Linear transformations

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Linear Transformation

Let V and W be vector spaces over the field F. A linear transformation from V into W is a function $T : V \longrightarrow W$ such that

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta) \text{ for all } \alpha, \beta \in V, \ c \in F$$

Examples of Linear Transformation

(1) Let V be a vector space over a field F. We define a function $I : V \longrightarrow V$ as $I(v) = v$ for all $v \in V$.

$$I(c\alpha + \beta) = c\alpha + \beta = cI(\alpha) + I(\beta) \quad \text{for all } \alpha, \beta \in V, c \in F$$

$\Longrightarrow I$ is a L.T.

(2) Let $V = \left\{ f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n \ : \ n \in \mathbb{N}, c_i \in F \right\}$. We define a function $D \ : \ V \longrightarrow V$ as $(Df)(x) = c_1 + 2c_2 x + \ldots + nc_n x^{n-1}$. Prove that D is a L.T.

Examples

(3) Let $A \in F^{m\times n}$. Define a function $T : F^{n\times 1} \longrightarrow F^{m\times 1}$ as $T(X) = AX$.

$$T(cX + Y) = A(cX + Y) = cAX + AY = cT(X) + T(Y)$$

$\Longrightarrow T$ is a L.T.

Prove that $T(0) = 0$

$$T(0) = T(0+0) = T(0) + T(0) \quad (T \text{ is a L.T. })$$

$$\implies T(0) = 0$$

$$T(c\alpha) = T(c\alpha + 0) = cT(\alpha) + T(0) = cT(\alpha) + 0 = cT(\alpha)$$

Note : Since T is a L.T.,

$$T(c_1\alpha_1 + c_2\alpha_2) = c_1 T(\alpha_1) + T(c_2\alpha_2) = c_1 T(\alpha_1) + c_2 T(\alpha_2)$$

Prove that if T is a L.T., then

$$T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n) = c_1 T(\alpha_1) + c_2 T(\alpha_2) + \ldots + c_n T(\alpha_n)$$

Problem 1

Verify which of the following functions $T : R^2 \longrightarrow R^2$ are linear transformations?

(1) $T(x_1, x_2) = (1 + x_1, x_2)$

$T(0,0) = (1,0) \Longrightarrow T(0) \neq 0$ (Not a L.T.)

(2) $T(x_1, x_2) = (x_2, x_1)$

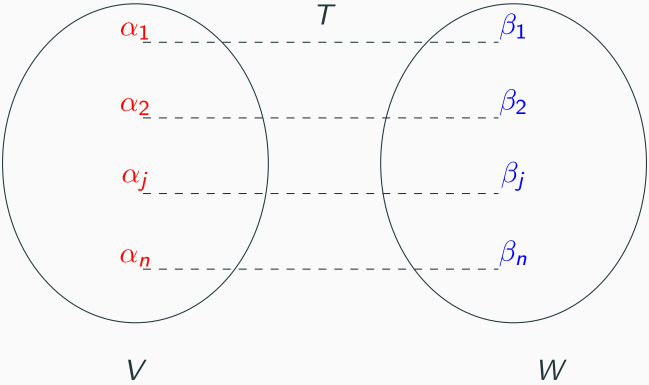
$$T(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow T(X) = AX$$ (Its a L.T.)

(3) $T(x_1, x_2) = (x_1^2, x_2)$

$\alpha = \beta = (1,0), \alpha + \beta = (2,0)$, $T(\alpha + \beta) \neq T(\alpha) + T(\beta)$

Not a L.T.

Linear transformations are special !!



Ordered basis, $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

β_j *'s* need not be distinct

T is a unique L.T. with $T(\alpha_j) = \beta_j$

Theorem 1

Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \beta_2, \ldots, \beta_n$ be any vectors in W. Then there is precisely one linear transformation $T : V \longrightarrow W$ such that $T(\alpha_j) = \beta_j$ for $j = 1, 2, \ldots, n$.

