FIELDS

if $\mathbb{F} \to$ field, then its a non-empty set with operations $+$ and $\cdot$, which follows some properties. Members of a field $\longrightarrow$ SCALARS.

OPERATIONS DEFINED :

(i) $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (Addn.)
(ii) $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (multipln.)

PROPERTIES FOLLOWED :

① Closure Axiom -

$\forall a, b \in \mathbb{F}$,

- $a+b \in \mathbb{F}$
- $a \cdot b \in \mathbb{F}$

② Associativity Axiom -

$\forall a, b, c \in \mathbb{F}$,

- $a+(b+c) = (a+b)+c$
- $a\cdot(b\cdot c) = (a\cdot b)\cdot c$

③ Identity Axiom -

$\exists\, 0, 1 \in \mathbb{F}$ s.t. $\forall a \in \mathbb{F}$,

- $a+0 = a$
- $a\cdot 1 = a$

④ Inverse Axiom -

- $\forall a \in \mathbb{F}$, $\exists\, b \in \mathbb{F}$ s.t. $a+b = b+a = 0$.
  $b \longrightarrow$ additive inverse of a.
- $\forall a \in \mathbb{F}-\{0\}$, $\exists\, c \in \mathbb{F}$ s.t. $a\cdot c = c\cdot a = 1$
  $c \longrightarrow$ multiplicative inverse of a.

⑤ Commutative Property -

$\forall a, b \in \mathbb{F}$,

- $a+b = b+a$
- $a\cdot b = b\cdot a$

⑥ Distributive property -

$\forall a, b, c \in \mathbb{F}$,

- $a\cdot(b+c) = a\cdot b + a\cdot c$
- $(a+b)\cdot c = a\cdot c + b\cdot c$

NOTATION :

A field w.r.t. operations $+, \cdot$ is usually denoted as $(F, +, \cdot)$

EXAMPLES :

① $(\mathbb{N}, +, \cdot) \longrightarrow$ NOT a field, as it doesn't satisfy identity for addition and inverse for both addition & multiplication.

$\{(-\infty, 0] \notin \mathbb{N}$ and $\exists\, a \in \mathbb{N}$ s.t. $\nexists\, 1/a \in \mathbb{N}.\}$

② $(\mathbb{Z}, +, \cdot) \longrightarrow$ NOT a field, as it doesn't satisfy inverse for multiplication.

$\{\exists\, a \in \mathbb{Z}$ s.t. $\nexists\, 1/a \in \mathbb{Z}\}$

③ $(\mathbb{Z}_2, +, \cdot) \longrightarrow$ is a Field, let's verify :

$\mathbb{Z}_2 = \{0, 1\} \longrightarrow$ set of all remainders possible $\forall\, a \in \mathbb{Z}_2$ when divided by 2.

* Closure :

$0 + 0 \equiv 0,\ 0 \% 2 = 0 \in \mathbb{Z}_2$

$0 + 1 \equiv 1,\ 1 \% 2 = 1 \in \mathbb{Z}_2$

$1 + 0 \equiv 1,\ 1 \% 2 = 1 \in \mathbb{Z}_2$

$1 + 1 \equiv 2,\ 2 \% 2 = 0 \in \mathbb{Z}_2$

$\therefore$, closure satisfied.

* Associativity : only 2 elements in $\mathbb{Z}_2$. & 4 choices – $0+0$, $0+1$, $1+0$, $1+1$, all of which have been proved to exist in $\mathbb{Z}_2$. $\therefore$, Associativity is satisfied.

* Identity : $0, 1 \in \mathbb{Z}_2$. $\therefore$, Identity satisfied. ($0 \to$ additive, $1 \to$ multiplicative)

* Inverse : from closure, its clear that add. inverse of – $0 \to 0$ and $1 \to 1$. mul. inverse of – $1 \to 1$.

add. inv exists $\forall a \in \mathbb{Z}_2$ & mul inv. exists $\forall a \in \mathbb{Z}_2 - \{0\}$

$\therefore$, inverse satisfied.

