Second Order Linear Equations

Definition

The general second order linear differential equation is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

or, simply,

$$y'' + P(x)y' + Q(x)y = R(x).$$

Here $P(x)$, $Q(x)$ and $R(x)$ are functions of x alone (or just constants).

Examples

- $y'' + y = 0$. Here $P(x) = 0, Q(x) = 1$ and $R(x) = 0$.
- $x^2y'' + 2xy' - 2y = 0$ is also a second order linear equation. This becomes obvious when we rewrite it in the standard form: $y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0$. Hence $P(x) = \frac{2}{x}, Q(x) = -\frac{2}{x^2}$ and $R(x) = 0$.
- $(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$. (What are $P(x), Q(x)$ and $R(x)$?)
- $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$. (What are $P(x), Q(x)$ and $R(x)$?)

An Existence and Uniqueness Theorem

Theorem

Let $P(x), Q(x)$ and $R(x)$ be continuous functions on a closed interval $[a, b]$. If x_0 is any point in $[a, b]$ and if y_0 and y_0' are any numbers, then the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

has one and only one solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$.

Note: The theorem implies that for any given point x_0 in $[a, b]$ and any choice of the numbers y_0 and y_0', the differential equation has a unique solution on $[a, b]$ that passes through the point (x_0, y_0) and has the slope y_0' at (x_0, y_0).

Example

Find the solution of the initial value problem

$$y'' + y = 0, \qquad y(0) = 0 \quad \text{and} \quad y'(0) = 1.$$

Solution: We see that $y = \sin x$ is a solution of the given differential equation:

$$y'' + y = -\sin x + \sin x = 0.$$

Similarly, $y = \cos x$ is a solution of the differential equation.

Indeed $y = c_1 \sin x + c_2 \cos x$ is also a solution of this differential equation for every choice of the constants c_1 and c_2.

We finally note that the solution $y = \sin x$ also satisfies the given initial conditions.

Therefore, by the theorem, $y = \sin x$ is the *unique* solution of the given initial value problem.

Example

Find the solution of the initial value problem

$$y'' + y = 0, \qquad y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

Solution: We note that $y = \cos x$ is a solution of this initial value problem.

From the theorem, we conclude that $y = \cos x$ is the *unique* solution of the initial value problem.

Homogeneous Equations

Definition

A differential equation of the form

$$y'' + P(x)y' + Q(x)y = 0$$

is called a **homogeneous equation**. In contrast, the differential equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

in which $R(x)$ is not identically zero is called a **non-homogeneous equation**. In this context, the equation $y'' + P(x)y' + Q(x)y = R(x)$ is called the **complete equation** and the corresponding homogeneous equation $y'' + P(x)y' + Q(x)y = 0$ that is obtained by replacing $R(x)$ with 0 is called the associated **reduced equation**.

Note

In obtaining the solution of a non-homogeneous equation $y'' + P(x)y' + Q(x)y = R(x)$, we will have to always consider the associated homogeneous equation also:

Suppose $y_g(x, c_1, c_2)$ is the general solution of the the corresponding reduced homogeneous equation $y'' + P(x)y' + Q(x)y = 0$. And suppose $y_p(x)$ is any one fixed particular solution of the given complete non-homogeneous equation.

Let $y(x)$ be any solution of the given non-homogeneous equation. It then follows that $y - y_p$ is a solution of the the reduced homogeneous equation:

$$(y - y_p)'' + (y - y_p)'P + (y - y_p) = [y'' + Py' + Qy] - [y_p'' + Py_p' + Qy_p] = R - R = 0.$$

But $y_g(x, c_1, c_2)$ is the general solution of $y'' + P(x)y' + Q(x)y = 0$.

$$\therefore \; y(x) - y_p(x) = y_g(x, c_1, c_2)$$

for some choice of the constants c_1 and c_2.

$$\therefore \; y(x) = y_g(x, c_1, c_2) + y_p(x)$$

for some choice of the constants c_1 and c_2.

Theorem

Consider the non-homogeneous equation $y'' + P(x)y' + Q(x)y = R(x)$*. If* y_g *is the general solution of the corresponding reduced homogeneous equation* $y'' + P(x)y' + Q(x)y = 0$ *and* y_p *is any particular solution of the given complete non-homogeneous equation, then* $y_g + y_p$ *is the general solution of the complete equation.*

Theorem

If $y_1(x)$ and $y_2(x)$ are any two solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0,$$

then

$$c_1 y_1(x) + c_2 y_2(x)$$

is also a solution for any constants c_1 and c_2.

