MA1002: Linear Algebra

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Text books

1. Linear Algebra, Kenneth Hoffman and Ray Kunze, Prentice-Hall, Second Edition (available online).
2. Topics in Algebra, I. N. Herstein, Wiley.
3. Linear Algebra and its Applications, Gilbert Strang, 4th edition.
4. Introduction to Linear Algebra, Krishnamurthi (BITS Pilani).

Evaluation Scheme (Tentative)

Quiz 1	25
Quiz 2	25
End Semester Examination	50

Applications of Linear Algebra

- To summarize and manipulate data (Machine Learning, Image Processing)
- To model satellites, jet engines (Eigen values)
- Many more!
- Prepare a detailed report of an application of linear algebra tools in CSE/ECE/ME and Design

You have first assignment !

Mathematical Structures

1. Field (Members are called scalars)
2. Vector Space (Members are called vectors)

A non empty set F together with operations

(i)$+ : F \times F \to F$ (addition) and

(ii) $\cdot : F \times F \to F$ (multiplication)

is said to be a field if the following axioms and properties are satisfied.

Closure axiom: For every $a, b \in F \implies$

$$a + b \in F \quad \text{and} \quad a \cdot b \in F$$

Associative axiom: For all $a, b, c \in F \implies$

$$a + (b + c) = (a + b) + c \text{ and}$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

Identity axiom: There exist elements $0, 1 \in F$ such that

$$a + 0 = 0 + a = a, \qquad \forall a \in F$$
$$a \cdot 1 = 1 \cdot a = a, \quad \forall a \in F$$

Inverse axiom:(i) For every $a \in F$, there exists $b \in F$ such that

$$a + b = b + a = 0$$

(b is called additive inverse of a)

and (ii) for every $a \in F - \{0\}$, there exists $c \in F - \{0\}$ such that

$$a \cdot c = c \cdot a = 1$$

(c is called multiplicative inverse of a)

Commutative Property:

$$a + b = b + a \quad \forall a, b \in F \quad \text{and}$$
$$a \cdot b = b \cdot a \quad \forall a, b \in F$$

Distributive Property: For every $a, b, c \in F$

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and}$$
$$(a + b) \cdot c = a \cdot c + b \cdot c$$

Notation: A field F with respect to operations $+, \cdot$ is usually denoted as $(F, +, \cdot)$

$\mathbb{N} = \{1, 2, 3, \ldots, \}$ - set of all natural numbers

Is $(\mathbb{N}, +, \cdot)$ a field?

Closure axiom:

We know that

for all $a, b \in \mathbb{N}, \quad a + b$ and $a \cdot b$ are also elements of $\mathbb{N}$

Associative axiom:

Associative property is true for $\mathbb{N}$ for both $+$ and $\cdot$

Identity axiom:

$0 \notin \mathbb{N}$ and hence identity axiom is not satisfied

Thus $(\mathbb{N}, +, \cdot)$ is not a field.

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ - set of all integers

Is $(\mathbb{Z}, +, \cdot)$ a field?

Closure axiom: For all $a, b \in \mathbb{Z}, \quad a + b$ and $a \cdot b$ are also elements of $\mathbb{Z}$

Associative axiom: Associative property is true for $\mathbb{Z}$ for both $+$ and $\cdot$

Identity axiom: There exists $0, 1 \in \mathbb{Z}$ such that $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$, $\forall a \in \mathbb{Z}$

Inverse axiom: For every $a \in \mathbb{Z}$, there exists $-a \in \mathbb{Z}$ such that $a + (-a) = -a + a = 0$. But for $2 \in \mathbb{Z}$ there doesnot exist a multiplicative inverse in $\mathbb{Z}$

Thus $(\mathbb{Z}, +, \cdot)$ is not a field

$\mathbb{Z}_2 = \{0, 1\}$ - congruence class modulo 2.

Is $(\mathbb{Z}_2, +, \cdot)$ a field?

Closure axiom:

$$0 + 0 = 0, \quad 1 + 0 = 1, \quad 0 + 1 = 1, \quad 1 + 1 = 0$$
$$0 \cdot 0 = 0, \quad 1 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 1 = 1$$

Thus, the closure axiom is true.

Associative axiom:

From closure axiom we can see that associative axiom also holds true.

Identity axiom:

$0 \in \mathbb{Z}_2$ is the additive identity and $1 \in \mathbb{Z}_2$ is the multiplicative identity.

Inverse axiom:

Additive inverse of 0 is 0 and 1 is 1.

Multiplicative inverse of 1 is 1.

Commutative property:

From closure axiom, we can see that addition and multiplication are commutative in $\mathbb{Z}_2$

Distributive property: For every $a, b \in \mathbb{Z}_2$

$$(a+b)\cdot 0 = 0 = a\cdot 0 + b\cdot 0$$
$$(a+b)\cdot 1 = a+b = a\cdot 1 + b\cdot 1$$

Thus distributive law holds.

Therefore, $(\mathbb{Z}_2, +, \cdot)$ is a field.

$\mathbb{R}$ - Set of real numbers.

Is $(\mathbb{R}, +, \cdot)$ a field?

Closure axiom:

Addition of two real numbers is a real number. Similarly,

Multiplication of two real numbers is a real number.

Thus, the closure axiom is true.

Associative axiom:

Addition and Multiplication are always associative in $\mathbb{R}$.

Identity axiom:

$0 \in \mathbb{R}$ is the additive identity and $1 \in \mathbb{R}$ is the multiplicative identity.

Inverse axiom:

Additive inverse of an element $a \in \mathbb{R}$ is $-a$.

Multiplicative inverse of an element $a \in \mathbb{R} - \{0\}$ is $\frac{1}{a}$.

Commutative property:

It is true that for every $a, b \in \mathbb{R}$

$$a + b = b + a \text{ and } a \cdot b = b \cdot a$$

Distributive property:

Distributive property is true for real numbers with respect to $+$ and $\cdot$

Therefore, $(\mathbb{R}, +, \cdot)$ is a field.

