Linear transformations

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(2) Let $V = \left\{ f(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n \; : \; n \in \mathbb{N}, c_i \in F \right\}$. We define a function $D \; : \; V \longrightarrow V$ as $(Df)(x) = c_1 + 2c_2x + \ldots + nc_nx^{n-1}$. Prove that D is a L.T.

Examples

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Prove that if T is a L.T., then

$$T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n) = c_1 T(\alpha_1) + c_2 T(\alpha_2) + \ldots + c_n T(\alpha_n)$$

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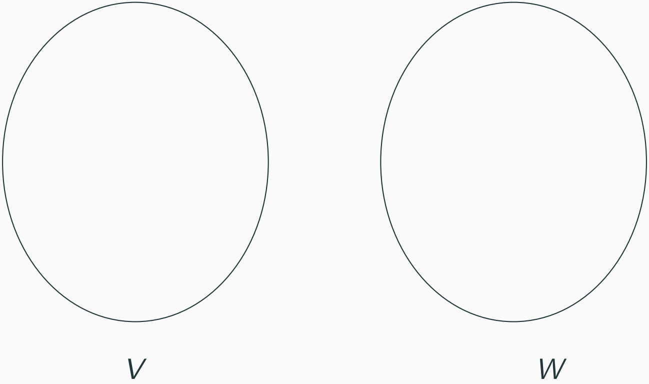
$\alpha = \beta = (1,0), \alpha + \beta = (2,0), T(\alpha + \beta) \neq T(\alpha) + T(\beta)$

Not a L.T.

Linear transformations are special !!



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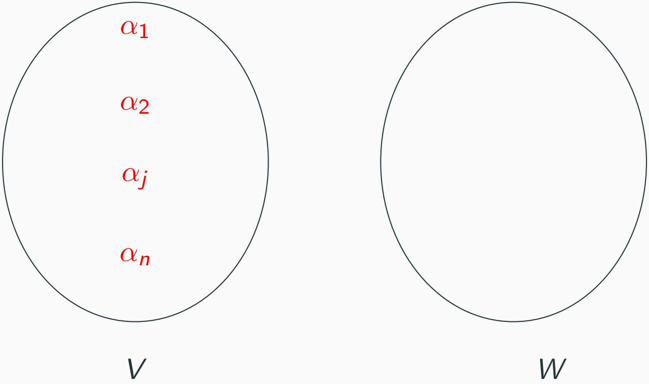

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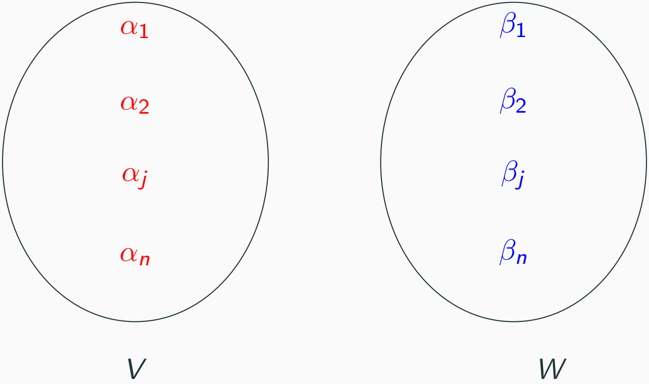

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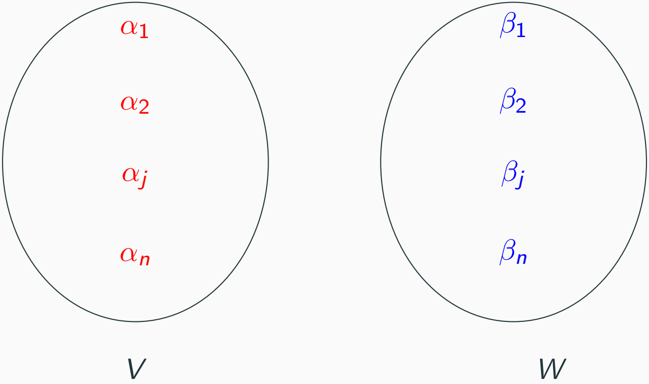

