Electrical Circuits for Engineers (EC1000)

Lecture 06 (a)
First-Order Circuits
Source Free (R-L & R-C)
(Chapter 7)

Introduction

we have studied the three passive elements (resistors,capacitors, and inductors) individually.

- We shall examine two types of simple circuits: a circuit comprising a resistor (R) and capacitor (C) and a circuit comprising a resistor (R) and an inductor (L).
- These are called *RC* and *RL* circuits, respectively.
- As simple as these circuits are, they find applications in electronics, communications, and control systems.

- Analysis of *RC* and *RL* circuits by Kirchhoff's laws, as we did for resistive circuits. Applying Kirchhoff's laws to purely resistive circuits results in algebraic equations,
- Applying the laws to *RC* and *RL* circuits produces differential equations, which are difficult to solve than algebraic equations

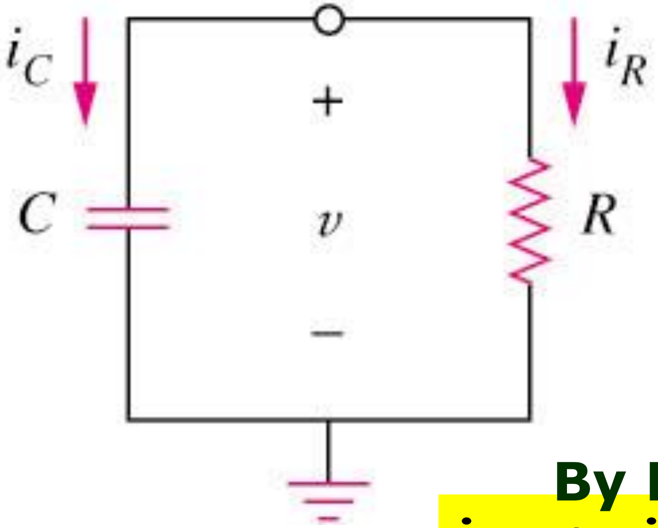
The differential equations resulting from analyzing *RC* and *RL* circuits are of the First order

7.1 The Source-Free RC Circuit (1)

A **first-order circuit** is characterized by a first-order differential equation.

Two types of first-order circuits (*RC* and *RL*), there are two ways to excite the circuits.

1. By initial conditions of the storage elements in the circuits. In these so-called *source-free circuits (i.e.* energy is initially stored in the capacitive or inductive element).
2. Exciting first-order circuits is by independent sources.



A source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Our objective is to determine the circuit response

By KCL

$$i_R + i_C = 0 \rightarrow \frac{v}{R} + C\frac{dv}{dt} = 0$$

Ohms law

Capacitor law

7.1 The Source-Free RC Circuit (1)

we assume to be the voltage $v(t)$ across the capacitor. Since the capacitor is initially charged, we can assume that at time t-0, the initial voltage is

$$v(0) = V_0$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2} C V_0^2$$

Applying KCL at the top node of the circuit in Fig. '

$$i_C + i_R = 0$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

This is a *first-order differential equation*, since only the first derivative of v is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

