Systems of Linear Equations

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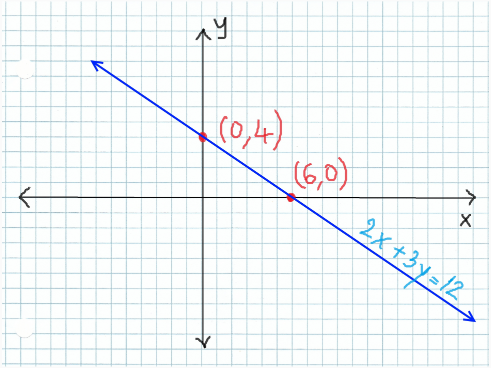
Linear Equations

Equation of a line :

$$y = mx + c$$

$$ax + by = c$$

Example of a line

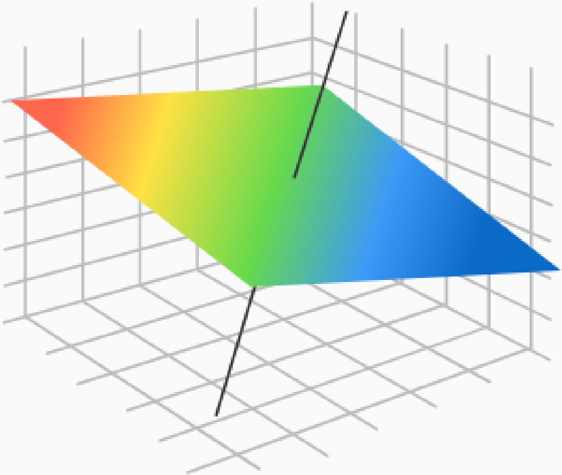


Equation of a plane :

$$z = ax + by + c$$

$$Ax + By + Cz = D$$

Example of a plane



Systems of linear equations

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n &= y_1 \\
A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n &= y_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\
A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n &= y_m
\end{aligned}
$$

- m equations
- n varaibles $(x_1, x_2, \ldots, x_n)$
- $A_{ij}, y_i \in F \quad$ (F is a field)

Matrix form

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_m \end{bmatrix}$$

$$AX = Y$$

$A = [A_{ij}]_{m \times n}$, $1 \leq i \leq m$, $1 \leq j \leq n$
$X = [x_j]_{n \times 1}$ $\qquad Y = [y_i]_{m \times 1}$
The solution set of the linear system $AX = Y$ is

$$S = \{X \in R^n \ : \ AX = Y\}$$

Problem 1

Solve the following system of linear equations

$$4x - y = 5$$

$$2x + y = 7$$

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$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

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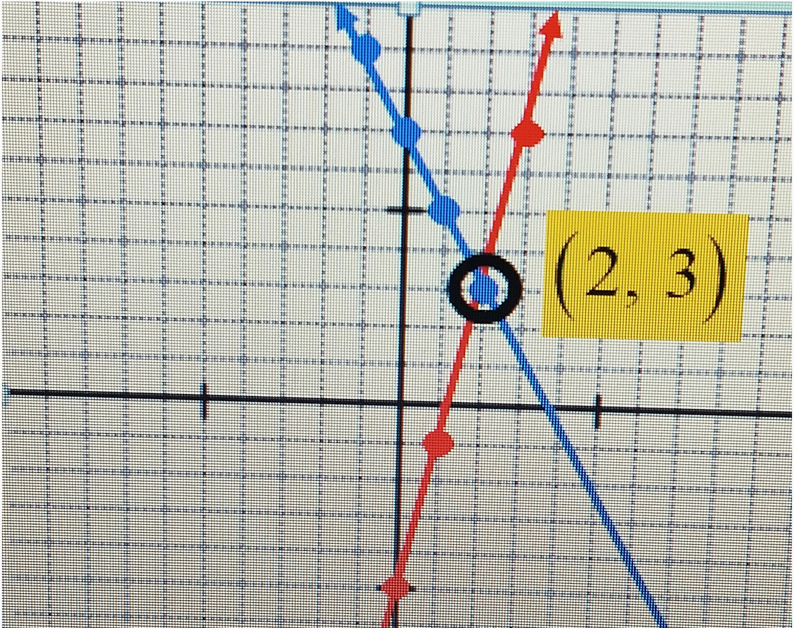
Solve graphically (sketch it!)

Problem 1 (solution)



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Matrix form

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Solve graphically (sketch it!)

Solution $S = \{(2, 3)\}$

Our objective is to design an efficient machinery to solve $AX = B$

Solution of Problem 1 through elementary row operations

$$4x - y = 5 \quad (Eq1)$$

$$2x + y = 7 \quad (Eq2)$$

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(Eq2)-(Eq1) $\Longrightarrow$

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We have three equivalent systems say red, blue and green

contd.

Let us express red system in augmented matrix form

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$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right]$$

contd.

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Multiply second row by 2 $\quad (R_2 \longleftarrow 2 \times R_2)$

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Substract first row from second row $\qquad (R_2 \longleftarrow R_2 - R_1)$

contd.

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$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & 3 \end{array}\right]$$

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$R_1 \longleftarrow R_1 + \frac{1}{4}R_2 \qquad \Longrightarrow$

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Let us write it in the matrix form

$$x + 0y = 2$$

$$0x + y = 3$$

Salient points

- Multiplying an equation by a non-zero scalar preserves the solution space ($R_i \longleftarrow cR_i, \quad c \neq 0$)

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- Multiplying an equation by a non-zero scalar preserves the solution space ($R_i \longleftarrow cR_i, \quad c \neq 0$)
- Replacing i^{th} equation by sum of i^{th} equation and constant multiple of j^{th} equation preserves the solution space. ($R_i \longleftarrow R_i + cR_j$)
- Interchanging two equations preserves the solution space. ($R_i \longleftrightarrow R_j$)

Problem 2

Solve the following system of linear equations.

$$3x - 2y = -6, \quad x + 2y = -10$$

Solve graphically !

