Engineering Electromagnetics

Lecture 5

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by

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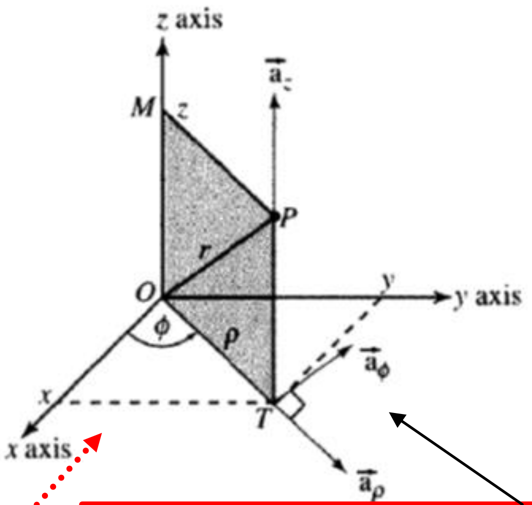
Transformations

- $\hat{\rho}$, $\hat{\varphi}$, and $\hat{\boldsymbol{z}}$ → **unit vectors** parallel to the ρ, φ and z axes
- Properties: $\hat{\rho}.\hat{\rho} = \hat{\varphi}.\hat{\varphi} = \hat{z}.\hat{z} = 1$; $\hat{\rho}.\hat{\varphi} = \hat{\varphi}.\hat{z} = \hat{\rho}.\hat{z} = 0$
- $\hat{\rho} \times \hat{\varphi} = \hat{z}$; $\hat{\varphi} \times \hat{z} = \hat{\rho}$; $\hat{z} \times \hat{\boldsymbol{\rho}} = \hat{\varphi}$
- **Conversion from cartesian to cylindrical coordinates:**
- $\hat{\boldsymbol{x}}.\hat{\rho} = \boldsymbol{Cos}\varphi$ **and** $\hat{\boldsymbol{y}}.\hat{\rho} = \boldsymbol{Sin}\varphi$
- $\hat{\boldsymbol{x}}.\hat{\varphi} = -\boldsymbol{Sin}\varphi$ **and** $\hat{\boldsymbol{y}}.\hat{\varphi} = \boldsymbol{Cos}\varphi$

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}. \end{aligned}$$



If $\hat{\rho}$ makes an angle φ with x axis, what about $\hat{\varphi}$?→ x and y components of $\hat{\varphi}$?

- **In a simple matrix form:**

$$\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

Q: For any vector A:

$$\boldsymbol{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

How to convert it to cylindrical coordinates? $\boldsymbol{A} = A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{z}$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Conversion cylindrical ↔ cartesian coordinates

Cartesian to cylindrical

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Cylindrical to cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

- Conversion to cartesian coordinates (Hint)
- From $A = A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{\mathbf{z}}$ to $A = A_x\hat{x} + A_y\hat{y} + A_z\hat{\mathbf{z}}$
- $A_x = \mathrm{A}.\,\hat{x} = (A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{\mathbf{z}}).\,\hat{x} = A_\rho\hat{\rho}.\hat{x} + A_\varphi\hat{\varphi}.\hat{x} + A_z\hat{\mathbf{z}}.\,\hat{x}$; $\hat{\boldsymbol{x}}.\hat{\rho} = \boldsymbol{Cos}\varphi$; $\hat{\boldsymbol{y}}.\hat{\rho} = \boldsymbol{Sin}\varphi$;
- $\hat{\boldsymbol{x}}.\hat{\varphi} = -\boldsymbol{Sin}\varphi$ and $\hat{\boldsymbol{y}}.\hat{\varphi} = \boldsymbol{Cos}\varphi$; $A_x = A_\rho\ \boldsymbol{Cos}\varphi - A_\varphi\boldsymbol{Sin}\varphi$; $A_y = \mathrm{A}.\ \hat{y} = A_\rho\boldsymbol{Sin}\varphi + A_\varphi\boldsymbol{Cos}\varphi$ and $A_z = \mathrm{A}.\hat{z}$ = A_z

Problem 2

Express the vector $\vec{\mathbf{A}} = \dfrac{k}{\rho^2}\vec{\mathbf{a}}_\rho + 5\sin 2\phi\, \vec{\mathbf{a}}_z$ in the rectangular coordinate system.

