Invertible matrices

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If $AB = I = BA$, then B is called a two-sided inverse of A and A is said to be invertible.

Note: If A is an invertible matrix, then A has no zero row.

Lemma

If A has a left inverse B and a right inverse C, then $B = C$.

Proof: Suppose that $BA = I$ and $AC = I$.

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Note (Theorem 10)

(i): If A has a left inverse and a right inverse, then A is invertible and the inverse of A is denoted by A^{-1}.

$$AA^{-1} = I = A^{-1}A$$

By symmetry of the definition, $(A^{-1})^{-1} = A$

(ii) Suppose that A and B are invertible $n \times n$ matrices.

$$(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$$

$\Longrightarrow$ **Corollary : Product of invertible matrices is invertible.**

Theorem 11 : An elementary matrix is invertible.

Proof : Let E be an $m \times m$ elementary matrix corresponding to the elementary row operation e. Thus $E = e(I)$. By Theorem 2, there exists an elementary operation e_1, same type as e, such that

$$e(e_1(A)) = A = e_1(e(A)) \text{ for every matrix } A.$$

Let $E_1 = e_1(I)$ where I is the $m \times m$ identity matrix.

$$EE_1 = e(I)E_1 = e(E_1) = e(e_1(I)) = I$$

$$E_1 E = e_1(I)E = e_1(E) = e_1(e(I)) = I$$

$EE_1 = I = E_1 E$ Hence E is an invertible matrix.

Thus an elementary matrix is invertible.

Find inverses of all 2×2 elementary matrices

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{c} \end{bmatrix}$$

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find inverses of all 3×3 elementary matrices.(Assignment)

Theorem 12

If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

Proof : Let R be a row-reduced echelon $n \times n$ matrix which is row-equivalent to A. By Corollary to Theorem 9,

$$R = E_k E_{k-1} \dots E_2 E_1 A \qquad ------ \quad (a)$$

where E_i is an elementary matrix. Note that the inverse of E_i is also an elementary matrix. Since $E_i's$ are invertible,

$$A = E_1^{-1} E_2^{-1} \dots E_k^{-1} R \quad ---- \ (b)$$

Theorem 12 contd.

$(i) \Longrightarrow (ii)$ Suppose that A is invertible. Using (a), R is a product of invertible matrices and by corolllary to Theorem 10, R is invertible. Note that an invertible matrix has no zero-row. So R is an $n \times n$ row-reduced echelon matrix with no zero row and $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$. Hence R is the $n \times n$ identity matrix. A is row-equivalent to $R = I$.
$(ii) \Longrightarrow (iii)$ Suppose that A is row-equivalent to $R = I$. By (b)

$$A = E_1^{-1} E_2^{-1} \ldots E_k^{-1} R = E_1^{-1} E_2^{-1} \ldots E_k^{-1} I$$

$$A = E_1^{-1} E_2^{-1} \ldots E_k^{-1}, \text{a product of elementary matrices.}$$

Theorem 12 contd.

$(iii) \implies (i)$ **Suppose that A is a product of elementary matrices.** **By Theorem 11, an elementary matrix is invertible. By Corollary to Theorem 10, a product of invertible matrices is invertible. Hence A is invertible.**

Corollaries (to Theorem 12)

Consider an $n \times n$ matrix A and the $n \times n$ identity matrix I. Next we consider the augmented matrix $[A|I]$. Suppose that

$$[A|I] \sim [I|B]$$

Note that A is row equivalent to I (thus A is invertible) and I is row equivalent to B. By Corollary to Theorem 9, there exists an $n \times n$ matrix P such that $I = PA \;\; and \;\; B = PI \quad \Longrightarrow B = P \;\; and \; I = BA$
$\Longrightarrow \quad A$ **is invertible and** $B = A^{-1}$

Corollary (12.1)

If A is an $n \times n$ matrix and if a sequence of elementary row operations reduces A to the indentity matrix, then that same sequence of operations when applied to I yields A^{-1}.

Corollary (12.2)

Let A and B be two $m \times n$ matrices. Then B is row-equivalent to A if and only if $B = PA$ where P is an $m \times m$ invertible matrix.

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Why?

Problem

Find the inverse of

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

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Solution : Consider

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right]$$

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$$R_2 \longleftarrow R_2 - \frac{1}{2}R_1, \quad R_3 \longleftarrow R_3 - \frac{1}{3}R_1$$

Solution contd.

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right]$$

Solution contd.

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right]$$

$$R_2 \longleftarrow 12R_2$$

Solution contd.

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$$R_2 \longleftarrow 12R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right]$$

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$$R_1 \longleftarrow R_1 - \frac{1}{2}R_2, \quad R_3 \longleftarrow R_3 - \frac{1}{12}R_2$$

Solution contd.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right]$$

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$$R_3 \longleftarrow 180R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$$

Solution contd.

