Fourier Series:

According to the Fourier theorem, any periodic function can be expressed as sum of sine and cosine functions whose frequencies increase in the ratio of natural numbers.

$\mathbf{f}(t) = \mathbf{f}(t + \mathrm{nT})$, **n=1,2,3,4,5,……, T is the period**

$$\mathbf{f}(t) = \sum_{n=0}^{\infty} a_n cos\left(n\frac{2\pi}{T}t\right) + \sum_{n=0}^{\infty} b_n sin\left(n\frac{2\pi}{T}t\right)$$

$$\mathbf{f}(t) = a_0 + \sum_{n=1}^{\infty} a_n cos\left(n\frac{2\pi}{T}t\right) + \sum_{n=0}^{\infty} b_n sin\left(n\frac{2\pi}{T}t\right)$$

$$\int_{t_0}^{t_0+T} f(t) cos(m\omega t)dt = \sum_{n=0}^{\infty} a_n \int_{t_0}^{t_0+T} cos(m\omega t)\, cos(n\omega t)dt + \sum_{n=0}^{\infty} b_n \int_{t_0}^{t_0+T} cos(m\omega t)\, sin(n\omega t)dt$$

$$\int_{t_0}^{t_0+T} f(t) cos(m\omega t)dt = \sum_{n=0}^{\infty} a_n \int_{t_0}^{t_0+T} cos(m\omega t)\, cos(n\omega t)dt \;= \frac{a_m}{2}\int_{t_0}^{t_0+T} \{1 + cos(2m\omega t)\}\, dt = \frac{a_m T}{2}$$

For, n = m

$$= \frac{1}{2}\int_{t_0}^{t_0+T} [cos(\{m-n\}\omega t) + cos(\{m+n\}\omega t)]dt = \frac{1}{2}\left[-\frac{cos(\{m+n\}\omega t)}{2\{m+n\}\omega} + \frac{cos(\{m-n\}\omega t)}{2\{m-n\}\omega}\right]_{t_0}^{t_0+T} = 0$$

$$\int_{t_0}^{t_0+T} cos(m\omega t) sin(n\omega t)dt = \frac{1}{2}\int_{t_0}^{t_0+T} [sin(\{m+n\}\omega t) - sin(\{m-n\}\omega t)]dt$$

$$= \frac{1}{2}\left[-\frac{sin(\{m-n\}\omega t)}{2\{m-n\}\omega} + \frac{sin(\{m+n\}\omega t)}{2\{m+n\}\omega}\right]_{t_0}^{t_0+T} = 0$$

$$a_m = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) cos(m\omega t)dt$$

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$$\int_{t_0}^{t_0+T} f(t)sin(m\omega t)dt = \sum_{n=0}^{\infty} a_n \int_{t_0}^{t_0+T} sin(m\omega t)\, cos(n\omega t)dt + \sum_{n=0}^{\infty} b_n \int_{t_0}^{t_0+T} sin(m\omega t)\, sin(n\omega t)dt$$

$$\int_{t_0}^{t_0+T} f(t)sin(m\omega t)dt = \sum_{n=0}^{\infty} b_n \int_{t_0}^{t_0+T} sin(m\omega t)\, sin(n\omega t)dt = \frac{b_n}{2}\int_{t_0}^{t_0+T}\{1 - cos(2m\omega t)\}\, dt = \frac{b_n T}{2}$$

$$= \frac{1}{2}\int_{t_0}^{t_0+T}[cos(\{m-n\}\omega t) - cos(\{m+n\}\omega t)]dt = \frac{1}{2}\left[\frac{sin(\{m+n\}\omega t)}{2\{m+n\}\omega} - \frac{cos(\{m-n\}\omega t)}{2\{m-n\}\omega}\right]_{t_0}^{t_0+T} = 0$$

$$b_m = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)sin(m\omega t)dt$$

$$\int_{t_0}^{t_0+T} cos(m\omega t)sin(n\omega t)dt = 0$$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=0}^{\infty} a_n cos\left(n\frac{2\pi}{T}t\right) + \sum_{n=0}^{\infty} b_n sin\left(n\frac{2\pi}{T}t\right)$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)cos(n\omega t)dt$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)sin(n\omega t)dt$$

