Engineering Electromagnetics

Lecture 7

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by

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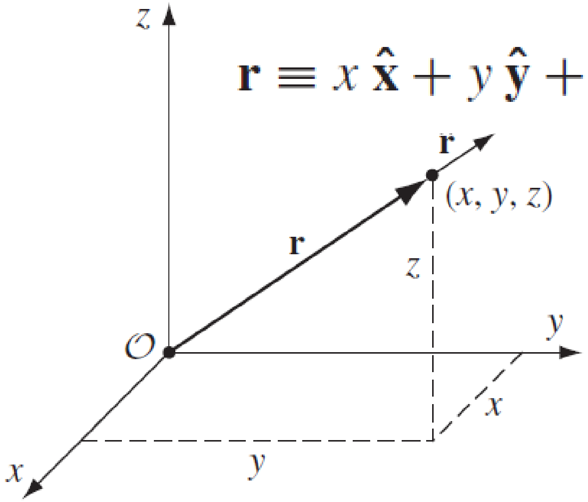
Diff. line, surface and volume elements

Cartesian

$$\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}.$$



$$r = \sqrt{x^2 + y^2 + z^2},$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

The **infinitesimal displacement vector**, from *(x, y, z)* to *(x + dx, y + dy, z + dz)*

$$\boxed{d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}}$$

Differential volume element

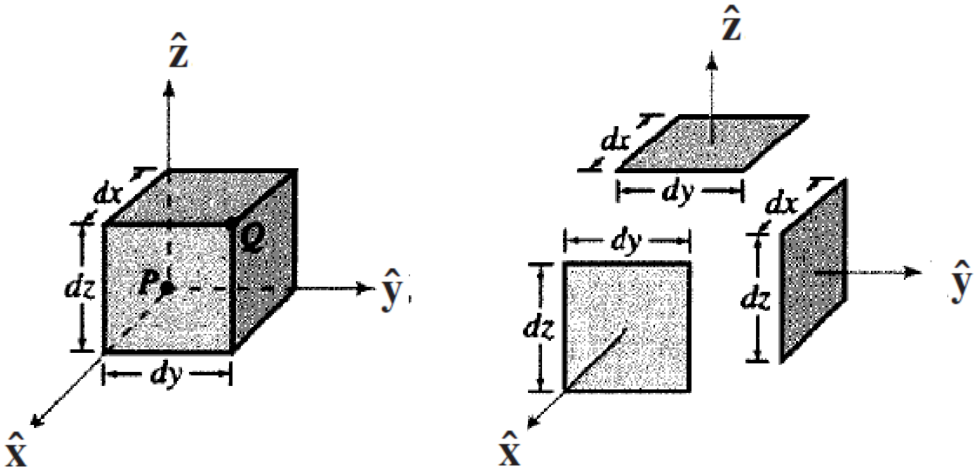
$$dv = dx\,dy\,dz$$

This volume is surrounded by six differential surfaces each expressed in the direction of unit vectors

$$\overrightarrow{d\mathbf{s}}_x = dy\,dz\,\hat{\mathbf{x}}$$

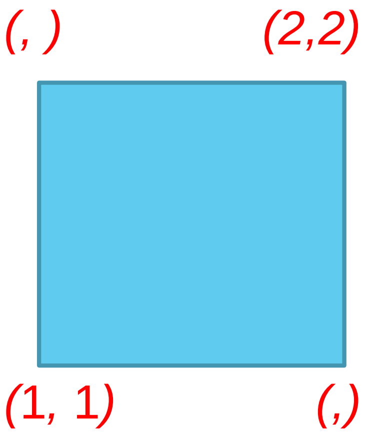
$$\overrightarrow{d\mathbf{s}}_y = dx\,dz\,\hat{\mathbf{y}}$$

