Engineering Electromagnetics

Lecture 12 and 13

13/12/2023 and 15/12/2023

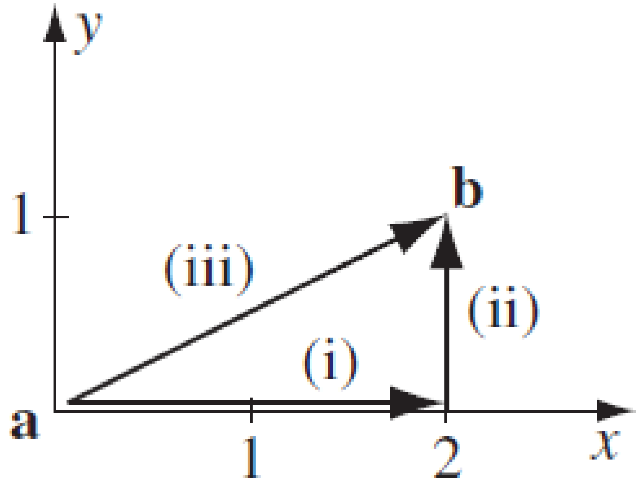
by

Debolina Misra

Assistant Professor in Physics
IIITDM Kancheepuram, Chennai, India

Problem-1

Example 1.9. Let $T = xy^2$, and take point **a** to be the origin $(0, 0, 0)$ and **b** the point $(2, 1, 0)$. Check the fundamental theorem for gradients.

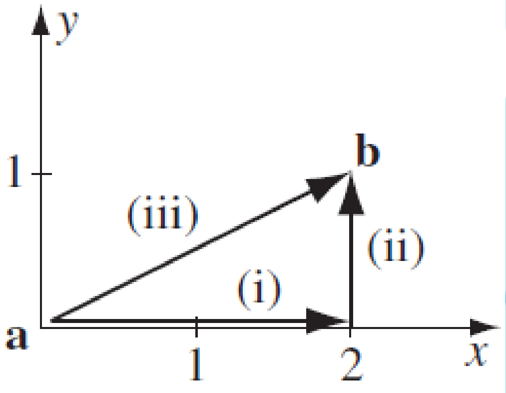


Solution

Although the integral is independent of path, we must *pick* a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii) (Fig. 1.27). As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\nabla T = y^2\,\hat{\mathbf{x}} + 2xy\,\hat{\mathbf{y}}$.

(i) $y = 0$; $d\mathbf{l} = dx\,\hat{\mathbf{x}}$, $\nabla T \cdot d\mathbf{l} = y^2\,dx = 0$, so

$$\int_{\mathrm{i}} \nabla T \cdot d\mathbf{l} = 0.$$



(ii) $x = 2$; $d\mathbf{l} = dy\,\hat{\mathbf{y}}$, $\nabla T \cdot d\mathbf{l} = 2xy\,dy = 4y\,dy$, so

$$\int_{\mathrm{ii}} \nabla T \cdot d\mathbf{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2.$$

The total line integral is 2. Is this consistent with the fundamental theorem? Yes: $T(\mathbf{b}) - T(\mathbf{a}) = 2 - 0 = 2$.

