Tutorial on Laplace Transforms

1) Change of scale property

If $L^{-1}\{F(P)\} = f(x)$, then $L^{-1}\{F(ap)\} = \frac{1}{a} f(x/a)$, $a > 0$

Proof. Since $F(P) = \int_0^\infty e^{-Px} f(x)\,dx$, we have

$$F(ap) = \int_0^\infty e^{-apx} f(x)\,dx \quad ; \quad \text{let } ax = t$$

$$\Rightarrow a\,dx = dt \quad \text{or} \quad dx = \frac{dt}{a} \quad \text{and } x = \frac{t}{a}$$

then, $F(ap) = \int_0^\infty e^{-pt} f(t/a) \cdot \frac{dt}{a} = \int_0^\infty e^{-px}\left[\frac{1}{a} f(t/a)\right] dt$

$$\Rightarrow F(ap) = L\left\{\frac{1}{a} f(x/a)\right\} \quad \text{or} \quad L^{-1}\{F(ap)\} = \frac{1}{a} f(x/a)$$

Use this property

to find $L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\}$

$a = 3$ $\quad 3P \leftarrow P$

First find $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\}$

$L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\}$. let $F(P) = \frac{p^2-1}{(p^2+1)^2}$

But $F(P) = -G'(P)$, where $G(P) = \frac{p}{p^2+1}$

$$L^{-1}\{G(P)\} = \cos x \Rightarrow L^{-1}\{\underbrace{-G'(P)}_{F(P)}\} = x\cos x \text{ (why?)}$$

$L[f(x)] = F(P) \Rightarrow L[-xf(x)] = F'(P)$ / Final Answer ??

2.) Prove that $\mathcal{L}^{-1}\left\{\dfrac{e^{-1/p}}{\sqrt{p}}\right\} = \dfrac{\cos 2\sqrt{x}}{\sqrt{\pi x}}$

$e^x = ?$

Since
$$\frac{e^{-1/p}}{\sqrt{p}} = \frac{1}{\sqrt{p}}\left[1 - \frac{1}{p} + \frac{1}{2!\,p^2} - \frac{1}{3!\,p^3} + \cdots\right]$$

$$= \frac{1}{p^{1/2}} - \frac{1}{p^{3/2}} + \frac{1}{2!\,p^{5/2}} - \frac{1}{3!\,p^{7/2}} + \cdots\cdots$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-1/p}}{p}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{p^{1/2}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{p^{3/2}}\right\} + \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{1}{p^{5/2}}\right\}$$
$$- \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{1}{p^{7/2}}\right\} - - - - - - - - - -$$

$$\mathcal{L}\{x^n\} = \frac{\Gamma(n+1)}{p^{n+1}}\ ;\ \Gamma(n+1) = n! = n\cdot(n-1)! = n\cdot\Gamma(n)$$

$$\Gamma(x) = \int_0^\infty t^{x-1}\cdot e^{-t}\,dt \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi}$$

Now
$$\mathcal{L}^{-1}\left\{\frac{e^{-1/p}}{p}\right\} = \frac{1}{\Gamma(1/2)}\mathcal{L}^{-1}\left\{\frac{\Gamma(1/2)}{p^{1/2}}\right\} - \frac{1}{\Gamma(3/2)}\mathcal{L}^{-1}\left\{\frac{\Gamma(3/2)}{p^{3/2}}\right\}$$
$$+ \frac{1}{2!\,\Gamma(5/2)}\mathcal{L}^{-1}\left\{\frac{\Gamma(5/2)}{p^{5/2}}\right\} - \frac{1}{3!\,\Gamma(7/2)}\mathcal{L}^{-1}\left\{\frac{\Gamma(7/2)}{p^{7/2}}\right\}\cdots$$

$$= \frac{1}{\sqrt{\pi}}\cdot x^{-1/2} - \frac{1}{\frac{1}{2}\cdot\sqrt{\pi}}x^{1/2} + \frac{1}{2\cdot\frac{3}{2}\cdot\frac{1}{2}}\cdot x^{3/2} - \frac{1}{3\,2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}\cdot x^{5/2}\cdots$$

$$= \frac{1}{\sqrt{\pi x}}\left\{1 - \frac{(2\sqrt{x})^2}{2!} + \frac{(2\sqrt{x})^4}{4!} - \frac{(2\sqrt{x})^6}{6!} + \cdots\right\} = \frac{\cos 2\sqrt{x}}{\sqrt{\pi x}}$$

