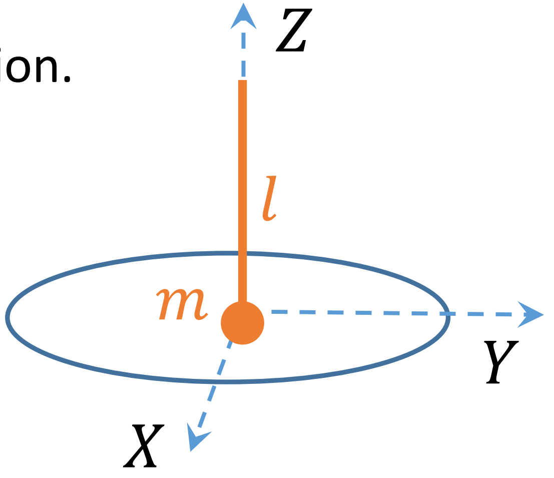
Introduction: Superposition

$$Resultant: f(t) = f_1 + f_2$$

Spherical Pendulum: Pendulum ($m, l: x = y = 0$) which is free to swing in any direction.



Uncoupled eqn.

$$m\frac{d^2x}{dt^2} = -\frac{mg}{l}x \qquad x(t) = A_1\cos(\omega t + \phi_1)$$

$$m\frac{d^2y}{dt^2} = -\frac{mg}{l}y \qquad y(t) = A_2\cos(\omega t + \phi_2)$$

$$\boldsymbol{\omega} = \sqrt{\frac{\boldsymbol{g}}{\boldsymbol{l}}}$$

$$Resultant: \phi(t) = \hat{\imath}x(t) + \hat{\jmath}y(t)$$

One solves for each direction (x, y) separately and then superposes the two motions to get total solution.

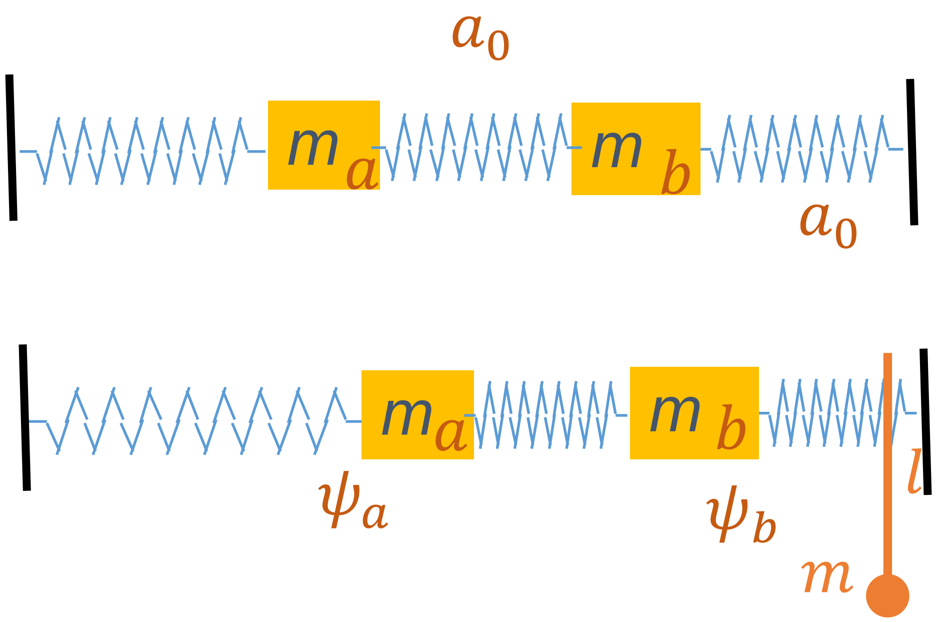
Motion of a system with more than one degree of freedom may appear to be very complicated one. However, it is sometimes possible to express it as the superposition of few independent simple harmonic motions.

These SHMs are known as the **normal modes**.

