x_3 \geq 0$$

Initial table

| | | $c_j$ | 5 | -6 | -7 | 0 | 0 | M | M |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ |
| M | $A_1$ | 15 | 1 | 5 | -3 | -1 | 0 | 1 | 0 |
| 0 | $s_2$ | 00 | -5 | $\boxed{6}$ | -10 | 0 | 1 | 0 | 0 |
| M | $A_2$ | 05 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| | $z_j$ | 20M | 2M | 6M | -2M | -M | 0 | 0 | 0 |
| | $z_j - c_j$ | | 2M-5 | 6M+6 | -2M+7 | -M | 0 | 0 | 0 |

In case of minimization problem we take the maximum of $z_j - c_j$
Thus, enter the variable $x_2$ and the variable $s_2$ leaves the base. The pivot element is 6

| | | $c_j$ | 5 | -6 | -7 | 0 | 0 | M | M |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ |
| M | $A_1$ | 15 | 31/6 | 0 | 16/3 | -1 | -5/6 | 1 | 0 |
| -6 | $x_2$ | 0 | -5/6 | 1 | -5/3 | 0 | 1/6 | 0 | 0 |
| M | $A_2$ | 5 | 11/6 | 0 | $\boxed{8/3}$ | 0 | -1/6 | 0 | 1 |
| | $z_j$ | | 7M+5 | -6 | 8M+10 | -M | -M-1 | M | M |
| | $z_j - c_j$ | | 7M | 0 | (8M+17) | -M | -M-1 | 0 | 0 |

Enter the variable $x_3$ and the variable $A_2$ leaves the base. The pivot element is 8/3

| | | $c_j$ | 5 | -6 | -7 | 0 | 0 | M | M |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ |
| M | $A_1$ | 5 | 3/2 | 0 | 0 | -1 | -1/2 | 1 | -2 |
| -6 | $x_2$ | 25/8 | 5/16 | 1 | 0 | 0 | 1/16 | 0 | 5/8 |
| -7 | $x_3$ | 15/8 | $\boxed{11/16}$ | 0 | 1 | 0 | -1/16 | 0 | 3/8 |
| | $z_j$ | | $\frac{3}{2}M - \frac{107}{16}$ | 6 | 7 | -M | $\frac{-1}{2}M + \frac{1}{16}$ | M | -2M-$\frac{51}{8}$ |
| | $z_j - c_j$ | | $\frac{3}{2}M - \frac{187}{16}$ | 0 | 0 | -M | $\frac{-1}{2}M + \frac{1}{16}$ | 0 | -3M-$\frac{51}{8}$ |

Enter the variable $x_1$ and the variable $x_3$ leaves the base. The pivot element is 11/16

|  |  | $c_j$ | 5 | -6 | -7 | 0 | 0 | M | M |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{C}_B$ | $\mathbf{y}_B$ | $\mathbf{x}_B$ | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $A_1$ | $A_2$ |
| M | $A_1$ | 10/11 | 0 | 0 | -24/11 | -1 | -4/11 | 1 | -31/11 |
| -6 | $x_2$ | 25/11 | 0 | 1 | -5/11 | 0 | 1/11 | 0 | 5/11 |
| 5 | $x_1$ | 30/11 | 1 | 0 | 16/11 | 0 | -1/11 | 0 | 6/11 |
|  | $z_j$ | $\frac{10}{11}M$ | 5 | -6 | $\frac{-24}{11}M+10$ | -M | $\frac{-4}{11}M-1$ | M | $\frac{-31}{11}M$ |
|  | $z_j - c_j$ |  | 0 | 0 | $\frac{-24}{11}M+17$ | -M | $\frac{-4}{11}M-1$ | 0 | $\frac{-42}{11}M$ |

The iterations have been completed and there are artificial variables in the base with values strictly greater than 0, so the problem has no solution (infeasible).

# Duality in LPP

## General Primal-Dual pair

### Definition (1)

$$\text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

subject to the constraints:

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i; &\qquad i = 1, 2, \cdots, m \\ x_j \geq 0; &\qquad j = 1, 2, \cdots, n \end{aligned}$$

Dual problem

$$\text{Minimize } z^* = b_1w_1 + b_2w_2 + \cdots + b_mw_m$$

$$\begin{aligned} a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m \geq c_i; &\qquad j = 1, 2, \cdots, n \\ w_i(i = 1, 2, \cdots, m), \text{unrestricted} \end{aligned}$$

Note: $x_j$'s are $n$ primal variables and $w_i$'s are $m$ dual variables.

## Definition (2)

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

subject to the constraints:

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i; \qquad & i = 1, 2, \cdots, m \\ x_j \geq 0; \qquad & j = 1, 2, \cdots, n \end{aligned}$$

Dual problem

$$\text{Maximize } z^* = b_1w_1 + b_2w_2 + \cdots + b_nw_n$$

$$\begin{aligned} a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m \leq c_i; \qquad & j = 1, 2, \cdots, n \\ w_i(i = 1, 2, \cdots, m), \text{unrestricted} \end{aligned}$$

## Definition (matrix form)

**Primal Form**: Find $x^T \in \mathbb{R}^n$ so as to

$$\text{Maximize } z = cx,\ \ c \in \mathbb{R}^n \text{ subject to } \mathbf{A}x = b,\ \text{and } x \geq 0,\ \ b^T \in \mathbb{R}^m$$

where $\mathbf{A}$ is an $m \times n$ real matrix.
**Dual Form**: Find $w^T \in \mathbb{R}^m$ so as to

$$\text{Minimize } z^* = b^T w, b \in \mathbb{R}^m \text{ subject to } \mathbf{A}^T w \geq c^T,\ \ c \in \mathbb{R}^n$$

where $A^T$ transpose of an $n \times m$ real matrix $\mathbf{A}$ and $w$ is unrestricted in sign

## Definition

**Primal Form**: Find $x^T \in \mathbb{R}^n$ so as to

$$\text{Minimize } z = cx,\ \ c \in \mathbb{R}^n \text{ subject to } \mathbf{A}x = b,\ \text{and } x \geq 0,\ \ b^T \in \mathbb{R}^m$$

where $\mathbf{A}$ is an $m \times n$ real matrix.
**Dual Form**: Find $w^T \in \mathbb{R}^m$ so as to

$$\text{Maximize } z^* = b^T w, b \in \mathbb{R}^m \text{ subject to } \mathbf{A}^T w \leq c^T,\ \ c \in \mathbb{R}^n$$

where $A^T$ transpose of an $n \times m$ real matrix $\mathbf{A}$ and $w$ is unrestricted in sign

| Primal Problem | Dual Problem | | |
|---|---|---|---|
| Objective | Objective | Constraint Type | Variables sign |
| Maximization | Minimization | $\geq$ | Unrestricted |
| Minimization | Maximization | $\leq$ | Unrestricted |

Table: Rules for constructing the dual problem.

