Electrical Circuits for Engineers (EC1000)

Lecture-3 (a)
Methods of Analysis (Ch-3)

Objectives

1. Develop an understanding of how to use *Kirchhoff's current law (KCL)* to write *Nodal equations* and then to solve for unknown *Node voltages*.
2. Develop an understanding of how to use *Kirchhoff's voltage Law (KVL)* to write mesh equations and then to solve for unknown *Loop currents*.

Apply these laws to develop two powerful techniques for circuit analysis:

1. Nodal analysis, which is based on a systematic application of Kirchhoff's current law (KCL), and
2. Mesh analysis, which is based on a systematic application of Kirchhoff's voltage law (KVL).

Circuit Analysis

Cramer's rule

we can analyse any **linear circuit** by obtaining a set of simultaneous equations that are then solved to obtain the required values of current or voltage. One method of solving simultaneous equations involves Cramer's rule, which allows us to calculate circuit variables as a quotient of determinants.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = y_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$AX = Y$$

$$Det(A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Det(A_1) = \begin{bmatrix} y_1 & a_{12} & a_{13} \\ y_2 & a_{22} & a_{23} \\ y_3 & a_{32} & a_{33} \end{bmatrix}$$

$$Det(A_2) = \begin{bmatrix} a_{11} & y_1 & a_{13} \\ a_{21} & y_2 & a_{23} \\ a_{31} & y_3 & a_{33} \end{bmatrix}$$

$$Det(A_3) = \begin{bmatrix} a_{11} & a_{12} & y_1 \\ a_{21} & a_{22} & y_2 \\ a_{31} & a_{32} & y_3 \end{bmatrix}$$

