Power Series Solutions to the Bessel Equation

The Bessel equation:

$$x^2 y'' + x y' + (x^2 - p^2) y = 0 \quad - (1)$$

where p is a ~~nonnegative~~ real (+ve) constant

Power series solution with $x_0 = 0$

Here $x_0 = 0$ is a regular singular point of the given Bessel differential equation. (check it)

We shall use the method of Frobenius to solve this equation.

Thus, we seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad x > 0, \quad - (2)$$

with $a_0 \neq 0$.

Differentiation of (2) term by term yields

$$y'(x) = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} \quad - (3)$$

Similarly, we obtain

$$y''(x) = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2} \quad - (4)$$

Substituting (4) and (3) in (1), we get

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+m} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+2} - \sum_{n=0}^{\infty} p^2 a_n x^{n+m} = 0$$

$$\Rightarrow \quad x^m \left(\sum_{n=0}^{\infty} \left([(n+m)^2 - p^2] a_n x^n + a_n x^{n+2} \right) \right) = 0 \qquad -(5)$$

Now we try to determine a_n's using the fact that coefficient of each power of x will vanish

For the constant term, we require $(m^2-p^2)a_0 = 0$

Since $a_0 \neq 0$, it follows that

$m^2 - p^2 = 0$ which is the indicial equation.

Thus $m = \pm p$.

Case 1: $m = +p$

For $m = p$, the equations for determining the coefficients are: (from (5))

$$[(1+p)^2 - p^2]\, a_1 = 0 \quad \text{and,} \qquad -(6)$$

$$[(n+p)^2 - p^2]\, a_n + a_{n-2} = 0, \quad n \geq 2. \qquad -(7)$$

Since $p \geq 0$, we have $a_1 = 0$. The equation (7) (from (6))

yields

$$a_n = -\frac{a_{n-2}}{(n+p)^2 - p^2} = -\frac{a_{n-2}}{n(n+2p)} \qquad (7.2)$$

Since $a_1 = 0$, we immediately obtain

$a_3 = a_5 = a_7 = \cdots = 0.$

For the coefficients with even subscripts, we have

$$a_2 = \frac{-a_0}{2(2+2p)} = \frac{-a_0}{2^2(1+p)}$$

$$a_4 = \frac{-a_2}{4(4+2p)} = \frac{(-1)^2 \cdot a_0}{2^4 \cdot 2!\,(1+p)(2+p)}$$

$$a_6 = \frac{-a_4}{6(6+2p)} = \frac{(-1)^3 a_0}{2^6 \cdot 3!\,(1+p)(2+p)(3+p)}$$

and, in general

$$a_{2n} = \frac{(-1)^n \cdot a_0}{2^{2n} \cdot n!\,(1+p)(2+p)\cdots(n+p)}.$$

Therefore, the choice $m=p$ yields the solution

$$y_p(x) = a_0 \cdot x^p \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot x^{2n}}{2^{2n} \cdot n!\,(1+p)\cdot(2+p)\cdots(n+p)}\right) \quad -(8)$$

Note: The ratio test shows that the power series formula converges for all $x \in \mathbb{R}$ (check it).

For $x<0$ also, we obtain same solution.

Solution $y_p(x)$ (8) is valid for all real $x \neq 0$.

$y_p(x) =$

Case II:

For $m=-p$, determine the coefficients from

$$[(1-p)^2-p^2]a_1=0 \text{ and } [(n-p)^2-p^2]a_n+a_{n-2}=0.$$

These equations become

$$(1-2p)a_1=0 \text{ and } n(n-2p)a_n+a_{n-2}=0$$

If $p\neq\frac{1}{2}$, these equations become

$$a_1=0 \text{ and } a_n=\frac{-a_{n-2}}{n(n-2p)}, n\geq 2 \quad -(9)$$

Note that (9) is same as (7.2), with p replaced by $-p$. Thus the solution is given by

$$y_{-p}(x)=a_0\cdot x^{-p}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot x^{2n}}{2^{2n}\cdot n!\,(1-p)(2-p)\cdots(n-p)}\right) \quad -(10)$$

valid for all real $x\neq 0$.

$\therefore$ The general solution is

$y(x)=c_1\cdot y_p(x)+c_2\cdot y_{-p}(x)$ where

$$y_p(x)=a_0x^p\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot x^{2n}}{2^{2n}\cdot n!\,(1+p)(2+p)\cdots(n+p)}\right)$$

$$y_{-p}(x)=a_0\cdot x^{-p}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot x^{2n}}{2^{2n}\cdot n!\,(1-p)(2-p)\cdots(n-p)}\right).$$

Euler's gamma function and its properties

For $s \in \mathbb{R}$ with $s > 0$, we define $\Gamma(s)$ by

$$\Gamma(s) = \int_{0+}^{\infty} t^{s-1} e^{-t} \, dt. \quad \text{— ⓐ}$$

The integral converges if $s > 0$ and diverges if $s \leq 0$.

