Elementary row operations

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The i^{th} row of A is $R_i = [A_{i1}, A_{i2}, \ldots, A_{in}]$

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Type 2: Replacement of i^{th} row by row i plus c times of row j **where** $c \in F$ $\quad (e \ : \ R_i \longleftarrow R_i + cR_j)$

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Proof :

Type 1	*Inverse of the Type* 1
$e : R_i \longleftarrow cR_i,\ c \neq 0$	$e_1 : R_i \longleftarrow \frac{1}{c} R_i$

Proof of Theorem 2 contd.

Type 2	*Inverse of the Type* 2
$e : R_i \longleftarrow R_i + cR_j$	$e_1 : R_i \longleftarrow R_i - cR_j$

Proof of Theorem 2 contd.

Type 2	*Inverse of the Type* 2
$e : R_i \longleftarrow R_i + cR_j$	$e_1 : R_i \longleftarrow R_i - cR_j$

Type 3	*Inverse of the Type* 3
$e : R_i \longleftrightarrow R_j$	$e_1 : R_i \longleftrightarrow R_j$

Proof of Theorem 2 contd.

Type 2	*Inverse of the Type* 2
$e : R_i \longleftarrow R_i + cR_j$	$e_1 : R_i \longleftarrow R_i - cR_j$

Type 3	*Inverse of the Type* 3
$e : R_i \longleftrightarrow R_j$	$e_1 : R_i \longleftrightarrow R_j$

Note that for an $m \times n$ matrix A, $e(e_1(A)) = A = e_1(e(A))$

Note

$e_1 \ : \ R_2 \longleftarrow 2R_2$ and $e_2 \ : \ R_2 \longleftarrow R_2 - R_1$

$$A = \begin{bmatrix} 4 & -1 & 5 \\ 2 & 1 & 7 \end{bmatrix}$$

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$$e_2(e_1(A)) = B$$

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$$e_1^{-1}(e_2^{-1}(B)) = A$$

A and B are called row-equivalent matrices.

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Definition : If A and B are $m \times n$ matrices over the field F, we say B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

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Definition : **If** A **and** B **are** $m \times n$ **matrices over the field** F**, we say** B **is row-equivalent to** A **if** B **can be obtained from** A **by a finite sequence of elementary row operations.**

Qn. Let $\mathrm{M}_{m\times n}$ **be the set of all** $m \times n$ **matrices defined on a field** F**. For** $A, B \in \mathrm{M}_{m\times n}$**, we say** $A \sim B$ **if** B **is row - equivalent to** A**. Show that** $\sim$ **is an equivalence relation defined on** $\mathrm{M}_{m\times n}$

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If A and B are row-equivalent $m \times n$ matrices, the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have exactly same solutions.

Proof: Suppose that we pass A to B by a finite sequence of elementary row operations :

$$A = A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \ldots \longrightarrow A_k = B$$

Note: If

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then $AX = 0$ and $BX = 0$ have same solutions.

Proof of Theorem 3 contd.

It is enough to prove that $A_j X = 0$ and $A_{j+1} X = 0$ have exactly the same solutions(that is one elementary row operation doesn't disturb the set of solutions).

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Problem 1

Show that the following systems are equivalent.

$AX = 0$	$BX = 0$
$2x_1 - x_2 + 3x_3 + 2x_4 = 0$	$x_3 - \frac{11}{3}x_4 = 0$
$x_1 + 4x_2 - x_4 = 0$	$x_1 + \frac{17}{3}x_4 = 0$
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Solution at page number 8(Hoffman and Kunz)

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Note that solving the second system is easy !

Note

Let us consider the matrix B from the previous problem.

$$B = \begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \end{bmatrix}$$

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Row-reduced matrix

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad , \quad \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 1

Find all solutions of the following system of equations by row-reducing the coefficient matrix.

$$\begin{aligned}
\tfrac{1}{3}x_1 + 2x_2 - 6x_3 &= 0 \\
-4x_1 \qquad\quad + 5x_3 &= 0 \\
-3x_1 + 6x_2 - 13x_3 &= 0 \\
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Solution: The coefficient matrix of the system is

$$\begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix}$$

Problem 1 contd.

$R_1 \longleftarrow 3R_1$, $R_4 \longleftarrow 3R_4$

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Thus

$$x_1 + 0x_2 - \frac{5}{4}x_3 = 0$$
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Note that
(i) x_3 **is a called free variable and**
(ii) x_1, x_2 **are called pivot variables.**

Theorem 4

Every $m \times n$ matrix over the field F is row-equivalent to a row-reduced matrix.

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Proof : (Assignment)

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Find all solutions of the systems of linear equations $AX = 2X$ and $AX = 3X$ where

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

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Solution : (i) The system $AX = 2X$ **is**

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$$\implies \left. \begin{aligned} 6x - 4y &= 2x \\ 4x - 2y &= 2y \\ -x + 3z &= 2z \end{aligned} \right\}$$

Problem 2 contd.

$$\Longrightarrow \qquad \left.\begin{aligned} 4x - 4y &= 0 \\ 4x - 4y &= 0 \\ -x + z &= 0 \end{aligned}\right\}$$

$$\implies \left.\begin{array}{r} 4x - 4y = 0 \\ 4x - 4y = 0 \\ -x + z = 0 \end{array}\right\}$$

The coefficient matrix is

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 4 & -4 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

Problem 2 contd.