where $\ln A$ is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC}$$

Taking powers of e produces

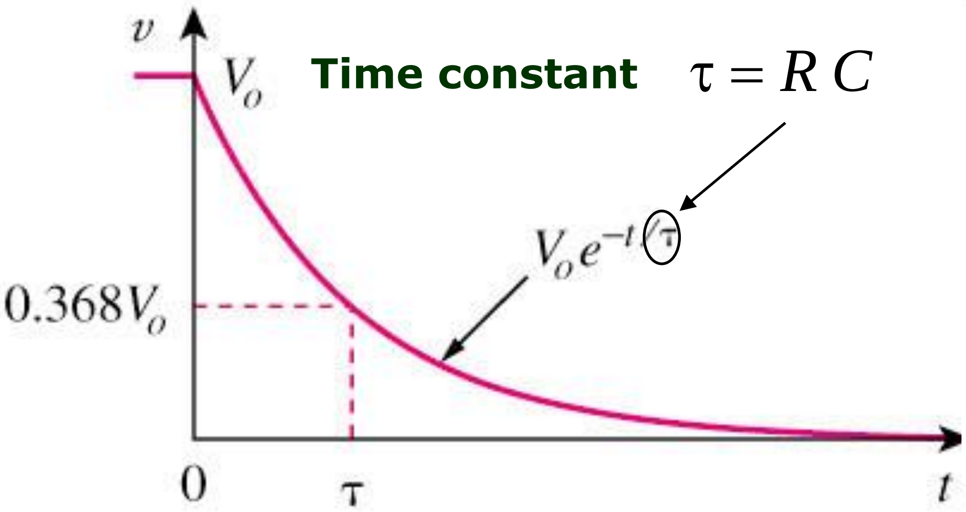
$$v(t) = Ae^{-t/RC}$$

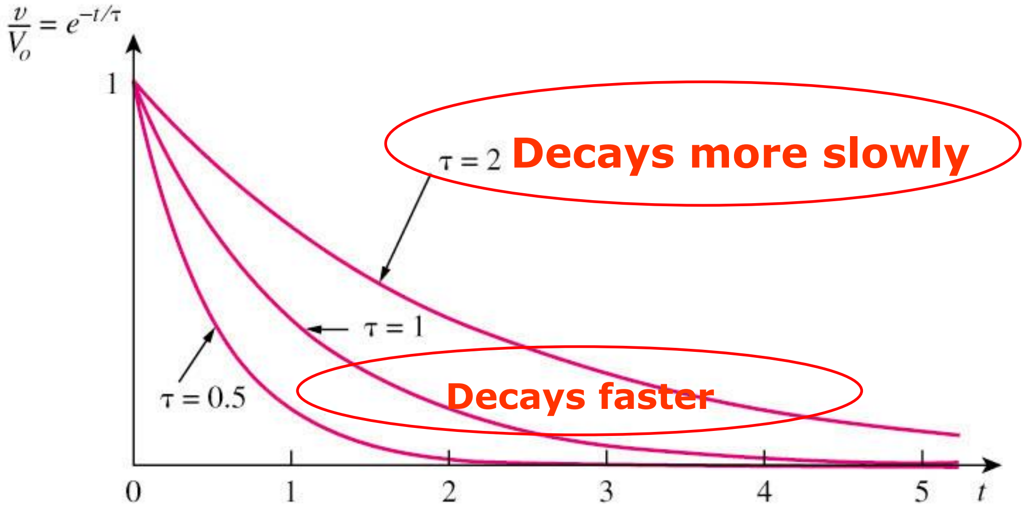
But from the initial conditions, $v(0) = A = V_0$. Hence,

$$v(t) = V_0 e^{-t/RC}$$

This shows that the voltage response of the *RC* circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the *Natural response* of the circuit.

The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with <u>no external sources of excitation</u>.





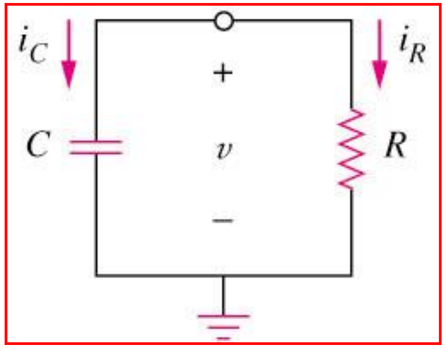
at $t = \tau$,

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$$

- The **<u>time constant</u>** τ of a circuit is the time required for the response to decay by a factor of <u>1/*e* or 36.8%</u> of its initial value.
- v decays faster for small τ and slower for large τ.

The key to working with a source-free RC circuit is finding:

$$v(t) = V_0\, e^{-t/\tau}$$ where $$\tau = RC$$



1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ = RC.

For Capacitor C

$$v_C(t) = v(t) = v(0)e^{-t/\tau}$$

Once the capacitor voltage is first obtained, other variables (capacitor current i_C, resistor voltage V_R and resistor current i_R) can be determined.

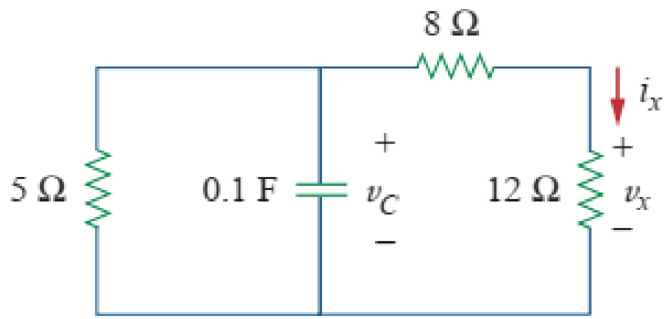
For Resistor R

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R}e^{-t/\tau} \qquad p(t) = v i_R = \frac{V_0^2}{R}e^{-2t/\tau} \qquad w_R(\infty) \to \tfrac{1}{2}CV_0^2$$

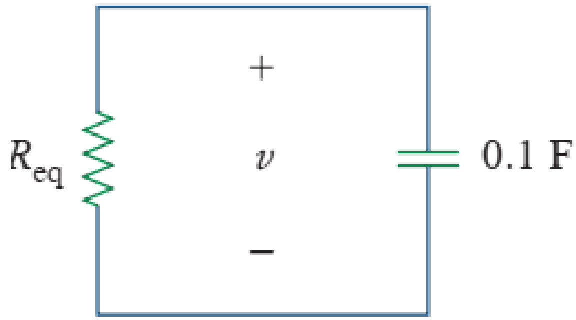
The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

Example Problem 1

let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.