$$x_1 = \frac{Det(A_1)}{Det(A)}$$

$$x_2 = \frac{Det(A_2)}{Det(A)}$$

$$x_3 = \frac{Det(A_3)}{Det(A)}$$

Circuit Analysis

- $A.X = Y$

- Using Ohm's Law
- $R.I = V$ Mesh current analysis

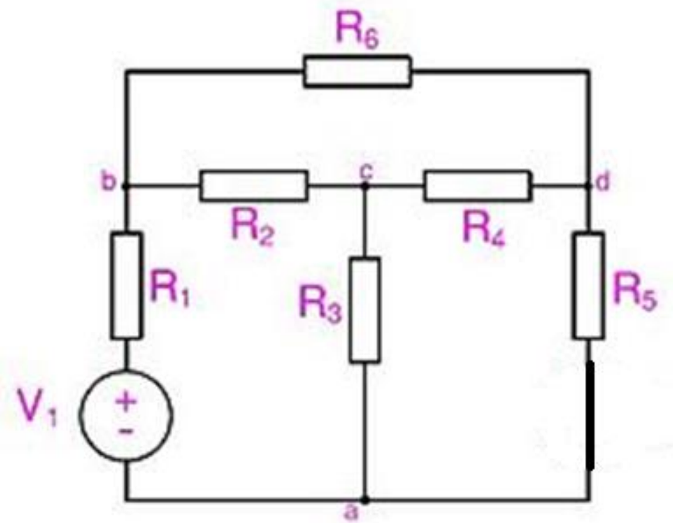
- $I = V\left(\frac{1}{R}\right)$

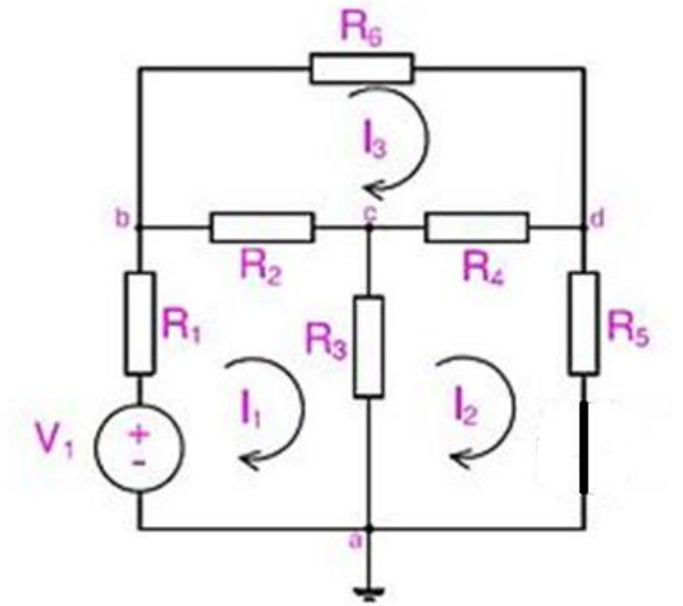
- $I = V.G$

- $G.V = I$ Node voltage analysis

Mesh Analysis

Mesh current analysis

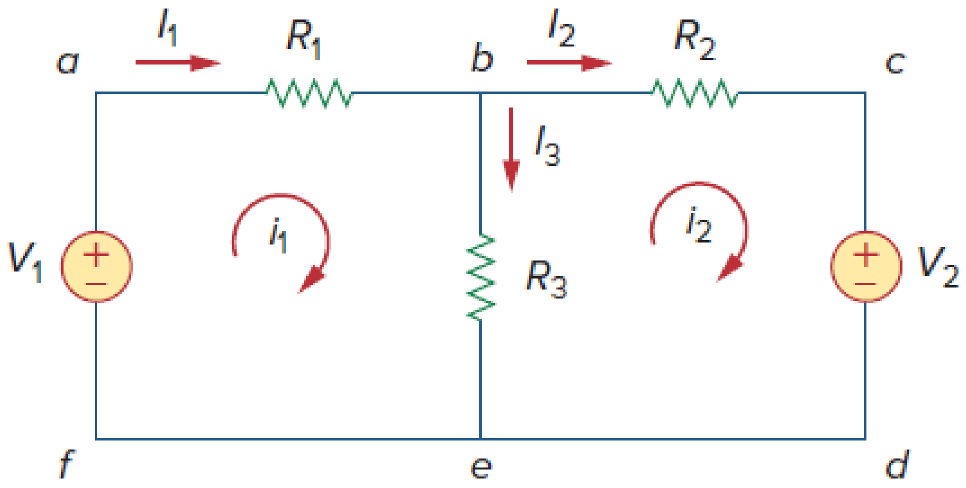




$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} (R_1 + R_2 + R_3) & (-R_3) & (-R_2) \\ (-R_3) & (R_3 + R_4 + R_5) & (-R_4) \\ (-R_2) & (-R_4) & (R_2 + R_4 + R_6) \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Mesh Analysis



<u>Steps to Determine Mesh Currents:</u>

1. Assign mesh currents $i_1, i_2, \ldots, i_n$ to the n meshes.
2. Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the resulting n simultaneous equations to get the mesh currents.

- The first step requires that mesh currents i_1 and i_2 are assigned to meshes 1 and 2. Assume the directions of currents (i.e. each mesh current flows clockwise).
- we apply KVL to each mesh. Applying KVL to mesh 1, we obtain

