Tutorial : Differential equations

Module 3: Power series techniques and special functions

1.) Find Ordinary and singular (regular/ irregular) points of the following differential equations.

a) $x^2(x^2-1)^2 y'' - x(1-x)y' + 2y = 0$.

given differential equation in standard form

$$y'' + \frac{1}{x(x-1)(x+1)^2} y' + \frac{2}{x^2(x^2-1)^2} y = 0$$

$$P(x) = \frac{1}{x(x-1)(x+1)^2} \text{ and } Q(x) = \frac{2}{x^2(x^2-1)^2}$$

$P(x)$ and $Q(x)$ are not analytic at $x = 0, -1, 1$

$\therefore \{0, -1, 1\}$ are singular points for given differential equation

To find if they are regular or irregular

1) $\underline{x=0}$ $\quad x P(x)$ and $x^2 Q(x)$ are both analytic

$\lim\limits_{x\to 0} x P(x) = -1 \qquad \lim\limits_{x\to 0} x^2 Q(x) = 2$

finite

so $\boxed{x=0}$ is a singular point

2) $x = -1$

$$\lim_{x\to -1} (x - \overset{=-1}{x_0}) P(x) = \lim_{x\to -1} (x+1) \frac{1}{x(x-1)(x+1)^2} \text{ is } \infty$$

1 is irregular singular point

3.) $x=1$

$$\lim_{x\to 1} (x-1)P(x) = \lim_{x\to 1} \frac{1}{x(x+1)^2} = \frac{1}{4} \to \text{finite}$$

$$\lim_{x\to 1} (x-1)^2 Q(x) = \lim_{x\to 1} \frac{2}{x^2(x+1)^2} = \frac{1}{2} \to \text{finite}$$

Since both $(x-1)P(x)$ and $(x-1)^2 Q(x)$ are both analytic

$x=1$ is a regular singular point

So $x=0,1$ are regular singular point and $x=-1$ is an irregular singular point of given differential equation.

b) $x^4 y'' + (\sin x) y = 0$

$$y'' + \frac{\sin x}{x^4} y = 0$$

$P(x) = 0 \qquad Q(x) = \dfrac{\sin x}{x^4}$

$\mathbb{R} - \{0\} \to$ ordinary points

0 is a singular point

$$\lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} \frac{\sin x}{x^2} = \infty$$

So $(x-0)^2 Q(x)$ is not analytic at $x=0$

$\Rightarrow$ $x=0$ is an irregular singular point of given differential equation.

c) $x^2y'' + (\sin x)\, y' + (\cos x)\, y = 0.$

$$y'' + \frac{\sin x}{x^2} y' + \frac{\cos x}{x^2} y = 0$$

$\mathbb{R} - \{0\} \rightarrow$ ordinary points

0 is a singular point

$$\lim_{x\to 0} (x-0)\frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \rightarrow \text{finite}$$

$$\lim_{x\to 0} (x-0)^2 \frac{\cos x}{x^2} = \lim_{x\to 0} \cos x = 1 \rightarrow \text{finite}$$

So $(x-0)\,P(x)$ and $(x-0)^2 Q(x)$ are analytic at $x=0$, hence $x=0$ is a regular singular point

2) Find the roots of the indicial equation of the following differential equation about $x=0$

a) $x^3y'' + (\cos 2x - 1)\, y' + 2xy = 0.$

$$y'' + \frac{\cos 2x - 1}{x^3} y' + \frac{2}{x^2} y = 0$$

$$P(x) = \frac{\cos 2x - 1}{x^3} = -\frac{2\sin^2 x}{x^3} \quad \text{and} \quad Q(x) = \frac{2}{x^2}$$

$x=0$ is a singular pt (why?)

$$\lim_{x\to 0} (x-0)\, P(x) = \lim_{x\to 0} \frac{-2\sin^2 x}{x^2} = \boxed{-2} \text{ (finite)} \rightarrow P_0$$

$$\lim_{x\to 0} (x-0)^2 Q(x) = \lim_{x\to 0} 2 = \boxed{2} \text{ (finite)} \rightarrow Q_0.$$

since $x P(x)$ and $x^2 Q(x)$ are analytic at $x=0$,

~~Ero~~ $x=0$ is ~~an~~ a regular singular point, hence Frobenius technique/method can be used to solve given differential equation to get a power series solution.