$$\overrightarrow{d\mathbf{s}}_z = dx\,dy\,\hat{\mathbf{z}}$$

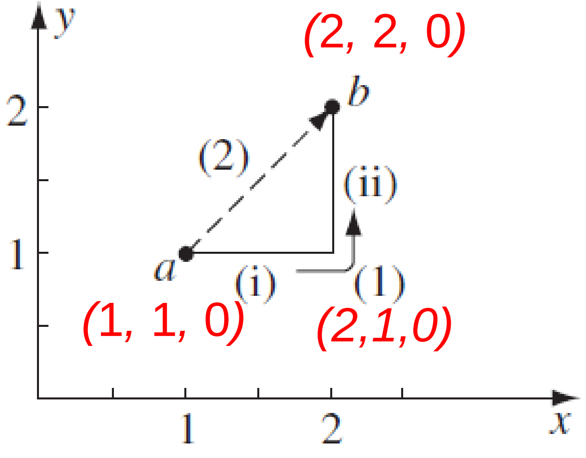


a) Differential volume   b) Exploded view

Example 1.6. Calculate the line integral of the function $\mathbf{v} = y^2\,\hat{\mathbf{x}} + 2x(y+1)\,\hat{\mathbf{y}}$ from the point $\mathbf{a} = (1, 1, 0)$ to the point $\mathbf{b} = (2, 2, 0)$, along the paths (1) and (2) in Fig. 1.21. What is $\oint \mathbf{v} \cdot d\mathbf{l}$ for the loop that goes from $\mathbf{a}$ to $\mathbf{b}$ along (1) and returns to $\mathbf{a}$ along (2)?



Circulation of **v** along the square



For the loop that goes *out* (1) and *back* (2), then,

$$\oint \mathbf{v} \cdot d\mathbf{l} = 11 - 10 = 1.$$

Solution

As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$. Path (1) consists of two parts. Along the "horizontal" segment, $dy = dz = 0$, so

(i) $d\mathbf{l} = dx\,\hat{\mathbf{x}},\ y = 1,\ \mathbf{v} \cdot d\mathbf{l} = y^2\,dx = dx$, so $\int \mathbf{v} \cdot d\mathbf{l} = \int_1^2 dx = 1$.

On the "vertical" stretch, $dx = dz = 0$, so

(ii) $d\mathbf{l} = dy\,\hat{\mathbf{y}},\ x = 2,\ \mathbf{v} \cdot d\mathbf{l} = 2x(y+1)\,dy = 4(y+1)\,dy$, so

$$\int \mathbf{v} \cdot d\mathbf{l} = 4 \int_1^2 (y+1)\,dy = 10.$$

By path (1), then,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l} = \underline{1 + 10 = 11.}$$

Meanwhile, on path (2) $x = y,\ dx = dy$, and $dz = 0$, so $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dx\,\hat{\mathbf{y}},\ \mathbf{v} \cdot d\mathbf{l} = x^2\,dx + 2x(x+1)\,dx = (3x^2 + 2x)\,dx$, and

$$\underline{\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l} = \int_1^2 (3x^2 + 2x)\,dx = (x^3 + x^2)\Big|_1^2 = 10.}$$

(The strategy here is to get everything in terms of one variable; I could just as well have eliminated x in favor of y.)

Surface Integrals. A surface integral is an expression of the form

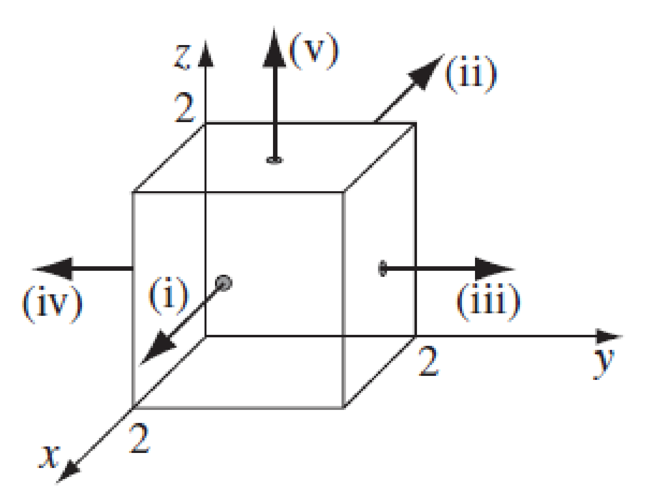
$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a},$$

where $\mathbf{v}$ is again some vector function, and the integral is over a specified surface $\mathcal{S}$. Here $d\mathbf{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface

then tradition dictates that "outward" is positive,

$$\boxed{\oint \mathbf{v} \cdot d\mathbf{a}}$$

Example 1.7. Calculate the surface integral of $\mathbf{v} = 2xz\,\hat{\mathbf{x}} + (x+2)\,\hat{\mathbf{y}} + y(z^2-3)\,\hat{\mathbf{z}}$ over five sides (excluding the bottom) of the cubical box (side 2) in Fig. 1.23. Let "upward and outward" be the positive direction, as indicated by the arrows.



