Subspaces

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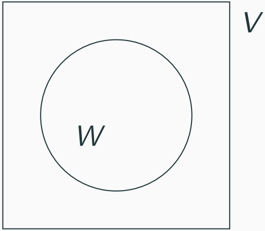

Remark

If $\langle V, F, +, . \rangle$ is a vector space, then

(i) $\forall \alpha, \beta \in V$, $\alpha + \beta \in V$ (V is closed under vector addition)

(ii) $\forall c \in F$ and $\alpha \in V$, $c\alpha \in V$ (V is closed under scalar multiplication)

(iii) If $\alpha_1, \ldots, \alpha_n \in V$, then $c_1\alpha_1 + \ldots + c_n\alpha_n \in V$ where $c_i \in F$.

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Hence

$$\forall\ \alpha, \beta \in W, c \in F \Longrightarrow c\alpha + \beta \in W.$$

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Examples of subspaces

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Reason : $0 = (0, 0, \ldots, 0) \notin W$

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Hence, S **is a subspace of** $F^{n\times 1}$.

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By Theorem 1, $W_1 \cap W_2$ is a subspace of V.

Corollary : Intersection of any collection of subspaces of a vector space V is a subspace of V

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Note (1) : Subspace spanned by S is the smallest subspace which contains S.

Note (2) : If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we call the subspace spanned by S as the subspace spanned by the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

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Lemma

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$\implies$ $L(S)$ is a subspace of V by Theorem 1.

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By (i) and (ii),

$$W^* = \cap\{W \ : \ S \subseteq W, \ \ W \text{ is a subspace of } V\} = L(S) - - - (a)$$

Row space and Column space of a matrix

Let $A \in F^{m \times n}$ with rows $\{R_1, R_2, \ldots, R_m\}$ and columns $\{C_1, C_2, \ldots, C_n\}$.

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Note : Row space of $A \subseteq F^{1\times n}$ **and Column space of** $A \subseteq F^{m\times 1}$**.**

Example

$$\text{Let} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where

$$R_1 = (1,0,0), R_2 = (0,1,0), C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Column Space of A

$$= \left\{ x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad : \quad x, y, z \in F \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

Assignment

Prove or disprove that

(i) column space of AB is same as column space of A and

(ii) row space of AB is same as row space of B.

Note 1: (Visit previous lecture notes)

Find the solution space of the system $RX = 0$

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\}$$

No. of non-zero rows of R, $r = 2$, **No. of variables,** $n = 5$

$k_1 = 2, k_2 = 4 \Longrightarrow$ Pivot variables $= \{x_{k_1}, x_{k_2}\} = \{x_2, x_4\}$

No. of free variables $= n - r = 5 - 2 = 3$,

Free variables $= \{u_1, u_2, u_3\} = \{x_1, x_3, x_5\}$

$$\left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0 \\ x_{k_2} + \sum_{j=1}^{n-r} C_{2j} u_j = 0 \end{array}\right\} \textit{(general expression)}$$

Note 1 contd.

$$\left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\} \implies \left.\begin{array}{c} x_{k_1} + \displaystyle\sum_{j=1}^{n-r} C_{1j}u_j = 0 \\ x_{k_2} + \displaystyle\sum_{j=1}^{n-r} C_{2j}u_j = 0 \end{array}\right\} \textit{general expression})$$

Set the free variables as :

$u_1 = x_1 = a,\ u_2 = x_3 = b,\quad u_3 = x_5 = c$

$\implies \quad x_2 = 3b - \frac{1}{2}c,\quad x_4 = -2c$

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c)\ :\ a, b, c \in \mathrm{R}\right\}$

Note 1 contd. (back to chapter one !)

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c) \ : \ a, b, c \in \mathrm{R}\right\}$

$$S = \left\{ a(1,0,0,0,0) + b(0,3,1,0,0) + c(0,-\frac{1}{2},0,-2,1) \ : \ a,b,c \in \mathrm{R} \right\}$$

$$= \text{Span of } \left\{ (1,0,0,0,0), (0,3,1,0,0), (0,-\frac{1}{2},0,-2,1) \right\}$$

Dimension of $S = dim\ S = 3 = n - r$ **(Information for future)**

Problem

Let W be set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfies

$$\begin{aligned}
2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \qquad &= 0 \\
x_1 \qquad + \tfrac{2}{3}x_3 \qquad - x_5 &= 0 \\
9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0
\end{aligned}$$

Find a finite set of vectors which spans W.

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Since $3\alpha + (-1)\beta + 0\gamma = 0$, $\{\alpha, \beta, \gamma\}$ is a linearly dependent set.