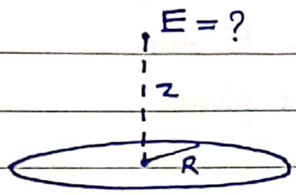
PH1000 Assignment 1

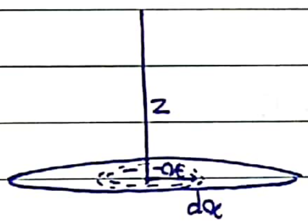
1. Flat disk of radius R and surface charge density σ.



Prerequisite,
we know that
$\bar{E}$ due to ring at distance x
from its center is

$$\bar{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{xQ}{(x^2+R^2)^{3/2}}$$

$\Rightarrow$ given disk can be thought of as collection of concentric loops / rings with infinitesimal thickness dx



$\therefore$ dE (due to one such ring)

$$= \frac{k\,dQ\,z}{(x^2+z^2)^{3/2}}\hat{z}$$

where dQ = charge on ring
$= \sigma \times dA$
$= \sigma \times 2\pi x\,dx$

$$\therefore \bar{E}_{net} = \int_0^R \frac{k\,dQ\,z}{(x^2+z^2)^{3/2}}$$

$$= \int_0^R \frac{k\,z}{(x^2+z^2)^{3/2}} \cdot \sigma \times 2\pi x\,dx$$

$$= (k\sigma 2\pi z)\int_0^R \frac{x\,dx}{(x^2+z^2)^{3/2}}$$

Let $(x^2+z^2) = t$

$$\therefore x\,dx = \frac{dt}{2}$$

$$= \frac{k\sigma 2\pi z}{2}\int \frac{dt}{t^{3/2}}$$

$$= \left(\frac{k\sigma 2\pi z}{2}\right)\left[\left(\frac{-1}{2}\right)^{-1}\cdot\frac{1}{\sqrt{t}}\right] = \frac{\sigma 2\pi z}{4\pi\varepsilon_0 2}\left[\left(\frac{-1}{2}\right)^{-1}\frac{1}{\sqrt{x^2+z^2}}\right]_0^R$$

$$= \frac{\sigma z}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right)\hat{z}$$

$$\therefore \quad \bar{E} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\hat{k}$$

now, if $\lim R \to \infty$,

$$\bar{E} = \frac{\sigma}{2\varepsilon}(1-0)$$

$$= \frac{\sigma}{2\varepsilon_0}\hat{k}$$ → same as $\bar{E}$ due to infinite plane.

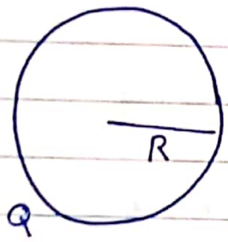
(as $R \to \infty$, the disk depicts an infinitely long plane)

if $z >> R$,

$$\bar{E} = \frac{\sigma}{2\varepsilon}\left(1 - \frac{z}{\sqrt{z^2}}\right) = \frac{\sigma}{2\varepsilon}(1-1) = 0$$

Thus if $z >> R \Rightarrow \bar{E} = 0$

2.

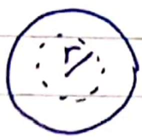


given sphere of radius R and charge Q.

$\Rightarrow$ Energy = W.D = $V \cdot q$

So $dE = V \cdot dq$

now,



at $r<R$, our $dq = \rho \, d\tau$

where ρ is the volume charge density = constant

(since uniformly dist.)

and V due to q charged sphere at r will be

$$V(r) = \frac{kq}{r}$$

now, $d\tau$ in spherical is $(dr)(r\sin\theta \, d\phi)(r\,d\theta)$

$$= r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$\text{So} \int dE = \int V \cdot dq$$

$$= \int_v \left(\frac{kq}{r}\right)\rho \, d\tau$$

$$= \iiint \left(\frac{kq}{r}\right)\rho \left(r^2 \sin\theta \, d\theta \, dr \, d\phi\right)$$

$\Rightarrow$ q at any r, will be

$$q = \rho \cdot \left(\tfrac{4}{3}\pi r^3\right)$$

$$\Rightarrow \int dE = \int_V \frac{k\rho\left(\frac{4}{3}\pi r^3\right)}{r} \rho \cdot r^2 \sin\theta \, d\theta \, d\phi \, dr$$