## Theorem (Dual of Dual)

*The Dual of the Dual is primal*

## Theorem (weak duality theorem)

*Let $x_0$ be a feasible solution to the primal problem*

$$Maximize\ f(x) = cx \text{ subject to } \mathbf{A}x \leq b, \quad x \geq 0$$

*where $x^T$ and $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $\mathbf{A}$ is an $m \times n$ real matrix. if $w_0$ be a feasible solution to the dual of primal, namely*

$$Minimize\ g(x) = b^T w \text{ subject to } \mathbf{A}^T w \geq c^T, \quad w \geq 0$$

*where $w^T \in \mathbb{R}^m$, then $cx_0 \leq b^T w_0$*

## Theorem (Existence theorem)

*If either the primal or the dual problem has an unbounded objective function value, then the other problem has no feasible solution.*

## Theorem (Fundamental theorem of Duality)

*If the primal or dual has a finite optimum solution, then the other problem also possesses a finite optimum solution and the optimum values of the objective functions of two problem are equal.*

## Theorem (Basic duality theorem)

*Let a primal problem be*

$$\text{Maximize } f(x) = cx \text{ subject to } \mathbf{A}x \leq b, \quad x \geq 0$$

*and the associated dual be*

$$\text{Minimize } g(w) = b^T w, \text{ subject to } \mathbf{A}^T w \geq c^T, w \geq 0, \quad w^T, b^T \in \mathbb{R}^m$$

*If $x_0(w_0)$ be the optimum solution to the primal (dual),then there exist a feasible solution $w_0(x_0)$ to the dual(primal) such that*

$$cx_0 = b^T w_0$$

# Formulation: Dual problem

Steps involved in the formulation of dual problem are:

**Step.1.** Put the linear programming problem into its standard form. Consider it as primal form.

**Step.2.** Identify the variable to be used in the Dual form. The number of these variable is equal to the number of constraints equation in the primal.

**Step.3.** Write down the objective function of the dual, using the right hand side constraint of the primal constraint.
If the primal problem is of maximization type, the dual will be a minimization problem and vice-versa.

**Step.4.** Making use of the dual variable identified in **Step 2**, write the constraint of the dual problem.

- If primal is a maximization problem, the dual constraints must be all of $' \geq'$ type. If the primal is minimization,he dual constrain must be all of $' \leq'$ type.
- The column coefficients of the primal constraints becomes the row coefficients of the dual constraints.

- The coefficients of the primal objective function becomes the right hand side constant of the dual constraints
- The dual variables are defined to be unrestricted in sign.

**Step.5.** Using **Step.3** and **4**, write down the dual of LPP

---

**Example** Formulate the dual of the following LPP:

$$\text{Maximize } z = 5x_1 + 3x_2$$

subject to the constraints:

$$3x_1 + 5x_2 \leq 15, 5x_1 + 2x_2 \leq 10, x_1 \geq 0, x_2 \geq 0$$

SOLUTION:
**Standard Primal.**Introducing the slack variables $s_1 \geq 0$ and $s_2 \geq 0$ the standard LPP:

$$\text{Maximize } z = 5x_1 + 3x_2 + 0s_3 + 0s_4$$

subject to the constraints:

$$
\begin{aligned}
3x_1 + 5x_2 + s_1 + 0s_2 &= 15, \\
5x_1 + 2x_2 + 0s_1 + s_2 &= 10, \\
x_1, x_2, s_1, s_2 &\geq 0
\end{aligned}
$$

**Dual.** Let $w_1$ and $w_2$ be the dual variables corresponding to the primal constraints. then the dual problem:

$$\text{Minimize } z^* = 15w_1 + 10w_2$$

subject to the constraints:

$$
\begin{aligned}
3w_1 + 5w_2 &\geq 5 \\
5w_1 + 2w_1 &\geq 3 \\
w_1 + 0w_2 &\geq 0 \Rightarrow w_1 \geq 0 \\
0w_1 + w_2 &\geq 0 \Rightarrow w_2 \geq 0
\end{aligned}
$$

$w_1$ and $W_2$ unrestricted

## Example

Write down the dual of LPP:

$$\text{minimize } z = 4x_1 + 6x_2 + 18x_3$$

subject to the constraints:

$$x_1 + 3x_2 \geq 3, x_2 + 2x_3 \geq 5, \text{ and } x_j \geq 0$$

SOLUTION:
**Standard Primal.**Introducing the slack variables $s_1 \geq 0$ and $s_2 \geq 0$ the standard LPP:

$$\text{minimize } z = 4x_1 + 6x_2 + 18x_3 + 0s_1 + 0s_2$$

subject to the constraints:

$$\begin{aligned} x_1 + 3x_2 - s_1 &= 3, \\ 0x_1 + x_2 + 2x_3 - s_2 &= 5, \\ x_j &\geq 0 \end{aligned}$$

**Dual.** Let $w_1$ and $w_2$ be the dual variables corresponding to the primal constraints. then the dual problem:

$$\text{Maximize } z^* = 3w_1 + 5w_2$$

subject to the constraints:

$$\begin{aligned} w_1 + 0w_2 &\leq 4 \\ 3w_1 + w_2 &\leq 6 \\ 0w_1 + 2w_2 &\leq 18 \\ -w_1 + 0w_1 &\leq 0 \\ 0w_1 - w_2 &\leq 0 \\ w_1 \text{ and } w_2 &\text{ unrestricted} \end{aligned}$$

The dual becomes:

$$\text{Maximize } z^* = 3w_1 + 5w_2$$

subject to the constraints:

$$\begin{aligned} &w_1 \leq 4, 3w_1 + w_2 \leq 6, w_2 \leq 9 \\ &w_1 \geq 0, w_2 \geq 0 \end{aligned}$$

# General non-Linear programming problem

Let $z$ be a real valued function of $n$ variables defined by

$$z = f(x_1, x_2, \cdots, x_n).$$

Let $b_1, b_2, \cdots, b_m$ be a set of constants such that

$$g_i(x_1, x_2, \cdots, x_n) \quad \{\geq, \leq, or =\} \quad b_i,\ i = 1, 2, \cdots, m$$

where $g_i$'s are real valued functions of $n$ variables. Finally let

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

If either $f$ or $g_i$, $i = 1, 2, \cdots, m$; or both are non linear, then the problem to determine the variables $(x_1, x_2, \cdots, x_n)$ which makes $z$ to be maxima/minima and satisfies the conditions (1) and (2) is called a **general non-Linear programming problem(GNLPP)**

### Matrix Representation

Determine $\mathbf{x}^T \in \mathbb{R}^{\mathrm{n}}$, maximize/minimize the objective function $z = f(\mathbf{x})$, subject to the constraints:

$$g_i(\mathbf{x}) \; \{\geq, \leq, or =\} \; b_i, \mathbf{x} \geq 0$$

where either $f(\mathbf{x})$ or $g_i$ or both are non linear in $\mathbf{x}$

Note: The constraint $g_i(\mathbf{x}) \; \{\geq, \leq, or =\} \; b_i$ can be convenient to write as $h_i(\mathbf{x})\{\geq, \leq, or =\}0$ where $h_i(\mathbf{x}) = g_i(\mathbf{x}) - b_i$

# Convex Optimization

A convex optimization form is one of the form

$$\min f_0(x), x \in \mathbb{R}^m$$

such that:

$$\begin{aligned} f_i(x) &\leq 0, i = 1, 2, \cdots, m \\ a_i^T x &= b_i, i = 1, 2, \cdots, p. \end{aligned}$$

where $f_0, f_1, \cdots, f_m$ are convex functions. Comparing with the general standard form, the convex problem has three additional requirement

- the objective function must be convex,
- the inequality constraint function must be convex,
- the equality constraint function $h_i(x) = a_i^T x - b_i$ must be affine

# Constrained Optimization with Equality Constraints

If the non linear programming problem is composed of some differentiable objective function and equality constraints, the optimisation may be achieved by use of Lagrange multipliers. The method of Lagrange multipliers is used to optimize a function subject to constraints.

