# Fourier Series

## Definition (Fourier Series)

Let $a_0, a_1, b_1, a_2, b_2, \ldots a_n, b_n, \ldots$ be any sequence of real numbers. Then the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called a Fourier Series.

**Applications:** Solving important partial differential equations arising in the theory of sound, heat conduction, electromagnetic waves, and mechanical vibrations.

**It is More Powerful than Power Series:** Can represent very general functions with many discontinuities, like the impulse function.

# Fourier Series: Its Relation to Power Series

**Recall:** A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots.$$

**A Generalization of Power Series:**

$$\sum_{n=-\infty}^{\infty} a_n x^n = \ldots + a_{-n} x^{-n} + \ldots + a_{-2} x^{-2} + a_{-1} x^{-1} + a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots.$$

**A Further Generalization to Complex Numbers:**

$$\sum_{n=-\infty}^{\infty} c_n z^n = \ldots + c_{-n} z^{-n} + \ldots + c_{-2} z^{-2} + c_{-1} z^{-1} + c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots.$$

**A Further Generalization to Complex Numbers:**

$$\sum_{n=-\infty}^{\infty} c_n z^n = \ldots + c_{-n} z^{-n} + \ldots + c_{-2} z^{-2} + c_{-1} z^{-1} + c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots.$$

Taking $z = e^{ix}$, we obtain

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = \ldots + c_{-n} e^{-inx} + \ldots + c_{-2} e^{-i2x} + c_{-1} e^{-ix} + c_0 + c_1 e^{ix} + c_2 e^{i2x} + \ldots + c_n e^{inx} + \ldots.$$

This is **Fourier Series**! Why? (**Our Fourier series is only a *special case* of this series!**)

# A Bit of Complex Analysis

Let $a$ and $b$ be any real numbers. Consider the linear combination

$$a \cos x + b \sin x.$$

There is a unique pair of complex numbers $c$ and $d$ such that the linear combination

$$ce^{ix} + de^{-ix} = a \cos x + b \sin x.$$

(Prove!)

Let $n$ be any integer. For any pair of real numbers $a$ and $b$, there is a unique choice of complex numbers $c$ and $d$ such that

$$ce^{inx} + de^{-inx} = a \cos nx + b \sin nx.$$

Consider a Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where $a_0, a_1, b_1, a_2, b_2, \ldots a_n, b_n, \ldots$ is a sequence of real numbers.

Then, for each $n \geq 1$, we can find complex numbers $c_n$ and $c_{-n}$ such that

$$a_n \cos nx + b_n \sin nx = c_n e^{inx} + c_{-n} e^{-inx}.$$

Hence we have

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} c_n \left(e^{ix}\right)^n.$$

# Properties of Sine and Cosine Functions

Let $n$ be any integer.

**Case 1:** $n = 0$: Then

$$\int_{-\pi}^{\pi} \sin nx \, dx = \int_{-\pi}^{\pi} 0 \, dx = 0.$$

**Case 2:** $n \neq 0$: Then

$$\int_{-\pi}^{\pi} \sin nx \, dx = \left[ \frac{-\cos nx}{n} \right]_{-\pi}^{\pi} = \frac{-1}{n} [\cos n\pi - \cos n\pi] = 0$$

and

$$\int_{-\pi}^{\pi} \cos \; nxdx = \left[ \frac{\sin nx}{n} \right]_{-\pi}^{\pi} = \frac{1}{n} [\sin n\pi + \sin n\pi] = 0.$$

# Properties of Sine and Cosine Functions

$$
\begin{aligned}
\sin mx \cos nx &= \frac{1}{2}[\sin(m+n)x + \sin(m-n)x] \\
\cos mx \cos nx &= \frac{1}{2}[\cos(m+n)x + \cos(m-n)x] \\
\sin mx \sin nx &= \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]
\end{aligned}
$$

Let $m, n \geq 1$ be integers. Then

$$
\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \cos nx \, dx &= 0 \\
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= 0 \quad (m \neq n) \\
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= 0 \quad (m \neq n)
\end{aligned}
$$

Let $n$ be any nonzero integer. Then

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi$$

and

$$\int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi.$$

# Note

Suppose the Fourier series on the RHS of the equation below converges and has sum $f(x)$:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad -\pi \leq x \leq \pi.$$

Then

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\, dx &= \int_{-\pi}^{\pi} \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right] dx \\
&= \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos nx\, dx + b_n \int_{-\pi}^{\pi} \sin nx\, dx \right)
\end{aligned}$$

But

$$\int_{-\pi}^{\pi} \sin nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \cos nx\, dx = 0$$

for $n = 1, 2, 3, \ldots$.

