10/11/23

EM ASSIGNMENT - I

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CS23I1027

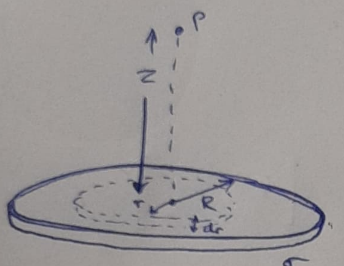
① Given :

Flat Circular Disk – Radius 'R'
└ Surface Charge 'σ'

To Find :

'E' at a distance z above the centre of the disk

Solution :

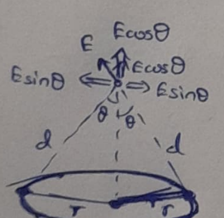


Let us consider a small disc of radius $r < R$ and thickness 'dr'.

$$dq = \sigma \, dA$$

$$\Rightarrow dq = \sigma \, 2\pi r \, dr \quad -①$$

E due to the small disc :



Horizontal components of E i.e. 'Esinθ', get cancelled and Vertical Components of E get added.

$$\therefore E_{net} = \int E \cos\theta$$

$$\Rightarrow E_{net} = \int \frac{k \cdot dq}{d^2} \cos\theta$$

$$= \int \frac{k\sigma \cdot 2\pi r \, dr}{d^2} \cos\theta \quad \text{[from ①]}$$

$$= k\sigma\pi \int \frac{2rz \, dr}{(z^2+r^2)^{3/2}}$$

$$d = \sqrt{x^2+z^2}$$

$$\& \cos\theta = \frac{z}{d} = \frac{z}{\sqrt{x^2+z^2}}$$

$$= k\sigma\pi z \int \frac{2r \, dr}{(z^2+r^2)^{3/2}}$$

Let $t = (z^2+r^2)^{3/2}$

$$\Rightarrow dt = \frac{3}{2}(z^2+r^2)^{1/2} \cdot 2r \, dr$$

$$\Rightarrow 2r \, dr = \frac{2}{3} t^{-1/3} \, dt$$

~~As $r \to 0$ to R~~
~~$t \to z^2$ to~~

$$\Rightarrow E_{net} = \frac{2}{3} k\sigma\pi z \int \frac{t^{-1/3}\,dt}{t}$$

$$= \frac{2}{3} k\sigma\pi z \left(-3t^{-1/3}\right]$$

$$= \frac{2}{3} k\sigma\pi z \left(-3\left(\sqrt{z^2+r^2}\right)^{-1}\right]_0^R$$

$$= -2k\sigma\pi z\left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{z}\right)$$

$$= 2k\sigma\pi\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)$$

$$\boxed{\therefore E_{net} = 2k\sigma\pi\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)}$$

(OR) $\boxed{E_{net} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)}$ $\left[k = \frac{1}{4\pi\varepsilon_0}\right]$

When $R \longrightarrow \infty$,

$$E_{net} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\infty}\right) = \frac{\sigma}{2\varepsilon_0}(1-0) = \frac{\sigma}{2\varepsilon_0}$$

$$\boxed{\therefore E_{net} = \frac{\sigma}{2\varepsilon_0}}$$

When $z >> R$,

$$E_{net} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{z\sqrt{1+\left(\frac{R}{z}\right)^2}}\right)$$

$$\left[\text{if } z >> R \Rightarrow \frac{R}{z} \to 0\right]$$

$$\Rightarrow E_{net} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{z}\right) = 0$$

$$\boxed{\therefore E_{net} = 0}$$

② Given:

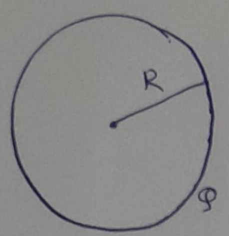
Sphere of Radius R

↳ Contains charge q (Uniformly Distributed)

To find:

Work done to assemble these charges

Solution:



$$W = \int V\, dq$$

$$= \underbrace{\int V_{out}\, dq}_{W_1} + \underbrace{\int V_{in}\, dq}_{W_2}$$

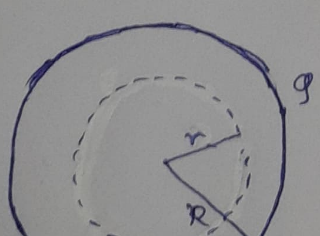
$$W_1 = \int (V_{out})\, dq = \int_0^q \frac{kq}{R}\, dq = \frac{kq^2}{2R}$$

$$\boxed{\therefore W_1 = \frac{q^2}{8\pi \varepsilon_0 R} \text{ J}}$$

$$W_2 = \int (V_{in})\, dq = \int \frac{1}{2} \varepsilon_0 E^2\, \cancel{dq}\, dV$$

↳ Volume

Let's Consider a Gaussian surface,



$$\oint E \cdot ds = \frac{q_{enc}}{\varepsilon_0} = \frac{\rho V}{\varepsilon_0}$$

$$\Rightarrow E_{in} \cdot 4\pi r^2 = \frac{\rho \cdot \frac{4}{3}\pi r^3}{\varepsilon_0}$$

$$\Rightarrow \boxed{E_{in} = \frac{\rho r}{3\varepsilon_0}}$$

$$W = \frac{1}{2}\varepsilon_0 \int (E_{in})^2\, d\tau$$

$$= \frac{1}{2}\varepsilon_0 \int \frac{\rho^2 r^2}{3^2 \varepsilon_0^2} \cdot 4\pi r^2\, dr$$

$$= \frac{2\pi\rho^2}{9\varepsilon_0}\int_0^R r^4\,dr$$

$$= \frac{2\pi\rho^2}{9\varepsilon_0} \times \frac{R^5}{5}$$

$$= \frac{2\pi R^5}{\cancel{9}\times 5\varepsilon_0} \times \frac{\cancel{9}q^2}{16\pi^2 R^6}$$

$$= \frac{q^2}{40\pi R\varepsilon_0}$$

$\rho V = q$

$\Rightarrow \rho\left(\frac{4}{3}\pi R^3\right) = q$

$\Rightarrow \rho^2\left(\frac{16}{9}\pi^2 R^6\right) = q^2$

$\Rightarrow \rho^2 = \frac{9q^2}{16\pi^2 R^6}$

$$\boxed{\therefore W_2 = \frac{q^2}{40\pi\varepsilon_0 R}\ \text{J}}$$

$$\therefore W = W_1 + W_2$$

$$= \frac{q^2}{\pi\varepsilon_0 R}\left(\frac{1}{8} + \frac{1}{40}\right)$$

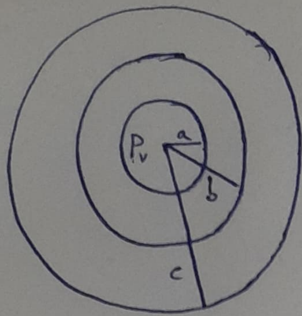
$$= \frac{6q^2}{40\pi\varepsilon_0 R}$$

$$\boxed{\therefore W = \frac{3q^2}{20\pi\varepsilon_0 R}\ \text{J}}$$

(OR)

$$\boxed{W = \frac{3kq^2}{5R}\ \text{J}}$$

(3) Given :



To find :

(i) $E_{r<a}$

(ii) $E_{a<r<b}$

(iii) $E_{b<r<c}$

(iv) $E_{r>c}$

Solution :

(iii) $E_{b<r<c} = 0$

Straight Away we can say because Electric field Inside a conductor = 0.