Solution Using the transformation matrix

$$A_\rho = \frac{k}{\rho^2}, \quad A_\phi = 0, \quad \text{and} \quad A_z = 5\sin 2\phi$$

we obtain

$$A_x = \frac{k\cos\phi}{\rho^2}, \quad A_y = \frac{k\sin\phi}{\rho^2}, \quad \text{and} \quad A_z = 10\cos\phi\sin\phi$$

Substituting $\rho = \sqrt{x^2 + y^2}$, $\cos\phi = \dfrac{x}{\rho}$, and $\sin\phi = \dfrac{y}{\rho}$, we obtain the desired transformation of vector $\vec{\mathbf{A}}$ as

$$\vec{\mathbf{A}} = \frac{kx}{[x^2 + y^2]^{3/2}}\vec{\mathbf{a}}_x + \frac{ky}{[x^2 + y^2]^{3/2}}\vec{\mathbf{a}}_y + \frac{10xy}{x^2 + y^2}\vec{\mathbf{a}}_z$$

...

Problem 3

If $\vec{\mathbf{A}} = 3\vec{\mathbf{a}}_\rho + 2\vec{\mathbf{a}}_\phi + 5\vec{\mathbf{a}}_z$ and $\vec{\mathbf{B}} = -2\vec{\mathbf{a}}_\rho + 3\vec{\mathbf{a}}_\phi - \vec{\mathbf{a}}_z$ are given at points $P(3, \pi/6, 5)$ and $Q(4, \pi/3, 3)$, find $\vec{\mathbf{C}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$ at point $S(2, \pi/4, 4)$.

The two vectors are not defined in the same ϕ = constant plane, so we cannot sum them directly in the cylindrical system. Conversion to the rectangular system is therefore necessary. For vector $\vec{\mathbf{A}}$ given at point $P(3, \pi/6, 5)$, the transformation matrix becomes

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$\vec{\mathbf{A}} = 1.598\vec{\mathbf{a}}_x + 3.232\vec{\mathbf{a}}_y + 5\vec{\mathbf{a}}_z$$

Similarly, with $\phi = \pi/3$, the transformed vector $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{B}} = -3.598\vec{\mathbf{a}}_x - 0.232\vec{\mathbf{a}}_y - \vec{\mathbf{a}}_z$$

Not correct.
C=A-B

$$\boxed{\vec{\mathbf{C}} = -2\vec{\mathbf{a}}_x + 3\vec{\mathbf{a}}_y + 4\vec{\mathbf{a}}_z}$$

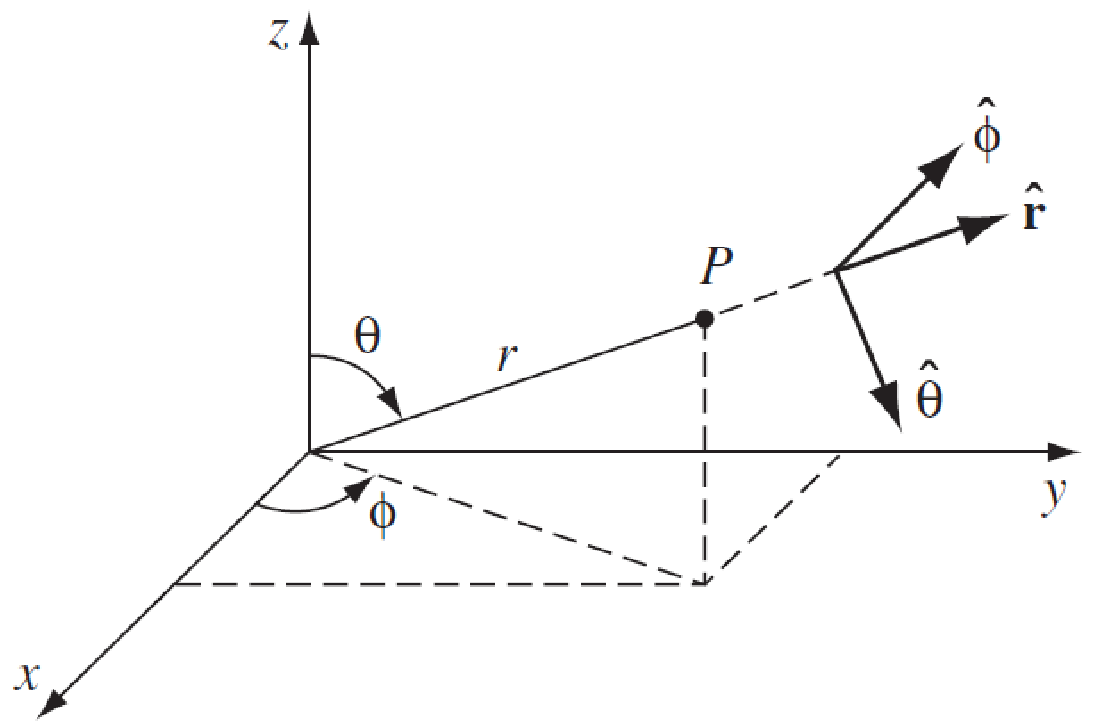
Vector $\vec{\mathbf{C}}$ can now be transformed into its components at point $S(2, \pi/4, 4)$ in the cylindrical system by making use of the transformation matrix given in (2.39). That is