The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.}$$

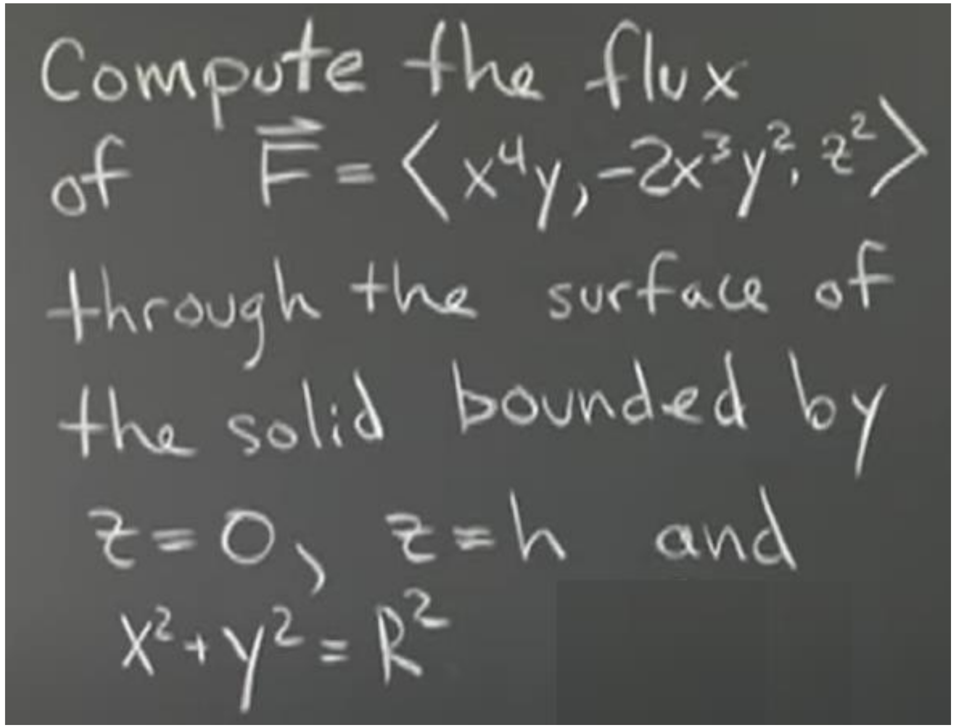
Gauss's theorem/divergence theorem

integral of a *derivative* (in this case the *divergence*) over a *region* (in this case a *volume*, *V*) =value of the function at the *boundary* (in this case the *surface S* that bounds the volume).

Notice that the boundary term is itself an integral (specifically, a surface integral). This is reasonable: the boundary of a *volume* is a (closed) surface.

From Griffith Book

Div. theorem: example (MIT open course)



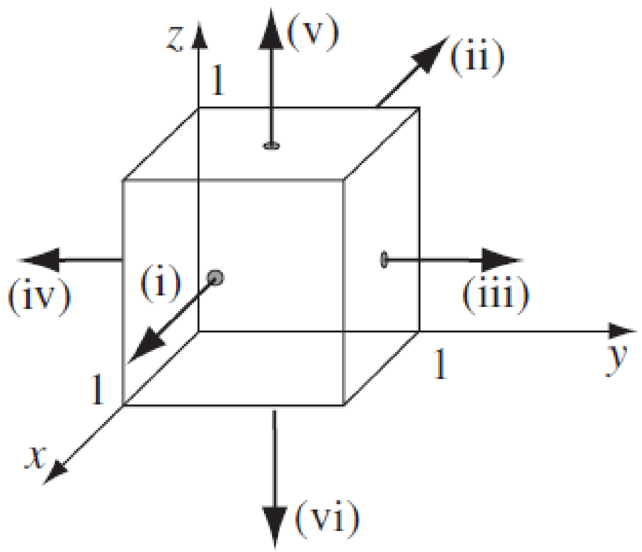
- https://www.youtube.com/watch?v=CCoTAyZ14XM

Problem-2 (Cartesian)

Check the divergence theorem using the function

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

and a unit cube at the origin.

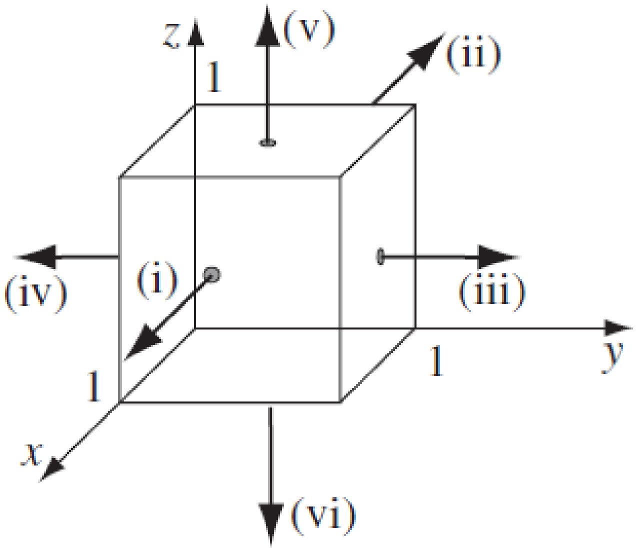


$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.}$$

From Griffith Book

Solution

For the LHS



$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{v} = 2(x + y)$$

$$\int_{\mathcal{V}} 2(x + y)\,d\tau = 2\int_0^1\int_0^1\int_0^1 (x + y)\,dx\,dy\,dz,$$