Questions

$\mathbb{Q}$- Set of all rational numbers

$\mathbb{C}$- Set of all complex numbers

1. Is $(\mathbb{Q}, +, \cdot)$ a field?
2. Is $(\mathbb{C}, +, \cdot)$ a field?

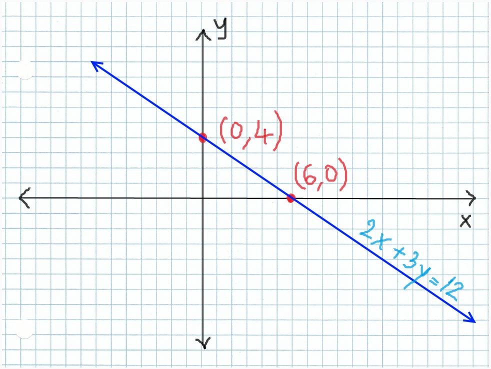
Linear Equations

Equation of a line :

$$y = mx + c$$

$$ax + by = c$$

Example of a line

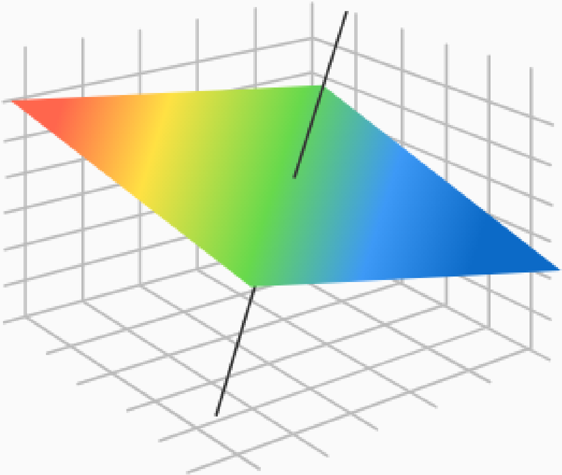


Equation of a plane :

$$z = ax + by + c$$

$$Ax + By + Cz = D$$

Example of a plane



Systems of linear equations

$$\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n &= b_1 \\
A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad\qquad & \vdots \\
A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n &= b_m
\end{aligned}$$

- m equations
- n varaibles $(x_1, x_2, \ldots, x_n)$
- $A_{ij}, b_i \in F \quad (F \text{ is a field})$

Matrix form

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

$$AX = B$$

$A = [A_{ij}]_{m \times n}$, $1 \leq i \leq m$, $1 \leq j \leq n$
$X = [x_j]_{n \times 1}$, $\qquad B = [b_i]_{m \times 1}$
The solution set of the linear system $AX = B$ is

$$S = \{X \in R^n \ : \ AX = B\}$$

Problem 1

Solve the following system of linear equations

$$4x - y = 5$$

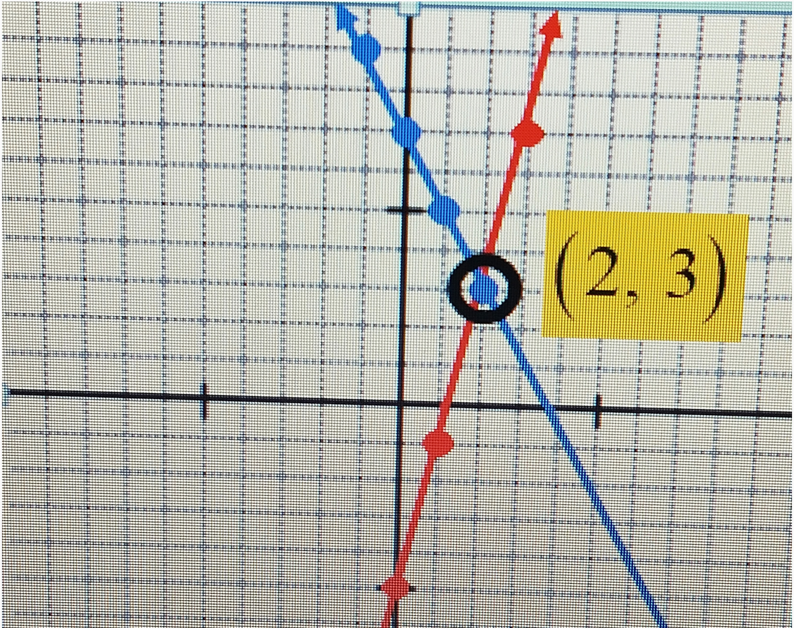
$$2x + y = 7$$

Matrix form

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Solve graphically (sketch it!)

Problem 1 (solution)



Problem 1

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$$4x - y = 5$$

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Matrix form

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Solve graphically (sketch it!)

Solution $S = \{(2, 3)\}$

Our objective is to design an efficient machinery to solve $AX = B$

Solution of Problem 1 through elementary row operations

$$4x - y = 5 \quad (Eq1)$$

$$2x + y = 7 \quad (Eq2)$$

Let us employ a high school technique

Multiply (Eq2) by 2

$$4x - y = 5 \quad (Eq1)$$

$$4x + 2y = 14 \quad (Eq2)$$

(Eq2)-(Eq1) $\Longrightarrow$

$$4x - y = 5 \quad (Eq1)$$

$$0x + 3y = 9 \quad (Eq2)$$

We have three equivalent systems say red, blue and green

Let us express red system in augmented matrix form

$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right]$$

Multiply second row by 2 $\quad (R_2 \longleftarrow 2 \times R_2)$

$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 4 & 2 & 14 \end{array}\right]$$

Substract first row from second row $\quad (R_2 \longleftarrow R_2 - R_1)$

$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 4 & 2 & 14 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 0 & 3 & 9 \end{array}\right]$$

Contd.

$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 4 & 2 & 14 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 0 & 3 & 9 \end{array}\right]$$

$$R_2 \longleftarrow \tfrac{1}{3}R_2 \qquad \Longrightarrow$$

$$\sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 0 & 1 & 3 \end{array}\right]$$

$$R_1 \longleftarrow \tfrac{1}{4}R_1 \qquad \Longrightarrow$$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & 3 \end{array}\right]$$

Contd.

$R_1 \longleftarrow \frac{1}{4}R_1 \qquad \Longrightarrow$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & 3 \end{array}\right]$$

$R_1 \longleftarrow R_1 + \frac{1}{4}R_2 \qquad \Longrightarrow$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Let us write it in the system of equations form

$$x + 0y = 2$$

$$0x + y = 3$$

Salient points

- Multiplying an equation by a non-zero scalar preserves the solution space ($R_i \longleftarrow cR_i, \quad c \neq 0$)
- Replacing i^{th} equation by sum of i^{th} equation and constant multiple of j^{th} equation preserves the solution space. ($R_i \longleftarrow R_i + cR_j$)
- Interchanging two equations preserves the solution space. ($R_i \longleftrightarrow R_j$)

Problem 2

Solve the following system of linear equations.

$$3x - 2y = -6, \quad x + 2y = -10$$

Solve graphically !

Augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -2 & -6 \\ 1 & 2 & -10 \end{array}\right]$$

Interchange first and second rows ($R_1 \longleftrightarrow R_2$)

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -10 \\ 3 & -2 & -6 \end{array}\right]$$

Problem 2 contd.

