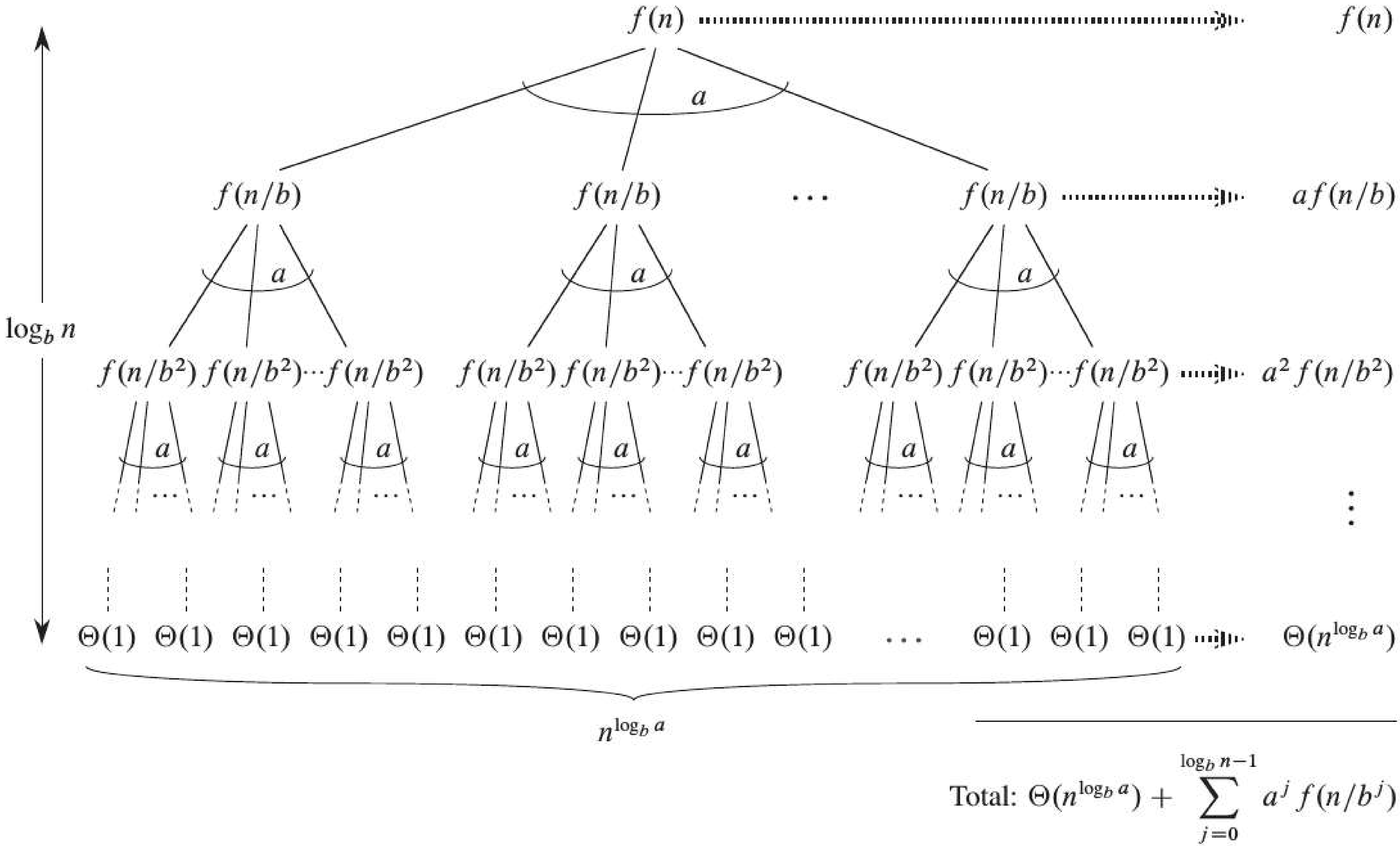
f(n)
a
f(n/b) f(n/b) … f(n/b)
af(n/b)
f(n/b²) f(n/b²)…f(n/b²)
a² f(n/b²)
log_b n
Θ(1)
Θ(n^{log_b a})
n^{log_b a}
Total: Θ(n^{log_b a}) + Σ_{j=0}^{log_b n−1} a^j f(n/b^j)

Master Theorem

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad T(1) = \text{Constant}$$

Subproblem of $\frac{n}{b}$ — Combining Cost

Case 1:- If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$, for some $\epsilon > 0$ then $T(n) = \Theta\left(n^{\log_b a}\right)$

Ex 1:- $T(n) = 3T\left(\frac{n}{2}\right) + n \Rightarrow n = O\left(n^{\log_2 3 - \epsilon}\right) \Rightarrow T(n) = \Theta\left(n^{\log_2 3}\right)$

