Inner Product Spaces

(Hoffman - Kunze and C.D Meyer)

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Definition

Definition

- Let $\mathbb{F}$ be the field of real numbers $\mathbb{R}$ or field of complex numbers $\mathbb{C}$, and V be the vector space over the field $\mathbb{F}$.
- An **inner product** on V is a function which assigns to each ordered pair of vectors α, β in V a scalar $\langle \alpha, \beta \rangle$ in $\mathbb{F}$ in such a way that for all α, β, γ in V and all scalar c

1. $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$
2. $\langle c\alpha, \beta \rangle = c\langle \alpha, \beta \rangle$
3. $\langle \alpha, \beta \rangle = \langle \overline{\beta, \alpha} \rangle$
4. $\langle \alpha, \alpha \rangle > 0$ if $\alpha \neq 0$

- One may use $\langle \alpha | \beta \rangle$ or $(\alpha | \beta)$ or (α, β) instead of $\langle \alpha, \beta \rangle$.

Observations

1. $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$,
2. $\langle c\alpha, \beta \rangle = c\langle \alpha, \beta \rangle$,
3. $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$
4. $\langle \alpha, \alpha \rangle > 0$ if $\alpha \neq 0$

Observation-1 :
$$\begin{aligned} \langle c\alpha + \beta, \gamma \rangle &= \langle c\alpha, \gamma \rangle + \langle \beta, \gamma \rangle && \text{by condition-1} \\ &= c\langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle && \text{by condition-2} \end{aligned}$$

Observation-2 :
$$\begin{aligned} \langle \gamma, c\alpha + \beta \rangle &= \overline{\langle c\alpha + \beta, \gamma \rangle} && \text{by condition-3} \\ &= \overline{c\langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle} && \text{by condition-1} \\ &= \overline{c\langle \alpha, \gamma \rangle} + \overline{\langle \beta, \gamma \rangle} && \overline{a+b} = \bar{a} + \bar{b} \\ &= \bar{c}\,\overline{\langle \alpha, \gamma \rangle} + \overline{\langle \beta, \gamma \rangle} && \overline{a * b} = \bar{a} * \bar{b} \\ &= \bar{c}\langle \gamma, \alpha \rangle + \langle \gamma, \beta \rangle && \text{by condition-3} \end{aligned}$$

Example-1

1. Given $V = \mathbb{F}^n$, $\alpha = (x_1, x_2, \ldots, x_n)$, $\beta = (y_1, y_2, \ldots, y_n)$, and $\mathbb{F} = \mathbb{R}/\mathbb{C}$

- If $\mathbb{F} = \mathbb{C}$, (standard norm on $\mathbb{C}$)

$$\begin{aligned} \langle \alpha, \beta \rangle &= \sum_{i=1}^{n} x_i \, \overline{y_i} \\ &= x_1 \, \overline{y_1} + x_2 \, \overline{y_2} + \cdots + x_n \, \overline{y_n} \end{aligned}$$

- If $\mathbb{F} = \mathbb{R}$, (standard norm on $\mathbb{R}$)

$$\begin{aligned} \langle \alpha, \beta \rangle &= \sum_{i=1}^{n} x_i \, y_i \qquad (\overline{y} = y) \\ &= x_1 \, y_1 + x_2 \, y_2 + \cdots + x_n \, y_n \end{aligned}$$

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Example-2(1/2)

2. Verify, is $\langle\alpha,\beta\rangle$ inner product?, where $V=\mathbb{F}^2$, $\alpha=(x_1,x_2)$, $\beta=(y_1,y_2)$, $\mathbb{F}=\mathbb{R}$, and given

$$\langle\alpha,\beta\rangle = x_1\,y_1 - x_2\,y_1 - x_1\,y_2 + 4x_2\,y_2$$

Let $\gamma=(w_1,w_2)$

Cond.1 $\alpha+\beta=(x_1+y_1,x_2+y_2)$, then

$$\begin{aligned}\langle\alpha+\beta,\gamma\rangle =& (x_1+y_1)w_1-(x_2+y_2)w_1-(x_1+y_)w_2\\ &+4(x_2+y_2)w_2\\ \langle\alpha,\gamma\rangle =& x_1\,w_1-x_2\,w_1-x_1\,w_2+4x_2\,w_2\\ \langle\beta,\gamma\rangle =& y_1\,w_1-y_2\,w_1-y_1\,w_2+4y_2\,w_2\end{aligned}$$

- It is straight-forward, $\langle\alpha+\beta,\gamma\rangle=\langle\alpha,\gamma\rangle+\langle\beta,\gamma\rangle$ (Cond.1 is verified)

Example-2(2/2)

Cond.2 & 3 Easy to verify.