Fourier Series: Applications

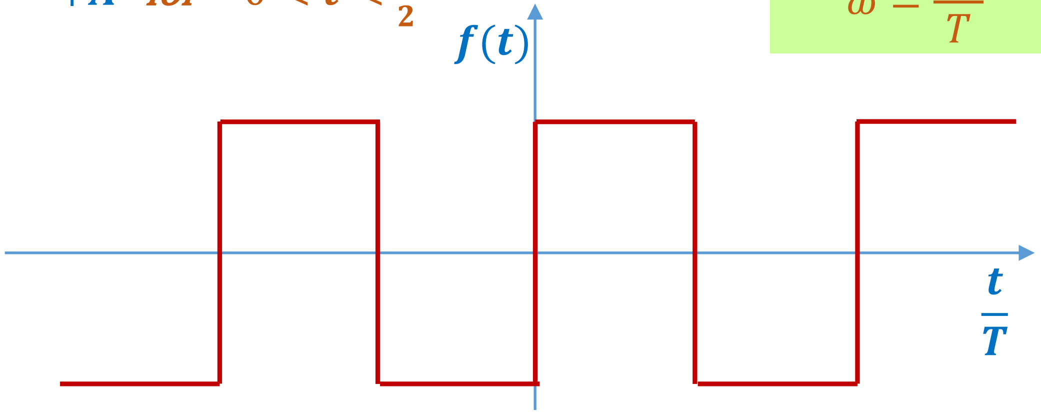
$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(n\omega t) + \sum_{n=0}^{\infty} b_n sin(n\omega t)$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt, \quad a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)sin(n\omega t)dt$$

$$\omega = \frac{2\pi}{T}$$

$$f(t) = -A \quad for \; -\frac{T}{2} < t < 0$$
$$= +A \quad for \quad 0 < t < \frac{T}{2}$$



$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{2}{T}\int_{-T/2}^{0} -Adt + \frac{2}{T}\int_{0}^{T/2} Adt$$
$$= \frac{2A}{T}\{-\int_{-\frac{T}{2}}^{0} dt + \int_{0}^{\frac{T}{2}} dt\}$$
$$= 0$$

As, $f(t)$ *is an odd function of t*

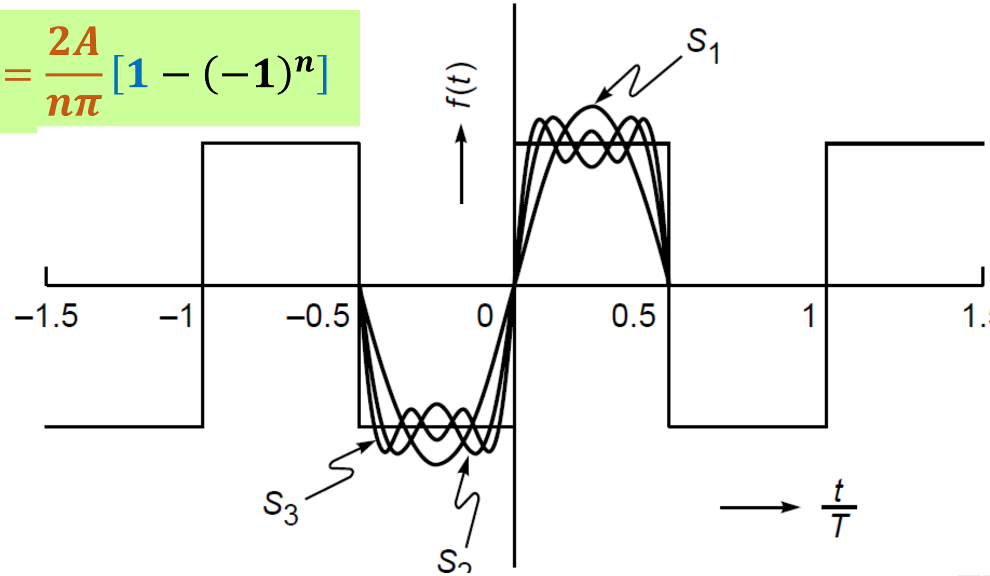
$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)cos(n\omega t)dt \Rightarrow even\ fn.\ Of\ t$$