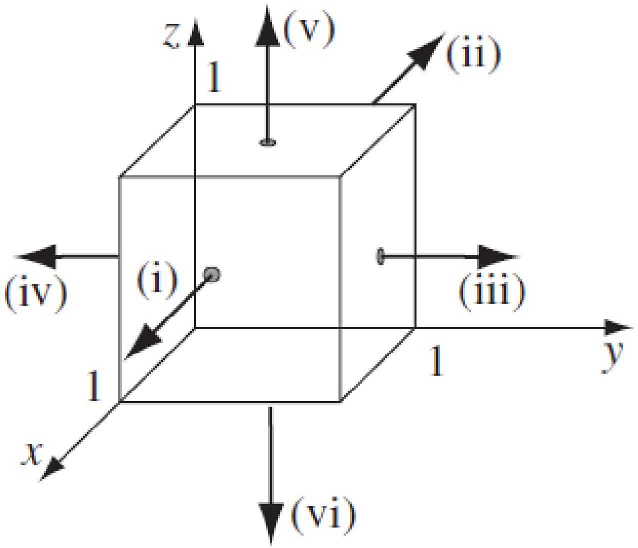
$$\int_0^1 (x + y)\,dx = \tfrac{1}{2} + y, \quad \int_0^1 (\tfrac{1}{2} + y)\,dy = 1, \quad \int_0^1 1\,dz = 1$$

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{v}\,d\tau = 2$$

From Griffith Book

For the RHS

So much for the left side of the divergence theorem. To evaluate the surface integral we must consider separately the six faces of the cube:



(i) $$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^1 \int_0^1 y^2 dy\, dz = \tfrac{1}{3}.$$

(ii) $$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^1 \int_0^1 y^2\, dy\, dz = -\tfrac{1}{3}.$$

(iii) $$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^1 \int_0^1 (2x + z^2)\, dx\, dz = \tfrac{4}{3}.$$

(iv) $$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^1 \int_0^1 z^2\, dx\, dz = -\tfrac{1}{3}.$$

(v) $$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^1 \int_0^1 2y\, dx\, dy = 1.$$

(vi) $$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^1 \int_0^1 0\, dx\, dy = 0.$$

So the total flux is:

$$\oint \mathbf{v}\cdot d\mathbf{a} = \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{4}{3} - \tfrac{1}{3} + 1 + 0 = 2,$$

From Griffith Book

We can also obtain the expressions for the divergence of a vector field in cylindrical and spherical coordinates as

$$\nabla \cdot \vec{\mathbf{F}} = \frac{1}{\rho}\frac{\partial}{\partial \rho}[\rho F_\rho] + \frac{1}{\rho}\frac{\partial}{\partial \phi}[F_\phi] + \frac{\partial}{\partial z}[F_z] \tag{2.87}$$

and

$$\nabla \cdot \vec{\mathbf{F}} = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2 F_r] + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}[\sin\theta\, F_\theta] + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}[F_\phi] \tag{2.88}$$

respectively.

Problem 3 (cylindrical)

How to calculate total flux of $\vec{V} = \rho z \hat{z}$ for a solid cylinder bounded by $z=0$, $z=2$ and radius $= 2$.