* Commutative : same argument as associativity. $\therefore$, commutativity satisfied.

* distributive : $\forall a, b \in \mathbb{Z}_2$,

$$(a+b) \cdot 0 = a \cdot 0 + b \cdot 0$$
$$(a+b) \cdot 1 = a \cdot 1 + b \cdot 1.$$
$$1 \cdot (a+b) = 1 \cdot a + 1 \cdot b.$$
$$0 \cdot (a+b) = 0 \cdot a + 0 \cdot b.$$

$\therefore$, distributive satisfied.

As all properties are satisfied, $(\mathbb{Z}_2, +, \cdot)$ is a field.

X ———————— X ———————— X

VECTOR SPACES

Vector Space $\longrightarrow \langle V, F, \cdot, + \rangle$ over field F.

(i) $F \to$ field of scalars.
(ii) Vector Space $\to$ set of objects V, called vectors.
(iii) two operations : $+$, $\cdot$

The operation $+ : V \times V \longrightarrow V$, is called vector addition, if -

a) Addition is commutative -

$$\alpha + \beta = \beta + \alpha, \quad \forall \alpha, \beta \in V$$

b) Addition is associative -

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma, \quad \forall \alpha, \beta, \gamma \in V$$

c) $\exists$ a unique vector $\vec{0} \in V$, called the zero-vector s.t.

$$\alpha + \vec{0} = \alpha, \quad \forall \alpha \in V$$

d) $\exists$ a unique vector $-\alpha \in V$ for each $\alpha \in V$ s.t.

$$\alpha + (-\alpha) = 0$$

The operation $\cdot : F \times V \to V$, is called scalar multiplication if -

a) $1 \cdot \alpha = \alpha, \quad \forall \alpha \in V$
b) $(c_1 c_2)\alpha = c_1 \cdot (c_2 \alpha), \quad \forall \alpha \in V, \quad \forall c_1, c_2 \in F$
c) $c \cdot (\alpha + \beta) = c \cdot \alpha + c \cdot \beta, \quad \forall \alpha, \beta \in V, \quad \forall c_1, c_2 \in F$
d) $(c_1 + c_2)\alpha = c_1 \alpha + c_2 \alpha, \quad \forall \alpha \in V, \quad \forall c_1, c_2 \in F$.

EXAMPLE 1: The Euclidean Space

let $V = \mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}\}$

and $F = \mathbb{R}$

let $\alpha = (x_1, x_2, \dots, x_n)$ and
$\beta = (y_1, y_2, \dots, y_n)$ where $\alpha, \beta \in V$

VECTOR ADDITION – $+ : V \times V \longrightarrow V$

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

(a)
$$\begin{aligned} \beta + \alpha &= (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n) \\ &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ &= \alpha + \beta. \end{aligned}$$
(COMMUTATIVITY)

(b) let $\gamma = (z_1, z_2, \dots, z_n)$, $\gamma \in V$
$$\begin{aligned} \alpha + (\beta + \gamma) &= (x_1, x_2, \dots, x_n) + (y_1 + z_1, y_2 + z_2, \dots, y_n + z_n) \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \dots, x_n + y_n + z_n) \\ &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) + (z_1, z_2, \dots, z_n) \\ &= (\alpha + \beta) + \gamma \end{aligned}$$
(ASSOCIATIVITY)

(c) if $\vec{0} = (0, 0, \dots, 0) \in V = \mathbb{R}^n$ s.t.
$$\begin{aligned} \Rightarrow \alpha + \vec{0} &= (x_1, x_2, \dots, x_n) + (0, 0, \dots, 0) \\ &= (x_1 + 0, x_2 + 0, \dots, x_n + 0) \\ &= (x_1, x_2, \dots, x_n) \\ &= \alpha \end{aligned}$$
(ZERO VECTOR)