Proof.

$$(c_1 y_1 + c_2 y_2)'' + P(x)(c_1 y_1 + c_2 y_2)' + Q(x)(c_1 y_1 + c_2 y_2)$$

$$\begin{aligned} &= (c_1 y_1'' + c_2 y_2'') + P(x)(c_1 y_1' + c_2 y_2') + Q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1[y_1'' + P(x)y_1' + Q(x)y_1] + c_2[y_2'' + P(x)y_2' + Q(x)y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

$\square$

Note

1. $c_1 y_1(x) + c_2 y_2(x)$ is called a **linear combination** of $y_1(x)$ and $y_2(x)$.
2. If we have managed to find two solutions of the equation $y'' + P(x)y' + Q(x)y = 0$, then this theorem provides us another solution which involves two arbitrary constants and thus may be the general solution.
3. But if either y_1 or y_2 is a constant multiple of the other, say $y_1 = ky_2$, then

$$c_1 y_1 + c_2 y_2 = c_1(ky_2) + c_2 y_2 = (c_1 k + c_2)y_2 = cy_2$$

   and thus $c_1 y_1 + c_2 y_2$ has essentially only one arbitrary constant.
4. Thus it may be expected that if neither y_1 nor y_2 is a constant multiple of the other, then

$$c_1 y_1(x) + c_2 y_2(x)$$

   is the general solution of $y'' + P(x)y' + Q(x)y = 0$.
5. This is indeed true and will be proved shortly!

Example

Solve $y'' + y' = 0$.

Solution: By inspection, we see that $y_1 = 1$ and $y_2 = e^{-x}$ are solutions of the given differential equation.

It is also obvious that neither function is a constant multiple of the other.

Thus by the theorem to be proved

$$y = c_1 + c_2 e^{-x}$$

is the general solution.

Example

Solve $x^2y'' + 2xy' - 2y = 0$.

Solution: The form of the differential equation suggests that it might possibly have solutions of the of the form $y = x^n$. On substituting this in the differential equation, we obtain

$$x^n(n(n-1) + 2n - 2) = 0 \quad \text{or} \quad n(n-1) + 2n - 2 = 0 \quad \text{or} \quad n^2 + n - 2 = 0.$$

The last equation above has roots $n = 1$ and $n = -2$. Thus $y_1 = x$ and $y_2 = x^{-2}$ are solutions of the given differential equations.

Since neither of this function is a constant multiple of the other, it follows that

$$y = c_1x + c_2x^{-2}$$

is the general solution of the equation on any interval not containing 0.

Definition

Let $f(x)$ and $g(x)$ be two functions defined on an interval $[a, b]$. If any one of these functions is a constant multiple of the other, then they are said to be **linearly dependent** on $[a, b]$. Otherwise - that is, if neither is a constant multiple of the other - they are called **linearly independent**.

Example

Suppose $f(x) \equiv 0$ on $[a, b]$; i.e., it is identically zero on $[a, b]$. Then $f(x)$ and $g(x)$ are linearly dependent for any function $g(x)$ as $f(x) = 0 \cdot g(x)$.

The General Solution of the Homogeneous Equation

Theorem

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0$$

on the interval $[a, b]$. Then

$$c_1 y_1(x) + c_2 y_2(x)$$

is the general solution of the differential equation on $[a, b]$.

The theorem will follow from an observation and two lemmas that we will be proving now.

Note

Recall: A solution $y(x)$ of a second order linear equation on an interval $[a, b]$ is uniquely determined by $y(x_0)$ and $y'(x_0)$, where x_0 is any point from the interval $[a, b]$.

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous equation $y'' + P(x)y' + Q(x)y = 0$ on the interval $[a, b]$.

Our goal is to prove that $c_1y_1(x) + c_2y_2(x)$ is the general solution of the differential equation on $[a, b]$.

That is, we want to prove that if $y(x)$ is any solution of the equation on $[a, b]$, then there is a choice of the constants c_1 and c_2 such that

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

Note...

Let x_0 be any point from the interval $[a, b]$. By the theorem stated above such a choice of constants c_1 and c_2 will be possible if

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= y(x_0) \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= y'(x_0) \end{aligned}$$

has a solution for c_1 and c_2.
And the system will have a solution if the determinant

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) \neq 0.$$

The Wronskian

Definition

Let y_1 and y_2 be any functions of x. Then their **Wronskian** is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

Lemma 1

If $y_1(x)$ and $y_2(x)$ are any two solutions of the homogeneous linear equation $y'' + P(x)y' + Q(x)y = 0$ on the interval $[a, b]$, then their Wronskian $W(y_1, y_2)$ is either identically zero or never zero on $[a, b]$.