$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$



$$\tau = R_{eq}C = 4(0.1) = 0.4 \text{ s}$$

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\text{ V}, \qquad v_C = v = 15e^{-2.5t}\text{ V}$$

voltage division to get v_x; so

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\text{ V} \qquad i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\text{ A}$$

Example Problem 2

The switch in the circuit in Fig. 7.8 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

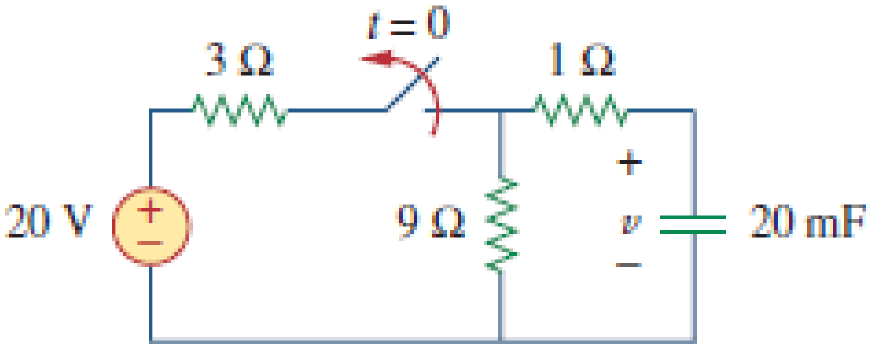
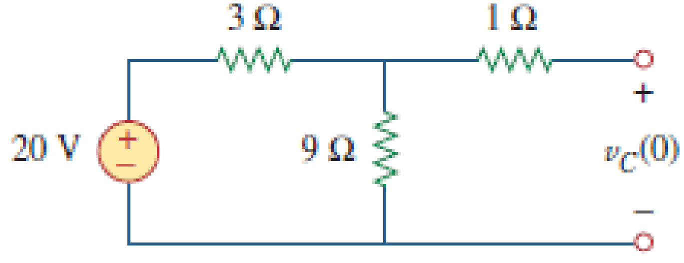


Figure 7.8



(a)

Solution:

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in Fig. 7.9(a). Using voltage division

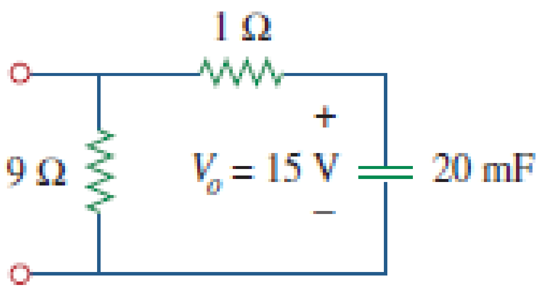
$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15 \text{ V}$$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig. 7.9(b). [Notice that the RC circuit in Fig. 7.9(b) is source free; the independent source in Fig. 7.8 is needed to provide V_0 or the initial energy in the capacitor.] The 1-Ω and 9-Ω resistors in series give

$$R_{eq} = 1 + 9 = 10\ \Omega$$



(b)

The time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2 \text{ s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} \text{ V}$$

or

$$v(t) = 15e^{-5t} \text{ V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$$

Practice Problem 1

Refer to the circuit in Fig. 7.7. Let $v_C(0) = 60$ V. Determine v_C, v_x, and i_o for $t \geq 0$.

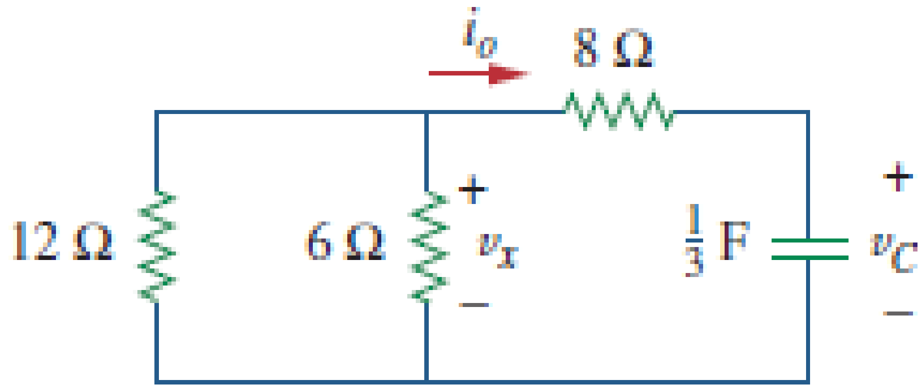


Figure 7.7

Answer: $60e^{-0.25t}$ V, $20e^{-0.25t}$ V, $-5e^{-0.25t}$ A

Practice Problem 2

If the switch in Fig. 7.10 opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

