24/4/24

DE Assignment 3

CS23I1027

Q1) $y' = \cos(x+y)$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=0}^{\infty} n \cdot a_n \cdot x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} \cdot x^n$$

$$\cos(x+y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x+y)^{2k}$$

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{(2k)!} \left(\sum_{m=0}^{2k} {}^{2k}C_m \, x^{2k-m} \cdot y^m \right) \right]$$

But $y = \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{(2k)!} \left(\sum_{m=0}^{2k} {}^{2k}C_m \cdot x^{2k-m} \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right)^m \right) \right]$$

Finding coefficients of x^n

$$x^{2k-m} \cdot y^m = x^{2k-m} \cdot \left(\sum_{i=0}^{m} a_i x^i \right)$$

$$= \sum_{i=0}^{m} a_i \cdot x^{2k-m+i}$$

If $2k-m+i = n$

Not Sure

(OR)

Q1) $y' = \cos(x+y)$

As $y(x) = \sum a_n x^n$

$\Rightarrow y'(x) = \sum n \cdot a_n \cdot x^{n-1}$

$y'(x) = \cos(x+y(x))$

As we know $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

$\Rightarrow \cos(x+y(x)) = \cos(x + a_0 + a_1 x + a_2 x^2 + \cdots)$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n \cdot x^{n-1} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(x+y)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\left(x + \sum a_n x^n\right)^{2n}}{(2n)!}$$

...

(OR)

$$\Rightarrow a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \cdots$$

$$= 1 - \frac{(a_0 + x(a_1+1) + a_2 x^2 + \cdots)^2}{2!} + \cdots$$

$$a_1 = 1 - \frac{a_0^2}{2!} + \frac{a_0^4}{4!} - \frac{a_0^6}{6!} + \cdots$$

$$\boxed{\therefore a_1 = \cos(a_0)}$$

$$2a_2 = \frac{-2a_0(a_1+1)}{2!}$$

$$\boxed{\therefore a_2 = \frac{-a_0(1+\cos(a_0))}{2!}} \quad \cdots$$

Not Sure

Correct Method but it is said to use power series

$$y' = \cos(x+y)$$

$$t = x+y$$

$$dt = (1+y')\,dx$$

$$\frac{dt}{dx} - 1 = y'$$

$$\Rightarrow \frac{dt}{dx} - 1 = \cos(t)$$

$$\Rightarrow \frac{dt}{dx} = \cos(t) + 1$$

$$\Rightarrow \frac{dt}{1+\cos(t)} = dx$$

$$\Rightarrow \int \frac{dt}{1+\cos(t)} = \int dx$$

$$\Rightarrow \tan\left(\frac{t}{2}\right) + c = x$$

$$\Rightarrow \tan\left(\frac{x+y}{2}\right) = x - c$$

$$\Rightarrow \frac{x+y}{2} = \tan^{-1}(x-c)$$

$$\Rightarrow x+y = 2\tan^{-1}(x-c)$$

$$\Rightarrow y = 2\tan^{-1}(c+x) - x$$

Q1) $y' = \cos(x+y)$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad — ①$$

$$\therefore\ y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} \quad — ②$$

Now, $y' = \cos(x+y)$

$$\Rightarrow y' = \cos x \cdot \cos y - \sin x \cdot \sin y$$

Substitute ① and ② in the above equation

$$\Rightarrow \sum n \cdot a_n \cdot x^{n-1} = \cos x \cdot \cos\left(\sum a_n x^n\right) - \sin x \cdot \sin\left(\sum a_n x^n\right)$$

As we cannot simplify further,

$$\sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

$$\sum n \cdot a_n \cdot x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Now, $\cos\left(\sum a_n x^n\right) = \cos\left(a_0 + a_1 x + a_2 x^2 + \cdots\cdots\right)$

$$= 1 - \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^2}{2!} + \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^4}{4!} - \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^6}{6!} + \cdots$$

And, $\sin\left(\sum a_n x^n\right) = \sin\left(a_0 + a_1 x + a_2 x^2 + \cdots\cdots\right)$

$$= (a_0 + a_1 x + a_2 x^2 + \cdots) - \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^3}{3!} + \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^5}{5!} - \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^7}{7!} + \cdots$$

Looking at the power series $\left(\sum a_n x^n\right)^m$ i.e. $(a_0 + a_1 x + a_2 x^2 + \cdots)^m$,

Constant term $= a_0^m$

Coefficient of x $= m \cdot a_1 \cdot a_0^{m-1}$

Coefficient of x^2 $= m \cdot a_2 \cdot a_0^{m-1} + \frac{m(m-1)}{2} a_1^2 \cdot a_0^{m-2}$

Coefficient of x^3 $= m \cdot a_3 \cdot a_0^{m-1} + m(m-1)\, a_2 \cdot a_1 \cdot a_0^{m-2} + \frac{m(m-1)(m-2)}{6} (a_1)^3 \cdot a_0^{m-3}$

Coefficient of x^4 $= m \cdot a_4 \cdot a_0^{m-1} + m(m-1)\, a_3 \cdot a_1 \cdot a_0^{m-2} + \frac{m(m-1)}{2} (a_2)^2 \cdot a_0^{m-2}$

$$+\ m(m-1)\, a_2 \cdot (a_1)^2 \cdot a_0^{m-3} + \frac{m(m-1)(m-2)(m-3)}{24} (a_1)^4 \cdot a_0^{m-4}$$

And so-on

$$\Rightarrow \cos\left(\sum a_n x^n\right) = 1 - \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^2}{2!} + \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^4}{4!} - \frac{(a_0 + a_1 x + a_2 x^2 + \cdots)^6}{6!} + \cdots$$

$$= 1 - \frac{a_0^2 + (2a_1a_0)x + (2a_2a_0 + a_1^2)x^2 + (2a_3a_0 + 2a_2a_1)x^3 + (a_2^2 + 2a_3a_1 + 2a_4a_0 + 2a_2a_1^2)x^4 + \cdots}{2!}$$

$$+ \frac{a_0^4 + (4a_1a_0^3)x + (4a_2a_0^3 + 6a_1^2a_0^2)x^2 + (4a_3a_0^3 + 12a_2a_1a_0^2 + 4a_1^3a_0)x^3 + (4a_4a_0^3 + 12a_3a_1a_0^2 + 6a_2^2a_0^2 + 12a_2a_1^2a_0 + a_1^4)x^4 + \cdots}{4!}$$

$$- \frac{a_0^6 + (6a_1a_0^5)x + (6a_2a_0^5 + 15a_1^2a_0^4)x^2 + (6a_3a_0^5 + 30a_2a_1a_0^4 + 20a_1^3a_0^3)x^3 + (6a_4a_0^5 + 30a_3a_1a_0^4 + 15a_2^2a_0^4 + 30a_2a_1^2a_0^3 + 15a_1^4a_0^2)x^4 + \cdots}{6!}$$

$$\therefore \cos\left(\sum a_n x^n\right) = \left(1 - \frac{a_0^2}{2!} + \frac{a_0^4}{4!} - \frac{a_0^6}{6!} + \dots\right) + \left(\frac{-2a_1a_0}{2!} + \frac{4a_1a_0^3}{4!} - \frac{6a_1a_0^5}{6!} + \dots\right)x + \left(\frac{-(2a_2a_0 + a_1^2)}{2!} + \frac{4a_2a_0^3 + 6a_1^2a_0^2}{4!} - \frac{6a_2a_0^5 + 15a_1^2a_0^4}{6!}\right)x^2$$

$$+ \left(\frac{-(2a_3a_0 + 2a_2a_1)}{2!} + \frac{4a_3a_0^3 + 12a_2a_1a_0^2 + 4a_1^3a_0}{4!} - \frac{6a_3a_0^5 + 30a_2a_1a_0^4 + 20a_1^3a_0^3}{6!} + \dots\right)x^3$$