So, we have

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\pi$$

$$\implies \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Now

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos x\,dx &= \int_{-\pi}^{\pi}\left[\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right]\cos x\,dx\\ &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\cos x\,dx+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos x\cos nx\,dx+b_n\int_{-\pi}^{\pi}\cos x\sin nx\,dx\right)\end{aligned}$$

But

$$\int_{-\pi}^{\pi}\cos x\,dx=0,\quad \int_{-\pi}^{\pi}\cos x\cos nx\,dx=0\quad (n\geq 2)\quad \text{and}\quad \int_{-\pi}^{\pi}\cos x\sin nx\,dx=0.$$

So, we have

$$\int_{-\pi}^{\pi} f(x)\cos x\,dx = \int_{-\pi}^{\pi} a_1 \cos^2 x\,dx = a_1\pi$$

$$\implies \quad a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

Similarly, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n \geq 1)$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (n \geq 1).$$

# Summary

Suppose

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad -\pi \leq x \leq \pi.$$

Then, if the above convergence is uniform, we have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n \geq 1) \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (n \geq 1) \end{aligned}$$

# Remarks

- In the preceding discussion, we assumed that we are given a convergent Fourier series whose sum is $f(x)$.
- In this case, if the convergence is uniform, then the coefficients $a_n$ and $b_n$ can be derived from the sum function $f(x)$ by the formulas given for them.
- In the following slides, we assume that we are given an integrable function $f(x)$ defined on the interval $-\pi \leq x \leq \pi$. We use it find *its Fourier Series*, by finding the coefficients $a_n$ and $b_n$ from $f(x)$.

# The Fourier Series of a Function

## Definition

Let $f(x)$ be an integrable function defined on the interval $-\pi \leq x \leq \pi$. Then the coefficients

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx \quad (n \geq 1) \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \quad (n \geq 1)
\end{aligned}
$$

are called the *Fourier coefficients* of the function $f(x)$ and the corresponding trigonometric series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called the *Fourier series* of the function $f(x)$.

# Important Note

- The Fourier series of $f(x)$ need not converge on the interval $-\pi \leq x \leq \pi$. Even if it converges, its limit need not be $f(x)$.
- If the Fourier series of $f(x)$ converges *uniformly* to $f(x)$, then its Fourier coefficients can *obviously* be recovered from the sum function $f(x)$.
- If the Fourier series converges, it defines a periodic function of period $2\pi$ over the entire real line.

# Example 1

Let $f(x)$ be a function of period $2\pi$ such that

$$f(x) = \begin{cases} 1, & -\pi \leq x < 0 \\ 0, & 0 \leq x < \pi \end{cases}$$

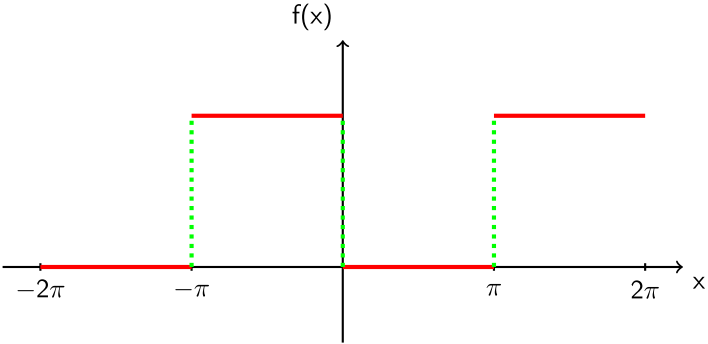
1. Sketch the graph of $f(x)$ in the interval $-2\pi < x < 2\pi$.
2. Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

3. By giving an appropriate value of $x$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

**Solution:** (1) The graph of the function:



(2) The Fourier series of the function:
**Step 1:**

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot dx + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot dx \\ &= \frac{1}{\pi}\big[x\big]_{-\pi}^{0} = 1. \end{aligned}$$

**Step 2:**

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos nx\,dx \\
&= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} \\
&= \frac{1}{n\pi}\big[\sin 0 - \sin(-n\pi)\big] = 0.
\end{aligned}$$

**Step 3:**

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \sin nx\,dx \\
&= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} \\
&= -\frac{1}{n\pi}[\cos 0 - \cos(-n\pi)] \\
&= -\frac{1}{n\pi}[1-(-1)^n].
\end{aligned}$$

Thus we have $a_0 = 1, a_n = 0$ and $b_n = 0$ if $n$ is even and $b_n = -\dfrac{2}{n\pi}$ if $n$ is odd.

