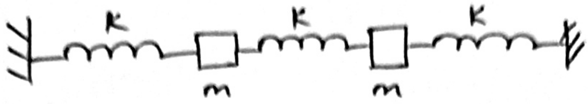
Q1) Consider two masses of 'm' connected by three spring of spring constant 'K' as shown in figure. Find normal modes ?



where x_1 & x_2 are the displacement of masses 'm' from the equilibrium position.

The Lagrangian & equation of motion of system is given by,

$$L = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - \frac{Kx_1^2}{2} - \frac{Kx_2^2}{2} - \frac{K(x_2-x_1)^2}{2}$$

$$m\ddot{x}_1 + K(2x_1 - x_2) = 0$$

$$m\ddot{x}_2 + K(2x_2 - x_1) = 0$$

Sol. $T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2)$; $V = \frac{K}{2}(x_1^2 + x_2^2 + (x_2 - x_1)^2)$

$$T = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad V = \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}$$

$$|V - \omega^2 T| = 0$$

$$\begin{vmatrix} 2K - \omega^2 m & -K \\ -K & 2K - \omega^2 m \end{vmatrix} = 0 \Rightarrow (2K - \omega^2 m)^2 - K^2 = 0$$

$$(2K - \omega^2 m - K)(2K - \omega^2 m + K) = 0$$

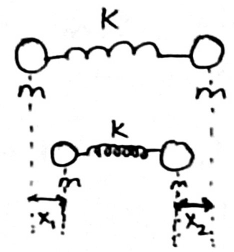
$$(K - \omega^2 m)(3K - \omega^2 m) = 0$$

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{K}{m}} \\ \omega_2 &= \sqrt{\frac{3K}{m}} \end{aligned}}$$

Q2) The Lagrangian of a diatomic molecule is given by

$$L = \frac{m}{2}(\dot{x}_1 + \dot{x}_2) - \frac{k}{2}(x_1 x_2)$$

where m is the mass of each atom and x_1 & x_2 are the displacement of atoms measured from the equilibrium position and $k>0$ (k is spring constant.) . Find the normal modes frequecies ?



Sol.

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2}(x_1 x_2) \quad : \text{(given)}$$

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2)$$

$$V = \frac{k}{2}(x_1 x_2)$$

$$T = \frac{m}{\cancel{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; ; \; V = \frac{k}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|V - \omega^2 T| = 0 \quad : \text{secular eqn.}$$

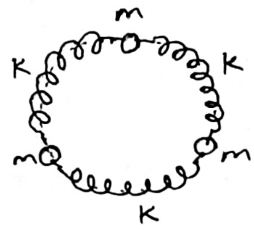
$$\left| \frac{k}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \omega^2 \frac{m}{\cancel{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -\omega^2 m & \frac{k}{2} \\ \frac{k}{2} & -\omega^2 m \end{vmatrix} = 0 \implies \omega^4 m^2 - \frac{k^2}{4} = 0$$

$$\boxed{\omega = \pm\sqrt{\frac{k}{2m}}}$$

Q3) Three masses 'm', initially located equidistant from one another on a horizontal circle of radius 'R'. They are connected in pairs by three spring of force constant 'K' each and of unstretched length '$\frac{2\pi R}{3}$'.

The spring threads the circular tract so that the mass is constrained to move on the circle.



The lagrangian & equation of motion of system is given by,

$$L = \frac{mR^2}{2}(\dot{\theta}_1^2+\dot{\theta}_2^2+\dot{\theta}_3^2) - \frac{KR^2}{2}\left[(\theta_1-\theta_2)^2+(\theta_2-\theta_3)^2+(\theta_3-\theta_1)^2\right]$$

$$mR^2\ddot{\theta}_1 + KR^2\left[(\theta_1-\theta_2)-(\theta_3-\theta_1)\right]=0$$

$$mR^2\ddot{\theta}_2 + KR^2\left[(\theta_2-\theta_3)-(\theta_1-\theta_2)\right]=0$$

$$mR^2\ddot{\theta}_3 + KR^2\left[(\theta_3-\theta_1)-(\theta_2-\theta_3)\right]=0$$

Find the normal mode frequency.

