MA1000: Calculus

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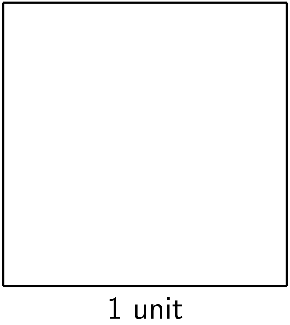
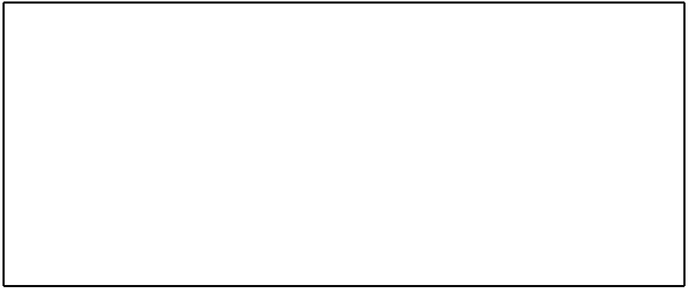
Riemann Integral: Motivation

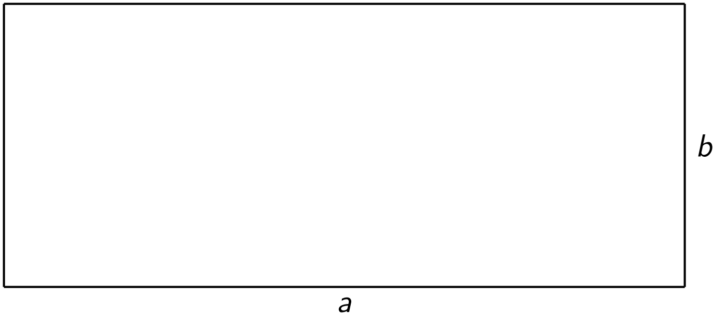


The area of a unit square is 1 square unit.

Area of a Rectangle



The Area of a Rectangle



Divide into unit squares? Yes!

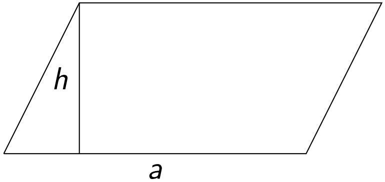
The Area of a Rectangle

Divide into unit squares and count them!

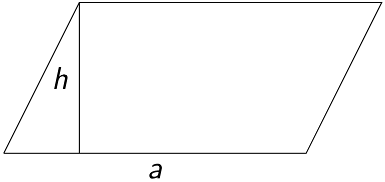
Area of the rectangle is ab square units.

Note: The formula is valid for all non-negative real numbers.

The Area of a Parallelogram

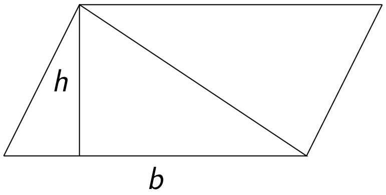


The Area of a Parallelogram

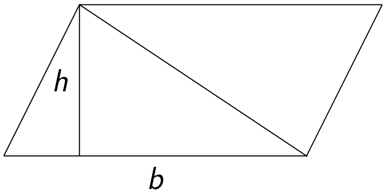


The area is ah square units.

The Area of a Triangle



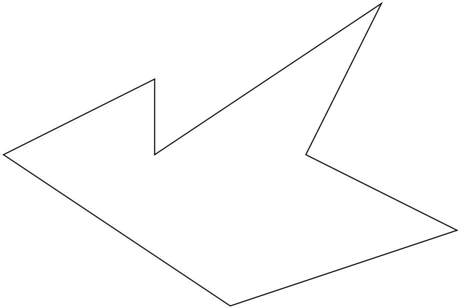
The Area of a Triangle



Area of the parallelogram is bh square units.

Hence the area of the triangle is $\frac{1}{2}bh$ square units.

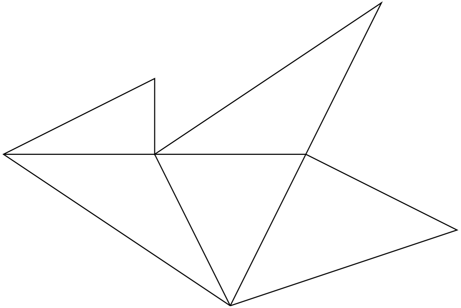
The Area of a Polygonal Region



Triangulate!