Hence the Fourier series of the given function $f(x)$ is

$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right].$$

# Dirichlet Conditions

Under what conditions does the Fourier series of $f(x)$ converge to $f(x)$?

German mathematician Dirichlet gave a sufficiently broad **sufficient** conditions in 1829. These conditions are called *Dirichlet conditions*.

## Dirichlet Conditions

- $f(x)$ is defined and bounded on $-\pi \leq x < \pi$.
- $f(x)$ has a finite number of discontinuities and a finite number of maxima and minima.
- $f(x)$ is defined for other values of $x$ by the periodicity condition $f(x+2\pi) = f(x)$.

# Examples

Functions which pass/fail the Derichlet conditions:

**Pass:** $\sin x$, $x^2$ and step functions on $[-\pi, \pi)$.

**Fail:** $\tan x$, $\sin \frac{1}{x}$ and $\frac{1}{x}$ on $[-\pi, \pi)$.

Classwork:

1. Explain explicitly which properties the above functions fail to satisfy.
2. Give an example of a function which has an infinite number of maxima.

# Note

If a bounded function $f(x)$ defined on $-\pi \leq x \leq \pi$ has *only a finite number of discontinuities* and *only a finite number of maxima and minima*, then all its discontinuities are simple. This means that $f(x-)$ and $f(x+)$ exist at every $x$ and the points of continuity are those for which $f(x-) = f(x+)$.

### Theorem (Dirichlet Theorem)

*Assume that the Dirichlet conditions hold for $f(x)$ on the interval $-\pi \leq x < \pi$. Then the Fourier series of $f(x)$ converges to*

$$\frac{1}{2}[f(x-) + f(x+)]$$

*at every point $x$ and therefore it converges to $f(x)$ at every point $x$ of continuity of the function.*

# Example 1 *Contd.*

(3) To show that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

**Recall:** For the periodic function $f(x)$ defined by
$$f(x) = \begin{cases} 1, & -\pi \leq x < 0 \\ 0, & 0 \leq x < \pi \end{cases}$$

we obtained the Fourier series
$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right].$$

The function $f(x)$ is continuous at $x = \pi/2$.

So, by Dirichlet theorem, the Fourier of $f(x)$ above converges to $f(x)$ at $x = \pi/2$.

So, on substituting $x = \pi/2$ in the Fourier series of $f(x)$ below,

$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right],$$

we have

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2} - \frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \frac{1}{7}\sin\left(\frac{7\pi}{2}\right) + \cdots\right]$$

or, since $f(\pi/2) = 0$, we have

$$0 = \frac{1}{2} - \frac{2}{\pi}\left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \cdots\right] = \frac{1}{2} - \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right].$$

This implies that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

# A Final Note on Example 1

**Recall:** The periodic function of period $2\pi$ defined by

$$f(x) = \begin{cases} 1, & -\pi \leq x < 0 \\ 0, & 0 \leq x < \pi \end{cases}$$

has the integer multiples of $\pi$ as the only points of discontinuity.

Thus, by Dirichlet theorem, at these points of discontinuity, the Fourier series of $f(x)$ has sum