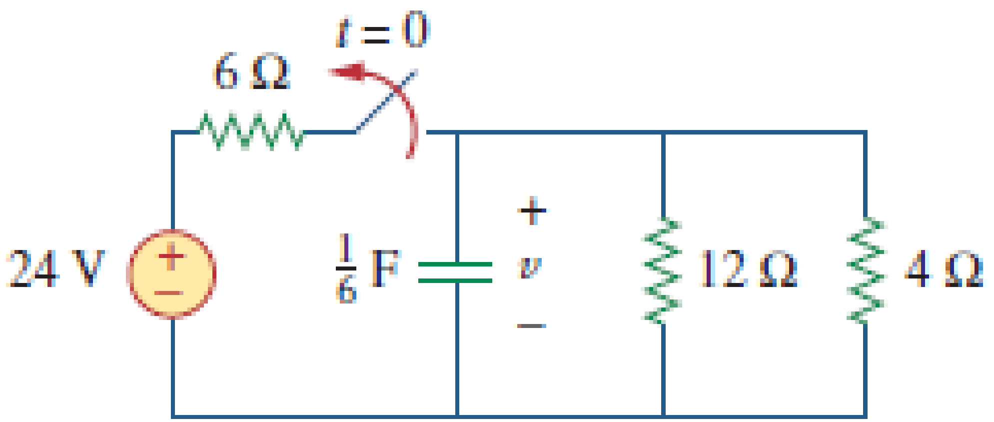
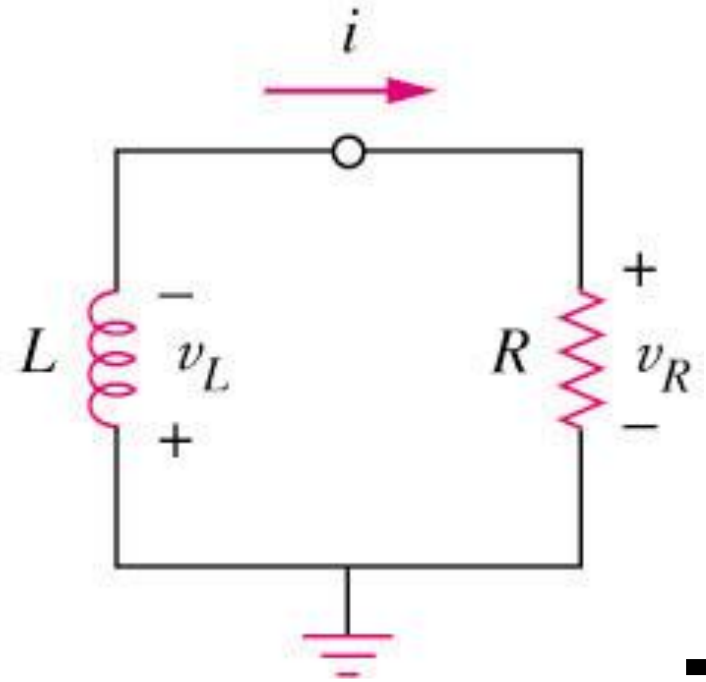


Figure 7.10

Answer: $8e^{-2t}$ V, 5.333 J.

7.2 The Source-Free RL Circuit

A **first-order RL circuit** consists of a inductor L (or its equivalent) and a resistor (or its equivalent)