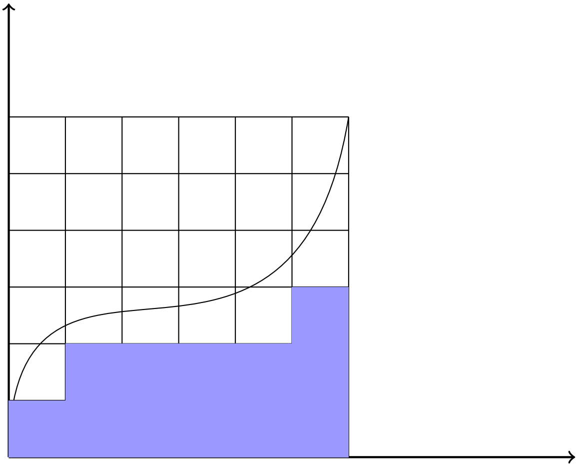
The Area of a Polygonal Region

Triangulate and add the areas of the triangles!

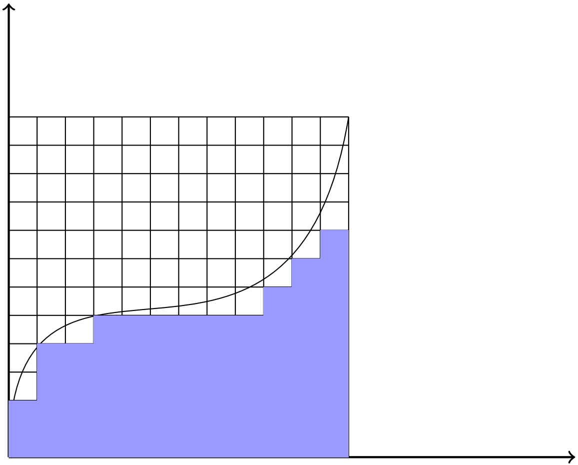


The area of the polygon is the sum of the areas of the triangles!

Computing area of an irregular object



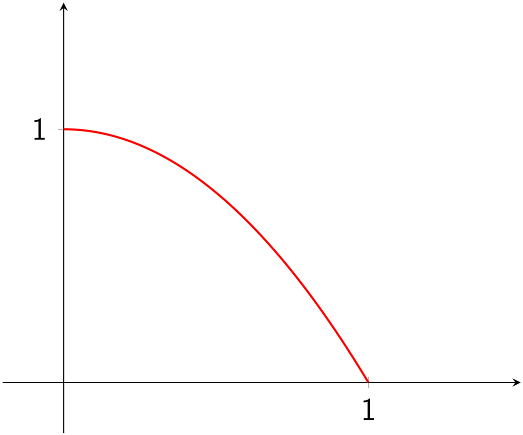
Increasing the Number of Cuts



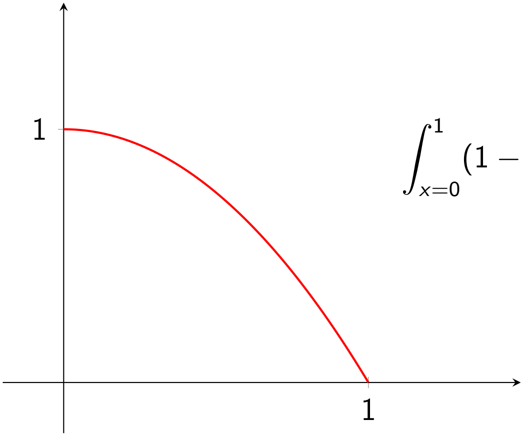
Note

1. Thus increase in number of cuts leads to a better estimate of the required area.
2. How to find the exact area? By cutting into infinitely many small square? Does it make sense? It does! Riemann Integration does exactly that!

Area Under the Curve $y = 1 - x^2$ in the First Quadrant

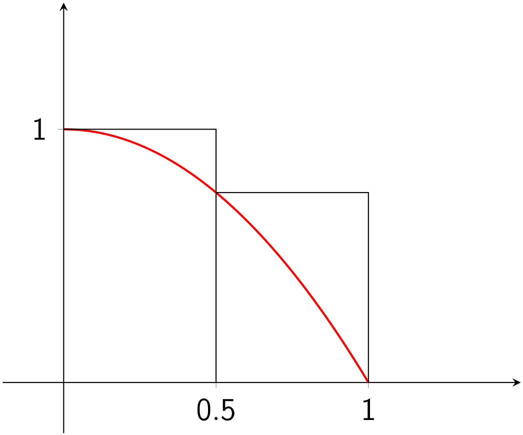


Area Under the Curve $y = 1 - x^2$ in the First Quadrant



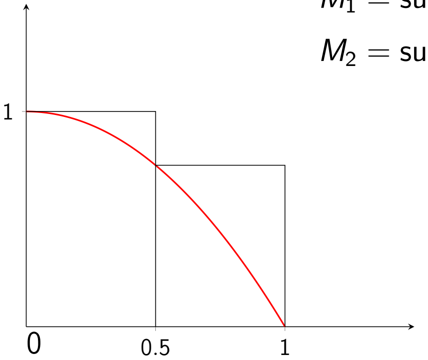
$$\int_{x=0}^{1} (1 - x^2) dx = \frac{2}{3} \approx 0.667$$

