Engineering Electromagnetics

Lecture 4

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by

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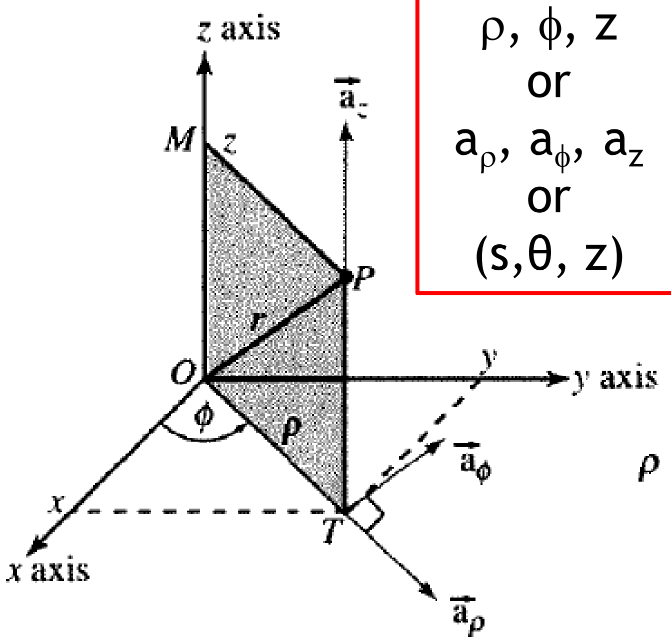
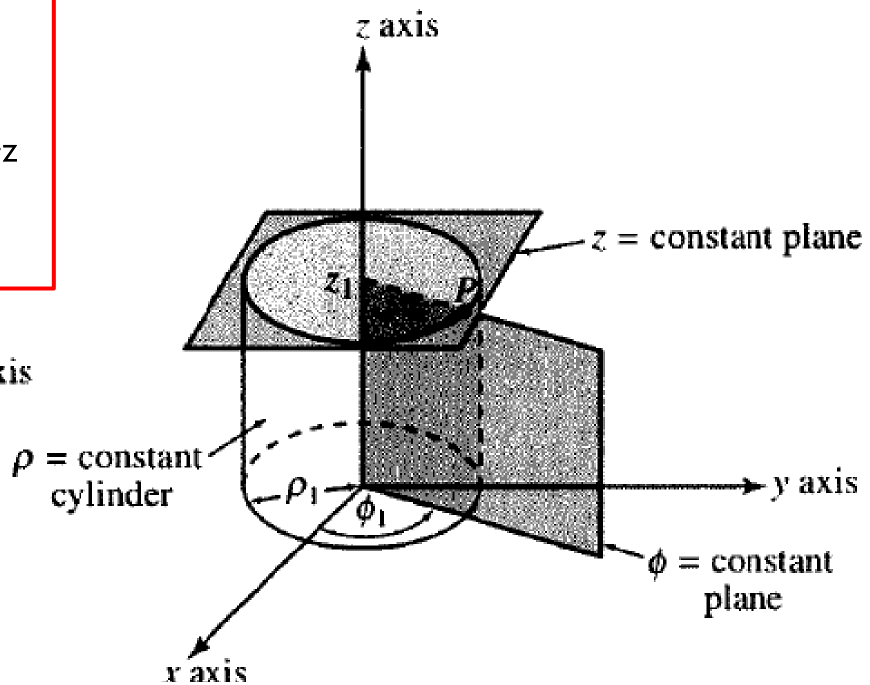
Cylindrical coordinate system

Unit vectors (same meaning, different notations)

ρ, ϕ, z
or
a_ρ, a_ϕ, a_z
or
(s,θ, z)





$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$

The coordinate surface

$$\rho = \sqrt{x^2 + y^2} = \text{constant}$$

is a cylinder of radius ρ with the z axis as its axis,

$\hat{\rho}$, $\hat{\varphi}$, and $\hat{\boldsymbol{z}}$ → unit vectors

Properties:

$\hat{\rho}.\hat{\rho} = \hat{\varphi}.\hat{\varphi} = \hat{\boldsymbol{z}}.\hat{\boldsymbol{z}} = 1$; $\hat{\rho}.\hat{\varphi} = \hat{\varphi}.\hat{\boldsymbol{z}} = \hat{\rho}.\hat{\boldsymbol{z}} = 0$

$\hat{\rho} \times \hat{\varphi} = \hat{\boldsymbol{z}}$; $\hat{\varphi} \times \hat{\boldsymbol{z}} = \hat{\rho}$; $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{\rho}} = \hat{\varphi}$