Consider a two variable function

$$\max / \min \ f(x_1, x_2),$$

subject to

$$h(x_1, x_2) = 0$$

Taking the total derivative of the function at $(x_1, x_2)$

$$\partial f = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0 \tag{1}$$

If $(x_1^*, x_2^*)$ be the solution of constrained optimization problem, then

$$h(x_1^*, x_2^*) = 0$$

Now the variation $dx_1$ and $dx_2$ is admissible only if

$$h(x_1^* + dx_1, x_2^* + dx_2) = 0,$$

which can be expanded as

$$h(x_1^* + dx_1, x_2^* + dx_2) = h(x_1^*, x_2^*) + \frac{\partial h(x_1^*, x_2^*)}{\partial x_1} dx_1 + \frac{\partial h(x_1^*, x_2^*)}{\partial x_2} dx_2 = 0$$

$$\partial h = \frac{\partial h}{\partial x_1} dx_1 + \frac{\partial h}{\partial x_2} dx_2 = 0 \Rightarrow dx_2 = -\frac{\frac{\partial h}{\partial x_1}}{\frac{\partial h}{\partial x_2}} dx_1$$

Putting in (1)

$$\begin{aligned} \partial f = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} \Big( -\frac{\frac{\partial h}{\partial x_1}}{\frac{\partial h}{\partial x_2}} \Big) dx_1 &= 0 \\ \Big( \frac{\partial f}{\partial x_1} \frac{\partial h}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial h}{\partial x_1} \Big) dx_1 &= 0 \\ \frac{\partial f}{\partial x_1} - \left( \frac{\frac{\partial f}{\partial x_2}}{\frac{\partial h}{\partial x_2}} \right) \frac{\partial h}{\partial x_1} &= 0 \end{aligned}$$

Thus,

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0,$$

We can also write

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0,$$

where $\lambda = (\lambda_1, \lambda_2) = \left( \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial h}{\partial x_1}}, \frac{\frac{\partial f}{\partial x_2}}{\frac{\partial h}{\partial x_2}} \right)$

Hence, we need to find where the gradient of $f$ and the gradient of $h$ are aligned

$$\Delta f(\mathbf{x}) = \lambda \Delta h(\mathbf{x})$$

for some Lagrange multiplier $\lambda$. We need the scalar $\lambda$ because the magnitudes of the gradients may not be the same.

Let's consider a maximization/minimization problem

$$\min /(\max) f(x_1, x_2)$$

subject to the constraints:

$$g(x_1, x_2) = c, x_1, x_2 \geq 0$$

where $c$ is a constant.

To find the necessary condition for maximum/minimum value of $z$, a new function is formed by using the Lagrange multipliers, $\lambda$ as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(g(x_1, x_2) - c)$$

where the function $L(x_1, x_2, \lambda)$ is known as Lagrange function with Lagrange multipliers, $\lambda$

The necessary condition are:

$$
\begin{aligned}
\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} =& 0 \Rightarrow \frac{\partial f}{\partial x_1} - \frac{\partial (g(x_1, x_2) - c)}{\partial x_1} = 0 \\
\frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} =& 0 \Rightarrow \frac{\partial f}{\partial x_2} - \frac{\partial (g(x_1, x_2) - c)}{\partial x_2} = 0 \\
\frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} =& 0 \Rightarrow g(x_1, x_2) = c
\end{aligned}
$$

Thus the necessary condition for maximum/minimum of $f(x_1, x_2)$ are thus given by

$$
\begin{gathered}
\Delta f(x_1, x_2) = \lambda \Delta h(x_1, x_2), \\
h(x_1, x_2) = 0
\end{gathered}
$$

where $h(x_1, x_2) = g(x_1, x_2) - c$

# Necessary condition for general NLPP

Consider the general NLPP:

Maximize or minimize $z = f(x_1, x_2, \cdots, x_n)$ subject to the constraints: $g_i(x_1, x_2, \cdots, x_n) = b_i$ and, $x_i \geq 0\, i = 1, 2, \cdots, m$.

The constraints are reduced to $h_i(x_1, x_2, \cdots, x_n) = 0$ for $i = 1, 2, \cdots n$ by the transformation $g_i(x_1, x_2, \cdots, x_n) - b_i = h_i(x_1, x_2, \cdots, x_n)$

Thus the problem in the matrix form as

$$\text{Maximize/minimize } z = f(\mathbf{x}); \mathbf{x} \in \mathbb{R}^n$$

subject to

$$h_i(\mathbf{x}) = 0, \mathbf{x} \geq 0$$

To find the necessary condition, the Lagrangian function $L(\mathbf{x}, \lambda)$, is formed by introducing $m$ Lagrange multipliers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$.

The function is defined as

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h_i(x)$$

If $L, f, h$ are differentiable partially with respect to $x_1, x_2, \cdots, x_n$ and $\lambda_1, \lambda_2, \cdots, \lambda_m$, the necessary condition for maxima/minima of $z$ are

$$\frac{\partial L}{\partial x_i} = 0 \Rightarrow \frac{\partial f}{\partial x_i} - \sum_{i=1}^{m} \lambda_i \frac{\partial h}{\partial x_i} = 0$$
$$\frac{\partial L}{\partial \lambda_i} = 0 \Rightarrow h_i(\mathbf{x}) = 0$$

That is the conditions are :

$$\Delta f(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i \Delta h(\mathbf{x})$$
$$h(\mathbf{x}) = 0$$

# Example - 01

Consider the minimization problem:

$$\min_{x} -exp\left( -(x_1x_2 - \frac{3}{2})^2 - (x_2 - \frac{3}{2})^2 \right)$$

subject to : $x_1 - x_2^2 = 0$

SOLUTION:

We can use the method of Lagrange multipliers to solve the problem,

$$L(x_1, x_2, \lambda) = -exp\left( -(x_1x_2 - \frac{3}{2})^2 - (x_2 - \frac{3}{2})^2 \right) - \lambda(x_1 - x_2^2)$$

and compute the gradient as

$$\frac{\partial L}{\partial x_1} = 2x_2\left\{ -exp\left( -(x_1x_2 - \frac{3}{2})^2 - (x_2 - \frac{3}{2})^2 \right) \right\}\left( -x_1x_2 + \frac{3}{2} \right) - \lambda$$

$$\frac{\partial L}{\partial x_2} = 2\lambda x_2 + \left\{ -exp\left( -(x_1x_2 - \frac{3}{2})^2 - (x_2 - \frac{3}{2})^2 \right) \right\} \times \square$$

$$\text{where } \square = \left( -2x_1(x_1x_2 - \frac{3}{2}) - 2(x_2 - \frac{3}{2}) \right)$$

$$\text{and } \frac{\partial L}{\partial \lambda} = x_1 - x_2^2$$

Setting these derivatives to zero and solving yields $x_1 \approx 1.358, x_2 \approx 1.165$, and $\lambda \approx 0.170$.