Area under the curve $y = 1 - x^2$: An Upper Estimate



An Upper Estimate

$$M_1 = \sup\{f(x) : x \in [0, 0.5]\} = 1$$
$$M_2 = \sup\{f(x) : x \in [0.5, 1]\} = \frac{3}{4}$$

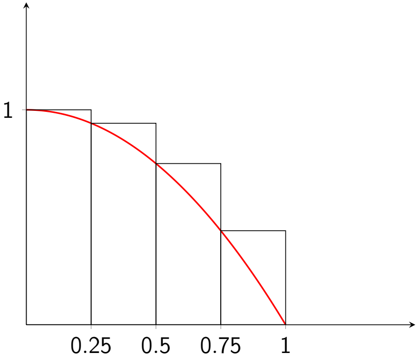


The total area of the two rectangles is

$$1 \cdot 0.5 + \frac{3}{4} \cdot 0.5 = 0.875.$$

This is an upper bound on the required area. It is called an **upper sum**.

Area Under the Curve $y = 1 - x^2$: A Better Upper Estimate

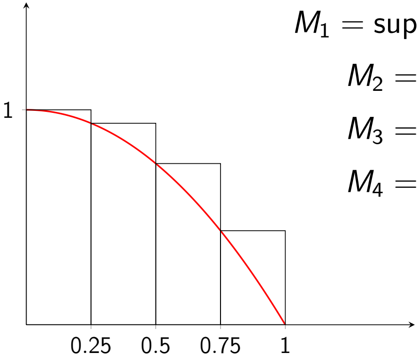


The total area of the four rectangles is

$$1 \cdot 0.25 + \frac{15}{16} \cdot 0.25 + \frac{3}{4} \cdot 0.25 + \frac{7}{16} \cdot 0.25 = 0.78125.$$

This is a better upper bound on the required area. Moreover, it is an **upper sum**.

Area Under the Curve $y = 1 - x^2$: A Better Upper Estimate



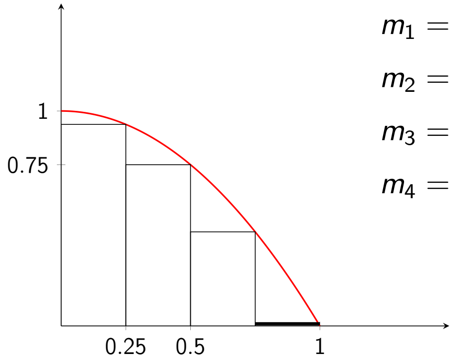
$$M_1 = \sup\{1 - x^2 : x \in [0, 0.25]\} = 1$$
$$M_2 = \sup\{1 - x^2 : x \in [0.25, 0.5]\}$$
$$M_3 = \sup\{1 - x^2 : x \in [0.5, 0.75]\}$$
$$M_4 = \sup\{1 - x^2 : x \in [0.75, 1]\} =$$

The total area of the four rectangles is

$$1 \cdot 0.25 + \frac{15}{16} \cdot 0.25 + \frac{3}{4} \cdot 0.25 + \frac{7}{16} \cdot 0.25 = 0.78125.$$

This is a better upper bound on the required area. Moreover, it is an **upper sum**.

Area Under the Curve $y = 1 - x^2$: A Lower Estimate



$m_1 = \inf \{1 - x^2 : x \in [0, 0.25]\} =$

$m_2 = \inf \{1 - x^2 : x \in [0.25, 0.5]\}$

$m_3 = \inf \{1 - x^2 : x \in [0.5, 0.75]\}$

$m_4 = \inf \{1 - x^2 : x \in [0.75, 1]\} =$

The total area of the four rectangles is

$$\frac{15}{16} \cdot 0.25 + \frac{3}{4} \cdot 0.25 + \frac{7}{16} \cdot 0.25 + 0 \cdot 0.25 = 0.53125.$$

It is called a **lower sum**.

Definition (Partition)

Let $[a, b]$ be a closed interval. Then a set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is called a partition of the interval if

$$a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = b.$$

Notation

If $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of an interval $[a, b]$, we denote the length of the ith subinterval $[x_{i-1}, x_i]$ by

$$\Delta x_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n.$$

Note

$$\sum_{i=1}^{n} \Delta x_i = (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1}) = x_n - x_0 = b - a.$$

Examples

Consider the interval $[0, 1]$. The following are some partitions of it:

(1) $P_1 = \{0, 0.5, 1\}$.

