Engineering Electromagnetics

Lecture 16

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by

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Work done and potential

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})].$$

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}$$

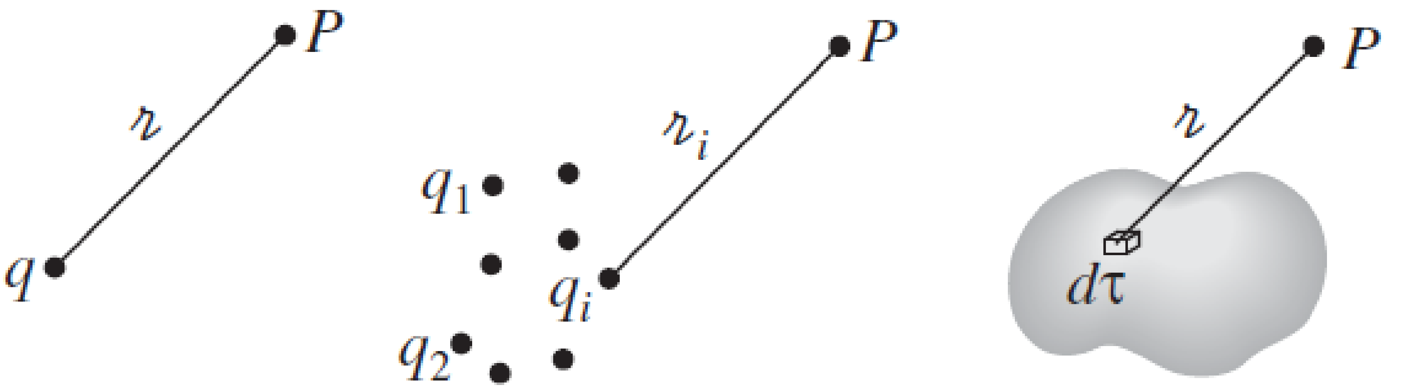
$$W = Q[V(\mathbf{r}) - V(\infty)]$$

so, if you have set the reference point at infinity,

$$W = QV(\mathbf{r}). \tag{2.39}$$

In this sense, *potential* is potential *energy* (the work it takes to create the system) *per unit charge* (just as the *field* is the *force* per unit charge).

Potential for discrete and continuous charge



For a single point charge:

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}}$$

For a set of point charges

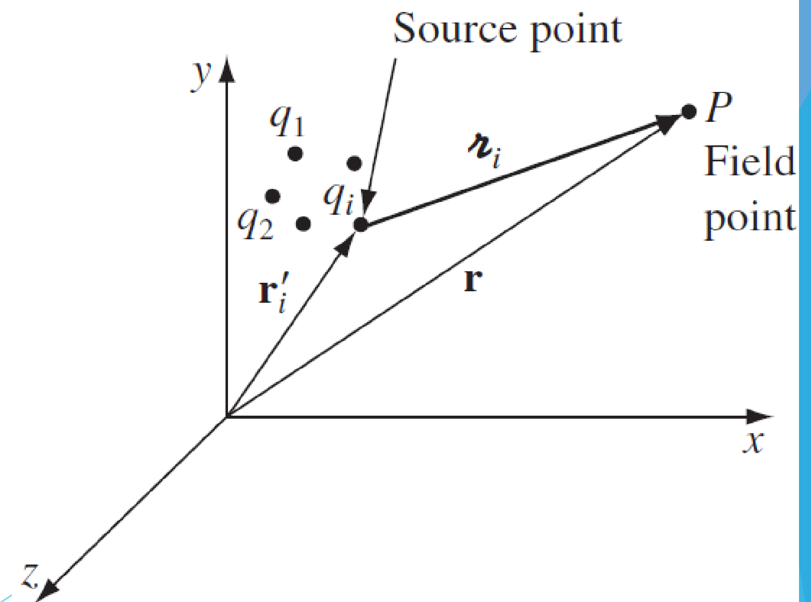
$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}}$$

For continuous charge distribution:

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\,dq}$$

For volume charge:

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'}$$



From Griffith Book

Poisson and Laplace's equation

electric field can be written as the gradient of a scalar potential.

$$\mathbf{E} = -\nabla V.$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla \times \mathbf{E} = \mathbf{0}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}.}$$

Poisson's equation

$$\rho = 0,$$

$$\nabla^2 V = 0.$$

Laplace's equation

A charge is uniformly distributed over a spherical surface of radius a, as illustrated in Figure 3.18. Determine the electric field intensity everywhere in space.

