Evaluation

Continuous Assessment: 25 Marks

Mid Semester: 25 Marks

End Semester: 50 Marks

Textbooks and References

Textbooks:

1. G.F. Simmons. Differential Equations, Tata McGraw Hill, 2003.
2. E. Kreyszig. Advanced Engineering Mathematics, Wiley, 2007.

References:

1. William. E. Boyce and R. C. Diprima, Elementary Differential Equations and Boundary Value Problems, John Wiley, 8 Edn, 2004.
2. L.S. Ross. Differential Equations, Wiley, 2007.
3. W. Trench. Elementary Differential Equations, http://digitalcommons.trinity.edu/mono

Syllabus

First Order Equations.

Linear ordinary differential equations with constant coefficients, method of variation of parameters, Linear systems of ordinary differential equations.

Power series solution of ordinary differential equations. Singular points.

Bessel and Legendre differential equations; properties of Bessel functions and Legendre Polynomials.

Laplace transforms elementary properties of Laplace transforms, inversion by partial fractions, convolution theorem and its applications to ordinary differential equations.

Fourier series.

Introduction to partial differential equations, wave equation, heat equation, diffusion equation

Differential Equations is the natural goal of elementary calculus and the most important part of mathematics for understanding the physical sciences.

G.F. Simmons

The Definition

An *equation* involving one dependent variable and its derivatives with respect to one or more independent variables is called a *differential equation*.

Applications

- Physics
- Chemistry
- Biology
- Astronomy
- Engineering
- Economics

Why is It so Ubiquitous?

- Suppose $y = f(x)$ is a given function.
- Then its derivative $\dfrac{dy}{dx}$ can be interpreted as the rate of change of y with respect to x.
- In many natural processes, the variables involved and their rates of change are related by means of certain basic scientific principles governing the respective processes.

An Example

Newton's second law: The acceleration a of a body of mass m is proportional to the total force F acting on it, with $1/m$ as the constant of proportionality. That is,

$$a = \frac{1}{m}F.$$

Equivalently,

$$F = ma.$$

An Example..

Suppose, for example, that a body of mass m falls freely under the influence of gravity alone. Then the force acting on the body is $F = mg$.

If y is the distance down to the body from a fixed height, then its velocity is $v = dy/dt$ and its acceleration is $a = dv/dt = d^2y/dt^2$.

So, by Newton's second law, we then have

$$m\frac{d^2y}{dt^2} = F \quad \text{or} \quad m\frac{d^2y}{dt^2} = mg \quad \text{or} \quad \frac{d^2y}{dt^2} = g.$$

An Example..

If we assume that air exerts a resisting force proportional to the velocity, then the total force acting on the object is $F = mg - k\dfrac{dy}{dt}$.

Then Newton's second law ($F = ma$) implies that

$$m\frac{d^2y}{dt^2} = mg - k\frac{dy}{dt}$$

.

More Examples

- $\dfrac{dy}{dt} = -ky$
- $m\dfrac{d^2y}{dt^2} = -ky$
- $\dfrac{dy}{dx} + 2xy = e^{-x^2}$
- $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$
- $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + p(p+1)y = 0$
- $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - p^2)y = 0$

More Examples...

- $$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$
- $$a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) = \frac{\partial w}{\partial t}$$
- $$a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) = \frac{\partial^2 w}{\partial t^2}$$

These equations are called the Laplace's equation, the heat equation, and the wave equation, respectively.

Definition

- An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a **differential equation**.
- A differential equation with only one independent variable is called an **ordinary differential equation** (so that all the derivatives occurring in it are ordinary derivatives).
- A differential equation with more than one independent variables is called a **partial differential equation** (so that all the derivatives occurring in it are partial derivatives).

Ordinary Differential Equations

The general ordinary differential equation of the *n*th **order** is

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0.$$

Or, using the prime notation for derivatives, it is

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0.$$

Question: How to check whether a function $y = y(x)$ is a solution of the given differential equation?

Example

$y = e^{2x}$ is a solution of the second order equation

$$y'' - 5y' + 6y = 0.$$

Why? Because it satisfies the equation! Substituting $y = e^{2x}$ on the left-hand side of the equations gives 0: Here we have $y = e^{2x}$, $y' = 2e^{2x}$ and $y'' = 4e^{2x}$.

$\therefore$ LHS $= y'' - 5y' + 6y = 4e^{2x} - 5(2e^{2x}) + 6(e^{2x}) = 10e^{2x} - 10e^{2x} = 0 =$ RHS.

Similarly, $y = e^{3x}$ is also a solution of this equation.