Hence Proved.

then $$\mathcal{L}^{-1}\left\{\frac{e^{-1/k\cdot p}}{\sqrt{k\cdot p}}\right\} = ?$$

3) $$\mathcal{L}^{-1}\left\{\frac{1}{p^2-6p+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(p-3)^2+1}\right\}$$

$$= \underline{e^{+3x}\mathcal{L}^{-1}\left\{\frac{1}{p^2+1}\right\}} \,(?) = e^{3x}\cdot\sin x$$

4) $$\mathcal{L}^{-1}\left\{\frac{p}{(p+3)^{7/2}}\right\} = \mathcal{L}^{-1}\left\{\frac{p+3-3}{(p+3)^{7/2}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(p+3)^{5/2}}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{(p+3)^{7/2}}\right\}$$

$$= e^{-3x}\mathcal{L}^{-1}\left\{\frac{1}{p^{5/2}}\right\} - 3\cdot e^{-3x}\mathcal{L}^{-1}\left\{\frac{1}{p^{7/2}}\right\}$$

$$= \frac{e^{-3x}}{\Gamma(5/2)}\cdot\mathcal{L}^{-1}\left\{\frac{\Gamma(5/2)}{p^{5/2}}\right\} - \frac{3\cdot e^{-3x}}{\Gamma(7/2)}\cdot\mathcal{L}^{-1}\left\{\frac{\Gamma(7/2)}{p^{7/2}}\right\}$$

$$= \frac{4\cdot x^{3/2}}{\sqrt{\pi}\cdot 15}\cdot(5-3x) \quad (\text{solve})$$

5) $$\mathcal{L}^{-1}\left\{\frac{6p-4}{p^2-4p+20}\right\} = \mathcal{L}^{-1}\left\{\frac{6(p-2)+2\cdot 4}{(p-2)^2+16}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{p-2}{(p-2)^2+(4)^2}\right\} + 2\cdot\mathcal{L}^{-1}\left\{\frac{4}{(p-2)^2+(4)^2}\right\} = 2e^{2x}(3\cos 4x + \sin 4x)$$

6.) $\mathcal{L}^{-1}\left\{\dfrac{4p+12}{p^2+8p+16}\right\} = \mathcal{L}^{-1}\left\{\dfrac{4(p+4)-4}{(p+4)^2}\right\} = ?$

$= 4e^{-4x}(1-x)$. (Check ??)

7.) $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{2p+3}}\right\} = \dfrac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{p+3/2}}\right\}$

$= \dfrac{1}{\sqrt{2}} \cdot e^{-3/2\,x} \cdot \mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{p}}\right\} = \dfrac{1}{\sqrt{2}}\dfrac{e^{-3/2\,x}}{\Gamma(1/2)}\mathcal{L}^{-1}\left\{\dfrac{\Gamma(1/2)}{p^{1/2}}\right\}$

$= \dfrac{e^{-3/2\,x} \cdot x^{-1/2}}{\sqrt{2\pi}}$

8.) $\mathcal{L}^{-1}\left\{\dfrac{p}{(p+1)^5}\right\} = e^{-x}\dfrac{x^3}{3!} - e^{-x}\dfrac{x^4}{4!}$ (Check)

9.) Given $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{p^2+a^2}}\right\} = J_0(ax)$

Bessel function | first kind order zero

then prove that $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{p^2-4p+20}}\right\} = e^{2x} \cdot J_0(4x)$

$$\boxed{J_0'(x) = -J_1(x)}$$

ADDITIONAL

Diff $\mathcal{L}\{J_0(ax)\}$ w.r.t 'a'

$\Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{(p^2+a^2)^{3/2}}\right\} = -\dfrac{x}{a}J_0'(ax) = \dfrac{x}{a}J_1(ax)$

NOT COMPULSORY – LEAVE IT IF CONFUSING !!