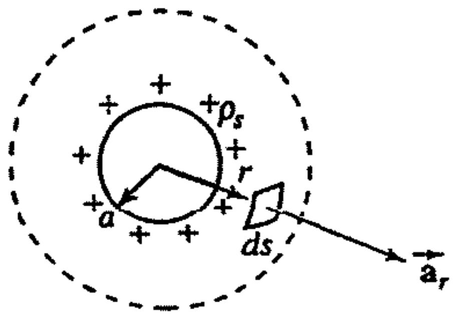


Figure 3.18 A spherical (Gaussian) surface at a radius r enclosing a surface charge distribution ρ_s on a sphere of radius a

Solution A spherical charge distribution suggests the selection of a spherical Gaussian surface of radius r on which the electric field intensity will be constant. If the surface is of radius $r < a$, the electric field intensity must be zero owing to the absence of charge enclosed. However, for the Gaussian surface when $r > a$, the total charge enclosed is

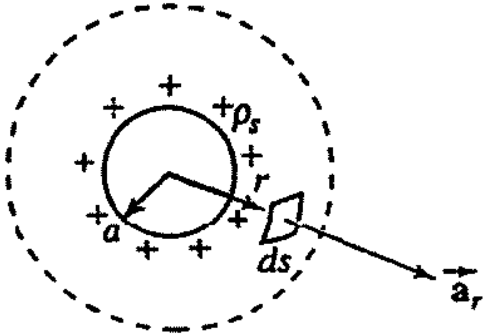


Figure 3.18 A spherical (Gaussian) surface at a radius r enclosing a surface charge distribution ρ_s on a sphere of radius a

$$Q = 4\pi a^2 \rho_s$$

where ρ_s is the uniform surface charge density. Once again,

$$\oint_s \vec{\mathbf{E}} \cdot \vec{ds} = 4\pi r^2 E_r$$

Thus, from Gauss's law, we have

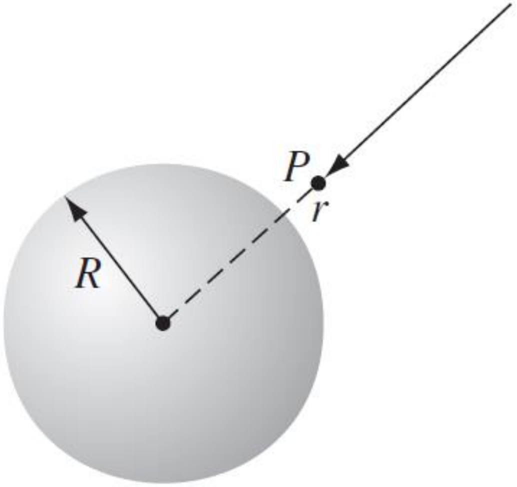
$$E_r = \frac{Q}{4\pi \epsilon_0 r^2} = \frac{\rho_s a^2}{\epsilon_o r^2} \text{ for } r \geq a$$

If charge density is given instead of net charge Q • • •
Then express Q in terms of Charge density ρ_s