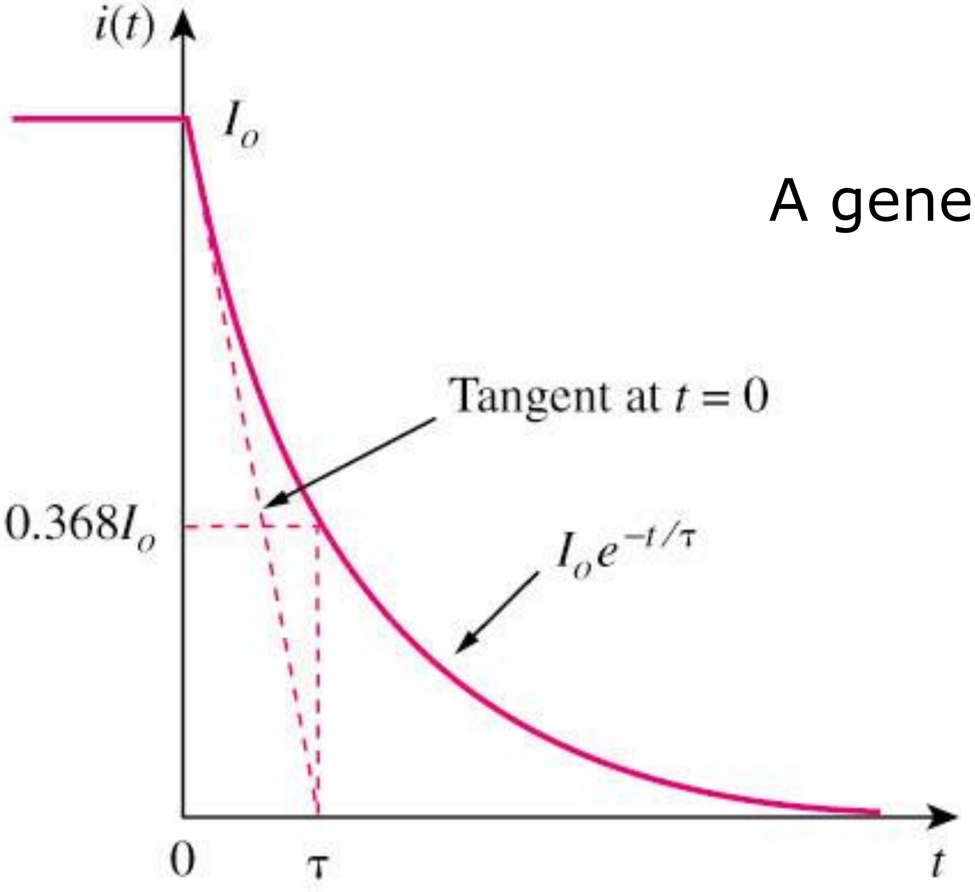
By KVL $$v_L + v_R = 0$$

$$\rightarrow \quad L\frac{di}{dt} + iR = 0$$

Inductors law

Ohms law

$$\rightarrow \quad \frac{di}{i} = -\frac{R}{L}dt \quad \rightarrow \quad i(t) = I_0\, e^{-Rt/L}$$



A general form representing a RL

$$i(t) = I_0\, e^{-t/\tau}$$

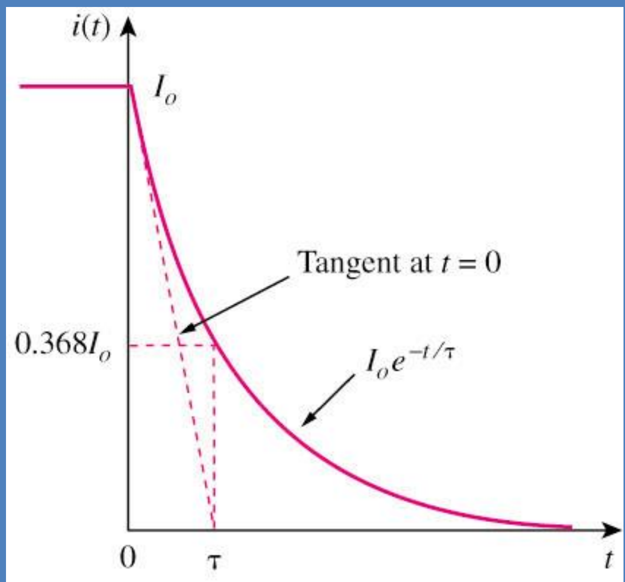
where $$\tau = \frac{L}{R}$$

- The **time constant** τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- $i(t)$ decays faster for small τ and slower for large τ.
- The general form is very similar to a RC source-free circuit.