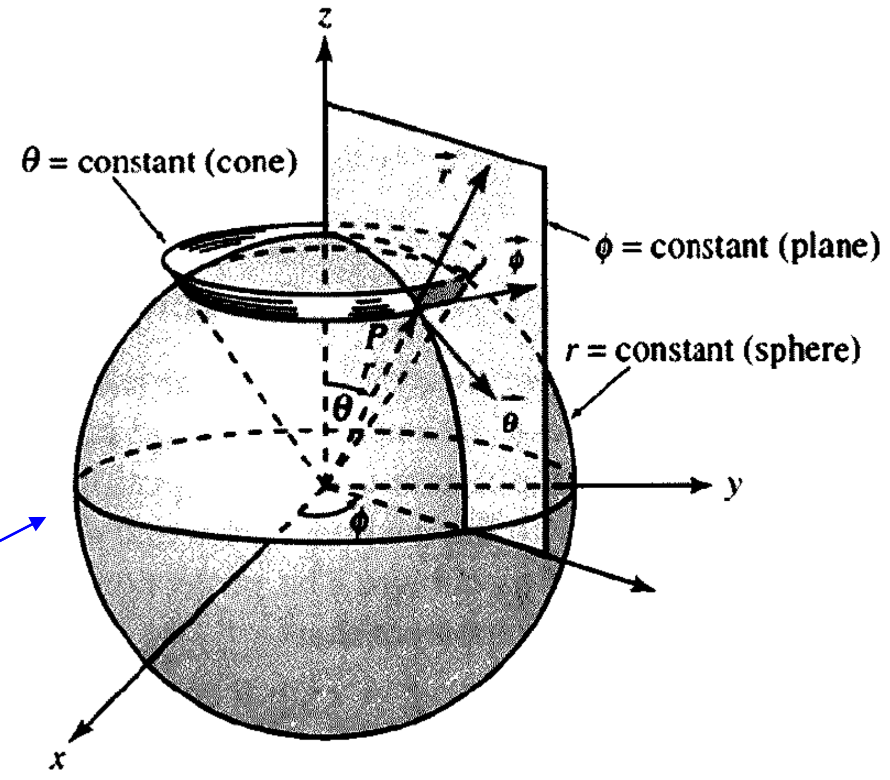
$$\begin{bmatrix} C_\rho \\ C_\phi \\ C_z \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

Thus, $\vec{\mathbf{C}} = 0.707\vec{\mathbf{a}}_\rho + 3.535\vec{\mathbf{a}}_\phi + 4\vec{\mathbf{a}}_z$ • • •

Note that the transformation of a vector from one coordinate system to another neither changes its magnitude nor its direction.

Spherical Coordinates





P: Cartesian coordinates *(x, y, z)*

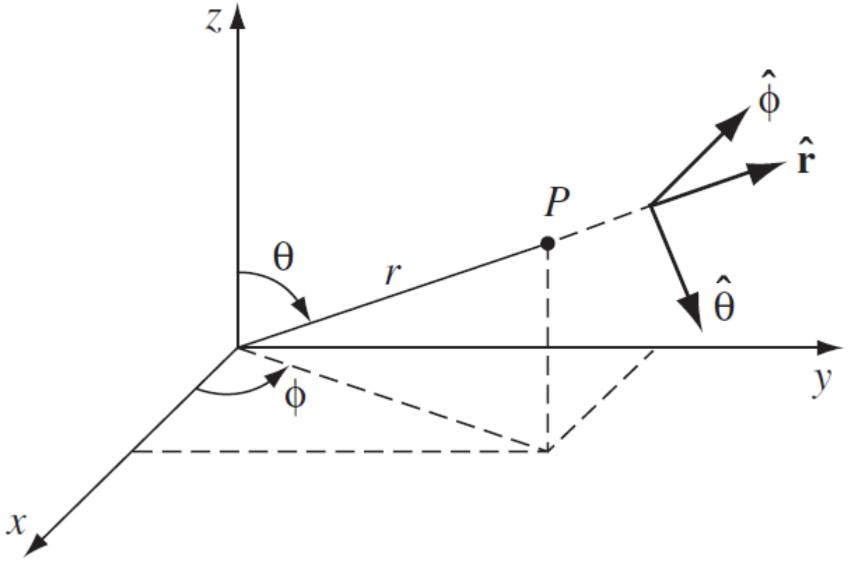
Spherical coordinates *(r, θ, φ)*;