$$\frac{1}{2}[f(x-) + f(x+)] = \frac{1}{2}.$$

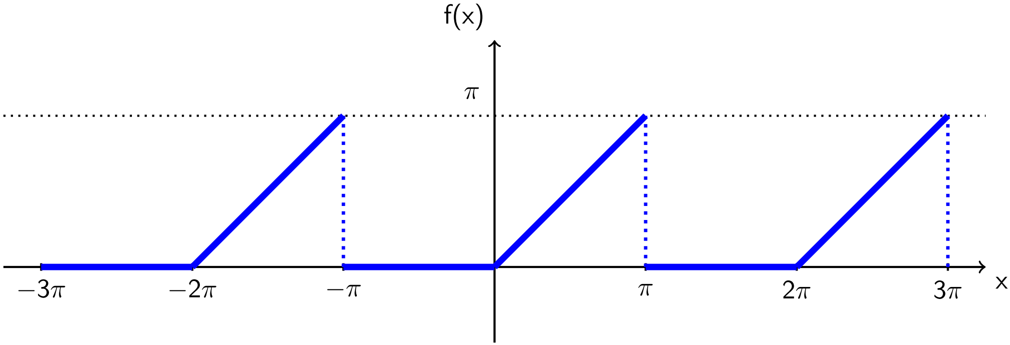
# Example 2

Let $f(x)$ be a function of period $2\pi$ such that

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases} .$$

1. Sketch the graph of $f(x)$ on the interval $-3\pi \le x < 3\pi$.
2. Find the Fourier series representation of $f(x)$ on the interval $-\pi \le x < \pi$.
3. By giving appropriate values of $x$, show that
   - $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
   - $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

**Solution:** (1) The Graph of the Function:



(2) The Fourier Series of the Function:
**Step 1:**

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot dx + \frac{1}{\pi}\int_{0}^{\pi} x \cdot dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} xdx \\
&= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \\
&= \frac{\pi}{2}.
\end{aligned}
$$

**Step 2:**

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\cdot\cos nx dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx \\
&= \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx\right\} \\
&= \frac{1}{\pi}\left\{0 - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right\} \\
&= \frac{1}{\pi n^2}[(-1)^n - 1].
\end{aligned}
$$

**Step 3:**

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\cdot\sin nx dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx \\
&= \frac{1}{\pi}\left\{\left[-x\frac{\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos nx}{n}dx\right\} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}[\pi\cos n\pi - 0] + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\} \\
&= -\frac{1}{n\pi}\pi(-1)^n + 0 \\
&= \frac{1}{n}(-1)^{n+1}.
\end{aligned}
$$

Thus the values of $a_n, b_n$ for different values of $n$ are as follows:

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | $-\dfrac{2}{\pi}$ | 0 | $-\dfrac{2}{\pi}(\dfrac{1}{3^2})$ | 0 | $-\dfrac{2}{\pi}(\dfrac{1}{5^2})$ |
| $b_n$ | 1 | $-\dfrac{1}{2}$ | $\dfrac{1}{3}$ | $-\dfrac{1}{4}$ | $\dfrac{1}{5}$ |

Hence the required Fourier series is

$$\begin{aligned} f(x) &= \frac{1}{2}\left(\frac{\pi}{2}\right) + \left(-\frac{2}{\pi}\right)\cos x + 0\cos 2x + \left(-\frac{2}{\pi}.\frac{1}{3^2}\right)\cos 3x + 0.\cos 4x \\ &\quad + \left(-\frac{2}{\pi}.\frac{1}{5^2}\right)\cos 5x + \cdots + \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots \\ &= \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right] + \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right] \end{aligned}$$

3(a) A series for $\dfrac{\pi}{4}$:

The given function satisfies the *Dirichlet conditions* on $-\pi \leq x < \pi$ and is continuous at $x = \pi/2$. So, by Dirichlet theorem, the Fourier series of $f(x)$ converges to $f(\pi/2) = \pi/2$ at $x = \pi/2$.

So, we have

$$
\begin{aligned}
f\left(\frac{\pi}{2}\right) &= \frac{\pi}{4} - \frac{2}{\pi}\left[\cos\left(\frac{\pi}{2}\right) + \frac{1}{3^2}\cos 3\left(\frac{\pi}{2}\right) + \frac{1}{5^2}\cos 5\left(\frac{\pi}{2}\right) + \cdots\right] \\
&\quad + \left[\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin 3\left(\frac{\pi}{2}\right) - \cdots\right] \\
&= \frac{\pi}{4} - \frac{2}{\pi}\left[0 + 0 + 0 + \cdots\right] + \left[\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} - \frac{1}{2}\underbrace{\sin 2\left(\frac{\pi}{2}\right)}_{0} + \frac{1}{3}\underbrace{\sin 3\left(\frac{\pi}{2}\right)}_{-1} - \cdots\right]
\end{aligned}
$$

