MA1000: Calculus

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Functions of Several Variables

In this Module:

- Limit and Continuity of Functions of Two Variables
- Partial Derivatives
- Differentiability of Functions of Two Variables
- The Chain Rule
- Directional Derivatives and Gradient Vectors

Functions of Several Variables: Examples

Functions of two variables:

1. $z = \sqrt{y - x^2}$ (real-valued for $y \geq x^2$).
2. $z = \dfrac{1}{xy}$ $(xy \neq 0)$.
3. $z = \sin xy$.

Functions of three variables:

1. $w = \sqrt{x^2 + y^2 + z^2}$.
2. $w = \dfrac{1}{x^2 + y^2 + z^2}$ $((x, y, z) \neq (0, 0, 0))$.
3. $w = xy \ln z$.

Understanding Regions in Higher Dimensions

Definition (Interior and Boundary Points, Open, Closed)

Let R be a region (set) in the xy-plane. A point (x_0, y_0) in R is called an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R. A point (x_0, y_0) is called a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The set of all interior points of a region is called the **interior** of the region. The set of all boundary points of a region is called its **boundary**. A region is called **open** if it consists entirely of interior points. A region is called **closed** if it contains all its boundary points.

Examples: Consider the following sets:

$$\begin{aligned} R_1 &= \{(x,y) \mid x^2+y^2<1\} && \text{Open unit disk.} \\ R_2 &= \{(x,y) \mid x^2+y^2=1\} && \text{The unit circle.} \\ R_3 &= \{(x,y) \mid x^2+y^2\leq 1\} && \text{Closed unit disk.} \end{aligned}$$

The region R_1 is open. The set R_2 consists of its boundary points. Thus R_3, that contains all its boundary points, is a closed region.

Definition (Bounded and Unbounded Regions in the Plane)

A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Examples:

Bounded Sets: Line segments, triangles, interiors of triangles, rectangles, circles and disks.

Unbounded Sets: Lines, the graphs of functions defined on infinite intervals, quadrants, half-planes and the plane itself.

Definition (Level Curve, Graph, Surface)

The set of all points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Example: For the function $f(x, y) = x^2 + y^2$, each circle in the xy-plane with origin as the center is a level curve. For example, the circle $x^2 + y^2 = 100$ is a level curve of this function.

Limits and Continuity in Higher Dimensions

Definition (Limit of a Function of Two Variables)

We say that $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \quad \Rightarrow \quad |f(x, y) - L| < \epsilon.$$

Examples

1. $\lim_{(x,y)\to(x_0,y_0)} x = x_0$
2. $\lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. $\lim_{(x,y)\to(x_0,y_0)} k = k$ (any number k).

Solution(1):

Here $f(x,y) = x$ and $L = x_0$. Let $\epsilon > 0$ be given. We choose $\delta = \epsilon$. Then

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta = \epsilon$$

implies that

$$0 < \sqrt{(x-x_0)^2} < \epsilon \quad \Rightarrow \quad |x - x_0| < \epsilon \quad \Rightarrow \quad |f(x,y) - x_0| < \epsilon.$$

Thus for $\delta = \epsilon$, we have

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \quad \Rightarrow \quad |f(x,y) - x_0| < \epsilon.$$

This proves that

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0.$$

Properties of Limits of Functions of Two Variables

Theorem

Let L, M *and* k *be real numbers such that*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

Then

1. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)+g(x,y)) = L+M$ *and* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)-g(x,y)) = L-M$
2. $\lim_{(x,y)\to(x_0,y_0)} (kf(x,y)) = kL$
3. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M.$
4. $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}, \quad M \neq 0$
5. *If* r *and* s *are integers with no common factors and* $s \neq 0$, *then*

$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^{r/s} = L^{r/s}, \quad \text{provided } L^{r/s} \text{ is a real number.}$$

Examples

1. $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0^2)(1) + 5(0)(1) - (1^3)} = -3$$

2. $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3^2) + (-4)^2} = \sqrt{25} = 5.$$

Homework

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Example

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists.

