Engineering Electromagnetics

Lecture 19

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by

Debolina Misra

Dept. of Physics
IIITDM Kancheepuram, Chennai, India

Bound charge

Polarization creates → accumulation of **bound charges**

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

With these definitions, Eq. 4.10 becomes

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}\, d\tau'}.$$

Now suppose the material has free charges too! → ?

Griffith book

Gauss's Law in the Presence of Dielectrics

Polarization creates → accumulation of **bound charges**

$\boxed{\rho_b = -\nabla \cdot \mathbf{P} \text{ and } \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}}$

We are now ready to put it all together: the field attributable to bound charge plus the field due to everything *else* (which, for want of a better term, we call **free charge**, ρ_f). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, that is *not* a result of polarization. Within the dielectric, the total charge density can be written:

$$\boxed{\rho = \rho_b + \rho_f}$$

and Gauss's law reads $\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$

Gauss's Law in the Presence of Dielectrics

$\rho_b = -\nabla \cdot \mathbf{P}$ and $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$

Total charge density:

$$\rho = \rho_b + \rho_f$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

electric displacement. $\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}}$

In terms of $\mathbf{D}$, Gauss's law reads $\boxed{\nabla \cdot \mathbf{D} = \rho_f}$

or, in integral form, $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$

where $Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume.

Permittivity

For *linear* dielectrics **P** **E**

$$\vec{\mathbf{P}} = \epsilon_0 \chi \vec{\mathbf{E}} \tag{3.57}$$

where the proportionality constant χ is called the *electric susceptibility*, and the factor ϵ_0 is included to make it a dimensionless quantity.

Equation (3.56) can now be expressed as

$$\vec{\mathbf{D}} = \epsilon_0 (1 + \chi) \vec{\mathbf{E}} \tag{3.58a}$$

The quantity $(1 + \chi)$ is called the *relative permittivity* or the *dielectric constant* of the medium and is symbolized as ϵ_r. Thus, the general expression for the electric flux density finally becomes

$$\vec{\mathbf{D}} = \epsilon_0 \epsilon_r \vec{\mathbf{E}} = \epsilon \vec{\mathbf{E}} \tag{3.58b}$$

where $\epsilon = \epsilon_0 \epsilon_r$ is the permittivity of the medium.

Problem-1

A point charge q is enclosed in a linear, isotropic, and homogeneous dielectric medium of infinite extent. Calculate the $\vec{\mathbf{E}}$ field, the $\vec{\mathbf{D}}$ field, the polarization vector $\vec{\mathbf{P}}$, the bound surface charge density ρ_{sb}, and the bound volume charge density ρ_{vb}.

Since $\vec{\mathbf{E}}$, $\vec{\mathbf{D}}$, and $\vec{\mathbf{P}}$ are all parallel to one another in a linear medium, we still expect that the $\vec{\mathbf{E}}$ field would be in the $\vec{\mathbf{a}}_r$ direction. Thus, from Gauss's law, where q is the only free charge in the medium, we have

$$\oint_s \vec{\mathbf{D}} \cdot \vec{d\mathbf{s}} = q$$

or

$$4\pi r^2 D_r = q$$

Therefore,

$$\vec{\mathbf{D}} = \frac{q}{4\pi r^2}\, \vec{\mathbf{a}}_r$$

The electric field intensity, from (3.59), is

$$\vec{\mathbf{E}} = \frac{q}{4\pi \epsilon_0 \epsilon_r r^2} \vec{\mathbf{a}}_r$$

Thus, the presence of a dielectric material has reduced the $\vec{\mathbf{E}}$ field by a factor of ϵ_r but has left the $\vec{\mathbf{D}}$ field unchanged.

From (3.56), we can compute $\vec{\mathbf{P}}$ as

$$\begin{aligned}\vec{\mathbf{P}} &= \vec{\mathbf{D}} - \epsilon_0 \vec{\mathbf{E}} \\ &= \frac{q}{4\pi \epsilon_r r^2}(\epsilon_r - 1)\vec{\mathbf{a}}_r\end{aligned}$$

Note that $\nabla \cdot \vec{\mathbf{P}} = 0$. Therefore, the bound volume charge density, from (3.53), is zero.

