Sequences

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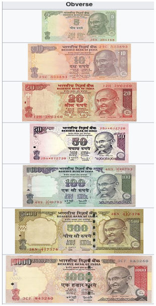
- Everyone knows how to add two numbers together, or even several.
- How do you add infinitely many numbers together ?

What is a sequence?

Motivation:



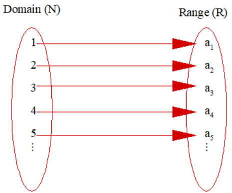
The solution of the problems is be organised. i.e. a list of things.
A list of thing is called a sequence or an ordered list thing is called a sequence (to be more precious)

Definition

A sequence of real numbers is a function from the set $\mathbb{N}$ of natural numbers to the set $\mathbb{R}$ of real numbers.

If $f : \mathbb{N} \to \mathbb{R}$ is a sequence, and if $a_n = f(n)$ for $n \in \mathbb{N}$, then we write the sequence f as $\{a_n\}$.

A sequence of real numbers is also called a real sequence.



Note: It is not mandatory to start with 1.

Remark

Notation: The domain for a sequence is always $\mathbb{N}$ *, a sequence is specified by the value of* S_n*,* $n \in N$*. Thus a sequence may be denoted as*

$$\{S_n\}, \ \ n \in \mathbb{N}, \quad or \ \{S_1, S_2, S_3, \dots\}$$

or

$$\{a_n\}, \ \ n \in \mathbb{N}, \quad or \ \{a_1, a_2, a_3, \dots\}$$

Examples

(i) $\{a_n\}$ with $a_n = 1$ for all $n \in \mathbb{N}$-a constant sequence.

(ii) $\{a_n\} = \{\sqrt{n}\} = \{1, \sqrt{2}, \sqrt{3}, \cdots\}$,

(iii) $\{a_n\} = \{\frac{n-1}{n}\} = \{0, \frac{1}{2}, \frac{2}{3}, \cdots\}$,

(iv) $\{a_n\} = \{(-1)^{n+1}\frac{1}{n}\} = \{1, -\frac{1}{2}, \frac{1}{3}, \cdots\}$,

(v) $\{a_n\} = \{(-1)^{n+1}\} = \{1, -1, 1, ..\}$.

(vi) $\{a_n\} = \left\{\dfrac{1}{n}\right\} = \left\{1, \ \dfrac{1}{2}, \ \dfrac{1}{3}, \ \dfrac{1}{4}, ..\right\}$.

Definition

Range Set: Range set is the set containing of all distinct elements of a sequence, without repetition and without regards to the position of a term.

Range set may be finite of infinite set.

Example

(i) $\{a_n\}$ with $a_n = 1$ for all $n \in \mathbb{N}$. Range Set: $\{1\}$

(ii) $\{a_n\} = \left\{\frac{1}{n}\right\}$ Range Set: $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..\right\}$

(iii) $\{a_n\} = \{(-1)^{n+1}\}$ Range Set: $\{1, -1\}$

(iv) $\{a_n\} = \{1 + (-1)^n\}$ Range Set: $\{0, 2\}$

(v) $\{a_n\} = \{\frac{n-1}{n}\}$ Range Set: $\{0, \frac{1}{2}, \frac{2}{3}, \cdots\}$

(vi) $\{a_n\} = \{(-1)^{n+1}\frac{1}{n}\}$ Range Set: $\{1, -\frac{1}{2}, \frac{1}{3}, \cdots\}$

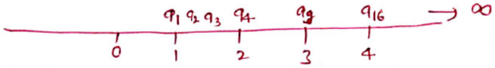
Note: For instance, a number may be repeated in a sequence $\{a_n\}$, but it need not be written repeatedly in the range set.