1. $P_2 = \{0, 0.25, 0.5, 1\}$.
2. $P_3 = \{0, 0.25, 0.5, 0.75, 1\}$.
3. $P_4 = \{0, 0.1, 0.3, 0.7, 1\}$.
4. $P_5 = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

For partition P_1,

$$x_0 = 0, x_1 = 0.5, x_2 = 1 \quad \text{and} \quad \Delta x_1 = 0.5, \Delta x_2 = 0.5.$$

For partition P_2,

$$x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 1 \quad \text{and} \quad \Delta x_1 = 0.25, \Delta x_2 = 0.25, \Delta x_3 = 0.5.$$

Definition (Upper Riemann Sum, Lower Riemann Sum)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$ and let

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \quad \text{and} \quad m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

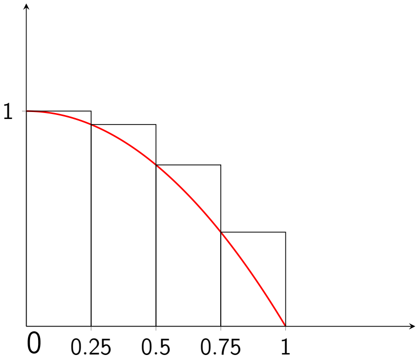
Then the **upper Riemann sum** corresponding to the partition P is

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Similarly, the **lower Riemann sum** corresponding to P is

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

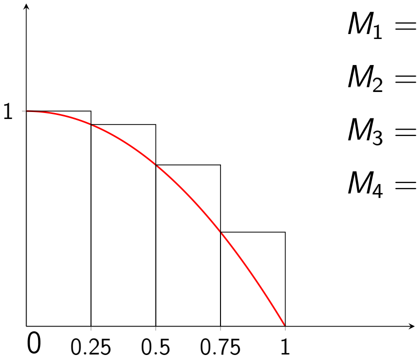
Example



The upper Riemann sum with $P = \{0, 0, 25, 0.5, 0.75, 1\}$ is

$$
\begin{aligned}
U(P, f) &= M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + M_4 \Delta x_4 \\
&= 1 \cdot 0.25 + \frac{15}{16} \cdot 0.25 + \frac{3}{4} \cdot 0.25 + \frac{7}{16} \cdot 0.25 = 0.78125.
\end{aligned}
$$

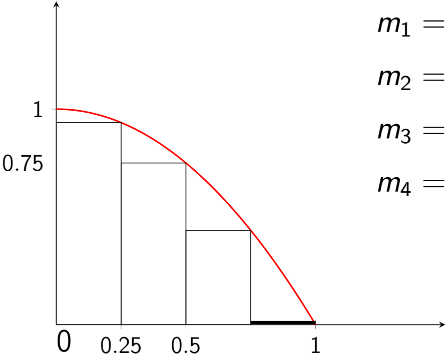
Example



$$M_1 = \sup\{1 - x^2 : x \in [0, 0.25]\}$$
$$M_2 = \sup\{1 - x^2 : x \in [0.25, 0.5]\}$$
$$M_3 = \sup\{1 - x^2 : x \in [0.5, 0.75]\}$$
$$M_4 = \sup\{1 - x^2 : x \in [0.75, 1]\}$$

The upper Riemann sum with $P = \{0, 0, 25, 0.5, 0.75, 1\}$ is

$$\begin{aligned} U(P, f) &= M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + M_4 \Delta x_4 \\ &= 1 \cdot 0.25 + \frac{15}{16} \cdot 0.25 + \frac{3}{4} \cdot 0.25 + \frac{7}{16} \cdot 0.25 = 0.78125. \end{aligned}$$



$$m_1 = \inf\{1 - x^2 : x \in [0, 0.25]\}$$
$$m_2 = \inf\{1 - x^2 : x \in [0.25, 0.5]\}$$
$$m_3 = \inf\{1 - x^2 : x \in [0.5, 0.75]\}$$
$$m_4 = \inf\{1 - x^2 : x \in [0.75, 1]\}$$

The lower Riemann sum corresponding to the partition $P = \{0, 0, 25, 0.5, 0.75, 1\}$ is

$$\begin{aligned} L(P, f) &= m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + m_4 \Delta x_4 \\ &= \frac{15}{16} \cdot 0.25 + \frac{3}{4} \cdot 0.25 + \frac{7}{16} \cdot 0.25 + 0 \cdot 0.25 = 0.53125. \end{aligned}$$

Note

In the preceding example, $L(P, f) \le U(P, f)$. It is true always:

Lemma

Let $f : [a, b] \to \mathbb{R}$ *be a bounded function with*

$$m \le f(x) \le M.$$

Then for any partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ *of* $[a, b]$

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a).$$

Proof:

Let

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \quad \text{and} \quad m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \quad (i = 1, 2, \ldots, n).$$

Then

$$\begin{aligned}
m \le m_i \le M_i \le M \quad &\Rightarrow \quad m\Delta x_i \le m_i\Delta x_i \le M_i\Delta x_i \le M\Delta x_i \\
&\Rightarrow \quad \sum_{i=1}^{n} m\Delta x_i \le \sum_{i=1}^{n} m_i\Delta x_i \le \sum_{i=1}^{n} M_i\Delta x_i \le \sum_{i=1}^{n} M\Delta x_i \\
&\Rightarrow \quad m\sum_{i=1}^{n} \Delta x_i \le \sum_{i=1}^{n} m_i\Delta x_i \le \sum_{i=1}^{n} M_i\Delta x_i \le M\sum_{i=1}^{n} \Delta x_i \\
&\Rightarrow \quad m(b-a) \le L(P, f) \le U(P, f) \le M(b-a).
\end{aligned}$$

Corollary

Let f be a bounded real-valued function on $[a, b]$. Then the set of all lower sums is bounded above. And the set of all upper sums is bounded below:

$$L(P, f) \leq M(b-a) \qquad \text{and} \qquad m(b-a) \leq U(P, f).$$

Refinement of a Partition and Common Refinedment

Definition (Refinement, Common Refinement)

1. Let P_1 be a partition of $[a, b]$. Then a partition P_2 of $[a, b]$ is called a **refinement** of P_1 if $P_1 \subseteq P_2$.
2. Let P_1 and P_2 be partitions of $[a, b]$. Then $P_1 \cup P_2$ is called a **common refinement** of P_1 and P_2.

Examples:

1. Consider the interval $[0, 1]$ and its partitions $P_1 = \{0, 0.5, 1\}$ and $P_2 = \{0, 0.5, 0.75, 1\}$. Here $P_1 \subseteq P_2$. So, P_2 is a refinement of P_1.
2. Consider the interval $[0, 1]$ and its partitions $P_1 = \{0, 0.25, 0.5, 1\}$ and $P_2 = \{0, 0.5, 0.75, 1\}$. Then their common refinement is $P_1 \cup P_2 = \{0, 0.25, 0.5, 0.75, 1\}$.

Lemma

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let P_1 and P_2 be partitions of $[a, b]$ such that P_2 is a refinement of P_1. Then

$$L(P_1, f) \leq L(P_2, f) \leq U(P_2, f) \leq U(P_1, f).$$

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let P_1 and P_2 be any partitions of $[a, b]$. Then

$$L(P_1, f) \leq U(P_2, f).$$

Proof: Let $Q = P_1 \cup P_2$ be the common refinement of P_1 and P_2. Then by the above lemma

$$L(P_1, f) \leq L(Q, f) \leq U(Q, f) \leq U(P_2, f).$$

Recall

Corollary

Let f be a bounded real-valued function on $[a, b]$. Then the set of all lower sums

$$\{L(P, f) \ : \ P \text{ is a partition of } [a, b]\}$$

is bounded above by $M(b - a)$. And the set of all upper sums

$$\{U(P, f) \ : \ P \text{ is a partition of } [a, b]\}$$

is bounded below by $m(b - a)$.

What is the improvement on this corollary due to the preceding theorem?

Definition

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The **upper Riemann integral** of f over $[a, b]$ is

$$\overline{\int_a^b} f(x)dx = \inf \{U(P, f) \; : \; P \text{ is a partition of } [a, b]\} = \inf U(P, f).$$

The **lower Riemann integral** of f over $[a, b]$ is

$$\underline{\int_a^b} f(x)dx = \sup \{L(P, f) \; : \; P \text{ is a partition of } [a, b]\} = \sup L(P, f).$$

Homework: Prove the following:

$$\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx.$$

Definition

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We say **f is Riemann integrable on $[a.b]$** if the upper and lower Riemann integrals are equal:

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$$

In this case, we say f is Riemann integrable on $[a, b]$ ($f \in \mathcal{R}$) and denote the common value by

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Example

Show that $f(x) = k$ (a constant function) on $[a, b]$ is Riemann integrable.