Proof: Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is an ordered basis for V, for a given vector $\alpha \in V$, there is a unique $n-$tuple $(x_1, x_2, \ldots, x_n)$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n$$

Theorem 1 contd.

We define a function $T : V \longrightarrow W$ as

$T(\alpha) = T(x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n) = x_1\beta_1 + x_2\beta_2 + \ldots + x_n\beta_n$.

Claim 1 : $T(\alpha_j) = \beta_j$

$$\begin{aligned} T(\alpha_j) &= T(0\alpha_1 + 0\alpha_2 + \ldots + 1.\alpha_j + \ldots + 0\alpha_n) \\ &= 0\beta_1 + 0\beta_2 + \ldots + 1.\beta_j + \ldots + 0\beta_n \\ &= \beta_j \end{aligned}$$

Claim 2 : T is a linear transformation.

Show that $T(c\alpha + \beta) = cT(\alpha) + T(\beta)$ for all $\alpha, \beta \in V$, $c \in F$.

Let $\beta = y_1\alpha_1 + y_2\alpha_2 + \ldots + y_n\alpha_n$.

$$c\alpha + \beta = (cx_1 + y_1)\alpha_1 + (cx_2 + y_2)\alpha_2 + \ldots + (cx_n + y_n)\alpha_n$$

Theorem 1 contd.

$\Longrightarrow$ (by the definition of T)

$$\begin{aligned} T(c\alpha+\beta) &= (cx_1+y_1)\beta_1+(cx_2+y_2)\beta_2+\ldots+(cx_n+y_n)\beta_n \\ &= cT(\alpha)+T(\beta)(\text{Prove it !}) \end{aligned}$$

Claim 3 : T is unique.

It is enough to prove that if $U : V \longrightarrow W$ is a L.T. with $U(\alpha_j)=\beta_j$ for $j=1,2,\ldots,n$, then $T(\alpha)=U(\alpha)$ for all $\alpha \in V$.

Consider

$$\begin{aligned} U(\alpha) &= U(x_1\alpha_1+x_2\alpha_2+\ldots+x_n\alpha_n) \\ &= x_1U(\alpha_1)+x_2U(\alpha_2)+\ldots+x_nU(\alpha_n) \quad (U \text{ is a L.T.}) \\ &= x_1\beta_1+x_2\beta_2+\ldots+x_n\beta_n \quad (\ U(\alpha_j)=\beta_j) \\ &= T(\alpha) \end{aligned}$$

It completes the proof.

Problem 2

Let $B = \{\alpha_1 = (1,2), \alpha_2 = (3,4)\}$ be an ordered basis for R^2. Let $\beta_1 = (3,2,1)$, $\beta_2 = (6,5,4) \in R^3$. Find a unique L.T.
$T : R^2 \longrightarrow R^3$ such that $T(\alpha_j) = \beta_j$ for $j = 1,2$.

Solution : $T(\alpha_1) = T(1,2) = (3,2,1) = \beta_1$

$T(\alpha_2) = T(3,4) = (6,5,4) = \beta_2$

Let $\alpha = (x,y) \in R^2$

$\alpha = a\alpha_1 + b\alpha_2 \Longrightarrow (x,y) = a(1,2) + b(3,4)$

$(x,y) = (-2x + \frac{3}{2}y)\alpha_1 + (x - \frac{1}{2}y)\alpha_2$

$T(x,y) = (-2x + \frac{3}{2}y)\beta_1 + (x - \frac{1}{2}y)\beta_2$

$T(x,y) = (-2x + \frac{3}{2}y)(3,2,1) + (x - \frac{1}{2}y)(6,5,4)$

$T(x,y) = (\frac{3}{2}y, x + \frac{1}{2}y, 2x - \frac{1}{2}y)$

Problem 2 contd.

$$T(x,y) = \left(\tfrac{3}{2}y, x + \tfrac{1}{2}y, 2x - \tfrac{1}{2}y\right) = \begin{bmatrix} 0 & \frac{3}{2} \\ 1 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

