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Indian Institute of Information Technology, Design and Manufacturing, Kancheepuram

End Semester Examination – July 2023

Course Code: MA1002
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Course Title: Linear Algebra
Category: Core
Instructors: M Subramani / S Vijayakumar
Maximum Marks: 60

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1. Prove that equivalent systems of linear equations have exactly the same solutions. (4)

2. Let $A_1, \ldots, A_k$ be $n \times n$ matrices. Prove that if $A = A_1 \ldots A_k$ is an invertible matrix, then each of $A_1, \ldots, A_k$ is invertible. (3)

3. Let A be an $n \times n$ matrix. Prove that the following are equivalent: (4)
   (i) A is invertible.
   (ii) The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.
   (iii) The system of equations $AX = Y$ has a solution X for each $n \times 1$ vector Y.

4. Let F be a field and let S be any nonempty set. Let V be the set of all functions from S to F. Show that V is a vector space under the operations of addition and scalar multiplication of functions. (3)

5. Show that the vectors $\alpha_1 = (1, 1, 0, 0)$, $\alpha_2 = (0, 0, 1, 1)$, $\alpha_3 = (1, 0, 0, 4)$ and $\alpha_4 = (0, 0, 0, 2)$ form a basis for $\mathbb{R}^4$. Express each of the standard basis vectors of $\mathbb{R}^4$ as a linear combination of $\alpha_1, \ldots, \alpha_4$. Hence express any vector (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ as a linear combination of $\alpha_1, \ldots, \alpha_4$. (5)

6. Prove that every linearly independent subset of a finite dimensional vector space V is a part of a basis for V. (2)

7. Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$? (2)

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of the xy-plane about the line through the points $(0, 0)$ and $(3, 4)$. Find a formula for $T(x, y)$. Show also that it is a linear transformation. (4)

9. State and prove the rank-nullity theorem (dimension theorem). (5)

10. Let V be a vector space with $\dim V = n$. Let W_1 and W_2 be any subspaces of V such that $\dim W_1 + \dim W_2 = n$. Prove that there is a linear transformation $T : V \to V$ such that the null space $N(T) = W_1$ and the range $R(T) = W_2$. (3)

11. Let A be any matrix over a field F. Prove that the row rank of A equals its column rank. (3)

12. Let V be the space of all polynomials of degree at most three with real coefficients. Let $D : V \to V$ be the differentiation operator. Find the matrix representation of D in the
Standard Basis $B = \{1, x, x^2, x^3\}$ (2)

13. Find the eigenvalues and the corresponding eigenspaces of the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. (3)

14. Diagonalize the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. (6)

15. Show that the dot product of vectors in $\mathbb{R}^3$ is an inner product. (2)

16. State and prove the Cauchy-Schwarz inequality. (3)

17. Prove that any set of nonzero orthogonal vectors is linearly independent. (2)

18. Apply the Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$ and $\beta_3 = (0, 3, 4)$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product. (3)