---

# Example - 02

Consider the problem,

$$\min f(X) = x_1^2 + x_2^2 + x_3^2$$

subject to

$$\begin{aligned} g_1(X) &= x_1 + x_2 + 3x_3 - 2 \\ g_2(X) &= 5x_1 + 2x_2 + x_3 - 5 \end{aligned}$$

SOLUTION:

The Lagrangian is defined as,

$$L(X,\lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

This yields the following necessary conditions:

$$\begin{aligned}
\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 =&0 \\
\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 =&0 \\
\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 =&0 \\
\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) =&0 \\
\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) =&0
\end{aligned}$$

The solution to these equations yields,

$$\begin{aligned}
X_0 =&(x_1, x_2, x_3) = (0.8043, 0.3478, 0.2826) \\
\lambda =&(\lambda_1, \lambda_2) = (0.0870, 0.3043)
\end{aligned}$$

# Example-03

(For a rectangle whose perimeter is 20 m, use the Lagrange multiplier method to find the dimensions that will maximize the area

SOLUTION:

Let, with $x$ and $y$ representing the width and height, respectively, of the rectangle, this problem can be stated as:

$$\max \; f(x, y) = xy,$$

given: $g(x, y) = 2x + 2y = 20$
Then solving the equation, $\Delta f(x, y) = \lambda \Delta g(x, y)$, for some $\lambda$, by solving the equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$\Rightarrow y = 2\lambda, x = 2\lambda$$

The general idea is to solve for $\lambda$ in both equations, then set those expressions equal (since they both equal $\lambda$ ) to solve for $x$ and $y$ . Doing this we get

$$\frac{x}{2} = \lambda = \frac{y}{2} \Rightarrow x = y$$

so now substitute either of the expressions for $x$ or $y$ into the constraint equation to solve for $x$ and $y$ :

$$g(x, y) = 2x + 2y = 20$$

$$\Rightarrow 4x = 20 \Rightarrow x = 5$$

There must be a maximum area, since the minimum area is 0 and $f(5, 5) = 25$, so the point $(5, 5)$ that we found (called a constrained critical point) must be the constrained maximum.
The maximum area occurs for a rectangle whose width and height both are $5m$.

# Sufficient condition for GNLPP to have extrema

## Sufficient condition for an $n$ variable function to have extrema

**Theorem**

A sufficient condition for a stationary point $X_0$, to be an extremum is that the Hessian $\mathbf{H}$ satisfies the following condition:

1. $\mathbf{H}$ is positive definite if $X_0$ is a minimum point.
2. $\mathbf{H}$ is negative definite if $X_0$ is a maximum point.

**Example**

Consider the function $f(x, y, z) = x + 2z + yz - x^2 - y^2 - z^2$

SOLUTION:

The necessary conditions are $\Delta f = 0$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 1 - 2x = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow z - 2x = 0$$

$$\frac{\partial f}{\partial z} = 0 \Rightarrow 2 + y - 2z = 0$$

Thus the critical point is $X_0 = (1/2, 2/3, 4/3)$

To determine the stationary points, consider the Hessian matrix:

$$H|_{X_0} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Since the principle minor determinants of $H|_{X_0}$ are $-2, 4, -6$ respectively. Thus $H$ is negative definite, and $X_0 = (1/2, 2/3, 4/3)$ represent the maximum point

# Sufficient Condition: Single constraint GNLPP

Let the Lagrangian function for general NLPP involving $n$ variable, one constraint be:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda h(\mathbf{x})$$

The necessary condition for stationary points to be maximum/minimum:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0, \qquad j = 1, 2, \cdots n$$
$$\frac{\partial L}{\partial \lambda} = -h(\mathbf{x}) = 0,$$

the value $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is obtained from

$$\lambda_i = \frac{\partial f / \partial x_j}{\partial h / \partial x_j}, \;\; j = 1, 2, \cdots n$$

The sufficient condition for a maximum or minimum require the evaluation at each stationary points, of $(n-1)$ principle minor of the determinant given below:

$$H^B = \begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ . & . & . & & . \\ . & . & . & & . \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n^2} \end{bmatrix}$$

where $H^B$ is known as Bordered Hessian matrix, and $\Delta_{n+1} = det(H^B)$

- If $\Delta_3 > 0, \Delta_4 < 0, \Delta_5 > 0 \cdots$, the sign pattern changes alternate, the stationary point is local maximum
- If $\Delta_3 < 0, \Delta_4 < 0, \Delta_5 < 0 \cdots$, the sign pattern being always negative, the stationary point is local minimum

# Example

Solve the non-linear programming problem :
Minimize $z = 2x^2 - 24x + 2y^2 - 8y + 2z^2 - 12z + 200$ subject to the constraint,

$$x + y + z = 11, x, y, z \geq 0$$

SOLUTION:

We formulate the Lagrangian as:

$$L(x, y, z, \lambda) = 2x^2 - 24x + 2y^2 - 8y + 2z^2 - 12z + 200 - \lambda(x + y + z - 11)$$

The necessary condition for stationary points are

$$\frac{\partial L}{\partial x} = 4x - 24 - \lambda = 0 \qquad \frac{\partial L}{\partial y} = 4y - 8 - \lambda = 0$$

$$\frac{\partial L}{\partial z} = 4z - 12 - \lambda = 0 \qquad \frac{\partial L}{\partial \lambda} = x + y + z - 11 = 0$$

The solution of these equation yields the stationary points $x_0 = (6, 2, 3); \lambda = 0$

The sufficient condition for the stationary point to be minimum is that $\Delta_3 and \Delta_4$ both negative.Thus

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -8 \qquad \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -48$$

which both are negative. Thus $x_0 = (6, 2, 3)$; provides the solution for NLPP

# Sufficient Condition: $m(< n)$ constraint GNLPP

Introducing $m$ Lagrange multipliers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$, the Lagrangian function can be defined as:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h_i(x)$$

The necessary conditions for stationary points are

$$\frac{\partial L}{\partial x_i} = 0 \qquad \frac{\partial L}{\partial \lambda_j} = 0 \qquad (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$$

Thus the optimization of $f(\mathbf{x})$ subject to the constraints $h(\mathbf{x})$ is equivalent to the optimization of $L(\mathbf{x}, \lambda)$.
Let's assume that the functions $L(\mathbf{x}, \lambda), f(\mathbf{x}), h(\mathbf{x})$ all posses partial derivatives of order one and two with respect to the decision variables.