$$a_n = 0$$

$$b_n = \frac{2}{T}2\int_{0}^{T/2} Asin(n\omega t)dt = \frac{4A}{T}\frac{1}{n\omega}[-cos(n\omega t)]_0^{T/2} = \frac{2A}{n\pi}[1 - cos(n\pi)] = \frac{2A}{n\pi}[1-(-1)^n]$$

$$f(t) = \sum_{n=0}^{\infty} \frac{2A}{n\pi}[1-(-1)^n]sin(n\omega t)$$

$$\boxed{f(t) = \frac{2A}{\pi}\left\{sin(\omega t) + \frac{1}{3}sin(3\omega t) + \frac{1}{5}sin(5\omega t) + \cdots \dots \dots\right\}}$$



Fourier Series: Triangular wave

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=0}^{\infty} b_n \sin(n\omega t)$$

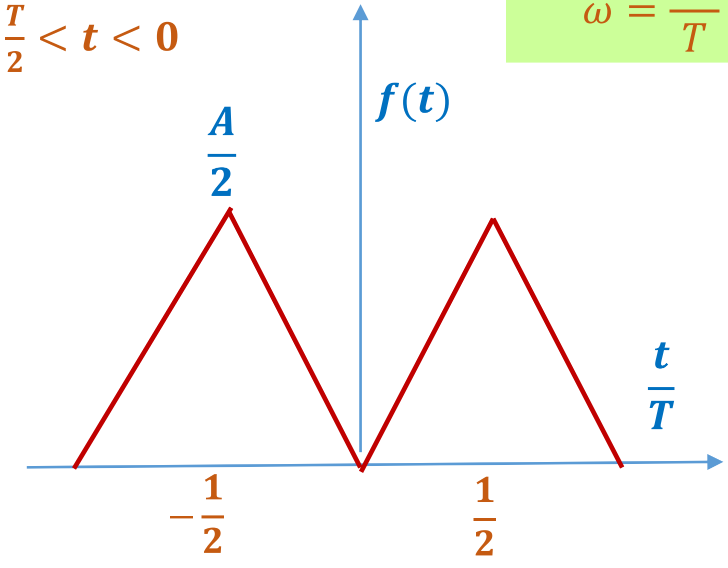
$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt, \quad a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega t)dt$$

$$\omega = \frac{2\pi}{T}$$

As, $f(t)$ *is an even function*

$$f(t) = At \;\; for\; 0 < t < \frac{T}{2}$$
$$= -At \;\; for\; -\frac{T}{2} < t < 0$$



$$a_0 = \frac{2}{T}\int_{-T/2}^{0}(-A)tdt + \frac{2}{T}\int_{0}^{T/2} Atdt = \frac{2A}{T}\left\{-\left[\frac{t^2}{2}\right]_{-\frac{T}{2}}^{0} + \left[\frac{t^2}{2}\right]_{0}^{T/2}\right\}$$

$$= \frac{A}{T}\left\{-\left[0-\left(\frac{T^2}{4}\right)\right] + \left[\left(\frac{T^2}{4}\right)-0\right]\right\} = \frac{AT}{2}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{0}(-A)t\cos(n\omega t)dt + \frac{2}{T}\int_{0}^{T/2} At\cos(n\omega t)dt$$

$$\int_{0}^{T/2} t\cos(n\omega t)dt = \left[\frac{t}{n\omega}\sin(n\omega t) - \frac{1}{n\omega}\left\{\frac{-1}{n\omega}\cos(n\omega t)\right\}\right]_0^{T/2} = \frac{-1}{n^2\omega^2}\,[1-\cos(n\pi)]$$