Use above formula

$$\mathcal{L}^{-1}\left\{\frac{1}{(p^2+2p+5)^{3/2}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{((p+1)^2+4)^{3/2}}\right\}$$

$$= e^{-x}\cdot \mathcal{L}^{-1}\left\{\frac{1}{(p^2+4)^{3/2}}\right\} = e^{-x}\cdot\frac{x}{2}\cdot J_1(2x)$$

10.) $\mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}$

let $F(p) = \log\left(1+\frac{1}{p^2}\right)$ and $\mathcal{L}^{-1}\{F(p)\} = f(x)$

$$= -\log\left(\frac{p^2}{p^2+1}\right) = -2\log p + \log(p^2+1)$$

$$\Rightarrow F'(p) = -2\cdot\left(\frac{1}{p} - \frac{p}{p^2+1}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\{F'(p)\} = -2\,\mathcal{L}^{-1}\left\{\frac{1}{p} - \frac{p}{p^2+1}\right\}$$

$$\Rightarrow \quad -x\cdot f(x) = -2(1-\cos x)$$

$$\Rightarrow \quad f(x) = \frac{2(1-\cos x)}{x}$$

$$\therefore \mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\} = \frac{2(1-\cos x)}{x}$$

10. b) Similarly prove

SAME METHOD

$$\mathcal{L}^{-1}\left\{\log\left(\frac{p^2+a^2}{p^2+b^2}\right)\right\} = \frac{2(\cos bx - \cos ax)}{x}$$

11.) $\mathcal{L}^{-1}\left\{\cot^{-1}(1+P)\right\} = f(x)$ (let)

Let $F(P) = \cot^{-1}(1+P)$

$$F'(P) = \frac{-1}{1+(P+1)^2} \Rightarrow \mathcal{L}^{-1}\left\{F'(P)\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{1+(P+1)^2}\right\}$$

$$\Rightarrow -x \cdot f(x) = \mathcal{L}^{-1}\left\{F'(P)\right\} = -e^{-x} \cdot \mathcal{L}^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$\Rightarrow f(x) = \frac{e^{-x} \cdot \sin x}{x}$$

12.) Let $L^{-1}\left\{F(P)\right\} = f(x)$; then prove that

$$\mathcal{L}^{-1}\left\{\frac{F(P)}{P^2}\right\} = \int_0^x \int_0^v f(u)\, du\, dv$$

Proof: let $g(x) = \int_0^x \int_0^v f(u)\, du\, dv$ fundamental theorem of Calculus

$\therefore \quad g'(x) = \int_0^x f(u)\, du$ and $g''(x) = f(x)$

Also $g(0) = g'(0) = 0$.

But $\mathcal{L}\left\{g''(x)\right\} = p^2 \mathcal{L}\left\{g(x)\right\} - P\, g(0) - g'(0)$

$$\Rightarrow \mathcal{L}\left\{g''(x)\right\} = p^2 \mathcal{L}\left\{g(x)\right\} \Rightarrow \mathcal{L}\left\{f(x)\right\} = p^2 \cdot \mathcal{L}\left\{g(x)\right\}$$

$$\Rightarrow F(P) = p^2 \cdot \mathcal{L}\left\{g(x)\right\} \Rightarrow \mathcal{L}\left\{g(x)\right\} = \frac{F(P)}{p^2}$$

$$\Rightarrow \int_0^x \int_0^v f(u)\, du\, dv = \mathcal{L}^{-1}\left\{\frac{F(P)}{p^2}\right\}$$ Hence Proved

In general $\mathcal{L}^{-1}\left\{\frac{F(p)}{p^n}\right\} = \underbrace{\int_0^x\int_0^{u_1}\int_0^{u_2}\cdots\int_0^{u_{n-1}}}_{n\text{-integrals}} f(u_n)\,du_1\,du_2\cdots du_n$