Cond.4

$$\begin{aligned}\langle \alpha, \alpha \rangle =& x_1^2 - x_2 x_1 - x_1 x_2 + 4x_2^2 \\ =& (x_1^2 - 2x_1 x_2 + x_2^2) + 3x_2^2 \\ =& (x_1 - x_2)^2 + 3x_2^2 > 0.\end{aligned}$$

So, Cond.4 is verified.

- Hence, $\langle \alpha, \beta \rangle$ is an inner product.

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Example-3

3. If V be $\mathbb{F}^{n \times n}$, the space of all $n \times n$ matrices over $\mathbb{F}$. Then V is isomorphic to $\mathbb{F}^{n^2}$ in natural way. Then from the Example-1, inner product is

$$\langle A, B \rangle = \sum_{j,k=1}^{n} A_{jk} \overline{B}_{jk} = \mathrm{tr}(AB^*) = \mathrm{tr}(BA^*)$$

4. If V be the vector spaces of all continuous complex valued functions on $[0, 1]$. The inner product is give by

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt.$$

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Relation between Norm & Inner product

- The euclidean norm on $\mathbb{C}^{n\times 1}$ defined by $||x|| = \sqrt{x^*x}$
- where $x = (x_1, x_2, \ldots, x_n)^T$
- The standard inner product $\langle x, x\rangle = x^*x$
- Then,

$$||x|| = \sqrt{\langle x, x\rangle}$$

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CBS Inequality (Cauchy–Bunyakovsky–Schwarz inequality)

or

Cauchy-Schwarz Inequality

If V is an inner-product space, and if we set $||\cdot|| = \sqrt{\langle \cdot\,,\cdot \rangle}$, then

$$|\langle x, y\rangle| \leq ||x|| \cdot ||y||$$

The equality holds iff $x = \alpha y$ for $\alpha = \frac{\langle x,y\rangle}{||y||^2}$

Proof.

- Set $\alpha = \frac{\langle x,y\rangle}{||y||^2}$ $\quad y \neq 0$, otherwise it is trivial case
- **Observe.** $\langle \alpha y - x, y\rangle = \alpha\langle y, y\rangle - \langle x, y\rangle = 0$ (substitute α)

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CBS Inq.

- So, $\boxed{\langle \alpha y - x, y\rangle = 0}$

$$
\begin{aligned}
0 \le ||\alpha y - x||^2 &= \langle \alpha y - x, \alpha y - x\rangle \\
&= \overline{\alpha}\langle \alpha y - x, y\rangle - \langle \alpha y - x, x\rangle \\
&= -\langle \alpha y - x, x\rangle \\
&= -\alpha\langle y, x\rangle + \langle x, x\rangle \\
&= -\frac{\langle x, y\rangle}{||y||^2}\langle y, x\rangle + ||x||^2 \quad \text{substitute } \alpha, \text{ and use the relation} \\
&= \frac{-|\langle x, y\rangle|^2 + ||x||^2||y||^2}{||y||^2} \qquad \text{by cond.2 } \langle y, x\rangle = \overline{\langle x, y\rangle}, a\overline{a} = |a|^2
\end{aligned}
$$

- Since $||y||^2 \neq 0$

$$
-|\langle x, y\rangle|^2 + ||x||^2||y||^2 \ge 0 \Rightarrow ||x||^2||y||^2 \ge |\langle x, y\rangle|^2|
$$

- Hence,

$$
|\langle x, y\rangle| \le ||x|| \cdot ||y|| \qquad \textbf{Proved.}
$$

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Norms in Inner-Product Spaces