Thus

$$\frac{\pi}{2} = \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

3(b) A series for $\frac{\pi^2}{8}$:

Again, by Dirichlet theorem, the Fourier series at $x = 0$ converges to $f(0) = 0$. So,

$$\begin{aligned} f(0) &= \frac{\pi}{4} - \frac{2}{\pi}\left[\cos(0) + \frac{1}{3^2}\cos(0) + \frac{1}{5^2}\cos(0) + \cdots\right] \\ &\quad + \left[\sin(0) - \frac{1}{2}\sin(0) + \frac{1}{3}\sin(0) - \cdots\right] \end{aligned}$$

That is,

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right].$$

Hence

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

# Homework

Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

# Even and Odd Functions: Cosine and Sine Series

Let $f(x)$ be a function defined on an evenly placed interval $-a \leq x \leq a$.

- The function $f$ is said to be **even** if $f(-x) = f(x)$ for all $x$.

  **Examples:** $1, x^2, x^4$ and $\cos x$

- The function $f$ is said to be **odd** if $f(-x) = -f(x)$ for all $x$.

  **Example:** $x, x^3, \sin x$

**Note:** If $f(x)$ is an odd function, then $f(0) = 0$.

# Note

- If $f(x)$ and $g(x)$ are both even, then $f(x)g(x)$ is even.
- If $f(x)$ and $g(x)$ are both odd, then $f(x)g(x)$ is even.
- If one of $f(x)$ and $g(x)$ is even and the other is odd, then $f(x)g(x)$ is odd.

**Example:** $x^3 \cos nx$ is odd as $x^3$ is odd and $\cos nx$ is even. So $x^3 \cos nx$ is odd.

# Recall

- If $f$ is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

- If $f$ is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

**Example:** The function $x^3 \cos nx$ is odd. So

$$\int_{-\pi}^{\pi} x^3 \cos nx\,dx = 0.$$

# Sine and Cosine series

**Theorem**

*Let $f(x)$ be a function defined and integrable on $-\pi \leq x \leq \pi$.*

- *If $f(x)$ is even, then its Fourier series has only cosine terms and the coefficients are given by*

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad \textit{and} \qquad b_n = 0.$$

- *If $f(x)$ is odd, then its Fourier series has only sine terms and the coefficients are given by*

$$a_n = 0 \qquad \textit{and} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

**Proof:** *Homework.*

# Examples

**Example 1:** The function $f(x) = x$ is an odd function on $-\pi \leq x \leq \pi$. So, its Fourier series is automatically a *sine* series. Indeed

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right), \quad -\pi < x < \pi.$$

(Prove!)

**Example 2:** The function $f(x) = |x|$ is an even function on $-\pi \leq x \leq \pi$. So, its Fourier series is automatically a *cosine* series. Indeed

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right), \quad -\pi \leq x \leq \pi.$$

(Prove!)

# Extending any Function on $0 \leq x \leq \pi$ to an Even or Odd Function on $-\pi \leq x \leq \pi$

Let $f(x)$ be a function defined for $0 \leq x \leq \pi$.

- It can be extended to an *even* function on $-\pi \leq x \leq \pi$ by defining $f(x) = f(-x)$ for $-\pi \leq x < 0$.
- It can be extended to an *odd* function on $-\pi \leq x \leq \pi$ by defining $f(x) = -f(-x)$ for $-\pi \leq x < 0$ and *redefining* $f(x) = 0$ if necessary.

The above observations imply the following theorem:

**Theorem**

*If $f(x)$ is an integrable function on the interval $0 \leq x \leq \pi$, then it can be expanded both as a sine series and as a cosine series on* this *interval.*

## Example

Find the sine series, and also the cosine series, for the function $f(x) = \cos x$, $0 \leq x \leq \pi$.

**Solution:** To obtain the *sine* series, we just *assume* that $f(x)$ has been extended to an odd function (We do not really bother to extend!). So, we will have

$$a_n = 0 \quad \text{and} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.$$

For $n = 1$, we have $b_1 = 0$. (Why?)

For $n > 1$, we obtain $b_n = \dfrac{2n}{\pi}\left[\dfrac{1 + (-1)^n}{n^2 - 1}\right]$. (Prove!)

