Power Series Solutions

Motivation:

The differential equation

$$y'' + y = 0$$

can be solved easily:
The familiar functions $y = \sin x$ and $y = \cos x$ from elementary calculus provide us the solutions of the equation.

But consider the simple differential equation

$$xy'' + y' + xy = 0.$$

It is known that this equation cannot be solved in terms of any known elementary functions.

This is the case of many important differential equations.

Fortunately, for most of them power series solutions are possible.

A Brief Review of Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is called a **power series** in x. The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

is called a power series in $x - x_0$.

Note: The second series above can always be reduced to the first by replacing $x - x_0$ by another variable or x itself.

Convergence of a Power Series

A power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is said to **converge** at a point x if the limit

$$\lim_{m\to\infty} (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$$

exists. In this case, the limit is called the **sum** of the power series.

Note: The power series above surely converges at $x = 0$.

Note: Any power series has one of the **three convergence possibilities**.

Examples

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- The first series diverges (fails to converge) for all $x \neq 0$.
- The second series converges for all x.
- The third series converges for $|x| < 1$ and diverges for $|x| > 1$.

Note: We say that the third series has **radius of convergence** $R = 1$, the second has radius of convergence $R = \infty$ and the first has radius of convergence $R = 0$.

Radius of Convergence

Each power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

has a **radius of convergence** R, where $0 \leq R \leq \infty$, with the property that the series converges for $|x| < R$ and diverges for $|x| > R$.

$R = 0$ signifies that it converges only at $x = 0$.

$R = \infty$ signifies that it converges for all x.

Finding the Radius of Convergence

The Ratio Test

Let

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

be a series of nonzero constants.
Let the limit

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = L.$$

Then the series converges if $L < 1$ and diverges if $L > 1$ or $L = \infty$.

Now consider a power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Suppose $a_n \neq 0$ for all n.
Then, for a fixed $x \neq 0$, it is a series of numbers. Let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}x}{a_n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then, by the ratio test, the series converges if $L < 1$ and diverges if $L > 1$. But

$$L < 1 \quad \Leftrightarrow \quad |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \quad \Leftrightarrow \quad |x| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| .$$

Hence the radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| .$$

Examples

We now find the radius of convergence of each series below by using the formula:

$$
\begin{aligned}
\sum_{n=0}^{\infty} n! x^n &= 1 + x + 2! x^2 + 3! x^3 + \cdots \\
\sum_{n=0}^{\infty} \frac{x^n}{n!} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sum_{n=0}^{\infty} x^n &= 1 + x + x^2 + x^3 + \cdots
\end{aligned}
$$

- $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$
- $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} (n+1) = \infty.$
- $R = \lim_{n\to\infty} \left| \frac{1}{1} \right| = \lim_{n\to\infty} 1 = 1.$

The Sum Function and its Properties

Suppose that a power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

converges for $|x| < R$ with $R > 0$. Let us denote its sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

This gives us a function $f(x)$ that is defined for $|x| < R$.
It is continuous and has derivatives of all orders for $|x| < R$:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2 a_3 x + \cdots \\ &\vdots \end{aligned}$$

These equations give the following basic formula:

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

So, we have that the given power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is identically equal to the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots .$$

Note: The sum function can also be integrated within the interval of convergence.

Addition of Power Series

Suppose we have two power series that converge for $|x| < R$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \\ g(x) &= \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots \end{aligned}$$

Then they can be added or subtracted termwise. And we have

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n = (a_0 \pm b_0) + (a_1 \pm b_1)x + (a_2 \pm b_2)x^2 + \cdots .$$

Multiplication of Power Series

The two power series can also be multiplied as we multiply polynomials. And we have

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0.$$

Note

We have that the two series below converge for $|x| < R$ with $R > 0$.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \\ g(x) &= \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots \end{aligned}$$

As previously noted, we have

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{and} \quad b_n = \frac{g^{(n)}(0)}{n!}.$$

Consequences:

- If the two power series converge to the same function, i.e., if $f(x) = g(x)$, then $a_n = b_n$ for all n.
- If $f(x) = 0$, then $a_n = 0$ for all n.