Introduction: Normal coordinate and normal mode

$$m\frac{d^2}{dt^2}\psi_a = -k\psi_a + k(\psi_b - \psi_a)$$

$$m\frac{d^2}{dt^2}\psi_b = -k(\psi_b - \psi_a) - k\psi_b$$



Adding

$$m\frac{d^2}{dt^2}(\psi_a+\psi_b) = -k(\psi_a + \psi_b)$$

Subtracting

$$m\frac{d^2}{dt^2}(\psi_a-\psi_b) = -3k(\psi_a - \psi_b)$$

$$\psi_1 = \psi_a + \psi_b \qquad m\frac{d^2\psi_1}{dt^2} = -k\psi_1 \qquad \psi_1 = A_1\cos(\omega_1 t + \phi_1)$$

$$\psi_2 = \psi_a - \psi_b \qquad m\frac{d^2\psi_2}{dt^2} = -k\psi_2 \qquad \psi_2 = A_2\cos(\omega_2 t + \phi_1)$$

$$\psi_a = \frac{A_1}{2}\cos(\omega_1 t + \phi_1) + \frac{A_2}{2}\cos(\omega_2 t + \phi_1)$$

$$\psi_a = \frac{A_1}{2}\cos(\omega_1 t + \phi_1) - \frac{A_2}{2}\cos(\omega_2 t + \phi_1)$$

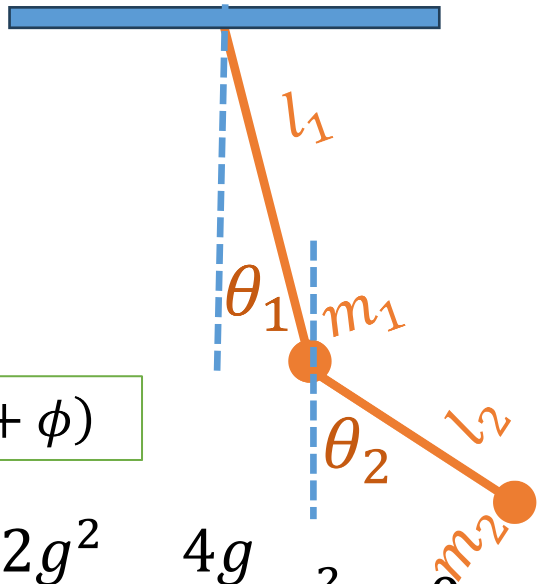
$$\boxed{\omega_1 = \sqrt{\frac{k}{m}}} \qquad \boxed{\omega_2 = \sqrt{\frac{3k}{m}}}$$

TPS

Introduction: Normal Modes and Normal coordinates

$$(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1-\theta_2)\,\ddot{\theta}_2 + m_2 l_1 l_2 sin(\theta_1-\theta_2)\dot{\theta}_2^2 + (m_1+m_2) l_1 g sin\theta_1 = 0$$

$$l_2\ddot{\theta}_2 + l_1 \cos(\theta_1-\theta_2)\,\ddot{\theta}_1 - l_1 sin(\theta_1-\theta_2)\dot{\theta}_1^2 + g sin\theta_2 = 0$$



$$m_1 = m_2 = m, \qquad l_1 = l_2 = l, \qquad \theta_1, \theta_2 \to 0$$

$$2\ddot{\theta}_1 + \ddot{\theta}_2 + \frac{2g}{l}\theta_1 = 0$$
$$\ddot{\theta}_1 + \ddot{\theta}_2 + \frac{g}{l}\theta_2 = 0$$

Let us consider single mode solution for form:

$$\theta_1(t) = A_1\cos(\omega t+\phi); \theta_2(t) = A_2\cos(\omega t+\phi)$$

$$\frac{2g}{l}A_1 - 2\omega^2 A_1 - \omega^2 A_2 = 0$$
$$-\omega^2 A_1 + \frac{g}{l}A_2 - \omega^2 A_2 = 0$$

$$\begin{vmatrix} \frac{2g}{l} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \frac{g}{l} - \omega^2 \end{vmatrix} = 0$$

$$\omega^4 + \frac{2g^2}{l^2} - \frac{4g}{l}\omega^2 = 0$$
$$\omega^2 = (2 \pm \sqrt{2})\frac{g}{l}$$

$$\frac{A_2}{A_1} = \frac{\omega^2}{\frac{g}{l} - \omega^2} = \frac{\pm\sqrt{2}}{-1}$$

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = C\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}\cos(\omega_+ t + \phi_1) + D\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}\cos(\omega_- t + \phi_2)$$