The sufficient condition for the Lagrange multiplier method of stationary point of $f(\mathbf{x})$ to be maxima or minima.
Let,

$$\mathbf{V} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{n \times n},$$

be the matrix of all second order partial derivative of $L(\mathbf{x}, \lambda)$, w.r.t decision variables

$$\mathbf{U} = \left[\frac{\partial h_j(\mathbf{x})}{\partial x_i},\right]_{m \times n}, \quad (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$$

Define the matrix

$$\mathbf{H}^B = \begin{bmatrix} \mathbf{0} & \mathbf{U} \\ \cdots & \cdots \\ \mathbf{U}^T & \mathbf{V} \end{bmatrix}_{(m+n) \times (m+n)}$$

where $\mathbf{H}^B$ is the Bordered Hessian matrix.

Then the sufficient condition for a maximum or minimum require the evaluation at each stationary points are given:

Let $(x_0, \lambda_0)$ be the stationary point of a function $L(\mathbf{x}, \lambda)$, let $\mathbf{H}_0^B$ be the Bordered Hessian matrix calculated at the stationary point. Then $x_0$ is a

- If starting with the principle minor of order $(2m+1)$, the last $(n-m)$ principle minor of $\mathbf{H}_0^B$ forms alternate sign pattern $(-1)^{m+n}$, the stationary point is local maximum
- And if starting with the principle minor of order $(2m+1)$, the last $(n-m)$ principle minor of $\mathbf{H}_0^B$ has sign of $(-1)^m$, the stationary point is local minimum

# Example

Optimize $z = 3x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$ subject to the constraints:

$$x_1 + x_2 + x_3 = 15, 2x_1 - x_2 + 2x_3 = 20$$

SOLUTION:
we have

$$f(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2, h_1(\mathbf{x}) = x_1 + x_2 + x_3 - 15$$

$$h_2(\mathbf{x}) = 2x_1 - x_2 + 2x_3 - 20$$

Construct the Lagrangian function as:

$$L(\mathbf{x}, \lambda) = 3x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

Now the stationary point can be obtained as

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \qquad \frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 \qquad \frac{\partial L}{\partial \lambda_1} = -[x_1 + x_2 + x_3 - 15] = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -[2x_1 - x_2 + 2x_3 - 20] = 0$$

The solution to these equations yields

$$x_0 = (x_1, x_2, x_3) = (33/9, 10/3, 8), \lambda = (\lambda_1, \lambda_2) = (40/9, 52/9)$$

The bordered Hessian matrix at the point $(x_0, \lambda_0)$

$$H_0^B = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Here since $n = 3$ and $m = 2$, therefore $n - m = 1$, $2m + 1 = 5$. This means that one needs to check the determinant of $H_0^B$ only it must have sign $(-1)^2$.

Since $\det(H_0^B) = 72 > 0$, $x_0$ is the minimum point

# Constrained optimization with inequality constraints

Karush–Kuhn–Tucker (KKT)conditions, the necessary and sufficient condition for optimal solution of General NLPP.

**One inequality constraint GNLPP**

Optimize: $z = f(x_1, x_2, \cdots, x_n)$ subject to the constraints:

$$g(x_1, x_2, \cdots, x_n) \leq C, x_i \geq 0\, i = 1, 2, \cdots, m$$

. The constraints are reduced to $h_i(x_1, x_2, \cdots, x_n) \leq 0$ for $i = 1, 2, \cdots n$ by the transformation $g(x_1, x_2, \cdots, x_n) - C = h(x_1, x_2, \cdots, x_n)$

Thus the problem in the matrix form as

$$\text{Optimize } z = f(\mathbf{x}); \mathbf{x} \in \mathbb{R}^{\mathrm{n}}$$

subject to

$$h(\mathbf{x}) \leq 0, \mathbf{x} \geq 0$$

Modify slightly by introducing the slack variable $S$, defined by $h(\mathbf{x}) + S^2 = 0$;($S^2$ is taken so as to ensure its non-negative)

Now the problem is restarted as

$$\text{Maximize/minimize } z = f(\mathbf{x}); \mathbf{x} \in \mathbb{R}^{\mathrm{n}}$$

subject to

$$h(\mathbf{x}) + S^2 = 0, \mathbf{x} \geq 0$$

To determine the stationary points, consider the Lagrangian function $L(\mathbf{x}, S, \lambda)$,

$$L(\mathbf{x}, S, \lambda) = f(\mathbf{x}) - \lambda(h(\mathbf{x}) + S^2),$$

where $\lambda$ is the Lagrange multipliers.The necessary condition for stationary points to be Optimize:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0, \qquad j = 1, 2, \cdots n \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = -[h(\mathbf{x}) + S^2] = 0, \tag{3}$$

$$\frac{\partial L}{\partial S} = -2\lambda S = 0 \tag{4}$$

The equation states that $\frac{\partial L}{\partial S} = 0$, which requires either $\lambda = 0$ or $S = 0$.
If $S = 0$ implies that $h(\mathbf{x}) = 0$.
Thus from (2) and (3), $\Rightarrow \lambda h(\mathbf{x}) = 0$

The variable $S$ was introduced merely to convert the inequality constraint to an equality one, therefore may be discarded. Moreover since $S^2 \geq 0$, (2) gives $h(\mathbf{x}) \leq 0$.

Whenever $h(\mathbf{x}) < 0$, we get $\lambda = 0$ and whenever $\lambda > 0$, $h(\mathbf{x}) = 0$. However, $\lambda$ is unrestricted in sign whenever $h(\mathbf{x}) = 0$

The necessary condition for a point to be a point of maximum are thus restated as

## KKT condition

$$\begin{aligned} \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} &= 0 \\ \lambda h &= 0 \\ h &\leq 0 \\ \lambda &\geq 0 \end{aligned}$$

Similar argument for minimization NLPP also,

Minimize $z = f(\mathbf{x}); \mathbf{x} \in \mathbb{R}^n$

subject to

$$h(\mathbf{x}) \geq 0, \mathbf{x} \geq 0$$

The constraints are reduced to $h_i(x_1, x_2, \cdots, x_n) \leq 0$ for $i = 1, 2, \cdots n$ by the transformation $g_i(x_1, x_2, \cdots, x_n) - C = h_i(x_1, x_2, \cdots, x_n)$

Modify slightly by introducing the surplus variable $S_0$, defined by $h(\mathbf{x}) - S_0^2 = 0$;($S_0^2$ is taken so as to ensure its non-negative)

Now the problem is restarted as

$$\text{minimize } z = f(\mathbf{x}); \mathbf{x} \in \mathbb{R}^n$$

subject to

$$h(\mathbf{x}) - S_0^2 = 0, \mathbf{x} \geq 0$$

To determine the stationary points, consider the Lagrangian function $L(\mathbf{x}, S_0, \lambda)$,

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda(h(\mathbf{x}) - S_0^2),$$

where $\lambda$ is the Lagrange multipliers.The necessary condition for stationary

## KKT condition

$$\begin{aligned} \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} &= 0 \\ \lambda h &= 0 \\ h &\geq 0 \\ \lambda &\geq 0 \end{aligned}$$

# Sufficient of KKT conditions

## Theorem

*The KKT condition for a maximization NLPP of maximizing $f(\mathbf{x})$ subject to the constraints $h(\mathbf{x}) \leq 0$ and $\mathbf{x} \geq 0$, are sufficient for a condition of $f(\mathbf{x})$, if $f(\mathbf{x})$ is concave and $h(\mathbf{x})$ is convex.*

## Proof.

The result follows if we are able to show that the Lagrange function

$$L(\mathbf{x}, S, \lambda) = f(\mathbf{x}) - \lambda(h(\mathbf{x}) + S^2),$$

where $S$ is defined by $h(\mathbf{x}) + S^2 = 0$, is concave in $\mathbf{x}$ under given conditions. In that case the stationary points from KKT condition must be the global maximum point.