$R_2 \longleftarrow R_2 - 3R_1 \implies$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -10 \\ 0 & -8 & 24 \end{array}\right]$$

$R_2 \longleftarrow -\frac{1}{8}R_2$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -10 \\ 0 & 1 & -3 \end{array}\right]$$

Problem 2 contd.

$R_1 \longleftarrow R_1 - 2R_2 \implies$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & -3 \end{array}\right]$$

$$x = -4, \quad y = -3$$

Solution $S = \{(-4, -3)\}$

Problem 3

Solve the system of linear equations

$$x + y = 2, \quad 2x + 2y = 5$$

Solve graphically!

Augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 5 \end{array}\right] \quad (R_2 \longleftarrow R_2 - 2R_1)$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]$$

Second row is $0x + 0y = 1$. No solution

Problem 4

Solve the system

$$x + 2y = 5, \quad 2x + 4y = 10$$

Solve graphically! **Augmented matrix is**

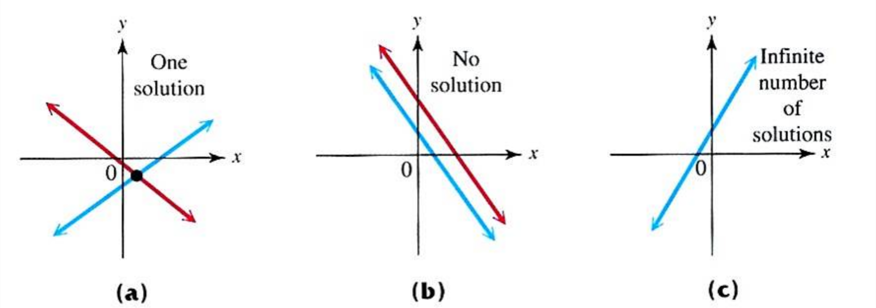
$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 4 & 10 \end{array}\right] \quad (R_2 \longleftarrow R_2 - 2R_1)$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \end{array}\right] \quad \Longrightarrow \quad x + 2y = 5$$

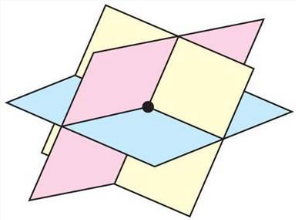
Let $y = c \quad \Longrightarrow \quad x = 5 - 2c$**.**

The solution $S = \{(5 - 2c, c) \ : \ c \in \mathbf{R}\}$

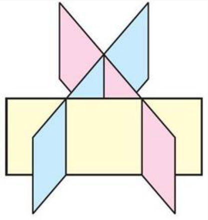
two dimensional problem and possible solutions

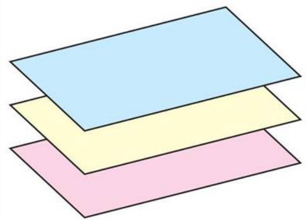


3-dimensional problem with a unique solution

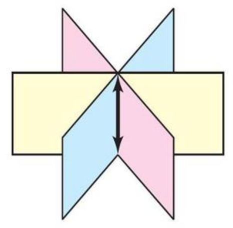


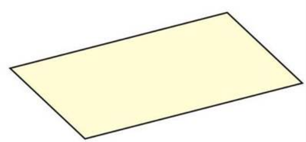
3-dimensional problem with no solutions





3-dimensional problem with infinite number of solutions





Linear combination of equations

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1 \qquad (1)$$

$$A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \qquad (2)$$

Consider $c_1(1) + c_2(2)$(a linear combination) $\Longrightarrow$

$$c_1\left(A_{11}x_1 + A_{12}x_2 + A_{13}x_3\right) + c_2\left(A_{21}x_1 + A_{22}x_2 + A_{23}x_3\right)$$

$$= c_1 b_1 + c_2 b_2 \qquad (3)$$

Suppose that $x_1 = a, x_2 = b, x_3 = c$ **is solution of (1) and (2)**

Show that above solution is also a solution of (3).

Consider L.H.S. of (3),

$$c_1 (A_{11} a + A_{12} b + A_{13} c) + c_2 (A_{21} a + A_{22} b + A_{23} c)$$

$$= c_1 b_1 + c_2 b_2$$

So $x_1 = a, x_2 = b, x_3 = c$ **is solution of (3)**

Converse need not be true (Try !)

Note: If X^* is a solution of k linear equations, then X^* is also a solution of a linear combination of those k equations.

Equivalent systems

We say two sytems are equivalent if each equation in each system is a linear combination of equations in the other system.

Why do we focus on equivalent systems?

Theorem 1: Equivalent systems of linear equations have exactly same solutions.

Proof:

Let (A) and (B) be two equivalent systems with solution sets S_A and S_B respectively. Prove that $S_A = S_B$.

Let $X \in S_A$. $\implies X$ satisfies every equation in (A), and every equation in (B) is a linear combination of equations in (A). $\implies$ X satisfies every equation in (B). $\implies X \in S_B$

$$\text{Hence} \quad S_A \subseteq S_B$$

$$\text{Similarly,} \quad S_B \subseteq S_A$$

$$\implies \qquad S_A = S_B$$

Problem

Show that the following systems of linear equations are equivalent.

$$\left.\begin{aligned} x - y &= 0 \\ 2x + y &= 0 \end{aligned}\right\} \quad ------ \quad (I)$$

$$\left.\begin{aligned} 3x + y &= 0 \\ x + y &= 0 \end{aligned}\right\} \quad ------ \quad (II)$$

Solution

$$
\begin{aligned}
3x + y &= \tfrac{1}{3}(x - y) + \tfrac{4}{3}(2x + y) \\
x + y &= -\tfrac{1}{3}(x - y) + \tfrac{2}{3}(2x + y) \\
x - y &= (3x + y) - 2(x + y) \\
2x + y &= \tfrac{1}{2}(3x + y) + \tfrac{1}{2}(x + y)
\end{aligned}
$$

Note : $AX = 0$ is called a homogeneous system.