OR the same function from MIT open course (slide 5)→ there you need to do a cartesian → cylindrical conversion

$\vec{ds}_1 = \hat{\rho}\ \rho\, d\phi\, dz$, ~~$\vec{V} = (\ \rho z)\hat{z}$~~ $V = \rho z\ \hat{z}$

$\therefore \iint_S \bar{V}\cdot \bar{ds}_1 = 0$

For $z=2$, $\vec{ds}_2 = \hat{z}\ d\rho\ \rho\, d\phi$ $\therefore \iint_S \vec{V}\cdot\vec{ds}_2 = \iint_S (\ \rho z)\cdot d\rho\ \rho\, d\phi$

$= \iint_S (2\ 2\rho)\ \rho\, d\rho\, d\phi$

$= \int_0^2 2\rho^2\, d\rho \cdot 2\pi$

$= \frac{2}{3}\cdot 2^3 \cdot 2\pi = \frac{32\pi}{3}$ —— (2)

For $z=0$, $\vec{ds}_3 = -\hat{z}\ d\rho\ \rho\, d\phi$

$\therefore \iint_S \vec{V}\cdot\vec{ds}_3 = -\iint_S \rho z\cdot\rho\cdot d\rho\cdot d\phi$

$= 0$ as $z=0$ —— (3)

$\therefore \iint_S \bar{F}\cdot\bar{ds} = \iint_{S_1}\bar{F}\cdot\bar{ds}_1 + \iint_S \bar{F}\cdot\bar{ds}_2 + \iint_S \bar{F}\cdot\bar{ds}_3 = 0 + \frac{32\pi}{3} +$

$= \frac{32\pi}{3}$

$= \iiint_V (\bar{\nabla}\cdot\bar{v})\, d\tau$

See ex. 2.20 of
Bhag Guru book
for Cartesian coord.

The Fundamental Theorem for Curls

The fundamental theorem for curls, which goes by the special name of **Stokes' theorem**, states that

$$\boxed{\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}.}$$

As always, the *integral* of a *derivative* (here, the *curl*) over a *region* (here, a patch of *surface*, $\mathcal{S}$) is equal to the value of the function at the *boundary* (here, the perimeter of the patch, $\mathcal{P}$). As in the case of the divergence theorem, the boundary term is itself an integral—specifically, a closed line integral.

From Griffith Book

Interpretation

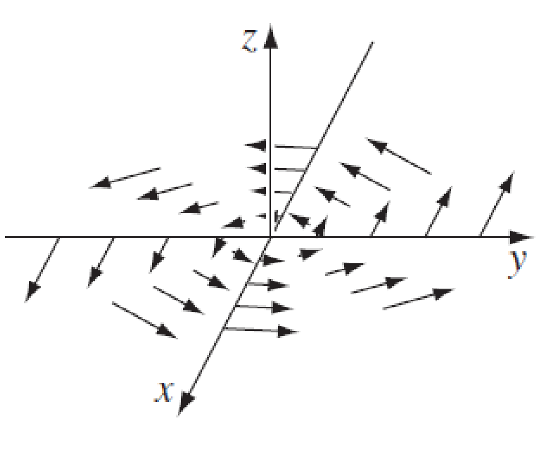
- Geometrical Interpretation: Recall that the curl measures the “twist” of the vectors **v**; a region of high curl is a whirlpool—if you put a tiny paddle wheel there, it will rotate.
- Now, the integral of the curl over some surface (or, more precisely, the flux of the curl through that surface) represents the “total amount of swirl,” and we can determine that just as well by going around the edge and finding how much the flow is following the boundary.

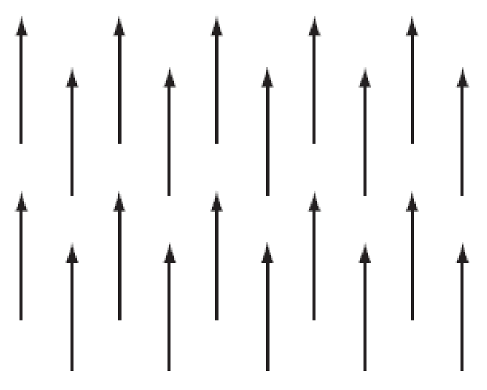
$\oint \mathbf{v} \cdot d\mathbf{l}$ is sometimes called the **circulation** of **v**.

From Griffith Book

Rotation/Irrotational fields

The physical significance of the curl of a vector field is that it represents the circulation per unit area of the vector field taken around a small area of any shape. Its direction is normal to the plane of the surface. Stated differently, if the line integral of a vector field about a closed elementary path is nonzero, the curl of the vector field is also nonzero and we say that the vector field is *rotational*. The flow of water out of a tub or a sink provides an excellent example of a rotational velocity field of the flow. On the other hand, if the curl of a vector field is zero, the vector field is said to be *irrotational* or *conservative*. A common example of a conservative field is the work done by a force acting on a body.





