Recurrence Relations

1. Substitution method
2. Recurrence Tree Method
3. Master Theorem Approach

Linear search

$$T(n) = 1 + T(n-1), \quad T(1) = 1$$

$$T(n) = 1 + T(n-1)$$

$$= \underbrace{1+1}_{2} + T(n-2)$$

$$= \underbrace{1+1+1}_{3} + T(n-3)$$

$$\vdots$$

$$= \underbrace{1+1+\cdots+1}_{n-1} + T(n-(n-1))$$

$$= n-1+T(1)$$

$$= n-1+1 = n$$

$$T(n) = n, \quad n \geq 1$$

\# Comp to search $x \in A$ = 'n'

Best Case:- $\Theta(1) = 1, 2, 3, \ldots k$

Worst Case:- $\Theta(n)$

Recurrence Relations

Binary Search

$$T(n) = 1 + T\left(\frac{n}{2}\right), \quad T(1) = 1$$

$$= 1 + 1 + T\left(\frac{n}{4}\right)$$

$$= 1 + 1 + 1 + T\left(\frac{n}{8}\right)$$

$$= 1 + 1 + 1 + 1 + T\left(\frac{n}{2^4}\right)$$

$$= 1 + 1 + 1 + 1 + 1 + T\left(\frac{n}{2^5}\right)$$

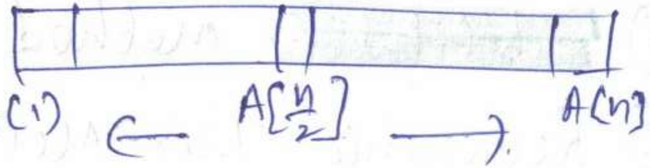
After k iterations

$$= 1 + 1 + \cdots + 1 + T\left(\frac{n}{2^k}\right)$$

Assume $n = 2^k$ for some k

$$= \underbrace{1 + 1 + \cdots + 1}_{k \text{ 1's}} + T\left(\frac{2^k}{2^k}\right)$$

$$= k + T(1) = k + 1 = \log_2 n + 1$$



$$\log_2 n \le \log_2 n + 1 \le 2 \cdot \log_2 n$$

$$T(n) = \Theta(\log_2 n)$$

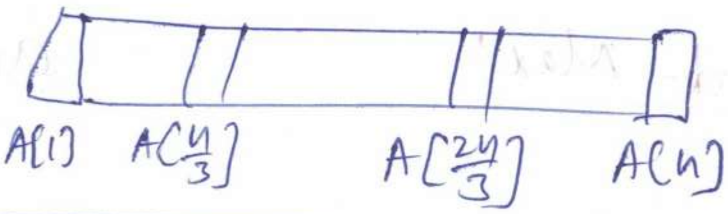
Best Case:- $\frac{n}{2}, \frac{n}{4}, \frac{3n}{4}, \frac{n}{8} \ldots$

$$= \Theta(1)$$

Worst Case:- $= \Theta(\log_2 n)$

Recurrence Relations

Ternary search



$T(n) = T\left(\frac{n}{3}\right) + 2, \quad T(1) = 1$

Substitution method

$$T(n) = T\left(\frac{n}{3}\right) + \overbrace{2}^{1}$$

$$= T\left(\frac{n}{3^2}\right) + \overbrace{2+2}^{2}$$

$$= T\left(\frac{n}{3^3}\right) + \overbrace{2+2+2}^{3}$$

After k-steps

$$= T\left(\frac{n}{3^k}\right) + \overbrace{2+2+ \cdots 2}^{k}$$

Assume $n = 3^k \Rightarrow T\left(\frac{3^k}{3^k}\right) + 2k$

$$T(n) = T(1) + 2k$$

$$= 1 + 2\log_3 n$$

Binary search	Ternary Search
$1 + \log_2 n$	$1 + 2\log_3 n$
$n = 81$	
$\rightarrow 1 + \log_2 81 \le 1 + \log_2 128$ ≤ 8 Comp	$1 + 2\log_3 81 = 9$ Comp
$n = 1024$	
$1 + \log_2 2^{10} = 11$ Comp	$1 + 2\log_3 1024 \ge 1 + 2\log_3 3^6$ ≥ 13 Comp

As per Step Count Anlaysis

$$1 + \log_2 n \le 1 + 2\log_3 n$$

As per Asymptotic Sense.

$$\log_2 n = \log_3 n \times \underbrace{\log_2 3}_{c} = \log_3 n \times C$$

$$\log_2 n \le C \cdot \log_3 n \Rightarrow \boxed{\log_2 n = \Theta(\log_3 n)}$$

Recurrence Relations

Find-Max!- $a_1, a_2, a_3, a_4 --- a_n$

Fundamental op
① Comparison
② Swap

$T(n) = 1 + T(n-1), T(1) = 0$

Substitution method

$$T(n) = 1 + T(n-1)$$
$$= 1 + 1 + T(n-2)$$
$$= \underbrace{1+1+1+\cdots+1}_{(n-1)\ 1's} + T(1)$$

$T(n) = n-1$

pruning, Incremental Design

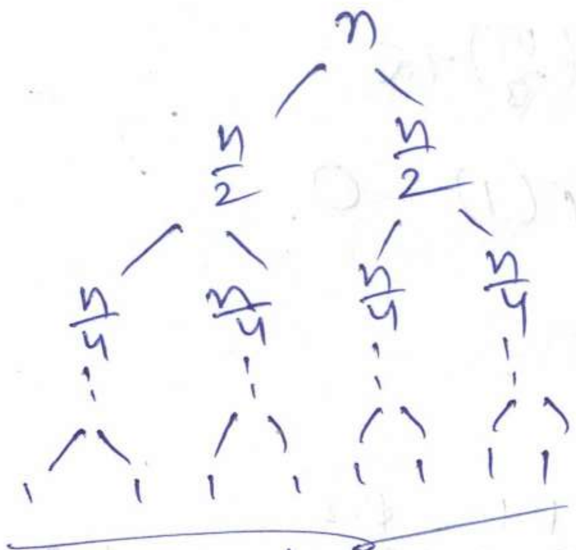
primitive op: # Comparisons.