Solution

(i) $x = 2$, $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, $\mathbf{v}\cdot d\mathbf{a} = 2xz\,dy\,dz = 4z\,dy\,dz$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = 4\int_0^2 dy \int_0^2 z\,dz = 16.$$

(ii) $x = 0$, $d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}}$, $\mathbf{v}\cdot d\mathbf{a} = -2xz\,dy\,dz = 0$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = 0.$$

(iii) $y = 2$, $d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}}$, $\mathbf{v}\cdot d\mathbf{a} = (x+2)\,dx\,dz$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^2 (x+2)\,dx \int_0^2 dz = 12.$$

(iv) $y = 0$, $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}}$, $\mathbf{v}\cdot d\mathbf{a} = -(x+2)\,dx\,dz$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^2 (x+2)\,dx \int_0^2 dz = -12.$$

(v) $z = 2$, $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}$, $\mathbf{v}\cdot d\mathbf{a} = y(z^2-3)\,dx\,dy = y\,dx\,dy$, so

$$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^2 dx \int_0^2 y\,dy = 4.$$

The *total* flux is

$$\int_{\text{surface}} \mathbf{v}\cdot d\mathbf{a} = 16 + 0 + 12 - 12 + 4 = 20.$$

Cylindrical coordinate

the differential volume bounded by the surfaces at $\rho, \rho + d\rho, \phi, \phi + d\phi, z,$ and $z + dz$. The differential volume enclosed is

$$dv = \rho \, d\rho \, d\phi \, dz$$

The differential surfaces in the positive direction of the unit vectors (Fig. 2.19b) are

$$\overrightarrow{d\mathbf{s}}_\rho = \rho \, d\phi \, dz$$

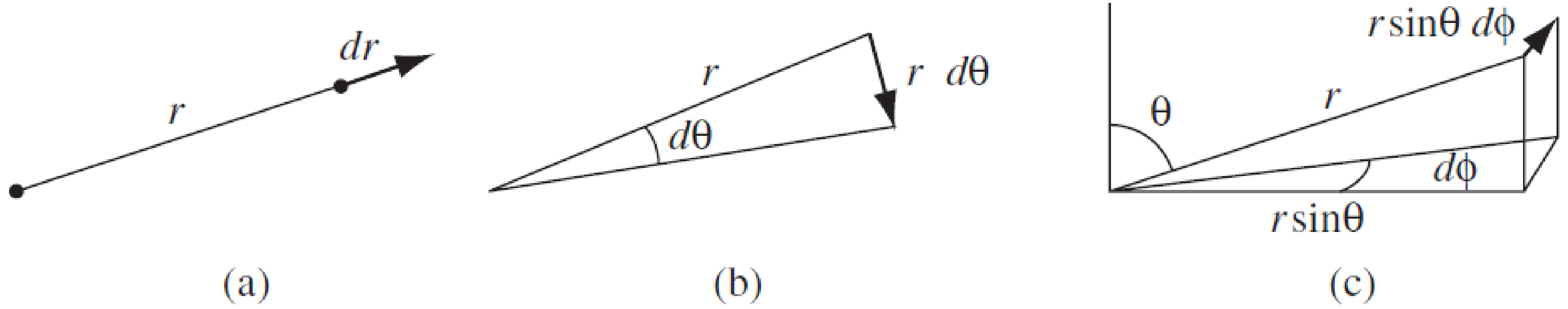
$$\overrightarrow{d\mathbf{s}}_\phi = d\rho \, dz$$

$$\overrightarrow{d\mathbf{s}}_z = \rho \, d\rho \, d\phi$$

The differential length vector from P to Q is

$$\overrightarrow{d\ell} = d\rho \qquad + \rho \, d\phi \qquad + dz$$

Diff. element of length: spherical coord.