Note

Recall: If a power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

converges for $|x| < R$ with $R > 0$ and $f(x)$ is the corresponding sum function, then

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{for all} n.$$

Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots .$$

Question: Suppose we have a continuous function $f(x)$ that has derivatives of all orders for $|x| < R$ with $R > 0$. Can $f(x)$ be represented by a power series?
In particular, is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots ?$$

For example, it is true for $f(x) = e^x$: It is continuous and has derivatives of all orders.

In fact, $f^{(n)}(0) = e^0 = 1$. And

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Definition (Analytic Function)

A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

is valid in some neighbourhood of the point x_0 is said to be **analytic at** x_0**.**

Note: It follows, as earlier, that

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The above series is called the *Taylor series* of $f(x)$ at x_0.

Examples

1. Polynomials and the functions e^x, $\sin x$ and $\cos x$ are analytic at all points.
2. If $f(x)$ and $g(x)$ are analytic at x_0, then $f(x) + g(x)$, $f(x)g(x)$ and $f(x)/g(x)$ [if $g(x_0) \neq 0$] are also analytic at x_0.
3. The sum function of a power series is analytic at all points inside the interval of convergence.

Homework

1. If p is not zero or a positive integer, show that the series

$$\sum_{n=0}^{\infty} \frac{p(p-1)(p-2)\dots(p-n+1)}{n!} x^n$$

   converges for $|x| < 1$ and diverges for $|x| > 1$.
2. Prove that each of the following series has radius of convergence $R = \infty$:
   2.1 $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$
   2.2 $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$

3. Prove that the following are true for $|x| < 1$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

and

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

Now use the fact that a power series can be integrated termwise within the interval of convergence to conclude that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdot$$

and

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

4. Use the first in the preceding problem to find the power series for $\dfrac{1}{(1-x)^2}$

(a) by squaring; (b) by differentiating.

Power Series Solutions of First Order Equations: An Example

Let us consider the equation

$$y' = y.$$

Assume that this equation has a power series solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

that converges for $|x| < R$ with $R > 0$.
This means that we assume that the differential equation has a solution that is analytic at the origin.
A power series can be differentiated term by term in its interval of convergence. So,

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots (n+1)a_{n+1} x^n + \cdots .$$

But the differential equation says that $y' = y$. So, the above series must be equal: the series must have the same coefficients:

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2, \ldots, (n+1)a_{n+1} = a_n, \ldots .$$

We have

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2, \ldots, (n+1)a_{n+1} = a_n, \ldots.$$

Using these equations, we obtain each a_n in terms of a_0:

$$a_1 = a_0, \quad , a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3}, \ldots, a_n = \frac{a_0}{n!}, \ldots.$$

We now substitute all these into

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

and get the power series solution of the differential equation:

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots + \frac{a_0}{n!}x^n + \cdots$$

or

$$y = a_0(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots).$$

We note that this implies that we can choose a_0 to be any constant.

We obtain the solution

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$

for

$$y' = y$$

by *assuming* that the differential equation has a power series solution with a positive radius of convergence.

But does it have one? Yes.

The power series

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$

converges for all x and so it can be differentiated term by term to obtain y'.

It also satisfies the equation. Hence

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$

is indeed the power series solution of the differential equation.

Note: We may recognize this solution as

$$y = a_0 e^x.$$

Homework

For each of the following differential equations, find a power series solution of the form $\sum a_n x^n$, try to recognize the resulting series as the expansion of a familiar function, and verify your conclusion by solving the equation directly.

1. $y' = 2xy$.
2. $y' + y = 1$.