Now to find indicial equation

$$\boxed{m(m-1) + \overset{=-2}{P_0} \cdot m + \overset{=2}{Q_0} = 0}$$

$$m^2 - m - 2m + 2 = 0 \Rightarrow m^2 - 3m + 2 = 0$$

$\Rightarrow$ $m = 1, 2$ are roots of ~~given~~ indicial equation for the given differential equation

b) $4x^2 y'' - 4x \cdot e^x y' + 3\cos x \, y = 0$.

$$y'' - \frac{e^x}{x} y' + \frac{3}{4} \frac{\cos x}{x^2} y = 0$$

$x=0$ is a regular singular point

$$\lim_{x \to 0} (x-0) P(x) = \lim_{x \to 0} -e^x = -1 \quad — \quad P_0$$

$$\lim_{x \to 0} (x-0)^2 Q(x) = \lim_{x \to 0} \tfrac{3}{4} \cos x = \frac{3}{4} \quad — \quad Q_0$$

Indicial equation

$$m(m-1) - m + \frac{3}{4} = 0$$

$$\Rightarrow 4m^2 - 8m + 3 = 0$$

$m = \frac{1}{2}, \frac{3}{2}$ are roots of indicial equation

3.) Show that $x=0$ is an irregular singular point of the following differential equation

$$y'' + \frac{1}{x^2}y' - \frac{1}{x^3}y = 0,$$ also find the general

— ① $\equiv \boxed{x^3y'' + xy' - y = 0}$

Solution

$$P(x) = \frac{1}{x^2} \quad ; \quad Q(x) = \frac{-1}{x^3}$$

$\lim_{x\to 0}(x-0)P(x)$ and $\lim_{x\to 0}(x-0)^2 \cdot \frac{-1}{x^3}$ are infinite

Hence $xP(x)$ and $x^2Q(x)$ are not analytic at $x=0$

Therefore $x=0$ is an irregular singular point.

Using Frobenius technique/method

$\Rightarrow$ $y = \sum_{n=0}^{\infty} a_n x^{m+n}$ be soln of ①

$$\Rightarrow \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n+1} + \sum_{n=0}^{\infty} a_n(m+n)x^{m+n} - \sum a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \underbrace{\cancel{a_n(m+n-1)(m+n-2)}}_{a_{n-1}(m+n-1)(m+n-2)} x^{m+n} + \sum_{n=0}^{\infty} a_n(m+n)x^{m+n}$$

$$- \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$a_0(m-1) + \sum_{n=1}^{\infty} \left[a_{n-1}(m+n-1)(m+n-2) + a_n(m+n-1)\right] \times x^n = 0$$

$$\Rightarrow m = 1, \quad a_n = -\frac{n(n-1)}{n}a_{n-1} = -(n-1)a_{n-1}$$

$$\Rightarrow a_{n+1} = -n\,a_n$$

$$a_1 = -0 \cdot a_0 = 0$$

$$a_2 = -1 \cdot a_1 = 0$$

$$\vdots$$

$$a_n = 0$$

$\Rightarrow$ $\boxed{y = a_0}$ is the solution of given differential equation.

let $y_1 = a_0$

let $y_2 = v \cdot y_1$

$$V(x) = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx = \int \frac{e^{-\int \frac{1}{x^2}dx}}{a_0^2}$$

$$= \int \frac{e^{1/x}}{a_0^2}\,dx.$$

$$y_2(x) = a_0 \cdot \frac{1}{a_0^2} \int e^{1/x}\,dx = \frac{1}{a_0}\int e^{1/x}\,dx$$

general solution

$$y(x) = c_1 \cdot y_1(x) + c_2 \cdot y_2(x)$$

$$= c_1 \cdot a_0 + \frac{c_2}{a_0} \int e^{1/x}\,dx.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

4.) Find the general solution of the following differential equation about point $x=0$.

$$(1-x^2)\,y'' - 2x\,y' + 2y = 0 \quad -①$$

$$P(x) = \frac{-2x}{1-x^2}$$

$$y'' - \frac{2x}{(1-x^2)}\,y' + \frac{2}{(1-x^2)}\,y = 0 \quad -$$

$$Q(x) = \frac{2}{1-x^2}$$

Let the soln be of the form $y = \sum_{n=0}^{\infty} a_n \cdot x^n$

($\because x=0$ is an ordinary point)

$$y' = \sum_{n=0}^{\infty} a_n \cdot n\,x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} a_n \cdot n(n-1)\,x^{n-2}$$