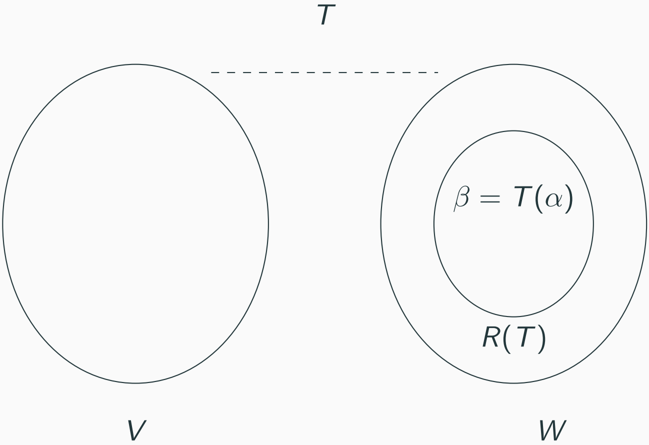
T is a unique L.T. thanks to Theorem 1.

Range of T

Let V, W be vector spaces over a field F. Let $T : V \longrightarrow W$ be a linear transformation.

Range of T, $\quad R(T) = \{\beta \in W \ : \ T(\alpha) = \beta \text{ for some } \alpha \in V\}$

Range of T



Show that $R(T)$ is a subspace of W.

Proof: Let $T : V \longrightarrow W$ is a L.T.

Note that $T(0) = 0.$ $\Longrightarrow 0 \in R(T) \neq \phi.$

Let $\beta_1, \beta_2 \in R(T), c \in F.$ There exist $\alpha_1, \alpha_2 \in V$ such that $T(\alpha_1) = \beta_1,\ T(\alpha_2) = \beta_2.$

Clearly $c\alpha_1 + \alpha_2 \in V.$ $\Longrightarrow T(c\alpha_1 + \alpha_2) \in R(T).$

$\Longrightarrow cT(\alpha_1) + T(\alpha_2) \in R(T) \quad (T \text{ is a L.T.})$

$\Longrightarrow c\beta_1 + \beta_2 \in R(T).$

Hence $R(T)$ is a subspace of W.

Rank of T = dim $R(T)$

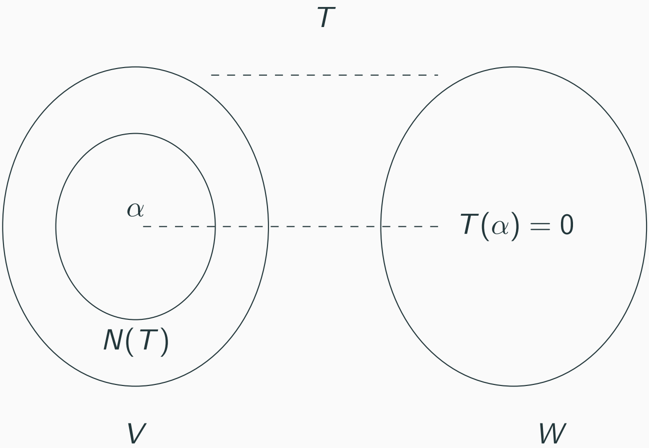
(provided V is a finite-dimensional vector space.)

The null space of T.

Let V, W be vector spaces over a field F. Let $T : V \longrightarrow W$ be a linear transformation.

$$\text{Null space of } T, \quad N(T) = \{\alpha \in V \ : \ T(\alpha) = 0\}$$

Null space of T.



Show that $N(T)$ is a subspace of V.

Proof: Let $T : V \longrightarrow W$ is a L.T.

Note that $T(0) = 0. \implies 0 \in N(T) \neq \phi$.

Let $\alpha_1, \alpha_2 \in N(T), c \in F$. Then $T(\alpha_1) = T(\alpha_2) = 0$. Since T is a L.T.,

$$T(c\alpha_1 + \alpha_2) = cT(\alpha_1) + T(\alpha_2) = c0 + 0 = 0$$

$\implies c\alpha_1 + \alpha_2 \in N(T)$.

