MA1001: Differential Equations

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Partial Differential Equations

Laplace, Heat and Wave Equations

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One-Dimensional Wave Equation

$$a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2}.$$

A Revisit to Ordinary Differential Equations

Let λ be any constant.

Find a nontrivial solution $y(x)$ of the ordinary differential equation

$$y'' + \lambda y = 0$$

satisfying the **boundary conditions**

$$y(0) = 0 \quad \text{and} \quad y(\pi) = 0.$$

Homework: If either $\lambda < 0$ or $\lambda = 0$, then this boundary value problem has only the trivial solution.

If $\lambda > 0$, then the general solution of $y'' + \lambda y = 0$ is

$$y(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x.$$

$$y(0) = 0 \quad \Longrightarrow \quad c_2 = 0 \quad \Longrightarrow \quad y(x) = c_1 \sin \sqrt{\lambda} x.$$

$$y(\pi) = 0 \quad \Longrightarrow \quad \sin \sqrt{\lambda} \pi = 0 \quad \Longrightarrow \quad \sqrt{\lambda} = 1, 2, 3, \ldots, n, \ldots.$$

$$\Longrightarrow \lambda = 1, 4, 9, \ldots, n^2, \ldots.$$

Corresponding solutions:

$$\sin x, \sin 2x, \sin 3x, \ldots, \sin nx, \ldots.$$

One-Dimensional Wave Equation

$$a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

Goal: A solution $y(x,t)$ satisfying the **boundary conditions**

$$\begin{aligned} y(0,t) &= 0, \\ y(\pi,t) &= 0 \end{aligned}$$

and the **initial conditions**

$$\begin{aligned} \left.\frac{\partial y}{\partial t}\right]_{t=0} &= 0, \\ y(x,0) &= f(x) \end{aligned}$$

Physical Significance of the Problem

The problem models a string that vibrates along the xy-plane and the solution $y(x, t)$ provides the shape of the string at any time instant t.

- The string is tied to the points $x = 0$ and $x = \pi$ on the x-axis. So, at any time t, we have

$$y(0, t) = 0 \quad \text{and} \quad y(\pi, t)0.$$

- Initially, the string is static. So, we have

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0.$$

- The initial shape of the string is given by the function $y = f(x)$. So, we have

$$y(x, 0) = f(x).$$

Separation of Variables

We find a solution of $a^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$ of the form

$$y(x,t) = u(x)v(t).$$

Put $y(x,t) = u(x)v(t)$ in the equation:

$$a^2 u''(x)v(t) = u(x)v''(t)$$

$$\implies \frac{u''(x)}{u(x)} = \frac{1}{a^2}\frac{v''(t)}{v(t)}.$$

$\implies$ Both sides are constant.

Denote this constant by $-\lambda$. The equation **splits** as:

$$u'' + \lambda u = 0$$
$$v'' + a^2 \lambda v = 0$$

Summary

$y(x,t) = u(x)v(t)$ is a solution of $a^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$ if

$$u'' + \lambda u = 0$$
$$v'' + a^2 \lambda v = 0$$

Also need: $y(0,t) = 0$ and $y(\pi,t) = 0$.

Hold true if: $u(0) = 0$ and $u(\pi) = 0$.

Leads to: The boundary value problem

$$u'' + \lambda u = 0, \quad u(0) = 0 \text{ and } u(\pi) = 0.$$

Recall

The boundary value problem

$$u'' + \lambda u = 0, \quad u(0) = 0 \text{ and } u(\pi) = 0$$

has nontrivial solutions if and only if $\lambda = n^2$ for some positive integer n.

$$\lambda = n^2 \Longrightarrow u_n(x) = \sin nx$$

is a solution of the boundary value problem.

The general solution of

$$v'' + \lambda a^2 v = 0 \quad \text{or} \quad v'' + n^2 a^2 v = 0 :$$

$$v(t) = c_1 \sin nat + c_1 \cos nat$$

But $y(x,t) = u(x)v(t)$ must also satisfy $\left.\dfrac{\partial y}{\partial t}\right]_{t=0} = 0.$

This holds true if $v'(0) = 0$.

$$v'(0) = 0 \Longrightarrow v_n(t) = \cos nat$$

Summary: For each positive integer n,

$$y_n(x,t) = u_n(x)v_n(t) = \sin nx \cos nat$$

is a solution of

$$a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

and satisfies the first three conditions below:

$$\begin{aligned} y(0,t) &= 0, \\ y(\pi,t) &= 0 \end{aligned}$$

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0,$$

$$y(x,0) = f(x)$$

$y_n(x,t) = u_n(x)v_n(t) = \sin nx \cos nat$ is a solution for each a positive integer n

$$\Downarrow$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cos nat = b_1 \sin x \cos at + b_2 \sin 2x \cos 2at + \ldots$$

is also a solution and satisfies the first three conditions.

We also neeed: $y(x,0) = f(x)$.

Holds true if:

$$f(x) = b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots.$$

The Solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cos nat = b_1 \sin x \cos at + b_2 \sin 2x \cos 2at + \ldots$$

Here

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

Homework

Learn about the solutions for

- One-dimensional heat equation: $a^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial y}{\partial t}$.
- Two-dimensional Laplace equation: $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0$.