$$+ \left(\frac{-(a_2^2 + 2a_3a_1 + 2a_4a_0 + 2a_2a_1)}{2!} + \frac{4a_4a_0^3 + 12a_3a_1a_0^2 + 6a_2^2a_0^2 + 12a_2a_1^2a_0 + a_1^4}{4!} - \frac{6a_4a_0^5 + 30a_3a_1a_0^4 + 15a_2^2a_0^4 + 30a_2a_1^2a_0^3 + 15a_1^4a_0^2}{6!} + \dots\right)x^4$$

$$+ \dots$$

$$\Longrightarrow \sin\left(\sum a_n x^n\right) = (a_0 + a_1x + a_2x^2 + \dots) - \frac{(a_0 + a_1x + a_2x^2 + \dots)^3}{3!} + \frac{(a_0 + a_1x + a_2x^2 + \dots)^5}{5!} - \frac{(a_0 + a_1x + a_2x^2 + \dots)^7}{7!} + \dots$$

$$= (a_0 + a_1x + a_2x^2 + \dots) - \frac{a_0^3 + (3a_1a_0^2)x + (3a_2a_0^2 + 3a_1^2a_0)x^2 + (3a_3a_0^2 + 3a_2a_1a_0 + a_1^3)x^3 + (3a_4a_0^2 + 3a_3a_1a_0 + 3a_2^2 + 3a_2a_1^2)x^4 + \dots}{3!}$$

$$+ \frac{a_0^5 + (5a_1a_0^4)x + (5a_2a_0^4 + 10a_1^2a_0^3)x^2 + (5a_3a_0^4 + 10a_2a_1a_0^3 + 10a_1^3a_0^2)x^3 + (5a_4a_0^4 + 10a_3a_1a_0^3 + 10a_2^2a_0^3 + 30a_2a_1^2a_0^2 + 5a_1^4a_0)x^4 + \dots}{5!}$$

$$- \frac{a_0^7 + (7a_1a_0^6)x + (7a_2a_0^6 + 21a_1^2a_0^5)x^2 + (7a_3a_0^6 + 21a_2a_1a_0^5 + 35a_1^3a_0^4)x^3 + (7a_4a_0^6 + 21a_3a_1a_0^5 + 21a_2^2a_0^5 + 105a_2a_1^2a_0^4 + 35a_1^4a_0^3)x^4 + \dots}{7!}$$

$$+ \dots$$

$$\therefore \sin\left(\sum a_n x^n\right) = \left(a_0 - \frac{a_0^3}{3!} + \frac{a_0^5}{5!} - \frac{a_0^7}{7!} + \dots\right) + \left(a_1 - \frac{3a_1a_0^2}{3!} + \frac{5a_1a_0^4}{5!} - \frac{7a_1a_0^6}{7!} + \dots\right)x + \left(a_2 - \frac{3a_2a_0^2 + 3a_1^2a_0}{3!} + \frac{5a_2a_0^4 + 10a_1^2a_0^3}{5!} - \frac{7a_2a_0^6 + 21a_1^2a_0^5}{7!} + \dots\right)x^2$$

$$+ \left(a_3 - \frac{3a_3a_0^2 + 3a_2a_1a_0 + a_1^3}{3!} + \frac{5a_3a_0^4 + 10a_2a_1a_0^3 + 10a_1^3a_0^2}{5!} - \frac{7a_3a_0^6 + 21a_2a_1a_0^5 + 35a_1^3a_0^4}{7!} + \dots\right)x^3$$

$$+ \left(a_5 - \frac{3a_4a_0^2 + 3a_3a_1a_0 + 3a_2^2 + 3a_2a_1^2}{3!} + \frac{5a_4a_0^4 + 10a_3a_1a_0^3 + 10a_2^2a_0^3 + 30a_2a_1^2a_0^2 + 5a_1^4a_0}{5!} - \frac{7a_4a_0^6 + 21a_3a_1a_0^5 + 21a_2^2a_0^5 + 105a_2a_1^2a_0^4 + 35a_1^4a_0^3}{7!} + \dots\right)x^4$$

$$+ \dots$$

Now,

$$\Longrightarrow a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots = \cos x \cdot \cos\left(\sum a_n x^n\right) - \sin x \cdot \sin\left(\sum a_n x^n\right)$$

$$\therefore a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)\Bigg[\left(1 - \frac{a_0^2}{2!} + \frac{a_0^4}{4!} - \frac{a_0^6}{6!} + \dots\right) + \left(\frac{-2a_1a_0}{2!} + \frac{4a_1a_0^3}{4!} - \frac{6a_1a_0^5}{6!} + \dots\right)x + \left(\frac{-(2a_2a_0 + a_1^2)}{2!} + \frac{4a_2a_0^3 + 6a_1^2a_0^2}{4!} - \frac{6a_2a_0^5 + 15a_1^2a_0^4}{6!} + \dots\right)x^2$$

$$+ \left(\frac{-(2a_3a_0 + 2a_2a_1)}{2!} + \frac{4a_3a_0^3 + 12a_2a_1a_0^2 + 4a_1^3a_0}{4!} - \frac{6a_3a_0^5 + 30a_2a_1a_0^4 + 20a_1^3a_0^3}{6!} + \dots\right)x^3$$

$$+ \left(\frac{-(a_2^2 + 2a_3a_1 + 2a_4a_0 + 2a_2a_1)}{2!} + \frac{4a_4a_0^3 + 12a_3a_1a_0^2 + 6a_2^2a_0^2 + 12a_2a_1^2a_0 + a_1^4}{4!} - \frac{6a_4a_0^5 + 30a_3a_1a_0^4 + 15a_2^2a_0^4 + 30a_2a_1^2a_0^3 + 15a_1^4a_0^2}{6!} + \dots\right)x^4$$

$$+ \dots\Bigg]$$

$$- \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)\Bigg[\left(a_0 - \frac{a_0^3}{3!} + \frac{a_0^5}{5!} - \frac{a_0^7}{7!} + \dots\right) + \left(a_1 - \frac{3a_1a_0^2}{3!} + \frac{5a_1a_0^4}{5!} - \frac{7a_1a_0^6}{7!} + \dots\right)x + \left(a_2 - \frac{3a_2a_0^2 + 3a_1^2a_0}{3!} + \frac{5a_2a_0^4 + 10a_1^2a_0^3}{5!} - \frac{7a_2a_0^6 + 21a_1^2a_0^5}{7!} + \dots\right)x^2$$

$$+ \left(a_3 - \frac{3a_3a_0^2 + 3a_2a_1a_0 + a_1^3}{3!} + \frac{5a_3a_0^4 + 10a_2a_1a_0^3 + 10a_1^3a_0^2}{5!} - \frac{7a_3a_0^6 + 21a_2a_1a_0^5 + 35a_1^3a_0^4}{7!} + \dots\right)x^3$$

$$+ \left(a_5 - \frac{3a_4a_0^2 + 3a_3a_1a_0 + 3a_2^2 + 3a_2a_1^2}{3!} + \frac{5a_4a_0^4 + 10a_3a_1a_0^3 + 10a_2^2a_0^3 + 30a_2a_1^2a_0^2 + 5a_1^4a_0}{5!} - \frac{7a_4a_0^6 + 21a_3a_1a_0^5 + 21a_2^2a_0^5 + 105a_2a_1^2a_0^4 + 35a_1^4a_0^3}{7!} + \dots\right)x^4$$

$$+ \dots\Bigg]$$

Comparing coefficients :

$$(1) \quad a_1 = \left(1 - \frac{a_0^2}{2!} + \frac{a_0^4}{4!} - \frac{a_0^6}{6!} + \dots\right)$$

$$\boxed{\therefore\ a_1 = \cos(a_0)}$$

If $a_0 = 0 \Rightarrow a_1 = 1$

$$(2) \quad 2a_2 = \left(\frac{-2a_1a_0}{2!} + \frac{4a_1a_0^3}{4!} - \frac{6a_1a_0^5}{6!} + \dots\right) + \left(a_0 - \frac{a_0^3}{3!} + \frac{a_0^5}{5!} - \frac{a_0^7}{7!} + \dots\right)$$