$$-\int_{-T/2}^{0} t\cos(n\omega t)dt = -\left[\frac{t}{n\omega}\sin(n\omega t) - \frac{1}{n\omega}\left\{\frac{-1}{n\omega}\cos(n\omega t)\right\}\right]_{-T/2}^{0} = \frac{-1}{n^2\omega^2}\,[1-\cos(n\pi)]$$

$$a_n = -\frac{2A}{T}\frac{2}{n^2\omega^2}\,[1-\cos(n\pi)] = \frac{-AT}{\pi^2 n^2}[1-(-1)^n]$$

$$\boxed{\begin{aligned} a_n &= 0 \quad for\ \ even\ \ values\ of\ n \\ &= \frac{-2AT}{\pi^2 n^2} \quad for\ odd\ values\ of\ n \end{aligned}}$$

Fourier Series: Applications

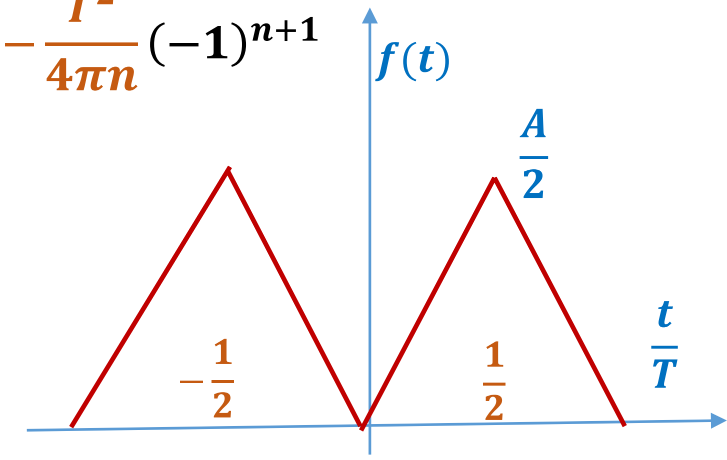
$$\omega = \frac{2\pi}{T}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{0}(-A)t\sin(n\omega t)dt + \frac{2}{T}\int_{0}^{T/2} At\sin(n\omega t)dt$$

$$a_n = 0 \quad \text{for even values of } n$$
$$= \frac{-2AT}{\pi^2 n^2} \quad \text{for odd values of } n$$

$$\int_0^{T/2} t\sin(n\omega t)dt = \left[\frac{-t}{n\omega}\cos(n\omega t) - \frac{1}{n\omega}\left\{\frac{-1}{n\omega}\sin(n\omega t)\right\}\right]_0^{T/2} = \frac{1}{n\omega}\left[-\frac{T}{2}\cos(n\pi) + 0\right] = \frac{T^2}{4\pi n}(-1)^{n+1}$$

$$-\int_{-T/2}^{0} t\sin(n\omega t)dt = \left[\frac{t}{n\omega}\cos(n\omega t) + \frac{1}{n\omega}\left\{\frac{-1}{n\omega}\sin(n\omega t)\right\}\right]_{-T/2}^{0} = \frac{1}{n\omega}\left[0 - \frac{-T}{2}\cos(n\pi)\right] = -\frac{T^2}{4\pi n}(-1)^{n+1}$$



$$b_n = 0$$

$$f(t) = \frac{AT}{2} + \sum_{n=0}^{\infty}\frac{-AT}{\pi^2 n^2}\left[1 - (-1)^n\right]\cos(n\omega t)$$

$$f(t) = At \quad \text{for } 0 < t < \frac{T}{2}$$
$$= -At \quad \text{for } -\frac{T}{2} < t < 0$$

$$\boxed{f(t) = \frac{AT}{2} - \frac{AT}{\pi^2}\left\{\cos(\omega t) + \frac{1}{3}\cos(2\omega t) + \frac{1}{5}\cos(3\omega t) - \cdots \ldots \ldots\right\}}$$