Integration by parts yields the functional equation

$$\boxed{\Gamma(s+1) = s \cdot \Gamma(s)} \to \text{imp. property of } \Gamma(s)$$

In general,

$$\Gamma(s+n) = (s+n-1) \cdot \ldots \cdot (s+1) \cdot s \cdot \Gamma(s), \text{ for } n \in \mathbb{Z}^{+}).$$

Since $\Gamma(1) = 1$ ($\because$ ⓐ), we find that $\Gamma(n+1) = n!$

Thus, the gamma function is an extension of the factorial function from integers to positive real numbers. Therefore, we write

$$\boxed{\Gamma(s) = \frac{\Gamma(s+1)}{s}, \quad s \in \mathbb{R}.}$$

Using this gamma function, we shall simplify the form of the solutions of the Bessel equation.

With $s = 1+p$, we note that

$$(1+p)(2+p)\ldots(n+p) = \frac{\Gamma(n+1+p)}{\Gamma(1+p)}$$

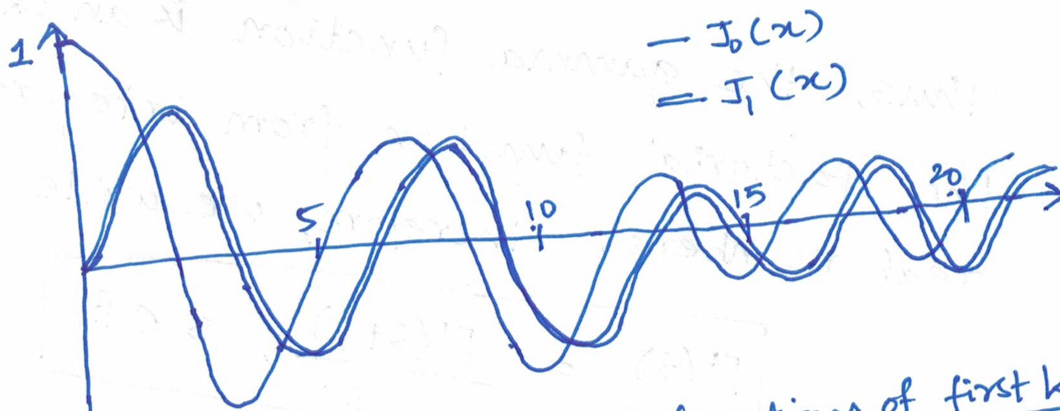
choose $a_0 = \dfrac{2^{-p}}{\Gamma(1+p)}$ in 8, the solution

$$y_p(x) = J_p(x) = \left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n}.$$

The function $J_p(x)$ (defined usually for $x>0$) and $p \geq 0$ is called the Bessel function of the first kind of order p.

when p is an integer, $p = m$ (integer), the Bessel function $J_m(x)$ is given by

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m)!} \left(\frac{x}{2}\right)^{2n+m}, \quad (m = 0, 1, 2, \cdots).$$



Plot of Bessel functions of first kind.

$$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

If $p \notin \mathbb{Z}^+$, defining a new function $J_{-p}(x)$ $\quad p \to -p$

$$\boxed{J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1-p)} \left(\frac{x}{2}\right)^{2n}.}$$

with $s = 1-p$, we note that

$$\Gamma(n+1-p) = (1-p)(2-p)\cdots(n-p)\,\Gamma(1-p).$$

Thus, the series for $J_p(x)$ is the same as for $y_p(x)$

with $a_0 = \frac{2^p}{\Gamma(1+p)}$

& $J_p(x) = y_{-p}(x)$ with $a_0 = \frac{2^p}{\Gamma(1-p)}$

If $p \notin \mathbb{Z}^+$, $J_p(x)$ and $J_{-p}(x)$ are linearly independent on $x>0$. The general solution of the Bessel equation is

$$y(x) = c_1 \cdot J_p(x) + c_2 J_{-p}(x).$$

Some important relations involving Bessel functions:

- $\frac{d}{dx}\left(x^{-p} J_p(x)\right) = -x^{-p} J_{p+1}(x)$
- $\frac{p}{x} J_p(x) + J_p'(x) = J_{p-1}(x)$
- $\frac{p}{x} J_p(x) - J_p'(x) = J_{p+1}(x)$
- $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$
- $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$

Work out these relations using general expression for $J_p(x)$. An example is provided in the next page.

Useful recurrence relations for $J_p(x)$

- $$\frac{d}{dx}\left(x^p J_p(x)\right) = x^p J_{p-1}(x)$$

$$\frac{d}{dx}\left(x^p. J_p(x)\right) = \frac{d}{dx}\left\{ x^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+p+n)} \left(\frac{x}{2}\right)^{2n+p} \right\}$$

$$= \frac{d}{dx}\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n.\, x^{2n+2p}}{n!\,\Gamma(1+p+n)\, 2^{2n+p}} \right\}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n.\,(2n+2p).\, x^{2n+2p-1}}{n!\,\Gamma(1+p+n)\, 2^{2n+p}}$$

Since $\Gamma(1+p+n) = (p+n)\,\Gamma(p+n)$, we have

$$\frac{d}{dx}\left(x^p J_p(x)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n.\, 2 \cdot x^{2n+2p-1}}{n!\,\Gamma(p+n)\, 2^{2n+p}}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+(p-1)+n)} \left(\frac{x}{2}\right)^{2n+p-1}$$

$$= x^p. J_{p-1}(x).$$

(Hence Proved).