Representation of Transformations by Matrices

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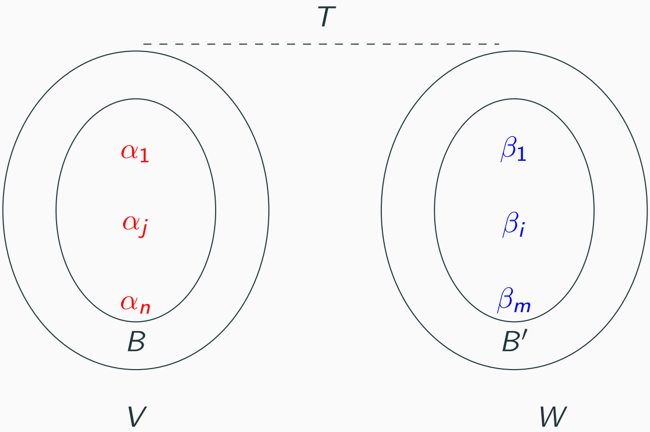
Linear transformations and Matrices



Ordered basis, $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

Ordered basis, $B' = \{\beta_1, \beta_2, \ldots, \beta_m\}$

Note that (i) $T : V \longrightarrow W$ is a L.T.

L.T. and Matrix

Note that $T(\alpha_j) \in W =$ Span $B' =$ Span $\{\beta_1, \dots, \beta_m\}$.

$$\begin{aligned} T(\alpha_j) &= A_{1j}\beta_1 + A_{2j}\beta_2 + \ldots + Amj\beta_m \\ &= \sum_{i=1}^{m} A_{ij}\beta_i \quad \text{for } 1 \leq j \leq n \end{aligned}$$

$$[T(\alpha_j)]_{B'} = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \ldots \\ Amj \end{bmatrix}$$

Let $\alpha \in V$. Then there exist unique scalars $x_1, x_2, \dots, x_n$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = \sum_{j=1}^{n} x_j\alpha_j$$

Since T is a L.T.,

$$T(\alpha) = T\left(\sum_{j=1}^{n} x_j \alpha_j\right) = \sum_{j=1}^{n} x_j T(\alpha_j)$$

$$\begin{aligned} T(\alpha) &= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} A_{ij} \beta_i\right) \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} x_j\right) \beta_i \end{aligned}$$

$$[T(\alpha)]_{B'} = \begin{bmatrix} \sum_{j=1}^{n} A_{1j}x_j \\ \sum_{j=1}^{n} A_{2j}x_j \\ \dots \\ \sum_{j=1}^{n} A_{mj}x_j \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ & & \dots & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

$$\implies \quad [T(\alpha)]_{B'} = A\,[\alpha]_B$$

where $A = ([T(\alpha_1)]_{B'}, [T(\alpha_2)]_{B'}, \dots, [T(\alpha_n)]_{B'})$

Theorem 11

Let V be an n-dimensional vector space over the field F and W an m-dimensional vector space over F. Let B be an ordered basis for V and B' an ordered basis for W. For each linear transformation $T : V \longrightarrow W$, there is an $m \times n$ matrix A with entries in F such that

$$[T(\alpha)]_{B'} = A\,[\alpha]_B$$

for every vector $\alpha \in V$. Furthermore, $T \longrightarrow A$ is a one-one correspondence between the set of all linear transformation from V into W and the set of all $m \times n$ matrices over the field F.

Proof (See the previous slides)

Note : Let V be a finite dimensional vector space and B an ordered basis for V. If $T : V \longrightarrow V$ a linear operator, then A is denoted as $[T]_B$.

$$[T(\alpha)]_B = [T]_B\,[\alpha]_B$$

Problem 1

Let $T : R^2 \longrightarrow R^3$ be a linear transformation defined as

$$T(x_1, x_2) = (x_2, x_1 - x_2, x_1 + x_2).$$

Let $B = \{\alpha_1 = (1,0), \alpha_2 = (0,1)\}$ and $B' = \{\beta_1 = (1,1,1), \beta_2 = (1,1,0), \beta_3 = (1,0,0)\}$ be respective ordered bases for R^2 and R^3. Find a 3×2 matrix A such that

$$[T(\alpha)]_{B'} = A\,[\alpha]_B \quad \text{for all } \alpha \in R^2$$

Solution :

$$\begin{aligned} T(\alpha_1) &= T(1,0) = (0,1,1) \\ &= (1,1,1) + 0(1,1,0) - (1,0,0) \\ &= \beta_1 + 0\beta_2 - \beta_3 \end{aligned}$$

Problem 1 contd.

$$[T(\alpha_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned} T(\alpha_2) &= T(0,1) = (1,-1,1) \\ &= (1,1,1) - 2(1,1,0) + 2(1,0,0) \\ &= \beta_1 - 2\beta_2 + 2\beta_3 \end{aligned}$$

$$[T(\alpha_2)]_{B'} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Problem 1 contd.