Substituting y, y' and y'' in ①, we get

$$\Rightarrow \cancel{\sum_{n=0}^{\infty} n(n-1)}$$

$$+y'' = \sum (n+1)(n+2)\,a_{n+2}\,x^n$$

$$-x^2 y'' = -\sum n(n-1)\,a_n\,x^n$$

$$-2xy' = -\sum 2n\,a_n\,x^n$$

$$2y = \sum 2 \cdot a_n\,x^n$$

recursion relation

$$(n+1)(n+2)\,a_{n+2} - n(n-1)\,a_n - 2n a_n + 2a_n = 0$$

$$\Rightarrow \quad a_{n+2} = \frac{(n-1)(n+2)}{(n+1)(n+2)}\,a_n = \frac{(n-1)}{(n+1)}\,a_n$$

$$\boxed{a_{n+2} = \frac{(n-1)}{(n+1)} a_n}$$

$a_2 = -a_0$; $\boxed{\underset{1+2}{a_3} = 0 = ? \left(\frac{1-1}{1+1}\right) \cdot a_1}$

$$a_4 = a_{2+2} = \frac{(2-1)}{2+1} a_2 = \frac{a_2}{3} = \frac{-a_0}{3}$$

$a_5, a_7 \ldots = 0$

$a_{2n+1} = 0$

$$a_6 = a_{4+2} = \frac{4-1}{4+1} a_4 = \frac{3}{5} a_4 = \frac{-a_0}{5}$$

$$\Rightarrow y(x) = a_0 \left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} + \cdots \right] + \cancel{a_1 \cdot x}$$

is the required solution.

5.) Find a solution of the following differential equation about point $x=0$

$$x^2 y'' + xy' + (x^2 - p^2) y = 0,$$ where $p \geq 0$ is a real number.

$x=0$ is a regular singular point

let $y(x) = \sum_{n=0}^{\infty} a_n x^{m+n}$

$$x^2 y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n}$$

$$xy' = \sum_{n=0}^{\infty} (m+n) a_n \cdot x^{m+n}$$

$$x^2 y = \sum a_{n-2}\, x^{m+n}$$

$$-p^2 = -p^2 \sum a_n x^{m+n}$$

$$(m+n)(m+n-1)\, a_n + (m+n)\, a_n + a_{n-2} - p^2 a_n = 0$$

Indicial equation

$$(m^2 - p^2)\, a_0 = 0$$

1 $\boxed{m = \pm p}$ (root of Indicial equation)

2 $$\boxed{\left((m+n)^2 - p^2\right) a_n + a_{n-2} = 0}$$

3 $\Rightarrow$ $$\boxed{a_n = -\frac{a_{n-2}}{(m+n)^2 - p^2}}$$

from 1, 2, 3 $a_1 = a_3 = a_5 \cdots = 0$

$a_{2n+1} = 0$ why?

or $$a_{n+2} = -\frac{a_n}{(m+n+2)^2 - p^2}$$

$n = 0, 1, 2, 3, 4, \cdots$ | let $m = p$:

$$\Rightarrow a_2 = \frac{-a_0}{(p+2)^2 - p^2} = \frac{-a_0}{4(p+1)}$$

$$\Rightarrow a_4 = \frac{-a_2}{(p+4)^2 - p^2} = \frac{+a_0}{4 \cdot 8 \cdot (p+1)(p+2)}$$

A

$$\Rightarrow y(x) = a_0 x^p \left[1 - \frac{x^2}{\quad} + \frac{x^4}{\cdots(p+1)(p+2)} \cdots\right]$$

let $m = -p$

$$a_{n+2} = \frac{-a_n}{(n+2-p)^2 - p^2} \qquad \left(a_{n+2} = \frac{-a_n}{(m+n+2)^2 - p^2} \right)$$

$$a_2 = \frac{-a_0}{4(1-p)} \; ; \quad a_4 = \frac{+a_0}{4 \times 8 \times (1-p) \times (2-p)}$$

$$\Rightarrow y_{-p}(x) = a_{0_2} \cdot x^{-p} \left[1 - \frac{x^2}{4(1-p)} + \frac{x^4}{4 \times 8 \times (1-p)(2-p)} + \cdots \right]$$

B

$\therefore$ the general solution

$$\boxed{y(x) = A \cdot y_p(x) + B \cdot y_{-p}(x)}$$

In $y_p(x)$

$$a_6 = \frac{-a_4}{(p+6)^2 - p^2} = \frac{-a_4}{12(p+3)} = \frac{(-1)^3 \cdot a_0}{4 \cdot 8 \cdot 12 \cdot (p+1)(p+2)(p+3)}$$

$$= \frac{(-1)^3 \cdot a_0}{(2^2)^3 \cdot 1 \cdot 2 \cdot 3 \cdot (p+1)(p+2)(p+3)}$$

$$a_{2n} = \frac{(-1)^n \cdot a_0}{2^{2n} \cdot n! \prod_{m=1}^{n} (p+m)} \qquad \left(\frac{(-1)^n \cdot a_0}{2^{2n} \cdot n! \prod_{m=1}^{n} (p+m)} \right)$$

$$\boxed{\boxed{\Gamma(n+1) = n!}} = \cancel{n!} \cdot (p+1) \cdot (p+2)(p+3) \cdots (p+n)$$

$$y_p(x) = \sum \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,\Gamma(p+n+1)} = J_p(x)$$

