Set A - Problems on Alternating Series and Power Series Concepts

1. Test the convergence (absolute/conditional) of the following series:

   (i) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$

   (ii) $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^{n+2}(1-n)}{3n-n^2}$

   (iii) $x + \dfrac{(a-b)}{2!}x^2 + \dfrac{(a-b)(a-2b)}{3!}x^3 + \dots$ (x any real number)

   (iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$

2. Find the radius and interval of convergence of the following power series.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$

   (ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{\sqrt{n}+3}$

   (iii) $\dfrac{1}{2}x + \dfrac{1.3}{2.5}x^2 + \dfrac{1.3.5}{2.5.8}x^3 + \dots$

   (iv) $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{\log(n)}$

3. Find the power series expansion of the function $f(x) = \dfrac{x}{1+2x+x^2}$ about the origin and find its radius of convergence.

4. Find the Taylor series of the following functions.

   (i) $f(x) = x^2\cos(x^2)$ about $x = 0$

   (ii) f(x) $= \cos\left(2x + \dfrac{\pi}{2}\right)$ about $x = \dfrac{\pi}{4}$

Set B - Problems on Limits Concepts

1. If $\lim_{x \to x_0} f(x) = L$, then show that $\lim_{x \to x_0} |f(x)| = |L|$. Is the converse true?

2. Evaluate:

   (i) $\lim_{x \to 0} \dfrac{\sin x}{\sqrt{x}}$ (ii) $\lim_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ (iii) $\lim_{x \to 0} \dfrac{3x + |x|}{7x - 5|x|}$ (iv) $\lim_{x \to 0} \dfrac{1 - 2\cos x + \cos 2x}{x^2}$

3. Prove the following using the $\epsilon - \delta$ definition.

   (i) $\lim_{x \to 0^+} \dfrac{1}{e^{-1/x} + 1} = 1$ (ii) $\lim_{x \to 2} \dfrac{x^3 - 4}{x^2 + 1} = \dfrac{4}{5}$

   (iii) $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ (iv) $\lim_{x \to -3} \sqrt{1 - 5x} = 4$

4. Determine a condition on $|x - 4|$ that will ensure that:

   (i) $|\sqrt{x} - 2| < \dfrac{1}{2}$ (ii) $|\sqrt{x} - 2| < 0.01$

Set C - Problems on Continuity Concepts

1. (a) Show that the function $f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ -1 & \text{if } x \text{ is rational} \end{cases}$ is discontinuous everywhere on the number line.

   (b) For what values of a and b the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

   is continuous at every x?

2. Test the continuity of the following functions at the given point(s).

   (i) $f(x) = \begin{cases} [x+1] \sin \dfrac{1}{x} & \text{if } x \in (-1, 0) \cup (0, 1) \\ 0 & \text{otherwise} \end{cases}$ at $x = 0$ and $x = 1$

   (ii) $f(x) = x - |x|$ at $x = 0$.

   (iii) $f(x) = \begin{cases} \dfrac{1 - x}{1 - \sqrt[3]{x}} & \text{when } x \neq 1 \\ 3 & \text{when } x = 1 \end{cases}$ at $x = 1$

3. (a) A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$?
(b) Show that $f(x) = 2\ln(x) + \sqrt{x} - 2$ has a root in the interval $[1, 2]$.

4. Prove that $h(t) = \dfrac{t^2 + 3t - 10}{t - 2}$ has a continuous extension to the point $t = 2$. Also find it.

5. Let $g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) = 2x$ for x rational and $g(x) = x + 3$ for x irrational. Discuss the continuity of g on $\mathbb{R}$.

6. Prove that if a function f is continuous at a, then prove that $|f|$ is also continuous at a but not conversely.

7. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in \mathbb{R}$ *if and only if* for every $\epsilon > 0$ there corresponds a $\delta > 0$ such that
$$|x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \epsilon.$$