Define: (a) Convergence of a sequence. (b) Divergence of a sequence to infinity. Prove using the definitions that $\left\{\frac{1}{n^p}\right\}$ converges if $p \geq 0$ and diverges to infinity if $p < 0$. (5)

Find $\lim_{n\to\infty}(n!)^{1/n^2}$. (2)

Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges. (3)

Present series of nonzero terms with sums (a) 0 and (b) π^2. (2)

Prove or disprove:

(a) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ converges; (b) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converges. (3)

Consider the series $\sum a_n$, where $a_n = \begin{cases} n/2^n & \text{if } n \text{ is prime;} \\ 1/2^n & \text{otherwise.} \end{cases}$

Does it converge? Give reasons. (2)

Prove that the alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if $p > 0$ and diverges if $p \leq 0$. (3)

Consider a power series $\sum a_n x^n$. Prove: (a) If the power series converges for $x = c \neq 0$, then it converges absolutely for $|x| < |c|$. (b) (a) If the power series diverges for $x = d$, then it diverges for $|x| > |d|$. (4)

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n^{10}}$. Where does it converge absolutely? Where does it converge conditionally? (3)

Find the Maclaurin series of $f(x) = \sqrt{1+x}$. In particular, find its general term. (3)