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$$S = \left\{X \;\in \mathrm{R}^3 \;:\; AX = 2X\right\} = \{(a, a, a) \;:\; a \in \mathrm{R}\}$$

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(ii) Find all solutions of $AX = 3X$

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Note

$$A = \begin{bmatrix} 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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1. A is a row-reduced matrix.

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1. A is a row-reduced matrix.
2. All non-zero rows are above zero rows.
3. The k_i denotes the column which contains leading one(if exists) of R_i(row i).
   $k_1 = 2$, $k_2 = 4$,and $k_3 = 5$.
   Note that $k_1 < k_2 < k_3$

Row-reduced echelon matrix

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blue zeros forms a staircase (echelon) from right to left

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(c) if rows $1, 2, \ldots, r$ are the non-zero rows of R, and if the leading non-zero entry of row i occurs in column k_i, $i = 1, 2, \ldots, r$, then $k_1 < k_2 < \ldots < k_r$.

B is a row-reduced matrix, but not a row-reduced echelon matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Problem

1. Prove or disprove that there is only one 3×3 row-reduced echelon matrix with 3 non-zero rows.

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Theorem 5

Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

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Assignment

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Set the free variables as :

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Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c) \ :\ a, b, c \in \mathrm{R}\right\}$

Note 1 contd. (a visit to future !)

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$$S = \left\{ a(1,0,0,0,0) + b(0,3,1,0,0) + c(0,-\frac{1}{2},0,-2,1) \ : \ a, b, c \in \mathrm{R} \right\}$$

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$$= \text{Span of} \quad \left\{ (1,0,0,0,0), (0,3,1,0,0), (0,-\frac{1}{2},0,-2,1) \right\}$$

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Dimension of $S = dim\ S = 3 = n - r$ **(Information for future)**

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Consider an $m \times n$ row-reduced echelon matrix R with r non-zero rows.

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Note 2 contd.

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0$$

$$x_{k_2} + \sum_{j=1}^{n-r} C_{2j} u_j = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0$$

Note 2 contd.

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$$\cdots\cdots\cdots\cdots\cdots\cdots$$
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All the solutions of the system of equations $RX = 0$ are obtained by assigning any value whatsoever to $u_1, u_2, \ldots, u_{n-r}$, and then computing the corresponding values of $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$.

Remarks (Note 2 contd.)

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(ii) **If $n - r = 0$, then the system $RX = 0$ has no free variable and thus it has only trivial solution**

Theorem 6

If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

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Proof: Let R be a row-reduced echelon matrix which is row-equivalent to A.

Theorem 6

If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

Proof: Let R be a row-reduced echelon matrix which is row-equivalent to A. Then the systems $AX = 0$ and $RX = 0$ have same solutions by Theorem 3.

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If R is an $n \times n$ (square) row-reduced echelon matrix with n non-zero rows,

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Hint : (i) Every row has a leading one and (ii)
$k_1 = 1 < k_2 = 2 < \ldots < k_n = n$

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Case 1. **Suppose that A is row-equivalent to the $n \times n$ identity matrix I.** By Theorem 3, $AX = 0$ and $IX = 0$ have the same solution set. Thus the solution set of $AX = 0$ is

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Hence the system $AX = 0$ has only the trivial solution.

Proof of Theorem 7 contd.

Case 2. **Suppose that the system $AX = 0$ has only the trivial solution.**

Proof of Theorem 7 contd.

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Proof of Theorem 7 contd.

Case 2. **Suppose that the system $AX = 0$ has only the trivial solution.** Prove that A is row-equivalent to the $n \times n$ identity matrix. Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A. By Theorem 3, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions.

Proof of Theorem 7 contd.

Case 2. Suppose that the system $AX = 0$ has only the trivial solution. Prove that A is row-equivalent to the $n \times n$ identity matrix. Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A. By Theorem 3, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions. Since $AX = 0$ has only the trivial solution, $RX = 0$ has only the trivial solution.

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Case 2. Suppose that the system $AX = 0$ has only the trivial solution. Prove that A is row-equivalent to the $n \times n$ identity matrix. Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A. By Theorem 3, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions. Since $AX = 0$ has only the trivial solution, $RX = 0$ has only the trivial solution. Hence the system $RX = 0$ has no free variables.

Proof of Theorem 7 contd.

Case 2. Suppose that the system $AX = 0$ has only the trivial solution. Prove that A is row-equivalent to the $n \times n$ identity matrix. Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A. By Theorem 3, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions. Since $AX = 0$ has only the trivial solution, $RX = 0$ has only the trivial solution. Hence the system $RX = 0$ has no free variables. Thus the number of free variables (of the system $RX = 0$), $n - r = 0$ where r is the number of non-zero rows of R.

Proof of Theorem 7 contd.

Case 2. Suppose that the system $AX = 0$ has only the trivial solution. Prove that A is row-equivalent to the $n \times n$ identity matrix. Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A. By Theorem 3, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions. Since $AX = 0$ has only the trivial solution, $RX = 0$ has only the trivial solution. Hence the system $RX = 0$ has no free variables. Thus the number of free variables (of the system $RX = 0$), $n - r = 0$ where r is the number of non-zero rows of R. So R is an $n \times n$ row-reduced echelon matrix with $n(= r)$ non-zero rows and thus $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$. This proves that $R = I$, an identity matrix.

Hence A is row-equivalent to $R = I$.

Reading assignment

Section 1.5 Matrix multiplication

Assignment

Solve all exercise problems in section 1.4 (pages 15-16)