r → distance from the origin (the magnitude of the position vector *r)*

θ → the angle with z axis → polar angle

φ → the angle around from the x axis → azimuthal angle

Spherical Coordinates



$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

$\hat{r}.\hat{r} = \hat{\theta}.\hat{\theta} = \hat{\varphi}.\hat{\varphi} =$

$\hat{r}.\hat{\theta} = \hat{\theta}.\hat{\varphi} = \hat{r}.\hat{\varphi} =$

$\hat{r}\mathrm{x}\hat{r} = \hat{\theta}\,\mathrm{x}\hat{\theta} = \hat{\varphi}\mathrm{x}\hat{\varphi} =$

$\hat{r}\mathrm{x}\hat{\theta} =$; $\hat{\theta}\mathrm{x}\hat{\varphi} =$; $\hat{\varphi}\mathrm{x}\ \hat{r} =$

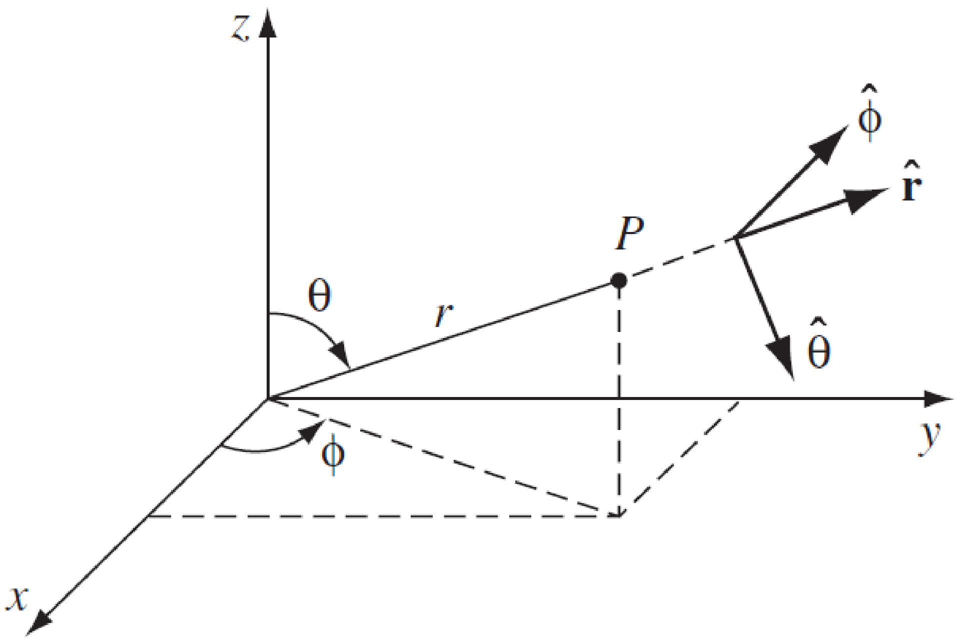
$\hat{\varphi}\mathrm{x}\ \hat{\theta} =$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left[\frac{z}{r}\right]$$
$$\phi = \tan^{-1}\left[\frac{y}{x}\right]$$

