MA1001: Differential Equations

Dr Sai Prashanth

IIITDM Kancheepuram

Problems on Laplace Transformations

Find the Laplace transformation of

$$f(x) = \sin 5x \cos 3x.$$

Solution:

$$\sin 5x \cos 3x = \frac{1}{2}(\sin 8x + \sin 2x).$$

$$\begin{aligned} L[\sin 5x \cos 3x] &= L[\frac{1}{2}(\sin 8x + \sin 2x)] \\ &= \frac{1}{2}\left\{L[\sin 8x] + L[\sin 2x]\right\} \\ &= \frac{1}{2}\left(\frac{8}{p^2+8^2} + \frac{2}{p^2+2^2}\right) \\ &= \frac{4}{p^2+64} + \frac{1}{p^2+4} \end{aligned}$$

Find the Laplace transformation of

$$f(x) = \cosh^2 4x.$$

Solution:

$$\cosh^2 4x = \left(\frac{e^{4x} + e^{-4x}}{2}\right)^2 = \frac{1}{4}\left(e^{8x} + 2 + e^{-8x}\right).$$

$$\begin{aligned}
L[\cosh^2 4x] &= \frac{1}{4}\left\{L[e^{8x}] + L[2] + L[e^{-8x}]\right\} \\
&= \frac{1}{4}\left(\frac{1}{p-8} + \frac{2}{p} + \frac{1}{p+8}\right) \\
&= \frac{1}{4}\left(\frac{2p}{p^2-64} + \frac{2}{p}\right) \\
&= \frac{p^2-32}{p(p^2-64)}
\end{aligned}$$

Find the Laplace transformation of

$$f(x) = \cos ax \sinh bx.$$

Solution:

$$\begin{aligned} \cos ax \sinh bx &= \cos ax \left(\frac{e^{bx} - e^{-bx}}{2} \right) \\ &= \frac{1}{2} \left(e^{bx} \cos ax - e^{-bx} \cos ax. \right) \end{aligned}$$

Hence

$$\begin{aligned} L[\cos ax \sinh bx] &= \frac{1}{2} \left\{ L[e^{bx} \cos ax] - L[e^{-bx} \cos ax] \right\} \\ &= \frac{1}{2} \left(\frac{p-b}{(p-b)^2 + a^2} - \frac{p+b}{(p+b)^2 + a^2} \right). \end{aligned}$$

Find the Laplace transformation of

$$f(x) = xe^{ax} \sin bx.$$

Solution:

$$L[x \sin bx] = -\frac{d}{dp}\frac{b}{p^2 + b^2} = \frac{2bp}{(p^2 + b^2)^2}.$$

Hence

$$L[xe^{ax} \sin bx] = L[e^{ax} x \sin bx] = \frac{2b(p-a)}{((p-a)^2 + b^2)^2}.$$

Find the Laplace transformation of

$$f(x) = \int_0^x \frac{1-e^{-u}}{u}\,du.$$

Solution:

$$L[1-e^{-u}] = \frac{1}{p} - \frac{1}{p+1} = \frac{1}{p(p+1)}.$$

So,

$$\begin{aligned} L\left[\frac{1-e^{-u}}{u}\right] = \int_p^\infty \frac{1}{p(p+1)}dp &= \int_p^\infty \left(\frac{1}{p} - \frac{1}{p+1}\right) dp \\ &= [\log p - \log(p+1)]_p^\infty \\ &= \left[\log\left(\frac{p}{p+1}\right)\right]_p^\infty \\ &= \log\left(\frac{p+1}{p}\right) = \log\left(1+\frac{1}{p}\right) \end{aligned}$$

Hence

$$L\left[\int_0^x \frac{1-e^{-u}}{u}\,du\right] = \frac{1}{p}\log\left(1+\frac{1}{p}\right)$$

Find the inverse Laplace transform of

$$F(p) = \log\left(\frac{p^2+1}{p(p+1)}\right).$$

Solution:

$$\log\left(\frac{p^2+1}{p(p+1)}\right) = \log(p^2+1) - \log p - \log(p+1).$$

So,

$$\begin{aligned} L[-xf(x)] = F'(p) &= \frac{2p}{p^2+1} - \frac{1}{p} - \frac{1}{p+1} \\ &= L[2\cos x] - L[1] - L[e^{-x}] \\ &= L[2\cos x - 1 - e^{-x}]. \end{aligned}$$

We have

$$L[-xf(x)] = L[2\cos x - 1 - e^{-x}].$$

Hence

$$xf(x) = 1 + e^{-x} - 2\cos x$$

or

$$f(x) = \frac{1 + e^{-x} - 2\cos x}{x}.$$