Example

$$\{a_n\} = \{\sqrt{n}\} = \{1, \sqrt{2}, \sqrt{3}, \cdots\},$$

$\{a_n\} = \{\sqrt{n}\}$

$a_1 = 1,\ a_2 = \sqrt{2}\quad a_3 = \sqrt{3},\ a_4 = \sqrt{4} \cdots$

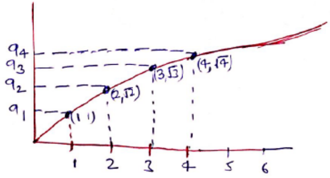


$a_n = \sqrt{n}$ i.e. $y = \sqrt{x}$

$\Rightarrow y^2 = x$

Divergent Sequence

$\lim_{n \to \infty} a_n = \infty$

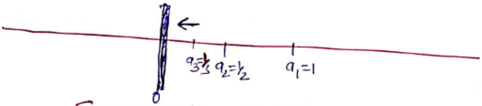


Example

$$\{a_n\} = \left\{\frac{1}{n}\right\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}, ..\right\}$$

$\{a_n\} = \{\frac{1}{n}\}$

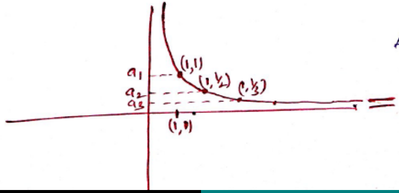
$a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3},\ a_4 = \frac{1}{4} \ldots$



Sequence approaches to 0

$a_n = \frac{1}{n}$

If we take as function form. i.e. $y = \frac{1}{x} \Rightarrow xy = 1$



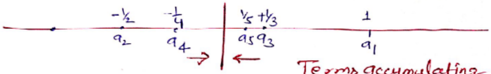
All the terms accumulating towards zero. (or one side)

Example

$$\{a_n\} = \left\{(-1)^{n+1}\frac{1}{n}\right\} = \{1, -\tfrac{1}{2}, \tfrac{1}{3}, \cdots\},$$

$$\{a_n\} = \left\{(-1)^{n+1}\frac{1}{n}\right\}$$

$a_1 = 1$ $a_2 = -\frac{1}{2}$, $a_3 = +\frac{1}{3}$ $a_4 = -\frac{1}{4}$, $a_5 = \frac{1}{5}$, $a_6 = -\frac{1}{6}$ ...



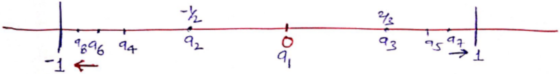
Terms accumulating near zero as $n \to \infty$ from both side.

Example

$$\{a_n\} = \left\{(-1)^{n+1}\left(\frac{n-1}{n}\right)\right\}$$

$$\{a_n\} = \left\{(-1)^{n+1}\left(\frac{n-1}{n}\right)\right\}$$

$a_1 = 0, \quad a_2 = -\frac{1}{2}, \quad a_3 = \frac{2}{3} \quad a_4 = \frac{-3}{4} \quad a_5 = \frac{4}{5} \quad a_6 = \frac{-5}{6} \quad a_7 = \frac{6}{7}$



Terms of the sequence are accumulating near 1 and -1 as $n \to \infty$

$\Downarrow$

Oscillatory sequence

$\downarrow$

Divergent sequence.

Definition

Limit: A sequence is said to tend to a limit l, if for every $\epsilon > 0$, a value N of n can be found such that $|a_n - l| < \epsilon$ for $n \geq N$.

We then write $\lim_{n\to\infty} a_n = l$ or simply $a_n \to l$ as $n \to \infty$

Definition

Bounded sequence: A sequence (a_n) is said to be bounded, if there exists a number k such that $a_n < k$ for every n

Definition

Monotonic sequence: The sequence (a_n) is said to increase steadily or decrease steadily according as $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$, for all values of n. Both incresing and decreasing sequences are called monotonic sequences.

We have observe that

1. $a_n = 1$: every term of the sequence is same,
2. $a_n = \sqrt{n}$: the terms becomes larger and larger,
3. $a_n = \frac{n-1}{n}$: the terms come closer to 1 as n becomes larger and larger,
4. $a_n = (-1)^{n+1}\frac{1}{n}$: the terms come closer to 0 as n becomes larger and larger,
5. $a_n = (-1)^{n+1}$: the terms of the sequence oscillates with values -1 and 1, and does not come closer to any number as n becomes larger and larger.

Now, we make precise the statement "a_n comes closer to a number L" as n becomes larger and larger.