Fourier Integral:

https://www.youtube.com/watch?v=spUNpyF58BY
https://www.youtube.com/watch?v=r18Gi8lSkfM

According to the Fourier theorem, any periodic function can be expressed as sum of sine and cosine functions whose frequencies increase in the ratio of natural numbers.

$$\mathbf{f}(t) = \frac{1}{2} a_0 + \sum_{n=0}^{\infty} a_n \cos(n\omega t) + \sum_{n=0}^{\infty} b_n \sin(n\omega t)$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt \qquad a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega t)dt \qquad b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega t)dt$$

$$\mathbf{f}(t) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t')dt' + \sum_{n=0}^{\infty} \cos(n\omega t)\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t')\cos(n\omega t')dt' + \sum_{n=0}^{\infty} \sin(n\omega t)\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t')\sin(n\omega t')dt'$$

$$\omega = \frac{2\pi}{T} = \Delta s$$

$$\mathbf{f}(t) = \frac{1}{T}\int_{-\frac{\pi}{\Delta s}}^{\frac{\pi}{\Delta s}} f(t')dt' + \sum_{n=0}^{\infty}\frac{\Delta s}{\pi}\int_{-\frac{\pi}{\Delta s}}^{\frac{\pi}{\Delta s}} f(t')\{\cos(n\Delta s t)\cos(n\Delta s t') + \sin(n\Delta s t)\sin(n\Delta s t')\}dt'$$

$T \to \infty, \omega(= \Delta s) \to 0$

1st term on RHS is non periodic

$$\mathbf{f}(t) = \frac{1}{T}\int_{-\infty}^{\infty} f(t')dt' + \sum_{n=0}^{\infty}\frac{\Delta s}{\pi}\int_{-\infty}^{\infty} f(t')\cos(n\Delta s\{t-t'\})\,dt'$$

$$\lim_{\Delta s \to 0}\sum_{n=1}^{\infty} F(n\Delta s)\Delta s = \frac{2}{T}\int_0^{\infty} F(s)ds$$

$$\mathbf{f}(t) = \frac{1}{\pi}\int_0^{\infty}\{\int_{-\infty}^{\infty} f(t')\cos(s\{t-t'\})\,dt'\}ds$$

$$\mathbf{f}(t) = \frac{1}{2\pi}\int_{\infty}^{\infty}\{\int_{-\infty}^{\infty} f(t')\cos(s\{t-t'\})\,dt'\}ds$$ **=> as cos is even of s**

$$\frac{i}{2\pi}\int_{\infty}^{\infty}\{\int_{-\infty}^{\infty} f(t')\sin(s\{t-t'\})\,dt'\}ds = 0 \Rightarrow$$ **as sine is odd of s**

$$\mathbf{f}(t) = \frac{1}{2\pi}\int_{\infty}^{\infty}\{\int_{-\infty}^{\infty} f(t')e^{\pm i\omega\{t-t'\}}\,dt'\}d\omega$$

Fourier theorem

$$\mathbf{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{\mp i\omega t}\,d\omega$$

Fourier transform (IFT)

$$\mathbf{F}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t')e^{\pm i\omega t}\,dt$$

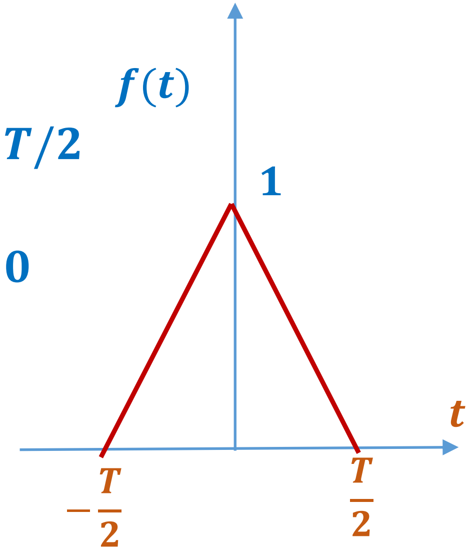
Inverse Fourier transform (FT)