Solution : Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Then

$$M_i = \sup\{f(x) \ : \ x_{i-1} \leq x \leq x_i\} = \sup\{k \ : \ x_{i-1} \leq x \leq x_i\} = k.$$

$$m_i = \inf\{f(x) \ : \ x_{i-1} \leq x \leq x_i\} = \inf\{k \ : \ x_{i-1} \leq x \leq x_i\} = k.$$

Thus the upper Riemann sum

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} k \Delta x_i = k \sum_{i=1}^{n} \Delta x_i = k(b - a).$$

That is, for any partition P of $[a, b]$:

$$U(P, f) = k(b - a).$$

Thus, the upper Riemann integral of f over $[a, b]$ is

$$\overline{\int_a^b} f(x)dx = \inf\{U(P, f) \ : \ P \text{ is a partition of } [a, b]\} = \inf\{k(b - a)\} = k(b-a).$$

And the lower Riemann sum

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} k \Delta x_i = k \sum_{i=1}^{n} \Delta x_i = k(b-a).$$

That is, for any partition P of $[a,b]$:

$$L(P,f) = k(b-a).$$

Thus the lower Riemann integral of f over $[a,b]$ is

$$\underline{\int_a^b} f(x)dx = \sup \{L(P,f) \ : \ P \text{ is a partition of } [a,b]\} = \sup \{k(b-a)\} = k(b-a).$$

Conclusion

The upper Riemann integral of the constant function $f(x) = k$ over $[a, b]$ is

$$\overline{\int_a^b} f(x)dx = k(b-a).$$

And the lower Riemann integral of $f(x) = k$ over $[a, b]$ is

$$\underline{\int_a^b} f(x)dx = k(b-a).$$

Thus, the upper and lower Riemann integrals are equal. Hence the function is Riemann integrable and

$$\int_a^b f(x)dx = \int_a^b kdx = k(b-a).$$

Example of a non-Riemann Integrable Bounded Function

Let $f : [0,1] \longrightarrow R$ be the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number} \end{cases}$$

Show that $f \notin \mathcal{R}$ (f is not a Riemann integrable function)

Solution : Note that $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. So, it is a bounded function on $[0,1]$.

Let $P = \{0 = x_0, x_1, \ldots, x_n = 1\}$ be any partition of $[0,1]$:

$$0 = x_0 \leq x_1 \leq \ldots \leq x_n = 1.$$

$$M_i = \sup\{f(x) \ : \ x_{i-1} \leq x \leq x_i\} = \sup\{1,0\} = 1.$$

$$m_i = \inf\{f(x) \ : \ x_{i-1} \leq x \leq x_i\} = \inf\{1,0\} = 0.$$

The upper Riemann sum :

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1 - 0 = 1$$

The lower Riemann sum :

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = 0.$$

Note that we computed $U(P,f)$ and $L(P,f)$ for an arbitrary partition P on $[0,1]$. Thus the upper Riemann integral of f over $[0,1]$ is

$$\overline{\int_0^1} f(x)dx = \inf \{U(P,f) \ : \ P \text{ is a partition of } [0,1]\} = \inf \{1\} = 1.$$

The lower Riemann integral of f over $[0,1]$ is

$$\underline{\int_0^1} f(x)dx = \sup\{L(P,f) \ : \ P \text{ is a partition of } [a,b]\} = \sup\{0\} = 0$$

Thus

$$\overline{\int_0^1} f(x)dx \neq \underline{\int_0^1} f(x)dx.$$

Hence f is not Riemann integrable on $[0,1]$.

Homework

Let $f : [0, 1] \rightarrow R$ be a function defined as

$$f(x) = \begin{cases} 1, & \text{if } \ x \ \text{ is a rational number} \\ \text{-}1, & \text{if } \ x \ \text{ is an irrational number} \end{cases}$$

Prove or disprove the following: (i) f is a Riemann integrable function and (ii) $|f|$ is a Riemann integrable function.

An Important Theorem

Theorem

Let $f : [a, b] \to \mathbb{R}$ *be a bounded function. Then* f *is Riemann integrable if and only if for every* $\epsilon > 0$, *there exists a partition* P *such that*

$$U(P, f) - L(P, f) < \epsilon.$$

Example

Let $f(x) = x$ be a function defined on $[a, b]$. Let $h = \frac{b-a}{n}$.
Let $P_n = \{a = x_0, x_1 = a + h, x_2 = a + 2h, \ldots, x_n = a + nh = b\}$.
Compute (i) $L(P_n, f)$ and (ii) $U(P_n, f)$.