Comparison between a RL and RC circuit

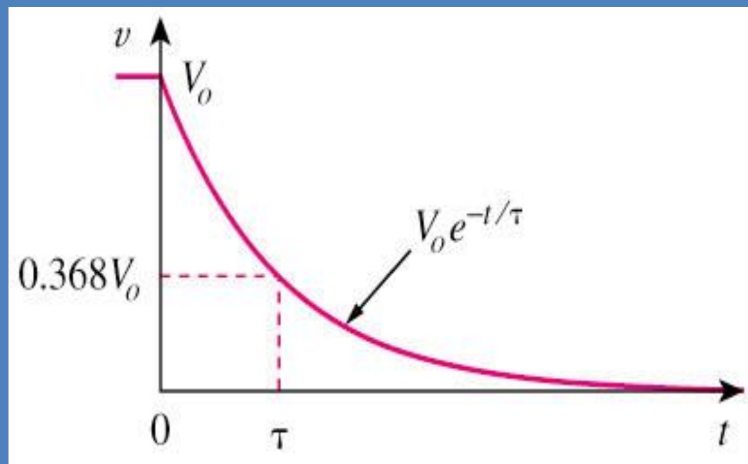
A RL source-free circuit

$i(t) = I_0\, e^{-t/\tau}$ **where** $\tau = \dfrac{L}{R}$



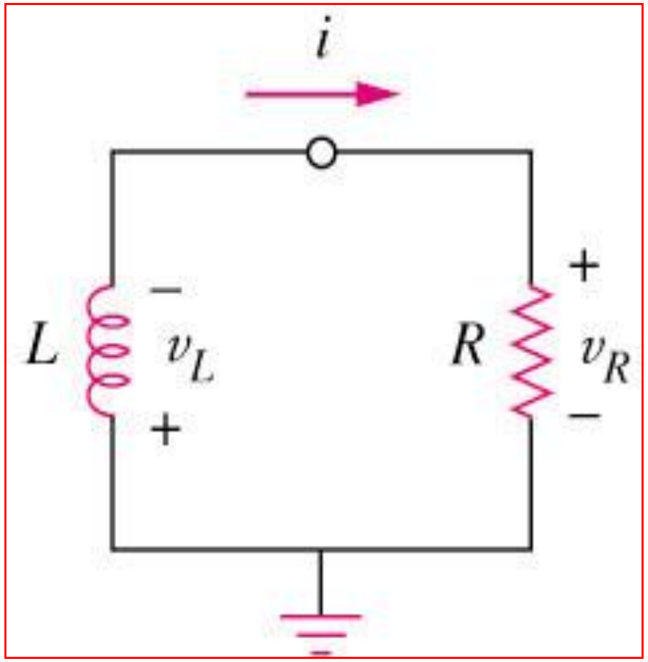
A RC source-free circuit

$v(t) = V_0\, e^{-t/\tau}$ **where** $\tau = RC$



The key to working with a source-free RL circuit is finding:

$$i(t) = I_0\, e^{-t/\tau} \quad \text{where} \quad \tau = \frac{L}{R}$$



The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

The energy absorbed by the resistor is

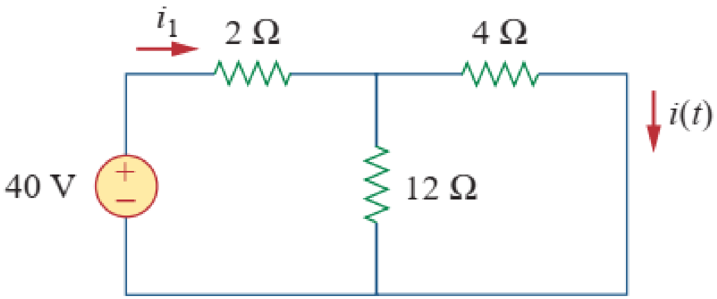
$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2 R e^{-2t/\tau}\,dt = -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

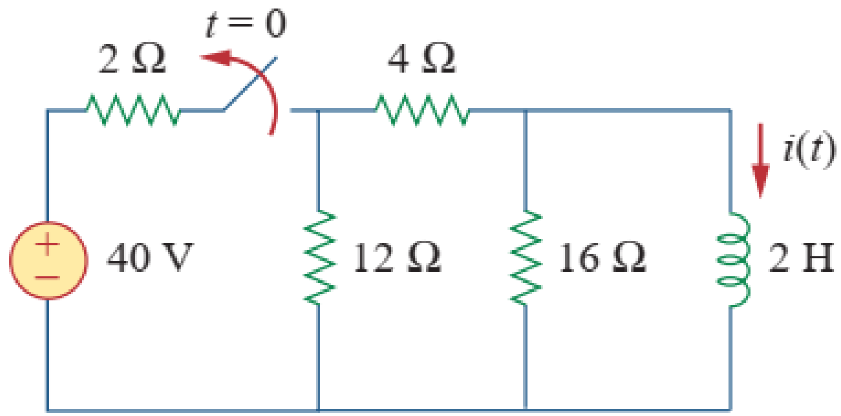
1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant $\tau = L/R$.