Fourier Series: Applications

$$f(t) = 1 + 2t/T \quad for -\frac{T}{2} < t < 0$$
$$= 1 - 2t/T \quad for \ \ 0 < t < \frac{T}{2}$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{0} (1 + 2t/T) e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{T/2} (1 - 2t/T) e^{-i\omega t}\, dt$$



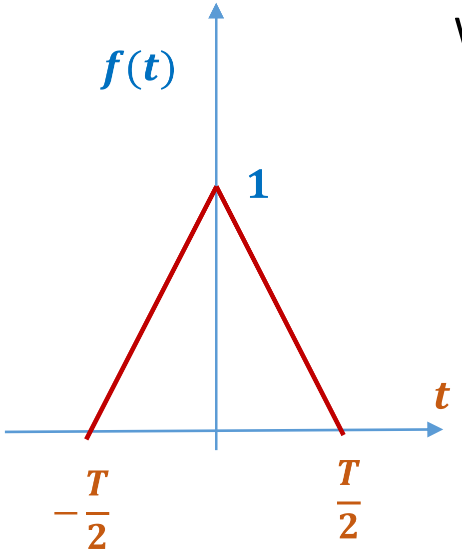
$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \left[\left(1 + \frac{2t}{T}\right) \frac{e^{-i\omega t}}{-i\omega} - \frac{2}{T} \frac{e^{-i\omega t}}{(-i\omega)^2} \right]_{-T/2}^{0} + \frac{1}{\sqrt{2\pi}} \left[\left(1 - \frac{2t}{T}\right) \frac{e^{-i\omega t}}{-i\omega} + \frac{2}{T} \frac{e^{-i\omega t}}{(-i\omega)^2} \right]_{0}^{T/2}$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{i\omega} - \frac{2}{T} \frac{1}{(i\omega)^2} + \frac{2}{T} \frac{e^{\frac{i\omega T}{2}}}{(i\omega)^2} + \frac{2}{T} \frac{e^{-\frac{i\omega T}{2}}}{(i\omega)^2} + \frac{1}{i\omega} - \frac{2}{T} \frac{1}{(i\omega)^2} \right]$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \left[\frac{2}{T} \frac{e^{\frac{i\omega T}{2}}}{(i\omega)^2} + \frac{2}{T} \frac{e^{-\frac{i\omega T}{2}}}{(i\omega)^2} - \frac{4}{T} \frac{1}{(i\omega)^2} \right]$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \left[\frac{4}{T(i\omega)^2} \left(\frac{e^{\frac{i\omega T}{2}}}{2} + \frac{e^{-\frac{i\omega T}{2}}}{2} \right) - \frac{4}{T} \frac{1}{(i\omega)^2} \right]$$

Fourier Series: Applications



$$f(t) = 1 + 2t/T \quad for -\frac{T}{2} < t < 0$$
$$= 1 - 2t/T \quad for \;\; 0 < t < \frac{T}{2}$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{4}{T(i\omega)^2}\left[cos(\frac{\omega T}{2}) - 1\right]$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{4}{T(\omega)^2}\left[1 - cos(\frac{\omega T}{2})\right]$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{4}{T(\omega)^2}\left[2sin^2(\frac{\omega T}{4})\right]$$

$$\mathrm{F}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{T}{2}\left[\frac{sin^2\left(\frac{\omega T}{4}\right)}{\left(\frac{\omega T}{4}\right)^2}\right] = \frac{T}{2\sqrt{2\pi}} sinc^2\left(\frac{\omega T}{4}\right)$$

$\mathrm{F}(\omega) = 0$, **when** $sin\left(\frac{\omega T}{4}\right) = 0$

$$\Rightarrow \omega T/4 = n\pi, n = \pm 1, \pm 2 \ldots$$

$$\Delta\omega = \frac{4\pi}{T} - \left(-\frac{4\pi}{T}\right) = \frac{8\pi}{T}$$

$$\Delta t = T$$