$$= \left(-a_1a_0 + \frac{a_1a_0^3}{3!} - \frac{a_1a_0^5}{5!} + \dots\right) + \sin(a_0)$$

$$= -a_1\left(a_0 - \frac{a_0^3}{3!} + \frac{a_0^5}{5!} - \frac{a_0^7}{7!} + \dots\right) + \sin(a_0)$$

$$= -a_1 \cdot \sin(a_0) + \sin(a_0)$$

$$\Rightarrow 2a_2 = \sin(a_0)\cdot\left[1 - a_1\right]$$

$$\boxed{\therefore\ a_2 = \frac{1-a_1}{2}\cdot\sin(a_0)}$$

If $a_0 = 0 \Rightarrow a_2 = 0$

$$(3) \quad 3a_3 = \left(\frac{-(2a_2a_0 + a_1^2)}{2!} + \frac{4a_2a_0^3 + 6a_1^2a_0^2}{4!} - \frac{6a_2a_0^5 + 15a_1^2a_0^4}{6!} + \dots\right) - \frac{1}{2!}\left(1 - \frac{a_0^2}{2!} + \frac{a_0^4}{4!} - \frac{a_0^6}{6!} + \dots\right) - \left(a_1 - \frac{3a_1a_0^2}{3!} + \frac{5a_1a_0^4}{5!} - \frac{7a_1a_0^6}{7!} + \dots\right)$$

$$= \left(-a_2a_0 - \frac{a_1^2}{2} + \frac{a_2a_0^3}{3!} + \frac{a_1^2a_0^2}{4} - \frac{a_2a_0^5}{5!} - \frac{a_1^2a_0^4}{48} + \dots\right) - \frac{1}{2}\cos(a_0) - a_1\left(1 - \frac{a_0^2}{2!} + \frac{a_0^4}{4!} - \frac{a_0^6}{6!} + \dots\right)$$

$$= (-a_2)\left(a_0 - \frac{a_0^3}{3!} + \frac{a_0^5}{5!} - \frac{a_0^7}{7!} + \dots\right) - \frac{a_1^2}{2}\left[1 - \frac{a_0^2}{2!} + \frac{a_0^4}{4!} - \frac{a_0^6}{6!} + \dots\right] - \frac{1}{2}\cos(a_0) - a_1\cos(a_0)$$

$$\Rightarrow 3a_3 = (-a_2)\cdot\sin(a_0) - \frac{a_1^2}{2}\cos(a_0) - \frac{1}{2}\cos(a_0) - a_1\cos(a_0)$$

$$\boxed{\therefore\ a_3 = \frac{-1}{3}\left[a_2\cdot\sin(a_0) + \frac{(a_1+1)^2}{2}\cos(a_0)\right]}$$

If $a_0 = 0 \Rightarrow a_3 = -2/3$

$$(4) \quad 4a_4 = \left(\frac{-(2a_2a_1 + 2a_3a_0)}{2!} + \frac{4a_3a_0^3 + 12a_2a_1a_0^2 + 4a_1^3a_0}{4!} - \frac{6a_3a_0^5 + 30a_2a_1a_0^4 + 20a_1^3a_0^3}{6!} + \dots\right) - \frac{1}{2!}\left(\frac{-2a_1a_0}{2!} + \frac{4a_1a_0^3}{4!} - \frac{6a_1a_0^5}{6!} + \dots\right)$$

$$- \left(a_2 - \frac{3a_2a_0^2 + 3a_1^2a_0}{3!} + \frac{5a_2a_0^4 + 10a_1^2a_0^3}{5!} - \frac{7a_2a_0^6 + 21a_1^2a_0^5}{7!} + \dots\right) + \frac{1}{3!}\left(a_0 - \frac{a_0^3}{3!} + \frac{a_0^5}{5!} - \frac{a_0^7}{7!} + \dots\right)$$

$$= \left(-a_2a_1 - a_3a_0 + \frac{a_3a_0^3}{3!} + \frac{a_2a_1a_0^2}{2!} + \frac{a_1^3a_0}{3!} - \frac{a_3a_0^5}{5!} - \frac{a_2a_1a_0^4}{4!} - \frac{a_1^3a_0^3}{36} + \dots\right) - \frac{1}{2!}\left(-a_1\cdot\sin(a_0)\right)$$

$$- \left(a_2 - \frac{a_2a_0^2}{2!} - \frac{a_1^2a_0}{2!} + \frac{a_2a_0^4}{4!} + \frac{a_1^2a_0^3}{12} - \frac{a_2a_0^6}{6!} + \frac{3a_1^2a_0^5}{6!} + \dots\right) + \frac{1}{3!}\left(\sin(a_0)\right)$$

$$= -a_3\left(\sin(a_0)\right) + \frac{a_1^3}{3!}\left(\sin(a_0)\right) - a_2a_1\left(\cos(a_0)\right) + \frac{1}{2!}a_1\cdot\sin(a_0) - \left(\frac{a_1^2}{2!}\left(\sin(a_0)\right) - a_2\cos(a_0)\right) + \frac{1}{3!}\sin(a_0)$$

$$\Rightarrow 4a_4 = \sin(a_0)\cdot\left[\frac{a_1^3}{3!} - a_3 + \frac{a_1}{2!} - \frac{a_1^2}{2!} + \frac{1}{3!}\right] + a_2\cos(a_0) - a_2a_1\cos a_0$$

$$\boxed{\therefore\ a_4 = \frac{\sin(a_0)}{4}\left[\frac{a_1^3}{3!} - \frac{a_1^2}{2!} + \frac{a_1}{2!} - a_3 + \frac{1}{3!}\right] + \frac{\cos(a_0)}{4}(a_2)(1 - a_1)}$$

If $a_0 = 0 \Rightarrow a_4 = 0$

$$(5)\quad 5a_5 = \left(\frac{-(a_2^2 + 2a_3a_1a_0 + 2a_4a_0 + 2a_2a_1^2)}{2!} + \frac{4a_4a_0^3 + 12a_3a_1a_0^2 + 6a_2^2a_0^2 + 12a_2a_1^2a_0 + a_1^4}{4!} - \frac{6a_4a_0^5 + 30a_3a_1a_0^4 + 15a_2^2a_0^4 + 30a_2a_1^2a_0^3 + 15a_1^4a_0^2}{6!} + \dots \right)$$

$$-\frac{1}{2!}\underline{\left(\frac{-(2a_2a_0 + a_1^2)}{2!} + \frac{4a_2a_0^3 + 6a_1^2a_0^2}{4!} - \frac{6a_2a_0^5 + 15a_1^2a_0^4}{6!} + \dots \right)} + \frac{1}{4!}\underline{\left(1 - \frac{a_0^2}{2!} + \frac{a_0^4}{4!} - \frac{a_0^6}{6!} + \dots \right)}$$

$$- \left(a_3 - \frac{3a_3a_0^2 + 3a_2a_1a_0 + a_1^3}{3!} + \frac{5a_3a_0^4 + 10a_2a_1a_0^3 + 10a_1^3a_0^2}{5!} - \frac{7a_3a_0^6 + 21a_2a_1a_0^5 + 35a_1^3a_0^4}{7!} + \dots \right)$$

$$+ \underline{\frac{1}{3!}\left(a_1 - \frac{3a_1a_0^2}{3!} + \frac{5a_1a_0^4}{5!} - \frac{7a_1a_0^6}{7!} + \dots \right)}$$

$$= \underline{\frac{-1}{2!}\left(-a_2\sin(a_0) - \frac{a_1^2}{2}\cos(a_0)\right)} + \underline{\frac{1}{4!}\left(\cos(a_0)\right)} + \underline{\frac{1}{3!}\left(a_1\cos(a_0)\right)}$$

$$- \left(a_3 - \frac{a_3a_0^2}{2!} - \frac{a_2a_1a_0}{2!} - \frac{a_1^3}{3!} + \frac{a_3a_0^4}{4!} + \frac{2a_2a_1a_0^3}{4!} + \frac{2a_1^3a_0^2}{4!} - \frac{a_3a_0^6}{6!} - \frac{3a_2a_1a_0^5}{6!} - \frac{5a_1^3a_0^4}{6!} + \dots \right)$$