Definition

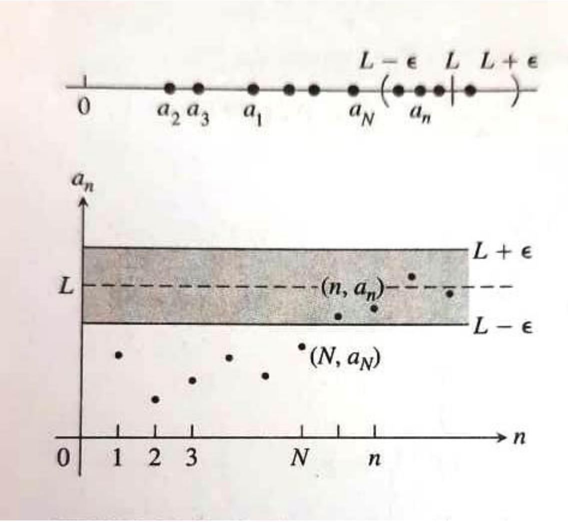
A sequence $\{a_n\}$ in $\mathbb{R}$ is said to converge to a real number L if for every $\epsilon > 0$, there exits positive integer N (in general depending on ϵ) such that

$$|a_n - L| < \epsilon, \qquad \forall\ n \geq N,$$

and in that case, the number L is called a limit of the sequence $\{a_n\}$, and $\{a_n\}$ is called a convergent sequence.

If no such number L exists, we say $\{a_n\}$ diverges.

Note: we say that a sequence sequence a_n converges to l then $\lim_{n \to \infty} a_n = l$.



$L-\epsilon$
L
$L+\epsilon$
0
a_2
a_3
a_1
a_N
a_n
a_n
$L+\epsilon$
L
(n, a_n)
$L-\epsilon$
(N, a_N)
0
1
2
3
N
n
n

This definition ensures that

- From some stage onwards the difference between a_n and L can be be made less than any preassigned positive number ϵ , however small, i.e. given any positive real number ϵ , no matter however small, $\exists$ a positive integer N such that Nth term onwards , a_n remains arbitrarily close to l
- At the most a finite number of terms of the sequence can lies outside $(L-\epsilon,\ L+\epsilon)$.
- If we find even one $\epsilon > 0$ for which infinitely many terms of the sequence lie outside $(L-\epsilon,\ L+\epsilon)$, then sequence cannot converge to L.

Example

Show that the sequences $\left\{\frac{1}{n}\right\}$, $\left\{\frac{(-1)^n}{n}\right\}$ and $\left\{1-\frac{1}{n}\right\}$ converges to the limits 0, 0 and 1, respectively.

Solution:

(i) Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, and let $\epsilon > 0$ be given. We have to identify an $N \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ for all $n \geq N$. Note that

$$\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}.$$

Thus, if we take $N = \left\lceil \frac{1}{\epsilon} \right\rceil + 1$, then we have

$$|a_n - 0| = \frac{1}{n} < \epsilon \quad \forall n \geq N.$$

Hence, $\left\{\frac{1}{n}\right\}$ converges to 0. Here $\lceil x \rceil$ denotes the integer part of x.

(ii) Next, Let $a_n = \dfrac{(-1)^n}{n}$ for all $n \in \mathbb{N}$. Since $|a_n| = 1$ for all $n \in \mathbb{N}$, in this case also, we see that

$$|a_n - 0| = \frac{1}{n} < \epsilon \quad \forall n \geq N := \left\lceil \frac{1}{\epsilon} \right\rceil + 1.$$

Hence, $\left\{\dfrac{(-1)^n}{n}\right\}$ converges to 0.

(iii) Now, let $a_n = 1 - \dfrac{1}{n}$ for all $n \in \mathbb{N}$. Since. $|a_n - 1| = \dfrac{1}{n}$ for all $n \in \mathbb{N}$, we have

$$|a_n - 1| < \epsilon \quad \forall n \geq N := \left\lceil \frac{1}{\epsilon} \right\rceil + 1.$$

Hence, $\left\{1 - \dfrac{1}{n}\right\}$ converges to 1.

Example

Find the N for a sequence $\{a_n\}$ where $a_n = k$.

According definition: $|a_n - l| < \epsilon \ \forall \ n \ \geq \ N$
Now $|k - k| < \epsilon \implies 0 < \epsilon$. This means that we can choose any value of n or any N. Hence, $|a_n - k| < \epsilon \ \forall \ n \geq 1$.
$\implies$

$$\lim_{n \to \infty} k = k$$

Example

Show that $\lim_{n \to \infty} = \dfrac{3 + 2\sqrt{n}}{\sqrt{n}} = 2$.