13.) Evaluate $\mathcal{L}^{-1}\left\{\frac{a^2}{p(p+a)^2}\right\}$

since $\mathcal{L}^{-1}\left\{\frac{a^2}{(p+a)^2}\right\} = a^2 \cdot e^{-ax}\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\} = a^2 \cdot e^{-ax} \cdot x$

same

or $\mathcal{L}^{-1}\left\{\frac{a^2}{(p+a)^2}\right\} = -a^2\mathcal{L}^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p+a}\right)\right\} = -a^2(-1)\cdot x \cdot e^{-ax}$

$\mathcal{L}^{-1}(F(p)) = f(x) = \underline{a^2 \cdot x \cdot e^{-ax}}$

$\mathcal{L}^{-1}\left\{\frac{F(p)}{p}\right\}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{p}\cdot\frac{a^2}{(p+a)^2}\right\} = a^2\int_0^x t\cdot e^{-at}\,dt$

$\int_0^x f(t)\,dt$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{p}\cdot\frac{a^2}{(p+a)^2}\right\} = a^2\int_0^x t\cdot e^{-at}\,dt$

$= a^2\left[t\,\frac{e^{-at}}{a} - \frac{e^{-at}}{a^2}\right]_0^x = 1 - e^{-ax}(ax+1)$

14.) $\mathcal{L}^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\}$

Since $\mathcal{L}^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin x$

$\rightarrow \mathcal{L}^{-1}\left\{\frac{1}{p(p^2+1)}\right\} = \int_0^x \sin t\,dt = 1-\cos x$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\} = \int_0^x (1-\cos t)\,dt = x - \sin x$$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\}} = \int_0^x t - \sin t \, dt = \boxed{\frac{1}{2}x^2 + \cos x - 1}$$

$$\Big\updownarrow \mathcal{L}$$

$$\frac{1}{p^3(p^2+1)} = \frac{p^2+1+p^4-p^4-p^2}{p^3(p^2+1)} = \frac{1}{2}\cdot\frac{2}{p^3} + \frac{p}{p^2+1} - \frac{1}{p}$$

15.) CONVOLUTION

$$f*g(x) = \int_0^x f(t)\,g(x-t)\,dt = \int_0^x f(x-t)\,g(t)\,dt$$

$$\underline{\mathcal{L}\{f*g(x)\} = \mathcal{L}\{f(x)\}\cdot\mathcal{L}\{g(x)\}}$$

CONVOLUTION THEOREM FOR LAPLACE TRANSFORMS

PROVE IT !!

16) Evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$

17) $$\mathcal{L}^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{p^2}\cdot\frac{p}{(p^2+4)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\} = x = g(x) \ ?$$

$$\mathcal{L}^{-1}\left\{\frac{p}{(p^2+4)^2}\right\} = \frac{x}{4}\sin 2x = f(x) \ ?$$ (check it/Prove)

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\} = \int_0^x \frac{t\sin 2t}{4}\,(x-t)\,dt$$

$$= \frac{x}{4}\int_0^x t\sin 2t \, dt - \frac{1}{4}\int_0^x t^2 \sin 2t \, dt$$

$$= \frac{x}{4}\left[-\frac{t\cos 2t}{2} + \frac{1}{4}\sin 2t\right]_0^x - \frac{1}{4}\left[-\frac{t^2}{2}\cos 2t + \frac{t\sin 2t}{2} + \frac{\cos 2t}{4}\right]_0^x$$

$$= \frac{x}{4}\left[-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right] - \frac{1}{4}\left[-\frac{x^2}{2}\cos 2x + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} - \frac{1}{4}\right]$$

$$= \frac{1}{16}\left[1 - \cos 2x - x\sin 2x\right]$$

17 Similarly use convolution theorem for L.T to find

(i) $\mathcal{L}^{-1}\left\{\frac{p^2}{(p^2+a^2)^2}\right\}$ $\quad \frac{p}{p^2+a^2} \quad \frac{p}{p^2+a^2}$ Ans: $\frac{1}{2}\left[x\cdot\cos ax + \frac{1}{a}\sin ax\right]$