Proof: The Wronskian of y_1 and y_2 is $W = W(y_1, y_2) = y_1 y_2' - y_2 y_1'$.

$$\therefore \ \frac{dW}{dx} = y_1 y_2'' + y_1' y_2' - y_2 y_1'' - y_2' y_1' = y_1 y_2'' - y_2 y_1''.$$

Also since y_1 and y_2 are solutions of $y'' + P(x)y' + Q(x)y = 0$, we have

$$y_1'' + Py_1' + Qy_1 = 0$$

$$y_2'' + Py_2' + Qy_2 = 0.$$

Now multiply the the first of the above equations by y_2 and the second by y_1:

$$y_2 y_1'' + Py_2 y_1' + Qy_2 y_1 = 0$$

$$y_1 y_2'' + Py_1 y_2' + Qy_1 y_2 = 0.$$

Next, subtract the first of the above equations from the second:

$$(y_1 y_2'' - y_2 y_1'') + P(y_1 y_2' - y_2 y_1') = 0.$$

We recognize the last equation as

$$\frac{dW}{dx} + PW = 0.$$

This is a first order linear equation and its general solution is

$$We^{\int P dx} = c \qquad \text{or} \qquad W = ce^{-\int P dx}.$$

The exponential function is never zero. Thus W is identically zero if $c = 0$ and is never zero if $c \neq 0$.

Thus we have proved that the Wronskian $W = W(y_1, y_2)$ is either identically zero or never zero.

Lemma 2

If $y_1(x)$ and $y_2(x)$ are any two solutions of the equation $y'' + P(x)y' + Q(x)y = 0$ on the interval $[a, b]$, then they are linearly dependent on this interval if and only if their Wronskian $W(y_1, y_2) = 0$ is identically zero on $[a, b]$.

Proof: Suppose y_1 and y_2 are linearly dependent. We must prove that $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \equiv 0$ on $[a, b]$.

If either y_1 or y_2 is identically zero on $[a, b]$, then $W(y_1, y_2) \equiv 0$.

Otherwise, since y_1 and y_2 are linearly dependent, each is a constant multiple of the other. So, $y_2 = cy_1$ for some constant c and hence $y_2' = cy_1'$. Hence

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = y_1(cy_1') - (cy_1)y_1' = cy_1 y_1' - cy_1 y_1' = 0.$$

This proves one direction of the lemma.

Proof:...

Converse: Suppose that the Wronskian $W(y_1, y_2)$ is identically zero on $[a, b]$. We must prove that the solutions are linearly dependent.

If y_1 is identically zero on $[a, b]$, it is obvious that y_1 and y_2 are linearly independent. So, let us suppose that y_1 is not identically zero on $[a, b]$.

Then the continuity of y_1 on $[a, b]$ implies that y_1 is *never* zero on some interval $[c, d] \subseteq [a, b]$.

Also the Wronskian is identically zero on $[a, b]$.

So, on the interval $[c, d] \subseteq [a, b]$, we have

$$\frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0 \quad \text{or} \quad \left(\frac{y_2}{y_1}\right)' = 0.$$

Thus, on the interval $[c, d]$, we have

$$\frac{y_2}{y_1} = k \qquad \text{or} \qquad y_2 = ky_1$$

for some constant k.
This means that, on the interval $[c, d]$, we also have

$$y_2' = ky_1'.$$

Thus we have that the solutions y_2 and ky_1 and their derivatives agree at some point in $[a, b]$.

Therefore, by a theorem, it follows that

$$y_2 = ky_1$$

on the entire interval $[a, b]$.

Thus we have proved that y_1 and y_2 are linearly independent.

Summary

Lemma

If $y_1(x)$ and $y_2(x)$ are any two solutions of the homogeneous linear equation $y'' + P(x)y' + Q(x)y = 0$ on the interval $[a, b]$, then their Wronskian $W(y_1, y_2)$ is either identically zero or never zero on $[a, b]$.

Lemma

If $y_1(x)$ and $y_2(x)$ are any two solutions of the equation $y'' + P(x)y' + Q(x)y = 0$ on the interval $[a, b]$, then they are linearly dependent on this interval if and only if their Wronskian $W(y_1, y_2) = 0$ is identically zero on $[a, b]$.