$$+ \left(\frac{-a_2^2}{2!} - a_3a_1a_0 - a_4a_0 - a_2a_1^2 + \frac{a_4a_0^3}{3!} + \frac{a_3a_1a_0^2}{2!} + \frac{a_2^2a_0^2}{4} + \frac{a_2a_1^2a_0}{2} + \frac{a_1^4}{4!} - \frac{a_4a_0^5}{5!} - \frac{a_3a_1a_0^4}{4!} - \frac{a_2^2a_0^4}{48} - \frac{a_2a_1^2a_0^3}{4!} - \frac{a_1^4a_0^2}{48} + \dots \right)$$

$$= \frac{-1}{2!}\left(-a_2\sin(a_0) - \frac{a_1^2}{2}\cos(a_0)\right) + \frac{1}{4!}\left(\cos(a_0)\right) + \frac{1}{3!}\left(a_1\cos(a_0)\right) - \left(a_3\cos(a_0) - \frac{a_2a_1}{2!}\sin(a_0) - \frac{a_1^3}{3!}\cos(a_0)\right)$$

$$+ \frac{a_2a_1^2(a_0-1)}{2a_0} - \frac{a_2a_1^2}{2}\cos(a_0) - a_4\sin(a_0) - \frac{a_2^2}{2!}\cos(a_0) + \frac{a_1^4}{4!}\cos(a_0) - a_3a_1\cos(a_0) + a_0a_1a_3$$

$$\Rightarrow 5a_5 = \sin(a_0)\left[\frac{a_2}{2!} + \frac{a_2a_1}{2!} - a_4\right] + \cos(a_0)\left[\frac{a_1^2}{4} + \frac{1}{4!} + \frac{a_1}{3!} - a_3 + \frac{a_1^3}{3!} - \frac{a_2a_1^2}{2} - \frac{a_2^2}{2!} + \frac{a_1^4}{4!} - a_3a_1\right] + a_0a_1a_3 + \frac{a_2a_1^2(a_0-1)}{2a_0}$$

$$\boxed{\therefore\ a_5 = \frac{\sin(a_0)}{5}\left[\frac{a_2}{2}(1+a_1) + a_4\right] + \frac{\cos(a_0)}{5}\left[\frac{a_1^4}{4!} + \frac{a_1^3}{3!} + \frac{a_1^2}{4} + \frac{a_1}{3!} - \frac{a_2^2}{2!} - a_3 - \frac{a_2a_1^2}{2} - a_3a_1 + \frac{1}{4!}\right] + \frac{a_0a_1a_3}{5} + \frac{a_2a_1^2(a_0-1)}{10a_0}}$$

If $a_0 = 0 \Rightarrow a_5 = 2/5$

$\therefore a_0 = 0,\ a_2 = 0,\ a_4 = 0, \dots$

$a_1 = 1,\ a_3 = -2/3,\ a_5 = 2/5, \dots$

$y = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots\right)$

$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots\right) + x - x$

$= 2\left(1 - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots\right) - x$

$\therefore y = 2\tan^{-1}(x) - x$

if $c_1 = 0$

$y = 2\tan^{-1}(x + c_1) - x + c_2$

if $c_1 \neq 0$

Now, $y = 2\left(1 - \frac{1}{3}(x+c_1)^3 + \frac{1}{5}(x+c_1)^5 - \dots\right) - x + c_2$

Also $y = a_0 + a_1x + a_2x^2 + \dots$

Assuming $c_1 \neq 0$:

$2\tan^{-1}(x+c_1) \longrightarrow$ By Using Taylor series,

Coefficient of x: $\frac{1}{1+c_1^2} \times 2$ & Constant $= 2\tan^{-1}(c_1)$

Comparing with ①:

$\frac{2}{1+c_1^2} - 1 = a_1 \Rightarrow \frac{2}{a_1+1} = 1 + c_1^2$

$\Rightarrow \frac{1-a_1}{1+a_1} = c_1^2$

$\boxed{\therefore c_1 = \sqrt{\frac{1-a_1}{1+a_1}}}$

This gives $1 - a_1 \geq 0 \Rightarrow a_1 \leq 1$

& $1 + a_1 < 0 \Rightarrow a_1 > -1$

Which is true $\because a_1 = \cos(a_0)$

& $-1 \leq \cos(a_0) \leq 1$

Comparing with ①:

$2\tan^{-1}(c_1) + c_2 = a_0$

$\boxed{\therefore c_2 = a_0 - 2\tan^{-1}\left(\sqrt{\frac{1-a_1}{1+a_1}}\right)}$

$$\boxed{\therefore y = 2\tan^{-1}\left(x + \sqrt{\frac{1-a_1}{1+a_1}}\right) - x + a_0 - 2\tan^{-1}\left(\sqrt{\frac{1-a_1}{1+a_1}}\right)}$$

Q2) $y'' - \sin(x)y = 0$

$y(\pi) = 1$

$y'(\pi) = 0$

Assume $t = x - \pi$

$\Rightarrow x = \pi + t$

$\therefore y(t) = \sum_{n=0}^{\infty} a_n . t^n$

General form : $y'' + P(x)y' + Q(x)y = 0$

Comparing, we get

$P(x) = 0$ — Analytical

$Q(x) = -\sin x$ — Analytical

@ $x_0 = 0$: x is a singular point

Both Exist & are finite @ $x = \pi$

$\therefore x_0 = 0$ is a Regular Point

We know that, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\therefore \sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}$$

The original equation:

$$y'' - (\sin x)y = 0$$

$$\Rightarrow y'' + (\sin t)y = 0$$

As $y = \sum a_n t^n$

$y' = \sum n . a_n . t^{n-1}$

$y'' = \sum n(n-1) a_n . t^{n-2}$

$$\Rightarrow \sum n.(n-1) a_n t^{n-2} + \left(t - \frac{t^3}{3!} + \ldots\right) \sum a_n . t^n = 0$$

$$\Rightarrow 2a_2 + 6a_3 t + 12a_4 t^2 + 20 a_5 t^3 + 30 a_6 t^4 + \ldots$$
$$+ \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\right)\left(a_0 + a_1 t + a_2 t^2 + \ldots\right) = 0$$

$$\Rightarrow 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 30a_6 t^4 + \dots$$
$$+ a_0 t + a_1 t^2 + \left(a_2 - \frac{a_0}{3!}\right)t^3 + \left(a_3 - \frac{a_1}{3!}\right)t^4 + \dots = 0$$

$$\Rightarrow 2a_2 + (6a_3 + a_0)t + (12a_4 + a_1)t^2 + \left(20a_5 + a_2 - \frac{a_0}{3!}\right)t^3$$
$$+ \left(30a_6 + a_3 - \frac{a_1}{3!}\right)t^4 + \dots = 0$$

Also from the Initial Conditions, we have

$$a_0 = 1$$
$$a_1 = 0$$

$$\therefore 2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$$

$$6a_3 - a_0 = 0 \Rightarrow \boxed{a_3 = -1/6}$$

Similarly,

$$a_4 = 0$$
$$a_5 = 1/120$$
$$a_6 = 1/180$$

Hence, from all the values above, we have:

$$y(t) = 1 - \frac{1}{6}t^3 + \frac{1}{120}t^5 + \frac{1}{180}t^6 + \dots$$

As $t = x - \pi$,

$$\boxed{\therefore y(x) = 1 - \frac{1}{6}(x-\pi)^3 + \frac{1}{120}(x-\pi)^5 + \frac{1}{180}(x-\pi)^6 + \dots}$$

Hence, we found a unique power series of the form $y = \sum a_n x^n$.