Example Problem.3

When $t < 0$, the switch is closed, At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

$\boxed{t < 0}$



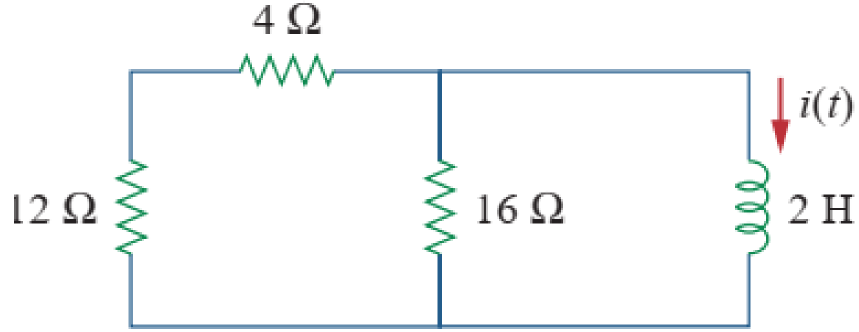


$$i_1 = \frac{40}{2 + 3} = 8 \text{ A} \qquad i(t) = \frac{12}{12 + 4} i_1 = 6 \text{ A}, \qquad t < 0$$

Since the current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = 6 \text{ A}$$

$\boxed{t > 0.}$



$$R_{eq} = (12 + 4) \parallel 16 = 8 \ \Omega$$

The time constant is

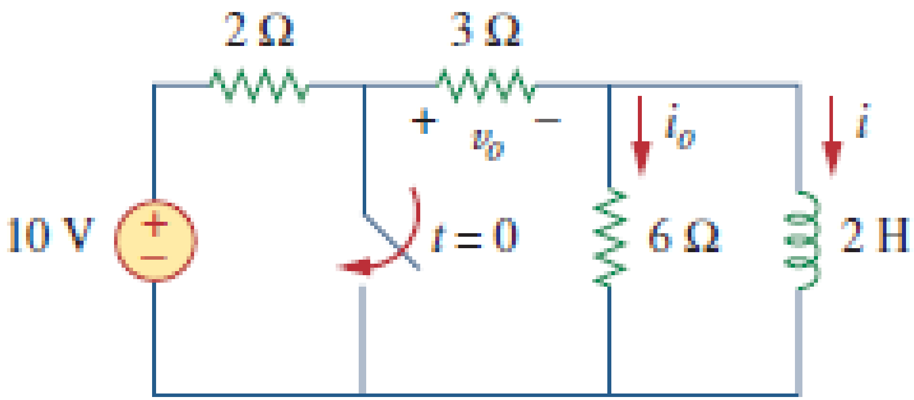
$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4} \text{ s}$$

Thus,

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t} \text{ A}$$

Example Problem.2

In the circuit shown in Fig. 7.19, find i_o, v_o, and i for all time, assuming that the switch was open for a long time.



Solution:

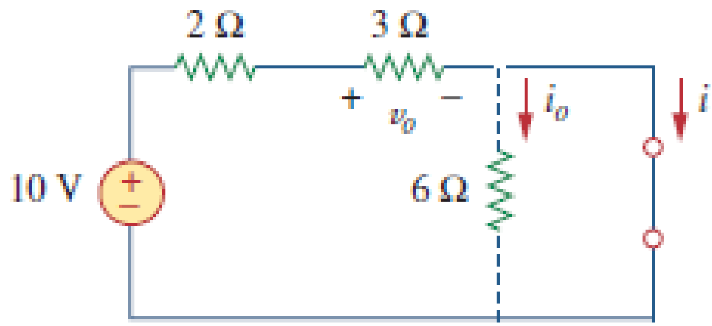
It is better to first find the inductor current i and then obtain other quantities from it.

For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, so that we have the circuit shown in Fig. 7.20(a). Hence, $i_o = 0$, and

$$i(t) = \frac{10}{2+3} = 2\text{ A}, \qquad t < 0$$

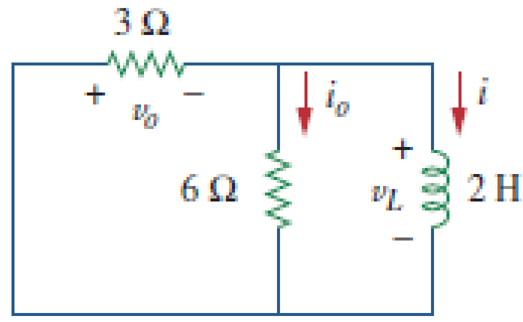
$$v_o(t) = 3i(t) = 6\text{ V}, \qquad t < 0$$

Thus, $i(0) = 2$.