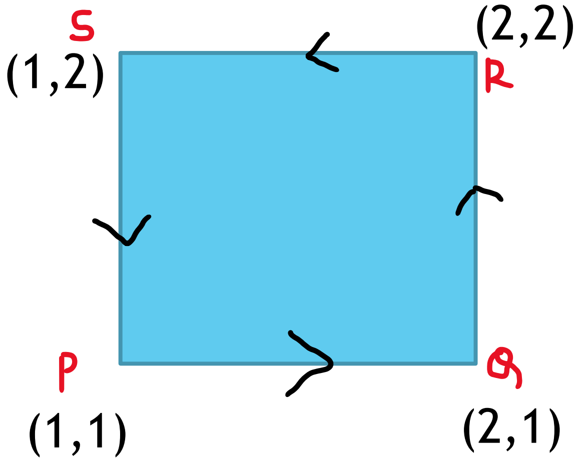
From BhagGuru and Griffith Books

Problem 4

- $\boldsymbol{F} = -2xy^2\hat{\boldsymbol{x}}$

verify stokes theorem

$$\int \nabla \times \boldsymbol{F} \,.\, \boldsymbol{dS} = \int_1^2\!\int_1^2 4xy \; dx \; dy = 9$$



$$\int \boldsymbol{F}.\boldsymbol{dl} = -\int 2xy^2 \; dx$$

$$\int \boldsymbol{F}.\boldsymbol{dl} = 0 \; along \; QR \; and \; SP \; (dx = 0), = -3 \; along \; \text{PQ and} = 12 \text{ along SP.}$$

$$Hence, \oint \boldsymbol{F}.\boldsymbol{dl} = -3 + 0 + 12 + 0 = 9 = \int \nabla \times \boldsymbol{F} \,.\, \boldsymbol{dS}$$

The expressions for the curl of the vector field $\vec{\mathbf{F}}$ in the cylindrical and spherical coordinate systems, respectively, are

$$\nabla \times \vec{\mathbf{F}} = \frac{1}{\rho} \begin{vmatrix} \vec{\mathbf{a}}_\rho & \rho \vec{\mathbf{a}}_\phi & \vec{\mathbf{a}}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \tag{2.100}$$

and

$$\nabla \times \vec{\mathbf{F}} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{\mathbf{a}}_r & r\vec{\mathbf{a}}_\theta & r\sin\theta\, \vec{\mathbf{a}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & r F_\theta & r\sin\theta\, F_\phi \end{vmatrix} \tag{2.101}$$

EXAMPLE 2.22

If $\vec{\mathbf{F}} = (2z+5)\vec{\mathbf{a}}_x + (3x-2)\vec{\mathbf{a}}_y + (4x-1)\vec{\mathbf{a}}_z$, verify Stokes' theorem over the hemisphere $x^2 + y^2 + z^2 = 4$ and $z \geq 0$.

Solution

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{a}}_x & \vec{\mathbf{a}}_y & \vec{\mathbf{a}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z+5 & 3x-2 & 4x-1 \end{vmatrix} = -2\vec{\mathbf{a}}_y + 3\vec{\mathbf{a}}_z$$

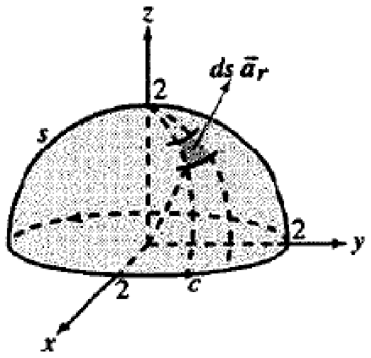


Figure 2.34

The unit normal over the surface of the hemisphere of radius 2 is $\vec{\mathbf{a}}_r$, as shown in Figure 2.34. Thus, the differential surface area is

$$\vec{ds} = 4\sin\theta\, d\theta\, d\phi\, \vec{\mathbf{a}}_r$$

Making the coordinate transformation from rectangular to spherical, the $\vec{\mathbf{a}}_r$ component of $\nabla \times \vec{\mathbf{F}}$ is

$$F_r = -2\sin\theta\sin\phi + 3\cos\theta$$

We can now evaluate the left-hand side of Stokes' theorem as

$$\int_s (\nabla \times \vec{\mathbf{F}}) \cdot \vec{ds} = -8\int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^{2\pi} \sin\phi\, d\phi + 12\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\phi = 12\pi$$