Elementary row operations

Consider a matrix $A = [A_{ij}]$, where $A_{ij} \in F$, a field.
The i^{th} row of A is $R_i = [A_{i1}, A_{i2}, \ldots, A_{in}]$
Type 1: Multiplication of one row of A by a non-zero scalar
$c \in F$ $\quad (e \;:\; R_i \longleftarrow cR_i)$

Type 2: Replacement of i^{th} row by row i plus c times of row j
where $c \in F$ $\quad (e \;:\; R_i \longleftarrow R_i + cR_j)$

Type 3: Interchange of two rows $\quad (e \;:\; R_i \longleftrightarrow R_j)$

Inverse of the Type 1 elementary row operation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Let $e \ : \ R_1 \longleftarrow cR_1, \quad c \neq 0$

$$e(A) = \begin{bmatrix} cA_{11} & cA_{12} & cA_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

We define $e_1 \ : \ R_1 \longleftarrow \frac{1}{c} R_1$

$$e_1(e(A)) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} = A$$

Prove that $e(e_1(A)) = A \implies e_1(e(A)) = A = e(e_1(A))$

Inverse of the Type 1 elementary row operation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Let $e \ : \ R_1 \longleftarrow cR_1, \quad c \neq 0$

$$e(A) = \begin{bmatrix} cA_{11} & cA_{12} & cA_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

We define $e_1 \ : \ R_1 \longleftarrow \frac{1}{c} R_1 \qquad \Longrightarrow$ **e_1 is the inverse elementary operation of e**

$$e_1(e(A)) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} = A$$

Prove that $e(e_1(A)) = A \implies e_1(e(A)) = A = e(e_1(A))$

Inverse of the Type 2 elementary row operation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Let $e \; : \; R_2 \longleftarrow R_2 + cR_1, \quad c \in F$

$$e(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} + cA_{11} & A_{22} + cA_{12} & A_{23} + cA_{13} \end{bmatrix}$$

We define $e_1 \; : \; R_2 \longleftarrow R_2 - cR_1$**.** $\implies \quad e_1(e(A)) = A$ **Similarly,** $e(e_1(A)) = A = e_1(e(A))$

Note: e_1 **is the inverse of** e

Inverse of the Type 3 elementary row operation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Let $e \;:\; R_1 \longleftrightarrow R_2.$

$$e(A) = \begin{bmatrix} A_{21} & A_{22} & A_{23} \\ A_{11} & A_{12} & A_{13} \end{bmatrix}$$

We define $e_1 \;:\; R_1 \longleftrightarrow R_2.$

$$e_1(e(A)) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} = A$$

Similarly, $e(e_1(A)) = A = e_1(e(A))$

Note: e_1 **is the inverse of** e

Theorem 2

To each elementary row operation e there corresponds an elementary operation e_1, of the same type as e, such that $e(e_1(A)) = A = e_1(e(A))$. In other words, inverse operation of an elementary operation exists and is of an elementary operation of the same type.

Proof :

Type 1	*Inverse of the Type* 1
$e : R_i \longleftarrow cR_i,\ c \neq 0$	$e_1 : R_i \longleftarrow \frac{1}{c} R_i$

Proof of Theorem 2 contd.

Type 2	*Inverse of the Type 2*
$e : R_i \longleftarrow R_i + cR_j$	$e_1 : R_i \longleftarrow R_i - cR_j$

Type 3	*Inverse of the Type 3*
$e : R_i \longleftrightarrow R_j$	$e_1 : R_i \longleftrightarrow R_j$

Note that for an $m \times n$ matrix A, $e(e_1(A)) = A = e_1(e(A))$

Note

$e_1 \ : \ R_2 \longleftarrow 2R_2$ and $e_2 \ : \ R_2 \longleftarrow R_2 - R_1$

$$A = \begin{bmatrix} 4 & -1 & 5 \\ 2 & 1 & 7 \end{bmatrix} \underset{(e_1)}{\longrightarrow} \begin{bmatrix} 4 & -1 & 5 \\ 4 & 2 & 14 \end{bmatrix} \underset{(e_2)}{\longrightarrow} \begin{bmatrix} 4 & -1 & 5 \\ 0 & 3 & 9 \end{bmatrix} = B$$

$$e_2(e_1(A)) = B$$

$e_1^{-1} \ : \ R_2 \longleftarrow \frac{1}{2}R_2$ and $e_2^{-1} \ : \ R_2 \longleftarrow R_2 + R_1$

$$A = \begin{bmatrix} 4 & -1 & 5 \\ 2 & 1 & 7 \end{bmatrix} \underset{(e_1^{-1})}{\longleftarrow} \begin{bmatrix} 4 & -1 & 5 \\ 4 & 2 & 14 \end{bmatrix} \underset{(e_2^{-1})}{\longleftarrow} \begin{bmatrix} 4 & -1 & 5 \\ 0 & 3 & 9 \end{bmatrix} = B$$

$$e_1^{-1}(e_2^{-1}(B)) = A$$

A and B are called row-equivalent matrices.

Row-equivalent matrices

Definition : If A and B are $m \times n$ matrices over the field F, we say B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

Theorem 3

If A and B are row-equivalent $m \times n$ matrices, the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have exactly same solutions.

Proof: Suppose that we pass A to B by a finite sequence of elementary row operations :

$$A = A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow A_k = B$$

Note: If

(1) $A_0X = 0$ and $A_1X = 0$ have same solutions,

(2) $A_1X = 0$ and $A_2X = 0$ have same solutions,

(j) $\dots, \dots,$

(k) $A_{k-1}X = 0$ and $A_kX = 0$ have same solutions,

then $AX = 0$ and $BX = 0$ have same solutions.

Proof of Theorem 3 contd.

It is enough to prove that $A_jX = 0$ and $A_{j+1}X = 0$ have exactly the same solutions(that is one elementary row operation doesn't disturb the set of solutions).

Suppose that B is obtained from A by a single elementary row operation, say $e \quad (i.e., e(A) = B)$. No matter which of the types the operation is : (1) , (2) or (3), each equation in the system $BX = 0$ is a linear combination of the equations in $AX = 0$. Since e^{-1} is an elementary row operation $(i.e., e^{-1}(B) = A)$, each equation in the system $AX = 0$ will also be a linear combination of equations in $BX = 0$. Hence these two systems are equivalent, and by Theorem 1, they have the same solutions.

Problem 1

Show that the following systems are row-equivalent.

$AX = 0$	$BX = 0$
$2x_1 - x_2 + 3x_3 + 2x_4 = 0$	$x_3 - \frac{11}{3}x_4 = 0$
$x_1 + 4x_2 - x_4 = 0$	$x_1 + \frac{17}{3}x_4 = 0$
$2x_1 + 6x_2 - x_3 + 5x_4 = 0$	$x_2 - \frac{5}{3}x_4 = 0$

Solution at page number 8(Hoffman and Kunz)

It's an assignment.

Note that solving the second system is easy !