If V be a inner-product space with an inner product $\langle x, y\rangle$, then

$$||x+y||^2 \leq \left(||x|| + ||y||\right)^2$$

Proof.

$$\begin{aligned}
||x+y||^2 &= \langle x+y, x+y\rangle, \qquad\qquad ||\cdot|| = \sqrt{\langle\cdot\rangle} \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle, \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle, \\
&\leq ||x||^2 + ||x||\cdot||y|| + ||y||\cdot||x|| + ||y||^2, \qquad\qquad \text{by CBS} \\
&= ||x||^2 + 2||x||\cdot||y|| + ||y||^2, \\
&= \left(||x|| + ||y||\right)^2 \qquad\qquad \textbf{Proved.}
\end{aligned}$$

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Parallelogram formula

If V be a inner-product space with an inner product $\langle x, y\rangle$, then

$$||x+y||^2 + ||x-y||^2 = 2\big(||x||^2 + ||y||^2\big)$$

Proof.

$$||x+y||^2 = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle, \quad \text{and}$$
$$||x-y||^2 = \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle,$$

- Adding this two relations, we get the required result.

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Orthogonal Vectors

Orthogonal: In inner-product space V, the vectors x and y in V are said to be **orthogonal** (to each other) iff $\langle x, y \rangle = 0$

If x and y are orthogonal, it is denoted by $x \perp y$

Therefore, $x \perp y \Leftrightarrow \langle x, y \rangle = 0$

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Orthogonal set & Orthonormal set

Let $\mathcal{S}$ be the set of vectors $\{u_1, u_2, \ldots, u_n\}$.

Orthogonal set: The set $\mathcal{S}$ is said to be orthogonal if

$$\langle u_i, u_j \rangle = 0, \text{ for } i \neq j, \quad 1 \leq i, j \leq n.$$

Orthonormal set: The set $\mathcal{S}$ is said to be orthogonal if

1. $\langle u_i, u_j \rangle = 0, \text{ for } i \neq j, \quad 1 \leq i, j \leq n.$
2. $\langle u_i, u_j \rangle = 1, \text{ for } i = j, \quad 1 \leq i, j \leq n.$

Example.

- Let $u_1 = (x_1, y_1)$ and $u_2 = (-y_1, x_1)$
- Therefore, $u_1 \perp u_2$ i.e. u_1 and u_2 are orthogonal to each other, since

$$\langle u_1, u_2 \rangle = x_1(-y_1) + y_1 x_1 = 0$$

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Orthogonal set $\longrightarrow$ Orthonormal set

- **Step-1:** (set, n=3) find the norm for all the orthogonal vectors i.e find $||u_1||, ||u_2||$ and $||u_3||$
- **Step-2:** $\left\{\frac{u_1}{||u_1||}, \frac{u_2}{||u_2||}, \frac{u_3}{||u_3||}\right\}$ is the required **orthonormal** set
- **Given:** the orthogonal set $(\langle u_i, u_j\rangle = 0, 1 \leq i, j \leq 3)$

$$\mathcal{U} \equiv \left\{ u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

- $||u_1|| = \sqrt{2}, ||u_2|| = \sqrt{3}$ and $||u_3|| = \sqrt{6}$

$$\left\{ \frac{u_1}{||u_1||} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{u_2}{||u_2||} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{u_3}{||u_3||} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

Theorem

Let $\mathcal{S}$ be the set of vectors $\{u_1, u_2, \ldots, u_n\}$.

Theorem

Every orthonormal $\mathcal{S}$ set is linearly independent.

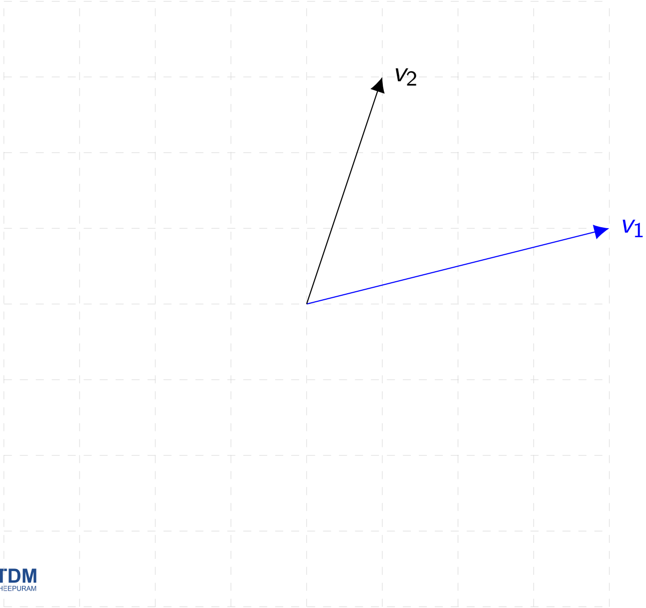
Proof.