$\epsilon > 0$

Ex 2:- $T(n) = 5T\left(\frac{n}{2}\right) + n \Rightarrow n = O\left(n^{\log_2 5 - \epsilon}\right) \Rightarrow T(n) = \Theta\left(n^{\log_2 5}\right)$

$\epsilon > 0$

Ex 3:- $T(n) = 9T\left(\frac{n}{2}\right) + n \Rightarrow n = O\left(n^{\log_4 9 - \epsilon}\right) \Rightarrow T(n) = \Theta\left(n^{\log_4 9}\right)$

$\epsilon > 0$

Master Theorem

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n), \quad T(1) = \text{Constant}$$

Case 2:- If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log_b n)$ / $\Theta(f(n) \log_b n)$

Ex 1:- $T(n) = 2T\left(\frac{n}{2}\right) + n \Rightarrow n = n^{\log_2 2} \Rightarrow T(n) = \Theta(n \log_2 n)$

Ex 2:- $T(n) = 4T\left(\frac{n}{2}\right) + n^2 \Rightarrow n^2 = n^{\log_2 4} \Rightarrow T(n) = \Theta(n^2 \log_2 n)$

Ex 3:- $T(n) = 8T\left(\frac{n}{2}\right) + n^3 \Rightarrow n^3 = n^{\log_2 8} \Rightarrow T(n) = \Theta(n^3 \log_2 n)$

Ex 4:- $T(n) = 16T\left(\frac{n}{4}\right) + n^2 \Rightarrow n^2 = n^{\log_4 16} \Rightarrow T(n) = \Theta(n^2 \log_4 n)$

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Master Theorem

$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$, $T(1) = \text{Constant}$

$a \geq 1$, $b > 1$

Regularity Condition (R.C)

Case 3:- If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$, $\epsilon > 0$ & $a f\left(\frac{n}{b}\right) \leq c \cdot f(n)$, $c < 1$

then $T(n) = \Theta(f(n))$

Ex1:- $T(n) = 3T\left(\frac{n}{2}\right) + n^2 \Rightarrow n^2 = \Omega\left(n^{\log_2 3 + \epsilon}\right) \Rightarrow T(n) = \Theta(n^2)$ $\epsilon = 0.1$

R.C. $3\frac{n^2}{4} \leq c \cdot n^2$, $c = \frac{3}{4} << 1$

Ex2:- $T(n) = T\left(\frac{n}{2}\right) + n \Rightarrow n = \Omega\left(n^{\log_2 1 + \epsilon}\right)$, $\epsilon = 0.5$

R.C. $1 \cdot \frac{n}{2} \leq c \cdot n$, $c = \frac{1}{2} << 1 \Rightarrow T(n) = \Theta(n)$

Master Theorem

$$T(n) = 2T\left(\frac{n}{2}\right) + 2^n \qquad f(n) = 2^n$$

$$n^{\log_b a} = n^{\log_2 2} = n$$

$$2^n = \Omega\left(n^{\log_2 2 + \epsilon}\right), \quad \epsilon = 0.5, 1, 2, \cdots \text{Constant} > 0$$

R.C. $\quad a\, f\left(\frac{n}{b}\right) \le c \cdot f(n), \quad c < 1$

$$2 \cdot 2^{\frac{n}{2}} \le c \cdot 2^n$$

$$\frac{2}{2^{\frac{n}{2}}} \le c \Rightarrow \frac{1}{2^{\frac{n}{2}-1}} \le c \Rightarrow T(n) = \Theta(2^n)$$

Master Theorem

$T(n) = a \cdot T(\frac{n}{b}) + f(n)$, $T(1) = \text{Constant}$

$a \geq 1$, $b > 1$

Case 1 :- $f(n) = O(n^{\log_b a - \epsilon})$ $\Rightarrow T(n) = \Theta(n^{\log_b a})$, $\epsilon > 0$

Case 2 :- $f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$

Case 3 :- $f(n) = \Omega(n^{\log_b a + \epsilon})$, $\epsilon > 0$

$a f(\frac{n}{b}) \leq c \cdot f(n) \Rightarrow T(n) = \Theta(f(n))$

$c < 1$

$T(n) = 2T(\frac{n}{2}) + n\log n$

$f(n) = n\log n \qquad n^{\log_b a} = n^{\log_2 2} = n^1$

Case 1 $\Rightarrow$ $n\log n \neq O(n^{1-\epsilon})$, $\epsilon > 0$

Case 2 $\Rightarrow$ $n\log n \neq \Theta(n)$

Case 3 $\Rightarrow$ $n\log n \neq \Omega(n^{1+\epsilon})$, $\epsilon > 0$

Is $\log n = \Omega(n^\epsilon)$, $\epsilon > 0$

$\epsilon = 0.01 \qquad n_0 = 2^{1000}$

$\log_2 2^{1000} \qquad n^\epsilon = (2^{1000})^{0.01}$

$= 1000 \qquad\qquad = 1024$

$1000 \ngeq 1024$

$\therefore \log n \ngeq c\, n^\epsilon$