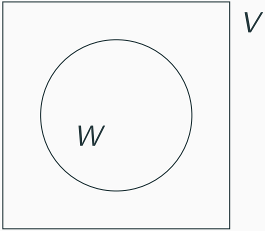
Subspaces

Nachiketa Mishra
IIITDM Kancheepuram, Chennai

Subspace

Let V be a vector space over a field F. A subspace of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication defined on V.



Remark

If $\langle V, F, +, . \rangle$ is a vector space, then

(i) $\forall \alpha, \beta \in V$, $\alpha + \beta \in V$ (V is closed under vector addition)

(ii) $\forall c \in F$ and $\alpha \in V$, $c\alpha \in V$ (V is closed under scalar multiplication)

(iii) If $\alpha_1, \ldots, \alpha_n \in V$, then $c_1\alpha_1 + \ldots + c_n\alpha_n \in V$ where $c_i \in F$.

Theorem 1

Let V be a vector space over the field F. A non-empty subset W of V is a subspace of V if and only if

$$\forall\ \alpha, \beta \in W, c \in F \Longrightarrow c\alpha + \beta \in W.$$

Proof:

Case 1 : Suppose that W is a subspace of V. $\Longrightarrow$ W is a vector space over the field F.

If $c \in F, \alpha, \beta \in W$, then $c\alpha \in W$ (closed under scalar multiplication) and $c\alpha + \beta \in W$ (closed under vector addition). Hence

$$\forall\ \alpha, \beta \in W, c \in F \Longrightarrow c\alpha + \beta \in W.$$

Theorem 1 contd.

Case 2 : Suppose that W is a non-empty subset of V such that

$$\forall\ \alpha, \beta \in W, c \in F \Longrightarrow c\alpha + \beta \in W. - - - - -(a)$$

Since $W \neq \phi$, there exists $\rho \in W$ and hence $(-1)\rho + \rho = 0 \in W$, by (a). For all $\alpha \in W \subseteq V$, $1.\alpha = \alpha$ (V is a vector space). For all $\alpha, \beta \in W, \quad 1.\alpha + \beta = \alpha + \beta \in W$ by (a). For all $c \in F$ and $\alpha \in W, \quad c\alpha + 0 = c\alpha \in W$, by (a). In addition, $(-1)\alpha + 0 = -\alpha \in W$ for all $\alpha \in W$ by (a). Since $W \subseteq V$, $\langle W, F, +, .\rangle$ satisfies the rest of the axioms (verify!) of a vector space and thus W is a subspace of V.

Examples of subspaces

(1) Let V be a vector space over the field F. Then the subset $\{0\}$ of V is a subspace of V and it is called the zero subspace.

(2) Note that $W = \{(0, x_2 \ldots, x_n) \quad : \quad x_i \in F\}$ is a subspace of F^n.

Proof: Clearly $0 = (0, 0, \ldots, 0) \in W$. So $\phi \neq W \subseteq F^n$. Let $\alpha = (0, x_2, \ldots, x_n)$, $\beta = (0, y_2, \ldots, y_n) \in W$ and $c \in F$.

$$c\alpha + \beta = (0, cx_2 + y_2, \ldots, cx_n + y_n) \in W$$

By Theorem 1, W is a subspace of F^n.

(3) Prove that $W = \{(1 + x_2, x_2, x_3, \ldots, x_n) \; : \; x_i \in F\}$ is not a subspace of F^n.

Reason : $0 = (0, 0, \ldots, 0) \notin W$

Examples contd.

(4) Prove that the solution set of the homogeneous system $AX = 0$ is subspace of $F^{n\times 1}$ where $A \in F^{m\times n}$.
Let $S = \{X \in F^{n\times 1} \ : \ AX = 0\}$. Clearly $0 \in S \neq \phi$.
Let $X_1, X_2 \in S$ and $c \in F$. $\implies \quad AX_1 = AX_2 = 0$.

$$\implies \quad A(cX_1 + X_2) = cAX_1 + AX_2 = 0, \qquad \implies \quad cX_1 + X_2 \in S$$

$$\forall\ X_1, X_2 \in S, c \in F \implies cX_1 + X_2 \in S$$

Hence, S is a subspace of $F^{n\times 1}$.