The $\vec{\mathbf{a}}_\phi$ component of $\vec{\mathbf{F}}$, using the rectangular-to-cylindrical coordinate transformation, is

$$F_\phi = -(2z + 5)\sin\phi + (3x - 2)\cos\phi$$

Substituting $z = 0$ and $x = 2\cos\phi$, we get

$$F_\phi = -5\sin\phi + 6\cos^2\phi - 2\cos\phi$$

Thus,

$$\oint_c \vec{\mathbf{F}} \cdot \vec{d\ell} = -10\int_0^{2\pi} \sin\phi\, d\phi + 12\int_0^{2\pi} \cos^2\phi\, d\phi - 4\int_0^{2\pi} \cos\phi\, d\phi = 12\pi$$

Flux of $\vec{A} = (1+r)\hat{z}$ for hemisphere, $r=1$

Also see

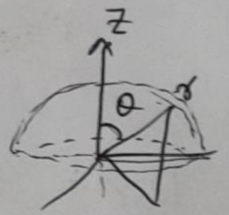
Bhagguru example 2.22 in cartesian coordinate

$r=1$

$\therefore \vec{ds} = \hat{r}\, dl_\theta\, dl_\phi = \hat{r}\, r\,d\theta\, r\sin\theta\, d\phi$

$= r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

$= \sin\theta\, d\theta\, d\phi\, \hat{r}\ (r=1)$



$\therefore \iint_S \bar{A}\cdot\bar{ds} = \iint_S (1+1)\sin\theta\, d\theta\, d\phi\ \hat{z}\cdot\hat{r}$

$z = r\cos\theta.$

$= \iint_S 2\sin\theta\cos\theta\, d\theta\, d\phi$

$\hat{z}\cdot\hat{r} = 1.1.\cos\theta = \cos\theta.$

$= \cancel{\cos 2\theta} \int_{\theta=0}^{\pi/2}\int_{0}^{2\pi} \sin 2\theta\, d\theta\, d\phi$

$= -\frac{\cos 2\theta}{2}\Big|_0^{\pi/2} \cdot 2\pi = -2\pi\cdot\frac{1}{2}(-1-1)$

$= \underline{\underline{2\pi}}$

(4) Unit vector normal to a surface $lx^2 \pm my^2 \pm nz^2 = \pm k$

let $f = lx^2 \pm my^2 \pm nz^2 \mp K$

$\therefore \hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$

(5) Source of a vector $\vec{V}$ at $(1,1,0) = \vec{\nabla}\cdot\vec{V}\,\big|_{(1,1,0)} > 0$

Sink $\Rightarrow \vec{\nabla}\cdot\vec{V} < 0$, Solenoidal if $\bar{\nabla}\cdot\bar{V} = 0$

Q. What kind of geometry the following equations (1) and (2) describe?

(1)
$$r^2 = 1$$
$$x^2 + y^2 + z^2 = 1 \rightarrow \text{Sphere}$$

(2)
$$r^2 = \operatorname{cosec}^2\theta = \frac{1}{\sin^2\theta}$$

$$r^2 \sin^2\theta = 1$$
$$r^2 \sin^2\theta \cdot 1 = 1 \Rightarrow r^2 \sin^2\theta\,(\cos^2\phi + \sin^2\phi) = 1$$
$$\Rightarrow (r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 = 1$$
$$\Rightarrow x^2 + y^2 = 1 \longrightarrow \text{Circle}$$

OR

$$r^2 (1 - \cos^2\theta) = 1$$
$$r^2 - r^2\cos^2\theta = 1$$
$$x^2 + y^2 + \cancel{z^2} - \cancel{z^2} = 1 \Rightarrow x^2 + y^2 = 1$$

Thank You