Energy stored in Electric field

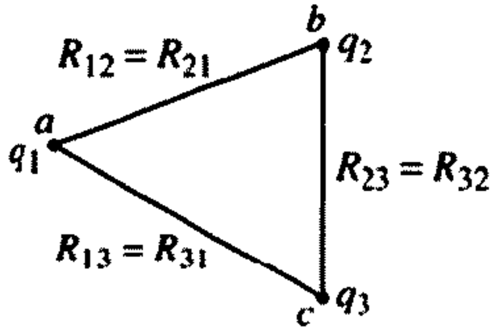


Figure 3.32 Potential energy in a system of three point charges

$$W = W_3 + W_2 + W_1 = 0 + q_2 V_{b,c} + q_1(V_{a,c} + V_{a,b})$$
$$= \frac{1}{4\pi\epsilon}\left[\frac{q_2 q_3}{R_{23}} + \frac{q_1 q_3}{R_{13}} + \frac{q_1 q_2}{R_{12}}\right]$$

$$W = \frac{1}{2}[q_1(V_{a,c} + V_{a,b}) + q_2(V_{b,a} + V_{b,c}) + q_3(V_{c,a} + V_{c,b})]$$

The total energy can now be written as

$$W = \frac{1}{2}[q_1 V_1 + q_2 V_2 + q_3 V_3] = \frac{1}{2}\sum_{i=1}^{3} q_i V_i$$

We can generalize this equation for a system of n point charges as

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V_i} \quad (3$$

Equation (3.64) allows us to compute the electrostatic potential energy for a group of point charges in their mutual field.

If the charges are continuously distributed, (3.64) becomes

$$\boxed{W = \frac{1}{2}\int_v \rho_v V\, dv} \quad (3.65)$$

where ρ_v is the volume charge density within v.

or

$$\boxed{W = \frac{1}{2}\int_s \rho_s V\, ds}$$

Let us now derive another expression for the energy in an electrostatic system in terms of the field quantities. Using Gauss's law, $\nabla \cdot \vec{\mathbf{D}} = \rho_v$, we can express (3.65) as

$$W = \frac{1}{2}\int_v V(\nabla \cdot \vec{\mathbf{D}})\,dv$$

However, using the vector identity, equation (2.126),

$$V(\nabla \cdot \vec{\mathbf{D}}) = \nabla \cdot (V\vec{\mathbf{D}}) - \vec{\mathbf{D}} \cdot \nabla V$$

we obtain the expression for the energy as

$$W = \frac{1}{2}\left[\int_v \nabla \cdot (V\vec{\mathbf{D}})\,dv - \int_v \vec{\mathbf{D}} \cdot (\nabla V)\,dv\right]$$

$$\int_v \nabla \cdot (V\vec{\mathbf{D}})\,dv = \oint_s V\vec{\mathbf{D}} \cdot \vec{ds}$$

Range of volume integral?

V and $\vec{\mathbf{D}}$ are negligibly small on the bounding surface,

$$\boxed{W = -\frac{1}{2}\int_v \vec{\mathbf{D}} \cdot (\nabla V)\,dv = \frac{1}{2}\int_v \vec{\mathbf{D}} \cdot \vec{\mathbf{E}}\,dv}$$

Electrostatic energy in terms of Field

If we define the **energy density**, the energy per unit volume

$$\boxed{w = \frac{1}{2}\vec{\mathbf{D}} \cdot \vec{\mathbf{E}} = \frac{1}{2}\epsilon E^2 = \frac{1}{2\epsilon}D^2}$$

$$W = \int_v w\,dv$$

or

$$\boxed{w = \frac{1}{2}\rho_v V}$$

as

$$W = \frac{1}{2}\int_v \rho_v V\,dv$$

An infinite plane carries a uniform surface charge σ. Find its electric field.