Sol.

$$T = \frac{mR^2}{2}(\dot{\theta}_1^2+\dot{\theta}_2^2+\dot{\theta}_3^2) \; ; \; V = \frac{KR^2}{2}\left[(\theta_1-\theta_2)^2+(\theta_2-\theta_3)^2+(\theta_3-\theta_1)^2\right]$$

$$T = mR^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \; ; \; V = KR^2\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$$

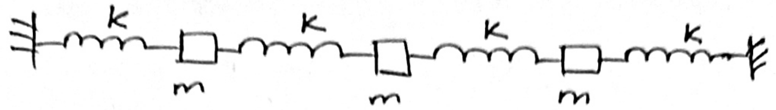
$$|V-\omega^2 T| = 0$$

$$\begin{vmatrix}2KR^2-mR^2 & -KR^2 & -KR^2\\ -KR^2 & 2KR^2-mR^2 & -KR^2\\ -KR^2 & -KR^2 & 2KR^2-mR^2\end{vmatrix} = 0$$

On solving the determinant, we get

$$\omega_1 = 0 \; ; \; \omega_2 = \omega_3 = \sqrt{\frac{3K}{m}}$$

Q4) Find the normal modes for following system.



The Kinectic & Potential energy of the system along with the equation of motion is given by,

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \; ; \; V = \frac{k}{2}\left[(x_2 - x_1)^2 + (x_3 - x_2)^2 + x_1^2 + x_3^2\right]$$

$$m\ddot{x}_1 + k(2x_1 - x_2) = 0$$

$$m\ddot{x}_2 + k(2x_2 - x_1 - x_3) = 0$$

$$m\ddot{x}_3 + k(2x_3 - x_2) = 0$$

Find the normal mode frepuencies?

Sol.

$$T = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; V = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

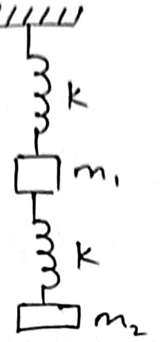
$$|V - \omega^2 T| = 0$$

$$\begin{vmatrix} 2k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 m & -k \\ 0 & -k & 2k - \omega^2 m \end{vmatrix} = 0$$

On solving the determinant, we get

$$\omega_1 = \sqrt{\frac{2k}{m}} \; ; \; \omega_2 = \sqrt{\frac{(2+\sqrt{2})k}{m}} \; ; \; \omega_3 = \sqrt{\frac{(2-\sqrt{2})k}{m}}$$

Q5) Find the normal modes for the given mass-spring system.



The kinectic & Potential energy of the system along with the eqn of motion is given by,

$$T = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2}$$

$$V = \frac{K x_2^2}{2} + \frac{K(x_1 - x_2)^2}{2}$$

$$m_1 \ddot{x}_1 + K(x_1 - x_2) = 0$$

$$m_2 \ddot{x}_2 + K(2x_2 - x_1) = 0$$

Sol.

$$T = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} ; \quad V = \begin{bmatrix} K & -K \\ -K & 2K \end{bmatrix}$$

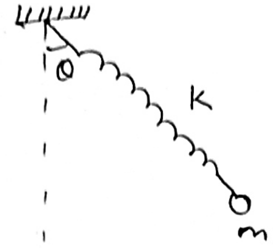
$$|V - \omega^2 T| = 0$$

$$\begin{vmatrix} K - \omega^2 m_1 & -K \\ -K & 2K - \omega^2 m_2 \end{vmatrix} = 0$$

$$\omega^4 (m_1 m_2) - \omega^2 (2K m_1 + K m_2) + K^2 = 0$$

$$\omega^2 = \frac{2Km_1 + Km_2 \pm K\sqrt{4m_1^2 + m_2^2}}{2m_1 m_2}$$

Q7) Find the normal mode frequency of the spring simple pendulum, as shown in figure.



b: unextended length of spring

r: extended length of spring

The kinetic & Potential energy of the system along with the eqn of motion is given by,

$$T = \frac{1}{2} m \cancel{x} (\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = \frac{1}{2} k (r-b)^2 - mgr\cos\theta$$

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + kr - kb = 0$$

$$\cancel{m\theta}\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0$$

Sol.