Q3) $xy'' + y' + xy = 0$

$$\Rightarrow y'' + \left(\frac{1}{x}\right) y' + y = 0$$

Form : $y'' + P(x) y' + Q(x) y = 0$

$$\therefore P(x) = 1/x$$
$$Q(x) = 1$$

@ $x_0 = 0$: x is a singular point

$$\lim_{x \to x_0} P(x) \cdot (x - x_0) = \lim_{x \to 0} \frac{1}{x} \cdot x = 1$$

$$\lim_{x \to x_0} Q(x) \cdot (x - x_0)^2 = \lim_{x \to 0} 1 \cdot x^2 = 0$$

Both Exist & are finite.

$\therefore x_0 = 0$ is a Regular Point

$P(x)$ is not Analytical as it cannot be expressed in terms of $\sum a_n x^n$, $n \geq 0$

But $(x - x_0) P(x) = 1$ — Analytical
$(x - x_0)^2 Q(x) = x^2$ — Analytical

Hence, we need to apply Frobenius method.

$$y(x) = x^m \cdot \sum a_n x^n = \sum a_n \cdot x^{m+n}$$

$$y'(x) = \sum (m+n) \cdot a_n \cdot x^{m+n-1}$$

$$y''(x) = \sum (m+n)(m+n-1) a_n \cdot x^{m+n-2}$$

Substituting these in Given Equation :

$$xy'' + y' + xy = 0$$

$$\Rightarrow x \sum (m+n)(m+n-1)\, a_n \cdot x^{m+n-2} + \sum (m+n) \cdot a_n \cdot x^{m+n-1} + x \sum a_n x^{m+n} = 0$$

$$\Rightarrow (m+n)(m+n-1) \sum a_n \cdot x^{m+n-1} + (m+n) \sum a_n \cdot x^{m+n-1} + \sum a_n \cdot x^{m+n+1} = 0$$

$$n \longrightarrow n-2$$

$$\Rightarrow (m+n)(m+n-1) \sum a_n \cdot x^{m+n-1} + (m+n) \sum a_n \cdot x^{m+n-1} + \sum a_{n-2} \cdot x^{m+n-1} = 0$$

$$\therefore (m+n)(m+n-1)\, a_n + (m+n)\, a_n + a_{n-2} = 0$$

$$\Rightarrow a_n (m+n) \left[\cancel{1} + m+n-\cancel{1} \right] + a_{n-2} = 0$$

$$\Rightarrow a_n (m+n)^2 + a_{n-2} = 0$$

$$\Rightarrow \boxed{a_n = \frac{-a_{n-2}}{(m+n)^2}}$$

From the Given Equation, Finding Indicial Eq.:

$$m(m-1) + m P_0 + Q_0 = 0$$

$$\Rightarrow m(m-1) + m = 0$$

$$\Rightarrow m^2 = 0$$

$$\boxed{\therefore m = 0}$$

$$\Rightarrow \boxed{\therefore a_n = \frac{-a_{n-2}}{n^2}}$$

$$\Longrightarrow (m+n)^2 a_n + a_{n-2} = 0$$

Sub $n = 1$:

$$\Longrightarrow (0+1)^2 \times a_1 + \cancelto{0}{a_{-1}} = 0$$

$$\Longrightarrow a_1 \times 1^2 = 0$$

$$\boxed{a_1 = 0} \Longrightarrow a_3 = \frac{-a_1}{3^2} = 0$$

$$a_5 = \frac{-a_3}{5^2} = 0$$

$$\therefore a_7 = a_9 = \cdots\cdots = 0$$

$$\boxed{\therefore a_{2n+1} = 0 \quad \forall n \geq 0}$$

$$\left.\begin{aligned}
a_2 &= \frac{-a_0}{2^2} \\
a_4 &= \frac{-a_2}{4^2} = \frac{a_0}{2^2 \times 4^2} \\
a_6 &= \frac{-a_4}{6^2} = \frac{-a_0}{2^2 \times 4^2 \times 6^2} \\
a_{2n} &= \frac{(-1)^n \cdot a_0}{2^2 \times 4^2 \times 6^2 \times \cdots (2n)^2}
\end{aligned}\right\} \quad \boxed{\therefore a_{2n} = \frac{(-1)^n \cdot a_0}{2^{2n} \times (n!)^2}}$$

Now, $y = \sum a_n x^n$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \cdots\cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots\cdots)$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \cdots\cdots)$$

$$= \sum_{n=0}^{\infty} a_{2n} \cdot x^{2n}$$

$$\therefore y = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \dots\dots = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot a_0 \cdot x^{2n}}{2^{2n} \times (n!)^2}$$

$$\boxed{\therefore y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{2^{2n} \times (n!)^2}}$$

As m has only one value, i.e. $m=0$

$$\therefore y_1(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{2^{2n} \cdot (n!)^2}$$

Expressing in terms of Bessel DE (zero order)

$$\boxed{y_1(x) = a_0 J_0(x)}$$

$$J_0(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$a_n = \begin{cases} \dfrac{(-1)^{n/2}}{2^n \cdot \left(\frac{n}{2}!\right)^2} & \text{, if } n = 2k \; [k \in W] \\ 0 & \text{, if } n = 2k+1 \; [k \in W] \end{cases}$$

The Indicial Equation has double root, i.e. $m=0$.
Hence, the second equation involves $\ln(x)$.

$$y_2(x) = \ln(x) \cdot y_1(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \dots\right)$$

$$\because y_2(x) = \ln(x) \cdot y_1(x) + \sum_{n=0}^{\infty} b_n \cdot x^{n+m'}$$

$$\boxed{\therefore y_2(x) = a_0 \ln(x) \cdot J_0(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \dots\right)}$$

Q4) $L\left[\dfrac{1-\cos x}{x^2}\right]$

As we know, $L[\cos x] = \dfrac{p}{p^2+1}$

$$L[1-\cos x] = L[1] - L[\cos x]$$

$$= \frac{1}{p} - \frac{p}{p^2+1}$$

$L\left[\dfrac{1-\cos x}{x}\right] \implies$ Let $f(x) = 1-\cos x$

$$L[f(x)] = F(p)$$

$$L\left[\frac{f(x)}{x}\right] = \int_p^\infty F(p)\,dp$$

As $F(p) = \dfrac{1}{p} - \dfrac{p}{p^2+1}$

$$\Rightarrow \int_p^\infty F(p)\,dp = \int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+1}\right)dp$$

$$= \log(p) - \frac{1}{2}\log(p^2+1)\Big]_p^\infty$$

$$= \log(p) - \log\left(\sqrt{p^2+1}\right)\Big]_p^\infty$$

$$= \log\left(\frac{p}{\sqrt{p^2+1}}\right)\Big]_p^\infty$$

$$\left[\text{As } \lim_{p\to\infty} \log\left(\frac{p}{p\sqrt{1+1/p^2}}\right) = \lim_{p\to\infty}\log\left(\frac{1}{\sqrt{1+1/p^2}}\right) = \log(1) = 0\right]$$

$$= \log\left(\frac{\sqrt{p^2+1}}{p}\right)$$

$$\boxed{\therefore \; L\left[\frac{1-\cos x}{x}\right] = \log\left(\frac{\sqrt{p^2+1}}{p}\right)}$$

$$L\left[\frac{1-\cos x}{x^2}\right] \implies \text{Let } f(x) = \frac{1-\cos x}{x}$$

$$L[f(x)] = F(p)$$

$$L\left[\frac{f(x)}{x}\right] = \int_p^\infty F(p)\,dp$$

$$\text{As } F(p) = \log\left(\frac{\sqrt{p^2+1}}{p}\right)$$

$$\Rightarrow \int_p^\infty F(p)\,dp = \int_p^\infty \log\left(\frac{\sqrt{p^2+1}}{p}\right)dp$$

$$= \int_p^\infty \left[\log\left(\sqrt{p^2+1}\right) - \log p\right] dp$$

$$\int \log\left(\sqrt{p^2+1}\right) dp = p\cdot\log\sqrt{p^2+1} - \int \frac{p^2}{p^2+1}\,dp$$

$$= p\cdot\log\sqrt{p^2+1} - p + \tan^{-1}(p) \quad -①$$

$$-\int \log(p)\,dp = -p\log(p) + p \quad -②$$

$$\therefore \int\left[\log\left(\sqrt{p^2+1}\right) - \log p\right] dp = ① + ②$$

$$= p\log\left(\frac{\sqrt{p^2+1}}{p}\right) + \tan^{-1}(p)$$

$$= p\log\left(\frac{\sqrt{p^2+1}}{p}\right) + \tan^{-1}(p)\Big]_p^\infty$$

$$= \frac{\pi}{2} - p\log\left(\frac{\sqrt{p^2+1}}{p}\right) - \tan^{-1}(p)$$

$$= \cot^{-1}(p) - p\log\left(\frac{\sqrt{p^2+1}}{p}\right)$$

$$\boxed{\therefore\ L\left[\frac{1-\cos x}{x^2}\right] = \cot^{-1}(p) - p\cdot\log\left(\frac{\sqrt{p^2+1}}{p}\right)}$$

$$L\left[\frac{1-\cos x}{x^2}\right] = \cot^{-1}(p) - \frac{1}{2}p\cdot\log\left(1+\frac{1}{p^2}\right)$$