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Interchange first and second rows ($R_1 \longleftrightarrow R_2$)

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$$\sim \left[\begin{array}{cc|c} 1 & 2 & -10 \\ 3 & -2 & -6 \end{array}\right]$$

Problem 2 contd.

$$R_2 \longleftarrow R_2 - 3R_1 \implies$$

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Solution $S = \{(-4, -3)\}$

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Second row is $0x + 0y = 1$. No solution **i.e.,** $S = \{\ \}$

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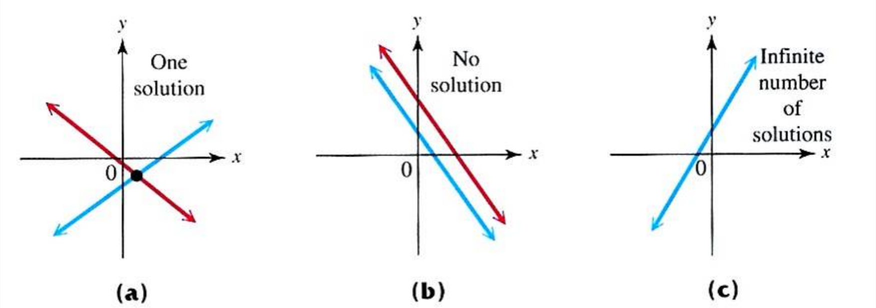
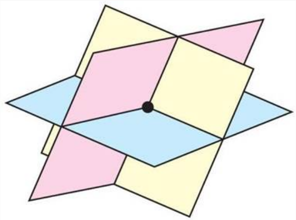
$$\sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \end{array}\right] \quad \Longrightarrow \quad x + 2y = 5$$

The solution set: $S = \{(5 - 2c, c) \ : \ c \in \mathrm{R}\}$

two dimensional problem and possible solutions



3-dimensional problem with a unique solution



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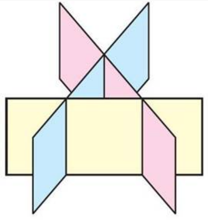
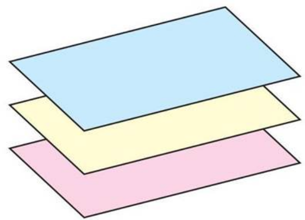
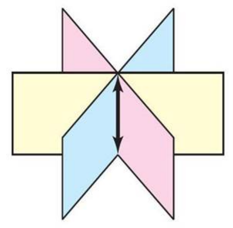
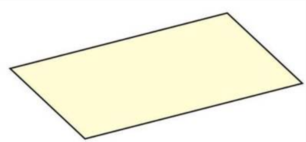




3-dimensional problem with infinite number of solutions





Linear combination of equations

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = y_1 \qquad (1)$$

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$$c_1\left(A_{11}x_1 + A_{12}x_2 + A_{13}x_3\right) + c_2\left(A_{21}x_1 + A_{22}x_2 + A_{23}x_3\right)$$

$$= c_1 y_1 + c_2 y_2 \qquad (3)$$

Suppose that $x_1 = a, x_2 = b, x_3 = c$ is solution of (1) and (2)

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Show that above solution is also a solution of (3).

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Converse need not be true (Try !)

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Show that above solution is also a solution of (3).

Consider L.H.S. of (3),

$$c_1 \left(A_{11} a + A_{12} b + A_{13} c \right) + c_2 \left(A_{21} a + A_{22} b + A_{23} c \right)$$

$$= c_1 y_1 + c_2 y_2$$

So $x_1 = a, x_2 = b, x_3 = c$ is solution of (3)
Converse need not be true (Try !)
Note: If X^* is a solution of k linear equations, then X^* is also a solution of a linear combination of those k equations.

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Why do we focus on equivalent systems?

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Problem

Show that the following systems of linear equations are equivalent.

$$\left.\begin{aligned} x - y &= 0 \\ 2x + y &= 0 \end{aligned}\right\} \quad ------ \quad (I)$$

$$\left.\begin{aligned} 3x + y &= 0 \\ x + y &= 0 \end{aligned}\right\} \quad ------ \quad (II)$$

Solution

$$3x + y = \frac{1}{3}(x - y) + \frac{4}{3}(2x + y)$$

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$$x + y = -\frac{1}{3}(x - y) + \frac{2}{3}(2x + y)$$

Solution

$$
\begin{aligned}
3x + y &= \tfrac{1}{3}(x - y) + \tfrac{4}{3}(2x + y) \\
x + y &= -\tfrac{1}{3}(x - y) + \tfrac{2}{3}(2x + y) \\
x - y &= (3x + y) - 2(x + y)
\end{aligned}
$$

Solution

$$\begin{gathered}
3x + y = \tfrac{1}{3}(x - y) + \tfrac{4}{3}(2x + y) \\
x + y = -\tfrac{1}{3}(x - y) + \tfrac{2}{3}(2x + y) \\
x - y = (3x + y) - 2(x + y) \\
2x + y = \tfrac{1}{2}(3x + y) + \tfrac{1}{2}(x + y)
\end{gathered}$$

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x - y = (3x + y) - 2(x + y) \\
2x + y = \tfrac{1}{2}(3x + y) + \tfrac{1}{2}(x + y)
\end{gathered}
$$

Note : $AX = 0$ is called a homogeneous system.