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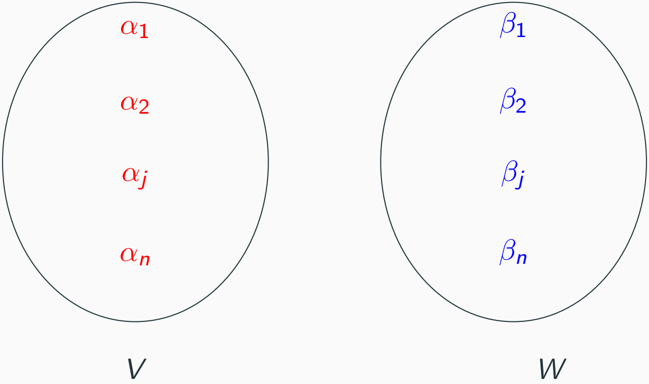

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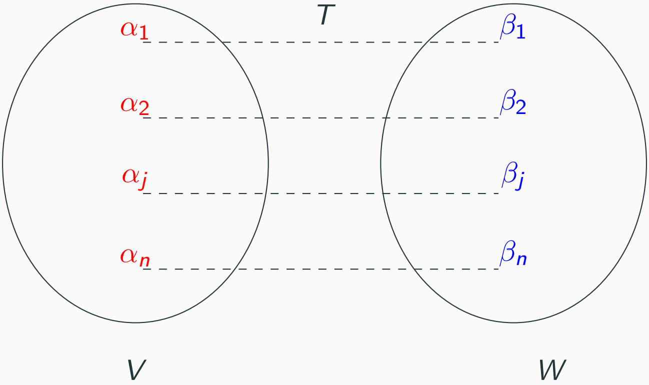

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Proof: Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is an ordered basis for V, for a given vector $\alpha \in V$, there is a unique $n-$tuple $(x_1, x_2, \ldots, x_n)$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n$$

Theorem 1 contd.

We define a function $T : V \longrightarrow W$ as
$T(\alpha) = T(x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n) = x_1\beta_1 + x_2\beta_2 + \ldots + x_n\beta_n$.

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$$c\alpha + \beta = (cx_1 + y_1)\alpha_1 + (cx_2 + y_2)\alpha_2 + \ldots + (cx_n + y_n)\alpha_n$$

Theorem 1 contd.

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It is enough to prove that if $U : V \longrightarrow W$ is a L.T. with $U(\alpha_j)=\beta_j$ for $j=1,2,\ldots,n$, then $T(\alpha)=U(\alpha)$ for all $\alpha \in V$.

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It completes the proof.

Problem 2

Let $B = \{\alpha_1 = (1, 2), \alpha_2 = (3, 4)\}$ be an ordered basis for R^2. Let $\beta_1 = (3, 2, 1)$, $\beta_2 = (6, 5, 4) \in R^3$. Find a unique L.T. $T : R^2 \longrightarrow R^3$ such that $T(\alpha_j) = \beta_j$ for $j = 1, 2$.

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Let $\alpha = (x,y) \in R^2$

$\alpha = a\alpha_1 + b\alpha_2 \Longrightarrow (x,y) = a(1,2) + b(3,4)$

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Let $B = \{\alpha_1 = (1,2), \alpha_2 = (3,4)\}$ be an ordered basis for R^2. Let $\beta_1 = (3,2,1)$, $\beta_2 = (6,5,4) \in R^3$. Find a unique L.T.
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Problem 2 contd.

$$T(x, y) = (\tfrac{3}{2}y, x + \tfrac{1}{2}y, 2x - \tfrac{1}{2}y) = \begin{bmatrix} 0 & \frac{3}{2} \\ 1 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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T is a unique L.T. thanks to Theorem 1.