$$-V_1 + R_1 i_1 + R_3 (i_1 - i_2) = 0 \quad \text{or}$$

$$(R_1 + R_3)\, i_1 - R_3 i_2 = V_1$$

- For mesh 2, applying KVL gives

$$R_2\, i_2 + V_2 + R_3 (i_2 - i_1) = 0$$

or

$$-R_3\, i_1 + (R_2 + R_3)\, i_2 = -V_2$$

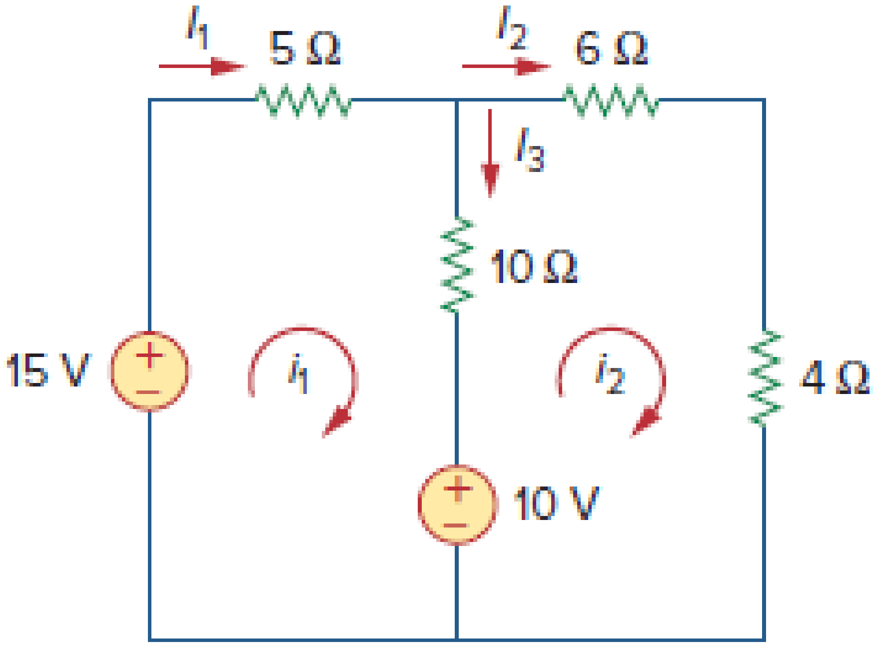
- The third step is to solve for the mesh currents.

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

Note: Branch currents are different from the mesh currents unless the mesh is isolated.
To distinguish between the two types of currents, we use *i* for a mesh current and *I* for a branch current.

Example Problem

1. For the circuit in Figure, find the branch currents I_1, I_2, and I_3 using mesh analysis.



Solution:

We first obtain the mesh currents using KVL. For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

or

$$3i_1 - 2i_2 = 1$$

For mesh 2,

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

or

$$i_1 = 2i_2 - 1$$

■ **METHOD 1** Using the substitution method, we substitute Eq. (3.5.2) into Eq. (3.5.1), and write

$$6i_2 - 3 - 2i_2 = 1 \quad \Rightarrow \quad i_2 = 1 \text{ A}$$

From Eq. (3.5.2), $i_1 = 2i_2 - 1 = 2 - 1 = 1$ A. Thus,

$$I_1 = i_1 = 1 \text{ A}, \qquad I_2 = i_2 = 1 \text{ A}, \qquad I_3 = i_1 - i_2 = 0$$

METHOD 2 To use Cramer's rule, we cast Eqs. (3.5.1) and (3.5.2) in matrix form as

$$\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We obtain the determinants

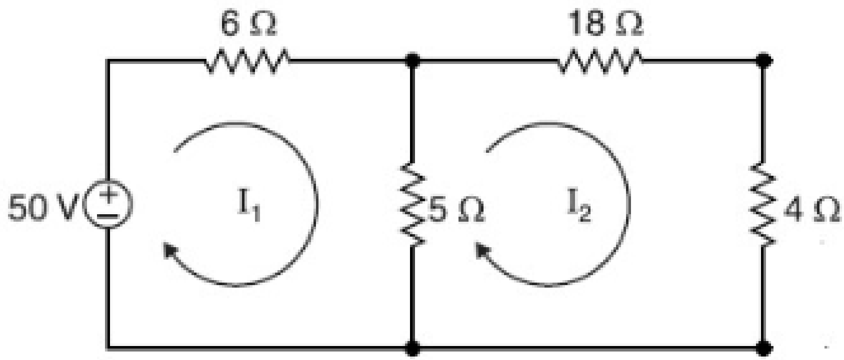
$$\Delta = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 6 - 2 = 4$$

$$\Delta_1 = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = 2 + 2 = 4, \qquad \Delta_2 = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 3 + 1 = 4$$

Thus,

$$i_1 = \frac{\Delta_1}{\Delta} = 1 \text{ A}, \qquad i_2 = \frac{\Delta_2}{\Delta} = 1 \text{ A}$$