(OR)

$$L\left[\frac{1-\cos x}{x^2}\right]$$

As we know, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$1-\cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$$

$$\Rightarrow \frac{1-\cos x}{x^2} = \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} \cdots}{x^2}$$

$$= \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots$$

$$L\left[\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots\right]$$

$$= \frac{1/2}{p} - \frac{1}{4!}\cdot\frac{2!}{p^3} + \frac{1}{6!}\cdot\frac{4!}{p^5} - \frac{1}{8!}\cdot\frac{6!}{p^7} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{1}{p^{2n+1}}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)(2n+1)\,p^{2n+1}}}$$

Test for Convergence

$$a_n = \frac{(-1)^n}{(2n+2)(2n+1)\,p^{2n+1}}$$

$$a_{n+1} = \frac{(-1)\cdot(-1)^n}{(2n+4)(2n+3)\,p^{2n+3}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)\cancel{(-1)^n}\cdot(2n+2)(2n+1)\,p^{2n+1}}{(2n+4)(2n+3)\,p^{2n+3}\cdot\cancel{(-1)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(2n+1)}{(n+2)(2n+3)}\cdot\frac{(-1)}{p^2}\right|$$

$$= \frac{1}{p^2}$$

Series converges for $p^2 > 1$ i.e. $p > 1$

Q5) **Given:**

$$f(x+a) = f(x) \quad \& \quad f(x-a) = f(x) \qquad \text{(i.e. period} = a)$$

To prove:

$$L[f(x)] = \frac{1}{1-e^{-ap}} \int_0^a e^{-px} f(x)\, dx \qquad \forall p > 0$$

Solution:

$$L[f(x)] = \int_0^\infty e^{-px} f(x)\, dx$$

Let $x = t - a$

$\Rightarrow t = x + a$

$dt = dx$

$x = 0 \Rightarrow t = x + a$

$\Rightarrow t = 0 + a$

$\therefore t = a$

$x = \infty \Rightarrow t = x + a$

$\Rightarrow t = \infty + a$

$\therefore t = \infty$

$$\Rightarrow L[f(x)] = \int_{x=0}^{x=\infty} e^{-px} \cdot f(x)\, dx$$

$$= \int_{t=a}^{t=\infty} e^{-p(t-a)} \cdot f(t-a)\, dt$$

$$= \int_{x=a}^{x=\infty} e^{-p(x-a)} \cdot f(x-a)\, dx$$

$$= \int_a^\infty e^{-p(x-a)} \cdot f(x)\, dx$$

$$= \int_a^\infty e^{-px} \cdot e^{ap} \cdot f(x)\, dx$$

$$\therefore L[f(x)] = e^{ap} \cdot \int_a^\infty e^{-px} \cdot f(x)\, dx \quad - (1)$$

As $\mathcal{L}[f(x)] = \int_0^\infty e^{-px} \cdot f(x)\, dx$ — ②

$$\Rightarrow e^{ap} \cdot \mathcal{L}[f(x)] = e^{ap} \int_0^\infty e^{-px} f(x)\, dx$$

$$= e^{ap} \left[\int_0^a e^{-px} f(x)\, dx + \int_a^\infty e^{-px} f(x)\, dx \right]$$

$$= e^{ap} \left[\int_0^a e^{-px} f(x)\, dx + \underbrace{\frac{\mathcal{L}[f(x)]}{e^{ap}}}_{\text{from ①}} \right]$$

$$= e^{ap} \int_0^a e^{-px} f(x)\, dx + \mathcal{L}[f(x)]$$

$$\Rightarrow \mathcal{L}[f(x)] \left(e^{ap} - 1\right) = e^{ap} \int_0^a e^{-px} f(x)\, dx$$

$$\Rightarrow \mathcal{L}[f(x)] = \frac{e^{ap}}{e^{ap} - 1} \int_0^a e^{-px} f(x)\, dx$$

$$\boxed{\therefore \mathcal{L}[f(x)] = \frac{1}{1 - e^{-ap}} \int_0^a e^{-px} f(x)\, dx}$$

$\therefore$ LHS = RHS

Hence Proved !

Q6) Given:

$$xy'' + y' + xy = 0 \quad , \quad y(0) = 1$$

<u>Solution</u>:

Apply Laplace Transform throughout.

$$xy'' + y' + xy = 0$$

$$\mathcal{L}[xy'' + y' + xy] = \mathcal{L}[0]$$

$$\Rightarrow \mathcal{L}[xy''] + \mathcal{L}[y'] + \mathcal{L}[xy] = 0$$

$$\Rightarrow \mathcal{L}[xy''] + p\mathcal{L}[y] - y(0) + \mathcal{L}[xy] = 0$$

$$\Rightarrow \mathcal{L}[xy''] + p\,\mathcal{L}[y] + \mathcal{L}[xy] = 1$$

Assume $\mathcal{L}[y] = F(p)$

$$\mathcal{L}[x \cdot y(x)] = -\mathcal{L}[(-x)\cdot y(x)]$$
$$= -F'(p)$$

$$\left[\begin{array}{l} \text{As } \mathcal{L}[y] = F(p) \\ \qquad \mathcal{L}[y'] = p\mathcal{L}[y] - 1 \\ \qquad \mathcal{L}[y''] = p^2 F(p) - p - y'(0) = G(p) \\ \mathcal{L}[x \cdot y''(x)] = -\mathcal{L}[(-x)\cdot y''(x)] \\ \qquad = -G'(p) \end{array}\right]$$

$$\Rightarrow -G'(p) + p\cdot F(p) - F'(p) = 1$$

$$\Rightarrow -\frac{d}{dp}\left[p^2 F(p) - p - \underbrace{(y'(0))}_{\text{constant}}\right] + pF(p) - F'(p) = 1$$

$$\Rightarrow -\left[p^2 F'(p) + 2pF(p) - 1\right] + p\cdot F(p) - F'(p) = 1$$

$$\Rightarrow F(p)\left[-2p+p\right] + F'(p)\left[-p^2-1\right] = 0$$

$$\Rightarrow (1+p^2)F'(p) + pF(p) = 0$$

$$\Rightarrow F'(p) + \frac{p}{1+p^2}F(p) = 0$$

[It is in the form of Linear DE

$$\frac{dy}{dx} + P(x)y = \varphi(x)$$

$\therefore$ The Integrating factor $= e^{\int P(x)\,dx}$]

$$\Rightarrow \text{Integrating factor} = e^{\int \frac{p}{1+p^2}dp}$$

$$= e^{\frac{1}{2}\int \frac{2p}{1+p^2}dp}$$

$$= e^{\frac{1}{2}\ln|1+p^2|}$$

$$= \sqrt{1+p^2}$$

$$\left[\text{Solution : } y\cdot e^{\int P(x)dx} = \int \varphi(x)\cdot e^{\int P(x)dx} + c,\ c\in\mathbb{R}\right]$$