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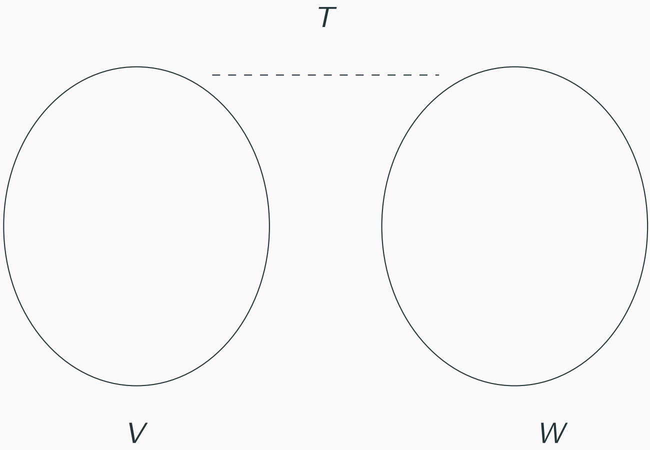

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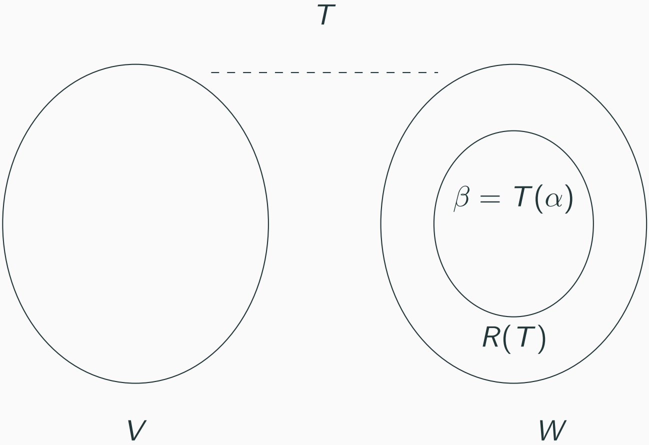

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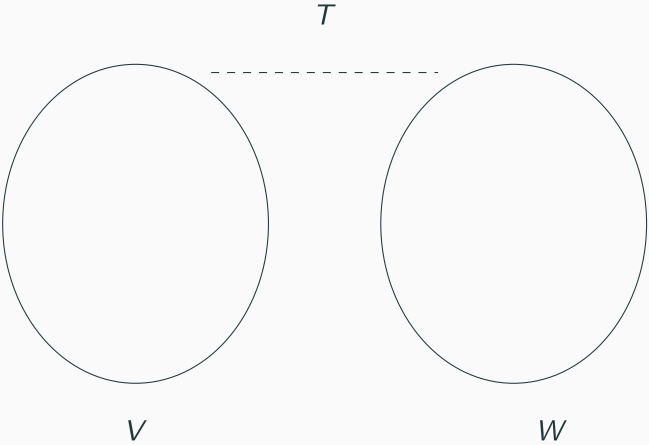

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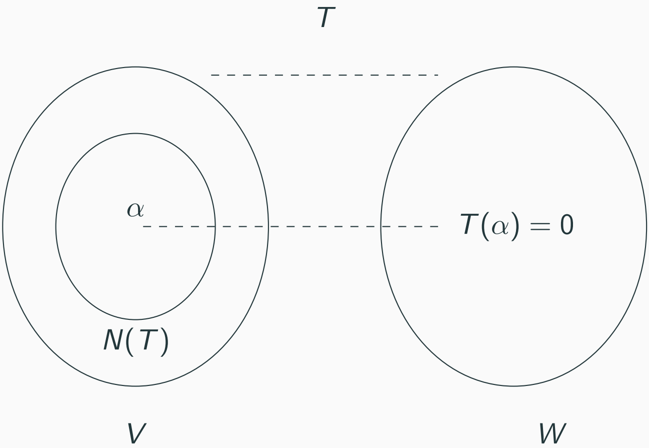

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Find the range and null space of the following linear transformations.