Hence $N(T)$ is a subspace of V.

Nullity of T = dim $N(T)$

(provided V is a finite-dimensional vector space.)

Examples:

Find the range and null space of the following linear transformations.

(1) Let $O : V \longrightarrow W$ be the zero linear transformation. That is $O(\alpha) = 0$ for all $\alpha \in V$.

$R(O) = \{\beta \in W \ : \ \beta = O(\alpha) \text{ for some } \alpha \in V\}$

$R(O) = \{\beta \in W \ : \ \beta = O(\alpha) = 0 \text{ for some } \alpha \in V\} = \{0\}$

$N(O) = \{\alpha \in V \ : \ O(\alpha) = 0\} = V$

Rank (O) **= dim** $R(O) = 0$ **and**

Nullity (O)**= dim** $N(O) =$ **dim** V**.**

Examples:

(2) Let $I : V \longrightarrow V$ be the identity linear transformation. That is $I(\alpha) = \alpha$ for all $\alpha \in V$.

$R(I) = \{\beta \in V : \beta = I(\alpha) \text{ for some } \alpha \in V\}$

$R(I) = \{\beta \in V : \beta = I(\alpha) = \alpha \text{ for some } \alpha \in V\} = V$

$N(I) = \{\alpha \in V : I(\alpha) = 0\} = \{\alpha \in V : \alpha = 0\} = \{0\}$

Rank (I) **= dim** $R(I) =$ **dim** V **and**

Nullity $(I) = 0$.

Problem 3

Find the rank and nullity of the linear transformation $T : R^2 \longrightarrow R^3$ defined as $T(x_1, x_2) = (x_1, 0, 0)$.

Solution : $T(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\implies TX = AX, \qquad \text{where } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$R(T) = \{Y \in R^3 \ : \ Y = TX \text{ for some } X \in R^2\}$
$R(T) = \{Y \in R^3 \ : \ Y = AX \text{ for some } X \in R^2\}$
$R(T) = \{AX : \ : \ X \in R^2\}$
$R(T) = \{ \text{ all linear combinations of columns of } A\}$

Problem 3 contd.

$\Longrightarrow R(T) =$ Column space of $A =$ Row space of A^t.

$\Longrightarrow R(T) = \{a(1,0,0) \ : \ a \in R\} \ = \$ Span of $\{(1,0,0)\}$

$$\Longrightarrow \quad \text{Rank}(T) = 1$$

$N(T) = \left\{X \in R^2 \ : \ TX = 0\right\} = \{X : \ AX = 0\}$

$AX = 0 \Longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow x_1 = 0, x_2 = a, a \in R$

$N(T) = \{(x_1, x_2) = (0, a) \ : \ a \in R\} = \{a(0,1) : \ a \in R\}$

$N(T) =$ Span $\{(0,1)\}$

$$\text{Nullity } (T) \ = 1$$

Problem 4

Show that

(i) $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$ is a linear transformation, and (ii)compute rank(T), nullity(T).

Solution :

$$T(x_1, x_2, x_3) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\implies T(X) = AX$$

Hence T is a linear transformation.

Range of T= Column space of A = Row space of A^t.

Problem 4 contd.

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Range of T **= Row space of** A^t **= Span** $\{(1,0,1),(0,1,-1)\}$

Range of T **=** $\{a(1,0,1)+b(0,1,-1) \ : \ a,b \in R\}$

Range of T **=** $\{(a,b,a-b) : a,b \in R\}$

rank $(T) =$ **dim** $R(T) = 2$

Problem 4 contd.

$$N(T) = \left\{X \in R^3 : \; TX = 0\right\} = \left\{X \in R^3 : \; AX = 0\right\}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$AX = 0 \Longrightarrow x_1 + \frac{2}{3}x_3 = 0, x_2 - \frac{4}{3}x_3 = 0$$

$$x_3 = a \Longrightarrow x_1 = -\frac{2}{3}a, x_2 = \frac{4}{3}a$$

Problem 4 contd.

$$N(T) = \left\{ \left(-\frac{2}{3}a, \frac{4}{3}a, a \right) \ : \ a \in R \right\} = \left\{ a \left(-\frac{2}{3}, \frac{4}{3}, 1 \right) \ : \ a \in R \right\}$$

$$N(T) = \text{ Span } \left\{ \left(-\frac{2}{3}, \frac{4}{3}, 1 \right) \right\}$$

Nullity (T) = 1.

