Engineering Electromagnetics

Lecture 3

25/08/2023

by

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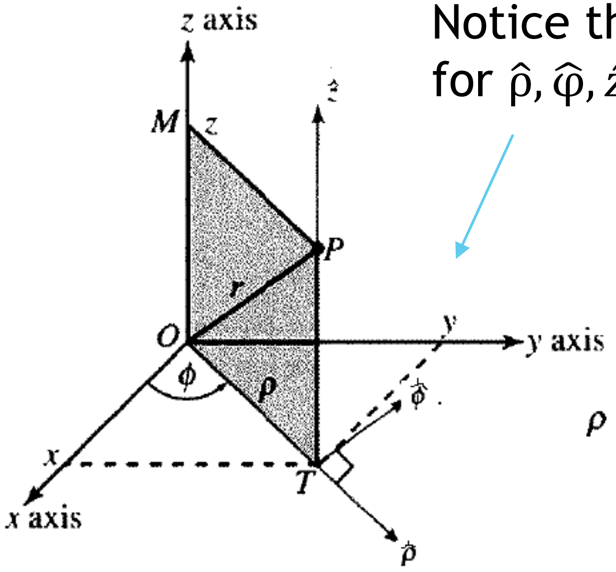
Cylindrical Coordinates

- Any point P (x,y,z) can also be expressed in cylindrical coordinates



Notice the directions for $\hat{\rho}, \hat{\varphi}, \hat{z}$

Figure 2.11 Projections of a point in a cylindrical coordinate system

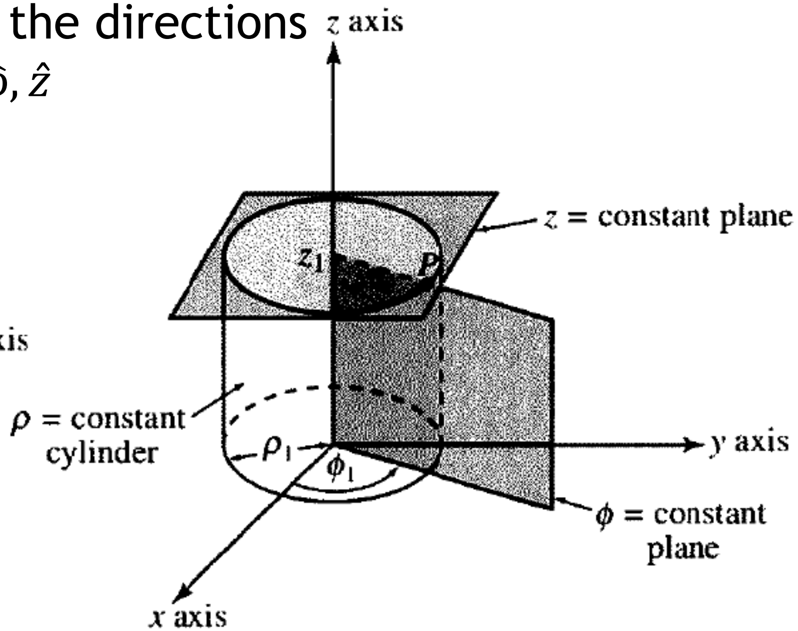


Figure 2.12 Three mutually perpendicular surfaces in the cylindrical coordinate system