(iv) $E_{r>c}$

Consider a Gaussian Surface of Radius > c,

$$\oint E.ds = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow E.4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \boxed{E = \frac{q}{4\pi\varepsilon_0 r^2}}$$

$$\boxed{\therefore E = \frac{kq}{r^2}}$$

(ii) $E_{a<r<b}$

Consider a Gaussian Surface of radius, r, such that $a<r<b$

$$\oint E.ds = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2}$$

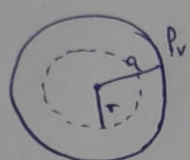
$$\boxed{\therefore E = \frac{q}{4\pi\varepsilon_0 r^2}}$$

(OR)

$$\boxed{E = \frac{kq}{r^2}}$$

(i) $E_{r<a}$

Consider a Gaussian Surface of Radius, $r<a$



$$\oint E \cdot ds = \frac{q_{enc}}{\varepsilon_0} = \frac{\rho_v V}{\varepsilon_0}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{\rho_v \cdot \frac{4}{3}\pi r^3}{\varepsilon_0}$$

$$\Rightarrow E = \frac{\rho r}{3\varepsilon_0}$$

$$\boxed{\therefore E = \frac{\rho_v r}{3\varepsilon_0}}$$

$\therefore$ $E \;\forall r<a = \frac{\rho_v r}{3\varepsilon_0}$ V/m

$E \;\forall r \in [a,b] = \frac{kq}{r^2}$ V/m

$E \;\forall r \in (b,c) = 0$ V/m

$E \;\forall r > c = \frac{kq}{r^2}$ V/m

④ Given :

Sphere of Radius R

$P(r) = kr$

k : constant

r : vector from centre

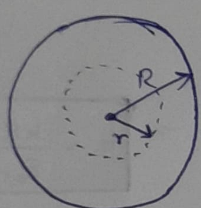
To find :

(i) Bound Charges

(ii) Field Inside and Outside Sphere

Solution :

(i) $P(r) = kr \Rightarrow \vec{P}(r) = k\vec{r}$



As we know,

$\sigma_b = \vec{P}(r) \cdot \hat{n}$ for Surface Bound Charge

[Here $\hat{n} = \hat{r}$]

$\Rightarrow \sigma_b = \vec{P}(r) \cdot \hat{r}$

$= k\vec{r} \cdot \hat{r}$

$= k(\vec{r}) \cdot \left(\frac{\vec{r}}{r}\right)$

$= \frac{k}{r}(\vec{r} \cdot \vec{r})$

$= \frac{k}{r} \cdot r^2 = kr$

$\boxed{\therefore \sigma_b = kr}$ [for $r \leq R$]

$\rho_b = -\vec{\nabla} \cdot \vec{P}$ for Volume Bound Charge

$= -\vec{\nabla} \cdot k\vec{r}$

$= -\frac{1}{r^2}\left(\frac{\partial}{\partial r}(r^2 P_r)\right) + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(P_\theta \sin\theta)\right) + \frac{1}{r\sin\theta}\left(\frac{\partial P_\phi}{\partial \phi}\right)$

$$\Rightarrow \rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 . kr)$$

$$= -\frac{1}{r^2}.(3kr^2)$$

$$= -3k$$

$$\boxed{\therefore \rho_b = -3k} \quad [\text{for } r \leq R]$$

(ii)

$$\sigma_b = \frac{q_b}{A}$$

$$\Rightarrow q_b = \sigma_b . A$$

$$= (kr)(4\pi r^2)$$

And $q_b = (kR)(4\pi R^2)$ [at surface]

$$\Rightarrow q_b = 4\pi k R^3$$

And $E = \frac{kq}{r^2} = \frac{kq_b}{r^2} = \frac{1}{4\pi\varepsilon_0} \times \frac{4\pi k R^3}{r^2}$

$$\therefore E = \frac{kR^3}{\varepsilon_0 r^2}$$

(OR)

$$\oint E . dA = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma_b A}{\varepsilon_0}$$

$$\Rightarrow E . 4\pi r^2 = \frac{q_b}{\varepsilon_0}$$

$$= \frac{4\pi k R^3}{\varepsilon_0}$$

$$\Rightarrow \boxed{E = \frac{kR^3}{\varepsilon_0 r^2}}$$

Inside, $\boxed{E = \frac{kR^3}{\varepsilon_0 r^2}}$ & $\boxed{E \propto \frac{1}{r^2}}$

$$\rho_b = \frac{q_b}{V}$$

$$\Rightarrow q_b = \rho_b . V$$

$$= (-3k)\left(\frac{4}{3}\pi r^3\right)$$

& $q_b = -4k\pi R^3$ [at surface]