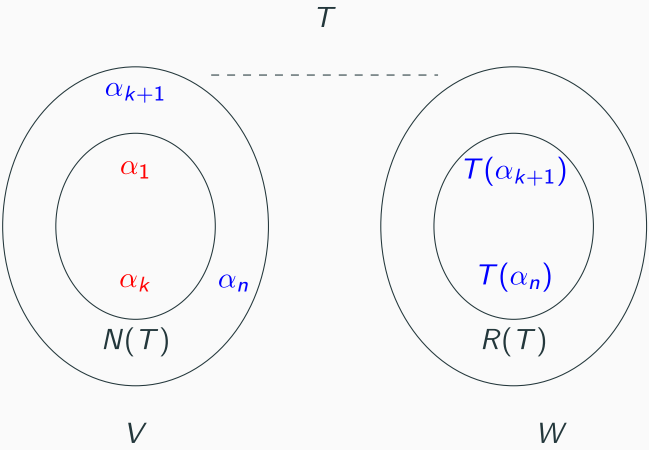
Theorem 2 (Rank-Nullity-Dimension Theorem)

Let V and W be vector spaces over the field F and let $T : V \longrightarrow W$ be a linear transformation. Suppose that V is finite-dimensional. Then

$$\textbf{rank}(T) + \textbf{nullity}(T) = \textbf{dim}\, V$$

Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis for $N(T)$ and let dim $V = n$. Note that nullity $(T) = k$. Since $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq V$ and V is finite-dimensional, there exist vectors $\alpha_{k+1}, \ldots, \alpha_n \in V$ such that $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V, thanks to Corollary 2 of Theorem 5. Next, we prove that $B = \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ is a basis for $R(T)$.

Theorem 2 contd.



Theorem 2 contd.

Claim 1: $R(T) = \text{Span } B = \text{Span } \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$

Let $\beta \in R(T)$. Then there exists $\alpha \in V$ such that $\beta = T(\alpha)$. Since $\alpha \in V = \text{Span } \{\alpha_1, \ldots, \alpha_n\}$, there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$\alpha = c_1\alpha_1 + \ldots + c_k\alpha_k + c_{k+1}\alpha_{k+1} + \ldots c_n\alpha_n$$

$$\beta = T(\alpha) = T(c_1\alpha_1 + \ldots + c_k\alpha_k + c_{k+1}\alpha_{k+1} + \ldots + c_n\alpha_n)$$

$$\beta = c_1 T(\alpha_1) + \ldots + c_k T(\alpha_k) + c_{k+1} T(\alpha_{k+1}) + \ldots + c_n T(\alpha_n)$$

$$\beta = c_1 0 + \ldots + c_k 0 + c_{k+1} T(\alpha_{k+1}) + \ldots + c_n T(\alpha_n)$$

$$\beta = c_{k+1}T(\alpha_{k+1}) + \ldots + c_nT(\alpha_n) \in \text{ Span } B$$

$\implies R(T) \subseteq$ Span B. Since $B \subseteq R(T)$, Span $B \subseteq R(T)$.

$$\implies \quad R(T) = \text{Span } B.$$

Claim 2 : $B = \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\}$ is a L.I. set. Consider

$$c_{k+1}T(\alpha_{k+1}) + \ldots + c_nT(\alpha_n) = 0$$

$$\implies T(c_{k+1}\alpha_{k+1} + \ldots + c_n\alpha_n) = 0$$

$$\implies c_{k+1}\alpha_{k+1} + \ldots + c_n\alpha_n \in N(T) = \text{Span } \{\alpha_1, \ldots, \alpha_k\}$$