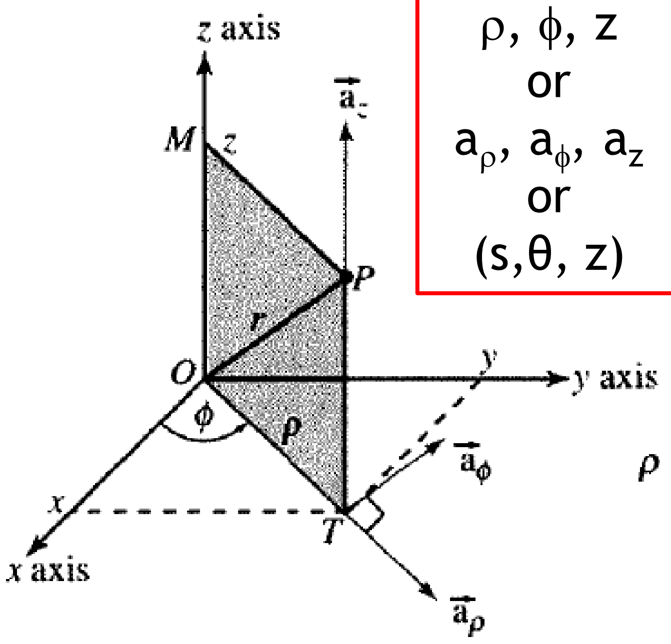
Cylindrical coordinates are useful in connection with objects and phenomena that have some rotational symmetry about the longitudinal axis, such as water flow in a straight pipe with round cross-section, heat distribution in a metal cylinder, electromagnetic fields produced by an electric current in a long, straight wire, accretion disks in astronomy, and so on.
https://en.wikipedia.org/wiki/Cylindrical_coordinate_system

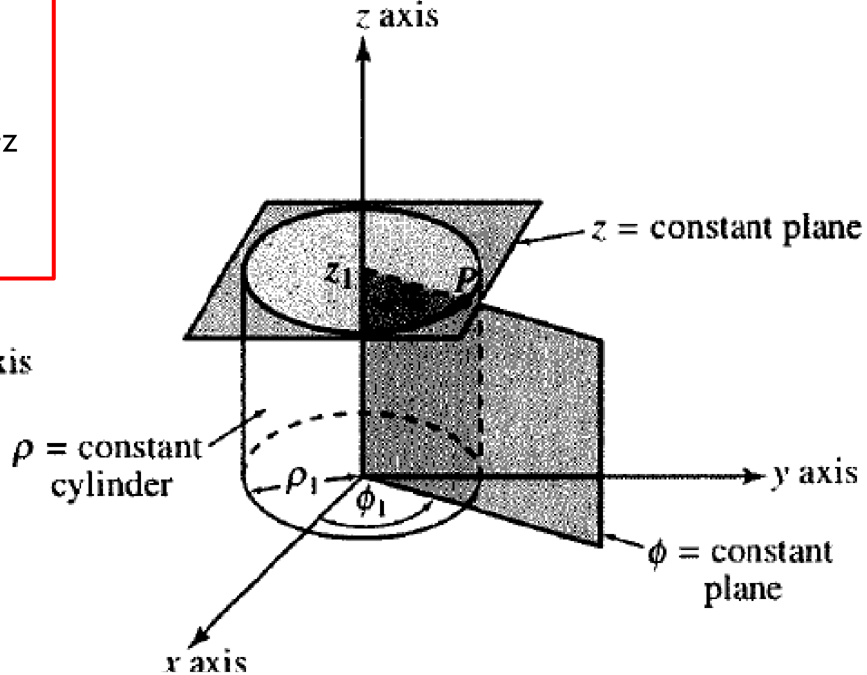
- Difference from cart. Coord.: $\rho \rightarrow$ projection of r on xy plane (not on an axis)
- Z: distance of P from z axis and ϕ: angle OT makes with x axis (or the plane OTPM)

Cylindrical coordinate system

Unit vectors (same meaning, different notations)

ρ, ϕ, z
or
a_ρ, a_ϕ, a_z
or
(s,θ, z)





$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

The coordinate surface

$$\rho = \sqrt{x^2 + y^2} = \text{constant}$$

is a cylinder of radius ρ with the z axis as its axis,

$\hat{\rho}$, $\hat{\varphi}$, and $\hat{\mathbf{z}}$ → unit vectors

Properties:

$\hat{\rho}.\hat{\rho} = \hat{\varphi}.\hat{\varphi} = \hat{\mathbf{z}}.\hat{\mathbf{z}} = 1$; $\hat{\rho}.\hat{\varphi} = \hat{\varphi}.\hat{\mathbf{z}} = \hat{\rho}.\hat{\mathbf{z}} = 0$

$\hat{\rho} \times \hat{\varphi} = \hat{\mathbf{z}}$; $\hat{\varphi} \times \hat{\mathbf{z}} = \hat{\rho}$; $\hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \hat{\varphi}$

