Indian Institute of Information Technology,
Design and Manufacturing Kancheepuram
MA1002 Linear Algebra

Date : 18/11/2022 | End Semester
Time : 09.30–12.30 | Marks : 50

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1. Determine whether the following system has a solution by using row reduced echelon form: [4]

$$
\begin{aligned}
x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1,\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1,\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2,\\
3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3.
\end{aligned}
$$

2. Suppose that A and B are 2×2 row-reduced echelon matrices and that the system $AX = 0$ and $BX = 0$ have exactly the same solutions. Prove or disprove that $A = B$. [3]

3. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible. [3]

4. Let $V = \{x \in \mathbf{R} : x \geq 0\}$ and $F = \mathbf{R}$. For $x, y \in V$, $\alpha \in \mathbf{R}$, we define $x + y := xy$, and $\alpha x := |\alpha| x$. Check whether V is a vector space over F with the given operations. [4]

5. An $m \times n$ matrix A is called an upper triangular if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the set of all upper triangular matrices form a subspace of $\mathbf{R}^{m\times n} = M_{m\times n}[\mathbf{R}]$ [3]

6. Let $W = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 - 2x_2 + x_3 = 0, \text{ and } 2x_1 - 3x_2 + x_3 = 0\}$. Prove that W is a subspace of $\mathbf{R}^3$. Find a basis of W and its dimension. [4]

7. State Rank-Nullity(Dimension) theorem. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation with $T(1, 0) = (1, 4)$, and $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Is T one-to-one? [4]

8. Let V and W be finite dimensional vector spaces over the field F such that dim V = dim W. If $T : V \longrightarrow W$ is a linear transformation, then prove that the following statements are equivalent. [6]

   (a) T is invertible.

   (b) T is non-singular.

   (c) T is onto, that is, $R(T) = W$.

9. Find the inverse of a linear operator T on R^3 defined as [3]

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

10. Let $A = \begin{pmatrix} -1 & -1 & -2 \\ 8 & -11 & -8 \\ -10 & 11 & 7 \end{pmatrix}$. [7]

   (a) Find the characteristic polynomial of A.

   (b) Find all eigenvalues and eigenvectors (or eigen spaces) of A .

   (c) Is matrix A diagonalizable? Justify your answer.

11. Define inner product on a real/complex vector space V. Let P be the vector space of all polynomials on R. Prove or disprove that

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)dt,$$

is an inner product on P [3]

12. Let V be an inner product space. Show that for $\alpha, \beta \in V$ [6]
(i) $||\alpha + \beta||^2 + ||\alpha - \beta||^2 = 2||\alpha||^2 + 2||\beta||^2$, and
(ii) $|\langle \alpha, \beta \rangle| \leq ||\alpha||\ ||\beta||$.