$J_m(x)$ / $J_\nu(x)$

Some text books $n \to r$; $p \to m$ or ν

Bessel function of first kind of order 'p' if not

gamma function

$\Gamma(m+1) = m!$ if m is integer, gamma function

$$\boxed{\Gamma(m+1) = m! = m\cdot(m-1)! = m\cdot\Gamma(m)}$$

$\Gamma(m) = \Gamma(m)$

similarly $J_{-p}(x) =$

$p \to -p$
$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-p}}{n!\,\Gamma(n+1-p)}$$

$y = A\cdot J_p(x) + B\cdot J_{-p}(x)$ is the gen. solution of Bessel differential equation if p is not an integer.

6.) Show that the indicial equation has only one root for $x^2y'' + xy' + x^2y = 0$ and that
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$$ is a corresponding particular soln.

$J_0(x) \to$ Bessel function of first kind of order 0.

$$x^2 y'' + x y' + x^2 y = 0.$$

$$y'' + \frac{1}{x} y' + y = 0$$

$$P(x) = \frac{1}{x}, \quad Q(x) = 0$$

$P_0 = 1$? $\quad Q_0 = 0$?

$x = 0$ is a regular singular point.

indicial equation

$$m(m-1) + m = 0 \quad (?)$$

$$m(m-1) + P_0 \cdot m + Q_0 = 0$$

$$\overset{\uparrow}{y''} + P^1(x) \cdot y' + Q^1(x) = 0$$

$$P_0 = \lim_{x \to x_0} \underbrace{(x - x_0) P(x)}_{P^1(x)}; \quad Q_0 = \lim_{x \to x_0} \underbrace{(x - x_0) Q(x)}_{Q^1(x)}$$

$m = 0$ is the root of indicial equation

$$x^2 y'' = \sum (m+n)(m+n-1) a_n x^{m+n}$$

$$x y' = \sum (m+n) a_n x^{m+n}$$

$$x^2 y = \sum \cancel{(m+n-2)} a_{n-2} x^{m+n}$$

$$\sum \Big[\big((m+n)(m+n-1) + (m+n)\big) a_n + a_{n-2} \Big] x^{m+n}$$

$$\sum \big[(m+n)^2 a_n + a_{n-2} \big] x^{m+n} = 0$$

$(m = 0$ since $a_0 \neq 0)$.

$$\Rightarrow a_{n+2} = \frac{-a_n}{(n+2)^2}$$

$$a_2 = \frac{-a_0}{2^2} \; ; \; a_4 = -\frac{a_2}{4^2} = \frac{(-1)^2 \cdot a_0}{4^2 \cdot 2^2}$$

$$a_6 = -\frac{a_4}{6^2} = \frac{(-1)^3 \cdot a_0}{6^2 \cdot 4^2 \cdot 2^2} = \frac{(-1)^3 \cdot a_0}{(2^2)^3 \cdot (3!)^2}$$

$$\boxed{a_{2n} = \frac{(-1)^n \cdot a_0}{2^{2n} \cdot (n!)^2}}$$

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^{m+n} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}}_{\to J_0(x)}$$

Hence proved.

Bessel function of first kind of order 0.

(récurrent neural networks)

Rodrigue's formula for legendre polynomials. (1/3)

$$\boxed{P_n(x) = \frac{1}{n!\, 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n}$$

orthogonal

Laguerre $[0, \infty)$

Hermite $(-\infty, \infty)$

$P_n(1) = 1$ (normalization condition)

generating function (connected to multipole expansion in electrostatics)

$$\frac{1}{|r - r'|} \quad \boxed{t = \frac{r'}{r}} \quad \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \quad \boxed{x = \cos\theta}$$

coefficient of $t^n \to P_n(x)$

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n \cdot P_{n-1}(x)$$

↓ Bonnet's recursion formula

$P_n(1) = 1$

orthogonalization condition

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2n+1}\, \delta_{mn} \quad -(1)$$

$\delta_{mn} = 1$ if $m = n$ & $\delta_{mn} = 0$ if $m \neq n$

a piecewise continuous function $f(x)$ in $[-1, 1]$

Now $$f_n(x) = \sum_{m=0}^{n} a_m P_m(x) \to f(x) \text{ as } n \to \infty \quad -(2)$$