(a)

For $t > 0$, the switch is closed, so that the voltage source is short-circuited. We now have a source-free RL circuit as shown in Fig. 7.20(b). At the inductor terminals,



(b)

$$R_{\mathrm{Th}} = 3 \parallel 6 = 2\ \Omega$$

so that the time constant is

$$\tau = \frac{L}{R_{\mathrm{Th}}} = 1\ \mathrm{s}$$

Hence,

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \mathrm{A}, \qquad t > 0$$

Since the inductor is in parallel with the 6-Ω and 3-Ω resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ \mathrm{V}, \qquad t > 0$$

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\ \mathrm{A}, \qquad t > 0$$

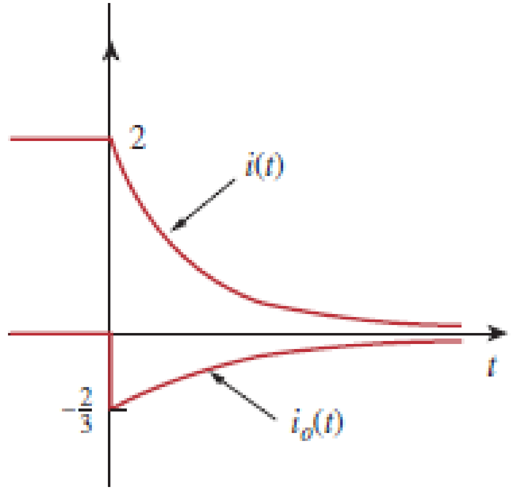


Figure 7.21
A plot of i and i_o.

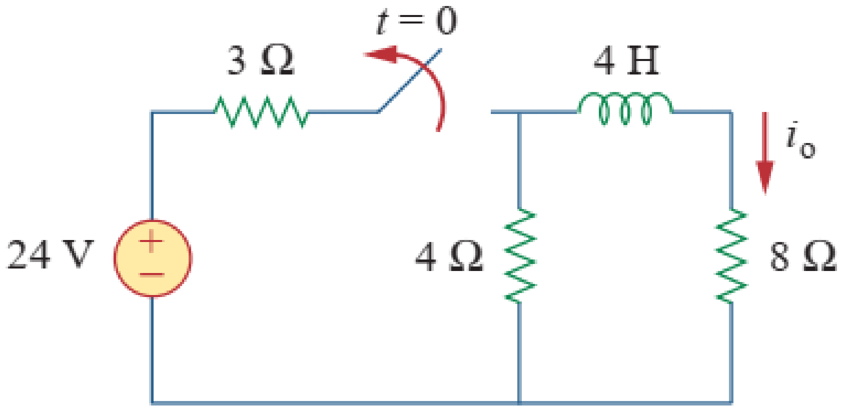
Thus, for all time,

$$i_o(t) = \begin{cases} 0\ \mathrm{A}, & t < 0 \\ -\frac{2}{3}e^{-t}\ \mathrm{A}, & t > 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6\ \mathrm{V}, & t < 0 \\ 4e^{-t}\ \mathrm{V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\ \mathrm{A}, & t < 0 \\ 2e^{-t}\ \mathrm{A}, & t \ge 0 \end{cases}$$

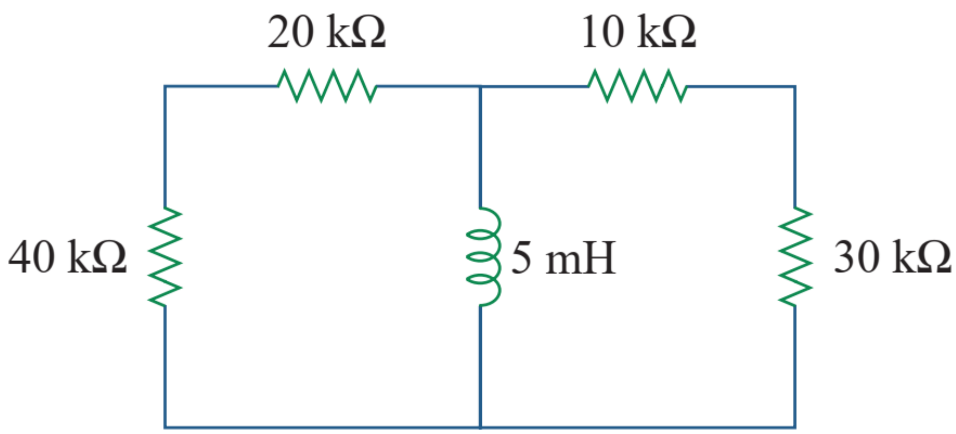
Practice Problem

find i_o for $t > 0$.



$$i_o(t) = i_o(0)e^{-t/\tau} = \underline{1.4118e^{-3t}\ \text{A}}$$

Calculate the time constant of the circuit



$\underline{0.2083\,\mu s}$