(d) $\forall \alpha = (x_1, x_2, \cdots, x_n)$, $-\exists -\alpha$ s.t.
$-\alpha = (-x_1, -x_2, \cdots, -x_n) \in V$ and.
$\alpha + (-\alpha) = 0$
$\Rightarrow (x_1, x_2, \cdots, x_n) + (-x_1, x_2, \cdots, -x_n) = 0$
$\Rightarrow (x_1 - x_1, x_2 - x_2, \cdots, x_n - x_n) = 0$
$\Rightarrow (0, 0, \cdots, 0) = 0$
$\Rightarrow 0 = 0$ (ANTI-PARALLEL VECTOR)

<u>SCALAR MULTIPLICATION</u> – $\cdot : F \times V \longrightarrow V$

$$c \cdot \alpha = (cx_1, cx_2, \cdots, cx_n)$$

(a) $1 \cdot \alpha = (1 \cdot x_1, 1 \cdot x_2, \cdots, 1 \cdot x_n)$
$= (x_1, x_2, \cdots, x_n)$
$= \alpha$. (IDENTITY)

(b) $(c_1 c_2)\alpha = (c_1 c_2 x_1, c_1 c_2 x_2, \cdots, c_1 c_2 x_n)$
$= (c_1 \cdot (c_2 x_1), c_1 \cdot (c_2 x_2), \cdots, c_1 \cdot (c_2 x_n))$
$= c_1 \cdot (c_2 x_1, c_2 x_2, \cdots, c_2 x_n)$
$= c_1 \cdot (c_2 \alpha)$ (ASSOCIATIVITY)

(c) let $\alpha = (x_1, x_2, \cdots, x_n)$ & $\beta = (y_1, y_2, \cdots, y_n)$
$\Rightarrow \alpha + \beta = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$
$\Rightarrow c \cdot (\alpha + \beta) = (c(x_1 + y_1), c(x_2 + y_2), \cdots, c(x_n + y_n))$
$= (cx_1 + cy_1, cx_2 + cy_2, \cdots, cx_n + cy_n)$
$= c\alpha + c\beta$.

(DISTRIBUTION w.r.t. VECTOR)

(d) let $c_1, c_2 \in \mathbb{F}$ and $\alpha = (x_1, x_2, \ldots, x_n) \in V$

$$(c_1 + c_2)\alpha = \big((c_1+c_2)x_1, (c_1+c_2)x_2, \ldots, (c_1+c_2)x_n\big)$$
$$= \big(c_1x_1 + c_2x_1, c_1x_2 + c_2x_2, \ldots, c_1x_n + c_2x_n\big)$$
$$= (c_1x_1, c_1x_2, \ldots, c_1x_n) + (c_1x_1, c_1x_2, \ldots, c_1x_n)$$
$$= c_1\alpha + c_2\alpha.$$

(DISTRIBUTION w.r.t. SCALAR)

$\because$ $V = \mathbb{R}^n$ follows ALL the 8 properties, its a vector space. $V = \langle \mathbb{R}^n, \mathbb{R}, +, \cdot \rangle$

EXAMPLE 2: The n-Tuple Space.

let $\mathbb{F}$ be a field.
let $V = \mathbb{F}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{F}\}$

let $\alpha = (x_1, x_2, \dots, x_n)$ & $\beta = (y_1, y_2, \dots, y_n) \in V = \mathbb{F}^n$

let $+ : V \times V \longrightarrow V$ be vector addition

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

let $\cdot : \mathbb{F} \times V \longrightarrow V$ be scalar multiplication

if $c \in \mathbb{F}$, then

$$c\alpha = (cx_1, cx_2, \dots, cx_n)$$

Prove that : $\langle \mathbb{F}^n, \mathbb{F}, + \cdot \rangle$ is a vector space.

proof :

Show that $\mathbb{F}^n$ satisfies the following :

a) commutativity in addition.
b) Associativity in addition.
c) has ZERO VECTOR.
d) has additive inverse.
e) has identity in multiplication.
f) Associativity in multiplication.
g) distribution over scalars.
h) distribution over vectors.

a) consider $\alpha, \beta \in \mathbb{F}^n$.

