Ordered Basis

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The coordinate matrix of the vector α relative to the ordered basis B is

$$[\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$$

Claim : $[\alpha]_B$ **is unique.**

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$$(x_1 - y_1)\alpha_1 + (x_2 - y_2)\alpha_2 + \ldots + (x_n - y_n)\alpha_n = 0$$

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Since B is L.I., $\implies \quad x_1 - y_1 = x_2 - y_2 = \ldots = x_n - y_n = 0.$
$\implies x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$ and thus $[\alpha]_B$ is unique.

Example

Find the coordinate matrix of the vector $\alpha = (1, 2, 3)$ w.r.t. the ordered basis $B = \{\epsilon_1 = (1, 0, 0), \epsilon_2 = (0, 1, 0), \epsilon_3 = (0, 0, 1)\}$ and $B_1 = \{\alpha_1 = (1, 1, 1), \alpha_2 = (0, 1, 1), \alpha_3 = (0, 0, 1)\}$

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$$\alpha = (1,2,3) = \epsilon_1 + 2\epsilon_2 + 3\epsilon_3, \quad \alpha = (1,2,3) = \alpha_1 + \alpha_2 + \alpha_3$$

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What is the relation between $[\alpha]_B$ and $[\alpha]_{B_1}$?

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Verify that the matrix P is invertible and $[\alpha]_{B_1} = P^{-1}[\alpha]_B$?

Relation between $[\alpha]_B$ and $[\alpha]_{B_1}$?

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $B_1 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be two ordered bases of a finite-dimensional vector space V.

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where

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \quad , \quad \alpha = \sum_{j=1}^{n} y_j \beta_j$$

contd..

Since $\beta_j \in V = \text{span}\ \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, there exists unique scalars P_{ij}, $1 \leq i \leq n$ such that

$$\beta_j = \sum_{i=1}^{n} P_{ij}\alpha_i \quad , 1 \leq j \leq n$$

where $$[\beta_j]_{B} = P_j = \begin{bmatrix} P_{1j} \\ P_{2j} \\ \dots \\ P_{nj} \end{bmatrix}$$

contd.

$$\alpha = \sum_{j=1}^{n} y_j \beta_j$$

contd.

$$\begin{aligned} \alpha &= \sum_{j=1}^{n} y_j \beta_j \\ &= \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{n} P_{ij} \alpha_i \right) \end{aligned}$$

contd.

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We have, $\alpha = \sum_{i=1}^{n} x_i \alpha_i$

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We have, $\alpha = \sum_{i=1}^{n} x_i \alpha_i \implies x_i = \sum_{j=1}^{n} P_{ij} y_j$, $1 \leq i \leq n$ (Thanks to unique coordinate matrix of α w.r.t. a basis B.)

contd.

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$$\implies \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

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$$\implies X = PX' - - - - - (1)$$

where $X = [\alpha]_B$, $X' = [\alpha]_{B_1}$ and $P = [P_1, P_2, \dots, P_n]$.

contd.

Note that $X = PX' - -(1)$, $\alpha = \sum_{i=1}^{n} x_i \alpha_i$, and $\alpha = \sum_{j=1}^{n} y_j \beta_j$

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Proof : $X = 0 \Longleftrightarrow x_1 = x_2 = \ldots = x_n = 0$

$\Longleftrightarrow \alpha = 0, (B$ is a L.I. set)

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Claim (2) : P is an invertible matrix.

Proof : $PX' = 0 \implies X = 0 \implies X' = 0$(*By Claim* (1))

Hence the homogeneous system $PX' = 0$ has only trivial solution $X' = 0$ and thus P is invertible.

Theorem 7

Let V be a n-dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $B_1 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be two ordered bases of V.

Theorem 7

Let V be a n-dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $B_1 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be two ordered bases of V. Then there is a unique, necessarily invertible, $n \times n$ matrix P with entries in F such that (i) $[\alpha]_B = P[\alpha]_{B_1}$ and (ii) $[\alpha]_{B_1} = P^{-1}[\alpha]_B$ for every $\alpha \in V$.

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The columns of P are given by $P_j = [\beta_j]_B$, $j = 1, 2, \ldots, n$.

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The columns of P are given by $P_j = [\beta_j]_B$, $j = 1, 2, \ldots, n$.
Proof : (See the previous slides.)

Theorem 8 (Assignment)

Note : For a given ordered basis B and an invertible matrix P, it is possible to contruct another ordered basis B_1 of a finite-dimensional vector space V.

An example (Theorem 8)

Find an ordered basis for R^4. Let $B = \{\alpha_1 = (0,1,1,1), \alpha_2 = (1,0,1,1), \alpha_3 = (1,1,0,1), \alpha_4 = (1,1,1,0)\}$ be an ordered basis for R^4 and let P be an invertible matrix, where

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}$$

Solution

$$[\beta_1]_B = P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [\beta_2]_B = P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, [\beta_3]_B = P_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix},$$

$$[\beta_4]_B = P_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

Solution

$$[\beta_1]_B = P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [\beta_2]_B = P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, [\beta_3]_B = P_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix},$$

$$[\beta_4]_B = P_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\beta_1 = 1\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\alpha_4 = (1, 1, 2, 2)$$

$$\beta_2 = 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 1\alpha_4 = (2, 2, 1, 1)$$

solution contd.

$$\beta_3 = 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 4\alpha_4 = (4, 5, 5, 1)$$

$$\beta_4 = 0\alpha_1 + 0\alpha_2 + 0\alpha_3 + 2\alpha_4 = (2, 2, 2, 0)$$

Row rank / Column rank

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Row-equivalent matrices admit same row space

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From (1) and (2), row space of $A =$ row space of B

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$$\text{Row rank of } \; R = \text{No. of non-zero rows of } R$$