More generally,

$$y = c_1 e^{2x} + c_2 e^{3x}$$

is a solution of the equation for every choice of the constants c_1 and c_2. (Prove!)

Homework

Prove that

$$xy = \log y + c$$

is a solution of

$$\frac{dy}{dx} = \frac{y^2}{1 - xy}$$

for every value of the constant c.

Hint: Differentiate $xy = \log y + c$ with respect to x and rearrange the terms.

Note

A solution of a differential equation usually contains one or more arbitrary constants, equal in number to the order of the equation.

Homework

Verify that the following functions (explicit or implicit) are solutions of the corresponding differential equations:

(a)	$y = x^2 + c$	$y' = 2x$
(b)	$y = cx^2$	$xy' = 2y$
(c)	$y^2 = e^{2x} + c$	$yy' = e^{2x}$
(d)	$y = ce^{kx}$	$y' = ky$
(e)	$y = c_1 e^{2x} + c_2 e^{-2x}$	$y'' - 4y = 0$
(f)	$y = c_1 \sin 2x + c_2 \cos 2x$	$y'' + 4y = 0$
(g)	$y = \sin^{-1} xy$	$xy' + y = y'\sqrt{1 - x^2y^2}$
(h)	$y = ce^{y/x}$	$y' = y^2/(xy - x^2)$
(i)	$x^2 = 2y^2 \log y$	$y' = \dfrac{xy}{x^2 + y^2}$
(j)	$x + y = \tan^{-1} y$	$1 + y^2 + y^2y' = 0$

Finding Solutions of Differential Equations

Equations of the form:

$$\frac{dy}{dx} = f(x).$$

Solution: Just integrate!

$$y = \int f(x)dx + c.$$

Example

Find the general solution of the differential equation $y' = e^{3x} - x$.

Solution: Here the differential equation is of the form $y' = f(x)$ with $f(x) = e^{3x} - x$.
So, the solution is

$$y = \int (e^{3x} - x)dx + c.$$

That is,

$$y = \frac{e^{3x}}{3} - \frac{x^2}{2} + c,$$

where c is an arbitrary constant.

Note

In some cases, finding $\int f(x)dx$ will not be possible (in terms of elementary functions); e.g.,

$$\int e^{-x^2}dx \quad \text{and} \quad \int \frac{\sin x}{x}dx$$

.

So, finding the solution even in the simplest case, $\frac{dy}{dx} = f(x)$, may not be easy.

Nevertheless, as long as a function $f(x)$ is continuous over the range of integration, we have

$$y = \int_{x_0}^{x} f(t)dt + c$$

defined and satisfies

$$\frac{dy}{dx} = f(x).$$

Homework

Find the general solution of each of the following differential equations:

(a) $xy' = 1$

(b) $y' = xe^{x^2}$

(c) $y' = \sin^{-1} x$

(d) $(1 + x^3)y' = x$

(e) $(1 + x^2)y' = \tan^{-1} x$

Separable Equations

These are equations of the form

$$\frac{dy}{dx} = f(x)g(y).$$

Such an equation can be rewritten in the form

$$\frac{dy}{g(y)} = f(x)dx.$$

The solution is now obtained by *simply integrating* both the sides:

$$\int \frac{dy}{g(y)} = \int f(x)dx + c.$$

Note: Equations which can be solved in this manner are called separable equations.

Example

Solve $xyy' = y - 1$.

Solution: The given equation is

$$xy\frac{dy}{dx} = y - 1.$$

This can be rewritten as

$$\frac{y}{y-1}dy = \frac{dx}{x}.$$

We now obtain the solution by simply integrating both sides:

$$\int \frac{y}{y-1}dy = \int \frac{dx}{x} + c.$$

Homework

Find the general solution of each of the following differential equations:

(a) $x^5 y' + y^5 = 0$

(b) $xy' = (1 - x^2) \tan y$

(c) $y' = 2xy$

(d) $(1 + x^2)dy + (1 + y^2)dx = 0$

(e) $y \log y dx - x dy = 0$

First Order Equations: A Quick Summary

- The general first order equation is

$$F\left(x, y, \frac{dy}{dx}\right) = 0.$$

- This equation cannot be solved in general, even in the simplest case $\frac{dy}{dx} = f(x)$.
- There are first order equations with no real-valued solutions: e.g.,

$$\left(\frac{dy}{dx}\right)^2 + 1 = 0.$$

- There are equations with the single solution $y = 0$ (not even with an arbitrary constant): e.g.,