Note

Let us consider the matrix B from the previous problem.

$$B = \begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \end{bmatrix}$$

- Note that the first non-zero entry of each non-zero row of B is 1.
- Note that each column of B which contains the leading non-zero entry of some row has all its other entries 0.

$$B = \begin{bmatrix} 0 & \color{red}{0} & 1 & -\frac{11}{3} \\ 1 & \color{red}{0} & 0 & \frac{17}{3} \\ 0 & \color{red}{1} & 0 & -\frac{5}{3} \end{bmatrix}$$

Row-reduced matrix

An $m \times n$ matrix R is called row-reduced if :

(a) the first non-zero entry of each non-zero row of R is 1;

(b) each column of R which contains the leading non-zero entry of some row has all its other entries 0.

Examples : (i) Identity matrix and (ii)the matrix B (previous problem)

Examples of non row-reduced matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 1

Find all solutions of the following system of equations by row-reducing the coefficient matrix.

$$\begin{aligned}
&\tfrac{1}{3}x_1 + 2x_2 - 6x_3 = 0\\
&-4x_1 \qquad\quad + 5x_3 = 0\\
&-3x_1 + 6x_2 - 13x_3 = 0\\
&-\tfrac{7}{3}x_1 + 2x_2 - \tfrac{8}{3}x_3 = 0
\end{aligned}$$

Solution: The coefficient matrix of the system is

$$\begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix}$$

Problem 1 contd.

$R_1 \longleftarrow 3R_1$, $R_4 \longleftarrow 3R_4$

$$\sim \begin{bmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -7 & 6 & -8 \end{bmatrix}$$

$R_2 \longleftarrow R_2 + 4R_1$, $R_3 \longleftarrow R_3 + 3R_1$, $R_4 \longleftarrow R_4 + 7R_1$

$$\sim \begin{bmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -67 \\ 0 & 48 & -134 \end{bmatrix}$$

$R_2 \longleftarrow \frac{1}{24} R_2$

$$\sim \begin{bmatrix} 1 & 6 & -18 \\ 0 & 1 & -\frac{67}{24} \\ 0 & 24 & -67 \\ 0 & 48 & -134 \end{bmatrix}$$

$R_1 \longleftarrow R_1 - 6R_2,\ R_3 \longleftarrow R_3 - 24R_2,\ R_4 \longleftarrow R_4 - 48R_2$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (a\ row - reduced\ matrix)$$

Thus

$$x_1 - \frac{5}{4}x_3 = 0$$
$$x_2 - \frac{67}{24}x_3 = 0$$

Problem 1 contd.

Let $x_3 = a$**.** $\implies x_1 = \frac{5}{4}a, x_2 = \frac{67}{24}a$
Solution set, $S = \{(\frac{5}{4}a, \frac{67}{24}a, a) \ : \ a \in \mathbf{R}\}$

Note that
(i) x_3 **is a called free variable and**
(ii) x_1, x_2 **are called pivot variables.**

Theorem 4

Every $m \times n$ matrix over the field F is row-equivalent to a row-reduced matrix.

Proof : (Assignment)

Problem 2

Find all solutions of the systems of linear equations $AX = 2X$ **and** $AX = 3X$ **where**

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$$

Solution : (i) The system $AX = 2X$ **is**

$$\begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Longrightarrow \qquad \left.\begin{aligned} 6x - 4y &= 2x \\ 4x - 2y &= 2y \\ -x + 3z &= 2z \end{aligned}\right\}$$

Problem 2 contd.

$$\Longrightarrow \qquad \left.\begin{aligned} 4x - 4y &= 0 \\ 4x - 4y &= 0 \\ -x + z &= 0 \end{aligned}\right\}$$

The coefficient matrix is

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let us find a row-reduced matrix which is row-equivalent to the above matrix. $R_3 \longleftarrow (-1)R_3$, $R_3 \longleftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 4 & -4 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

Problem 2 contd.

$R_2 \longleftarrow R_2 - 4R_1$ and $R_3 \longleftarrow R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & 4 \\ 0 & -4 & 4 \end{bmatrix}$$

$R_2 \longleftarrow -\frac{1}{4}R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{bmatrix}$$

$R_3 \longleftarrow R_3 + 4R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 2 contd.

The equivalent system is

$$\left.\begin{aligned} x - z &= 0 \\ y - z &= 0 \end{aligned}\right\}$$

Let $z = a$ (Note that z is a free variable). Thus $x = a = y$

(ii) Find all solutions of $AX = 3X$

The solution set is

$S = \left\{X \ \in \mathbf{R}^3 \ : \ AX = 3X\right\} = \{(0,0,a) \ : \ a \in \mathbf{R}\}.$

Note

$$A = \begin{bmatrix} 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. A is a row-reduced matrix.
2. All non-zero rows are above zero rows.
3. The k_i denotes the column which contains leading one (called pivot elements) (if exists) of R_i (row i).
   $k_1 = 2$, $k_2 = 4$, and $k_3 = 5$.
   Note that $k_1 < k_2 < k_3$.

Row-reduced echelon matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

blue zeros forms a staircase (echelon) from right to left.

An $m \times n$ **matrix** R **is called a** row-reduced echelon **matrix if:**

(a) R is row-reduced ;

(b) every row of R which has all its entries 0 occurs below every row which has a non-zero entry;

(c) if rows $1, 2, \ldots, r$ are the non-zero rows of R, and if the leading non-zero entry of row i occurs in column k_i, $i = 1, 2, \ldots, r$, then $k_1 < k_2 < \ldots < k_r$.

B is a row-reduced matrix, but not a row-reduced echelon matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Why?

$k_1 = 3, k_2 = 1, k_3 = 2$ **which violates the condition** (c)

Could you find a a row-reduced echelon matrix C **which is row-equivalent to** B**?** $(R_1 \longleftrightarrow R_2, R_2 \longleftrightarrow R_3)$

$$C = \begin{bmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(Is it unique?)}$$

Theorem 5

Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

Proof.

Assignment. $\square$

Problem 3

Solve the system of linear equations $AX = b$ where

$$A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{bmatrix} \textbf{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Note 1:

Consider a row-reduced echelon matrix R **and the system** $RX = 0$, where

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\}$$

No. of non-zero rows of R, $r = 2$, **No. of variables,** $n = 5$

$k_1 = 2, k_2 = 4 \Longrightarrow$ Pivot variables $= \{x_{k_1}, x_{k_2}\} = \{x_2, x_4\}$.

No. of free variables $= n - r = 5 - 2 = 3$,

Free variables $= \{u_1, u_2, u_3\} = \{x_1, x_3, x_5\}$.