$$A = ([T(\alpha_1)]_{B'}, \;\; [T(\alpha_2)]_{B'}) \; = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 2 \end{bmatrix}$$

Verification :

$$T(\alpha) = \; T(x_1, x_2) = (x_2, x_1 - x_2, x_1 + x_2)$$

$$T(\alpha) = (x_1 + x_2)\beta_1 - 2x_2\beta_2 + (-x_1 + 2x_2)\beta_3$$

Problem 1 contd.

$$[T(\alpha)]_{B'} = \begin{bmatrix} x_1 + x_2 \\ -2x_2 \\ -x_1 + 2x_2 \end{bmatrix}, \quad [\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \alpha = x_1\alpha_1 + x_2\alpha_2$$

$$A\,[\alpha]_B = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -2x_2 \\ -x_1 + 2x_2 \end{bmatrix} = [T(\alpha)]_{B'}$$

Note : determinant of T, $det(T) = det(A)$

Problem 2

Let $T : R^2 \longrightarrow R^2$ be a L.T. defined as $T(x_1, x_2) = (x_1, 0)$. Let $B = \{\alpha_1 = (1,1), \alpha_2 = (1,2)\}$ be an ordered basis for R^2. Find $[T]_B$.

$$\begin{aligned} T(\alpha_1) &= T(1,1) = (1,0) \\ &= 2\alpha_1 - \alpha_2 \end{aligned}$$

$$[T(\alpha_1)]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{aligned} T(\alpha_2) &= T(1,2) = (1,0) \\ &= 2\alpha_1 - \alpha_2 \end{aligned}$$

$$[T(\alpha_2)]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Problem 2 contd

$$[T]_B = ([T(\alpha_1)]_B, [T(\alpha_2)]_B) = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

Please verify the answer.

Theorem 14

Let V be a finite dimensional vector space over the field F and let

$$B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}. \quad \text{and } B' = \{\beta_1, \beta_2, \ldots, \beta_n\}$$

be two ordered bases for V. Suppose $T : V \longrightarrow V$ is a linear operator. If $P = [P_1, P_2, \ldots, P_n]$ is the $n \times n$ matrix with columns $P_j = [\beta_j]_B$, then

$$[T]_{B'} = P^{-1} [T]_B P$$

Proof (Reading assignment)

Similar matrices

Let A and B be $n \times n$ matrices over the field F. We say B is similar to A over F if there exists an invertible $n \times n$ matrix P over F such that

$$B = P^{-1}AP.$$

Show that similarity is an equivalence relation on $F^{n \times n}$.

$A \sim A$ for all $A \in F^{n \times n}$ since $A = I^{-1}AI$.

If $A \sim B$, then there exists P such that $B = P^{-1}AP$.

$\implies A = (P^{-1})^{-1}BP^{-1}$. $\implies B \sim A$.

If $A \sim B$ and $B \sim C$, then $B = P^{-1}AP$ and $C = Q^{-1}BQ$.

$\implies C = (PQ)^{-1}A(PQ)$ and thus $A \sim C$.

Eigen values/ Characteristic values

Let A be an $n \times n$ (square) matrix over the field F. A scalar $\lambda \in F$ is an eigen value of A if there exists a non-zero vector $X \in F^{n\times 1}$ such that

$$AX = \lambda X$$

Note :(1) The non-zero vector X such that $AX = \lambda X$ is called an eigen vector of A associated with λ.
(2) $E_A(\lambda) = \{X \ : \ AX = \lambda X\}$ is callled the eigen space of A associated with λ. (Prove that $E_A(\lambda)$ is a subspace.)
(3) λ is an eigen value of $A \iff$ There exists a non-zero vector X such that $AX = \lambda X$. $\iff (A - \lambda I)X = 0$ has a non-trivial solution. $\iff \det(A - \lambda I) = 0$. $\iff \det(\lambda I - A) = 0$. $\iff$ the matrix $(A - \lambda I)$ is singular (not invertible).

Charateristic Polynomial

Consider $f(x) = \det\,(xI - A)$.
If λ is an eigen value of A, then $f(\lambda) = 0$.
So $f(x) = \det\,(xI - A)$ is called the characteristic polynomial of A.