- three unit vectors: $\hat{r}, \hat{\theta}, \hat{\varphi}$ → pointing in the direction of increase of the corresponding coordinates.
- They constitute an orthogonal (mutually perpendicular) basis set
- any vector **A** can be expressed in terms of them, in the usual way:

$$\boxed{\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}}$$

Cartesian→ Spherical Coordinates



Cartesian → Spherical

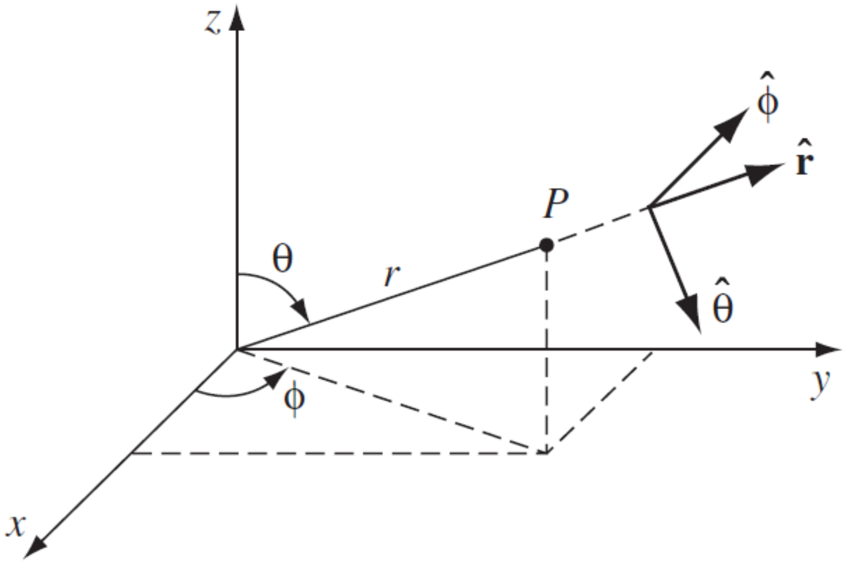
$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}
\end{aligned}$$

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix}$$

For any vector **A**

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Spherical → Cartesian Coordinates



From $A = A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}$ **to** $A = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$$A_x = A.\,\hat{x} = \left(A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}\right).\,\hat{x} = A_r\hat{r}.\hat{x} + A_\theta\hat{\theta}.\hat{x} + A_\varphi\hat{\varphi}.\,\hat{x}$$

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} &= \sin\theta\cos\phi, & \hat{\mathbf{r}}\cdot\hat{\mathbf{y}} &= \sin\theta\sin\phi, & \hat{\mathbf{r}}\cdot\hat{\mathbf{z}} &= \cos\theta \\
\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{x}} &= \cos\theta\cos\phi, & \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{y}} &= \cos\theta\sin\phi, & \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{z}} &= -\sin\theta \\
\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{x}} &= -\sin\phi, & \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{y}} &= \cos\phi, & \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} &= 0
\end{aligned}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Q: limits?
φ →
θ →?
r → ?

Problem -1

Two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are given at a point $P(r, \theta, \phi)$ in space as

$$\vec{\mathbf{A}} = 10\vec{\mathbf{a}}_r + 30\vec{\mathbf{a}}_\theta - 10\vec{\mathbf{a}}_\phi \quad \text{and} \quad \vec{\mathbf{B}} = -3\vec{\mathbf{a}}_r - 10\vec{\mathbf{a}}_\theta + 20\vec{\mathbf{a}}_\phi$$

Determine (a) $2\vec{\mathbf{A}} - 5\vec{\mathbf{B}}$, (b) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$, (c) $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$, (d) the scalar component of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$, (e) the vector projection of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$, and (f) a unit vector perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Solution Both vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are given at the same point P, so the rules of vector operations can be applied directly in the spherical coordinate system.

a) $2\vec{\mathbf{A}} - 5\vec{\mathbf{B}} = (20 + 15)\vec{\mathbf{a}}_r + (60 + 50)\vec{\mathbf{a}}_\theta + (-20 - 100)\vec{\mathbf{a}}_\phi$
$= 35\vec{\mathbf{a}}_r + 110\vec{\mathbf{a}}_\theta - 120\vec{\mathbf{a}}_\phi$

b) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = 10(-3) + 30(-10) + (-10)20 = -530$

c) $$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \vec{\mathbf{a}}_r & \vec{\mathbf{a}}_\theta & \vec{\mathbf{a}}_\phi \\ 10 & 30 & -10 \\ -3 & -10 & 20 \end{vmatrix} = 500\vec{\mathbf{a}}_r - 170\vec{\mathbf{a}}_\theta - 10\vec{\mathbf{a}}_\phi$$

d) The magnitude of $\vec{\mathbf{B}}$: $B = [(-3)^2 + (-10)^2 + (20)^2]^{1/2} = 22.561$
The scalar projection of $\vec{\mathbf{A}}$ onto $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{a}}_B = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{B} = \frac{-530}{22.561} = -23.492$$

Thank You