$$\left(\frac{dy}{dx}\right)^2 + y^2 = 0.$$

Gaining Some Understanding of the Solutions

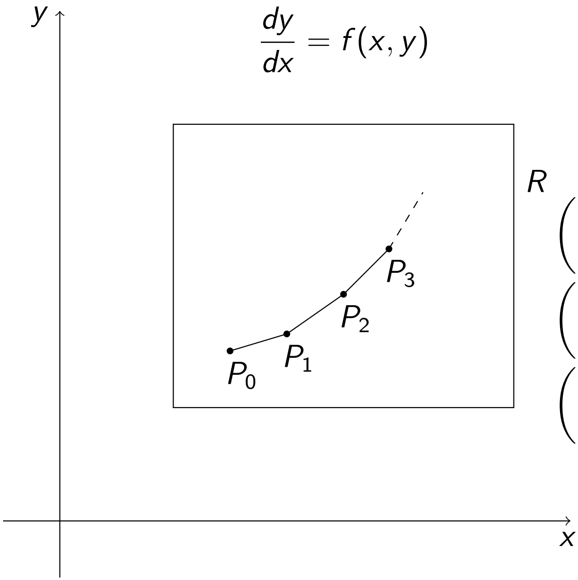
- Suppose the first order equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$ can be written in the form
$$\frac{dy}{dx} = f(x, y).$$
- What is the geometric meaning of a solution of $\dfrac{dy}{dx} = f(x, y)$?

Gaining Some Understanding of the Solutions of $\frac{dy}{dx} = f(x, y)$

- Suppose that $f(x, y)$ is continuous throughout some rectangle R in the xy-plane.
- Let $P_0 = (x_0, y_0)$ be a point in the region R. Then the number
$$\left(\frac{dy}{dx}\right)_{P_0} = f(x_0, y_0)$$
determines a direction at P_0.
- Let $P_1 = (x_1, y_1)$ be a point near P_0 in this direction. Then the number
$$\left(\frac{dy}{dx}\right)_{P_1} = f(x_1, y_1)$$
determines a new direction at P_1.
- Next let $P_2 = (x_2, y_2)$ be a point near P_1 in this new direction and let
$$\left(\frac{dy}{dx}\right)_{P_2} = f(x_2, y_2)$$

Gaining Some Understanding of the Solutions...

$$\frac{dy}{dx} = f(x, y)$$



$$\left(\frac{dy}{dx}\right)_{P_0} = f(x_0, y_0)$$

$$\left(\frac{dy}{dx}\right)_{P_1} = f(x_1, y_1)$$

$$\left(\frac{dy}{dx}\right)_{P_2} = f(x_2, y_2)$$

Gaining Some Understanding of the Solutions...

- Imagine that these successive points move closer to one another and become more numerous.
- Then the broken line approaches a *smooth curve* through the initial point P_0.
- This curve is a solution of the equation $\dfrac{dy}{dx} = f(x, y)$.
- For each point (x, y) on this curve, the slope is $f(x, y)$.
- This is precisely what the differential equation $\dfrac{dy}{dx} = f(x, y)$ demands.
- Indeed through each point in R, there passes a unique solution curve (also called the integral curve) of the differential equation.

Theorem (Picard's Theorem)

If $f(x, y)$ and $\partial f/\partial y$ are continuous functions on a closed rectangle R, then through each point (x_0, y_0) in the interior of R there passes a unique integral curve of the equation $dy/dx = f(x, y)$.

Note: If we consider a fixed x_0 in this theorem, then the integral curve that passes through (x_0, y_0) is fully determined by the choice of y_0. Thus the integral curves of the equation constitute a **one-parameter family of curves**.

Example

For the differential equation $y' = xe^x$, find the particular solution for which $y = 3$ when $x = 1$.

Solution: The general solution of this differential equation is

$$y = \int xe^x dx + c.$$

That is,

$$y = (x-1)e^x + c.$$

Substituting $x = 1$ in this solution gives $y = c$. But we need $y = 3$. So, we must take $c = 3$.

Thus the required solution is

$$y = (x-1)e^x + 3.$$

Homework

For each of the following differential equations, find the particular solution that satisfies the given initial condition:

(a) $y' = 2\sin x \cos x,\ y = 1$ when $x = 0$.

(b) $y' = \log x,\ y = 0$ when $x = e$.

(c) $x(x^2 - 4)y' = 1,\ y = 0$ when $x = 1$.

(d) $(x+1)(x^2+1)y' = 2x^2 + x,\ y = 1$ when $x = 0$.

Homework

For each of the following differential equations, find the integral curve that passes through the given point:

(a) $y' = e^{3x-2y}$, $(0,0)$.
(b) $xdy = (2x^2+1)dx$, $(1,1)$.
(c) $3\cos 3x \cos 2y \, dx - 2\sin 3x \sin 2y \, dy = 0$, $(\pi/12, \pi/8)$.
(d) $y' = e^x \cos x$, $(0,0)$.