- **Given:** $\langle u_i, u_j \rangle = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j \end{cases}$
- First set, $c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0$
- **Aim:** all c_i's are zero.
- Let, for each i,

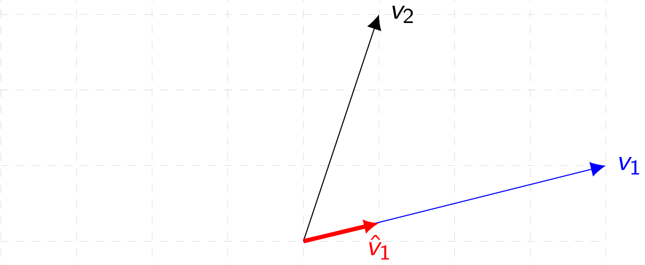
$$
\begin{aligned}
0 &= \langle u_i, 0 \rangle \\
&= \langle u_i, c_1 u_1 + \cdots + c_{i-1} u_{i-1} + c_i u_i + c_{i+1} u_{i+1} \cdots + c_n u_n \rangle \\
&= c_i
\end{aligned}
$$

- Hence proved.

Gram-Schmidt orthogonalization process

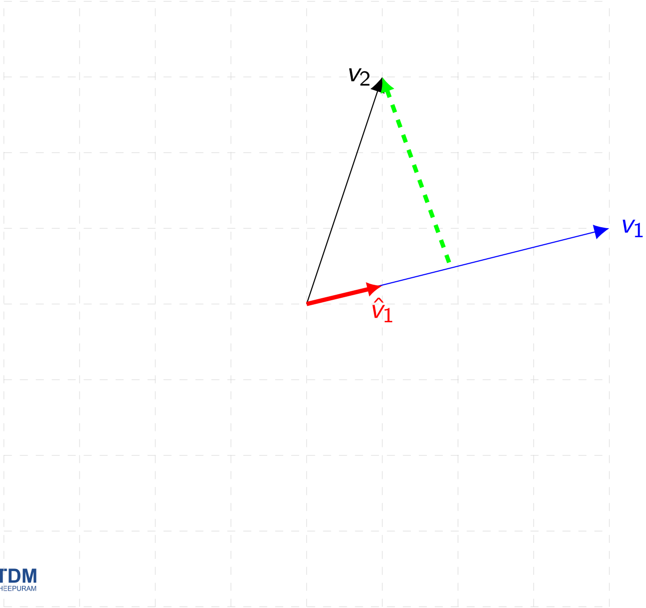


Gram-Schmidt orthogonalization process

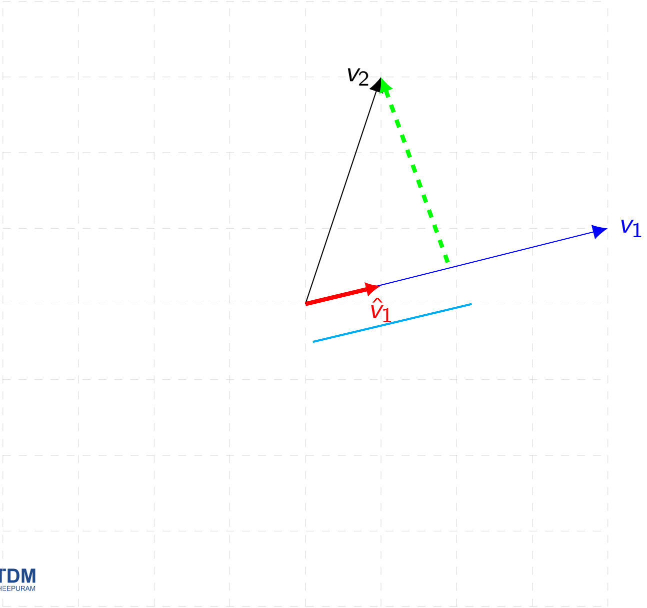


$$\boxed{\hat{v}_1 = \frac{v_1}{||v_1||}}$$

Gram-Schmidt orthogonalization process



Gram-Schmidt orthogonalization process

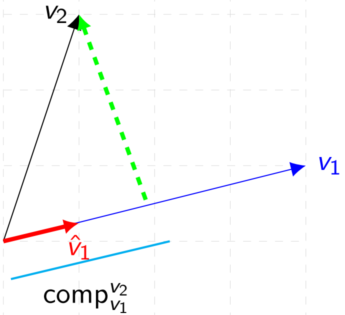


Gram-Schmidt orthogonalization process

Scalar projection

of v_2 onto v_1

$$\text{comp}_{v_1}^{v_2} = \frac{v_1 \cdot v_2}{||v_1||}$$

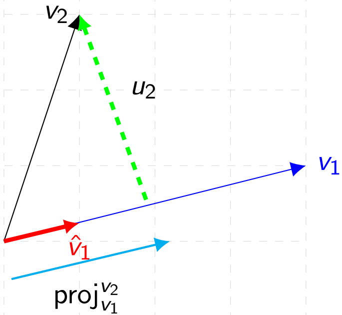


Gram-Schmidt orthogonalization process

Vector projection of v_2