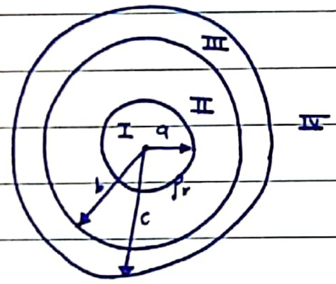
$$= k\rho^2 \cdot \frac{4\pi}{3} \int_0^R r^4 \cdot dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{1}{4\pi\varepsilon} \cdot \left(\tfrac{4}{3}\rho^2\right) \frac{4\pi}{3} \left[\frac{R^5}{5}\right] \left[\cos\theta\right]_\pi^0 \left[2\pi\right]$$

$$= \frac{1}{4\pi\varepsilon_0} \times \frac{Q^2}{\left(\frac{4}{3}\right)^2 \pi^2 R^6} \times \frac{4\pi}{3} \left(\frac{R^5}{5}\right) (2)(2\pi)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q^2}{R} \cdot \frac{3}{5}$$

$$\therefore \boxed{E = \frac{3}{5} \frac{kQ^2}{R}}$$ energ required to assemble the given system.

3. 

given system of uniformly distributed sphere with vol charge density ρ and radius a, encapsulated by spherical shell with no charge and $r_{in} = b$, $r_{out} = c$

E in I.

drawing a spherical gaussian surface, of radius r we use gauss law.

$$\Rightarrow \int E \cdot ds = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{\rho \cdot \left(\frac{4}{3}\pi r^3\right)}{\varepsilon_0} \hat{r}$$

$$\Rightarrow \vec{E} = \frac{\rho r \cdot \hat{r}}{3\varepsilon} \text{ for } r < a$$

E in II

using a spherical gaussian surface of $a < r < b$, we apply gauss law

$$\Rightarrow \int \vec{E} \cdot ds = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{\rho\left(\frac{4}{3}\pi R^3\right)}{\varepsilon_0}$$

$$\Rightarrow \bar{E} = \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{r}$$

$\bar{E}$ in III

since $\bar{E}$ inside conductor is zero,

$$\bar{E} = 0 \quad ; b<r<c$$

$\bar{E}$ in IV

using spherical surface(gaussian), $c<r$
we apply gauss law

$$\oint \bar{E}\cdot d\bar{s} = \frac{q_{enc}}{\varepsilon_0}$$

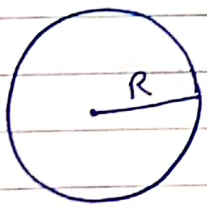
$$\Rightarrow \bar{E}\cdot 4\pi r^2 = \frac{\left(\frac{4}{3}\pi R^3\right)\rho}{\varepsilon_0}$$

$$\Rightarrow \bar{E} = \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{r} \quad , \quad c<r$$

4. given polarization $P(r) = kr\,\hat{r}$

, r is the distance from centre.

in a sphere of radius R.



To find σ_b and ρ_b, we use bound charges concept.

$$\bar{P}\cdot\hat{n} = \sigma_b \quad \text{and} \quad \bar{\nabla}\cdot\bar{P} = -\rho_b$$

Thus,

since $\hat{n}$ is always radially outwards, $\hat{r}$

$$\sigma_b = (k\bar{r})\cdot(\hat{r})$$

$$\sigma_b = kR$$

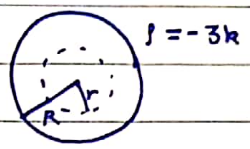
and in spherical coordinates

$$-\bar{\nabla}\cdot\bar{P} = \frac{-1}{r^2}\frac{\partial(r^2\cdot kr)}{\partial r}$$

$$\rho_b = \frac{-1}{r^2}\frac{\partial(kr^3)}{\partial r} \Rightarrow \frac{-3kr^2}{r^2} = -3k$$

$\therefore \sigma_b = kR$ and $\rho_b = -3k$

To find feild inside, we use



$$\int \bar{D}\cdot ds = Q_{free}$$

$$\Rightarrow \bar{D}\cdot(4\pi r^2) = (-3k)(4/3\pi r^3)$$

$$\Rightarrow \bar{D} = -kr\,\hat{r}$$

So $\bar{E} = \dfrac{\bar{D}}{\varepsilon} = \dfrac{-kr}{\varepsilon}\hat{r}$

To find field outside, we can use gauss law.