$$\alpha = (x_1, x_2, \dots, x_n)$$
$$\beta = (y_1, y_2, \dots, y_n)$$
$$\Rightarrow \alpha+\beta = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$$
$$= (y_1+x_1, y_2+x_2, \dots, y_n+x_n)$$
$$= \beta+\alpha$$
$$\Rightarrow \alpha+\beta = \beta+\alpha$$

as individual x_i & y_i's $\in \mathbb{F}$ and $\mathbb{F}$ demands commutativity

Hence, Commutativity is TRUE in Addition.

b) let $\gamma = (z_1, z_2, \dots, z_n)$

$$\Rightarrow \alpha+(\beta+\gamma) = (x_1+(y_1+z_1), x_2+(y_2+z_2), \dots, x_n+(y_n+z_n))$$
$$= ((x_1+y_1)+z_1, (x_2+y_2)+z_2, \dots, (x_n+y_n)+z_n)$$
$$= (\alpha+\beta)+\gamma$$
$$\Rightarrow \alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$$

$x_i, y_i, z_i \in \mathbb{F}$ which demands Associativity

Hence, Associativity is TRUE in Addition.

c) let $\vec{0} = (0,0,0,\dots,0) \in \mathbb{F}^n$ $\quad 0 \in \mathbb{F}$, as $\mathbb{F}$ demands identity.

$$\Rightarrow \alpha+\vec{0} = (x_1+0, x_2+0, \dots, x_n+0)$$
$$= (x_1, x_2, \dots, x_n)$$
$$= \alpha$$
$$\Rightarrow \alpha+\vec{0} = \alpha$$

Hence, $\vec{0}$ (ZERO-VECTOR) $\in \mathbb{F}^n$.

d) if $\alpha = (x_1, x_2, \dots, x_n) \in \mathbb{F}^n$ and each $x_i \in \mathbb{F}$, then take $-\alpha = (-x_1, -x_2, \dots, -x_n)$ Where each $-x_i$ ALSO $\in \mathbb{F}$ too, as $\mathbb{F}$ demands additive inverse. So, $-\alpha \in \mathbb{F}^n$.

$$\Rightarrow \alpha + (-\alpha) = (x_1 - x_1, x_2 - x_2, \cdots, x_n - x_n)$$
$$= (0, 0, 0, \cdots, 0)$$
$$= \vec{0}$$
$$\Rightarrow \alpha + (-\alpha) = \vec{0}$$

Hence, additive inverse exists.

e) $1 \in \mathbb{F}$, as $\mathbb{F}$ demands multiplicative identity.

$$\Rightarrow 1 \cdot \alpha = (1 \cdot x_1, 1 \cdot x_2, \cdots, 1 \cdot x_n)$$
$$= (x_1, x_2, \cdots, x_n)$$
$$= \alpha.$$

$x_i \in \mathbb{F}$, So, $1 \cdot x_i = x_i$

$$\Rightarrow 1 \cdot \alpha = \alpha.$$

Hence, identity in multiplication exists.

f) let $c_1, c_2 \in \mathbb{F}$

$$\Rightarrow (c_1 c_2)\alpha = ((c_1 c_2)x_1, (c_1 c_2)x_2, \cdots, (c_1 c_2)x_n)$$
$$= (c_1 \cdot (c_2 x_1), c_1 \cdot (c_2 x_2), \cdots, c_1 \cdot (c_2 x_n))$$

as $c_1, c_2 \in \mathbb{F}$, they follow associativity over multiplication.

$$= c_1 \cdot (c_2 \alpha)$$
$$\Rightarrow (c_1 c_2)\alpha = c_1 \cdot (c_2 \alpha)$$

Hence, associativity is TRUE in Multiplication.