Second Order Linear Equations

We now study power series solutions of the general homogeneous second order linear equation

$$y'' + P(x)y' + Q(x)y = 0.$$

Second Order Linear Equations. Ordinary Points

Definition

A point x_0 is called an **ordinary point** of the differential equation $y'' + P(x)y' + Q(x)y = 0$ if both $P(x)$ and $Q(x)$ are analytic at x_0. Any point that is not an ordinary point of the differential equation is called a **singular point**.

Fact

If x_0 is an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0$, then every solution of the equation is analytic at x_0 so that its Taylor series expansion about $x = x_0$ is valid in some neighbourhood of x_0.

Example

Find a power series solution of $y'' + y = 0$.

Solution: Here the coefficient functions are $P(x) = 0$ and $Q(x) = 1$.
These functions are analytic at all points. Hence the differential equation must have a power series solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

Differentiation of this power series successively twice gives

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots$$

$$y'' = 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + \cdots + (n+1)(n+2)a_{n+2}x^n + \cdots$$

We now substitute the series for y and y'' into the differential equation and add them term by term. This gives

$$(2a_2 + a_0) + (2 \cdot 3a_3 + a_1)x + (3 \cdot 4a_4 + a_2)x^2 + (4 \cdot 5a_5 + a_3)x^3 + \cdots = 0.$$

We have

$$(2a_2 + a_0) + (2 \cdot 3a_3 + a_1)x + (3 \cdot 4a_4 + a_2)x^2 + (4 \cdot 5a_5 + a_3)x^3 + \cdots = 0.$$

This implies that

$$2a_2 + a_0 = 0, \quad 2 \cdot 3a_3 + a_1 = 0, \quad 3 \cdot 4a_4 + a_2 = 0, \quad 4 \cdot 5a_5 + a_3 = 0, \ \ldots$$

Using these equations we now express each a_n in terms of a_0 or a_1 according as n is even or odd:

$$a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_1}{2 \cdot 3}, \quad a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{2 \cdot 3 \cdot 4}, \quad a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}, \ldots.$$

Substituting these into $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ we get

$$y = a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{2 \cdot 3}x^3 + \frac{a_0}{2 \cdot 3 \cdot 4}x^4 + \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}x^5 - \cdots$$

$$
\begin{aligned}
y &= a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{2\cdot 3}x^3 + \frac{a_0}{2\cdot 3\cdot 4}x^4 + \frac{a_1}{2\cdot 3\cdot 4\cdot 5}x^5 - \cdots \\
&= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)
\end{aligned}
$$

Let $y_1(x)$ and $y_2(x)$ denote the two series in parentheses.

We have shown that the power series y above satisfies $y'' + y = 0$ for any pair of constants a_0 and a_1.

Thus choosing $a_0 = 1$ and $a_1 = 0$, we see that $y_1(x)$ satisfies this equation.

And choosing $a_0 = 0$ and $a_1 = 1$, we see that $y_2(x)$ satisfies this equation.

We have already (homework) seen that each of the series $y_1(x) and\ y_2(x)$ converges for all x.

Also $y_1(x) and\ y_2(x)$ are obviously linearly independent.

Any particular solution is obtained by specifying $y(0) = a_0$ and $y'(0) = a_1$.

Example

Solve the Legendre's equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0,$$

where p is a constant.

Solution: Writing the equation in standard form, we see that

$$P(x) = \frac{-2x}{1-x^2} \quad \text{and} \quad Q(x) = \frac{p(p+1)}{1-x^2}.$$

Both of these functions are analytic at the origin.

Thus the origin is an ordinary point.
Hence it has a power series solution of the form $y = \sum a_n x^n$ (with a positive radius of convergence).

So, we have a solution of the form $y = \sum a_n x^n$.

Thus $y' = \sum (n+1) a_{n+1} x^n$.