Note 1 contd.

$$x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0$$
$$x_4 + 2x_5 = 0$$

Set the free variables as :

$u_1 = x_1 = a, \;\; u_2 = x_3 = b, \quad u_3 = x_5 = c$

$\implies \quad x_2 = 3b - \tfrac{1}{2}c, \quad x_4 = -2c$

Solution set $S = \left\{(a, 3b - \tfrac{1}{2}c, b, -2c, c) \;:\; a, b, c \in \mathbf{R}\right\}$

Observations from Note 1

$$\left.\begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} x_{k_1} + \sum_{j=1}^{n-r} C_{1j}u_j = 0 \\ x_{k_2} + \sum_{j=1}^{n-r} C_{2j}u_j = 0 \end{aligned}\right\} \textit{general expression})$$

Note 2

Consider an $m \times n$ row-reduced echelon matrix R with r non-zero rows. Let rows $1, 2, \ldots, r$ be the non-zero rows of R, and suppose that the leading non-zero entry of row i occurs in column k_i. The system $RX = 0$ has r (non-trivial) equations. Let x_{k_i}s are the pivot variables. Let $u_1, u_2, \ldots, u_{n-r}$ denote the $(n-r)$ unknowns which are different from $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$. Then r non-trivial equations of $RX = 0$ are of the form

Note 2 contd.

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0$$

$$x_{k_2} + \sum_{j=1}^{n-r} C_{2j} u_j = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0$$

All the solutions of the system of equations $RX = 0$ are obtained by assigning any value whatsoever to $u_1, u_2, \ldots, u_{n-r}$, and then computing the corresponding values of $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$.

Thus, we have

(i) **If $n > r$, then the system $RX = 0$ has at least one free variable and thus it has a non-trivial solution.**

(ii) **If $n = r$, then the system $RX = 0$ has no free variable and thus it has only trivial solution.**

Theorem 6

If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

Proof: Let R be a row-reduced echelon matrix which is row-equivalent to A. Then the systems $AX = 0$ and $RX = 0$ have same solutions by Theorem 3. If r is the number of non-zero rows of R, then $r \leq m$. Since $m < n$, we have $r < n$. Thus the system $RX = 0$ has $n - r(\geq 1)$ free variables and it admits a non-trivial solution. Hence $AX = 0$ has a non-trivial solution.

Note

If R is an $n \times n$ (square) row-reduced echelon matrix with n non-zero rows, then $R = I$ (the identity matrix).

Because : (i) Every row has a leading one and (ii) $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$**.**

Theorem 7

If A is an $n \times n$ (square) matrix, then A is row equivalent to the $n \times n$ identity matrix if and only if the system of equations $AX = 0$ has only the trivial solution.

Proof.

Case 1. Suppose that A is row-equivalent to the $n \times n$ identity matrix I. By Theorem 3, $AX = 0$ and $IX = 0$ have the same solution set. Thus the solution set of $AX = 0$ is

$$S = \{X \ : \ AX = 0\} = \{X \ : \ IX = 0\} = \{X \ : \ X = 0\} = \{0\}$$

Hence the system $AX = 0$ has only the trivial solution.

Proof of Theorem 7 contd.

Case 2. Suppose that the system $AX = 0$ has only the trivial solution. Prove that A is row-equivalent to the $n \times n$ identity matrix.

Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A. By Theorem 3, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions. Since $AX = 0$ has only the trivial solution, $RX = 0$ has only the trivial solution. Hence the system $RX = 0$ has no free variables. Thus the number of free variables (of the system $RX = 0$), $n - r = 0$ where r is the number of non-zero rows of R. So R is an $n \times n$ row-reduced echelon matrix with $n(= r)$ non-zero rows and thus $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$. This proves that $R = I$, an identity matrix.

Hence A is row-equivalent to $R = I$.

Theorem 8

Let $AX = b$ be a given system of equations, where A is an $m \times n$ matrix, X is an $n \times 1$ vector and b is an $m \times 1$ vector. Let $RX = f$ be the row-reduced-echelon form of the system $AX = b$. Let there are r non-zero rows in the augmented matrix $[R|f]$. Then the following are true.

1. No solution. If $r < m$ (meaning that R actually has at least one row of all 0s) and at least one of the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ is not zero, then the system $AX = b$ is inconsistent, i.e, no solution is possible.
2. Unique solution. If the system $AX = b$ is consistent and $r = n$, then the system $AX = b$ has a unique solution.
3. Infinitely many solutions. If the system $AX = b$ is consistent and $r < n$, then the system $AX = b$ has infinitely many solutions.

Reading assignment

Section 1.5 Matrix multiplication

Assignment

Solve all exercise problems in section 1.4 (pages 15-16)

Elementary matrices

An $m \times m$ matrix is said to be an elementary matrix if it can be obtained from the $m \times m$ identity matrix by means of a single elementary row operation.

Example :

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (e : R_1 \longleftarrow cR_1, \quad c \neq 0)$$

$$E = e(I) = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \quad (\text{ E is an elementary matrix})$$

Find all 2×2 elementary matrices

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \quad (Using \quad Type \ \ 1, \quad c \neq 0)$$

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \quad (Using \quad Type \ \ 2)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (Using \quad Type \ \ 3)$$

Find all 3×3 elementary matrices. (Assignment)

Properties of elementary matrices

Type 1

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left(e : R_1 \longleftarrow cR_1,\ c \neq 0, e_1 : R_1 \longleftarrow \frac{1}{c}R_1\right)$$

$$E = e(I) = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = e_1(I) = \begin{bmatrix} \frac{1}{c} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EE_1 = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{c} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly (verify), $E_1 E = I = EE_1$

Properties of elementary matrices

$$\text{Let} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad (e : R_1 \longleftarrow cR_1,\ c \neq 0)$$

$$e(A) = \begin{bmatrix} cA_{11} & cA_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} cA_{11} & cA_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = e(A)$$

$$e(I)A = e(A)$$

Properties of elementary matrices

Type 2

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (e : R_1 \longleftarrow R_1 + cR_2, e_1 : R_1 \longleftarrow R_1 - cR_2)$$

$$E = e(I) = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = e_1(I) = \begin{bmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EE_1 = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly (verify), $E_1 E = I = EE_1$

Properties of elementary matrices

Let $$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad (e : R_1 \longleftarrow R_1 + cR_2)$$

$$e(A) = \begin{bmatrix} A_{11} + cA_{21} & A_{12} + cA_{22} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} A_{11} + cA_{21} & A_{12} + cA_{22} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = e(A)$$

Properties of elementary matrices

Type 3

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (e : R_1 \longleftrightarrow R_2, e_1 : R_1 \longleftrightarrow R_2)$$

$$E = e(I) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = e_1(I) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EE_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly (verify), $E_1 E = I = EE_1$

Properties of elementary matrices

$$\text{Let} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad (e : R_1 \longleftrightarrow R_2)$$

$$e(A) = \begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = e(A)$$

Theorem 9

Let e be an elementary row operation and I be the $m \times m$ identity matrix. Then for every $m \times n$ matrix A,

$$e(I)A = e(A)$$

Proof: Assignment

Note: For every elementary row operation e, there exists an inverse elementary operation of the same type e_1 such that

$$e(I)e_1(I) = I = e_1(I)e(I) \qquad (EE_1 = I = E_1E)$$

Corollary (to Theorem 9)

Let A and B be $m \times n$ matrices over the field F. Then B is row-equivalent to A if and only if $B = PA$ where P is a product of $m \times m$ elementary matrices.

