IIITDM KANCHEEPURAM

MAT1001: DIFFERENTIAL EQUATIONS
ASSIGNMENT 2

APRIL 11, 2024

DUE DATE: APRIL 17, 2024 MARKS: 8

Instructions:

1. *It is a team assignment. Each team will consist of two consecutive students (as per the roll list of your class).*
2. *Identify your partner and work as a team. Each team needs to submit just one hand-written hard copy (not two) of the completed assignment.*

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1. It is clear that $\sin x$, $\cos x$ and $\sin x$, $\sin x - \cos x$ are two distinct pairs of linearly independent solutions of $y'' + y = 0$. Thus, if y_1 and y_2 are linearly independent solutions of the homogeneous equation $y'' + P(x)y' + Q(x)y = 0$, then y_1 and y_2 are not uniquely determined by the equation.

   (a) Show that

   $$P(x) = -\frac{y_1 y_2'' - y_2 y_1''}{W(y_1, y_2)} \qquad \text{and} \qquad Q(x) = \frac{y_1' y_2'' - y_2' y_1''}{W(y_1, y_2)}$$

   so that the differential equation is uniquely determined by any given pair of linearly independent solutions.

   (b) Use (a) to reconstruct the equation $y'' + y = 0$ from each of the two pairs of linearly independent solutions mentioned above.

   (c) Use (a) to reconstruct the equation $y'' - 4y' + 4y = 0$ from the pair of linearly independent solutions e^{2x} and xe^{2x}.

2. If y_1 is a nonzero solution of $y'' + P(x)y' + Q(x)y = 0$ and $v = \displaystyle\int \frac{1}{y_1^2} e^{-\int P dx} dx$, then we know that $y_2 = vy_1$ is also a solution of the differential equation. Show that the formula for v can also be obtained by differentiating $v = \dfrac{y_2}{y_1}$ and using the Abel's formula $W(y_1, y_2) = ce^{-\int P dx}$ (which is the general solution of $\dfrac{dW}{dx} + PW = 0$).

3. Let m_1 and m_2 be any real numbers.

   (a) If $m_1 \neq m_2$, prove that the differential equation $y'' - (m_1 + m_2)y' + m_1 m_2 y = 0$ has $y = \dfrac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$ as a solution.

(b) Think of m_2 as fixed and use l'Hospital's rule to find the limit of the solution in part (a) as $m_1 \to m_2$.

(c) Verify that the limit in part (b) satisfies the differential equation obtained from the equation in part (a) by replacing m_1 by m_2.

4. (a) Show that the method of variation of parameters applied to the equation $y'' + y = f(x)$ leads to the particular solution $y_p(x) = \int_0^x f(t)\sin(x-t)dt$.

   (b) Find a similar formula for a particular solution of the equation $y'' + k^2 y = f(x)$, where k is a positive constant.

5. Prove that if a function $f(x)$ is analytic at the origin (i.e., at $x = 0$) and $f(0) \neq 0$, then $\dfrac{1}{f(x)}$ is also analytic at the origin.

6. Prove that the function $(1+x)^p$, where p is any fixed real number, is analytic on the interval $(-1, 1)$.