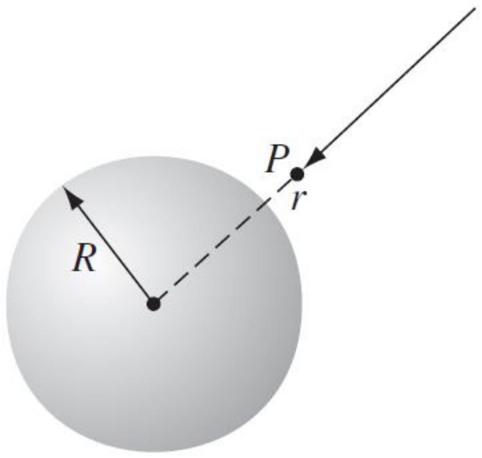
Problem-2

- Find the potential inside and outside a spherical shell of radius R that carries a uniform surface charge. Set the reference point at infinity.
- At any point (r>R and r<R)
- q is the total charge on the sphere (surface charge)

$$\boxed{V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}}$$



Solution



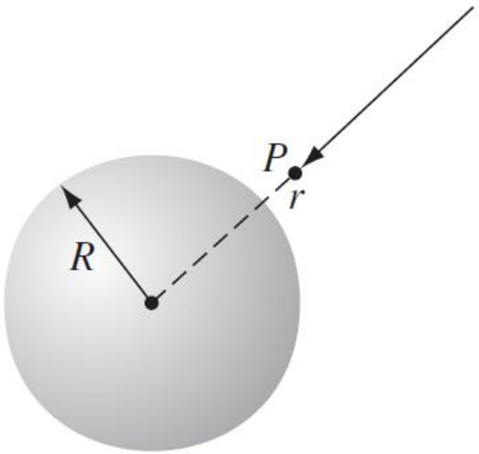
the field outside is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

where q is the total charge on the sphere. For points outside the sphere $(r > R)$,

$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Solution



To find the potential inside the sphere ($r < R$), we must break the integral into two pieces, using in each region the field that prevails there:

$$V(r) = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{R}\frac{q}{r'^2}\,dr' - \int_{R}^{r}(0)\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{R} + 0 = \frac{1}{4\pi\epsilon_0}\frac{q}{R}.$$

Q: Potential inside the shell?

Q: Field inside the shell ?

A charged ring of radius a carries a uniform charge distribution. Determine the potential and the electric field intensity at any point on the axis of the ring.

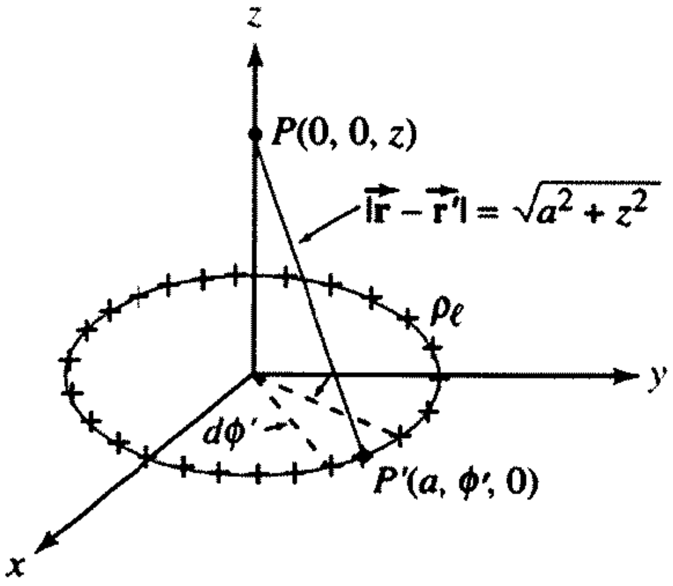
A charged ring bearing a uniform charge distribution is shown in Figure 3.23. The potential at point $P(0, 0, z)$ on the z axis, from (3.37c), is

$$V(z) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\rho_\ell a \, d\phi'}{[a^2 + z^2]^{1/2}}$$

$$= \frac{\rho_\ell a}{2\epsilon_0\sqrt{a^2+z^2}}$$

which reduces at the center of the ring to

$$V(z=0) = \frac{\rho_\ell}{2\epsilon_0}$$



The electric field intensity, from (3.33), is

$$\vec{\mathbf{E}} = -\nabla V = -\frac{\partial V(z)}{\partial z}\vec{\mathbf{a}}_z = \frac{\rho_\ell a}{2\epsilon_0}\left[\frac{z}{(a^2+z^2)^{3/2}}\right]\vec{\mathbf{a}}_z$$