Let ϵ be any positive number

$$\left| \frac{3 + 2\sqrt{n}}{\sqrt{n}} - 2 \right| \epsilon, \implies \left| \frac{3}{\sqrt{n}} \right| < \epsilon \quad or \quad n > \frac{9}{\epsilon^2}$$

so N be a positive integer grater than $9/\epsilon^2$.

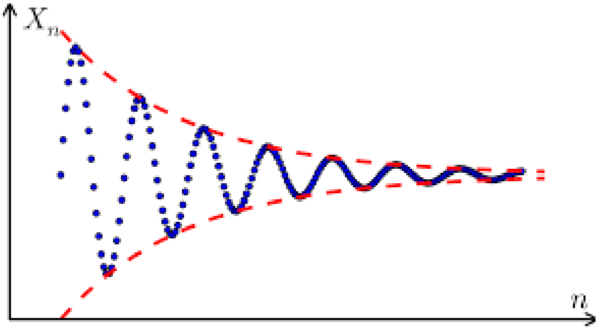
- Every convergent sequence is bounded.
- A sequence cannot converge to more than one limit.
- Every convergent sequence is bounded and has a unique limit.

When limit L is not known (nor can any guess be made of the same.)

Definition

A necessary and sufficient condition for the convergence of a sequence $\{a_n\}$ is that , for each $\epsilon > 0$, $\exists$ a positive integer m such that

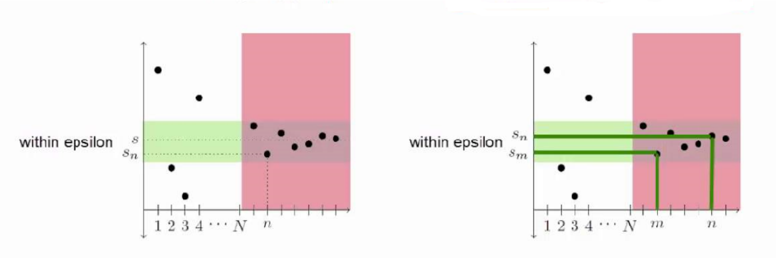
$$|a_{n+p} - a_n| < \epsilon \ \forall \, n \geq m, \ \ p \geq 1$$



Definition

A sequence $\{a_n\}$ is called a Cauchy sequence or a fundamental sequence if for each $\epsilon > 0$, $\exists$ a positive integer m such that

$$|a_{n+p} - a_n| < \epsilon \ \forall\, n \geq m, \ \ p \geq 1$$



Note that in the field of real numbers, a sequence is convergent iff it is a Cauchy sequence.

Definition

A real number α is said to be a limit point of a sequence $\{a_n\}$, if every neighbourhood of α contains an infinite numbers of the sequence.

Thus α is a limit point of a sequence if given any positive number ϵ, however small, $a_n \in \ (\alpha - \epsilon, \ \ \alpha + \epsilon)$ for an infinite number of values of n, i.e.

$|a_n - \alpha| < \epsilon$, for infinitely many values of n.

Remark

A limit of the range set of a sequence is also a limit point of the sequence but converse may not always be true.

- The constant sequence $\{a_n\}$, where $a_n = 1, \ \forall \ n \in \mathbb{N}$, has the only limit point 1. The range is 1 and has no limit point.
- The sequence $\left\{\dfrac{1}{n}\right\}$ has 0 as a limit point which is as well a limit point of the range $\left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dots\right\}$
- 0 and 2 are the only limit points of the sequence $\{1 + (-1)^n\}, \ \ n \in \mathbb{N}$. The range set $\{0, 2\}$ has no limit point.
- 1 and -1 are two limit points of the sequence $\{(-1)^n \left(1 + \dfrac{1}{n}\right)\}, \ \ n \in \mathbb{N}$ which are also limit points of the range set.

Theorem

Bolzano-Weierstrass Theorem: *Every bounded sequence has a limit point.*

Remark

The converse of the theorem is not always true, for there do exist unbounded sequence having only one real limit point.