Theorem 2

Let V be a vector space over the field F. Let W_1, W_2 be two subspaces of V. Then $W_1 \cap W_2$ is a subspace of V.

Proof : Since W_1 and W_2 are subspace of V, (a) $0 \in W_i \neq \phi$ and (b) $\forall\, \alpha, \beta \in W_i, c \in F \implies c\alpha + \beta \in W_i$ for $i = 1, 2$

By (a), $0 \in W_1 \cap W_2 \neq \phi$.

Let $\alpha, \beta \in W_1 \cap W_2$, $c \in F$. $\implies \alpha, \beta \in W_i$ for $i = 1, 2$.

$\implies c\alpha + \beta \in W_i$ for $i = 1, 2$ by (b).

$\implies c\alpha + \beta \in W_1 \cap W_2$.

By Theorem 1, $W_1 \cap W_2$ is a subspace of V.

Corollary : Intersection of any collection of subspaces of a vector space V is a subspace of V

The subspace spanned by S

Let S be a subset of a vector space V. The subspace spanned by S is defined as the intersection all subspaces of V which contains S.

$$\text{Subspace spanned by } S = \cap\{W \,:\, S \subseteq W,\ \ W \text{ is a subspace of } V\}$$

Note (1) : Subspace spanned by S is the smallest subspace which contains S.

Note (2) : If $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, we call the subspace spanned by S as the subspace spanned by the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$.

$L(S)$= the set of all linear combinations of vectors in S

Let S be a non-empty subset of a vector space V. The set of all linear combinations of vectors in S is denoted by $L(S)$.

$$L(S) = \left\{ \sum_{i=1}^{n} c_i \alpha_i \quad : \quad c_i \in F, \alpha_i \in S, n \in \mathbb{N} \right\}$$

(1) Let $S = \{(1,0,0),(0,0,1)\}$

$$L(S) = \{a(1,0,0) + b(0,0,1) \quad : \quad a, b \in \mathbb{R}\}$$

$$L(S) = \{(a,0,b) \quad : \quad a, b \in \mathbb{R}\}$$

Lemma

$L(S)$ is a subspace of V and $S \subseteq L(S)$.

Proof: Since $S \neq \phi$, there exists $\alpha \in S \subseteq V$. By Note 2, $0\alpha = 0 \in L(S)$ (0α is a linear combination of α). $L(S) \neq \phi$. In addition $\forall \alpha \in S$, $1.\alpha = \alpha \in L(S)$ and thus $S \subseteq L(S)$.

Let $x, y \in L(S)$. $\implies x = \sum_{i=1}^{m} c_i \alpha_i, y = \sum_{j=1}^{n} d_j \beta_j$

$\implies cx + y = \sum_{i=1}^{m} cc_i \alpha_i + \sum_{j=1}^{n} d_j \beta_j$ is a linear combination of vectors in S. Thus $cx + y \in L(S)$.

$\implies$ $L(S)$ is a subspace of V by Theorem 1.

Theorem 3

Let S be a non-empty subset of a vector space V over the field F. Then the subspace spanned by the set S is the set of all linear combinations of vectors in S.

Proof.

It is enough to prove that the subspace spanned by $S = L(S)$. Prove that

$$W^* = \cap\{W \ : \ S \subseteq W, \ \ W \text{ is a subspace of } V\} = L(S) - - - (a)$$

By the previous lemma, $S \subseteq L(S)$ and $L(S)$ is a subspace of V, and thus $W^* \subseteq L(S) - - - -(i)$.

Theorem 3 contd.