Families of Curves. Orthogonal Trajectories

- The general solution of a first order equation normally contains one arbitrary constant, called a *parameter*.
- Assigning various values to this parameter leads to a **one-parameter family of curves.** Each curve in the family is a particular solution of the differential equation.
- The converse also holds true: the curves of any **one-parameter family** are solution curves of a differential equation.

Families of Curves to Differential Equations

- Consider a one-parameter family of curves

$$f(x, y, c) = 0.$$

- Differentiate this equation implicitly with respect to x to get a relation of the form

$$g\left(x, y, \frac{dy}{dx}, c\right) = 0.$$

- Eliminate the parameter c between the above equations to obtain

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

as the differential equation of the given one-parameter family of curves.

Example

Find the differential equation of the family of straight lines

$$y = c.$$

Solution: Differentiating the equation of the given family of straight lines with respect to x gives

$$\frac{dy}{dx} = 0.$$

Since c is already absent in this differential equation, there is no need of eliminating it.

Hence it is the desired differential equation (as can be easily verified).

Example

Find the differential equation of the family of straight lines

$$y = cx.$$

Solution: Differentiating the equation of the given family of straight lines with respect to x gives

$$\frac{dy}{dx} = c.$$

Eliminating c between the above equations gives the required differential equation:

$$x\frac{dy}{dx} = y \qquad \text{or} \qquad \frac{dy}{dx} = \frac{y}{x}.$$

Example

Find the differential equation of the family of circles

$$x^2 + y^2 = 2cx.$$

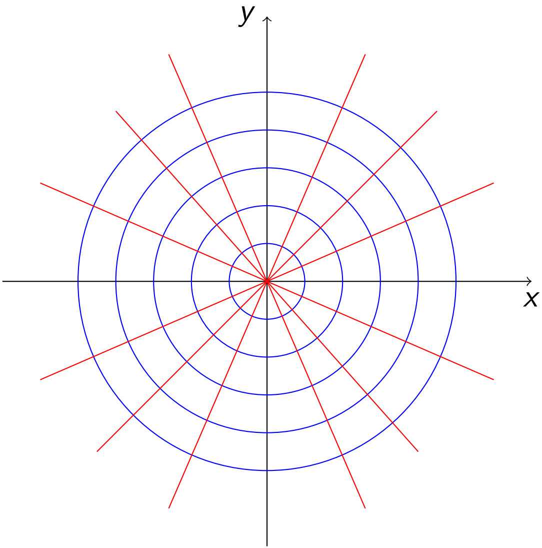
Solution: The given family of curves consists of all circles that are tangent to the y-axis at the origin.
On differentiation with respect to x, the given equation becomes

$$2x + 2y\frac{dy}{dx} = 2c \qquad \text{or} \qquad x + y\frac{dy}{dx} = c.$$

Eliminating c between the given and above equations, we obtain the differentiable equation of the family of circles:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

Orthogonal Trajectories



Definition

Consider two family of curves such that each curve in one family is orthogonal to each curve in the other family. Then each family is called a family of **orthogonal trajectories** of the other.

Example: As we have already seen, the family of concentric circles $x^2 + y^2 = c^2$ and the family of straight lines $y = cx$ are orthogonal trajectories of each other.

Finding Orthogonal Trajectories

$\frac{dy}{dx}$

Consider the differential equation

$$\frac{dy}{dx} = f(x, y).$$

It often represents a one-parameter family of curves, where each curve in the family, at a point (x, y) on the curve, has slope $f(x, y)$.

Thus for finding the corresponding orthogonal trajectories, we must consider

$$\frac{dy}{dx} = \frac{-1}{f(x, y)}.$$

More generally, the orthogonal trajectories of a one-parameter family of curves corresponding to a first order differential equation is obtained by replacing each occurrence of $\frac{dy}{dx}$ in the differential equation by $-\frac{dx}{dy}$ and solving the resultant differential equation.

Example

Find the orthogonal trajectories of the family of circles $x^2 + y^2 = c^2$.