The electric field intensity at the center of the ring, $z = 0$, is zero as expected from the symmetry of the charge distribution. • • •

4. An infinitely long cylinder extended along –ve z direction has ρ = 3 cm and contains a surface charge density, $\rho_s = 2\ e^z\ nC/m^2$.

a) What is the total charge?

b) How much flux leaves the surface ρ = 3 cm, -2 cm < z < -1 cm, $90^o < \varphi < 180^\circ$?

(iii)

$$\rho_s = 2e^{z} \quad nC/m^2$$

(i) $$q_{+} = \int \rho_s \cdot da = \int 2e^{z} \rho \, d\phi \cdot dz$$

$$= 6\int_{0}^{2\pi}\int_{-\infty}^{0} e^{z}\, d\phi\, dz = 12\pi(e^{0}-0)$$

$$= 12\pi$$

(ii) $$\frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho_s\, da \quad q = \int_{\pi/2}^{\pi}\int_{-2}^{-1} 2e^{z}\rho\, d\phi\, dz$$

$$= 2\cdot\frac{\pi}{2}\cdot 3\, e^{z}\Big|_{-2}^{-1} = 3\pi\left(\frac{1}{e}-\frac{1}{e^2}\right)$$

$$= 3\pi(0.367-0.135)$$

$$= 0.232\pi \times 3 = 0.696\pi$$

Work done and potential

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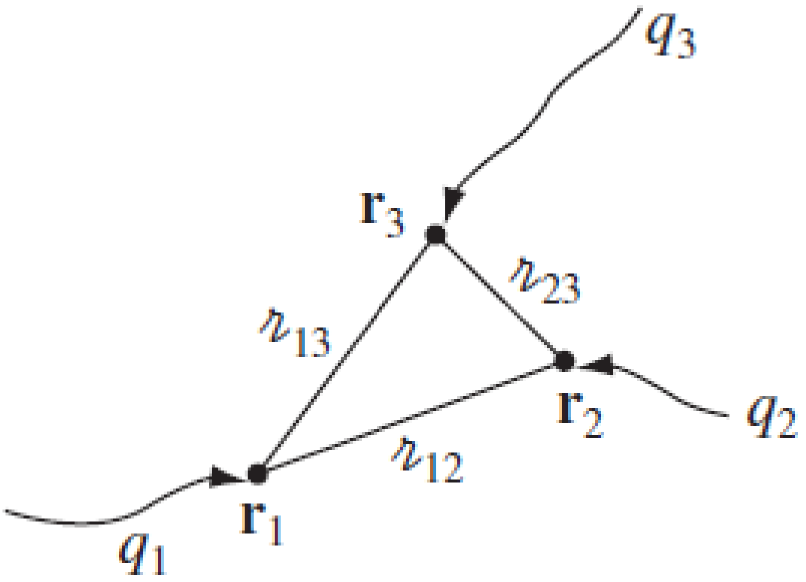
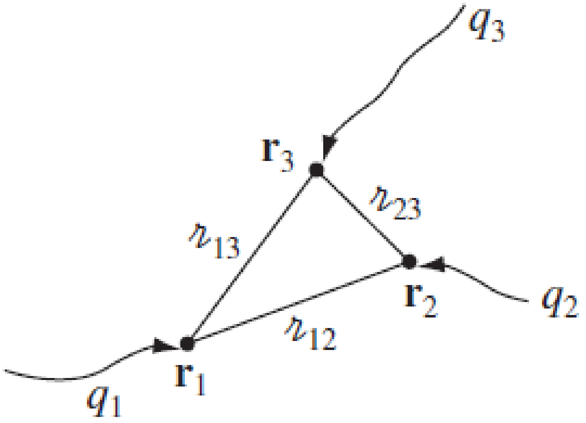
How much work would it take to assemble an entire collection of point charges?