Solution :

$$\begin{aligned} m_i &= \inf\{f(x) \ : \ x_{i-1} \le x \le x_i\} \\ &= \inf\{x \ : \ a + (i-1)h \le x \le a + ih\} \\ &= a + (i-1)h. \end{aligned}$$

$$
\begin{aligned}
L(P_n, f) &= \sum_{i=1}^{n} m_i \Delta x_i \\
&= \sum_{i=1}^{n} \left(a + (i-1)h\right) h \\
&= ah\sum_{i=1}^{n}(1) + h^2\sum_{i=1}^{n}(i-1) \\
&= ahn + h^2\left(0 + 1 + 2 + \ldots + (n-1)\right) \\
&= a \times nh + h^2 \tfrac{n(n-1)}{2} \\
&= a(b-a) + \tfrac{(b-a)^2}{n^2}\tfrac{n(n-1)}{2} = a(b-a) + \tfrac{(b-a)^2}{2}(1 - \tfrac{1}{n})
\end{aligned}
$$

We observe that

$$\lim_{n\longrightarrow\infty} L(P_n, f) = a(b-a) + \frac{(b-a)^2}{2}(1-0) = \frac{b-a}{2}(2a+b-a) = \frac{b^2-a^2}{2}.$$

$$
\begin{aligned}
M_i &= \sup \{ f(x) \ : \ x_{i-1} \leq x \leq x_i \} \\
&= \sup \{ x \ : \ a + (i-1)h \leq x \leq a + ih \} \\
&= a + ih
\end{aligned}
$$

$$
\begin{aligned}
U(P_n, f) &= \sum_{i=1}^{n} M_i \Delta x_i \\
&= \sum_{i=1}^{n} (a + ih)\, h \\
&= ah \sum_{i=1}^{n} (1) + h^2 \sum_{i=1}^{n} (i) \\
&= ahn + h^2 \left(1 + 2 + \ldots + n\right) \\
&= a \times nh + h^2 \tfrac{n(n+1)}{2} \\
&= a(b-a) + \tfrac{(b-a)^2}{n^2} \tfrac{n(n+1)}{2} \\
&= a(b-a) + \tfrac{(b-a)^2}{2} (1 + \tfrac{1}{n})
\end{aligned}
$$

Thus

$$
\lim_{n \longrightarrow \infty} U(P_n, f) = a(b-a) + \frac{(b-a)^2}{2}(1+0) = \frac{b-a}{2}\,(2a + b - a) = \frac{b^2 - a^2}{2}.
$$

We have

$$\lim_{n\longrightarrow\infty} L(P_n, f) = \frac{b^2 - a^2}{2} \qquad \text{and} \qquad \lim_{n\longrightarrow\infty} U(P_n, f) = \frac{b^2 - a^2}{2}.$$

Since both $L(P_n, f)$ and $U(P_n, f)$ converge have the same limit, we have

$$\lim_{n\longrightarrow\infty} [U(P_n, f) - L(P_n, f)] = 0.$$

Then, for each $\epsilon > 0$ there corresponds an integer N such that

$$n \geq N \Longrightarrow U(P_n, f) - L(P_n, f) < \epsilon.$$

So, corresponding to each $\epsilon > 0$, there is a partition P_n such that

$$U(P_n, f) - L(P_n, f) < \epsilon.$$

Hence, by theorem, the function is Riemann integrable.
Also

$$L(P_n, f) \leq \int_a^b f(x)dx \leq U(P_n, f).$$

Hence

$$\int_a^b f(x)dx = \int_a^b xdx = \frac{b^2 - a^2}{2}.$$

Homework

1. Let $f(x) = x^2$ be a function defined on $[a, b]$. Let $h = \frac{b-a}{n}$. Let $P_n = \{a = x_0, x_1 = a + h, x_2 = a + 2h, \ldots, x_n = a + nh = b\}$. Compute (i) $L(P_n, f)$ and (ii) $U(P_n, f)$. Find the limits of these lower and upper Riemann sums and conclude that the function is Riemann itegrable and find the Riemann integral.
2. Let $g(x) = x^2$ be a function defined on $[a, b]$ where $0 < a < b$. Let $h = \left(\frac{b}{a}\right)^{\frac{1}{n}}$. Let $Q_n = \{a = x_0, x_1 = ah, x_2 = ah^2, \ldots, x_n = ah^n = b\}$. Compute (i) $L(Q_n, g)$ and (ii) $U(Q_n, g)$. Find the limits of these lower and upper Riemann sums and conclude that the function is Riemann itegrable and find the Riemann integral.
3. Let $f(x) = 1 - x^2$ be a function defined on $[0, 1]$. Let $P_n = \left\{x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_n = \frac{n}{n} = 1\right\}$. Compute (i) $L(P_n, f)$ and (ii) $U(P_n, f)$. Argue that the function is Riemann integrable and find the Riemann integral.

Definition (Riemann sum)

Let $f(x)$ be a bounded real valued function defined on $[a, b]$. Let $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$. Let $c_i \in [x_{i-1}.x_i]$, $1 \leq i \leq n$. Then

$$S_P = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

is called a **Riemann sum** for f corresponding to the partition P.