Hence the expansions for the terms on the LHS of the Legendre's equantion $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ are

$$\begin{aligned} y'' &= \sum (n+1)(n+2) a_{n+2} x^n \\ -x^2 y'' &= \sum -(n-1) n a_n x^n \\ -2xy' &= \sum -2n a_n x^n \\ p(p+1)y &= \sum p(p+1) a_n x^n \end{aligned}$$

The Legendre's equation requires that the sum of these series be zero.
Thus, on summing these series, we have that the coefficient of x^n is zero for every n:

$$(n+1)(n+2)a_{n+2} - (n-1)na_n - 2na_n + p(p+1)a_n = 0.$$

We have

$$(n+1)(n+2)a_{n+2} - (n-1)na_n - 2na_n + p(p+1)a_n = 0.$$

Or, equivalently,

$$a_{n+2} = \frac{(n-1)n + 2n - p(p+1)}{(n+1)(n+2)} a_n.$$

But

$$(n-1)n + 2n - p(p+1) = n^2 - n + 2n - p(p+1) = n^2 + n - p^2 - p = n^2 - p^2 + n - p$$

$$= (n+p)(n-p) + (n-p) = (n-p)(n+p+1) = -(p-n)(p+n+1).$$

$$\therefore \; a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)} a_n.$$

Using this recursion formula, we can express each a_n in terms of a_0 or a_1.

We have

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)} a_n$$

Hence

$$a_2 = -\frac{p(p+1)}{1 \cdot 2} a_0 = -\frac{p(p+1)}{2!} a_0$$

$$a_3 = -\frac{(p-1)(p+2)}{2 \cdot 3} a_1 = -\frac{(p-1)(p+2)}{3!} a_1$$

$$a_4 = -\frac{(p-2)(p+3)}{3 \cdot 4} a_2 = \frac{(p-2)(p+3)}{3 \cdot 4} \cdot \frac{p(p+1)}{2!} a_0 = \frac{p(p-2)(p+1)(p+3)}{4!} a_0$$

$$a_5 = -\frac{(p-3)(p+4)}{4 \cdot 5} a_3 = \frac{(p-3)(p+4)}{4 \cdot 5} \cdot \frac{(p-1)(p+2)}{3!} a_1 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

Similarly, we obtain

$$a_6 = -\frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}a_0$$

$$a_7 = -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}a_1$$

(Homework!)

and so on.

Thus we have

$$a_2 = -\frac{p(p+1)}{2!}a_0$$

$$a_4 = \frac{p(p-2)(p+1)(p+3)}{4!}a_0$$

$$a_6 = -\frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}a_0$$

and so on. And

$$a_3 = -\frac{(p-1)(p+2)}{3!}a_1$$

$$a_5 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!}a_1$$

$$a_7 = -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}a_1$$

and so on.

Putting these values in $y = \sum a_n x^n$ gives us a power series solution of the Legendre's equation whose coefficients a_n are in terms of a_0 or a_1 depending whether n is even or odd.

On grouping the a_0 and a_1 terms separately, we obtain $y = \sum a_n x^n$ in the form

$$y = a_0 y_1(x) + a_1 y_2(x),$$

where

$$y_1(x) = 1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}x^6 + \cdots$$

and

$$y_2(x) = x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}x$$

Each of the above series has radius of converge $R = 1$: They converge for $|x| < 1$.

This can be seen by using the recursion for a_n:

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)} a_n.$$

Replacing n by $2n$ in the above recursion gives

$$a_{2n+2} = -\frac{(p-2n)(p+2n+1)}{(2n+1)(2n+2)} a_{2n} \qquad \text{or} \qquad \frac{a_{2n+2}}{a_{2n}} = -\frac{(p-2n)(p+2n+1)}{(2n+1)(2n+2)}.$$

Thus

$$\left|\frac{a_{2n+2}x^{2n+2}}{a_{2n}x^{2n}}\right| = \left|\frac{a_{2n+2}}{a_{2n}}\right| |x|^2 = \left|-\frac{(p-2n)(p+2n+1)}{(2n+1)(2n+2)}\right| |x|^2 = \left|\frac{(\frac{p}{2n}-1)(\frac{p+1}{2n}+1)}{(1+\frac{1}{2n})(1+\frac{2}{2n})}\right| |x|^2 \to |x|^2$$

as $n \to \infty$.