Solution: The differential equation of the given family of circles is

$$x + y\frac{dy}{dx} = 0.$$

Therefore, the differential equation of the orthogonal trajectories of the given family of circles is

$$x + y\left(-\frac{dx}{dy}\right) = 0$$

or

$$\frac{dy}{dx} = \frac{y}{x}.$$

Solving this, we obtain the equation of the orthogonal trajectories:

Example

Find the orthogonal trajectories of the family of circles

$$x^2 + y^2 = 2cx.$$

Solution: The differential equation of the given family of circles is

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

Therefore, the differential equation of the orthogonal trajectories of the given family of circles is

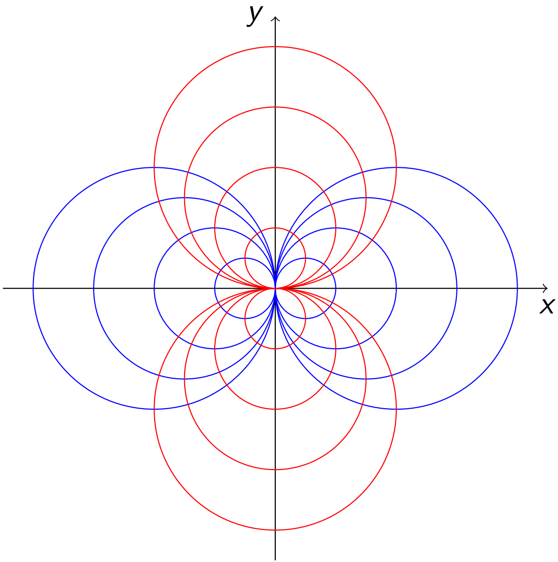
$$-\frac{dx}{dy} = \frac{y^2 - x^2}{2xy}$$

or

Solving this gives the equation of the orthogonal trajectories of the given family of circles:

$$x^2 + y^2 = 2cy.$$

Orthogonal Trajectories



Homework

1. Sketch each of the following families of curves, find the orthogonal trajectories, and add them to the sketch:
   (a) $xy = c$; (b) $y = cx^2$; (c) $y = ce^x$.
2. What are the orthogonal trajectories of the family of curves (a) $y = cx^4$; (b) $y = cx^n$ where n is any positive integer? In each case sketch both family of curves. What is the effect on the orthogonal trajectories of increasing the exponent n?

Solutions of First Order Equations: Homogeneous Equations

These are first order differential equations which can be transformed into separable equations.

Definition

A function $f(x, y)$ is called *homogeneous* of degree n if

$$f(tx, ty) = t^n f(x, y)$$

for all suitable values of x, y and t.

Examples:

- $x^2 + xy$ is a homogeneous function of degree 2.
- $\sqrt{x^2 + y^2}$ is a homogeneous function of degree 1.
- $\sin(x/y)$ is a homogeneous function of degree 0.

Homogeneous Equations

Definition

The differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is said be *homogeneous* if M and N are homogeneous functions of the *same* degree.

Note: The homogeneous differential equation above can be written in the form

$$\frac{dy}{dx} = f(x, y),$$

where $f(x, y) = -M(x, y)/N(x, y)$ is a homogeneous function of degree 0.

The Idea

A homogeneous differential equation can be written in the form

$$\frac{dy}{dx} = f(x,y)$$

where $f(x,y)$ is a homogeneous function of degree 0.
So, we have

$$f(tx,ty) = t^0 f(x,y) = f(x,y).$$

Thus if we set $t = 1/x$, we get

$$f(x,y) = f(1, y/x).$$

Introduce a new dependent variable $z = y/x$. Then the above identity becomes

$$f(x,y) = f(1,z).$$

Further, since $y = zx$, we have

The Idea...

So, the differential equation becomes

$$z + x\frac{dz}{dx} = f(1, z).$$

Now, the variables can be separated:

$$\frac{dz}{f(1, z) - z} = \frac{dx}{x}.$$

We complete the solution by integrating and replacing z by y/x.

Example

Solve $(x+y)dx - (x-y)dy = 0$.

Solution: We note that the given equation is a homogeneous equation.
We write the equation in the form

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$

And the function on the RHS is a homogeneous function of degree 0.
So, it can be made a function of y/x. We do this and replace y/x by z to obtain

$$\frac{dy}{dx} = \frac{1+y/x}{1-y/x} = \frac{1+z}{1-z}.$$

Also the substitution $y = zx$ gives $\dfrac{dy}{dx} = z + x\dfrac{dz}{dx}$.
Thus we get

Example...

On simplification followed by separation of variables, we obtain

$$\frac{(1-z)dz}{1+z^2} = \frac{dx}{x}.$$

On integration this yields

$$\tan^{-1} z - \frac{1}{2}\log(1+z^2) = \log x + c.$$

Finally, we replace z by y/x (and rearrange the terms) to obtain the desired solution:

$$\tan^{-1}\frac{y}{x} = \log\sqrt{x^2+y^2} + c.$$

Homework

Verify that the following equations are homogeneous and solve them:

(a) $(x^2 - 2y^2)dx + xydy = 0$

(b) $x^2y' - 3xy - 2y^2 = 0$

(c) $x^2y' = 3(x^2 + y^2)\tan^{-1}\frac{y}{x} + xy$

(d) $x\sin\frac{y}{x}\frac{dy}{dx} = y\sin\frac{y}{x} + x$

(e) $xy' = y + 2xe^{-y/x}$.