g) let $c_1, c_2 \in \mathbb{F}$ & $\alpha = (x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$

$$\Rightarrow (c_1 + c_2) \cdot \alpha = ((c_1 + c_2)x_1, (c_1 + c_2)x_2, \cdots, (c_1 + c_2)x_n)$$
$$= (c_1 x_1 + c_2 x_1, c_1 x_2 + c_2 x_2, \cdots, c_1 x_n + c_2 x_n)$$

as $c_1, c_2 \in \mathbb{F}$, they must follow distributive property.

$$= c_1 \alpha + c_2 \alpha$$
$$\Rightarrow (c_1 + c_2)\alpha = c_1 \alpha + c_2 \alpha.$$

Hence, distribution is TRUE over Scalars.

h) let $\alpha = (x_1, x_2, \ldots, x_n)$ and $\beta = (y_1, y_2, \ldots, y_n)$ $\in \mathbb{F}^n$. let $c \in \mathbb{F}$.

$$\Rightarrow c\cdot(\alpha+\beta) = (c\cdot(x_1+y_1), c(x_2+y_2), \ldots\ldots, c\cdot(x_n+y_n))$$

as $c, x_i, y_i \in \mathbb{F}$, they follow distribution.

$$= (cx_1 + cy_1, cx_2 + cy_2, \ldots\ldots, cx_n + cy_n)$$

$$= c\cdot\alpha + c\cdot\beta$$

$$\Rightarrow c(\alpha+\beta) = c\cdot\alpha + c\cdot\beta$$

Hence, distribution is TRUE over vectors.

$\therefore$, $\langle \mathbb{F}^n, \mathbb{F}, +, \cdot \rangle$ is a vector space !

HENCE PROVED.

X ———————— X ———————— X

EXAMPLE 3 : Space of $\mathbb{F}^{m \times n}$ matrices.

let $\mathbb{F}$ be a field.

let $V = \mathbb{F}^{m\times n} = \{ A = [a_{ij}]_{m\times n} : a_{ij} \in \mathbb{F} \}$

let $A = [a_{ij}]$ and $B = [b_{ij}] \in V$ and $c \in \mathbb{F}$.

then :

$+ : V \times V \longrightarrow V$, vector Addition is defined as -

$$[A+B]_{ij} = [a_{ij} + b_{ij}]$$

$\cdot : \mathbb{F} \times V \longrightarrow V$, scalar multiplication is defined as -

$$[cA]_{ij} = [c a_{ij}]$$

prove that : $\langle \mathbb{F}^{m\times n}, \mathbb{F}, +, \cdot \rangle$ is a field.

proof : prove the same properties as done for n-tuple space.

a) $$\begin{aligned}[A+B]_{ij} &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \qquad \{a_{ij}, b_{ij} \in \mathbb{F}\} \\ &= [B+A]_{ij}\end{aligned}$$

b) $$\begin{aligned}[A+[B+C]]_{ij} &= [a_{ij} + [B+C]_{ij}] \\ &= [a_{ij} + [b_{ij} + c_{ij}]] \qquad \{a_{ij}, b_{ij}, c_{ij} \in \mathbb{F}\} \\ &= [[a_{ij} + b_{ij}] + c_{ij}] \\ &= [[A+B]_{ij} + c_{ij}] \\ &= [[A+B]+C]_{ij}\end{aligned}$$

c) let $\vec{0} = [O]_{ij} = [O_{ij}]$ s.t. $O_{ij} = 0$.
$0 \in \mathbb{F}$, due to additive identity.

$$\begin{aligned} \Rightarrow [A+O]_{ij} &= [a_{ij} + O_{ij}] \\ &= [a_{ij} + 0] \\ &= [a_{ij}] \\ &= [A]_{ij} \end{aligned}$$

d) if $[A]_{ij} = [a_{ij}] \in \mathbb{F}^{m \times n}$, then

$[-A]_{ij} = [-a_{ij}] \in \mathbb{F}^{m \times n}$ too $\{ -a_{ij} \in \mathbb{F} \}$

$$\begin{aligned} \Rightarrow [A + [-A]]_{ij} &= [a_{ij} + [-A]_{ij}] \\ &= [a_{ij} + [-a_{ij}]] \\ &= [O_{ij}] \\ &= [O]_{ij} = \vec{0} \end{aligned}$$

e) $1 \in \mathbb{F}$.