Now since $h(\mathbf{x}) + S^2 = 0$, it follows from the necessary condition that $\lambda S^2 = 0$. since $h(\mathbf{x})$ is convex and $\lambda \geq 0$ it follows that $\lambda h(\mathbf{x})$ is convex and $-\lambda h(\mathbf{x})$ is concave. thus, we conclude that $f(\mathbf{x}) - \lambda h(\mathbf{x})$ and hence $f(\mathbf{x}) - \lambda(h(\mathbf{x}) + S^2) = L(\mathbf{x}, S, \lambda)$ is concave $\square$

## Theorem

*The KKT condition for a minimization NLPP of minimizing $f(\mathbf{x})$ subject to the constraints $h(\mathbf{x}) \geq 0$ and $\mathbf{x} \geq 0$, are sufficient for a condition of $f(\mathbf{x})$, if $f(\mathbf{x})$ is convex and $h(\mathbf{x})$ is concave.*

---

**Example**

Maximize $z = 3.6x - 0.4x^2 + 1.6y - 0.2y^2$ subject to the constraint :

$$2x + y \leq 10, \qquad x, y \geq 0$$

SOLUTION:

Here

$$\begin{aligned} f(\mathbf{x}) = 3.6x - 0.4x^2 + 1.6y - 0.2y^2 \\ h(\mathbf{x}) = 2x + y - 10 \end{aligned}$$

The KKT condition are:

$$\frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0, \quad \lambda h = 0, \quad h \geq 0, \quad \lambda \geq 0$$

That is,

$$\begin{aligned} 3.6 - 0.8x &= 2\lambda \\ 1.6 - 0.4y &= \lambda \\ \lambda(2x + y - 10) &= 0 \\ 2x + y - 10 &\leq 0 \\ \lambda &\geq 0 \end{aligned}$$

From the third equation either $\lambda = 0$ or $2x + y - 10 = 0$

Let $\lambda = 0 \Rightarrow x = 4.5, y = 4$, with these values the fourth equation don't satisfies. thus the optimum solution cannot be obtained for $\lambda = 0$.

Now, let $\lambda \neq 0 \Rightarrow 2x + y = 10$. thus we get $x_0 = (3.5, 3)$

Now it is easy to observe that $h(\mathbf{x})$ is convex, and $f(\mathbf{x})$ is concave. thus KKT conditions are sufficient for maximum.

maximum value of $z = 10.7$, at stationary point $(3.5, 3)$

# KKT condition for general NLPP: $m$ constraints

Introducing $S = (S_1, S_2, \cdots, S_m)$, the slack variables, the Lagrangian function for the GLNPP with $m(< n)$ constraints be

$$L(\mathbf{x}, S, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i (h_i(\mathbf{x}) + S_i^2)$$

where the Lagrange multipliers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$.

The necessary conditions for $f(\mathbf{x})$ to be maximum are

$$\begin{aligned}
\frac{\partial L}{\partial x_i} &= \frac{\partial f}{\partial x_i} - \sum_{i=1}^{m} \lambda_i \frac{\partial h}{\partial x_i} + S_i^2 = 0 \\
\frac{\partial L}{\partial \lambda_i} &= h_i(\mathbf{x}) = 0 \\
\frac{\partial L}{\partial S_i} &= -2\lambda_i S_i = 0
\end{aligned}$$

where $L(\mathbf{x}, S, \lambda) = L$, $f = f(\mathbf{x})$ and $h = h_i(\mathbf{x})$

The above equations states that $\lambda_i = 0$or $S_i = 0$. By an argument as done in case of single inequality constraint.

Thus the necessary condition for maximum is stated as:

## KKT condition

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \sum_{i=1}^{m} \lambda_i \frac{\partial h}{\partial x_i} \\ \lambda_i h_i &= 0 \\ h_i &\leq 0 \\ \lambda &\geq 0 \end{aligned}$$

Same pattern we can do for minimization problems also, where the constraints $h_i(\mathbf{x}) \geq 0$

# Sufficiency condition : KKT

## Theorem

*The KKT condition for a maximization NLPP of maximizing $f(\mathbf{x})\mathbf{x} \in \mathbb{R}^n$ subject to the constraints $h(\mathbf{x})_i \leq 0$ and $\mathbf{x} \geq 0 (i = 1, 2, \cdots, m)$, are sufficient for a condition of $f(\mathbf{x})$, if $f(\mathbf{x})_i$ is concave and $h(\mathbf{x})$ is convex.*

Note: It can be shown that for the minimization problem the KKT condition are also sufficient for minima if $f(\mathbf{x})$, if $f(\mathbf{x})_i$ is convex and $h(\mathbf{x})$ is concave or $-h(\mathbf{x})$ is convex

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**Example**

Maximize $z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$ subject to the constraints:

$$x_1 + x_2 \leq 2, 2x_1 + 3x_2 \leq 12, x_1, x_2 \geq 0$$

<u>SOLUTION</u> : Let,

$$f(\mathbf{x}) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$$

$$h_1(\mathbf{x}) = x_1 + x_2 - 2, h_2(\mathbf{x}) = 2x_1 + 3x_2 - 12$$

Clearly, $f(\mathbf{x})$ is concave and $h_1(\mathbf{x})$ and $h_2(\mathbf{x})$ are convex in $\mathbf{x}$. Thus the KKT condition will be the necessary and sufficient condition for maximum. These conditions are obtained by the partial differentiation of the Lagrange function

$$L(\mathbf{x}, S, \lambda) = f(\mathbf{x}) - \lambda_1(h_1(\mathbf{x}) + S_1^2) + \lambda_2(h_2(\mathbf{x}) + S_2^2)$$

where $S = (S_1, S_2)$ be the slack variables, $\lambda = (\lambda_1, \lambda_2)$ be the Lagrange multipliers.

The KKT conditions are given by

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \sum_{i=1}^{m} \lambda_i \frac{\partial h}{\partial x_i} \\ \lambda_i h_i &= 0 \\ h_i &\leq 0 \\ \lambda &\geq 0 \end{aligned}$$

In this problem we have,

$$
\begin{array}{llll}
(1) & 2x_1 + 4 = \lambda_1 + 2\lambda_2 & 2x_1 + 6 = \lambda_1 + 3\lambda_2 & -2x_3 = 0 \\
(2) & \lambda_1(x_1 + x_2 - 2) = 0 & \lambda_2(2x_1 + 3x_2 - 12) = 0 & \\
(3) & x_1 + x_2 - 2 \leq 0 & 2x_1 + 3x_2 - 12 \leq 0 & \\
(4) & \lambda_1 \geq 0, \lambda_2 \geq 0 & &
\end{array}
$$

Now there arises four cases:

**Case 1**: $\lambda_1 = 0$ and $\lambda_2 = 0 \Rightarrow x_1 = 2, x_2 = 3, x_3 = 0$
However the solution violates for (3).