Proof:

Case 1: Suppose that B is row-equivalent to A.

Then B can be obtained from A by a finite sequence of elementary row operations, say

$$A = A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \ldots \longrightarrow A_{k-1} \longrightarrow A_k = B$$

where $e_i(A_{i-1}) = A_i$, e_i is an elementary row operation for $1 \leq i \leq k$.

Note that $e_i(A_{i-1}) = e_i(I)A_{i-1}$, by Theorem 9 and $e_i(I)$ is an $m \times m$ elementary matrix.

Corollary contd.

Clearly, $A_1 = e_1(A) = e_1(I)A$ **,** $A_2 = e_2(A_1) = e_2(I)A_1$
$\implies A_2 = e_2(I)e_1(I)A$
Using similar arguments,

$$B = A_k = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(I)A = PA$$

where $P = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(I)$ is a product of $m \times m$ elementary matrices.

Case 2 : Suppose that $B = PA$**, where** P **is a product of** $m \times m$ **elementary matrices.**
Let $P = E_kE_{k-1}\dots E_2E_1$ **where** E_i **is an** $m \times m$ **elementary matrix for** $1 \leq i \leq k$**. Since** E_i **is an elementary matrix, there exists an elementary row operation** e_i **such that** $E_i = e_i(I)$**.**

$$B = PA = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(I)A$$

Corollary contd.

$$B = PA = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(I)A$$

$$B = PA = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(A)$$

$$B = PA = e_k(I)e_{k-1}(I)\dots e_2(e_1(A))$$

$$\dots\dots\dots\dots\dots\dots$$

$$B = PA = e_k(e_{k-1}(\dots e_2(e_1(A))))$$

Hence B can be obtained from A by a finite sequence of elementary row operations $e_1, e_2, \dots, e_k$. Then B is row-equivalent to A.

Problem

Show that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 8 & 10 \end{bmatrix}$ **are row-equivalent. Find a** 3×3 **matrix** P **such that** $B = PA$.

Solution : Let $e_1 \; : \; R_1 \longleftrightarrow R_2$ **and** $e_2 \; : \; R_3 \longleftarrow R_3 + R_1$

Clearly, $B = e_2(e_1(A)) = e_2(e_1(I)A) = e_2(I)e_1(I)A = PA$

$$P = e_2(I)e_1(I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Definition

Let A be an $n \times n$ matrix over the field F. An $n \times n$ matrix B such that $BA = I$ is called a left inverse of A; an $n \times n$ matrix B such that $AB = I$ is called a right inverse of A.
If $AB = I = BA$, then B is called a two-sided inverse of A or simply the inverse of A and A is said to be invertible.

Note: If A is an invertible matrix, then A has no zero row.

Lemma

If A has a left inverse B and a right inverse C, then $B = C$.

Proof Suppose that $BA = I$ and $AC = I$.

$$B = BI = B(AC) = (BA)C = IC = C$$

Note: If A has a left inverse and a right inverse, then A is invertible and the inverse of A is denoted by A^{-1}.

Theorem 10

(i). If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(ii). If both A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

$$(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB).$$

Note: Product of invertible matrices is invertible.

Theorem 11: An elementary matrix is invertible.

Proof : Let E be an $m \times m$ elementary matrix corresponding to the elementary row operation e. Thus $E = e(I)$. By Theorem 2, there exists an elementary row operation e_1, same type as e, such that

$$e(e_1(A)) = A = e_1(e(A)) \text{ for every matrix } A.$$

Let $E_1 = e_1(I)$ where I is the $m \times m$ identity matrix. Then

$$EE_1 = e(I)E_1 = e(E_1) = e(e_1(I)) = I$$

and

$$E_1E = e_1(I)E = e_1(E) = e_1(e(I)) = I.$$

$EE_1 = I = E_1E$. Hence E is an invertible matrix.

Thus an elementary matrix is invertible.

Find inverses of all 2×2 elementary matrices

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{c} \end{bmatrix}$$

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find inverses of all 3×3 elementary matrices. (Assignment)

Theorem 12

If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

Proof: Let R be a row-reduced echelon $n \times n$ matrix which is row-equivalent to A. By Corollary to Theorem 9,

$$R = E_k E_{k-1} \dots E_2 E_1 A \qquad ------ \quad (a)$$

where E_i is an elementary matrix. Note that the inverse of E_i is also an elementary matrix. Since $E_i' s$ are invertible,

$$A = E_1^{-1} E_2^{-1} \dots E_k^{-1} R \quad ---- \; (b)$$

Theorem 12 contd.

$(i) \Longrightarrow (ii)$ Suppose that A is invertible. Using (a), R is a product of invertible matrices and by corolllary to Theorem 10, R is invertible. Note that an invertible matrix has no zero-row. So R is an $n \times n$ row-reduced echelon matrix with no zero row and $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$. Hence R is the $n \times n$ identity matrix. A is row-equivalent to $R = I$.

$(ii) \Longrightarrow (iii)$ Suppose that A is row-equivalent to $R = I$. By (b)

$$A = E_1^{-1} E_2^{-1} \ldots E_k^{-1} R = E_1^{-1} E_2^{-1} \ldots E_k^{-1} I$$

$$A = E_1^{-1} E_2^{-1} \ldots E_k^{-1}, \text{a product of elementary matrices.}$$

$(iii) \implies (i)$ **Suppose that A is a product of elementary matrices.** By Theorem 11, an elementary matrix is invertible. By Corollary to Theorem 10, a product of invertible matrices is invertible. Hence A is invertible.

Corollaries (to Theorem 12)

Let A be an $n \times n$ matrix. Consider the augmented matrix $[A|I]$. Suppose that

$$[A|I] \sim [I|B].$$

Note that A is row equivalent to I (thus A is invertible) and I is row equivalent to B. By Corollary to Theorem 9, there exists an $n \times n$ matrix P such that $I = PA \; \textit{and} \; B = PI \implies B = P \; \textit{and} \; I = BA$ $\implies$ **A is invertible and $B = A^{-1}$.**

Corollary 12.1.