$$\begin{aligned} \Rightarrow [1 \cdot A]_{ij} &= [1 \cdot a_{ij}] \\ &= [a_{ij}] \\ &= [A]_{ij} \end{aligned}$$

f) $c_1, c_2 \in \mathbb{F}$.

$$\begin{aligned} \Rightarrow [c_1 c_2 A]_{ij} &= [c_1 c_2 a_{ij}] \\ &= [c_1 \cdot (c_2 a_{ij})] \\ &= [c_1 \cdot [c_2 A]_{ij}] \\ &= [c_1 \cdot [c_2 A]]_{ij} \end{aligned}$$

g) $c_1, c_2 \in \mathbb{F}$.

$$\Rightarrow [(c_1+c_2)A]_{ij} = [(c_1+c_2)a_{ij}]$$
$$= [c_1 \cdot a_{ij} + c_2 \cdot a_{ij}]$$
$$= [[c_1 A]_{ij} + [c_2 A]_{ij}]$$
$$= [[c_1 A] + [c_2 A]]_{ij}$$

h) if $c \in \mathbb{F}$ & $[A]_{ij} = [a_{ij}]$ and $[B]_{ij} = [b_{ij}] \in \mathbb{F}^{m \times n}$

$$\Rightarrow [c \cdot [A+B]]_{ij} = [c \cdot [A+B]_{ij}]$$
$$= [c \cdot [a_{ij} + b_{ij}]]$$
$$= [c \cdot a_{ij} + c \cdot b_{ij}]$$
$$= [[cA]_{ij} + c[B]_{ij}]$$
$$= [[cA] + [cB]]_{ij}$$

Hence Proved.

In the note, $\mathbb{F}^{n \times n}$ is NOT a field because it violates commutativity over multiplication.
if $A, B \in \mathbb{F}^{n \times n}$, its NOT necessary that:
$AB = BA$.

X ———— X ———— X

Example 4 : Set of all real-valued, continous fns. defined on $[0,1]$.

$$V = \{ f : f:[0,1] \to \mathbb{R} \text{ and } f \text{ is continuous on } [0,1] \}$$

$+ : V \times V \to V \equiv (f+g)s = f(s) + g(s) \ \forall s \in [0,1]$.

$\cdot : \mathbb{R} \times V \to V \equiv (cf)s = cf(s) \ \forall s \in [0,1] \ \& \ c \in \mathbb{R}$

Prove that : $\langle V, \mathbb{R}, +, \cdot \rangle$ is a vector space.

NOTE : the outputs / range of the fns. are all $\in \mathbb{R}$. as $\mathbb{R}$ is a field, we can apply its properties to them.

Proof :

a) $$(f+g)(s) = f(s) + g(s) = g(s) + f(s) = (g+f)(s)$$

b) $$(f+(g+h))s = f(s) + (g+h)(s) = f(s) + g(s) + h(s) = (f+g)(s) + h(s) = ((f+g)+h)(s)$$

c) $\exists$ a fn. 0 s.t. $0(s) = 0 \ \forall s \in [0,1]$.

$$\Rightarrow (f+0)(s) = f(s) + 0(s) = f(s)$$

d) if $f \in V$, $\exists\, h \in V$ s.t.

$h(s) = -f(s) \quad \forall\, s \in [0,1]$

$-f(s) \in \mathbb{R}$ as $\mathbb{R}$ follows additive inverse.

$$\Rightarrow (f+h)(s) = f(s) + h(s)$$
$$= f(s) - f(s)$$
$$= 0.$$

e) $1 \in \mathbb{R}$ {R follows multiplicative inverse}

$$\Rightarrow (1 \cdot f)(s) = 1 \cdot f(s)$$
$$= f(s)$$

f) $c_1, c_2 \in \mathbb{R}$.