- Find the mesh currents I_1 and I_2



- Loop ABDA

$$-50 + 6I_1 + 5(I_1 - I_2) = 0 \qquad (1)$$

$$11I_1 - 5I_2 = 50 \qquad (2)$$

- Loop BCDB

$$+18I_2 + 4I_2 + 5(I_2 - I_1) = 0 \qquad (3)$$

$$+27I_2 - 5I_1 = 0 \qquad (4)$$

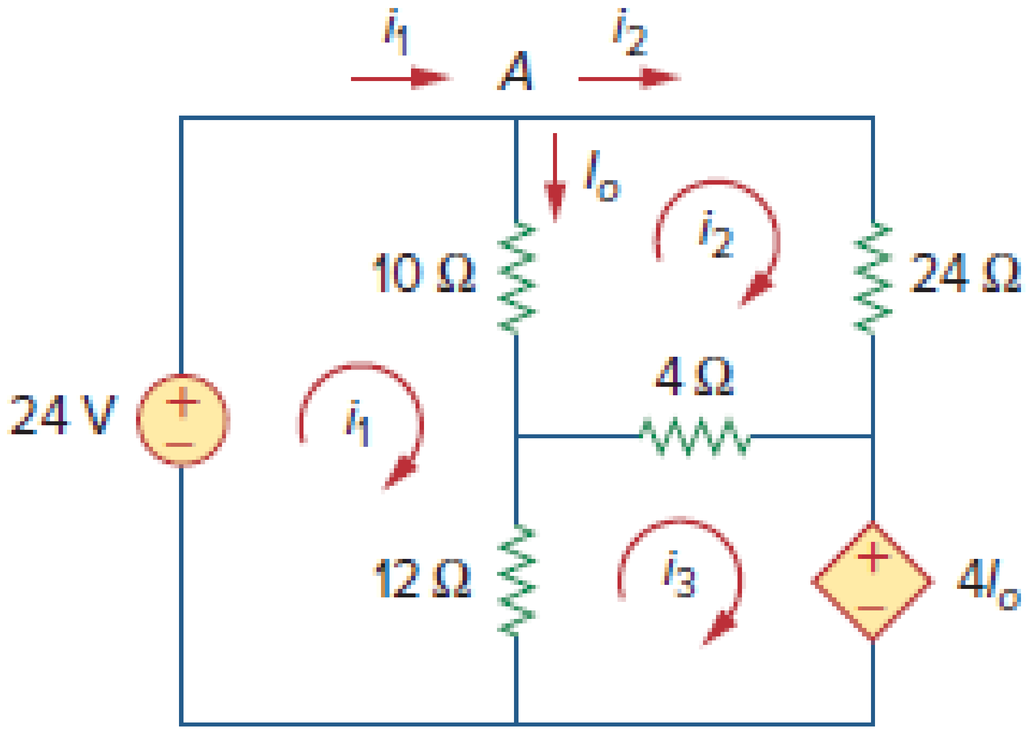
$$I_1 = \frac{27}{5} I_2$$

Sub I_1 in (2)

$$I_2 = 0.92\,A$$

$$I_1 = 4.96\,A$$

3. Find mesh currents and i_0 using Mesh Analysis



Mesh equations

$11i_1-5i_2-6i_3=12$

$-5i_1+19i_2-2i_3=0$

$-i_1-i_2+2i_3=0$

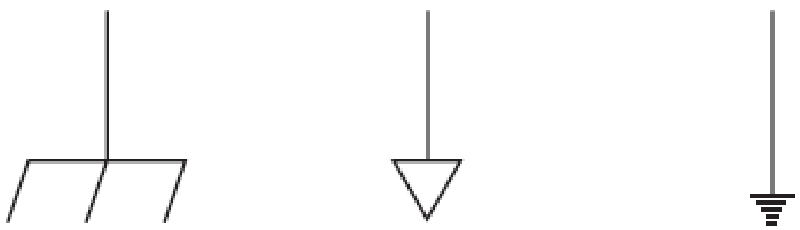
Ans: $i_1=2.25$ A, $i_2=0.75$ A, $i_3=1.5$ A, $i_0=1.5$ A

Nodal Analysis

Ground

- The voltage at one point in a circuit is always measured relative to another point in the circuit.
- For example, if we say that voltage at a point in a circuit is + 10V, we mean that the point is 10V more positive than some reference point in the circuit.
- This reference point in a circuit is usually called the *ground point*.
- Thus ground is used as reference point (zero reference point) for specifying voltages.