Thus if y_1 and y_2 are linearly independent solutions of the differential equation, then we have that their Wronskian is never zero on $[a, b]$. Hence if $y(x)$ is any solution of the equation, then the system below has a unique solution c_1 and c_2 for every choice of x_0 from $[a, b]$:

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= y(x_0) \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= y'(x_0) \end{aligned}$$

Theorem

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0$$

on the interval $[a, b]$. Then

$$c_1 y_1(x) + c_2 y_2(x)$$

is the general solution of the differential equation on $[a, b]$.

Example

Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval. Also find the particular solution for which $y(0) = 2$ and $y'(0) = 3$.

Solution: By substitution, we see that $y_1 = \sin x$ and $y_2 = \cos x$ are solutions of the given differential equation.

Also they are linearly independent since $y_1/y_2 = \tan x$ is not a constant or since their Wronskian is never zero:

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Thus $y = c_1 \sin x + c_2 \cos x$ is the general solution on any interval $[a, b]$.

Now, $y(0) = 2$ implies that $c_1 \sin 0 + c_2 \cos 0 = 2$ or $c_2 = 2$.

And $y'(0) = 3$ implies that $c_1 \cos 0 - c_2 \sin 0 = 3$ or $c_1 = 3$.

Thus $y = 3 \sin x + 2 \cos x$ is the required particular solution.

Homework

1. Show that e^x and e^{-x} are linearly independent solutions of $y'' - y = 0$ on any interval.
2. Show that $y = c_1 x + c_2 x^2$ is the general solution of $x^2 y'' - 2xy' + 2y = 0$ on any interval not containing 0. Also find the particular solution for which $y(1) = 3$ and $y'(1) = 5$.
3. Show that $y = c_1 e^x + c_2 e^{2x}$ is the general solution of $y'' - 3y' + 2y = 0$ on any interval. Also find the particular solution for which $y(0) = -1$ and $y'(0) = 1$.
4. Show that $y = c_1 e^{2x} + c_2 x e^{2x}$ is the general solution of $y'' - 4y' + 4y = 0$ on any interval.

The Use of a Known Solution to Find Another

Let $y_1(x)$ be a solution of

$$y'' + P(x)y' + Q(x)y = 0.$$

Then we know that $y_2 = cy_1$ is also a solution of the equation. But in this case y_1 and y_2 are linearly dependent.

So, instead we try to find a non-constant function $v(x)$ such that $y_2 = vy_1$ is also a solution.

In this case, $y_2/y_1 = v$ is not a constant and hence y_1 and y_2 are linearly independent.

Now, $y_2 = vy_1$ is a solution of the equation implies that

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0.$$

We now compute y_2' and y_2'' from $y_2 = vy_1$:

$$y_2' = vy_1' + v'y_1 \qquad \text{and} \qquad y_2'' = vy_1'' + v'y_1' + v'y_1' + v''y_1 = vy_1'' + 2v'y_1' + v''y_1.$$

On substituting these into the differential equation, we obtain

$$(vy_1'' + 2v'y_1' + v''y_1) + P(vy_1' + v'y_1) + Q(vy_1) = v(y_1'' + Py_1' + Qy_1) + v''y_1 + v'(2y_1' + Py_1) = 0.$$

But y_1 is a solution of the differential equation. So, we have

$$v''y_1 + v'(2y_1' + Py_1) = 0$$

or

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - P.$$

Integrating the above gives

$$\log v' = -2\log y_1 - \int P dx \qquad \text{or} \qquad v' = \frac{1}{y_1^2} e^{-\int P dx}.$$

So, we have

$$v = \int \frac{1}{y_1^2} e^{-\int P dx} dx.$$

Example

Solve $x^2y'' + xy' - y = 0$.

Solution: By inspection, we discover that $y_1 = x$ is a solution of the differential equation. We must find another solution y_2 that is linearly independent of y_1.

We begin by writing the differential equation in the standard form:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0.$$

Here $P = \dfrac{1}{x}$. Hence $y_2 = vy_1$ is a second linearly independent solution of the equation, where

$$v = \int \frac{1}{y_1^2} e^{-\int P dx} dx = \int \frac{1}{x^2} e^{-\int (1/x) dx} dx = \int \frac{1}{x^2} e^{-\log x} dx = \int x^{-3} dx = \frac{x^{-2}}{-2}.$$

Thus we have $y_2 = vy_1 = \frac{x^{-2}}{-2} \cdot x = (-1/2)x^{-1}$.