So,

$$b_{2n-1} = 0 \quad \text{and} \quad b_{2n} = \frac{8n}{\pi(4n^2 - 1)}.$$

Thus the sine series is

$$\cos x = \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{n\sin 2nx}{4n^2 - 1}, \qquad 0 < x < \pi.$$

To obtain the cosine series, assume that $f(x)$ has been extended to an even function. So,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = 0.$$

Now,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\,dx = \begin{array}{ll} 1, & n = 1 \\ 0, & n \neq 1 \end{array}$$

Thus the cosine series for $\cos x$ is *simply* $\cos x$.

# Homework

Find the sine series and the cosine series of the constant function $f(x) = \pi/4$.

# Extension to Arbitrary Intervals $-L \leq x \leq L$

In many applications, it is desirable to express a function $f(x)$ defined on an interval $-L \leq x \leq L$ as a trigonometric series where $L \neq \pi$.
This can be easily effected by a change of variable:

Introduce a new variable $t$ that varies from $-\pi$ to $\pi$ as $x$ varies from $-L$ to $L$:

$$t = \frac{\pi x}{L} \quad \Rightarrow \quad x = \frac{Lt}{\pi}.$$

Now put

$$x = \frac{Lt}{\pi}$$

in the function $f(x)$ to obtain a function

$$g(t) = f\left(\frac{Lt}{\pi}\right), \qquad -\pi \leq t \leq \pi$$

and proceed with finding the Fourier series of $g(t)$, $-\pi \leq t \leq \pi$:

$$g(t) = \frac{1}{2}a_0 + \sum_{n=0}^{\infty}(a_n \cos nt + b_n \sin nt)$$

And finally replace $t$ by

$$\frac{\pi x}{L}$$

in the Fourier series found:

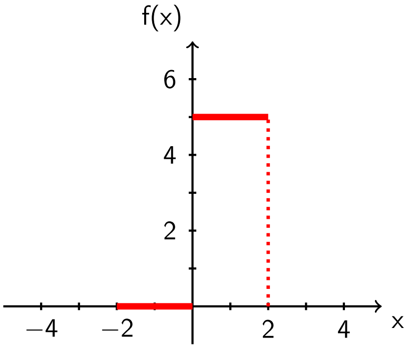
$$f(x) = \frac{1}{2}a_0 + \sum_{n=0}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

# Example

Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -2 \leq x < 0 \\ 5, & \quad 0 \leq x \leq 2. \end{cases}$$

**Solution:** The graph of the function:



Here

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{0}^{2} 5dx = 5.$$

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right) \\ &= \frac{1}{2}\int_{0}^{2} 5\cos\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{5}{2}\left[\frac{2\sin\frac{n\pi x}{2}}{n\pi}\right]_0^2 = \frac{5}{n\pi}\left[\sin\frac{2n\pi}{2} - \sin\frac{0n\pi}{2}\right] = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right) \\ &= \frac{1}{2}\int_{0}^{2} 5\sin\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{5}{2}\left[\frac{-2\cos\frac{n\pi x}{2}}{n\pi}\right]_0^2 \end{aligned}$$

# Example

Using these coefficient, we have the Fourier series

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \\
&= \frac{5}{2} + \sum_{n=0}^{\infty} \left( \frac{5}{n\pi}(1-(-1)^n) \right) \sin \frac{n\pi x}{2} \\
&= \frac{5}{2} + \frac{10}{\pi}\left[ \sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} + \frac{1}{5} \sin \frac{5\pi x}{2} + \frac{1}{7} \sin \frac{7\pi x}{2} + \cdots \right].
\end{aligned}
$$

# Homework

1. Find the Fourier series of the following functions:

   1.1
   $$f(x) = \begin{cases} -3, & -2 \leq x < 0 \\ 3, & 0 \leq x < 2 \end{cases}$$

   1.2
   $$f(x) = \begin{cases} 1+x, & -1 \leq x < 0 \\ 1-x, & 0 \leq x \leq 1 \end{cases}$$

   1.3
   $$f(x) = |x|, \quad -2 \leq x \leq 2.$$

2. Show that
   $$\frac{1}{2}L - x = \frac{L}{\pi}\sum_{n=1}^{\infty} \sin\frac{2n\pi x}{L}, \qquad 0 < x < L.$$

3. Find the cosine series for the function
   $$f(x) = \begin{cases} 2, & 0 \leq x \leq 1 \\ 0, & 1 < x \leq 2 \end{cases}$$