An infinitesimal displacement in the $\hat{\mathbf{r}}$ direction is simply dr (Fig. 1.38a), just as an infinitesimal element of length in the x direction is dx:

$$dl_r = dr.$$

an infinitesimal element of length in the $\hat{\boldsymbol{\theta}}$ direction

$$dl_\theta = r\, d\theta. \qquad dl_\phi = r \sin\theta\, d\phi.$$

general infinitesimal displacement $d\mathbf{l}$ is

$$d\mathbf{l} =$$

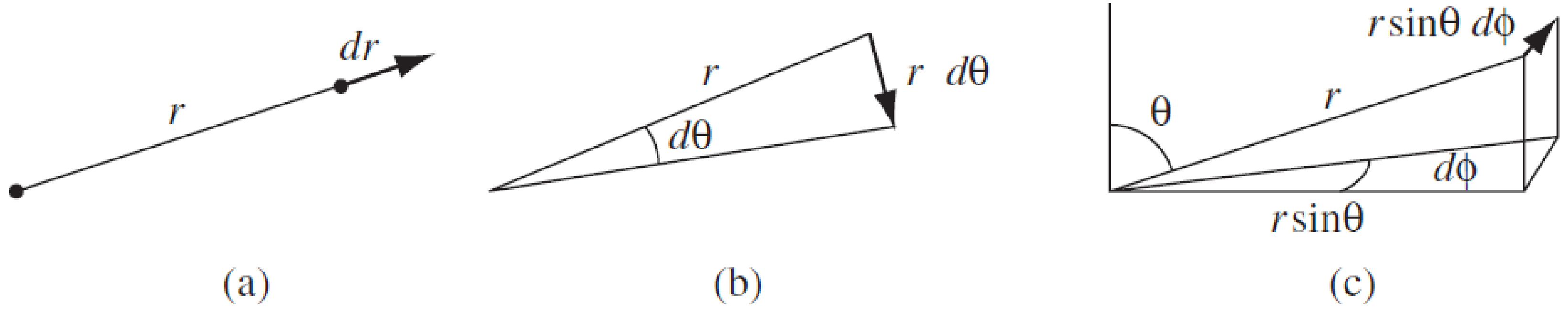
$$\overrightarrow{d\mathbf{s}_r} =$$

$$\overrightarrow{d\mathbf{s}_\theta} =$$

$$\overrightarrow{d\mathbf{s}_\phi} =$$

$$d\tau =$$

Diff. element of length: spherical coord.



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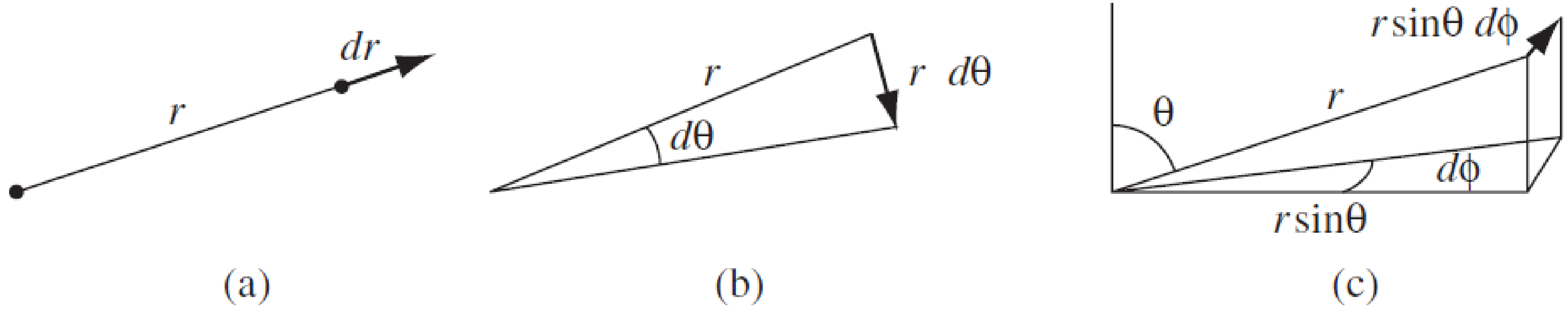
$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}.$$

Diff. element of length: spherical coord.



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$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}.$$

$$\overrightarrow{d\mathbf{s}_r} = r^2 \sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$$

$$\overrightarrow{d\mathbf{s}_\theta} = r\,dr\,\sin\theta\,d\phi\,\hat{\boldsymbol{\theta}}$$

$$\overrightarrow{d\mathbf{s}_\phi} = r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}.$$

$$d\tau = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Find the volume of a sphere of radius R

Example 1.13. Find the volume of a sphere of radius R.

Solution

$$V = \int d\tau = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \left(\int_0^R r^2 \, dr \right) \left(\int_0^{\pi} \sin\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right)$$

$$= \left(\frac{R^3}{3} \right) (2)(2\pi) = \frac{4}{3} \pi R^3$$

Thank You