This proves, by the ratio test, that the first series converges for $|x| < 1$.
By similar means, we can prove that the second series also converges for $|x| < 1$.

Thus we have that Legendre's equation has the power series solution

$$y = a_0 y_1(x) + a_1 y_2(x)$$

(with radius of convergence 1), where

$$y_1(x) = 1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}x^6 + \cdots$$

and

$$y_2(x) = x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}x$$

The functions $y_1(x)$ and $y_2(x)$ that are defined by the above series on the interval $(-1, 1)$ are called **Legendre functions**.

When p is a nonnegative integer, one of the series terminates and hence is a polynomial.
These are called **Legendre polynomials**.

A Property of Power Series Solutions

Theorem

Let x_0 be an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

and let a_0 and a_1 be arbitrary constants. Then there exists a unique function $y(x)$ that is analytic at x_0, is a solution of the differential equation in a certain neighbourhood of this point and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on an interval $|x - x_0| < R$, $R > 0$, then the power series expansion of this solution is also valid on the same interval.

Homework

1. Find the general solution of $(1+x^2)y'' + 2xy' - 2y = 0$ in terms of power series in x. Can you express this solution by means of elementary functions?
2. Consider the equation $y'' + xy' + y = 0$.
    2.1 Find the general solution $y = \sum a_n x^n$ in the form $y = a_0 y_1(x) + a_1 y_2(x)$, where $y_1(x)$ and $y_2(x)$ are power series.
    2.2 Use the ratio test to verify that the two series $y_1(x)$ and $y_2(x)$ converge for all x. Can you conclude the same by any other means?
    2.3 Show that $y_1(x)$ is the series expansion of $e^{-x^2/2}$, use this fact to find a second independent solution by the method known to us, and convince yourself that this second solution is the function $y_2(x)$.

Regular Singular Points. The Frobenius Series Solutions

Recall: A point x_0 is a **singular point** of the differential equation $y'' + P(x)y' + Q(x)y = 0$ if at least one of the coefficient functions $P(x)$ and $Q(x)$ is not analytic at x_0.

Definition

A singular point x_0 of the differential equation $y'' + P(x)y' + Q(x)y = 0$ is said to be a **regular singular point** if the functions

$$(x - x_0)P(x) \qquad \text{and} \qquad (x - x_0)^2 Q(x)$$

are analytic at x_0. Otherwise x_0 is called an **irregular singular point**.

Example

Consider the equation

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0.$$

Here we have $P(x) = \dfrac{2}{x}$ and $Q(x) = -\dfrac{2}{x^2}$.

These functions are not analytic at the origin, i.e, at $x = 0$. (Why?)
So, the origin is a singular point.

But $xP(x) = 2$ and $x^2Q(x) = -2$ are analytic at the origin. (Why?)

Hence it is a regular singular point.

Example

Consider the Legendre's equation

$$y'' - \frac{2x}{1-x^2} + \frac{p(p+1)}{1-x^2} = 0.$$

For this equation, $x = 1$ and $x = -1$ are singular points. But

$$(x-1)P(x) = \frac{2x}{x+1} \qquad \text{and} \qquad (x-1)^2 Q(x) = -\frac{(x-1)p(p+1)}{x+1}$$

are analytic at $x = 1$. (Why?)

Hence $x = 1$ is a regular singular point.

Similarly, wee can see that $x = -1$ is also a regular singular point. (Prove!)

Example

The *Bessel's equation of order p* (p a nonnegative constant) is

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

Let us write the equation in the form

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0.$$

Clearly, origin ($x = 0$) is a singular point.

We have

$$xP(x) = 1 \qquad \text{and} \qquad x^2Q(x) = x^2 - p^2 = -p^2 + x^2.$$

The above functions are analytic at the origin.

Hence the origin is a regular singular point.

