Problem Set-2

February 19, 2025

1. Suppose that we roll four fair six-sided dice. (a) What is the conditional probability that the first die shows 2, conditional on the the event that exactly three dice show 2? (b) What is the conditional probability that the first die shows 2, conditional on the event that at least three dice show 2?

2. Suppose a baseball pitcher throws fastballs 80% of the time and curve balls 20% of the time. Suppose a batter hits a home run on 8% of all fastball pitches and on 5% of all curve ball pitches. What is the probability that this batter will hit a home run on this pitcher's next pitch?

3. Suppose the probability of snow is 20%, and the probability of a traffic accident is 10%. Suppose further that the conditional probability of an accident, given that it snows, is 40%. What is the conditional probability that it snows, given that there is an accident?

4. Consider two urns, labelled urn #1 and urn #2. Suppose that urn #1 has 5 red and 7 blue balls, that urn #2 has 6 red and 12 blue balls, and that we pick three balls uniformly at random from each of the two urns. Conditional on the fact that all six chosen balls are the same color, what is the conditional probability that this color is red?

5. Suppose we roll a fair six-sided die and then flip a number of fair coins equal to the number showing on the die. (For example, if the die shows 4, then we flip 4 coins.) (a) What is the probability that the number of heads equals 3? (b) Conditional on knowing that the number of heads equals 3, what is the conditional the probability that the die showed the number 5?

6. Suppose we roll a fair six-sided die and then pick a number of cards from a well-shuffled deck equal to the number showing on the die. (For example, if the die shows 4, then we pick 4 cards.) (a) What is the probability that the number of jacks in our hand equals 2? (b) Conditional on knowing that the number of jacks in our hand equals 2, what is the the conditional probability that the die showed the number 3?

7. Suppose we flip three fair coins. (a) What is the probability that all three coins are heads? (b) What is the conditional probability that all three coins are heads, conditional on knowing that the number of heads is odd?

8. Consider three cards, as follows: One is red on both sides, one is black on both sides, and one is red on one side and black on the other. Suppose the cards are placed in a hat, and one is chosen at random. Suppose further that this card is placed flat on the table, so we can see one side only. (a) What is the probability that this one side is red? (b) Conditional on this one side being red, what is the probability that the card showing is the one that is red on both sides? (Hint: The answer is somewhat surprising.) (c) Suppose you wanted to verify the answer in part (b), using an actual, physical experiment. Explain how you could do this.

9. Prove that A and B are independent if and only if A^c and B are independent.

10. Let A and B be events of positive probability. Prove that $P(A|B) > P(A)$ if and only if $P(B|A) > P(B)$.

11. (The game of craps) The game of craps is played by rolling two fair, six-sided dice. On the first roll, if the sum of the two numbers showing equals 2, 3, or 12, then the player immediately loses. If the sum equals 7 or 11, then the player immediately wins. If the sum equals any other value, then this value becomes the player's "point." The player then repeatedly rolls the two dice, until such time as he or she either rolls the point value again (in which case he or she wins) or rolls a 7 (in which case he or she loses). (a) Suppose the player's point is equal to 4. Conditional on this, what is the conditional probability that he or she will win (i.e., will roll another 4 before rolling a 7)? (Hint: The final roll will be either a 4 or 7; what is the conditional probability that it is a 4?) (b) For $2 \leq i \leq 12$, let p_i be the conditional probability that the player will win, conditional on having rolled i on the first roll. Compute p_i for all i with $2 \leq i \leq 12$. (Hint: You've already done this for $i = 4$ in part (b). Also, the cases $i = 2, 3, 7, 11, 12$ are trivial. The other cases are similar to the $i = 4$ case.) (c) Compute the overall probability that a player will win at craps.

12. (The Monty Hall problem) Suppose there are three doors, labeled A, B, and C. A new car is behind one of the three doors, but you don't know which. You select one of the doors, say, door A. The host then opens one of doors B or C, as follows: If the car is behind B, then they open C; if the car is behind C, then they open B; if the car is behind A, then they open either B or C with probability $\frac{1}{2}$ each. (In any case, the door opened by the host will not have the car behind it.) The host then gives you the option of either sticking with your original door choice (i.e., A), or switching to the remaining unopened door (i.e., whichever of B or C the host did not open). You then win (i.e., get to keep the car) if and only if the car is behind your final door selection. (Source: Parade Magazine,

"Ask Marilyn" column, September 9, 1990.) Suppose for definiteness that the host opens door B.

(a) If you stick with your original choice (i.e., door A), conditional on the host having opened door B, then what is your probability of winning?

(b) If you switch to the remaining door (i.e., door C), conditional on the host having opened door B, then what is your probability of winning?

(c) Do you find the result of parts (a) and (b) surprising? How could you design a physical experiment to verify the result?

(d) Suppose we change the rules so that, if you originally chose A and the car was indeed behind A, then the host always opens door B. How would the answers to parts (a) and (b) change in this case?

(e) Suppose we change the rules so that, if you originally chose A, then the host always opens door B no matter where the car is. We then condition on the fact that door B happened not to have a car behind it. How would the answers to parts (a) and (b) change in this case?