$$\therefore E = \frac{kq}{r^2} = \frac{kq_b}{r^2} = \frac{1}{4\pi\varepsilon_0} \times \frac{-4k\pi R^3}{r^2}$$

$$= -\frac{kR^3}{\varepsilon_0 r^2}$$

(OR)

$$\oint E . dA = \frac{q_{enc}}{\varepsilon_0} = \frac{\rho_b V}{\varepsilon_0}$$

$$\Rightarrow E . 4\pi r^2 = \frac{q_b}{\varepsilon_0}$$

$$= \frac{-4k\pi r^3}{\varepsilon_0}$$

$$\Rightarrow \boxed{E = -\frac{kr}{\varepsilon_0}}$$

But we want to find Bound Charges density for $r = R$,

Outside, $\boxed{E = -\frac{kr}{\varepsilon_0}}$ & $\boxed{k \propto -r}$

⑤ Given :

Dielectric Cube , Side $a = 2$

Centre $O(0,0,0)$

$P = 6r$

To find :

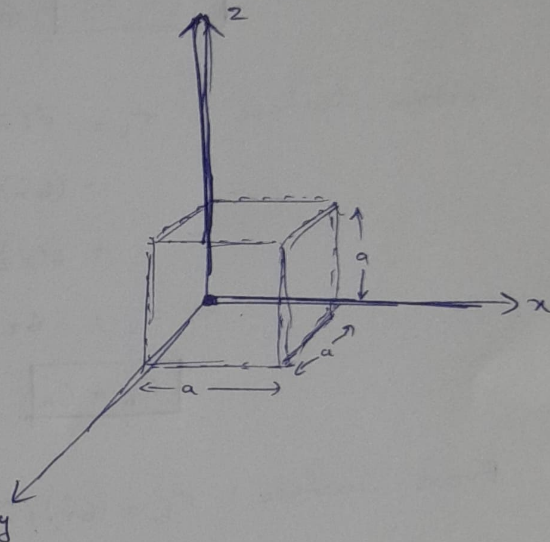
(i) Total Volume Bound Charge Density (ρ_b)

(ii) Total Surface Bound Charge Density (σ_b)

Solution :

$$P(r) = 6r$$

$$\Rightarrow \vec{P}(r) = 6\vec{r}$$



(i) $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$$= -\vec{\nabla}\cdot 6\vec{r}$$

$$= -\frac{1}{r^2}\left(\frac{\partial}{\partial r}(r^2 \cdot 6r)\right)$$

$$= -\frac{1}{r^2}\cdot(18r^3)$$

$$\Rightarrow \rho_b = -18$$

$$\boxed{\therefore \rho_b = -18}$$

(ii) $\sigma_b = \vec{P}(r)\cdot\hat{n}$

On each Surface , we will have same amount of σ_b (Magnitude)

for Ex. ,

Right Surface : $\sigma_b = \vec{P}(r)\cdot\hat{x}$

$$= 6\vec{r}\cdot\hat{x} \qquad \left[\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}\right]$$