Hence the general solution is $y = c_1x + c_2x^{-1}$.

Homework

1. Use the method of this section to find y_2 and the general solution for each of the following equations from the given solution y_1:
   (a) $y'' + y = 0, \; y_1 = \sin x$; (b) $y'' - y = 0, \; y_1 = e^x$.
2. The equation $xy'' + 3y' = 0$ has the obvious solution $y_1 = 1$. Find y_2 and the general solution.
3. Verify that $y_1 = x^2$ is one solution of $x^2y'' + xy' - 4y = 0$, and find y_2 and the general solution.
4. The equation $(1 - x^2)y'' - 2xy' + 2y = 0$ has $y_1 = x$ as an obvious solution. Find the general solution.

The Homogeneous Equation With Constant Coefficients

We now consider homogeneous equations

$$y'' + P(x)y' + Q(x)y = 0$$

in which $P(x)$ and $Q(x)$ are constants p and q:

$$y'' + py' + qy = 0.$$

And we look for solutions of the form

$$y = e^{mx}.$$

Note that

$$y = e^{mx} \implies y' = me^{mx} \quad \text{and} \quad y'' = m^2 e^{mx}.$$

Thus substituting $y = e^{mx}$ in the equation $y'' + py' + qy = 0$ gives

$$m^2 e^{mx} + pme^{mx} + qe^{mx} = 0$$

or

$$(m^2 + pm + q)e^{mx} = 0.$$

But e^{mx} is never zero. Thus the LHS of the above equation is zero if and only if

$$m^2 + pm + q = 0.$$

Hence $y = e^{mx}$ is a solution of $y'' + py' + qy = 0$ if and only if

$$m^2 + pm + q = 0.$$

Theorem

The homogeneous equation

$$y'' + py' + qy = 0$$

has

$$y = e^{mx}$$

as a solution if and only if m is a root of the **auxiliary equation**

$$m^2 + pm + q = 0.$$

The Roots a Quadratic Equation

The quadratic equation

$$m^2 + pm + q = 0$$

has roots

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

The roots could be real and distinct or real and equal or purely complex, depending on the sign of the discriminant $p^2 - 4q$.

Case 1: The Auxiliary Equation Has Two Distinct Real Roots

This is the case if and only if $p^2 - 4q > 0$.

Let m_1 and m_2 be the distinct real roots of the auxiliary equation.

Then $e^{m_1 x}$ and $e^{m_2 x}$ are two solutions of the differential equation. Also

$$\frac{e^{m_1 x}}{e^{m_2 x}} = e^{(m_1 - m_2)x}$$

is not a constant.

Thus the solutions $e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent and

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

is the general solution of the differential equation.

Case 2: The Auxiliary Equation Has Two Distinct Complex Roots

This is the case if and only if $p^2 - 4q < 0$.

Let the two distinct complex roots be $m_1 = a + ib$ and $m_2 = a - ib$. (Here a and b are real numbers.)

Recall *Euler's formula:*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus the two solutions are

$$e^{m_1 x} = e^{(a+ib)x} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx)$$

and

$$e^{m_2 x} = e^{(a-ib)x} = e^{ax}e^{-ibx} = e^{ax}(\cos bx - i\sin bx).$$

These are linearly independent solutions of the differential equation $y'' + py' + qy = 0$.

But complex solutions! We are interested in real solutions!

We have obtained the complex solutions

$$e^{ax}(\cos bx + i \sin bx) \qquad \text{and} \qquad e^{ax}(\cos bx - i \sin bx).$$

But we are interested in real solutions!

Add the above solutions and divide by 2.

Also subtract the second from the first and divide by $2i$.

This gives us the following **real** solutions:

$$e^{ax} \cos bx \qquad \text{and} \qquad e^{ax} \sin bx.$$

These solutions are obviously linearly independent. So, the general solution is

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx).$$

Case 3: The Auxiliary Equation Has Two Equal Real Roots

This is the case if and only if $p^2 - 4q = 0$.

In this case, the auxiliary equation has essentially only one root, namely $m = -p/2$.

So, we obtain only one solution for the differential equation, namely $y = e^{mx}$, where $m = -p/2$.