Note:

$$L(P, f) \leq S_P \leq U(P, f).$$

Theorem (Riemann Integrability of Continuous Functions)

If a function f is continuous on the interval $[a, b]$, then it is Riemann integrable. Moreover, if $h = \frac{b-a}{n}$ and $P_n = \{a = x_0, x_1 = a + h, x_2 = a + 2h, \ldots, x_n = a + nh = b\}$ is a partition of $[a, b]$ into equal subintervals, then

$$\lim_{n \longrightarrow \infty} L(P_n, f) = \lim_{n \longrightarrow \infty} U(P_n, f) = \int_a^b f(x)dx.$$

Hence if S_{P_n} is any Riemann sum corresponding to P_n, then

$$\lim_{n \longrightarrow \infty} S_{P_n} = \int_a^b f(x)dx.$$

Homework

For the following continuous functions , find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_i. Then take a limit of these sums as $n \longrightarrow \infty$ to compute the corresponding Riemnann integral (which is also the area under the curve $y = f(x)$, $[a, b]$, and above the x-axis).

1. $f(x) = x + x^2$ over the interval $[0, 1]$.
2. $f(x) = x^2 + 1$ over the interval $[0, 3]$.
3. $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.
4. $f(x) = 2x^3$ over the interval $[0, 1]$.

Solution:

(1) Consider $f(x) = x + x^2$ on the interval $[0, 1]$. Let us divide $[0, 1]$ into n equal subintervals, each of length $\frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$: That is, consider the partition

$$P_n = \left\{ x_0 = 0, x_1 = \frac{1}{n}, \ldots, x_i = \frac{i}{n}, \ldots, x_n = 1 \right\}.$$

For each $i, 1 \leq i \leq n$, let $c_i = x_i = \frac{i}{n}$. Then $f(c_i) = c_i + c_i^2 = \frac{i}{n} + \frac{i^2}{n^2}$.
Also note that $\Delta x_i = \frac{1}{n}$.

$$
\begin{aligned}
S_{P_n} &= \textstyle\sum_{i=1}^{n} f(c_i)\Delta x_i \\
&= \textstyle\sum_{i=1}^{n} \left(\frac{i}{n} + \frac{i^2}{n^2}\right) \frac{1}{n} \\
&= \tfrac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \tfrac{1}{n^2}\tfrac{n(n+1)}{2} + \tfrac{1}{n^3}\tfrac{n(n+1)(2n+1)}{6} \\
&= \tfrac{1}{2}\left(1 + \tfrac{1}{n}\right) + \tfrac{1}{6}\left(1 + \tfrac{1}{n}\right)\left(2 + \tfrac{1}{n}\right)
\end{aligned}
$$

$$\lim_{n \longrightarrow \infty} S_P = \frac{5}{6}.$$

Hence $\displaystyle\int_0^1 \left(x + x^2\right) dx = \frac{5}{6}.$

Properties of Riemann Integration

Theorem

Let f and g be integrable over the interval $[a,b]$. Then

(1) *Order of Integration:* $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$ *(definition)*

(2) *Zero Width Interval :* $\displaystyle\int_a^a f(x)dx = 0$ *(a definition when $f(a)$ exists)*

(3) *Constant Multiple:* $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$
(any constant k)

(4) *Sum:* $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
Difference: $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Theorem *Contd.*

(5) Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

(6) Max-Min Inequality: If f has maximum value M and minimum value m on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

(7) Domination : If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

Theorem (The Fundamental Theorem of Calculus)

If f is Riemann integrable on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof:

Let $\epsilon > 0$ be given.
Choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ so that $U(P, f) - L(P, f) < \epsilon$.

The mean value theorem implies that there is a t_i in $[x_{i-1}, x_i]$ such that

$$\frac{F(x_i) - F(x_{i-1})}{\Delta x_i} = F'(t_i) = f(t_i) \quad \text{or} \quad F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i \quad (1 \le i \le n).$$

Thus

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a).$$

We also note that

$$L(P, f) \le \sum_{i=1}^{n} f(t_i)\Delta x_i \le U(P, f) \quad \text{and} \quad L(P, f) \le \int_a^b f(x)dx \le U(P, f).$$

Hence

$$\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)dx\right| < \epsilon \quad \Rightarrow \quad \left|F(b) - F(a) - \int_a^b f(x)dx\right| < \epsilon.$$

Since this holds for every ϵ, the theorem follows.

Theorem

Let f be Riemann integrable on $[a, b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(x)dx.$$

Then F is continuous on $[a, b]$. Further, if f is continuous on $[a, b]$, then F is differentiable on $[a, b]$ and $F' = f$.