The Energy of a Point Charge Distribution-I

How much work would it take to assemble an entire *collection* of point charges?

Bringing in charges, one by one, from far away: q_1, takes *no* work. $q_2 \rightarrow q_2 V_1$



$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}}\right)$$

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$$

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}}\right)$$

$$\boxed{W = \frac{1}{2}\int \rho V \, d\tau}$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)}$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{\mathscr{r}_{ij}}\right)}$$

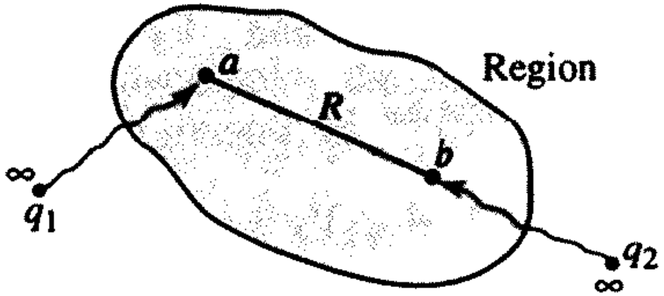
General rule: Take the product of each pair of charges, divide by their separation distance, and add it all up.

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}} \Rightarrow \boxed{W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}}$$

potential at point $\mathbf{r}_i$ (the position of q_i) due to all

- q_1 at a: $W_1 = 0$
- q_2 at b: $W_2 = q_2 V_{b,a}$ Net $W = W_1 + W_2 = \dfrac{q_1 q_2}{4\pi\epsilon R}$
- Reverse the order:
- 1st bring q_2 at b: $W_2 = 0$
- Then q_1 at a: $W_1 = q_1 V_{a,b}$
- $$W = W_1 + W_2 = \frac{q_1 q_2}{4\pi\epsilon R}$$

Same



Not the sequence but the arrangement matters!

For continuous charge distribution

$$\boxed{W = \frac{1}{2}\int \rho V \, d\tau}$$

OR

$$\boxed{W = \frac{1}{2}\int \sigma V \, da}$$

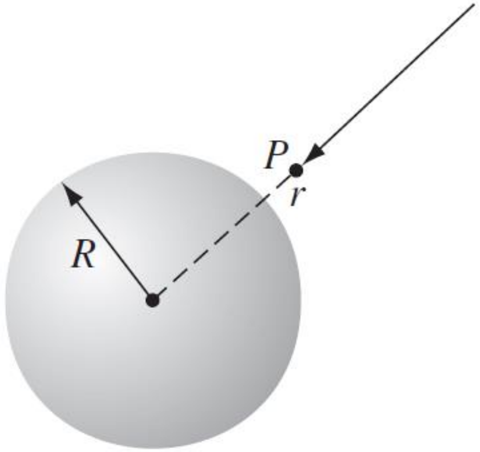
$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

Evaluate the integral

$$\boxed{W = \frac{\epsilon_0}{2}\int E^2 \, d\tau}$$

- $\lambda \rightarrow$ line charge density?

Find the energy of a uniformly charged spherical shell of total charge q and radius R.



From Griffith Book

Problem-2

A metallic sphere of radius 10 cm has a surface charge density of 10 nC/m^2. Calculate the electric energy stored in the system.

Solution

The potential on the surface of the sphere is

$$V = \int_s \frac{\rho_s \, ds}{4\pi\epsilon_0 R} = 9 \times 10^9 \times 10 \times 10^{-9} \times 0.1 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$
$$= 113.1 \text{ V}$$

where Q_t is the total charge on the sphere. For uniform charge distribution, the total charge is

$$Q_t = 4\pi R^2 \rho_s = 4\pi (0.1)^2 \, 10 \times 10^{-9} = 1.257 \text{ nC}$$

Thus,

$$W = 0.5 \times 1.257 \times 10^{-9} \times 113.1 = 71.08 \times 10^{-9} \text{ joules (J)}$$

• • •

Thank You