We take $y_1 = e^{-(p/2)x}$ and find a second linear independent solution $y_2 = vy_1$ by the method the preceding section:

$$v = \int \frac{1}{y_1^2} e^{-\int P dx} dx = \int \frac{1}{e^{-px}} e^{-\int p dx} dx = \int \frac{1}{e^{-px}} e^{-px} dx = x.$$

Hence

$$y_2 = xe^{mx}$$

and the general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

Homework

Find the general solution of each of the following equations:

1. $y'' + y' - 6y = 0$
2. $y'' + 2y' + y = 0$
3. $y'' + 8y = 0$
4. $2y'' - 4y' + 8y = 0$
5. $y'' - 4y' + 4y = 0$
6. $y'' - 9y' + 20y = 0$
7. $y'' + y' = 0$

The Method of Undetermined Coefficients

We now take up the task of finding a particular solution $y_p(x)$ of the non-homogeneous equation:

$$y'' + P(x)y' + Q(x)y = R(x).$$

The first method we are going to learn is called "The Method of Undetermined Coefficients."

The Method of Undetermined Coefficients

This method is applicable for equations of the form

$$y'' + py' + qy = R(x),$$

where p and q are constants and $R(x)$ is an exponential, a sine or cosine, a polynomial or some combination of such functions.

Equations of the Form $y'' + py' + qy = e^{ax}$

In this case, substituting $y = e^{ax}$ on the LHS of the equation gives a constant multiple of e^{ax}.

So, we guess that the equation perhaps has a particular solution

$$y_p = Ae^{ax}$$

for some constant A.

Here A is the *undetermined coefficient* that we want to determine in such a way that it is actually a solution of the differential equation.

On substituting $y_p = Ae^{ax}$ into $y'' + py' + qy = e^{ax}$, we get

$$A(a^2 + pa + q)e^{ax} = e^{ax}$$

so that

$$A = \frac{1}{a^2 + pa + q}.$$

This choice of A will make $y_p = Ae^{ax}$ a solution of the differential equation *provided* $a^2 + pa + q \neq 0$.
[$a^2 + pa + q = 0$ means that a is a root of the auxiliary equation $m^2 + pm + q = 0$. So, there cannot be a particular solution of the form Ae^{ax}.]

In this case, try a particular solution of the form

$$y_p = Axe^{ax}.$$

This indeed works if *a is not a double root* of the auxiliary equation.

If a is a double root of the auxiliary equation, then the nonhomogeneous equation has a particular solution of the form

$$y_p = Ax^2e^{ax}.$$

Summary

Goal: To find a particular solution of the equation of the form $y'' + py' + qy = e^{ax}$.

- If a is not root of the auxiliary equation $m^2 + pm + q = 0$, then the nonhomogeneous equation has a particular solution of the form Ae^{ax}.
- If a is simple root of the auxiliary equation, then the nonhomogeneous equation has a particular solution of the form Axe^{ax}.
- If a is double root of the auxiliary equation, then the nonhomogeneous equation has a particular solution of the form Ax^2e^{ax}.

In each case, we substitute the respective $y_p(x)$ in the nonhomogeneous equation and determine the coefficient A.

Equations of the Form $y'' + py' + qy = \sin bx$

The derivatives of $\sin bx$ are constant multiples of $\sin bx$ and $\cos bx$.

So, we take a trial solution of the form

$$y_p = A \sin bx + B \cos bx.$$

The undetermined coefficients A and B are now determined by substituting this trial y_p into the nonhomogeneous differential equation and equating the resulting coefficients of $\sin bx$ and $\cos bx$ on the left and right.

Note: This approach works when $R(x)$ on the RHS is $\cos bx$ or any linear combination $\sin bx$ and $\cos bx$: $\alpha \sin bx + \beta \cos bx$.

Equations of the Form $y'' + py' + qy = \sin bx$

This method breaks down if $A \sin bx + B \cos bx$ satisfies the reduced homogeneous equation $y'' + py' + qy = 0$.

In this case, the differential equation has a particular solution of the form

$$y_p = x(A \sin bx + B \cos bx).$$

Example

Find a particular solution of
$$y'' + y = \sin x.$$

Solution: The reduced homogeneous equation $y'' + y = 0$ has $y = c_1 \sin x + c_2 \cos x$ as a particular solution.

So, it is useless to take $y_p = A \sin x + B \cos x$ as a trial solution. So, we try
$$y_p = x(A \sin x + B \cos x).$$

This gives
$$y_p' = A \sin x + B \cos x + x(A \cos x - B \sin x)$$
and
$$y_p'' = 2A \cos x - 2B \sin x + x(-A \sin x - B \cos x).$$

Substituting these in $y'' + y = \sin x$ gives

$$2A\cos x - 2B\sin x = \sin x.$$

This implies that $A = 0$ and $B = -\dfrac{1}{2}$.
Thus the particular solution is

$$y_p = -\frac{1}{2}x\cos x.$$

Equations of the Form $y'' + py' + qy = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

In this case, we seek a particular solution of the form

$$y_p = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n.$$

We substitute this y_p into the differential equation and equate the like powers of x to determine the undetermined coefficients $A_0, A_1, \ldots, A_n$.