$$\Rightarrow (c_1 c_2 f)(s) = c_1 c_2 f(s)$$
$$= c_1 \cdot (c_2 f(s))$$
$$= c_1 \cdot (c_2 f)(s)$$
$$= (c_1 \cdot (c_2 f))(s).$$

g) $c_1, c_2 \in \mathbb{R}$

$$\Rightarrow ((c_1 + c_2) f)(s) = (c_1 + c_2) f(s)$$
$$= c_1 f(s) + c_2 f(s)$$
$$= (c_1 f)(s) + (c_2 f)(s).$$

h) $c \in \mathbb{R}$ and $f, g \in \mathbb{R}$

$$\Rightarrow (c \cdot (f+g))(s) = c \cdot (f+g)(s)$$
$$= c \cdot (f(s) + g(s))$$
$$= c \cdot f(s) + c \cdot g(s)$$
$$= (cf)(s) + (cg)(s)$$

Hence proved.

X ——————————— X ——————————— X

Example 5 : The space of polynomial fns. over a field.

if $f(x) \rightarrow$ polynomial fn., then:

$f: \mathbb{F} \rightarrow \mathbb{F}$, $f(x) = \sum_{k=0}^{n} a_k x^k$, where each

$a_i \in \mathbb{F}$.

$$V = \left\{ f : f: \mathbb{F} \rightarrow \mathbb{F},\ f(x) = \sum_{k=0}^{n} a_i x^i,\ a_i \in \mathbb{F} \right\}$$

define - $+ : V \times V \rightarrow V$ (vector addn.)
$(f+g)(s) = f(s) + g(s)$, $\forall s \in \mathbb{F}$.

$\cdot : F \times V \rightarrow V$ (scalar multiplication)
$(cf)(s) = c \cdot f(s)$, $\forall c, s \in \mathbb{F}$.

is the defined space a vector space? Proof - HOMEWORK for the reader. Its similar to previous ones. The Answer is yes, its a vector space.

X ———————— X ———————— X

Problem 1 : let $V = \{(x,y) : x, y \in \mathbb{R}\}$.

$+ : V \times V \to V$ defined as -

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$\cdot : \mathbb{F} \times V \to V$ defined as -

$$c(x,y) = (cx, y)$$

is V a vector space ?

Solution : All properties w.r.t. addition would be satisfied.
but, distribution over scalars would fail. Why ?

$$\text{LHS} \to (c_1 + c_2)\alpha = ((c_1 + c_2)x, y)$$

$$\begin{aligned} \text{RHS} \to c_1\alpha + c_2\alpha &= (c_1 x, y) + (c_2 x, y) \\ &= (c_1 x + c_2 x, y + y) \\ &= ((c_1 + c_2)x, 2y) \end{aligned}$$

LHS != RHS.

$\therefore$, distribution over scalar fails.

X ———————— X ———————— X

NOTE 1 : let V be a vector space over $\mathbb{F}$.

$$\Rightarrow \quad \vec{0} + \vec{0} = \vec{0}, \qquad 0 \in V$$
$$\Rightarrow \quad c\cdot(\vec{0}+\vec{0}) = c\cdot\vec{0}, \qquad c \in \mathbb{F}.$$
$$\Rightarrow \quad c\cdot\vec{0} + c\cdot\vec{0} = c\cdot\vec{0}, \qquad c(\alpha+\beta) = c\cdot\alpha + c\cdot\beta$$
$$\Rightarrow \quad (c\cdot\vec{0} + c\cdot\vec{0}) - c\cdot\vec{0} = c\cdot\vec{0} - c\cdot\vec{0}$$
$$\exists\,(-c\cdot 0) \in V$$
$$\Rightarrow \quad c\cdot\vec{0} + (c\cdot\vec{0} - c\cdot\vec{0}) = \vec{0}$$
$$\Rightarrow \quad c\cdot\vec{0} + \vec{0} = \vec{0}$$
$$\Rightarrow \quad c\cdot\vec{0} = \vec{0} \longrightarrow \forall\ c \in \mathbb{F}.$$