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Rank (O) **= dim** $R(O) = 0$ **and**

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Find the rank and nullity of the linear transformation
$T : R^2 \longrightarrow R^3$ defined as $T(x_1, x_2) = (x_1, 0, 0)$.

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Problem 3 contd.

$\implies R(T) =$ Column space of A

Problem 3 contd.

$\implies R(T) =$ Column space of $A =$ Row space of A^t.

Problem 3 contd.

$\implies R(T) =$ Column space of $A =$ Row space of A^t.

$\implies R(T) = \{a(1,0,0) \; : \; a \in R\}$

Problem 3 contd.

$\implies R(T) =$ Column space of $A =$ Row space of A^t.

$\implies R(T) = \{a(1,0,0) \ : \ a \in R\} =$ Span of $\{(1,0,0)\}$

Problem 3 contd.

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$\implies R(T) = \{a(1,0,0) \ : \ a \in R\} =$ Span of $\{(1,0,0)\}$

$$\implies \quad \text{Rank}(T) = 1$$

Problem 3 contd.

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$N(T) = \left\{X \in R^2 \ : \ TX = 0\right\}$

Problem 3 contd.

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Problem 3 contd.

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$$AX = 0 \implies \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$N(T) = \{(x_1, x_2) = (0, a) \ : \ a \in R\}$

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Problem 3 contd.

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$N(T) =$ Span $\{(0,1)\}$

$$\text{Nullity } (T) = 1$$

Problem 4

Show that
(i) $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$ is a linear transformation, and (ii)compute rank(T), nullity(T).

Problem 4

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(i) $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$ is a linear transformation, and (ii)compute rank(T), nullity(T).

Solution :

$$T(x_1, x_2, x_3) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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Hence T is a linear transformation.

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Range of T= Column space of A

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$$\implies T(X) = AX$$

Hence T is a linear transformation.

Range of T= Column space of A = Row space of A^t.

Problem 4 contd.

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}$$

Problem 4 contd.

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}$$

Problem 4 contd.

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}$$

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Problem 4 contd.

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Range of T = Row space of A^t = Span $\{(1,0,1),(0,1,-1)\}$

Problem 4 contd.

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}$$

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Range of T = Row space of A^t = Span $\{(1,0,1),(0,1,-1)\}$

Range of T = $\{a(1,0,1)+b(0,1,-1) \ : \ a,b \in R\}$

Problem 4 contd.

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}$$

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Problem 4 contd.

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rank $(T) =$ **dim** $R(T) = 2$

Problem 4 contd.

$$N(T) = \left\{X \in R^3 : \;\; TX = 0\right\}$$

Problem 4 contd.

$$N(T) = \left\{X \in R^3 : \;\; TX = 0\right\} = \left\{X \in R^3 : \;\; AX = 0\right\}$$

Problem 4 contd.

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Problem 4 contd.

$$N(T) = \{X \in R^3 : \; TX = 0\} = \{X \in R^3 : \; AX = 0\}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 4 contd.

$$N(T) = \left\{X \in R^3 : \; TX = 0\right\} = \left\{X \in R^3 : \; AX = 0\right\}$$

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Problem 4 contd.

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$$AX = 0 \Longrightarrow x_1 + \frac{2}{3}x_3 = 0, x_2 - \frac{4}{3}x_3 = 0$$

Problem 4 contd.

$$N(T) = \left\{X \in R^3 : \; TX = 0\right\} = \left\{X \in R^3 : \; AX = 0\right\}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}$$

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$$AX = 0 \Longrightarrow x_1 + \frac{2}{3}x_3 = 0, x_2 - \frac{4}{3}x_3 = 0$$

$$x_3 = a \Longrightarrow x_1 = -\frac{2}{3}a, x_2 = \frac{4}{3}a$$

Problem 4 contd.

$$N(T) = \left\{ \left(-\frac{2}{3}a, \frac{4}{3}a, a \right) \ : \ a \in R \right\}$$