Homework

1. (a) If $ae \neq bd$, show that constants h and k can be chosen in such a way that the substitutions $z = x - h$, $w = y - k$ reduce
$$\frac{dy}{dx} = F\left(\frac{ax + by + c}{dx + ey + f}\right)$$
to a homogeneous equation.
   (b) If $ae = bd$, discover a substitution that reduces the equation in (a) to one in which the variables are separable.
2. Solve the following equations:
   (a) $\dfrac{dy}{dx} = \dfrac{x + y + 4}{x - y - 6}$;
   (b) $\dfrac{dy}{dx} = \dfrac{x + y - 1}{x + 4y + 2}$;
   (c) $\dfrac{dy}{dx} = \dfrac{x + y + 4}{x + y - 6}$.

Solving First Order Equations: Exact Equations

Note: The differential equation of a family of curves $f(x, y) = c$ is $df = 0$ or

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

Example: The family $x^2y^3 = c$ has $2xy^3dx + 3x^2y^2dy = 0$ as its differential equation.

Consider the **reverse** scenario: Suppose for a differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

there is a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N.$$

Then the differential equation can be written in the form

Definition

Consider the differential equation $M(x, y)dx + N(x, y)dy = 0$. If there is a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N,$$

then $Mdx + Ndy$ is called an *exact differential* and $Mdx + Ndy = 0$ is called an *exact differential equation*.

Question: How do we recognize whether a differential equation is exact? If exact, how do we find the function f?

It is easy sometimes:

- The left side of $ydx + xdy = 0$ is the differential of xy and so the solution is $xy = c$.
- The left side of $\frac{1}{y}dx - \frac{x}{y^2}dy = 0$ is the differential of x/y and so the

A Necessary Condition for Exactness

Suppose the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is exact.

Then there is a function f such that

$$\frac{\partial f}{\partial x} = M \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N.$$

Then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

But the mixed second derivatives of a function are equal: $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that $ydx + (x^2y - x)dy = 0$ is not exact.

Solution: Here $M = y$ and $N = x^2y - x$.

So,

$$\frac{\partial M}{\partial y} = 1 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 2xy - 1.$$

Thus

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

.

Hence the equation is not exact.

But This Condition is Also Sufficient!

Proved: If $M(x,y)dx + N(x,y)dy = 0$ is exact, then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

To Prove: If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $M(x,y)dx + N(x,y)dy = 0$ is exact.

Theorem

The differential equation $M(x,y)dx + N(x,y)dy = 0$ *is exact* if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

The Condition is Sufficient!

Suppose

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

We will now prove that this condition enables us to find a function f such that

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N.$$

If there is a function f satisfying the above conditions, it must, by the first condition, have the form

$$f = \int M dx + g(y).$$

And, by the second condition, this f must be such that

$$\frac{\partial}{\partial y} \int M dx + g'(y) = N.$$

The Condition is Sufficient!

Suppose

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

We will now prove that this condition enables us to find a function f such that

$$\frac{\partial f}{\partial x} = M \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N.$$

If there is a function f satisfying the above conditions, it must, by the first condition, have the form

$$f = \int M dx + g(y).$$

And, by the second condition, this f must be such that

$$\frac{\partial}{\partial y} \int M dx + g'(y) = N.$$

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M dx\right) &= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x \partial y}\int M dx \\
&= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y \partial x}\int M dx \\
&= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \\
&= 0
\end{aligned}
$$

by the assumption. Thus the function $f = \int M dx + g(y)$ with $g(y) = \int \left(N - \frac{\partial}{\partial y}\int M dx\right) dy$ satisfies

$$
\frac{\partial f}{\partial x} = M \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N.
$$

Hence the given equation is exact.

Example

Test the equation $e^y dx + (xe^y + 2y)dy = 0$ for exactness, and solve it if it is exact.

Solution: Here

$$M = e^y \qquad \text{and} \qquad N = xe^y + 2y.$$

$$\therefore \frac{\partial M}{\partial y} = e^y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = e^y.$$

Thus the equation is exact. Hence there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = e^y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = xe^y + 2y.$$

$$\frac{\partial f}{\partial x} = e^y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = xe^y + 2y.$$

Integrating the first of these equations with respect to x gives

$$f = \int e^y dy + g(y) = xe^y + g(y).$$

So

$$\frac{\partial f}{\partial y} = xe^y + g'(y).$$

Also, $\frac{\partial f}{\partial y} = xe^y + 2y$. So, we have $g'(y) = 2y$ or $g(y) = y^2$ and $f(x, y) = xe^y + y^2$.
Thus the general solution of the given differential equation is

$$xe^y + y^2 = c.$$

Homework

Determine which of the following equation are exact, and solve the ones that are.