Example: $\{1, 2, 1, 4, 1, 6, \ldots\}$ has a unique limit point , but it is not bounded above.

Theorem

Every bounded sequence with a unique limit point is convergent.

1. A bounded sequence which does not converge, and has at least two limit points, is said to oscillate finitely.
2. An unbounded sequence is said to oscillate infinitely if it diverges neither to $+\infty$ nor to $-\infty$.
3. A bounded sequence either converges or else oscillates finitely but an unbounded sequence either diverges to $+\infty$ or $-\infty$ oscillates infinitely

Example

Test the convergence/divergence of sequence $\{r^n\}$.

Case:1 When $r > 1$

$\therefore \; r^n = (1+h)^n > 1 + nh \; \forall \; n \in \mathbb{N}$

If $\epsilon > 0$ be any number however large, we have

$$1 + nh > \epsilon, \;\; if \;\; n > \frac{\epsilon - 1}{h}$$

Let m is a positive integer greater than $\dfrac{\epsilon - 1}{h}$, therefore for $\epsilon > 0 \; \exists$ a positive integer m such that $r^n > \epsilon, \; \forall \; n \geq m$. Hence sequence diverges to ∞.

Case:2 When $r = 1$

In this case $\lim r^n = 1$. The sequence converge to 1.

Case:3 When $r = -1$

In this case the sequence $\{(-1)^n\}$ is bounded and has two limit points. Hence the sequence oscillates finitely.

Case:4 When $r < 1$

We take $|r| = \dfrac{1}{1+h}, \quad h > 0.$

$\therefore \quad |r^n| = |r|^n = \dfrac{1}{(1+h)^n} \leq \dfrac{1}{(1+nh)} \ \forall \ n \in \ \mathbb{N}.$

We take $\epsilon > 0$ then

$$\frac{1}{(1+nh)} < \epsilon, \quad when \quad n > \frac{\left(\frac{1}{\epsilon} - 1\right)}{h}$$

Let m be a positive integer greater than $\frac{\left(\frac{1}{\epsilon}-1\right)}{h} \implies$ for $\epsilon > 0$, there exist a positive integer m such that $|r^n| < \epsilon, \quad n \geq m$. Hence $\{r^n\}$ converges to 0 for $|r| < 1$. Case:5 When $r < -1$

Let $r = -t$ so that $t > 1$. Thus we get the sequence $\{(-1)^n t^n\}$. This sequence oscillates infinitely.

Remark

The sequence $\{r^n\}$ converges to zero iff $|r| < 1$.

- $\{1+(-1)^n\}$ oscillates finitely.
- $\left\{(-1)^n\left\{1+\frac{1}{n}\right\}\right\}$ oscillates finitely
- $\{-2^n\}$ diverges to $-\infty$
- $\left\{\frac{(-1)^{n-1}}{n!}\right\}$ converges to the limit 0.
- $\left\{1+\frac{1}{n}\right\}$ converges to the limit 1.
- $\left\{1, 2, \frac{1}{2}, 3, \frac{1}{3} \dots\right\}$ oscillates infinitely (bounded below and unbounded above)
- The sequence $\left\{m+\frac{1}{n}\right\}$, where $m, \ n$ are natural numbers, also oscillates infinitely, $1, 2, 3 \dots$ being its limit point

Theorem

If $\{a_n\}$ *and* $\{b_n\}$ *be two sequence such that* $\lim a_n = a$*,* $\lim b_n = b$ *then*
(i) $\lim(a_n \pm b_n) = \lim a_n \pm \lim b_n$
(ii) $\lim(a_n b_n) = \lim a_n \ \lim b_n$
(iii) $\lim(a_n/b_n) = \lim a_n / \lim b_n$ *if* $b \neq 0,\ b_n \neq 0\ \forall\ n.$

Remark

The converse may not be true, i.e. the sequence $\{a_n \pm b_n\},\ \{a_n b_n\}\ \{a_n/b_n\}$ *is convergent, the sequence* $\{a_n\}$ *and* $\{b_n\}$ *may not be convergent.*

Theorem

Sandwich theorem: *If* $\{a_n\}$ $\{b_n\}$ *and* $\{c_n\}$ *are three sequences such that (i)* $a_n \leq b_n \leq c_n \ \forall \ n$ *and (ii)* $\lim a_n = \lim c_n = l$ *then* $\lim b_n = l$

Example

Show that the sequence $\{b_n\}$, where
$b_n = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right]$ converges to zero.