Discussion

Determine the nature of the point $x = 0$ for the following equations:

1. $y'' + (\sin x)y = 0.$
2. $xy'' + (\sin x)y = 0.$
3. $x^2y'' + (\sin x)y = 0.$
4. $x^3y'' + (\sin x)y = 0.$
5. $x^4y'' + (\sin x)y = 0.$

Regular Singular Points: The Method of Frobenius

Definition (Frobenius Series)

A series of the form

$$x^m(a_0 + a_1x + a_2x^2 + \cdots),$$

where $a_0 \neq 0$ and the exponent m is any real number (an integer, a fraction or an irrational number), is called a **Frobenius series**.

Examples:

$$x^{1/2}(1 + x + x^2 + \cdots).$$

$$x^{-1}(1 + x + x^2 + \cdots).$$

Fact

Let $y'' + P(x)y' + Q(x)y = 0$ be a differential equation with origin as a regular singular point. Then it has a Frobenius series solution of the form

$$x^m(a_0 + a_1 x + a_2 x^2 + \cdots).$$

Note: The method of finding a Frobenius series solution that we will illustrate with examples is called the method of Frobenius.

Example: Regular Singular Points: The Method of Frobenius

Consider the differential equation

$$2x^2y'' + x(2x+1)y' - y = 0.$$

It can be written as

$$y'' + \frac{1/2 + x}{x}y' + \frac{-1/2}{x^2}y = 0.$$

Clearly, $x = 0$ is a singular point.
We have $xP(x) = \frac{1}{2} + x$ and $x^2Q(x) = -\frac{1}{2}$. These functions are analytic at $x = 0$.

So, $x = 0$ is a regular singular point.

Thus it has a Frobenius series solution

$$y = x^m(a_0 + a_1x + a_2x^2 + \cdots) = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots, \quad a_0 \neq 0.$$

We have

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots, a_0 \neq 0.$$

Hence

$$y' = a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots$$

and

$$y'' = a_0 m(m-1)x^{m-2} + a_1(m+1)m x^{m-1} + a_2(m+2)(m+1)x^m + \cdots.$$

We now substitute these series in $y'' + \frac{1/2 + x}{x} y' + \frac{-1/2}{x^2} y = 0$ and cancel the common factor x^{m-2}.

We obtain

$$a_0 m(m-1) + a_1(m+1)mx + a_2(m+2)(m+1)x^2 + \cdots$$
$$+\left(\frac{1}{2}+x\right)[a_0 m + a_1(m+1)x + a_2(m+2)x^2 + \cdots]$$
$$-\frac{1}{2}(a_0 + a_1 x + a_2 x^2 + \cdots) = 0.$$

Adding the above series and equating the corresponding powers of x to 0, we get

$$a_0\left[m(m-1) + \frac{1}{2}m - \frac{1}{2}\right] = 0$$

$$a_1\left[(m+1)m + \frac{1}{2}(m+1) - \frac{1}{2}\right] + a_0 m = 0$$

$$a_2\left[(m+2)(m+1) + \frac{1}{2}(m+2) - \frac{1}{2}\right] + a_1(m+1) = 0$$

$$\cdots$$

Since $a_0 \neq 0$, from the first equation we have

$$m(m-1) + \frac{1}{2}m - \frac{1}{2} = 0 \qquad \text{or} \qquad m^2 - \frac{1}{2}m - \frac{1}{2} = 0.$$

This is called the **indicial equation** of the differential equation.

Its roots are

$$m_1 = 1 \qquad \text{and} \qquad m_2 = -\frac{1}{2}.$$

These are the values for the exponents in the Frobenius solution.

For each of these values of m, we now use the other equations and determine $a_1, a_2, \ldots$ in terms of a_0.