Symbols for ground



Nodal Analysis

Steps to Determine Node Voltages:

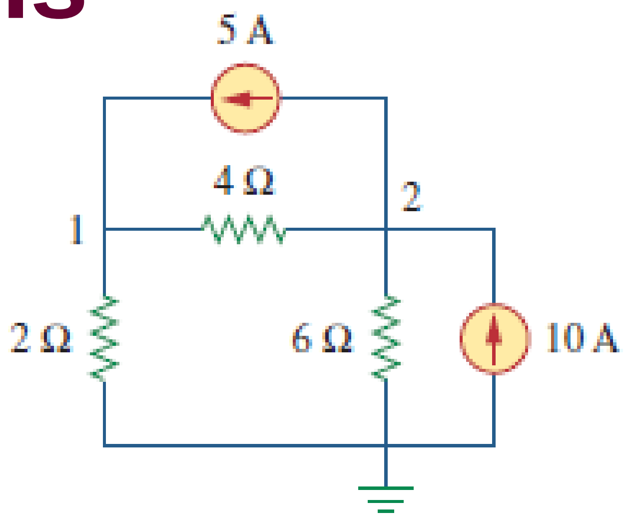
1. Select a node as the reference node. Assign voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining $n - 1$ nodes. The voltages are referenced with respect to the reference node.
2. Apply KCL to each of the $n - 1$ non-reference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

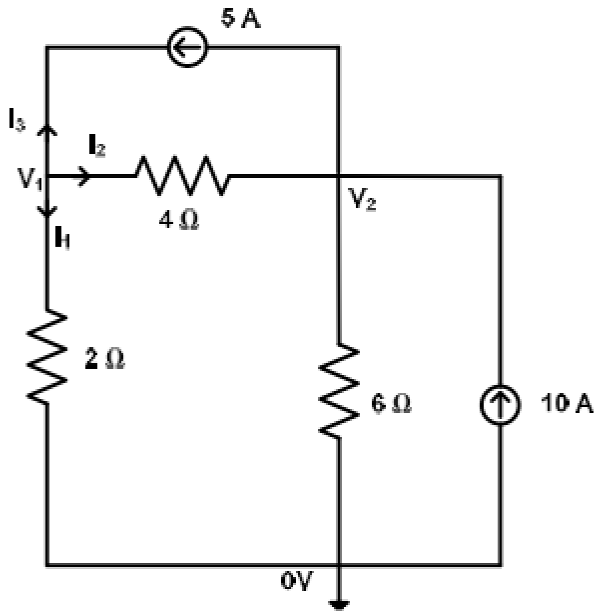
Current always flows from higher potential to lower potential