Comparisons : Best Case, Worst Case } n-1 Comps, $\Theta(n)$

Swaps: Best Case : '0' Swaps
Worst Case : $(n-1)$ Swaps

Recurrence Relations

Find-Max:- Divide and Conquer Paradigm



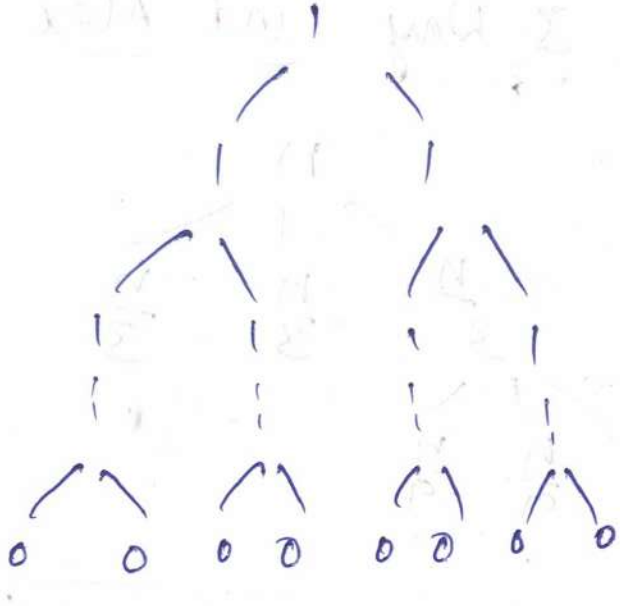
Divide

$T(n) =$ # Comparisons to find max

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + 1$$

Comp to find max in $A_1 \ldots A_{\frac{n}{2}}$ | Comp to find max in $A_{\frac{n}{2}} \ldots A_n$

$$= 2T\left(\frac{n}{2}\right) + 1, \quad T(1) = 0$$



Substitution method:-

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$

$$= 2\left[2T\left(\frac{n}{4}\right) + 1\right] + 1$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + 2 + 1$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 2^2 + 2 + 1$$

$$= 2^k T\left(\frac{n}{2^k}\right) + 2^{k-1} + \cdots + 2 + 1$$

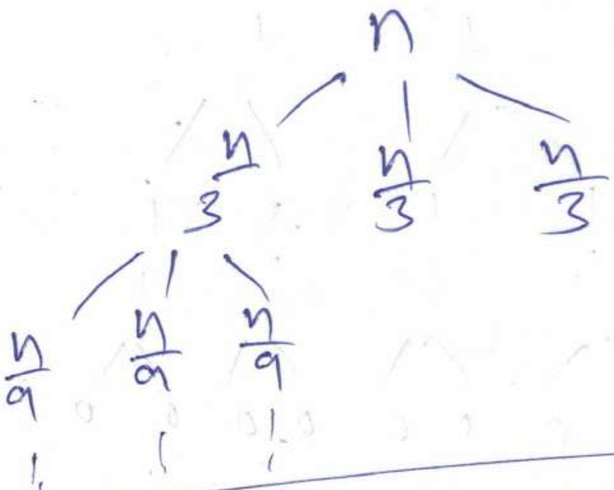
Assume $n = 2^k$

$$= 2^k T(1) + 2^{k-1} + \cdots + 2 + 1 = 2^k - 1 = n - 1$$

$$= 2^k - 1 = n - 1 = T(n)$$

Recurrence Relations

3-Way Find-Max



$T(n)$ = # Comps to find Max

$= T(\frac{n}{3}) + T(\frac{n}{3}) + T(\frac{n}{3}) + 2$

$= 3\,T(\frac{n}{3}) + 2, \quad T(1) = 0$

Substitution method

$T(n) = 3\,T(\frac{n}{3}) + 2$

$= 3\cdot[3\,T(\frac{n}{9}) + 2] + 2$

$= 3^2 \cdot T(\frac{n}{3^2}) + 3\cdot 2 + 2$

$= 3^3\,T(\frac{n}{3^3}) + 3^2\cdot 2 + 3\cdot 2 + 2$

After k-steps

$\Rightarrow 3^k \cdot T(\frac{n}{3^k}) + 3^{k-1}\cdot 2 + 3^{k-2}\cdot 2 + \cdots + 2$

Assume $n = 3^k$

$= 3^k \cdot \underbrace{T(1)}_{0} + 2(3^{k-1} + 3^{k-2} + \cdots + 3^1 + 3^0)$

$= 2(3^k - 1)$

$= n - 1$ Comparisons

$= 3^k - 1 = n-1$ Comparisons