If $q = 0$, then this procedure gives x^{n-1} as the highest power of x on the left. So, in this case, we take

$$y_p = x(A_0 + A_1x + A_2x^2 + \ldots + A_nx^n) = A_0x + A_1x^2 + A_2x^3 + \ldots + A_nx^{n+1}.$$

If both p and q are zero, then the equation can be solved by direct integration.

Example

Find the general solution of

$$y'' - y' - 2y = 4x^2.$$

Solution: Here the reduced homogeneous equation is $y'' - y' - 2y = 0$.
Its auxiliary equation is $m^2 - m - 2 = 0$ or $(m-2)(m+1) = 0$. So, its roots are $m_1 = -1$ and $m_2 = 2$.

Thus the general solution of the auxiliary equation is $y_g = c_1 e^{-x} + c_2 e^{2x}$.

The RHS of the differential equation is a polynomial of the second degree.
So, we take a trial solution of the form $y_p = A + Bx + Cx^2$ and substitute it in the differential equation:

$$2C - (B + 2Cx) - 2(A + Bx + Cx^2) = 4x^2.$$

$$2C - (B + 2Cx) - 2(A + Bx + Cx^2) = 4x^2.$$

Equating the coefficients of like powers of x gives

$$\begin{aligned} 2C - B - 2A &= 0 \\ -2C - 2B &= 0 \\ -2C &= 4 \end{aligned}$$

Solving this linear systems gives $C = -2$, $B = 2$ and $A = -3$.
Hence a particular solution of the given nonhomogeneous is $y_p = -3 + 2x - 2x^2$.

And hence its general solution is

$$y = c_1 e^{-x} + c_2 e^{2x} - 3 + 2x - 2x^2.$$

Homework

Find the general solution of each of the following equations:

1. $y'' + 3y' - 10y = 6e^{4x}$
2. $y'' + 4y = 3\sin x$
3. $y'' + 10y' + 25y = 14e^{-5x}$
4. $y'' - 2y' + 5y = 25x^2 + 12$
5. $y'' - y' - 6y = 20e^{-2x}$
6. $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$

The Principle of Superposition

If $y_1(x)$ and $y_2(x)$ are solutions of

$$y'' + P(x)y' + Q(x)y = R_1(x)$$

and

$$y'' + P(x)y' + Q(x)y = R_2(x),$$

respectively, then $y(x) = y_1(x) + y_2(x)$ is a solution of

$$y'' + P(x)y' + Q(x)y = R_1(x) + R_2(x).$$

This principle enables us to apply the method of undetermined coefficients to nonhomogeneous equations in which $R(x)$ is a sum of exponential, sine, cosine, and/or polynomial functions.

Homework

Use the principle of superposition to find the general solution of

1. $y'' + 4y = 4\cos 2x + 6\cos x + 8x^2 - 4x$
2. $y'' + 9y = 2\sin 3x + 4\sin x - 26e^{-2x} + 27x^3$

The Method Of Variation of Parameters

Goal: To find a particular solution $y_p(x)$ of the non-homogeneous equation:

$$y'' + P(x)y' + Q(x)y = R(x).$$

Note: The Method of Undetermined Coefficients does not solve this problem in general: This method is applicable only for equations of the form

$$y'' + py' + qy = R(x),$$

where p and q are constants and $R(x)$ is an exponential, a sine or cosine, a polynomial or some combination of such functions.

But The Method Of Variation of Parameters is much more powerful and works *almost* always!

The Method Of Variation of Parameters

This method always works, regardless the nature of $P(x), Q(x), R(x)$, *provided* a general solution of the corresponding homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0$$

is already known.

Idea: Suppose we have found the general solution of $y'' + P(x)y' + Q(x)y = 0$:

$$y = c_1 y_(x) + c_2 y_2(x).$$

We replace the constants c_1 and c_2 by two unknown functions $v_1(x)$ and $v_2(x)$. Subsequently, we determine $v_1(x)$ and $v_2(x)$ so that

$$y = v_1 y_1 + v_2 y_2$$

is a solution of the complete equation

$$y'' + P(x)y' + Q(x)y = R(x).$$

Finding $v_1(x)$ and $v_2(x)$

Idea: Obtain two equations relating $v_1(x)$ and $v_2(x)$ and solve them for $v_1(x)$ and $v_2(x)$.