Should be defined at a common point

If two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are defined either at a common point $P(\rho, \phi, z)$ or in a $\phi =$ constant plane, we can add, subtract, and multiply these vectors as we did in the rectangular coordinate system. For example, if the two vectors at point $P(\rho, \phi, z)$ are $\vec{\mathbf{A}} = A_\rho \vec{\mathbf{a}}_\rho + A_\phi \vec{\mathbf{a}}_\phi + A_z \vec{\mathbf{a}}_z$ and $\vec{\mathbf{B}} = B_\rho \vec{\mathbf{a}}_\rho + B_\phi \vec{\mathbf{a}}_\phi + B_z \vec{\mathbf{a}}_z$, then

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_\rho + B_\rho)\vec{\mathbf{a}}_\rho + (A_\phi + B_\phi)\vec{\mathbf{a}}_\phi + (A_z + B_z)\vec{\mathbf{a}}_z \tag{2.32a}$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z \tag{2.32b}$$

and

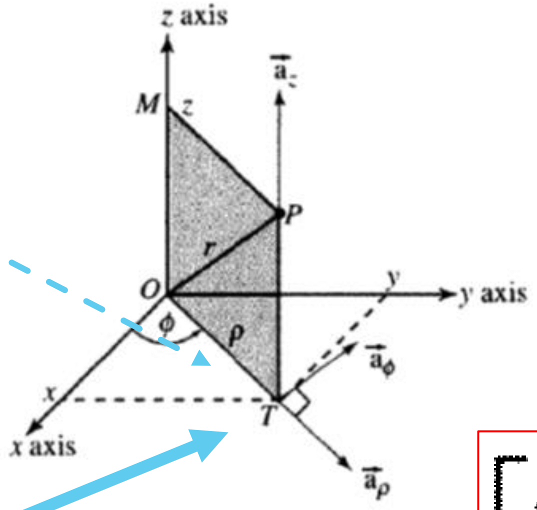
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \vec{\mathbf{a}}_\rho & \vec{\mathbf{a}}_\phi & \vec{\mathbf{a}}_z \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix} \tag{2.32c}$$

Transformations

- **Conversion from cartesian to cylindrical coordinates:**
- $\hat{x}.\hat{\rho} = Cos\varphi$ **and** $\hat{y}.\hat{\rho} = Sin\varphi$
- $\hat{x}.\hat{\varphi} = -Sin\varphi$ **and** $\hat{y}.\hat{\varphi} = Cos\varphi$

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}. \end{aligned}$$

If $\hat{\rho}(or\ \overrightarrow{a_\rho})$ makes an angle φ with x axis, **what about** $\hat{\boldsymbol{\varphi}}(\mathbf{or}\ \overrightarrow{\boldsymbol{a_\varphi}})$? And the **x and y components of** $\hat{\boldsymbol{\varphi}}$**?**



$$\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

Q: For any vector A:

$$A = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

How to convert it to cylindrical coordinates? $A = A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{z}$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Conversion cylindrical ↔ cartesian coordinates

Cartesian to cylindrical

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Cylindrical to cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

- Conversion to cartesian coordinates (Hint)
- From $A = A_\rho \hat{\rho} + A_\varphi \hat{\varphi} + A_z \hat{z}$ to $A = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$
- $A_x = A.\,\hat{x} = (A_\rho \hat{\rho} + A_\varphi \hat{\varphi} + A_z \hat{z}).\,\hat{x} = A_\rho \hat{\rho}.\hat{x} + A_\varphi \hat{\varphi}.\hat{x} + A_z \hat{z}.\,\hat{x}$; $\hat{x}.\hat{\rho} = Cos\varphi$; $\hat{y}.\hat{\rho} = Sin\varphi$;
- $\hat{x}.\hat{\varphi} = -Sin\varphi$ and $\hat{y}.\hat{\varphi} = Cos\varphi$; $A_x = A_\rho\, Cos\varphi - A_\varphi Sin\varphi$; $A_y = A.\ \hat{y} = A_\rho Sin\varphi + A_\varphi Cos\varphi$ and $A_z = A.\hat{z}$ = A_z

Thank You