onto v_1 in v_1 direction

$$\text{proj}_{v_1}^{v_2} = \frac{v_1 \cdot v_2}{||v_1||} \hat{v}_1$$



Where u_2 can be obtained by

$$\boxed{u_2 = v_2 - \text{proj}_{v_1}^{v_2} = v_2 - \frac{v_1 \cdot v_2}{||v_1||} \frac{v_1}{||v_1||}}$$

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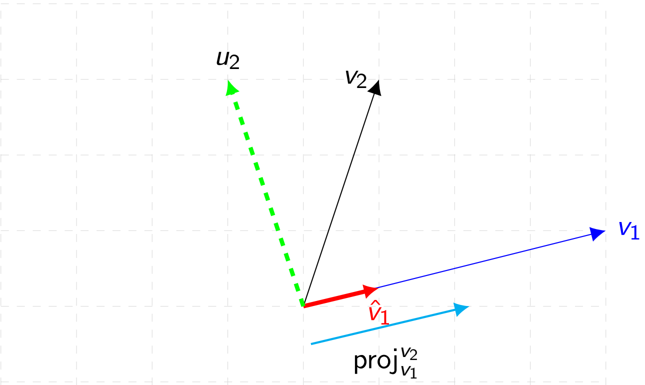
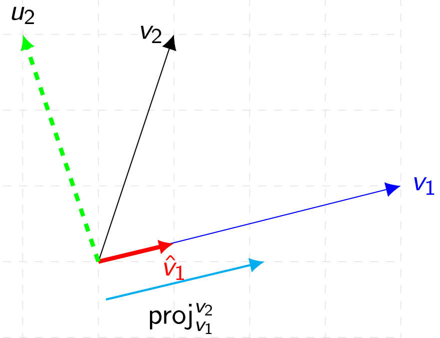
Gram-Schmidt orthogonalization process



$$\boxed{u_2 = v_2 - \text{proj}_{v_1}^{v_2} = v_2 - \frac{v_1 \cdot v_2}{||v_1||}\frac{v_1}{||v_1||}}$$

Gram-Schmidt orthogonalization process



Therefore, $u_1 (= v_1), u_2$ are **orthogonal**

Gram-Schmidt orthogonalization process(GSOP)

- Given two vectors v_1, v_2.
- **Aim:** to find orthonormal vectors u_1, u_2.
- Set, first orthonormal vector $u_1 = v_1$
- Second orthonormal vector is obtained from the previous result

$$\begin{aligned} u_2 &= v_2 - \text{proj}_{v_1}^{v_2} \\ &= v_2 - \frac{v_1 \cdot v_2}{||v_1||^2} v_1 \\ &= v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1, \qquad ||\cdot||^2 = \langle \cdot \rangle \end{aligned}$$