There exist scalars $b_1, \ldots, b_k \in F$ such that

$$c_{k+1}\alpha_{k+1} + \ldots + c_n\alpha_n = b_1\alpha_1 + \ldots + b_k\alpha_k$$

$$b_1\alpha_1 + \ldots + b_k\alpha_k - c_{k+1}\alpha_{k+1} - \ldots - c_n\alpha_n = 0$$

Since $\{\alpha_1, \ldots, \alpha_k, \ldots, \alpha_n\}$ is a L.I. set,
$b_1 = \ldots = b_k = -c_{k+1} = \ldots = -c_n = 0.$

$$c_{k+1}T(\alpha_{k+1}) + \ldots + c_nT(\alpha_n) = 0 \Longrightarrow c_{k+1} = \ldots = c_n = 0$$

This proves Claim 2. By Claims 1 and 2, B is a basis of $R(T)$ and dim $R(T) = |B| = n - k. \Longrightarrow$ dim $R(T) =$ dim $V-$ dim $N(T)$.
$\Longrightarrow$ rank$(T)+$ nullity $(T) =$ dim V.

Theorem 3

If $A \in F^{m \times n}$, then row rank $(A) =$ column rank (A).

Proof: We construct a linear transformation $T : F^{n\times 1} \longrightarrow F^{m\times 1}$ defined as $T(X) = AX$. By Rank-Nullity-Dimension Theorem, rank$(T)+$ nullity $(T) =$ dim $V=$ dim $F^{n\times 1} = n - - - -(1)$.

$$
\begin{aligned}
R(T) &= \{Y \in F^{m\times 1} : T(X) = Y \text{ for some } X \in F^{n\times 1}\} \\
&= \{Y \in F^{m\times 1} : AX = Y \text{ for some } X \in F^{n\times 1}\} \\
&= \{AX \ : \ X \in F^{n\times 1}\} \\
&= \{x_1 A_1 + \ldots + x_n A_n \ : \ A_i, i^{th}\text{column of } A, x_i \in F\} \\
&= \text{Column space } (A)
\end{aligned}
$$

rank $(T) =$ dim $R(T) =$ dim column space $(A) =$ column rank $(A) - (2)$

Theorem 3 contd.

$$\begin{aligned} N(T) &= \{X \in F^{n\times 1} \;:\; T(X) = 0\} \\ &= \{X \in F^{n\times 1} \;:\; AX = 0\} = S \end{aligned}$$

Let R be the row-reduced echelon matrix row-equivalent to A. Let r be the number of non-zero rows of R.

$$r = \text{ row rank } (R) = \text{ row rank } (A) - - - -(3)$$

$$RX = 0 \Longrightarrow x_{k_i} + \sum_{j=1}^{n-r} C_{ij} u_j = 0 \text{ for } 1 \leq i \leq r$$

The above system has $n - r$ free variables and it implies that

$$\dim\ S = n - r = \dim\ N(T) = \text{nullity}(T) - -(4)$$

Theorem 3 contd.

From (1), (2) and (4),

$$\text{column rank } (A) + n - r = n$$

$$\implies \text{column rank } (A) = r = \text{row rank } (A), \quad \text{by (3)}$$

It completes the proof.

Note : $\textbf{rank } (A) = \textbf{column rank } (A) = \textbf{row rank } (A)$

Problem 5

Describe explicitly a linear transformation from R^3 into R^3 which has as its range the subspace spanned by $(1,0,-1),(1,2,2)$.

Solution : From Theorem 3, if $T(X) = AX$, then $R(T) =$ Column space (A) (Note that $A \in R^{3\times 3}$).