**Case 2**: $\lambda_1 = 0$ and $\lambda_2 \neq 0$

(2) $\Rightarrow 2x_1 + 3x_2 = 12$ and (1) $\Rightarrow -2x_1 + 4 = 2\lambda_2$ and $-2x_2 + 6 = 3\lambda_2$

The solution to the above equations yields $x_1 = 2/13, x_2 = 3/13$ and $\lambda_2 = 24/13 > 0$

But the solution violates for (3)

**Case 3**: $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$

(2)$\Rightarrow$ $x_1 + x_2 = 2$ and $2x_1 + 3x_2 = 12 \Rightarrow x_1 = -6, x_2 = 8$

Thus (1)$\Rightarrow$ $x_3 = 0, \lambda_1 = 68$ and $\lambda_2 = -26$, which violates (4)

**Case 4**: $\lambda_1 \neq 0$ and $\lambda_2 = 0$

(2) $\Rightarrow x_1 + x_2 = 0$

Together with (1) $\Rightarrow x_1 = 1/2$ and $x_2 = 3/2$, $\lambda_1 = 3 > 0$ and also $x_3 = 0$.

We observe that the solution don't violate any of the KKT conditions.

Hence the optimum solution is at $(1/2, 3/2, 0)$ and also $\lambda_1 = 3$ and $\lambda_2 = 0$

The maximum value is $z = 17/2$

Lagrange Function and Duality: Consider the convex programming problem :

$$\begin{aligned} &\min f_0(x), x \in R^m \\ \text{such that:} \quad &f_i(x) \leq 0, i = 1, 2. \cdots, m \\ &h_i(x) = a_i^T x - b_i = 0, i = 1, 2, \cdots, p. \end{aligned} \tag{5}$$

where $f_i(x), i = 0, 1, \cdots, m$ are continuously differentiable and convex in $\mathbb{R}^n$. We start from estimating its optimal value $p^*$ defined by

$$p^* = \inf \{f_0(x) \mid x \in D\}$$

where.

$$D = \{x \mid f_i(x) \leqslant 0, i = 1, \ldots, m; h_i(x) = 0, i = 1, \cdots, j, x \in \mathbb{R}^n\}$$

Introduce the Lagrangian function

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \tag{6}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_p)^T$ are the Lagrangian multipliers.

Obviously, when $x \in D$, $\lambda \geq 0$, we have,

$$L(x, \lambda, \nu) \leq f_0(x)$$

thus

$$\inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu) \leq \inf_{x \in D} L(x, \lambda, \nu) \leq \inf_{x \in D} f_0(x) = p^* \tag{7}$$

Therefore, introducing the Lagrangian dual function

$$g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu)$$

yields

$$g(\lambda, \nu) \leq p^*$$

The above inequality indicates that, for any $\lambda \geq 0, g(\lambda, \nu)$ is a lower bound of $p^*$. Among these lower bounds, finding the best one leads to the optimisation problem

$$\max\ g(\lambda, \nu) = \inf_{x \in R^n} L(x, \lambda, \nu).$$

$$\text{such that} \qquad \lambda \geq 0.$$

where $L(x, \lambda, \nu)$ is the Lagrangian function given by Introduce the Lagrangian function

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

## Theorem (weak Duality theorem)

*Let $p^*$ be the optimal value of the primal problem*

$$\begin{aligned} & \min f_0(x), x \in R^m \\ \textit{such that.} \qquad & f_i(x) \leq 0, i = 1, 2, \cdots, m. \\ & h_i(x) = a_i^T x - b_i = 0, i = 1, 2, \cdots, p. \end{aligned}$$

*and $d^*$ be the optimal value of the dual problem*

$$\begin{aligned} & max\ g(\lambda, \nu) = \inf_{x \in R^n} L(x, \lambda, \nu). \\ \textit{such that} \qquad & \lambda \geq 0. \end{aligned}$$

*Then*

$$\begin{aligned} p^* &= \inf \left\{ f_0(x) \mid f_i(x) \leq 0, i = 1, \cdots, m; a_i^T x - b_i = 0, i = 1, 2, \cdots, p; x \in R^n \right\} \\ &\geq \sup\{g(\lambda, \mu) \mid \lambda \geqslant 0\} = d^* \end{aligned}$$

## Corollary (1)

*Let $\bar{x}$ be the feasible point of the problem*

$$\begin{aligned} & \min f_0(x), x \in R^m \\ \textit{such that.} \quad & f_i(x) \leqslant 0, i = 1, 2, \cdots, m. \\ & h_i(x) = a_i^T x - b_i = 0, i = 1, 2, \cdots, p. \end{aligned}$$

*and $(\bar{\lambda}, \bar{\nu})$ be the feasible point of the dual problem*

$$\begin{aligned} & max\ g(\lambda, \nu) = \inf_{x \in R^n} L(x, \lambda, \nu). \\ \textit{such that} \quad & \lambda \geq 0. \end{aligned}$$

*If $f_0(\bar{x}) = g(\bar{\lambda}, \bar{\nu})$, then $\bar{x}$ and $(\bar{\lambda}, \bar{\nu})$ are their solutions respectively.*

# Strong Duality conditions

## Definition (Slater's condition)

Convex programming problem (5) is said to satisfy Slater's condition if there exists a feasible point $x$ such that

$$f_i(x) < 0, i = 1, 2, \cdots, m; \qquad a_i^T x - b_i = 0, i = 1, 2, \cdots, p.$$

Or, when the first $k$ inequality constraints are linear constraints, there exists a feasible point $x$ such that

$$f_i(x) = \bar{a}_i^T x - \bar{b}_i \leq 0, i = 1, 2, \cdots, k; \qquad f_i(x) < 0, i = k+1, \cdots, m;$$

$$a_i^T x - b_i = 0, i = 1, \ldots, p.$$

## Theorem (Strong duality theorem)

*Consider the convex programming problem (5) satisfying Slater's condition. Let $p^*$ be the optimal value of the primal problem (5) and $d^*$ the optimal value of the dual problem (7). Then*

$$\begin{aligned} p^* &= \inf\{f_0(x) \mid f(x) \leq 0, i = 1, 2, \cdots, m; a_i^T x - b_i = 0, i = 1, 2, \cdots, p; x \in \mathbb{R}^n\} \\ &= \sup\{g(\lambda, \nu) \mid \lambda \geq 0\} = d^* \end{aligned}$$