(1) and (2)

$$\int_{-1}^{+1} f(x)\, P_m(x) = \frac{2}{2m+1}\, a_m$$

or

$$a_m = \frac{2m+1}{2} \int_{-1}^{+1} f(x)\, P_m(x)\, dx$$

7.) Find the first three terms of the legendre's series of the function $f(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \end{cases}$

From the completeness property of legendre polynomials in $[-1, 1]$

$$f(x) = \sum a_n P_n(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + \cdots$$

$$P_0(x) = 1, \quad P_1(x) = x \; ; \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \, .$$

$$a_0 = \frac{2(0)+1}{\cancel{2(0)}\; 2} \int_0^1 x \cdot \overset{P_0(x)}{1} \, dx = \frac{1}{4}$$

$$a_1 = \frac{2(1)+1}{2} \int_0^1 x \cdot \overset{P_1(x)}{x} \, dx = \frac{1}{2}$$

$$a_2 = \frac{2(2)+1}{2} \int_0^1 x \cdot \frac{1}{2}(3x^2 - 1) \, dx = \frac{5}{16}$$

Thus $f(x) = \frac{1}{4} + \frac{1}{2} \cdot x + \frac{5}{16} \cdot \frac{1}{2} \cdot (3x^2 - 1) + \cdots$

8.) Let $P_n(x)$ be the n^{th} legendre polynomial, where $n \geq 0$ is any integer. Prove the following

a) $P_n(1) = 1$ b) $P_n'(1) = \frac{1}{2} n(n+1)$.

$P_n(x)$ is the solution of

$\cancel{(1-x^2) P_n''(x)} -$

$$(1-x^2) y'' - 2xy' + n(n+1) y = 0$$

$$\Rightarrow (1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$$

if we put $x = 1$

we get $P_n'(x) = \frac{1}{2} n(n+1) P_n(x)$

Proving $P_n(1) = 1$ will prove both the results.

Now consider generating function for legendre polynomials

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Putting $x=1$ $(x=1)$

$$\frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{1-t} = 1+t+t^2+t^3+\dots = \sum_{n=0}^{\infty} P_n(1)t^n$$

$\therefore$ $P_n(1) = 1$ (Hence proved $\boxed{a)}$ & $\boxed{b)}$)

9.) Let y be a polynomial solution of the differential equation $(1-x^2)y'' - 2xy' + 12y = 0$

if $y(1) = 2$, then find the value of the integral

$$\int_{-1}^{1} y^2 \, dx$$

given differential equation is a legendre differential equation (on comparing with standard form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0)$$

Here $n = 3$ for which solution is

$k \cdot P_3(x)$ where k is arbitrary constant

$\therefore\ y(x) = k\cdot P_3(x)$ is the solution

given $y(1) = 2 = k\cdot P_3(1) = k \quad (\because P_n(1) = 1)$

$\Rightarrow k = 2$.

Now $\displaystyle\int_{-1}^{+1} y^2\,dx = \int_{-1}^{+1} k^2\cdot P_3^2(x)\,dx$

$$= (2)^2\cdot \frac{2}{2(3)+1} = \frac{8}{7} \quad \left(\because k = 2 \text{ and } \int_{-1}^{+1} P_n^2(x)\,dx = \frac{2}{2n+1}\right)$$

$$\boxed{\therefore \int_{-1}^{+1} y^2 = \frac{8}{7}}$$

10.) Suppose the legendre equation

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ — (1) has an n^{th} degree

Polynomial solution $y_n(x)$ such that $y_n(1) = 3$. If

$\displaystyle\int_{-1}^{1} y_n^2(x) + y_{n-1}^2(x)\,dx = \frac{144}{15}$, then find the

value of n.

eq. (1) is legendre differential equation

So the general solution is $y(x) = k\cdot P_n(x)$

given $y(1) = 3 \Rightarrow k = 3 \quad (\because P_n(1) = 1)$

Now

$$\int_{-1}^{+1} y_n^2(x) + y_{n-1}^2(x) = k^2 \int_{-1}^{+1} P_n^2(x) + P_{n-1}^2(x)\,dx$$

$$= 9 \cdot \frac{2}{2n+1} + 9 \cdot \frac{2}{2(n-1)+1}$$

$$= 18\left(\frac{2n-1+2n+1}{4n^2-1}\right)$$

$$= \frac{72n}{4n^2-1} = \frac{144}{15}$$

$$n = \frac{-1}{8} \text{ or } 2$$

$n = -\frac{1}{8} \rightarrow$ not acceptable

$$\boxed{\therefore n = 2}$$