$$F(p)\cdot\sqrt{1+p^2} = \int 0\cdot\sqrt{1+p^2}\,dp + C \quad [C\in\mathbb{R}]$$

$$= C$$

$$\boxed{\therefore F(p) = \frac{C}{\sqrt{1+p^2}}}$$

$$\mathcal{L}[y(x)] = F(p)$$

$$\Rightarrow y(x) = \mathcal{L}^{-1}[F(p)]$$

$$\therefore y(x) = \mathcal{L}^{-1}\left[\frac{c}{\sqrt{1+p^2}}\right]$$

$$\Rightarrow y(x) = \mathcal{L}^{-1}\left[\frac{c}{\sqrt{1+p^2}}\right]$$

$$= c\,J_0(x)$$

$$\therefore y_1(x) = c\,J_0(x)$$

Assuming $c = a_0$

$$\boxed{\therefore y_1(x) = a_0 \cdot J_0(x)}$$

(OR)

$$y(x) = \mathcal{L}^{-1}\left[c \cdot (1+p^2)^{-1/2}\right]$$

from $(1+x)^n = \sum_{r=0}^{\infty} {}^nC_r \cdot x^r \quad [n \in \mathbb{R}]$

$$= 1 + nx + \frac{n(n-1)}{2} x^2 + \cdots$$

$$\therefore (1+p^2)^{-1/2} = 1 - \frac{1}{2}p^2 + \frac{1\times 3}{2\times 2}p^4 - \frac{1\times 3\times 5}{2\times 2\times 2}p^6 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2n)!}{2^{2n}\cdot n!} p^{2n}$$

Now, Let $c = a_0$:

$$y(x) = \mathcal{L}^{-1}\left[a_0 \cdot (1+p^2)^{-1/2}\right]$$

$$\therefore\ y(x) = L^{-1}\left[a_0 \cdot \sum (-1)^n \cdot \frac{(2n)!}{2^{2n} \cdot n!} p^{2n}\right]$$

$$= a_0\, L^{-1}\left[\sum (-1)^n \cdot \frac{(2n)!}{2^{2n} \cdot n!} p^{2n}\right]$$

$$= a_0 \cdot L^{-1}\left[1 - \frac{1}{2} p^2 + \frac{3}{4} p^4 - \frac{15}{8} p^6 + \dots\right]$$

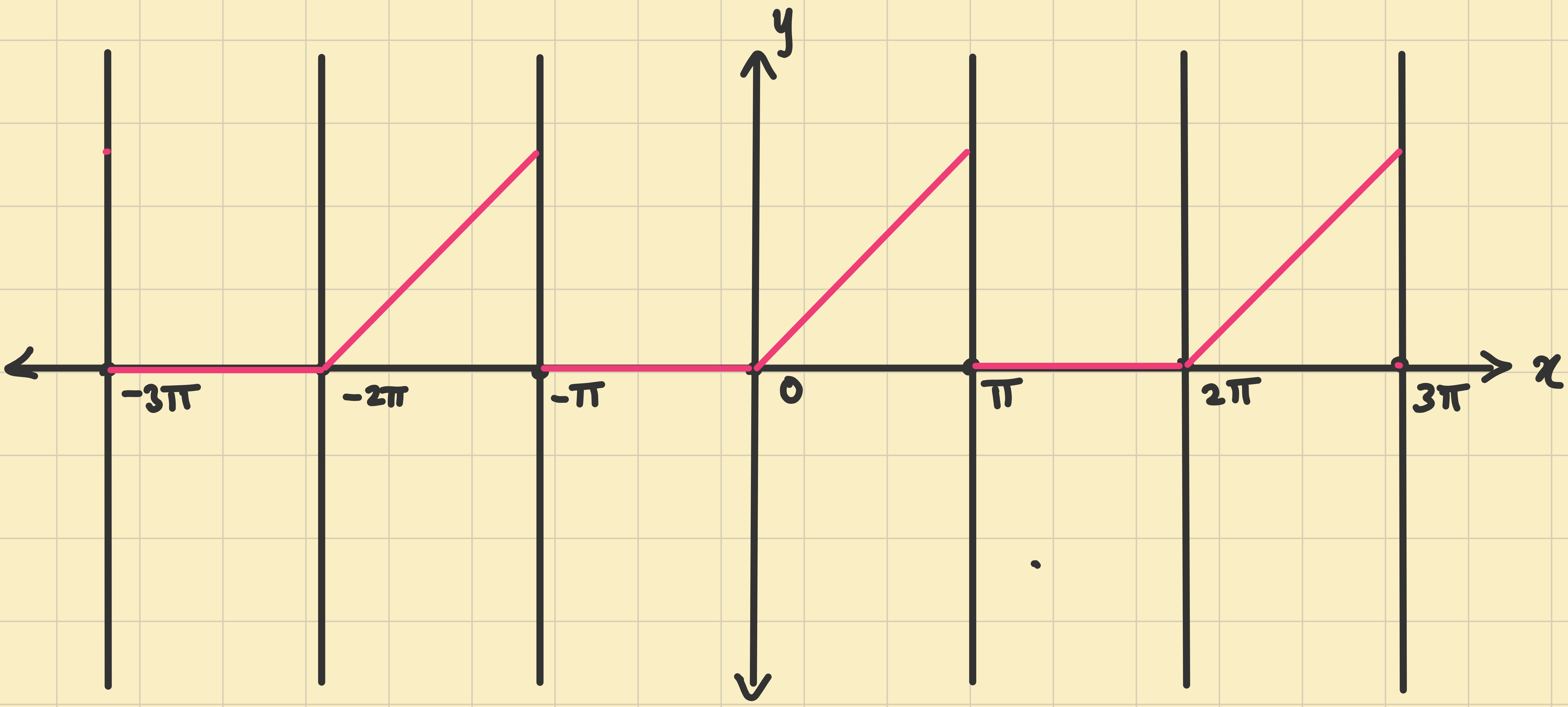
$$\therefore\ y(x) = a_0\left(\frac{1}{p} - \frac{1}{2}\delta''(x) + \frac{3}{4}\delta^{(4)}(x) - \frac{15}{8}\delta^{(6)}(x) + \dots\right)$$

Q7) <u>Given</u> :

$$f(x) = \begin{cases} 0 \ , & -\pi \leq x < 0 \\ x \ , & 0 \leq x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x)$$

(a) Sketching $f(x)$ in $[-3\pi, 3\pi)$



(b) Fourier Series for $f(x)$ in $[-\pi, \pi)$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\text{Where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx$$

First, checking if $f(x)$ can have fourier series using Dirichlet conditions

(1) $f(x) \longrightarrow$ Defined and Bounded

True in $[-\pi, \pi)$

(2) $f(x) \longrightarrow$ finite no. of discontinuous & finite no. of maximas and minimas

True in $[-\pi, \pi)$

(3) $f(x)$ is Defined

True in $[-\pi, \pi)$

Now,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} x\, dx \right]$$

0

$$= \frac{1}{\pi} \cdot \left(\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi} \cdot \frac{\pi^2}{2}$$

$$\boxed{\therefore\ a_0 = \frac{\pi}{2}}$$

AND $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx)\, dx$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \cdot \cos(nx)\, dx + \int_{0}^{\pi} x \cdot \cos(nx)\, dx \right]$$

0

$$= \frac{1}{\pi} \int_{0}^{\pi} x \cdot \cos(nx)\, dx$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2}\left(nx\sin(nx) + \cos(nx)\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{n\pi\sin(n\pi) + \cos(n\pi) - 1}{n^2}\right]$$

$$\boxed{\therefore \; a_n = \frac{n\pi\sin(n\pi) + \cos(n\pi) - 1}{\pi \cdot n^2}}$$

AND $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \int_{0}^{\pi} x\cdot\sin(nx)\,dx\right]$$