Since

$$R(T) = \textbf{Span } \{(1,0,-1),(1,2,2)\} \Longrightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

Problem 5 contd.

$$T(X) = AX = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_2 + 2x_3, -x_1 + 2x_2 + 2x_3)$$

Problem 6

Find a L.T. (if exists) $T : R^3 \longrightarrow R^3$ such that $N(T) = \text{Span } \{(1,1,1)\}$ and $R(T) = \text{Span}\{(1,0,-1),(1,2,2)\}$. Justify your answer.

Outline of the answer : Note that $\{\alpha_1 = (1,1,1)\}$ be a basis for $N(T)$. Using the basis of $N(T)$, we construct a basis for $V = R^3$, say $\{\alpha_1 = (1,1,1), \alpha_2 = (0,1,1), \alpha_3 = (0,0,1)\}$ (We have solved similar problems in the past!). Note that
$\beta_1 = (0,0,0), \beta_2 = (1,0,-1), \beta_3 = (1,2,2) \in R(T)$.
Let us contruct T such that $T(\alpha_1) = T(1,1,1) = \beta_1 = (0,0,0)$,
$T(\alpha_2) = T(0,1,1) = \beta_2 = (1,0,-1)$, and
$T(\alpha_3) = T(0,0,1) = \beta_3 = (1,2,2)$

Problem 6 contd.

$$(x, y, z) = a\alpha_1 + b\alpha_2 + c\alpha_3 = a(1,1,1) + b(0,1,1) + c(0,0,1)$$

$$\implies \quad (x, y, z) = x\alpha_1 + (y - x)\alpha_2 + (z - y)\alpha_3$$

$$\implies \quad T(x, y, z) = x\beta_1 + (y - x)\beta_2 + (z - y)\beta_3$$

$$\implies \quad T(x, y, z) = (-x + z, -2y + 2z, x - 3y + 2z)$$

$$T(x, y, z) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Problem 7

Let $T : V \longrightarrow V$ be a linear transformation. Prove that following statements are equivalent.

(a) $N(T) \cap R(T) = \{0\}$

(b) If $T(T(\alpha)) = 0$, then $T(\alpha) = 0$.

Solution : $(a) \Longrightarrow (b)$.
Suppose that $N(T) \cap R(T) = \{0\}$.
$T(T(\alpha)) = 0 \Longrightarrow T(\alpha) \in N(T)$. Note that $T(\alpha) \in R(T)$.
$\Longrightarrow T(\alpha) \in N(T) \cap R(T) = \{0\}$. $\Longrightarrow T(\alpha) = 0$.

$(b) \implies (a)$

Suppose that if $T(T(\alpha)) = 0$, then $T(\alpha) = 0$. Clearly $\{0\} \subseteq N(T) \cap R(T) - - - - - - (1)$.

Let $\beta \in N(T) \cap R(T)$. $\implies \beta \in N(T)$ and $\beta \in R(T)$.

$\implies T(\beta) = 0$ and there exists $\alpha \in V$ such that $\beta = T(\alpha)$.

$\implies T(\beta) = T(T(\alpha)) = 0$. $\implies T(\alpha) = 0$ (by hypothesis).

$\implies \beta = T(\alpha) = 0$. $\implies \beta \in \{0\}$.

$\implies N(T) \cap R(T) \subseteq \{0\} - - - (2)$.

From (1) and (2), $N(T) \cap R(T) = \{0\}$.

$L(V, W)$: set of all linear transfomrations from V into W.

Let V, W be vector spaces over the field F.

$$L(V, W) = \{ T \ : \quad T : V \longrightarrow W \text{ is a L.T. } \}$$

Observation 1 : $L(V, W)$ is a vector space under the opeartions

$$(T + U)(\alpha) = T(\alpha) + U(\alpha), \quad (cT)(\alpha) = cT(\alpha)$$

for all $T, U \in L(V, W)$, $c \in F$.

Observation 2 : If V and W are finite dimensional vector spaces, then $\dim L(V, W) = \dim V \dim W$.

Linear Operator : If V is a vector space over the field F, then a linear operator T is a linear transformation $T : V \longrightarrow V$.