We have

$$a_1 \left[(m+1)m + \frac{1}{2}(m+1) - \frac{1}{2} \right] + a_0 m = 0$$

$$a_2 \left[(m+2)(m+1) + \frac{1}{2}(m+2) - \frac{1}{2} \right] + a_1(m+1) = 0$$

$$\cdots$$

For $m_1 = 1$:

$$a_1 = -\frac{a_0}{2 \cdot 1 + \frac{1}{2} \cdot 2 - \frac{1}{2}} = -\frac{2}{5} a_0$$

$$a_2 = -\frac{2a_1}{3 \cdot 2 + \frac{1}{2} \cdot 3 - \frac{1}{2}} = -\frac{2}{7} a_1 = \frac{4}{35} a_0.$$

$$\cdots$$

We have

$$a_1\left[(m+1)m+\frac{1}{2}(m+1)-\frac{1}{2}\right]+a_0 m=0$$

$$a_2\left[(m+2)(m+1)+\frac{1}{2}(m+2)-\frac{1}{2}\right]+a_1(m+1)=0$$

$$\cdots$$

For $m_1=-\frac{1}{2}$:

$$a_1=\frac{\frac{1}{2}a_0}{\frac{1}{2}\left(-\frac{1}{2}\right)+\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}}=-a_0$$

$$a_2=-\frac{\frac{1}{2}a_1}{\frac{3}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{3}{2}-\frac{1}{2}}=-\frac{1}{2}a_1=\frac{1}{2}a_0$$

$$\cdots$$

Thus, taking $a_0 = 1$, we have the following Frobenius series solutions for the differential equation:

$$y_1 = x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots\right)$$

$$y_2 = x^{-1/2}\left(1 - x + \frac{1}{2}x^2 + \cdots\right)$$

These solutions are linearly independent. So, the general solution of the differential equation for $x > 0$ is

$$y = c_1 x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots\right) + c_2 x^{-1/2}\left(1 - x + \frac{1}{2}x^2 + \cdots\right).$$

Note: The series in parentheses converge for all x. (Why?)

Note

The equation from which we obtain the exponents of the Frobenius series solutions is called the **indicial equation**. It is indeed

$$m(m-1)+p_0 m+q_0=0,$$

where p_0 and q_0 are the constant terms in the power series expansions of $xP(x)$ and $x^2Q(x)$.

Homework

1. For each of the following differential equations, locate and classify its singular points on the x-axis:
   1.1 $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$
   1.2 $x^2(x^2-1)^2y'' - x(1-x)y' + 2y = 0$
   1.3 $x^2y'' + (2-x)y' = 0$
2. Find the indicial equation and its roots for each of the following differential equations:
   2.1 $x^2y'' + (\cos 2x - 1)y' + 2xy = 0$
   2.2 $4x^2y'' + (2x^4 - 5x)y' + (3x^2 + 2)y = 0$
3. For each of the following equations, verify that the origin is a regular singular point and calculate two independent Frobenius series solutions:
   3.1 $4xy'' + 2y' + y = 0$
   3.2 $2xy'' + (3-x)y' - y = 0$

The Existence of Frobenius Series Solutions and their Properties

Theorem

Assume that $x = 0$ *is a regular singular point of the differential equation* $y'' + P(x)y' + Q(x)y = 0$ *and that the power series expansions of* $xP(x)$ *and* $x^2Q(x)$ *are valid on an interval* $|x| < R$ *with* $R > 0$*. Let the indicial equation of the differential equation have roots* m_1 *and* m_2 *with* $m_2 \leq m_1$*. Then the equation has at least one Frobenius series solution*

$$y = x^{m_1} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0)$$

on the interval $0 < x < R$*, where* $\sum a_n x^n$ *converges for* $|x| < R$*. Furthermore, if* $m_1 - m_2$ *is not zero or a positive integer, then the equation has a second independent Frobenius series solution*

$$y = x^{m_2} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0)$$

on the same interval $0 < x < R$*, where* $\sum a_n x^n$ *converges for* $|x| < R$*.*

Homework

1. The equation $x^2y'' - 3xy' + (4x + 4)y = 0$ has only one Frobenius series solution. Find it.
2. Find two independent Frobenius series solutions of $xy'' + 2y' + xy = 0$.