$$\boxed{i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}}$$

Example Problems

1. Find the node voltages at nodes 1 and 2

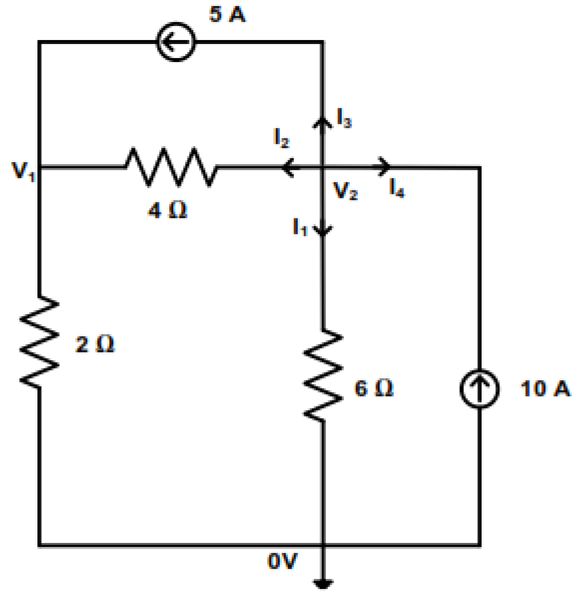




At node 1

$$\frac{V_1}{2}+\frac{V_1-V_2}{4}-5=0 \qquad (1)$$

$$3V_1-V_2=20 \qquad (2)$$



At node 2

$$\frac{V_2}{6}+\frac{V_2-V_1}{4}+5-10=0 \qquad (2)$$

$$-3V_1+5V_2=60 \qquad (4)$$

Solving Eqn. (2) and (4) we get

V_1=13.33 V

V_2 = 20V

Contd.,

- Using Cramer's rule
- $G.V = I$ Node voltage analysis

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

$$v_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12$$

$$v_2 = \frac{\Delta_2}{\Delta}$$

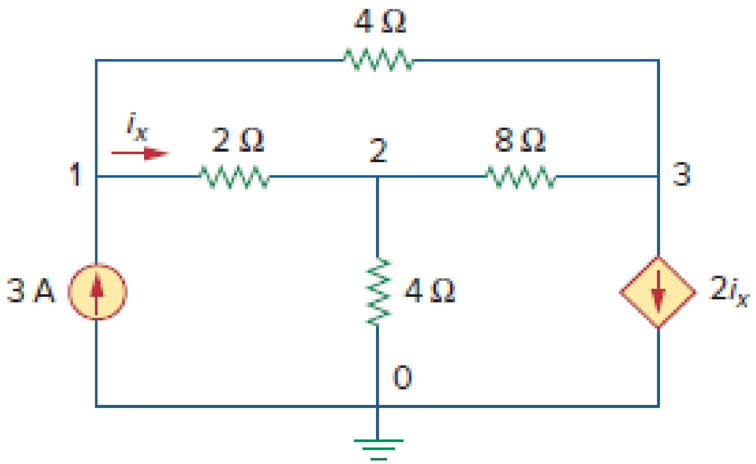
$$\Delta_1 = \begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}$$

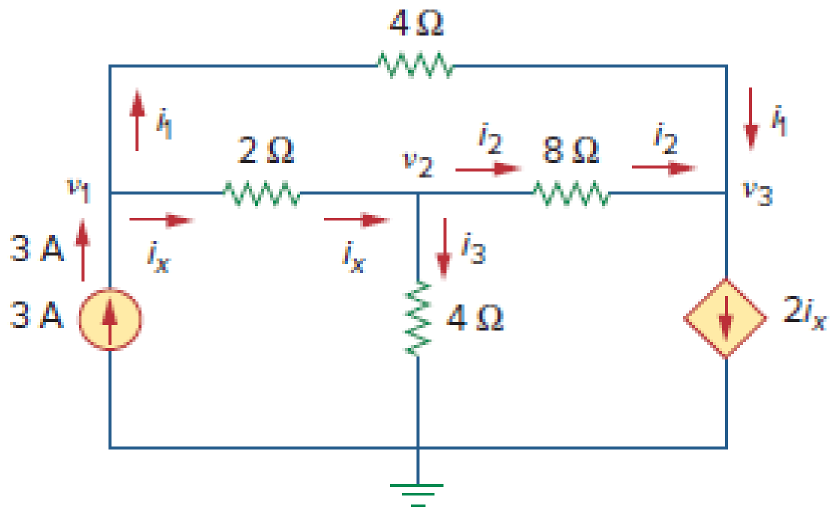
V_1=13.33 V

V_2 = 20V

Example Problems

2. Obtain the node voltages in the circuit of Fig





At node 1,

$$3 = i_1 + i_x \qquad \Rightarrow \qquad 3 = \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2}$$

Multiplying by 4 and rearranging terms, we get

$$3v_1 - 2v_2 - v_3 = 12$$

At node 2,

$$i_x = i_2 + i_3 \qquad \Rightarrow \qquad \frac{v_1 - v_2}{2} = \frac{v_2 - v_3}{8} + \frac{v_2 - 0}{4}$$

Multiplying by 8 and rearranging terms, we get

$$-4v_1 + 7v_2 - v_3 = 0$$

At node 3,

$$i_1 + i_2 = 2i_x \qquad \Rightarrow \qquad \frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} = \frac{2(v_1 - v_2)}{2}$$