Claim : If W is a subspace of V and $S \subseteq W$, then $L(S) \subseteq W$.
Let $x \in L(S)$. $\implies$ x is a linear combination of vectors in S. Since W is a subspace and $S \subseteq W$, every linear combination of vectors in S is also a member of W and thus $x \in W$.

$$x \in L(S) \Longrightarrow x \in W. \text{ Thus } L(S) \subseteq W.$$

By above claim,

$$L(S) \subseteq \cap\{W \ : \ S \subseteq W, \ \ W \text{ is a subspace of } V\} = W^* - -(ii)$$

By (i) and (ii),

$$W^* = \cap\{W \ : \ S \subseteq W, \ \ W \text{ is a subspace of } V\} = L(S) - - - (a)$$

Row space and Column space of a matrix

Let $A \in F^{m \times n}$ with rows $\{R_1, R_2, \ldots, R_m\}$ and columns $\{C_1, C_2, \ldots, C_n\}$. Then

Row space of $A =$ The subspace spanned by $R_1, R_2, \ldots, R_m$

Column space of $A =$ The subspace spanned by $C_1, C_2, \ldots, C_n$

Note : Row space of $A \subseteq F^{1 \times n}$ **and Column space of** $A \subseteq F^{m \times 1}$**.**

Example

$$\text{Let} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where

$$R_1 = (1,0,0), R_2 = (0,1,0), C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Row space of A

$$= \{x(1,0,0) + y(0,1,0) \quad : \quad x, y \in F\} = \{(x, y, 0) \quad : \quad x, y \in F\}$$

Column Space of A

$$= \left\{ x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad : \quad x, y, z \in F \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

Assignment

Prove or disprove that

(i) column space of AB is same as column space of A and

(ii) row space of AB is same as row space of B.

Note 1 contd.

$$\left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\} \Longrightarrow \left.\begin{array}{c} x_{k_1} + \sum_{j=1}^{n-r} C_{1j}u_j = 0 \\ x_{k_2} + \sum_{j=1}^{n-r} C_{2j}u_j = 0 \end{array}\right\} \textit{general expression})$$

Set the free variables as :

$u_1 = x_1 = a,\ u_2 = x_3 = b,\quad u_3 = x_5 = c$

$\Longrightarrow\quad x_2 = 3b - \frac{1}{2}c,\quad x_4 = -2c$

Solution set $S = \left\{\left(a, 3b - \frac{1}{2}c, b, -2c, c\right)\ :\ a, b, c \in \mathrm{R}\right\}$

Note 1 contd. (back to chapter one !)

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c) \;:\; a, b, c \in \mathrm{R}\right\}$

$$S = \left\{ a(1,0,0,0,0) + b(0,3,1,0,0) + c(0,-\frac{1}{2},0,-2,1) \;:\; a,b,c \in \mathrm{R} \right\}$$

$$= \text{Span of } \left\{ (1,0,0,0,0), (0,3,1,0,0), (0,-\frac{1}{2},0,-2,1) \right\}$$

Dimension of $S = dim\ S = 3 = n - r$ **(Information for future)**

Problem

Let W be set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfies

$$\begin{aligned} 2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \qquad &= 0 \\ x_1 \qquad + \tfrac{2}{3}x_3 \qquad - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0 \end{aligned}$$

Find a finite set of vectors which spans W.

Note

Let $\alpha = (2,3)$ and $\beta = (6,9)$. Then $\beta = 3\alpha$.

$\implies \quad 3\alpha + (-1)\beta = 0.$

$\implies \quad c_1\alpha + c_2\beta = 0$ where $c_i \neq 0$ for at least one i.

We say $\{\alpha, \beta\}$ is a linearly dependent set.

Let $\gamma = (3,4)$. Prove that there is no $c \in \mathbb{R}$ such that $\gamma = c\alpha$

$c_1\alpha + c_2\gamma = 0 \implies c_1 = c_2 = 0$

We say $\{\alpha, \gamma\}$ is a linearly independent set.

Since $3\alpha + (-1)\beta + 0\gamma = 0$, $\{\alpha, \beta, \gamma\}$ is a linearly dependent set.