(ii) $\mathcal{L}^{-1}\left\{\frac{p}{(p^2+a^2)^3}\right\}$ Ans: $\frac{x}{8a^3}\left[\sin ax - ax\cos ax\right]$

Hint: $\frac{p}{(p^2+a^2)^2} \cdot \frac{1}{(p^2+a^2)}$

$\downarrow \mathcal{L}^{-1}$ $\qquad g(x) \downarrow \mathcal{L}^{-1}$

$f(x) \to \frac{x\cdot\sin ax}{2a}$ $\qquad \to \frac{1}{a}\sin ax$

Perform $\int_0^x f(t)\, g(x-t)\, dt$

18.) $\mathcal{L}^{-1}\left\{\frac{5p^2-15p-11}{(p+1)(p-2)^3}\right\}$ use partial fractions

$$\frac{5p^2-15p-11}{\underbrace{(p+1)(p-2)^3}_{F(p)}} = \frac{A}{(p+1)} + \frac{B}{(p-2)^3} + \frac{C}{(p-2)^2} + \frac{D}{(p-2)} \quad -(1)$$

$A = \lim_{p\to -1} (p+1)\cdot F(p)$; $B = \lim_{p\to 2} (p-2)^3 \cdot F(p)$

$A = -1/3$; $B = -7$

To find C and D ; let $p=0$ and $p=1$ in (1)

we get $C=4$; $D = 1/3$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{5p^2-15p-11}{(p+1)(p-2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/3}{(p+1)} + \frac{-7}{(p-2)^3} + \frac{4}{(p-2)^2} + \frac{(1/3)}{p-2}\right\}$$

$$= \frac{-1}{3}e^{-x} - \frac{7}{2}x^2\cdot e^{2x} + 4x\cdot e^{2x} + \frac{1}{3}e^{2x} \quad \text{Ans:}$$

Similarly use partial fractions to find

19.) $$\mathcal{L}^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\} = \mathcal{L}^{-1}\{\ \ \}$$ simplify using partial fractions

$$\frac{A}{p-1} + \frac{Bp+C}{p^2+1} \Rightarrow A=2;\ B=-2;\ C=1$$

Ans: $2e^x - 2\cos x + \sin x$

20.) $\mathcal{L}^{-1}\left\{\frac{p^2-4}{(p^2+1)(p^2+4)^2}\right\}$; Ans: $\frac{5}{9}\sin x - \frac{1}{8}x\cos 2x + \frac{49}{144}\sin 2x$

Hint : may use first $p^2 = y$ then $\rightarrow$ easy to solve partial fractions

21.) Solve $\frac{d^2y}{dx^2} + a^2 y = f(x)$ $y(0)=1$; $y'(0) = -2$

Apply L.T

$$p^2 \cdot \mathcal{L}[y] - p\, y(0) - y'(0) + a^2 \cdot \mathcal{L}[y] = \mathcal{L}\{f(x)\}$$

Let $\mathcal{L}[y] = y_p$ and $\mathcal{L}[f(x)] = F(p)$

$$\Rightarrow p^2 \cdot y_p - p + 2 + a^2 y_p = F(p)$$

$$\Rightarrow (p^2 + a^2)\, y_p - p + 2 = F(p)$$

$$\Rightarrow y_p = \frac{p}{p^2+a^2} - \frac{2}{p^2+a^2} + \frac{F(p)}{p^2+a^2}$$

$$\Rightarrow y(x) = \mathcal{L}^{-1}\left\{\frac{p}{p^2+a^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{p^2+a^2}\right\} +$$

$$+ \mathcal{L}^{-1}\left\{F(p) \cdot \frac{1}{p^2+a^2}\right\}$$ ← Apply convolution theorm

$$= \underbrace{\cos ax - \frac{2}{a}\sin ax}_{\text{general solution}} + \underbrace{\frac{1}{a}\int_0^x f(t)\, \sin a(x-t)\, dt}_{y_p(x)}$$

general solution
NATURAL STATE OF A HARMONIC SYSTEM with out Source term $R(x)$ or $f(x)$

$y_p(x)$ ↓ Convolution of force & Natural state term