$$\ddot{r} - r\dot{\theta}^2 + \frac{k}{m}(r-b) - g\cos\theta = 0$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} + \frac{g}{r}\sin\theta = 0$$

if $r = \text{constant} = l$

$\dot{r} = 0$

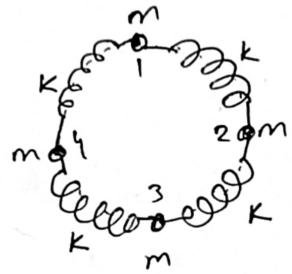
$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad (\sin\theta \approx \theta)$$

$$\ddot{\theta} + \omega^2\theta = 0$$

$$\Rightarrow \boxed{\omega^2 = \sqrt{\frac{g}{l}}}$$

Q8) Find the normal mode frequency for the given spring-m system arranged in a circle of radius 'R'. Each ~~spring~~ mass is coupled to its two neighbouring points by a spring constant 'k'.



The kinetic & Potential energy of a system along with the eqn of motion is given by,

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2)$$

$$V = +\frac{K}{2}(x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_1)$$

$$m\ddot{x}_1 + \frac{K}{2}(x_2 + x_4) = 0$$

$$m\ddot{x}_2 + \frac{K}{2}(x_1 + x_3) = 0$$

$$m\ddot{x}_3 + \frac{K}{2}(x_2 + x_4) = 0$$

$$m\ddot{x}_4 + \frac{K}{2}(x_3 + x_1) = 0$$

Sol.

$$T = m\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad V = \begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & 2k \end{bmatrix}$$

$$|V - \omega^2 T| = 0$$

$$\begin{vmatrix} 2K-\omega^2 m & -k & 0 & 0 \\ -k & 2k-\omega^2 m & -k & 0 \\ 0 & -k & 2k-\omega^2 m & -k \\ 0 & 0 & -k & 2k-\omega^2 m \end{vmatrix} = 0$$

On solving the determinant, we get

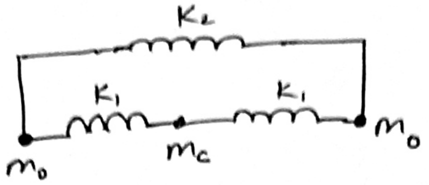
$\omega_1 = 0$

$\omega_2 = \sqrt{\frac{2k}{m}}$

$\omega_3 = \sqrt{\frac{2k}{m}}$

$\omega_4 = \sqrt{\frac{4k}{m}}$

Q9) Find the normal modes and normal frequencies for linear vibrations of the CO_2 molecules as shown in figure.



The kinectic & Potential energy of the system along with equation of motion is given by,

$$T = \frac{1}{2}\left(m_0 \dot{x}_1^2 + m_c \dot{x}_2^2 + m_0 \dot{x}_3^2\right)$$

$$V = \frac{1}{2}K_1\left[(x_2-x_1)^2 + (x_3-x_2)^2\right] + \frac{1}{2}K_2(x_3-x_1)^2$$

$$m_0 \ddot{x}_1 - K_1(x_2-x_1) - K_2(x_3-x_1) = 0$$

$$m_c \ddot{x}_2 + K_1\left[(x_2-x_1) + (x_3-x_2)\right] = 0$$

$$m_0 \ddot{x}_3 - K_1(x_3-x_2) + K_2(x_3-x_1) = 0$$

Sol.

$$T = \begin{bmatrix} m_0 & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_0 \end{bmatrix} \; ; \; V = \begin{bmatrix} K_1+K_2 & -K & -K_2 \\ -K & 2K_1 & -K_1 \\ -K_2 & -K_1 & K_1+K_2 \end{bmatrix}$$

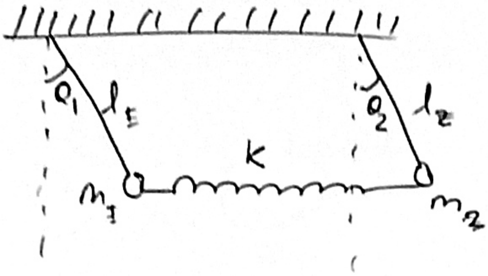
$$|V - \omega^2 T| = 0$$

$$\begin{vmatrix} K_1+K_2-m_0\omega^2 & -K_1 & -K_2 \\ -K_1 & 2K_1 - m_c\omega^2 & -K_1 \\ -K_2 & -K_1 & K_1+K_2-m_0\omega^2 \end{vmatrix} = 0$$