0

$$= \frac{1}{\pi}\cdot\int_0^{\pi} x\cdot\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2}\left(\sin(nx) - nx\cos(nx)\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\sin(n\pi) - n\pi\cos(n\pi)}{n^2}\right)$$

$$\boxed{\therefore \; b_n = \frac{\sin(n\pi) - n\pi\cos(n\pi)}{\pi \cdot n^2}}$$

$$\therefore f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$\Rightarrow f(x) = \frac{1}{2}\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty}\left(\frac{n\pi\sin(n\pi) + \cos(n\pi) - 1}{\pi \cdot n^2}\right)\cos(nx)$$

$$+ \sum_{n=1}^{\infty}\left(\frac{\sin(n\pi) - n\pi\cos(n\pi)}{\pi \cdot n^2}\right)\sin(nx)$$

$$\Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\left[\left(n\pi \sin(n\pi) + \cos(n\pi) - 1\right)\cos(nx) + \left(\sin(n\pi) - n\pi\cos(n\pi)\right)\sin(nx)\right]\right]$$

$$\Rightarrow f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left[\left(n\pi \sin(n\pi) + \cos(n\pi) - 1\right)\cos(nx) + \left(\sin(n\pi) - n\pi\cos(n\pi)\right)\sin(nx)\right]\right]$$

Putting $n = 1, 2, 3$ for the Series

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left[\left(n\pi \sin(n\pi) + \cos(n\pi) - 1\right)\cos(nx) + \left(\sin(n\pi) - n\pi\cos(n\pi)\right)\sin(nx)\right]\right]$$

When n is even

$$= \sum_{n=2}^{\infty}\left[\left(\frac{-\pi}{n}\right)\sin(nx)\right]$$

$$= \frac{-\pi}{2}\sin(2x) - \frac{\pi}{4}\sin(4x) - \frac{\pi}{6}\sin(6x) - \dots$$

When n is odd

$$= \sum_{n=1}^{\infty}\left[\frac{-2}{n^2}\cos(nx) - \frac{\pi}{n}\sin(nx)\right]$$

$$= -2\cos(x) - \pi\sin(x) - \frac{2}{9}\cos(3x) - \frac{\pi}{3}\sin(3x) - \frac{2}{25}\cos(5x) - \frac{\pi}{5}\sin(5x) - \dots$$

Combining both we get,

$$\Rightarrow f(x) = \frac{\pi}{4} - \frac{1}{\pi}\left[\frac{\pi}{2}\sin(2x) + 2\cos(x) + \pi\sin(x) + \frac{\pi}{4}\sin(4x) + \frac{2}{9}\cos(3x) + \frac{\pi}{3}\sin(3x) + \frac{\pi}{6}\sin(6x) + \frac{2}{25}\cos(5x) + \frac{\pi}{5}\sin(5x) - \dots\right]$$

Hence, the fourier series has been obtained.

(c) When $x = \pi/2$:

$$f(x) = \frac{\pi}{4} - \frac{1}{\pi}\Big[\overset{\nearrow 0}{\frac{\pi}{2}\sin(2x)} + \overset{\nearrow 0}{2\cos(x)} + \pi\sin(x) + \overset{\nearrow 0}{\frac{\pi}{4}\sin(4x)}$$
$$+ \overset{\nearrow 0}{\frac{2}{9}\cos(3x)} + \frac{\pi}{3}\sin(3x) + \overset{\nearrow 0}{\frac{\pi}{6}\sin(6x)}$$
$$+ \overset{\nearrow 0}{\frac{2}{25}\cos(5x)} + \frac{\pi}{5}\sin(5x) - \dots\Big]$$

[When $x = \pi/2$]

$$\Rightarrow f(\pi/2) = \frac{\pi}{4} + \frac{1}{\pi}\left[\pi - \frac{\pi}{3} + \frac{\pi}{5} - \frac{\pi}{7} + \dots\right]$$

$$\Rightarrow \pi/2 = \frac{\pi}{4} + \frac{1}{\pi}\left[\pi - \frac{\pi}{3} + \frac{\pi}{5} - \frac{\pi}{7} + \dots\right]$$

$$\Rightarrow \pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\boxed{\therefore \pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

$$f(x) = \frac{\pi}{4} - \frac{1}{\pi}\Big[\frac{\pi}{2}\sin(2x) + 2\cos(x) + \pi\sin(x) + \frac{\pi}{4}\sin(4x)$$
$$+ \frac{2}{9}\cos(3x) + \frac{\pi}{3}\sin(3x) + \frac{\pi}{6}\sin(6x)$$
$$+ \frac{2}{25}\cos(5x) + \frac{\pi}{5}\sin(5x) - \dots\Big]$$

Rough

From RHS, We can observe that we need only cosx terms and all be equal.

$$\because 2(1) + \frac{2}{3^2}(1) + \frac{2}{5^2}(1) + \dots$$

$$\Rightarrow 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

$\therefore$ All sin terms should be 0s.

$$f(x) = \frac{\pi}{4} - \frac{1}{\pi}\left[2 \times \text{RHS}\right]$$

$$= \frac{\pi}{4} - \frac{1}{\pi}\left[2 \times \frac{\pi^2}{8}\right]$$

$$= \frac{\pi}{4} - \frac{\pi}{4}$$

$$= 0$$

At $x = 0$, Left hand limit $= 0$
Right hand limit $= 0$

$\therefore$ Continuous
$\therefore$ Can be used

$$f(x) = \frac{\pi}{4} - \frac{1}{\pi}\Big[\frac{\pi}{2}\sin(2x) + 2\cos(x) + \pi\sin(x) + \frac{\pi}{4}\sin(4x)$$
$$+ \frac{2}{9}\cos(3x) + \frac{\pi}{3}\sin(3x) + \frac{\pi}{6}\sin(6x)$$
$$+ \frac{2}{25}\cos(5x) + \frac{\pi}{5}\sin(5x) - \ldots\Big]$$

[When $x = 0$]

$$f(0) = \frac{\pi}{4} - \frac{1}{\pi}\left[2(1) + \frac{2}{9}(1) + \frac{2}{25}(1) + \ldots\right]$$

$$\Rightarrow 0 = \frac{\pi}{4} - \frac{1}{\pi}\left[(2)\left(1 + \frac{1}{9} + \frac{1}{25} + \ldots\right)\right]$$

$$\Rightarrow 0 = \frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$\boxed{\therefore \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots}$$

Assume $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots = \omega = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Now, $\omega - \frac{\pi^2}{8}$

$$= \left(\cancel{1} + \frac{1}{2^2} + \cancel{\frac{1}{3^2}} + \frac{1}{4^2} + \cancel{\cdots}\right) - \left(\cancel{1} + \cancel{\frac{1}{3^2}} + \cancel{\frac{1}{5^2}} + \cancel{\cdots}\right)$$

$$= \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{4} \omega$$

$$\therefore \quad \omega - \frac{\pi^2}{8} = \frac{1}{4} \omega$$

$$\Rightarrow \frac{3}{4} \omega = \frac{\pi^2}{8}$$

$$\Rightarrow \omega = \frac{\pi^2}{6}$$

$$\Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{6}$$

$$\boxed{\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

$\therefore$ LHS = RHS

Hence Proved!

Now,

Using $f(x) = \begin{cases} 0 , & -\pi \le x < 0 \\ x , & 0 \le x < \pi \end{cases}$ in $[-\pi, \pi)$

Applying Parseval's Theorem,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

$$\Rightarrow \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{0} f^2(x)\, dx}_{0} + \int_{0}^{\pi} f^2(x)\, dx \right] = \frac{(\pi/2)^2}{2} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

$$\Rightarrow \frac{1}{\pi} \int_{0}^{\pi} x^2\, dx = \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

$$\Rightarrow \frac{1}{\cancel{\pi}} \times \frac{\pi^{\cancel{3}\,2}}{3} = \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

$$\Rightarrow \frac{\pi^2}{3} - \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

As $a_n = \frac{(-1)^n - 1}{\pi n^2}$

$b_n = \frac{-(-1)^n}{n}$

$$\Rightarrow \frac{5\pi^2}{12} = \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$= \frac{n^2 \pi^2 + 2}{n^4 \pi^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{2}{n^4 \pi^2}$$

Not Sure