1. $\left(x + \dfrac{2}{y}\right) dy + y dx = 0$
2. $(\sin x \tan y + 1) dx + \cos x \sec^2 y dy = 0$
3. $(y - x^3) dx + (x + y^3) dy = 0$
4. $(2y^2 - 4x + 5) dx = (4 - 2y + 4xy)$
5. $(y + y \cos xy) dx + (x + x \cos xy) dy = 0$

Integrating Factors

Motivation: The equation

$$ydx + (x^2y - x)dy = 0$$

is not exact:for $\partial M/\partial y = 1$ and $\partial N/\partial x = 2xy - 1$.

But multiplying this equation by $\dfrac{1}{x^2}$ results in

$$\frac{y}{x^2}dx + \left(y - \frac{1}{x}\right) dy = 0$$

which is exact.

If

$$M(x, y)dx + N(x, y)dy = 0$$

is not exact, when can we find a function $\mu(x, y)$ such that

$$\mu(Mdx + Ndy) = 0$$

exact?

Definition

Suppose $Mdx + Ndy = 0$ is a nonexact equation. Suppose $\mu(x, y)$ is a function such that

$$\mu(Mdx + Ndy) = 0$$

is exact. Then μ is called an *integrating factor* of the given differential equation.

Example: The equation

$$ydx + (x^2y - x)dy = 0$$

has $\dfrac{1}{x^2}$ as an integrating factor.

Question: When does a nonexact equation $Mdx + Ndy = 0$ have an integrating factor? Always! (As long as it has a general solution!)

Suppose $Mdx + Ndy = 0$ has a general solution

$$f(x, y) = c.$$

Eliminating c from the equation above, we obtain:

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0.$$

The above differential equations have the same solutions and so they are equivalent. Thus

$$\frac{dy}{dx} = -\frac{M}{N} = -\frac{\partial f / \partial x}{\partial f / \partial y}.$$

$$\therefore \ \frac{\partial f / \partial x}{M} = \frac{\partial f / \partial y}{N}.$$

Let us denote the common ratio above by $\mu(x, y)$. Then we have

Let us denote the common ratio above by $\mu(x, y)$. Then we have

$$\frac{\partial f}{\partial x} = \mu M \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \mu N.$$

Thus, on multiplying $Mdx + Ndy = 0$ by μ, it becomes

$$\mu M dx + \mu N dy = 0.$$

That is, it becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0,$$

which is an exact equation.

Note: If $Mdx + Ndy = 0$ has at least one integrating factor, it has infinitely many of them:
Let $F(f)$ be any function of f. Then

$$\mu F(f)(Mdx + Ndy) = F(f)df = d[\int F(f)df].$$

That is, $\mu F(f)(Mdx + Ndy)$ is an exact differential. This means that $\mu F(f)$ is also an integrating factor of $Mdx + Ndy = 0$.

Finding Integrating Factors

μ is an integrating factor of $Mdx + Ndy = 0$ if and only if $\mu Mdx + \mu Ndy = 0$ is exact. That is, if and only if

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$$

$$\Leftrightarrow \quad \mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}.$$

$$\Leftrightarrow \quad \frac{1}{\mu}\left(N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.$$

Suppose the differential equation has an integrating factor μ that is a function of x alone. Then with such a μ the above equation takes the form

Thus μ is an integrating factor of the differential equation if and only if

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.$$

Suppose the differential equation has an integrating factor μ that is a function of x alone. Then with such a μ the above equation takes the form

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}.$$

The left side of the above equation is a function of x only. So, the right side must also function of x only.

So, let

$$g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}.$$

Thus we have

$$\frac{1}{\mu}\frac{d\mu}{dx} = g(x).$$

Integrating, we obtain

$$\log \mu = \int g(x)dx$$

or

$$\mu = e^{\int g(x)dx}.$$

Example

Find an integrating factor of $ydx + (x^2y - x)dy = 0$.
Solution: Here we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1 - (2xy - 1)}{x^2y - x} = \frac{-2(xy-1)}{x(xy-1)} = -\frac{2}{x},$$

which is a function of x only.

$$\therefore\ \mu = e^{\int(-2/x)dx} = e^{\log(1/x^2)} = \frac{1}{x^2}$$

is an integrating factor of the given differential equation.