On solving the determinant, we get

$$\omega_1 = 0 \; ; \; \omega_2 = \sqrt{\frac{2K_1}{m_c} + \frac{K_1}{m_0}} \; ; \; \omega_3 = \sqrt{\frac{K_1}{m_0} + \frac{2K_2}{m_0}}$$

Q11)



Given, $T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$

$$V = \frac{mg}{2l}(x_1^2 + x_2^2) + \frac{k}{2}(x_2 - x_1)^2$$

$$m\ddot{x}_1 + \frac{mgx_1}{l} - k(x_2 - x_1) = 0$$

$$m\ddot{x}_2 + \frac{mgx_2}{l} + k(x_2 - x_1) = 0.$$

Find the normal mode frequencies?

Sol.

$$\begin{vmatrix} m\omega^2 - \frac{mg}{l} - k & k \\ k & m\omega^2 - \frac{mg}{l} - k \end{vmatrix} = 0$$

$$\left(m\omega^2 - \frac{mg}{l} - k\right)^2 - k^2 = 0$$

on solving we get,

$$\omega_1 = \sqrt{\frac{g}{l}} \quad \& \quad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

Q12)

$$\eta_1 = x_1 - x_{10} \;;\quad \eta_2 = x_2 - x_{20} \;;\quad \eta_3 = x_3 - x_{30}$$

$$T = \frac{1}{2} m (\dot{\eta}_1^2 + \dot{\eta}_3^2) + \frac{1}{2} M \dot{\eta}_2^2$$

$$V = \frac{k}{2} (\eta_2 - \eta_1)^2 + \frac{k}{2} (\eta_3 - \eta_2)^2$$

$$m\ddot{\eta}_1 + k(\eta_1 - \eta_2) = 0$$

$$M\ddot{\eta}_2 + k\left[(\eta_2 - \eta_1) + (\eta_2 - \eta_3)\right] = 0$$

$$m\ddot{\eta}_3 + k(\eta_3 - \eta_2) = 0$$

Find normal mode frequencies?

Sol.

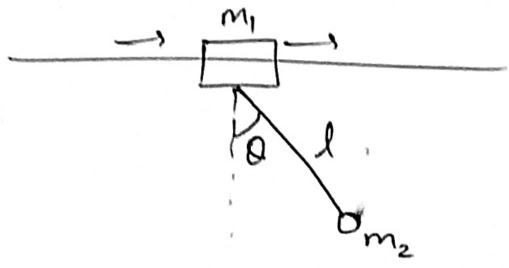
$$\begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

On solving the determinant, we get

$$(k - m\omega^2)\,\omega^2\,(\omega^2 mM - Mk - 2mk) = 0$$

$$\Rightarrow \omega_1 = 0 \;;\quad \omega_2 = \sqrt{\frac{k}{m}} \;;\quad \omega_3 = \sqrt{\frac{k(M+2m)}{mM}}$$

Q13)



$$T = \frac{(m_1+m_2)\dot{x}^2}{2} + l\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}m_2 l^2\dot{\theta}^2$$

$$V = -m_2 g l\cos\theta$$

$$(m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta = 0 \quad \text{— (1)}$$

$$\ddot{x}\cos\theta + l\ddot{\theta} + g\sin\theta = 0 \quad \text{— (2)}$$

Find the normal mode frequencies?

Sol.

$$\ddot{x} = -\frac{m_2 l}{m_1+m_2}\ddot{\theta} \qquad \left[\sin\theta \approx 0\ ;\ \cos\theta \approx 1 \text{ in eqn (1)}\right]$$

Put this value of $\ddot{x}$ in another eqn (2), we get,

$$\ddot{\theta} + \frac{g}{l}\left(\frac{m_1}{m_1+m_2}\right)\theta = 0$$

$$\Rightarrow \omega = \pm\sqrt{\frac{g}{l}\left(\frac{m_1}{m_1+m_2}\right)}$$