*Furthermore, if $p^*$ is attained, i.e there exists a solution $x^*$ to the primal problem, then $d^*$ is also attained, i.e there exists a global solution $(\lambda^*, \nu^*)$ to the dual problem such that*

$$p^* = f(x^*) = g(\lambda^*, \nu^*) = d^* < \infty.$$

# Optimality conditions

## Definition (KKT conditions)

Consider the convex programming problem (5). Point $x^*$ is said to satisfy the KKT conditions, if there exist the multipliers $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)^T$ and $w^* = (\nu_1^*, \nu_2^*, \cdots, \nu_p^*)^T$ are corresponding to constraints in (5), such that the Lagrangian function

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{n} \nu_i h_i(x)$$

satisfies,

$$f_i(x^*) \leq 0, \quad i = 1, 2, \cdots, m, \tag{8}$$

$$h_i(x^*) = 0, \quad i = 1, 2, \cdots, p, \tag{9}$$

$$\lambda_i^* \geq 0, \quad i = 1, 2, \cdots, m. \tag{10}$$

$$\lambda_i^* f_i(x^*) = 0, \quad i = 1, 2, \cdots, m \tag{11}$$

$$\Delta_x L(x, \lambda, \nu) = \Delta_x f_0(x) + \sum_{i=1}^{m} \lambda_i \Delta_x f_i(x) + \sum_{i=1}^{n} \nu_i \Delta_x h_i(x) = 0 \tag{12}$$

It is not difficult to show from strong duality theorem that for convex programming, the KKT conditions are the necessary condition of its solution:

## Theorem

*Consider the convex programming problem (5) satisfying Slater's condition. If $x^*$ is its solution, then $x^*$ satisfies the KKT conditions.*

## Proof.

Noticing that $x^*$ is a solution to the primal problem (5) where Slater's condition is satisfied, we conclude by strong duality theorem that there exists $(\lambda^*, \nu^*)$ such that $x^*$ and $(\lambda^*, \nu^*)$ are the solutions to the primal problem (5) and the solution to the dual problem (7) respectively, and their optimal values are equal. This means that

$$\begin{aligned} f_0(x^*) &= g(\lambda^*, \nu^*) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x)\right) \\ &\leq f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \\ &\leq f_0(x^*) \end{aligned}$$

□

## cts.

The second line and the third line follow from the definition.
The last line follows from $\lambda_i^*[. \geq 0, f_i\left(x^*\right) \leq 0, i = 1, 2, \cdots, m$ and $h_i(x^*) = 0, i = 1, 2, \cdots, p$.
We conclude that the two inequalities in this chain hold with equality. This yield

$$\begin{aligned} &\inf_x \left( f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right) \\ =& f_0(x^*) + \sum_{i=1}^{n} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) = f_0(x^*) \end{aligned} \tag{13}$$

Now we are in the position to prove the conclusions. First. equations (8),(9) and (10) are obvious. Second, equality (11) follows from the second equality in (13). At last, equality (12) is valid since $x^*$ is the minimal point of the Lagrangian function $L\left(x\lambda^*, \nu^*\right)$ by the first equality in (13). □

The next theorem shows that, for a convex programming, the KKT conditions arc also a sufficient condition of its solution.

## Theorem

*Consider the convex programming problem (5) If $x^*$ satisfies the KKT conditions, then $x^*$ is its solution.*

## Proof.

Suppose that $x^*$ and $(\lambda^*, \nu^*)$ satisfy conditions (8) $\sim$ (12). Note that the first two conditions (8) $\sim$ (9) state that $x^*$ is a feasible point of the primal problem and condition (10) states that $(\lambda^*, \nu^*)$ is a feasible point of the dud problem. Since $\lambda_i^* \geq 0, i = 1, 2, \cdots, m, L(x, \lambda^*, \nu^*)$ is convex in $x$. Therefore, condition (12) states that $x^*$ minimizes $L(x, \lambda^*, \nu^*)$ over $x$. From this we conclude that

$$\begin{aligned} g(\lambda^*, \nu^*) = & \inf_{x \in \mathbb{R}^n} L(x, \lambda^*, \nu^*) = L(x^*, \lambda^*, \nu^*) \\ = & f_0(x^*) + \sum_{i=1}^{n} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) = f_0(x^*) \end{aligned} \tag{14}$$

where in the last line we use conditions (11) and (9). Therefore. $x^*$ a solution to the primal problem by (14) and Corollary (27). □

The above two theorems are summarized in the following theorem.

**Theorem**

*Consider the convex programming problem (5) satisfying Slater's condition. Then for its solution $x^*$, it is necessary and sufficient condition that $x^*$ satisfies the KKT conditions given Definition of KKT conditions.*

# Example

Consider the problem:

$$\underset{\mathrm{x}}{\text{minimize}} \quad x_1 + x_2 + x_1 x_2$$

subject to $\quad x_1^2 + x_2^2 = 1$

The Lagrangian is

$$L\left(x_1, x_2, \lambda\right) = x_1 + x_2 + x_1 x_2 + \lambda\left(x_1^2 + x_2^2 - 1\right)$$

We apply the method of Lagrange multipliers:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 + x_2 + 2\lambda x_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 + x_1 + 2\lambda x_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x_1^2 + x_2^2 - 1 = 0$$

Solving yields four potential solutions, and thus four critical points:

| $x_1$ | $x_2$ | $\lambda$ | $x_1 + x_2 + x_1 x_2$ |
|---|---|---|---|
| $-1$ | $0$ | $1/2$ | $-1$ |
| $0$ | $-1$ | $1/2$ | $-1$ |
| $\frac{\sqrt{2}+1}{\sqrt{2}+2}$ | $\frac{\sqrt{2}+1}{\sqrt{2}+2}$ | $\frac{1}{2}(-1-\sqrt{2})$ | $\frac{1}{2}+\sqrt{2} \approx 1.914$ |
| $\frac{\sqrt{2}-1}{\sqrt{2}-2}$ | $\frac{\sqrt{2}-1}{\sqrt{2}-2}$ | $\frac{1}{2}(-1+\sqrt{2})$ | $\frac{1}{2}-\sqrt{2} \approx -0.914$ |

We find that the two optimal solutions are $[-1, 0]$ and $[0, -1]$.

The dual function has the form

$$\mathcal{D}(\lambda) = \underset{x_1, x_2}{minimize} \quad x_1 + x_2 + x_1 x_2 + \lambda \left( x_1^2 + x_2^2 - 1 \right)$$

The dual function is unbounded below when $\lambda$ is less than $1/2$ (consider $x_1 \to \infty$ and $x_2 \to -\infty$).

Setting the gradient to 0 and solving yields

$$x_2 = -1 - 2\lambda x_1$$

and

$$x_1 = (2\lambda - 1)/\left(1 - 4\lambda^2\right) \quad \text{for} \quad \lambda \neq \pm 1/2$$

. When $\lambda = 1/2$ $x_1 = -1 - x_2$ and $\mathcal{D}(1/2) = -1$.

Substituting these into the dual function yields:

$$D(\lambda) = \begin{cases} -\lambda - \frac{1}{2\lambda+1} & \lambda \geq \frac{1}{2} \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem maximize ${}_{\lambda}\mathcal{D}(\lambda)$ is maximized at $\lambda = 1/2$.