Similarly, if the differential equation has an integrating factor μ that is a function of y alone

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M}$$

is a function of y alone, say $h(y)$, and $\mu = e^{\int h(y)dy}$ is an integrating factor.

Note

Consider the equation $ydx + (x^2y - x)dy = 0$ rearranged as follows:

$$x^2ydy - (xdy - ydx) = 0.$$

The presence of $(xdy - ydx)$ in the differential equation reminds us the differential formula

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

and suggests us dividing the equation by x^2. This transforms the equation into

$$ydy - d\left(\frac{y}{x}\right) = 0.$$

And its general solution is

$$y^2/2 - y/x = c.$$

Useful Differential Formulas

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$d(xy) = xdy + ydx$$

$$d(x^2 + y^2) = 2(xdx + ydy)$$

$$d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$$

$$d\left(\log\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$$

Homework

Solve each of the following equations by finding an integrating factor:

(a) $(3x^2 - y^2)dy - 2xydx = 0$

(b) $(xy - 1)dx + (x^2 - xy)dy = 0$

(c) $e^x dx + (e^x \cot y + 2y \csc y)dy = 0$

(d) $(x + 2)\sin y dx + x \cos y dy = 0$

1. Show that if $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})/(Ny - Mx)$ is a function $g(z)$ of the product $z = xy$, then
$$\mu = e^{\int g(z)dz}$$
is an integrating factor for $Mdx + Ndy = 0$.
2. Under what circumstances will the equation $Mdx + Ndy = 0$ have an integrating factor that is a function of the sum $z = x + y$?

Linear Equations

Definition

A differential equation in which the derivative of highest order is a linear function of the lower order derivatives is called a linear equation.

Examples: The general first order linear equation is

$$\frac{dy}{dx} = p(x)y + q(x).$$

The general second order linear linear equation is

$$\frac{d^2y}{dx^2} = p(x)\frac{dy}{dx} + q(x)y + r(x).$$

Note

$$\frac{d}{dx}(e^{\int Pdx}y) = e^{\int Pdx}\frac{dy}{dx} + yPe^{\int Pdx} = e^{\int Pdx}\left(\frac{dy}{dx} + Py\right).$$

Thus, multiplying $\frac{dy}{dx} + P(x)y = Q(x)$ through by $e^{\int Pdx}$, it becomes

$$\frac{d}{dx}(e^{\int Pdx}y) = Qe^{\int Pdx}.$$

Integration now yields the general solution:

$$e^{\int Pdx}y = \int Qe^{\int Pdx}dx + c$$

or

$$y = e^{-\int Pdx}\left(intQe^{\int Pdx}dx + c\right).$$

Example

Solve $\dfrac{dy}{dx} + \dfrac{1}{x}y = 3x$.

Solution: The given equation is a linear equation with $P = 1/x$. So,

$$\int P dx = \int \frac{1}{x} dx = \log x \qquad \text{and} \qquad e^{\int P dx} = e^{\log x} = x.$$

Thus multiplying the given equation through by x, it becomes

$$\frac{d}{dx}(xy) = 3x^2.$$

Hence, the general solution is

$$xy = x^3 + c \qquad \text{or} \qquad = yx^2 + cx^{-1}.$$

Homework

Solve the following as linear equations.

1. $x\frac{dy}{dx} - 3y = x^4$
2. $y' + y = \frac{1}{1+e^{2x}}$
3. $(1 + x^2)dy + 2xydx = \cot x dx$
4. $(x \log x)y' + y = 3x^3$

Homework

Write the equation $\frac{dy}{dx} + P(x)y = Q(x)$ in the form $Mdx + Ndy = 0$ and show that it has integrating factor μ that is a function of x alone. Find μ and solve $\mu Mdx + \mu Ndy = 0$ as an exact equation.

Bernoulli's Equations

A first order equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a Bernoulli's equation.

Note that this is just a first order linear equation if $n = 0$ or 1.

In other cases, it can be reduced to a linear equation by the substitution $z = y^{1-n}$.

Multiplying the Bernoulli's equation through by y^{-n} gives

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

We further multiply it through by the constant $1 - n$ (???) to obtain

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)P(x)y^{1-n} = (1-n)Q(x). \tag{1}$$

Differentiating $z = y^{1-n}$ with respect to x gives

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

Appropriately replacing the terms in Equation 1 now gives a linear equation in z and x:

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

We complete the solution by solving the above as a linear equation and substituting y^{1-n} for z.

Homework

Solve the following as linear equations treating x as the dependent variable and y as the independent variable:

1. $(e^y - 2xy)y' = y^2$
2. $y - xy' = y'y^2e^y$
3. $f(y)^2\frac{dx}{dy} + 3f(y)f'(y)x = f'(y)$