Should be defined at a common point

If two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are defined either at a common point $P(\rho, \phi, z)$ or in a ϕ = constant plane, we can add, subtract, and multiply these vectors as we did in the rectangular coordinate system. For example, if the two vectors at point $P(\rho, \phi, z)$ are $\vec{\mathbf{A}} = A_\rho \vec{\mathbf{a}}_\rho + A_\phi \vec{\mathbf{a}}_\phi + A_z \vec{\mathbf{a}}_z$ and $\vec{\mathbf{B}} = B_\rho \vec{\mathbf{a}}_\rho + B_\phi \vec{\mathbf{a}}_\phi + B_z \vec{\mathbf{a}}_z$, then

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_\rho + B_\rho)\vec{\mathbf{a}}_\rho + (A_\phi + B_\phi)\vec{\mathbf{a}}_\phi + (A_z + B_z)\vec{\mathbf{a}}_z \tag{2.32a}$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z \tag{2.32b}$$

and

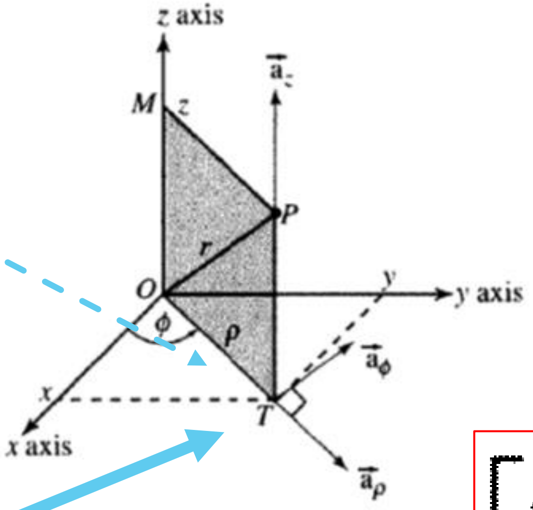
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \vec{\mathbf{a}}_\rho & \vec{\mathbf{a}}_\phi & \vec{\mathbf{a}}_z \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix} \tag{2.32c}$$

Transformations

- **Conversion from cartesian to cylindrical coordinates:**
- $\hat{x}.\hat{\rho} = \boldsymbol{Cos}\varphi$ **and** $\hat{y}.\hat{\rho} = \boldsymbol{Sin}\varphi$
- $\hat{x}.\hat{\varphi} = -\boldsymbol{Sin}\varphi$ **and** $\hat{y}.\hat{\varphi} = \boldsymbol{Cos}\varphi$

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}. \end{aligned}$$

If $\hat{\rho}(or\ \overrightarrow{a_\rho})$ makes an angle φ with x axis, **what about** $\hat{\boldsymbol{\varphi}}(\mathbf{or}\ \overrightarrow{\boldsymbol{a_\varphi}})$? And the **x and y components of** $\hat{\boldsymbol{\varphi}}$**?**



Q: For any vector A:

$$\boldsymbol{A} = \boldsymbol{A_x}\hat{\boldsymbol{x}} + \boldsymbol{A_y}\hat{\boldsymbol{y}} + \boldsymbol{A_z}\hat{\boldsymbol{z}}$$

How to convert it to cylindrical coordinates? $\boldsymbol{A} = \boldsymbol{A_\rho}\hat{\rho} + \boldsymbol{A_\varphi}\hat{\varphi} + \boldsymbol{A_z}\hat{\boldsymbol{z}}$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

Conversion cylindrical ↔ cartesian coordinates

Cartesian to cylindrical

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Cylindrical to cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

- Conversion to cartesian coordinates (Hint)
- From $A = A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{z}$ to $A = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$
- $A_x = \mathrm{A}.\,\hat{x} = (A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{z}).\,\hat{x} = A_\rho\hat{\rho}.\hat{x} + A_\varphi\hat{\varphi}.\hat{x} + A_z\hat{z}.\,\hat{x}$; $\hat{x}.\hat{\rho} = Cos\varphi$; $\hat{y}.\hat{\rho} = Sin\varphi$;
- $\hat{x}.\hat{\varphi} = -Sin\varphi$ and $\hat{y}.\hat{\varphi} = Cos\varphi$; $A_x = A_\rho\, Cos\varphi - A_\varphi Sin\varphi$; $A_y = \mathrm{A}.\ \hat{y} = A_\rho Sin\varphi + A_\varphi Cos\varphi$ and $A_z = \mathrm{A}.\hat{z}$ = A_z

Problem 1

Write an expression for a position vector at any point in space in the rectangular coordinate system. Then transform the position vector into a vector in the cylindrical coordinate system.