Draw a "Gaussian pillbox," extending equal distances above and below the plane (Fig. 2.22). Apply Gauss's law to this surface:

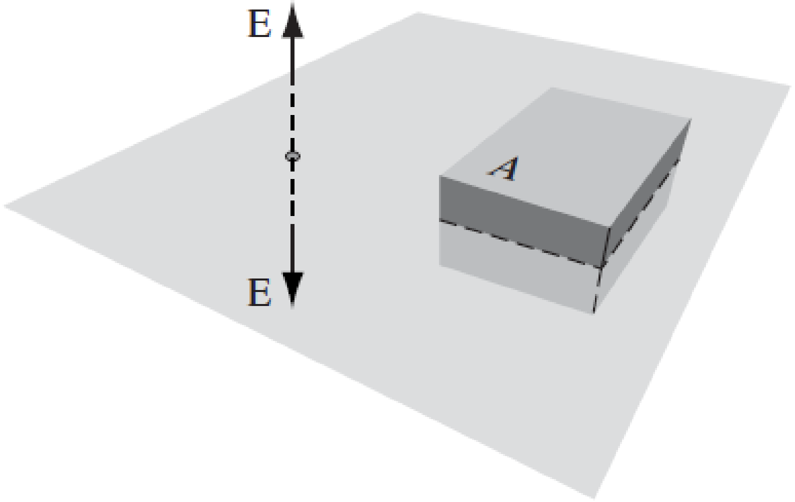
$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

In this case, $Q_{\text{enc}} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry, $\mathbf{E}$ points away from the plane (upward for points above, downward for points below). So the top and bottom surfaces yield

$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|,$$

whereas the sides contribute nothing. Thus

$$2A\,|\mathbf{E}| = \frac{1}{\epsilon_0}\sigma A,$$



Example 2.6. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ (Fig. 2.23). Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.

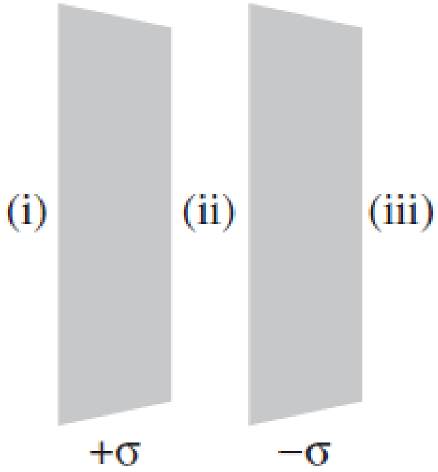
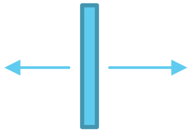


FIGURE 2.23

Hint: Fields due to a plane is on both the sides (See previous slide)



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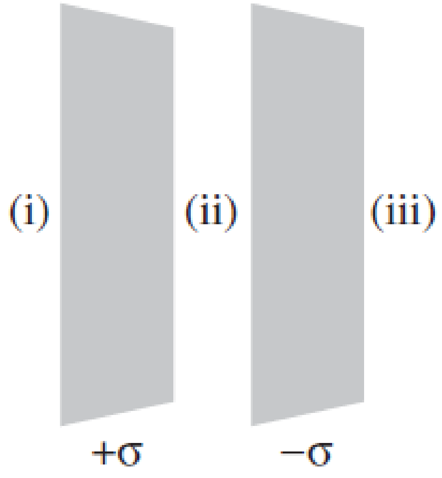

FIGURE 2.23

Solution

The left plate produces a field $(1/2\epsilon_0)\sigma$, which points away from it (Fig. 2.24)—to the left in region (i) and to the right in regions (ii) and (iii). The right plate, being negatively charged, produces a field $(1/2\epsilon_0)\sigma$, which points *toward* it—to the right in regions (i) and (ii) and to the left in region (iii). The two fields cancel in regions (i) and (iii); they conspire in region (ii). *Conclusion:* The field between the plates is σ/ϵ_0, and points to the right; elsewhere it is zero.

Q: What if the region between the plates is now filled with a dielectric with permittivity ε ?
What will be net **E** in region (ii)?

Capacitors

FIGURE 2.51

Since **E** is proportional to Q, so also is V. The constant of proportionality is called the **capacitance** of the arrangement:

$$\boxed{C \equiv \frac{Q}{V}}.$$

$$\boxed{\begin{aligned} Q &= CV \\ W &= \frac{1}{2}CV^2 \end{aligned}}$$

Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation of the two conductors. In SI units, C is measured in **farads** (F); a farad is a coulomb-per-volt. Actually, this turns out to be inconveniently large; more practical units are the microfarad (10^{-6} F) and the picofarad (10^{-12} F).

Griffith book

Thank You