Problem 4 contd.

$$N(T) = \left\{ \left(-\frac{2}{3}a, \frac{4}{3}a, a \right) \ : \ a \in R \right\} = \left\{ a \left(-\frac{2}{3}, \frac{4}{3}, 1 \right) \ : \ a \in R \right\}$$

Problem 4 contd.

$$N(T) = \left\{ \left(-\frac{2}{3}a, \frac{4}{3}a, a \right) \;:\; a \in R \right\} = \left\{ a \left(-\frac{2}{3}, \frac{4}{3}, 1 \right) \;:\; a \in R \right\}$$

$$N(T) = \text{ Span } \left\{ \left(-\frac{2}{3}, \frac{4}{3}, 1 \right) \right\}$$

Problem 4 contd.

$$N(T) = \left\{ \left(-\frac{2}{3}a, \frac{4}{3}a, a \right) \ : \ a \in R \right\} = \left\{ a \left(-\frac{2}{3}, \frac{4}{3}, 1 \right) \ : \ a \in R \right\}$$

$$N(T) = \text{ Span } \left\{ \left(-\frac{2}{3}, \frac{4}{3}, 1 \right) \right\}$$

Nullity (T) = 1.

Theorem 2 (Rank-Nullity-Dimension Theorem)

Let V and W be vector spaces over the field F and let $T : V \longrightarrow W$ be a linear transformation. Suppose that V is finite-dimensional. Then

$$\textbf{rank}(T) + \textbf{nullity}(T) = \textbf{dim}\, V$$

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Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis for $N(T)$ and let dim $V = n$.

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Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis for $N(T)$ and let dim $V = n$. Note that nullity $(T) = k$. Since $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq V$ and V is finite-dimensional, there exist vectors $\alpha_{k+1}, \ldots, \alpha_n \in V$ such that $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V, thanks to Corollary 2 of Theorem 5.

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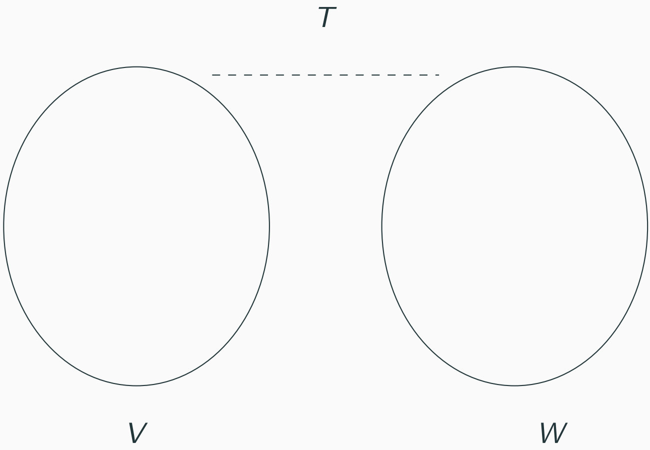
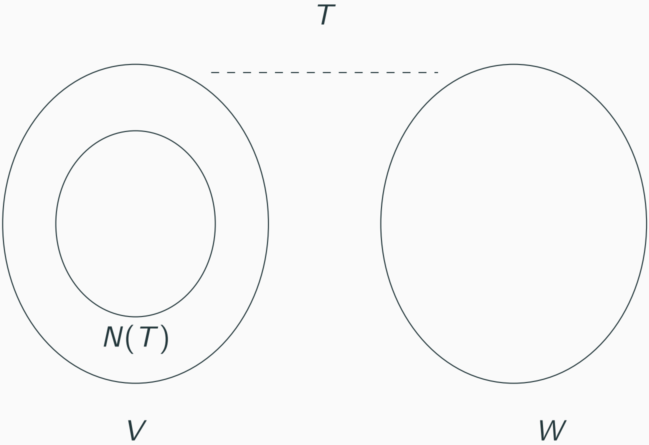
Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis for $N(T)$ and let dim $V = n$. Note that nullity $(T) = k$. Since $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq V$ and V is finite-dimensional, there exist vectors $\alpha_{k+1}, \ldots, \alpha_n \in V$ such that $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V, thanks to Corollary 2 of Theorem 5. Next, we prove that $B = \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ is a basis for $R(T)$.