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Gram-Schmidt orthogonalization process

Let V be an inner-product space. Let $v_1, v_2, v_3, \ldots, v_n$ independent vectors in V. Using these vectors on may construct orthogonal vectors $u_1, u_2, u_3, \ldots, u_n$ by Gram-Schmidt orthogonalization process as follows:

$$
\begin{aligned}
u_1 &= v_1, \qquad \text{set up} \\
u_2 &= v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 \\
u_3 &= v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 \\
&-- - - - - - - - - -- \\
&-- - - - - - - - - -- \\
u_n &= v_n - \frac{\langle v_n, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_n, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \cdots - \frac{\langle v_n, u_{n-1} \rangle}{\langle u_{n-1}, u_{n-1} \rangle} u_{n-1}
\end{aligned}
$$

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Example

Given: $v_1 = (3,0,4), v_2 = (-1,0,7), v_3 = (2,9,11)$
Find : orthogonal vectors u_1, u_2, u_3

$$
\begin{aligned}
u_1 =& v_1 = (3,0,4), \qquad \text{set up} \\
u_2 =& v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = (-1,0,7) - \frac{\langle(-1,0,7),(3,0,4)\rangle}{\langle(3,0,4),(3,0,4)\rangle}(3,0,4) \\
=& (-1,0,5) - \frac{-3+28}{9+16}(3,0,4) = (-4,0,3) \\
u_3 =& v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2 \\
=& (2,9,11) - \frac{\langle(2,9,11),(3,0,4)\rangle}{\langle(3,0,4),(3,0,4)\rangle}(3,0,4) \\
& - \frac{\langle(2,9,11),(-4,0,3)\rangle}{\langle(-4,0,3),(-4,0,3)\rangle}(-4,0,3)
\end{aligned}
$$

$$u_3 = (2, 9, 11) - 2(3, 0, 4) - (-4, 0, 3) = (0, 9, 0)$$

Therefore, the required orthogonal set is

$$\boxed{\left\{(3,0,4),\ (-4,\ 0,3),\ (0,9,0)\right\}}$$

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QR-Factorization

A QR factorization of a rectangular(square) matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n (m = n)$ is a factorization

$$A = QR$$

with $Q \in \mathbb{R}^{m \times n}$ orthonormal columns of A and $R \in \mathbb{R}^{n \times n}$ upper triangular matrix with positive diagonal entries.

$$A =: \begin{bmatrix} a_1 | a_2 | \cdots | a_n \end{bmatrix} = \begin{bmatrix} q_1 | q_2 | \cdots | q_n \end{bmatrix} \begin{bmatrix} r_{11} & r_{11} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}$$

- $\langle q_i, q_j \rangle = q_i \cdot q_j = q_i^T q_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

QR

- From GSOP,

$$u_1 = a_1, q_1 = \frac{u_1}{||u_1||}$$

$$u_2 = a_2 - \frac{\langle a_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = a_2 - \frac{\langle a_2, u_1\rangle}{||u_1||^2} u_1$$

$$= a_2 - \langle a_2, \frac{u_1}{||u_1||}\rangle \frac{u_1}{||u_1||} = a_2 - \langle a_2, q_1\rangle q_1, \quad q_2 = \frac{u_2}{||u_2||}$$

$$-- - - - - - -$$

Similarly,

$$u_n = a_n - \langle a_n, q_1\rangle q_1 - \langle a_n, q_2\rangle q_2 - \cdots - \langle a_n, q_{n-1}\rangle q_{n-1}, \text{ and}$$

$$q_n = \frac{u_n}{||u_n||}$$

QR

Therefore,

$$A =: \left[a_1 | a_2 | \cdots | a_n\right] = \left[q_1 | q_2 | \cdots | q_n\right] \begin{bmatrix} r_{11} & r_{11} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}$$

where $r_{i\,j} = \langle a_i, q_j \rangle \quad i \neq j$ and $r_{ii} = ||u_i||$

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