We can write the sequence $\{b_n\}$ in between two sequence such as

$$\frac{n}{(2n)^2} \leq b_n \leq \frac{n}{n^2} \implies \frac{1}{4n} \leq b_n \leq \frac{1}{n}$$

Now the sequence $\{a_n\}$ $\{c_n\}$, where $a_n = \frac{1}{4n}$ $c_n = \frac{1}{n}$ and both are having the limit zero.

Theorem

Cauchy's First theorem on limits: If $\lim_{n\to\infty} a_n = l$ *then*

$$\lim_{n\to\infty}\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) = l$$

Example

Show that $\left[\frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \cdots + \frac{1}{\sqrt{(n^2+n)}}\right] = 1.$

We consider $a_k = \frac{n}{\sqrt{(n^2+k)}}$, *where* $k = 1, 2, 3, \ldots n.$ and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{(1+k/n)}} = 1$ Now we apply the theorem we get

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{(n^2+1)}} + \frac{n}{\sqrt{(n^2+2)}} + \cdots + \frac{n}{\sqrt{(n^2+n)}}\right] = 1$$

$\Longrightarrow$

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{(n^2+1)}} + \frac{n}{\sqrt{(n^2+2)}} + \cdots + \frac{n}{\sqrt{(n^2+n)}}\right] = 1$$

Example

(i)

$$\lim_{n\to\infty} \frac{1}{n}\left[1+2^{1/2}+3^{1/3}\cdots+n^{1/n}\right]=1 \quad (a_n=n^{1/n} \quad \lim a_n=1)$$

(ii)

$$\lim_{n\to\infty} \frac{1}{n}\left[1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n}\right]=1(a_n=1/n \quad \lim a_n=0)$$

Theorem

Cauchy's Second theorem on limits: If all the terms of a sequence $\{a_n\}$ are positive and if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists the so dose $\lim\limits_{n\to\infty} (a_n)^{1/n}$ and the two limits are equal, i.e. $\lim\limits_{n\to\infty} (a_n)^{1/n} = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$, provided the later limit exist.

Example

Show that the sequence $\{a_n^{1/n}\}$ and $\{b_n^{1/n}\}$ where (i) $a_n = \frac{(3n)!}{(n!)^3}$ (ii) $b_n = \frac{n^n}{(n+1)(n+2)\ldots(n+n)}$ converges and find their limit.

(i) $a_n = \frac{(3n)!}{(n!)^3}$ and $a_{n+1} = \frac{(3n+3)!}{((n+1)!)^3} \Longrightarrow$

$\lim \frac{a_{n+1}}{a_n} = \lim \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} = 27$. Applying theorem, we get

$\lim (a_n)^{1/n} = \lim \frac{a_{n+1}}{a_n} = 27$. Hence convergent and limit is 27.

(ii) Limit is $e/4$

Theorem

If $\{a_n\}$ *be a sequence such that* $\lim \frac{a_{n+1}}{a_n} = l$ *where* $|l| < 1$*, then* $\lim a_n = 0$ *and* $|l| > 1$ *then* $\lim a_n = \infty$

Example

Show that for any real number x, $\lim \frac{x^n}{n!} = 0$

Here $\lim \frac{a_{n+1}}{a_n} = \frac{x}{n+1} = 0 < 1$. Hence by theorem the limit is zero.

Example

Test whether the sequence $a_n = \dfrac{2^n 3^n}{n!}$ is a non-decreasing or not.

We take the ratio

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2^{n+1}3^{n+1}}{(n+1)!}}{\dfrac{2^n 3^n}{n!}} = \frac{6}{n+1}$$

For increasing $a_{n+1} \geq a_n$ i.e.

$$\frac{a_{n+1}}{a_n} \geq 1 \quad \Longrightarrow \; n+1 \;\leq\; 6 \quad \Longrightarrow \quad n \leq 5$$

$$a_1 = 6,\; a_2 = 18,\; a_3 = 36,\; a_4 = 54,\; a_5 = 64.8,\; a_6 = 64.8,\; a_7 = 55.542$$

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