Theorem 2 contd.



Theorem 2 contd.



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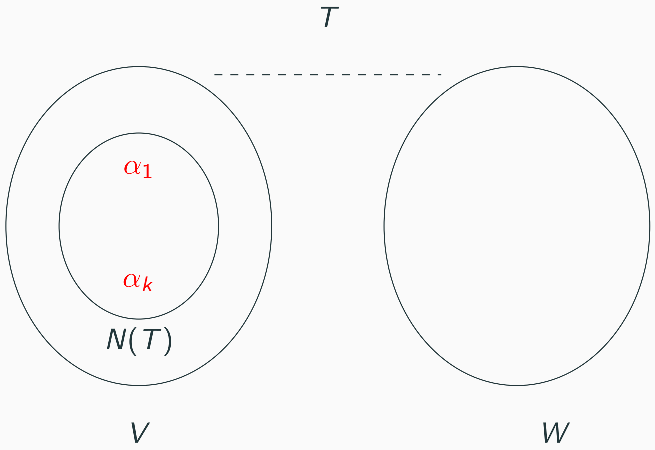

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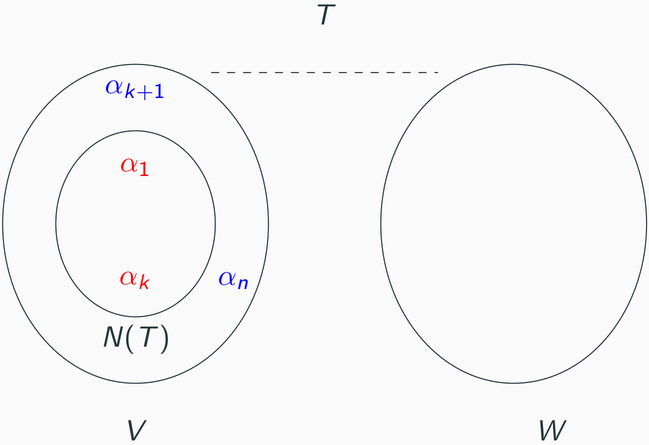
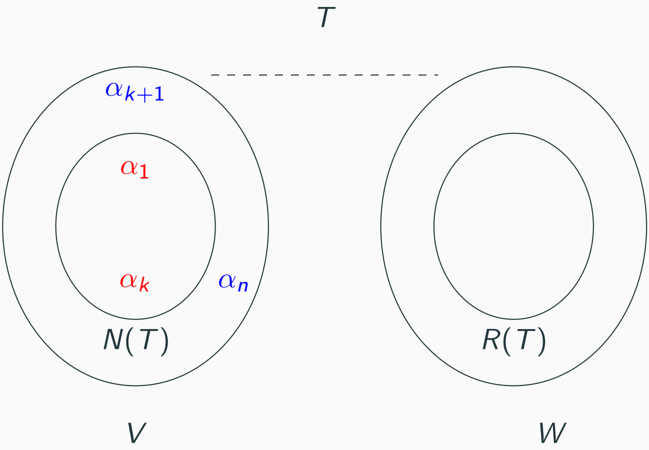
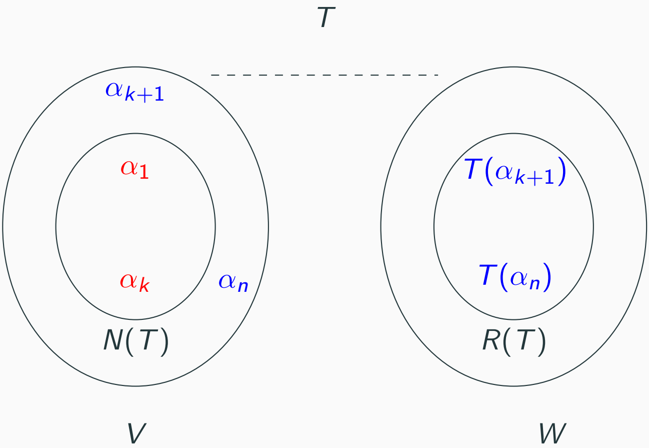


Theorem 2 contd.