Finding $Q_{enc} \Rightarrow \quad Q_{enc} = \oint_s \sigma\cdot ds + \oint_v \rho\cdot d\tau$

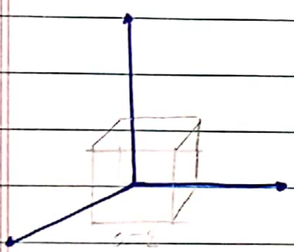
$$= kR \times 4\pi R^2 + (-3k)\frac{4}{3}\pi R^3$$

$$= 4\pi R^3 k - 4\pi R^3 k$$

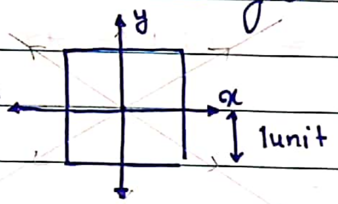
$$= 0$$

$\therefore$ Total charge inside $= 0$, $\therefore \bar{E} = 0$ for $r > R$

5. given $P(r) = 6r\hat{r}$, where r is distance from centee.



Top view:



$\bar{r}$ in cartesion is $x\hat{i}+y\hat{j}+z\hat{k}$

To find bound charges, we use:

$\sigma_b = \bar{P}\cdot\hat{n}$ and $\rho_b = -\bar{\nabla}\cdot\bar{P}$

for surface 1. (in $\hat{i}$ direction)

$$\Rightarrow \sigma_1 = \bar{P}\cdot\hat{n}$$

$$= 6(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i})$$

$$= 6x = 6\cdot 1 = 6$$

lly,

$$\sigma_2 = 6(x\hat{i}+y\hat{j}+z\hat{k})(-\hat{i})$$

$$= -6x = -6(-1) = 6$$

lly we get

$$\sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 6$$

To find ρ_b,

$$-\overline{\nabla}\cdot\overline{P} \Rightarrow -\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)6$$

$$= -6(1+1+1) = -18$$

now,

To find total Q_{enc} we use

$$Q_{enc} = \int \sigma\cdot ds + \int \rho\cdot d\tau$$

$$= \int_{s_1} \sigma_1\cdot ds_1 + \int_{s_2} \sigma_2\cdot ds_2 + \int_{s_3} \sigma_3\cdot ds_3$$

$$+ \int_{s_4} \sigma_4\cdot ds_4 + \int_{s_5} \sigma_5\cdot ds_5 + \int_{s_6} \sigma_6\cdot ds_6$$

$$+ \int \rho\cdot d\tau$$

$$\Rightarrow [6(2\times 2)]6 + \int \rho\cdot d\tau$$

as $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 6$

$$\Rightarrow 144 + \rho\int d\tau \quad (\text{as } \rho \text{ is const})$$

$$144 + (-18)\times[2^3]$$

$$= 144 - 144 = 0$$

$\therefore Q_{enc} = 0$

6. to find ε_r in parallel plate capacitor

a) $E = \frac{1}{2}CV^2$

$$C = \frac{A\varepsilon}{d} \Rightarrow \frac{0.12\times\varepsilon}{80\times 10^{-6}} = 1500\,\varepsilon F$$

so $\frac{1}{2}\times 1500\times 12^2 = 1\times 10^{-6}$

$$\Rightarrow \varepsilon = 111.1\times 10^{-12}/12 \Rightarrow 9.25\times 10^{-12}$$

now $\varepsilon = \varepsilon_0\cdot\varepsilon_r \Rightarrow \varepsilon_r = \varepsilon/\varepsilon_0$

$$\Rightarrow \frac{9.25\times 10^{-12}}{8.854\times 10^{-12}} = 1.045$$

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b) given energy density $= 100 J/m^3$, Energy E is

$\therefore E = 100 \times Vol$

$= 100 \times A \times d$

also, E in parallel plate capacitor $= \frac{1}{2} C V^2$

and $C = \frac{A\varepsilon}{d} = \frac{A\varepsilon_0\varepsilon_r}{d}$

$\therefore \quad 100 \times A \times d = \frac{1}{2} \cdot \frac{A\varepsilon_0\varepsilon_r}{d} \times 200^2$

$= \frac{100 \times 2 \times d^2}{200^2 \times 8.854 \times 10^{-12}}$

$= \frac{(45 \times 10^{-6})^2}{200 \times 8.854 \times 10^{-12}}$

$= \underline{1.14}$

c) $E = 200 \, kV/m$

$\sigma = 20 \mu C/m^2$

To find ε_r, we use $\bar{E}$ between parallel plate capacitor

$\bar{E} = \frac{\sigma}{\varepsilon}$

$\Rightarrow 200 \times 10^3 = \frac{20 \times 10^{-6}}{\varepsilon_0 \cdot \varepsilon_r}$

$200 \times 10^3 = \frac{20 \times 10^{-6}}{8.854 \times 10^{-12} \cdot \varepsilon_r}$

$= \varepsilon_r = \frac{20}{200} \frac{\times 10^{-9}}{8.854 \times 10^{-12}}$

$\varepsilon_r = \frac{100}{8.854} = \underline{11.29}$