Solution: We note that along the line $x = 0$, the function has value 0 when $y \neq 0$.
Similarly, along the line $y = 0$ (x-axis), the function has value 0 when $x \neq 0$.
So, if the limit exists as (x, y) ((y-axis)) approaches $(0, 0)$, it must be 0. We apply the definition to see if the limit is indeed 0.
Let $\epsilon > 0$ be given. We want to find a $\delta > 0$ such that

$$0 < \sqrt{x^2+y^2} < \delta \quad \Rightarrow \quad \left|\frac{4xy^2}{x^2+y^2} - 0\right| < \epsilon.$$

Now,

$$\left|\frac{4xy^2}{x^2+y^2}-0\right|<\epsilon \Leftrightarrow \frac{4|x|y^2}{x^2+y^2}<\epsilon.$$

But $y^2 \leq x^2+y^2$. So,

$$\frac{4|x|y^2}{x^2+y^2} \leq 4|x| \leq 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2}.$$

Let us choose $\delta = \epsilon/4$. Then

$$0<\sqrt{x^2+y^2}<\delta \quad \Rightarrow \quad \left|\frac{4xy^2}{x^2+y^2}-0\right| \leq 4\sqrt{x^2+y^2}<4\delta<4\left(\frac{\epsilon}{4}\right)=\epsilon.$$

Thus the function has limit 0 as $(x,y)\to(0,0)$.

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0), then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.

Example

Show that the function $f(x,y) = \frac{2x^2y}{x^4+y^2}$ has no limit as (x,y) approaches $(0,0)$.

Solution: Along the curve $y = kx^2$, $x \neq 0$, the function has a constant value:

$$f(x,y)\big|_{y=k^2} = \left.\frac{2x^2y}{x^4+y^2}\right|_{y=k^2} = \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \frac{2kx^4}{x^4+kx^4} = \frac{2k}{1+k^2}.$$

Therefore, as (x,y) approaches $(0,0)$ along the curve $y = kx^2$,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \frac{2k}{1+k^2}.$$

Thus the limit varies with the path of approach. (For instance, along the path $y = x^2$ (i.e., $k = 1$), we get the limit 1. Along the path $y = 0$ (i.e., $k = 0$) we get the limit 0.)
Thus, by the two-path test, the limit does not exist.

Continuous Functions of Two Variables

Definition

A function $f(x, y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0).
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists.
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Example

Show that

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at every point except at the origin.

Solution: The function f is continuous at every point $(x,y) \neq (0,0)$. The function fails to be continuous at $(0,0)$ because the limit does not exist as $(x,y) \to (0,0)$. This can be proved by the two path test:
For any value of m, the function f has a constant value on the punctured line $y = mx, x \neq 0$: $\dfrac{2m}{1+m^2}$.
So, as $(x,y) \to (0,0)$ along the line $y = mx$, the function has the limit $\dfrac{2m}{1+m^2}$. This limit changes as m changes. Hence, by the two-path test, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So, the function is not continuous at the origin.

Partial Derivatives

"The calculus of several variables is basically single-variable calculus applied to several variables one at a time." –Thomas' Calculus

The geometry:

1. Suppose (x_0, y_0) is a point in the domain of a function $f(x, y)$.
2. Then the intersection of the vertical plane $y = y_0$ and the surface $z = f(x, y)$ is the curve $z = f(x, y_0)$.
3. The horizontal in this plane is x and the vertical coordinate is z.
4. As the y-value is held constant at y_0, it is not a variable.
5. The partial derivative of f with respect to x at the point (x_0, y_0) is the ordinary derivative of $f(x, y_0)$ at the point x_0.

Definition (Partial Derivative with Respect to x)

The **partial derivative of** $f(x, y)$ **with respect to** x **at the point** (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

Definition (Partial Derivative with Respect to y)

The **partial derivative of** $f(x, y)$ **with respect to** y **at the point** (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

Example

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if $f(x, y) = x^2 + 3xy + y - 1$.

Solution: To find $\frac{\partial f}{\partial x}$, we treat y constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3y \cdot 1 + 0 - 0 = 2x + 3y.$$

Thus $\frac{\partial f}{\partial x}$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\frac{\partial f}{\partial y}$, we treat x constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3x \cdot 1 + 1 = 3x + 1.$$

Thus $\frac{\partial f}{\partial y}$ at $(4, -5)$ is $3(4) + 1 = 13$.