If A is an $n \times n$ matrix and if a sequence of elementary row operations reduces A to the indentity matrix, then that same sequence of operations when applied to I yields A^{-1}.

Problem

Find the inverse of

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

Solution : Consider

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \longleftarrow R_2 - \frac{1}{2}R_1, \quad R_3 \longleftarrow R_3 - \frac{1}{3}R_1$$

Solution contd.

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right]$$

$$R_2 \longleftarrow 12R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right]$$

$$R_1 \longleftarrow R_1 - \frac{1}{2}R_2, \quad R_3 \longleftarrow R_3 - \frac{1}{12}R_2$$

Solution contd.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right]$$

$$R_3 \longleftarrow 180R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$$

$$R_2 \longleftarrow R_2 - R_3, \quad R_1 \longleftarrow R_1 + \frac{1}{6}R_3$$

Solution contd.

$$[A|I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right] = [I|B]$$

By Corollary 12.1,

$$A^{-1} = B = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

Problem 2

Find the inverse of

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix}$$

Solution :

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 6 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

Take $R_3 \longleftarrow R_3 - R_1$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 4 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

Solution contd.

Now, take $R_3 \longleftarrow R_3 - R_2$. Then

$$[A|I] \sim \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

Thanks to the last zero row, A is not row-equivalent to I and A is not invertible.

Theorem 13

For an $n \times n$ matrix A the following are equivalent.

(i) A is invertible.

(ii) The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.

(iii) The system of equations $AX = B$ has a solution X for each $n \times 1$ matrix B.

Proof: $(i) \Longrightarrow (ii)$ Suppose that A is invertible. By Theorem 12, A is row-equivalent to I. By Theorem 3, $AX = 0$ and $IX = 0$ have exactly same solutions. The solution set of $AX = 0$ is

$$S = \{X \ : \ AX = 0\} = \{X \ : \ IX = 0\} = \{X \ : \ X = 0\} = \{0\}.$$

Hence the system $AX = 0$ has only the trivial solution $X = 0$.

Theorem 13 contd.

$(ii) \Longrightarrow (i)$ **Suppose that $AX = 0$ has only the trivial solution $X = 0$.** By Theorem 7, A is row-equivalent to I. By Theorem 12, A is invertible.

$(i) \Longrightarrow (iii)$ Suppose that A is invertible. That is A^{-1} exists. Consider the system $AX = B$. This implies that $X = A^{-1}B$ is a solution for the system $AX = B$ for each B.

Theorem 13 contd.

$(iii) \implies (i)$ Suppose that the system of equations $AX = B$ has a solution X for each $n \times 1$ matrix B. Let R be a row-reduced echelon matrix which is row-equivalent to A. By an corollary of Theorem 9, $R = PA$, where P is a product of elementary matrices. Since elementary matrices are invertible, so is their product P.

$AX = B$ has a solution X for each B.
$\iff$ $P(AX) = PB$ has a solution X for each B.
$\iff$ $RX = PB$ has a solution X for each B (Note that $R = PA$).
$\iff$ $RX = E$ has a solution X for each E $(= PB)$.

Theoem 13 contd.

Now, take

$$E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$RX = E$ has a solution X.

$\implies$ The last row of R is non-zero.

$\implies$ R is an $n \times n$ row-reduced echelon matrix with no zero rows.

$\implies$ $R = I$.

Hence A is row-equivalent to $R = I$.

By Theorem 12, A is invertible.

Corollary 13.1

A square matrix with either a left or right inverse is invertible.

Proof: Let A be an $n \times n$ matrix.

Case 1. Suppose A has a left inverse, say B. That is $BA = I$.

Consider the system $AX = 0$**. That implies** $B(AX) = B0 = 0$**.**

$\implies (BA)X = 0. \implies \quad IX = 0. \implies \; X = 0.$

$$\textbf{Thus} \quad AX = 0 \implies X = 0$$

By Theorem 13, A **is invertible.**

Case 2 : **Suppose that** A **has a right inverse say** C**. i.e.,** $AC = I$**. So** A **is a left inverse of** C**. By Case 1,** C **is invertible and** $C^{-1} = A$**. Hence** A **is invertible and** $A^{-1} = C$**.**

Corollary 13.2

Let $A = A_1 A_2 \dots A_k$ where $A_1, A_2, \dots, A_k$ are $n \times n$ (square) matrices. If A is invertible then each A_i is invertible.

Proof : $A = A_1 A_2 \dots A_k \quad - - - (a)$

Suppose that A is invertible. By Theorem 13, $AX = 0 \implies X = 0$. We want to show that each A_i is invertible. First, we prove that A_k is invertible. Consider the system $A_k X = 0$.

$\implies \quad A_1 A_2 \dots A_{k-1}(A_k X) = 0. \implies \quad AX = 0. \implies X = 0.$

$$\text{Thus } A_k X = 0 \implies X = 0$$

By Theorem 13, A_k is invertible. Since A and A_k are invertible, $AA_k^{-1} = A_1 A_2 \dots A_{k-1}$ is invertible. By preceeding argument, A_{k-1} is invertible. Continuing in this way, we conclude that each A_i is invertible.

Problem 1

Question. Prove or disprove that if A is an $m \times n$ matrix, B is an $n \times m$ matrix and $n < m$, then AB is not invertible.

Solution: Since B is an $n \times m$ matrix and $n < m$, by Theorem 6, the homogeneous system $BX = 0$ has a non-trivial solution, say $X^* \neq 0$.
i.e.

$$BX^* = 0.$$

Consider the system $(AB)X = 0$.

$$(AB)X^* = A(BX^*) = A0 = 0.$$

$\Longrightarrow X^*$ is a non-trivial solution of the homogeneous system $(AB)X = 0$. By Theorem 13, AB is not invertible.

Problem 2

$$\text{Let} \quad A = \begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix}$$

Does there exist a 3×4 matrix B such that (i) $AB = 0$ and (ii) $B \neq 0$?

Solution of Problem 2

Find a non-trivial solution of the sytem $AX = 0$. Solution set, $S = \{(\frac{5}{4}a, \frac{67}{24}a, a) \ : \ a \in \mathbf{R}\}$ (Visit previous lecture notes.)
Choose $a = 24 \implies (30, 67, 24)$ is a solution.

$$\implies \quad B = \begin{bmatrix} 30 & 30 & 30 & 30 \\ 67 & 67 & 67 & 67 \\ 24 & 24 & 24 & 24 \end{bmatrix}$$

Verify that $AB = 0, \quad and \quad B \neq 0$

Problem 3

Prove of disprove that A is invertible and find A^{-1} if it exists where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Solution to Problem 3

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$