Engineering Electromagnetics

Lecture 28

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by

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Magnetic vector potential

Just as $\nabla \times \mathbf{E} = \mathbf{0}$ permitted us to introduce a scalar potential (V) in electrostatics,

$$\mathbf{E} = -\nabla V,$$

so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a *vector* potential $\mathbf{A}$ in magnetostatics:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.} \tag{5.61}$$

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$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

$$\boxed{\nabla \cdot \mathbf{A} = 0.}$$

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

This *again* is nothing but Poisson's equation—or rather, it is *three* Poisson's equations, one for each Cartesian[19] component. Assuming $\mathbf{J}$ goes to zero at infinity, we can read off the solution:

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau'.}$$

How??

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_c \frac{\vec{d\ell}' \times \vec{\mathbf{R}}}{R^3}$$

$$\vec{\mathbf{R}} = (x - x')\vec{\mathbf{a}}_x + (y - y')\vec{\mathbf{a}}_y + (z - z')\vec{\mathbf{a}}_z$$

$$\nabla\left(\frac{1}{R}\right) = -\frac{\vec{\mathbf{R}}}{R^3} \Rightarrow \vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_c \nabla\left(\frac{1}{R}\right) \times \vec{d\ell}'$$

$$\nabla\left(\frac{1}{R}\right) \times \vec{d\ell}' = \boldsymbol{\nabla} \times \left[\frac{\vec{d\ell}'}{R}\right] - \frac{1}{R}[\boldsymbol{\nabla} \times \vec{d\ell}']$$

Because the curl operation is with respect to the unprimed coordinates of point $P(x, y, z)$, $\nabla \times \vec{d\ell}' = 0$. Thus, from (5.25), we have

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_c \boldsymbol{\nabla} \times \left[\frac{\vec{d\ell}'}{R}\right]$$

The integration and the differentiation are with respect to two different sets of variables, so we can interchange the order and write the preceding equation as

$$\vec{\mathbf{B}} = \boldsymbol{\nabla} \times \left[\frac{\mu_0 I}{4\pi} \int_c \frac{\vec{d\ell}'}{R}\right] \tag{5.26}$$

Comparing (5.24) and (5.26), we obtain an expression for the magnetic vector potential $\vec{\mathbf{A}}$ as

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi} \int_c \frac{I\,\vec{d\ell}'}{R} \tag{5.27a}$$

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi} \oint_c \frac{I\,\vec{d\ell}'}{R}$$

A i.t.o J ?

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_c \frac{\vec{d\ell}' \times \vec{\mathbf{R}}}{R^3}$$

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$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi} \int_v \frac{\vec{\mathbf{J}}_v\,dv'}{R}$$

Magnetic flux i.t.o. A

We can also express the magnetic flux Φ in terms of $\mathbf{A}$ as

$$\Phi = \int_s \vec{\mathbf{B}} \cdot \overrightarrow{d\mathbf{s}} = \int_s (\nabla \times \vec{\mathbf{A}}) \cdot \overrightarrow{d\mathbf{s}}$$

A direct application of Stokes' theorem yields

$$\Phi = \oint_c \vec{\mathbf{A}} \cdot \overrightarrow{d\ell}$$

where c is the contour bounding the open surface s.

What is **B** for an infinite solenoid with no. of turns per unit length n carrying a current I?

Example 5.12. Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

Solution

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi,$$

$$\oint \mathbf{A} \cdot d\mathbf{l} = A(2\pi s) = \int \mathbf{B} \cdot d\mathbf{a} = \mu_0 n I (\pi s^2),$$

so

$$\mathbf{A} = \frac{\mu_0 n I}{2} s \, \hat{\boldsymbol{\phi}}, \quad \text{for } s \leq R.$$

Magnetic field intensity

- $D = \varepsilon E$
- Magnetic field intensity H in free space is $H = B/\mu_0$
- $B = \mu_0 H$
- What is Ampere's circuital law in terms of H then?

$$\oint_c \vec{\mathbf{H}} \cdot \vec{d\ell} = I$$

H rotational/irrotational? Value of curl?

$$\oint_c \vec{\mathbf{H}} \cdot \vec{d\ell} = I$$

$$I = \int_s \vec{\mathbf{J}}_v \cdot \vec{d\mathbf{s}}$$

the integral form of Ampère's law, from (5.34a), becomes

$$\oint_c \vec{\mathbf{H}} \cdot \vec{d\ell} = \int_s \vec{\mathbf{J}}_v \cdot \vec{d\mathbf{s}}$$

Stokes' theorem allows us to express the line integral in terms of the surface integral as

$$\int_s (\nabla \times \vec{\mathbf{H}}) \cdot \vec{d\mathbf{s}} = \int_s \vec{\mathbf{J}}_v \cdot \vec{d\mathbf{s}}$$

As s can be any arbitrary open surface bounded by a closed contour c, the preceding equation can be written in the general form as

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_v \tag{5.34b}$$

A very long, very thin, straight wire located along the z axis carries a current I in the z direction. Find the magnetic field intensity at any point in free space using Ampère's law.

Solution The symmetry arguments dictate that the magnetic field lines must be concentric circles, as shown in Figure 5.20. The magnetic field intensity will have a constant magnitude along each circle. Thus, at any radius ρ, we have

$$\oint_c \vec{\mathbf{H}} \cdot \vec{d\ell} = \int_0^{2\pi} H_\phi \rho \, d\phi = 2\pi \rho H_\phi$$

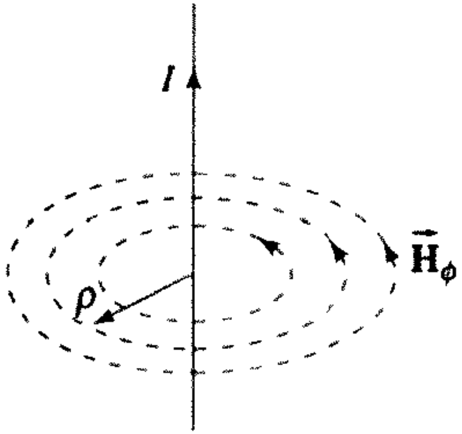

Figure 5.20 Magnetic field surrounding a very long current-carrying conductor

Since the current enclosed by the closed path is I, Ampère's law gives us

$$\vec{\mathbf{H}} = \frac{I}{2\pi\rho} \vec{\mathbf{a}}_\phi$$

Thus, Ampère's law yields the same result that was obtained earlier using the Biot–Savart law.

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Thank You