Multiplying by 8, rearranging terms, and dividing by 3, we get

$$2v_1 - 3v_2 + v_3 = 0$$

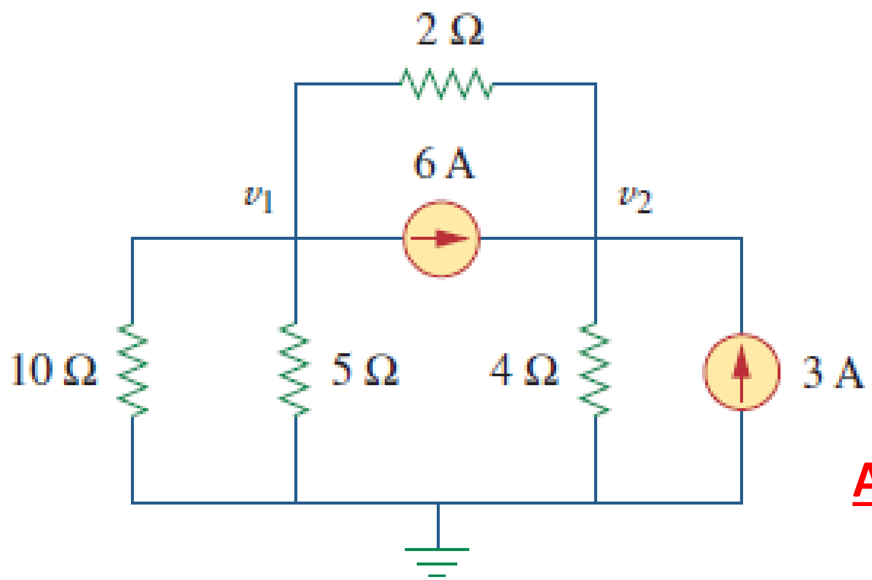
$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{48}{10} = 4.8 \text{ V}, \qquad v_2 = \frac{\Delta_2}{\Delta} = \frac{24}{10} = 2.4 \text{ V}$$

$$v_3 = \frac{\Delta_3}{\Delta} = \frac{-24}{10} = -2.4 \text{ V}$$

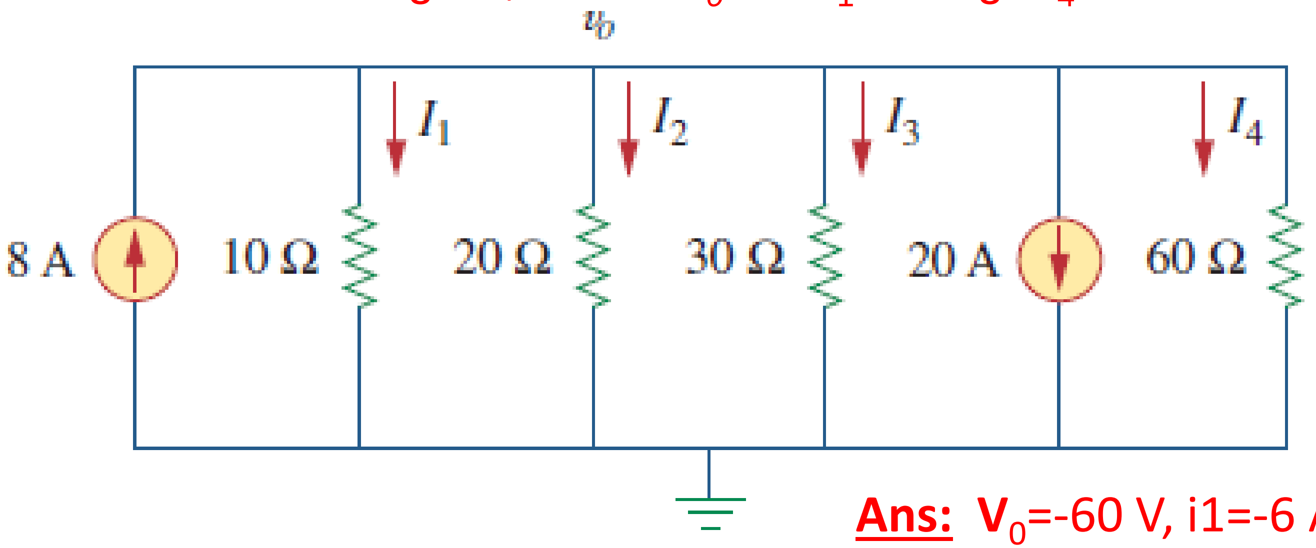
Practical Problems

1. For the circuit in Figure, obtain V_1 and V_2.



Ans: V_1=0 V, V_2=12 V

1. For the circuit in Figure, obtain V_0 and I_1 through I_4.



Ans: V_0=-60 V, i1=-6 A, i_2=-3 A, i_3=-2 A, i_4=-1 A,