Theorem 2 contd.



Theorem 2 contd.

Claim 1: $R(T) = \text{Span } B = \text{Span } \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$

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Let $\beta \in R(T)$.

Theorem 2 contd.

Claim 1: $R(T) = \text{Span } B = \text{Span } \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$

Let $\beta \in R(T)$. Then there exists $\alpha \in V$ such that $\beta = T(\alpha)$.

Theorem 2 contd.

Claim 1: $R(T) = \text{Span } B = \text{Span } \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$

Let $\beta \in R(T)$. Then there exists $\alpha \in V$ such that $\beta = T(\alpha)$. Since $\alpha \in V = \text{Span } \{\alpha_1, \ldots, \alpha_n\}$, there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$\alpha = c_1\alpha_1 + \ldots + c_k\alpha_k + c_{k+1}\alpha_{k+1} + \ldots c_n\alpha_n$$

Theorem 2 contd.

Claim 1: $R(T) = \text{Span } B = \text{Span } \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$

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$$\alpha = c_1\alpha_1 + \ldots + c_k\alpha_k + c_{k+1}\alpha_{k+1} + \ldots c_n\alpha_n$$

$$\beta = T(\alpha) = T(c_1\alpha_1 + \ldots + c_k\alpha_k + c_{k+1}\alpha_{k+1} + \ldots + c_n\alpha_n)$$

Theorem 2 contd.

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The above system has $n - r$ free variables and it implies that

$$\dim\ S = n - r = \dim\ N(T) = \text{nullity}(T) - -(4)$$

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Note : **rank** $(A) =$ **column rank** $(A) =$ **row rank** (A)

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$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_2 + 2x_3, -x_1 + 2x_2 + 2x_3)$$

Problem 6

Find a L.T. (if exists) $T : R^3 \longrightarrow R^3$ such that
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Justify your answer.

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Outline of the answer : Note that $\{\alpha_1 = (1, 1, 1)\}$ be a basis for $N(T)$. Using the basis of $N(T)$, we construct a basis for $V = R^3$, say $\{\alpha_1 = (1, 1, 1), \alpha_2 = (0, 1, 1), \alpha_3 = (0, 0, 1)\}$ (We have solved similar problems in the past!). Note that
$\beta_1 = (0, 0, 0), \beta_2 = (1, 0, -1), \beta_3 = (1, 2, 2) \in R(T)$.
Let us contruct T such that $T(\alpha_1) = T(1, 1, 1) = \beta_1 = (0, 0, 0)$,
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Problem 6 contd.

$$(x, y, z) = a\alpha_1 + b\alpha_2 + c\alpha_3 = a(1, 1, 1) + b(0, 1, 1) + c(0, 0, 1)$$

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$$\implies \quad T(x, y, z) = (-x + z, -2y + 2z, x - 3y + 2z)$$

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$$T(x, y, z) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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$\implies T(\beta) = T(T(\alpha)) = 0. \implies T(\alpha) = 0$ (by hypothesis).

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From (1) and (2), $N(T) \cap R(T) = \{0\}$.

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Observation 1 : $L(V, W)$ is a vector space under the opeartions

$$(T + U)(\alpha) = T(\alpha) + U(\alpha), \quad (cT)(\alpha) = cT(\alpha)$$

for all $T, U \in L(V, W)$, $c \in F$.

Observation 2 : If V and W are finite dimensional vector spaces, then $\dim L(V, W) = \dim V \dim W$.

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Linear Operator : If V is a vector space over the field F, then a linear operator T is a linear transformation $T : V \longrightarrow V$.