Solution The position vector of any point $P(x, y, z)$ in space is

$$\vec{\mathbf{A}} = x\vec{\mathbf{a}}_x + y\vec{\mathbf{a}}_y + z\vec{\mathbf{a}}_z$$

Using the transformation matrix as given in (2.39), we obtain

$$A_\rho = x\cos\phi + y\sin\phi$$

$$A_\phi = -x\sin\phi + y\cos\phi \quad \text{and} \quad A_z = z$$

Substituting $x = p\cos\phi$ and $y = \rho\sin\phi$, we obtain

$$A_\rho = \rho, \quad A_\phi = 0, \quad \text{and} \quad A_z = z$$

Thus, the position vector $\vec{\mathbf{A}}$ in the cylindrical coordinate system is

$$\vec{\mathbf{A}} = \rho\vec{\mathbf{a}}_\rho + z\vec{\mathbf{a}}_z$$

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Problem 2

Express the vector $\vec{\mathbf{A}} = \frac{k}{\rho^2}\vec{\mathbf{a}}_\rho + 5\sin 2\phi\, \vec{\mathbf{a}}_z$ in the rectangular coordinate system.

Solution Using the transformation matrix

$$A_\rho = \frac{k}{\rho^2}, \quad A_\phi = 0, \quad \text{and} \quad A_z = 5\sin 2\phi$$

we obtain

$$A_x = \frac{k\cos\phi}{\rho^2}, \quad A_y = \frac{k\sin\phi}{\rho^2}, \quad \text{and} \quad A_z = 10\cos\phi\sin\phi$$

Substituting $\rho = \sqrt{x^2+y^2}$, $\cos\phi = \frac{x}{\rho}$, and $\sin\phi = \frac{y}{\rho}$, we obtain the desired transformation of vector $\vec{\mathbf{A}}$ as

$$\vec{\mathbf{A}} = \frac{kx}{[x^2+y^2]^{3/2}}\vec{\mathbf{a}}_x + \frac{ky}{[x^2+y^2]^{3/2}}\vec{\mathbf{a}}_y + \frac{10xy}{x^2+y^2}\vec{\mathbf{a}}_z$$

...

Problem 3

If $\vec{\mathbf{A}} = 3\vec{\mathbf{a}}_\rho + 2\vec{\mathbf{a}}_\phi + 5\vec{\mathbf{a}}_z$ and $\vec{\mathbf{B}} = -2\vec{\mathbf{a}}_\rho + 3\vec{\mathbf{a}}_\phi - \vec{\mathbf{a}}_z$ are given at points $P(3, \pi/6, 5)$ and $Q(4, \pi/3, 3)$, find $\vec{\mathbf{C}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$ at point $S(2, \pi/4, 4)$.

The two vectors are not defined in the same ϕ = constant plane, so we cannot sum them directly in the cylindrical system. Conversion to the rectangular system is therefore necessary. For vector $\vec{\mathbf{A}}$ given at point $P(3, \pi/6, 5)$, the transformation matrix becomes

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$\vec{\mathbf{A}} = 1.598\vec{\mathbf{a}}_x + 3.232\vec{\mathbf{a}}_y + 5\vec{\mathbf{a}}_z$$

Similarly, with $\phi = \pi/3$, the transformed vector $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{B}} = -3.598\vec{\mathbf{a}}_x - 0.232\vec{\mathbf{a}}_y - \vec{\mathbf{a}}_z$$

$$\vec{\mathbf{C}} = -2\vec{\mathbf{a}}_x + 3\vec{\mathbf{a}}_y + 4\vec{\mathbf{a}}_z$$

Vector $\vec{\mathbf{C}}$ can now be transformed into its components at point $S(2, \pi/4, 4)$ in the cylindrical system by making use of the transformation matrix given in (2.39). That is

$$\begin{bmatrix} C_\rho \\ C_\phi \\ C_z \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

Thus, $\vec{\mathbf{C}} = 0.707\vec{\mathbf{a}}_\rho + 3.535\vec{\mathbf{a}}_\phi + 4\vec{\mathbf{a}}_z$ • • •

Note that the transformation of a vector from one coordinate system to another neither changes its magnitude nor its direction.

Thank You