Qn : Show that $0\alpha = \vec{0} \ \forall\ \alpha \in V$

$$\alpha = (x_1, x_2, \cdots, x_n) \qquad \{\vec{0} = (0,0,\cdots,0)\}$$
$$c\alpha = (cx_1, cx_2, \cdots, cx_n) \rightarrow \text{scalar multipln.}$$
$$\Rightarrow 0\alpha = (0\cdot x_1, 0\cdot x_2, \cdots, 0\cdot x_n)$$
$$= (0,0,\cdots,0) \rightarrow \text{pdt. of } 0 \text{ with scalar is } 0.$$
$$= \vec{0}$$
$$\Rightarrow 0\alpha = \vec{0}$$

NOTE 2 : $0\alpha = \vec{0}$

$$\Rightarrow (1+(-1))\alpha = \vec{0} \qquad \{\text{Additive inverse}\}$$
$$\Rightarrow 1\cdot\alpha + (-1)\alpha = \vec{0} \qquad \{\text{dist. over scalar}\}$$
$$\Rightarrow \alpha + (-1)\alpha = \vec{0} \qquad \{1\cdot\alpha = \alpha\}$$
$\Rightarrow$ additive inverse of α exists and its :
$$-\alpha = (-1)\alpha.$$

NOTE 3 : if $c\alpha = 0$, then $c = 0$ or $\alpha = \vec{0}$

* suppose $c = 0$.

then, $c \cdot \alpha = 0 \cdot \alpha = 0$.

* suppose $c \neq 0$, $c \in \mathbb{F}$, $\mathbb{F} \to$ field.

$\Rightarrow c^{-1} \in \mathbb{F} \to$ multiplicative inverse.

$\Rightarrow c^{-1} \cdot (c\alpha) = 0$

$\Rightarrow (c^{-1} \cdot c)\alpha = 0 \to$ Associativity.

$\Rightarrow 1 \cdot \alpha = 0 \to$ multiplicative inverse.

$\Rightarrow \alpha = 0 \to 1 \cdot \alpha = \alpha$.

X ———————— X ———————— X

LINEAR COMBINATION :

let $V \to$ vector space over field $\mathbb{F}$. A vector $\beta \in V$ is said to be a linear combination of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in V if $\exists$ scalars $c_1, c_2, \ldots, c_n$ s.t.

$$\beta = \sum_{i=1}^{n} c_i \alpha_i$$

problems 2 and 3 $\to$ HOMEWORK to the reader.

Problem 4 : let $\mathbb{R}$ be the real field. find all the vectors $\in \mathbb{R}^3$ that are a linear combo. of $(1,0,-1)$, $(0,1,1)$ and $(1,1,1)$.

Solution : find all $(x,y,z) \in \mathbb{R}^3$ s.t. $\exists\, a,b,c \in \mathbb{R}$ s.t.

$$a(1,0,-1) + b(0,1,1) + c(1,1,1) = (x,y,z)$$

$$\Rightarrow (a,0,-a) + (0,b,b) + (c,c,c) = (x,y,z)$$

① $a + 0 + c = x$ ② $0 + b + c = y$ ③ $-a + b + c = z$

$\Rightarrow a + c = x$ $\Rightarrow b + c = y$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. let $R \to$ RRE form of A.

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R.$$

$R = I$.

$\Rightarrow$ A is row-equivalent to I. (by defn.)

$\Rightarrow$ A is invertible (by theorem 12)

$\Rightarrow$ the system of eqns. $AX = B$ has a soln. for each B (by theorem 13)

$\Rightarrow$ $\therefore$, $\forall\, Y^t = (x, y, z) \in \mathbb{R}^3$, $\exists\, X^t = (a, b, c) \in \mathbb{R}^3$
s.t. $a(1, 0, -1) + b(0, 1, 1) + c(1, 1, 1) = (x, y, z)$.
We consider X^t and Y^t as originally X and Y are column matrices, and (a, b, c) and (x, y, z) are row matrices.