Application (Mechanical Engineering)

Eigenvalues and eigenvectors allow us to "reduce" a linear operation to separate, simpler, problems. For example, if a stress is applied to a "plastic" solid, the deformation can be dissected into "principle directions"- those directions in which the deformation is greatest. Vectors in the principle directions are the eigenvectors and the percentage deformation in each principle direction is the corresponding eigenvalue

Note

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A - \lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

The charecteristic polynomial of A,

$f_A(\lambda) = det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc$

$f_A(\lambda) = \lambda^2 - trace(A)\lambda + det(A)$

If λ_1, λ_2 are the roots of the charateristic polynomial, then

$f_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$ and

$\lambda_1 + \lambda_2 = trace(A)$ and $\lambda_1\lambda_2 = det(A)$

Problem

Find the eigen values and corresponding eigen spaces of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

Solution: $A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{pmatrix}.$

$det(A - \lambda I) = 0 \implies \lambda = 1, 2$

Consider $\lambda = 1 \implies E_A(1) = \{(a, 0) : a \in R\} = span\,\{(1, 0)\}$

Consider $\lambda = 2 \implies E_A(2) = \{(2a, a) : a \in R\} = span\,\{(2, 1)\}$

Note that the set of eigen vectors $\{(1,0),(2,1)\}$ is a L.I. set. Let us construct an invertible matrix $P = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Note that $P^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. Compute $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D$

Let us explore a simple case of diagonalization

Diagonalization (a simple case)

Let $A \in F^{n\times n}$ and let $AX_i = \lambda_i X_i$ for $i = 1, 2, \ldots, n$. Suppose that $\{X_1, X_2, \ldots, X_n\}$ is a L.I. subset of $F^{n\times 1}$. Clearly $P = [X_1, X_2, \ldots, X_n]$ is an invertible $n \times n$ matrix.

$$AP = A[X_1, X_2, \ldots, X_n] = [AX_1, AX_2, \ldots, AX_n]$$

$$= [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n] = [X_1, X_2, \ldots, X_n] \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

$AP = PD \Longrightarrow P^{-1}AP = D$

$D^2 = P^{-1}A^2P$, $D^k = P^{-1}A^kP$ and $A^k = PD^kP^{-1}$

$A^k \longrightarrow O$ as $k \longrightarrow \infty$ provided $|\lambda_i| < 1$ for $i = 1, 2, \ldots, n$

A few points

1 Find the eign values of the $D, D^2, D^3, \ldots,$ (see last page)

2 Find the eigen values and eigen spaces of $C = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

3 Find the eigen values and eigen spaces of $C^2, C^3, \ldots$

4 Suppose $P^{-1}AP = B$. Show that A and B have same eigen values. If λ is an eigen value of B, find an eigen vector of A corresponding to λ.

Problem 3

Find the eigen values and corresponding eigen spaces of the matrix

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

Solution : The characteristic polynomial of A

$$f_A(x) = \det\,(xI - A) = \begin{vmatrix} x-5 & 6 & 6 \\ 1 & x-4 & -2 \\ -3 & 6 & x+4 \end{vmatrix} = (x-2)^2(x-1)$$

$f_A(\lambda) = 0 \Longrightarrow \lambda = 1, 2, 2$
Hence eigen values of $A = \{1, 2\}$.

(a) The eigen space when $\lambda = 1$

$$E_A(\lambda) = E_A(1) = \{X : \quad AX = \lambda X = X\} = \{X : \ (A - I)X = 0\}$$

$$A - I = \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - I)X = 0 \Longrightarrow x_1 - x_3 = 0, \ x_2 + \frac{1}{3}x_3 = 0$$

Note that (i) pivot variables $= \{x_1, \ x_2\}$ and
(ii) free variables $= \{x_3\}$. Let $x_3 = a$. $\Longrightarrow x_1 = a, x_2 = -\frac{a}{3}$

$$E_A(1) = \left\{(a, -\frac{a}{3}, a) : \ a \in R\right\} = \left\{\frac{a}{3}(3, -1, 3) : a \in R\right\}$$

$$\implies \quad E_A(1) = \text{ span}\{(3,-1,3)\}$$

(b) The eigen space when $\lambda = 2$

$$E_A(\lambda) = E_A(2) = \{X : \quad AX = \lambda X = 2X\} = \{X : \ (A - 2I)X = 0\}$$

$$A - 2I = \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - 2I)X = 0 \implies x_1 - 2x_2 - 2x_3 = 0$$

Note that (i) pivot variables $= \{x_1\}$ and
(ii) free variables $= \{x_2, x_3\}$.

Problem 3 contd.

Let $x_2 = a$ and $x_3 = b$. $\implies x_1 = 2a + 2b$

$E_A(1) = \{(2a + 2b, a, b) : \ a, b \in R\} \ = \{a(2, 1, 0) + b(2, 0, 1) : a, b \in R\}$

$$E_A(2) = \text{ span } \{(2, 1, 0), (2, 0, 1)\}$$

Let us construct a diagonal matrix D with eigen values as diagonal entries.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 3 contd.

Let us construct an inverible matrix P using basis vectors (as columns) of $E_A(1)$ and $E_A(2)$.

$$P = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Prove that $AP = PD$

$$\implies D = P^{-1}AP$$

So A is similar to a diagonal matrix D and hence A is diagonalizable.