Bessel Function

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$$

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

$$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

Orthogonal properties of Bessel functions.

(i) $\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$ if $\alpha \neq \beta$,

(ii) $\int_0^1 x \{J_n(\alpha x)\}^2 dx = \frac{1}{2}\{J_n'(\alpha)\}^2$, where α and β are zeros of $J_n(x)$

$= \frac{1}{2}\{J_{n+1}(\alpha)\}^2$ i.e. $J_n(\alpha) = 0$ $J_n(\beta) = 0$

Fourier Bessel Expansion

$$f(x) = \sum_{i=1}^{\infty} a_i J_n(\alpha_i x), \quad 0 \leq x \leq 1$$

$$a_j = \frac{2}{\{J_{n+1}(\alpha_j)\}^2} \int_0^1 x f(x) J_n(\alpha_j x)\, dx$$

Prove that $J_n(-x) = (-1)^n J_n(x)$ for +ve as well as negative integer.

Prove that $J_0' = -J_1$, $4J_n'' = J_{n-2} - 2J_n + J_{n+2}$

Prove that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$, $J_{+1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

Prove that $\frac{d}{dx}\{x J_n J_{n+1}\} = x(J_n^2 - J_{n+1}^2)$

Legendre Polynomials

$y = \sum_{r=0} a_r x^{r}$

Legendre's diff eq^n

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ for large value of x, when n is +ve integer.

$\Downarrow$

$y = AP_n(x) + BQ_n(x)$

$$P_n(x) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$

$$Q_n(x) = \frac{n!}{1\cdot 3\cdot 5\cdots(2n+1)}\left\{x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}x^{-n-5} + \cdots\right\}$$

Few important Properties

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n \quad \text{(Rodrigues' Formula)}$$

Orthogonal Properties of Legendre Polynomials

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \text{ if } m \neq n$$

$$\int_{-1}^{1}\{P_n(x)\}^2 = \frac{2}{2n+1}$$

How, we find sol^n about $x = \pm 1$,

Ex $\int_{-1}^{1}\left(\sum_{j=1}^{n}\sqrt{j(2j+1)}\,P_j(x)\right)^2 dx = 20$

Find $n = ??$ with $P_n(1) = 1$

Fourier - Legendre expansion

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \qquad -1 < x < 1, \qquad a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx.$$

Recurrence relations.

$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

$nP_n = xP_n' - P_{n-1}'$

$(2n+1)P_n = P_{n+1}' - P_{n-1}'$

$(1-x^2)P_n' = n(P_{n-1} - xP_n)$

$P_n(-x) = (-1)^n P_n(x)$

$P_n(-1) = (-1)^n, \quad P_n(1) = 1$

$P_n'(1) = \frac{1}{2}n(n+1)$

Ex Let y be a polynomial sol^n of diff eq^n $(1-x^2)y'' - 2xy' + 6y = 0$

if $y(1) = 1$, then value of integral $\int_{-1}^{1} y^2 dx = ??$

if $y(1) = 2$ then ??

(Product→10)

Examples:

$x(1+x^2)\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 7y = 0$ Find the roots of indicial eq^n in a nbd of $x=0$

Find initial eq^n for $x^2 y'' + 3\sin x\, y' + y = 0$, about $x=0$

$x(x-1)\, y'' + \sin x\, y' + 2x(x-1)y = 0$

Ex Let $P_n(x)$ be the legendre polynomial of degree n, and let

$$P_{m+1}(0) = -\frac{m}{m+1} P_{m-1}(0) \quad , m\geq 1, 2, \dots$$

If $P_n(0) = -\frac{5}{16}$, then $\int_{-1}^{1} P_n^2(x)\,dx$

Ex $x^3 y'' + (\cos 2x - 1)y' + 2xy = 0$ find the roots of indicial eq^n.

Ex Consider the diff eq^n

$$y'' + \frac{1}{x^2} y' - \frac{1}{x^3} y = 0$$

(a) check the point $x=0$,

(b) Find the sol^n. of diff eq^n