How do we get them?

If $y = v_1 y_1 + v_2 y_2$ is a solution of the non-homogeneous equation, then it must satisfy the equation.

So, we can synthesize one equation by substituting $y = v_1 y_1 + v_2 y_2$ in the equation.

We get another equation by imposing one more condition on v_1 and v_2.

Finding $v_1(x)$ and $v_2(x)$: The Details

Differentiating $y = v_1 y_1 + v_2 y_2$ gives

$$y' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2 = (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2).$$

Hence computing y'' will introduce the second order derivatives of the unknown functions v_1 and v_2. And it is not desirable.

We avoid this by requiring that

$$v_1' y_1 + v_2' y_2 = 0.$$

Interestingly, this becomes one of the two equations we are looking for!
Thus we have

$$y' = v_1 y_1' + v_2 y_2'$$

We have

$$y' = v_1 y_1' + v_2 y_2'.$$

So,

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

We also have

$$y = v_1 y_1 + v_2 y_2.$$

We now substitute all these into

$$y'' + P(x)y' + Q(x)y = 0$$

to get another equation:

$$(v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2') + P(x)(v_1 y_1' + v_2 y_2') + Q(x)(v_1 y_1 + v_2 y_2) = R(x)$$

or

$$v_1(y_1'' + P(x)y_1' + Q(x)y_1) + v_2(y_2'' + P(x)y_2' + Q(x)y_2) + v_1' y_1' + v_2' y_2' = R(x).$$

We have

$$v_1(y_1'' + P(x)y_1' + Q(x)y_1) + v_2(y_2'' + P(x)y_2' + Q(x)y_2) + v_1'y_1' + v_2'y_2' = R(x).$$

But y_1 and y_2 are solutions of the homogeneous equation $y'' + P(x)y' + Q(x)y = 0$. So, the above equation becomes

$$v_1'y_1' + v_2'y_2' = R(x).$$

Thus we have the following two linear equations in the two unknown functions v_1' and v_2':

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= R(x) \end{aligned}$$

The above equations give

$$v_1'(y_1y_2' - y_2y_1') = -y_2R(x) \qquad \text{and} \qquad v_2'(y_1y_2' - y_2y_1') = y_1R(x)$$

We have

$$v_1'(y_1y_2' - y_2y_1') = -y_2R(x) \qquad \text{and} \qquad v_2'(y_1y_2' - y_2y_1') = y_1R(x).$$

From this, we get

$$v_1' = \frac{-y_2R(x)}{W(y_1, y_2)} \qquad \text{and} \qquad v_2' = \frac{y_1R(x)}{W(y_1, y_2)}.$$

Integration now gives

$$v_1 = \int \frac{-y_2R(x)}{W(y_1, y_2)}dx \qquad \text{and} \qquad v_2 = \int \frac{y_1R(x)}{W(y_1, y_2)}dx.$$

Hence the particular solution $y = v_1y_1 + v_2y_2$ we are after is

$$y = y_1\int \frac{-y_2R(x)}{W(y_1, y_2)}dx + y_2\int \frac{y_1R(x)}{W(y_1, y_2)}dx.$$

Note: This method will be successful only if the above integrals can be found. And a general solution for the associated homogeneous equation is available.

Example

Find a particular solution of $y'' + y = \csc x$.

Solution: The associated homogeneous equation $y'' + y = 0$ has the general solution

$$y(x) = c_1 \sin x + c_2 \cos x.$$

So, we have $y_1 = \sin x$ and $y_2 = \cos x$; hence $y_1' = \cos x$ and $y_2' = -\sin x$.
Thus the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1.$$

$$v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx = \int \frac{-\cos x \csc x}{-1} dx = \int \frac{\cos x}{\sin x} dx = \log(\sin x)$$

and

$$v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx = \int \frac{\sin x \csc x}{-1} dx = -x.$$

Homework

Find a particular solution of each of the following equations:

1. $y'' - 2y' + y = 2x$.
2. $y'' - y' - 6y = e^{-x}$.
3. $y'' + 4y = \tan 2x$.
4. $y'' + 2y' + 5y = e^{-x} \sec 2x$.
5. $y'' + 2y' + y = e^{-x} \log x$.
6. $y'' - 3y' + 2y = (1 + e^{-x})^{-1}$.