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$$R_2 \longleftarrow R_2 - R_3, \quad R_1 \longleftarrow R_1 + \frac{1}{6}R_3$$

Solution contd.

$$[A|I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$$

Solution contd.

$$[A|I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right] = [I|B]$$

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By Corollary 12.1,

$$A^{-1} = B = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

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Thanks to the last zero row, A is not row-equivalent to I

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Thanks to the last zero row, A is not row-equivalent to I and A is not invertible.

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For an $n \times n$ matrix A, the following are equivalent.

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Hence the system $AX = 0$ has only the trivial solution $X = 0$

Theorem 13 contd.

$(ii) \implies (i)$ **Suppose that** $AX = 0$ **has only the trivial solution** $X = 0$**.**

Theorem 13 contd.

$(ii) \implies (i)$ **Suppose that $AX = 0$ has only the trivial solution $X = 0$.** By Theorem 7, A is row-equivalent to I.

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$(ii) \Longrightarrow (i)$ **Suppose that $AX = 0$ has only the trivial solution $X = 0$.** By Theorem 7, A is row-equivalent to I. By Theorem 12, A is invertible.

$(i) \Longrightarrow (iii)$ Suppose that A is invertible.

Theorem 13 contd.

$(ii) \implies (i)$ **Suppose that $AX = 0$ has only the trivial solution $X = 0$.** By Theorem 7, A is row-equivalent to I. By Theorem 12, A is invertible.

$(i) \implies (iii)$ Suppose that A is invertible. That is A^{-1} exists.

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$(i) \implies (iii)$ Suppose that A is invertible. That is A^{-1} exists.
Consider the system $AX = Y$.

Theorem 13 contd.

$(ii) \implies (i)$ **Suppose that $AX = 0$ has only the trivial solution $X = 0$.** By Theorem 7, A is row-equivalent to I. By Theorem 12, A is invertible.

$(i) \implies (iii)$ Suppose that A is invertible. That is A^{-1} exists. Consider the system $AX = Y$. This implies that $X = A^{-1}Y$ is a solution for the system $AX = Y$ for each Y.

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$(iii) \implies (i)$ Suppose that the system of equations $AX = Y$ has a solution X for each $n \times 1$ matrix Y. Let R be a row-reduced echelon matrix which is row-equivalent to A.

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$(i) \implies (iii)$ Suppose that A is invertible. That is A^{-1} exists. Consider the system $AX = Y$. This implies that $X = A^{-1}Y$ is a solution for the system $AX = Y$ for each Y.

$(iii) \implies (i)$ Suppose that the system of equations $AX = Y$ has a solution X for each $n \times 1$ matrix Y. Let R be a row-reduced echelon matrix which is row-equivalent to A. By Corollary 12.2, $R = PA$ where P is an $n \times n$ invertible matrix.

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$AX = Y$ has a solution X for each Y. $\iff$ $P(AX) = PY$ has a solution X for each Y.

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Let

$$E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Theoem 13 contd.

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Hence A is row-equivalent to $R = I$. By Theorem 12, A is invertible.

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A square matrix with either a left or right inverse is invertible.

Proof: Let A be an $n \times n$ matrix.

Case 1. Suppose A has a left inverse, say B. That is $BA = I$.

Consider the system $AX = 0$. That implies $B(AX) = B0 = 0$.

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Consider the system $AX = 0$**. That implies** $B(AX) = B0 = 0$**.**

$\implies (BA)X = 0$**.**

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$\implies (BA)X = 0$. $\implies \quad IX = 0$.

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A square matrix with either a left or right inverse is invertible.

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Corollary 13.2

Let $A = A_1 A_2 \ldots A_k$ **where** $A_1, A_2, \ldots, A_k$ **are** $n \times n$ **(square) matrices.**

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Let $A = A_1 A_2 \dots A_k$ where $A_1, A_2, \dots, A_k$ are $n \times n$ (square) matrices. Then A is invertible if and only if each A_i is invertible.

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Case 2. **Suppose that each A_i is invertible.**

Contd.

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Qn. **Prove or disprove that if A is an $m \times n$ matrix, B is an $n \times m$ matrix and $n < m$, then AB is not invertible.**

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$(AB)X^* = A(BX^*) = A0 = 0$. $\implies X^*$ is a non-trivial solution of the homogeneous system $(AB)X = 0$. By Theorem 7, AB is not invertible.

Problem 2

$$\text{Let} \quad A = \begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix}$$

Does there exist a 3×4 matrix B such that (i) $AB = 0$, a zero matrix, and (ii) $B \neq 0$?

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Choose $a = 24 \implies (30, 67, 24)$ is a solution.

$$\implies B = \begin{bmatrix} 30 & 30 & 30 & 30 \\ 67 & 67 & 67 & 67 \\ 24 & 24 & 24 & 24 \end{bmatrix}$$

Verify that $AB = 0, \quad and \quad B \neq 0$

Problem 3

Prove of disprove that A is invertible and find A^{-1} if it exists where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

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Solution to Problem 3

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$