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MA1000 Calculus (B Batch)
Assignment 2

February 18, 2021

Submit by: February 23, 2021 — Marks: 10

1. Prove that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$

2. A function $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$$

   Show that

   (a) f is injective (one-to-one) on $[0,1]$.

   (b) f assumes every real numbers in $[0,1]$ (i.e., $[0,1]$ is a subset of the range of f).

   (c) f is continuous at $\frac{1}{2}$ and is discontinuous at every other point in $[0,1]$.

3. Find the points of discontinuities of the function f defined by $f(x) = \lim_{n\to\infty}\frac{(1+\sin\pi x)^n - 1}{(1+\sin\pi x)^n + 1}$, $x \in \mathbb{R}$.

4. If $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are both continuous functions on $[a,b]$ having the same range $[0,1]$, prove that $f(c) = g(c)$ for some $c \in [a,b]$.

5. Give an example of a function f such that $f(0) = 0$ and f and $|f|$ both are differentiable for every $x \in \mathbb{R}$. Justify your answer.

6. Give an example of a function which is not differentiable exactly at two points. Prove the correctness of your answer.

7. Prove that between any two real roots of the equation $e^x \cos x + 1 = 0$ there is at least one real root of the equation $e^x \sin x + 1 = 0$.

8. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

9. Evaluate:

   (a) $\lim_{y\to 0}\frac{\sin 3y \cot 5y}{y \cot 4y}$.

   (b) $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$.

(c) $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$.

10. For what values of a and b is $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?.

11. Prove that if the funtions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous on $\mathbb{R}$, then the set $S = \{x \in \mathbb{R} : f(x) = g(x)\}$ is a closed set in $\mathbb{R}$.

12. Prove that $\dfrac{x}{1+x} < \ln(1+x) < x$ for all $x > 0$.

13. Divide the number 10 into two parts such that the sum of their cubes is the least possible. Justify your answer.

14. Determine a, b, c such that $\displaystyle\lim_{x\to 0} \frac{x(a + b\cos x) + c\sin x}{x^5} = \frac{1}{60}$.

15. Use Taylor's theorem to prove that $1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$, if $x > 0$.

16. A function f is defined on $[0, 1]$ by

$$f(x) = \begin{cases} x, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

Find $\underline{\displaystyle\int_0^1} f(x)dx$ and $\overline{\displaystyle\int_0^1} f(x)dx$. Deduce that f is not Riemann integrable on $[0, 1]$.

17. Prove or disprove: If $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to \mathbb{R}$ are Riemann integrable and if the range of f is contained in $[c, d]$, then the composite function $(g \circ f)(x) = g(f(x))$ is Riemann integrable on $[a, b]$.

18. Let a function $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose $f(x) \geq 0$ for all $x \in [a, b]$. If there exist point c in $[a, b]$ such that f is continuous at c and $f(c) > 0$, then prove that $\displaystyle\int_a^b f(x)dx > 0$.

19. Let $f(x) = [x]$ for $x \in [0, 3]$. Prove that f is Riemann integrable on $[0, 3]$. Also evaluate $\displaystyle\int_0^3 f(x)dx$.

20. Let $f : [a, b] \to \mathbb{R}$ be a bounded function with a finite number of points of discontinuity. Prove that f is Riemann integrable on $[a, b]$.