$$= 6(x\hat{x}\cdot\hat{x})$$

$$\Rightarrow \sigma_b = 6x$$

$$\boxed{\sigma_b = 6a} \ @\ x = a$$

Left Surface : $\sigma_b = \vec{P}(\vec{r}).(-\hat{x})$

$= (6\vec{r}).(-\hat{x})$ $\quad [r = x\hat{x} + y\hat{y} + z\hat{z}]$

$= 6(x\hat{x}).(-\hat{x})$

$\Rightarrow \sigma_b = -6x$

$\boxed{\sigma_b = +6a}$ @ $x = -a$

Top Surface : $\sigma_b = \vec{P}(\vec{r}).\hat{z}$

$= (6\hat{r}).\hat{z}$ $\quad [r = x\hat{x} + y\hat{y} + z\hat{z}]$

$= 6(z\hat{z}).(\hat{z})$

$\Rightarrow \sigma_b = 6z$

$\boxed{\sigma_b = 6a}$ @ $z = a$

Bottom Surface : $\sigma_b = \vec{P}(\vec{r}).(-\hat{z})$

$= (6\hat{r}).(-\hat{z})$ $\quad [r = x\hat{x} + y\hat{y} + z\hat{z}]$

$= 6(z\hat{z}).(-\hat{z})$

$\Rightarrow \sigma_b = -6z$

$\boxed{\sigma_b = 6a}$ @ $z = -a$

Front Surface : $\sigma_b = (6\hat{r}).(\hat{y})$

$= 6(y\hat{y}).\hat{y}$ $\quad [r = x\hat{x} + y\hat{y} + z\hat{z}]$

$\Rightarrow \sigma_b = 6y$

$\boxed{\sigma_b = 6a}$ @ $y = a$

Back Surface : $\sigma_b = (6\hat{r})(-\hat{y})$

$= 6(y\hat{y}).(-\hat{y})$ $\quad [r = x\hat{x} + y\hat{y} + z\hat{z}]$

$\Rightarrow \sigma_b = -6y$

$\boxed{\sigma_b = 6a}$ @ $y = -a$

$\therefore$ Total Surface Bound Charge Density $= \sum_{i=1}^{6} \sigma_b$

$= 6(\sigma_b) = 6(6a) = 36a$

$\boxed{\therefore \sigma_{b\,tot} = 36(a)}$

(6) To Find:

Relative Permittivity (ε_r)

Given:

(a) $A = 0.12\ m^2$

$d = 80\ \mu m$

$V_0 = 12\ V$

$E_c = 1\ \mu J$

(b) $d = 45\ \mu m$

$V_0 = 200\ V$

$U_E = 100\ J/m^3$

(c) $E = 200\ kV/m$

$\sigma = 20\ \mu C/m^2$

Solution:

(a) $C_0 = \frac{\varepsilon_0 A}{d}$

$C = \varepsilon_r \cdot C_0$

$\Rightarrow C = \varepsilon_r \cdot \frac{\varepsilon_0 A}{d}$

$\Rightarrow \varepsilon_r = \frac{Cd}{\varepsilon_0 A} = \frac{C \times 80 \times 10^{-6}}{8.854 \times 10^{-12} \times 0.12}$ $\quad [\varepsilon_0 = 8.854 \times 10^{-12}\ F/m]$

$\left[\text{Also } E_0 = \frac{1}{2} C V_0^2 \Rightarrow C = \frac{2E_0}{V_0^2} \right]$

$$= \frac{\frac{2E_0}{V_0^2} \times 80 \times 10^{-6}}{8.854 \times 10^{-12} \times 0.12}$$

$$= \frac{2E_0}{V_0^2} \times 10^{6} \times 75.2955$$

$$= \frac{2 \times 1 \times 10^{-6}}{(12)^2} \times 10^{6} \times 75.3$$

$= 1.04577\ F/m$

$\Rightarrow \boxed{\varepsilon_r = 1.0458\ F/m}$

(b) $U = \left(\frac{1}{2} E^2 \varepsilon_0 \varepsilon_r\right) / \text{Volume}$

$\Rightarrow 100 = \frac{1}{2} \times (8.854 \times 10^{-12}) \times \left(\frac{V_0}{d}\right)^2 \cdot \varepsilon_r$ $\quad [V = E \cdot d]$

$$\Rightarrow \varepsilon_r = \frac{\cancel{100} \times 2 \times 45 \times \cancel{10^{-6}} \times 45 \times \cancel{10^{-6}}}{\cancel{300} \times 200 \times 8.854 \times \cancel{10^{-12}}}$$

$$= 1.14355 \ F/m$$

$$\boxed{\therefore \varepsilon_r = 1.14355 \ F/m}$$

(c) As $E = \dfrac{\sigma}{\varepsilon}$

$$\Rightarrow E = \frac{\sigma}{\varepsilon_0 \varepsilon_r}$$

$$\Rightarrow \varepsilon_r = \frac{\sigma}{\varepsilon_0 E} = \frac{20 \times 10^{-6}}{8.854 \times 10^{-12} \times 200 \times 10^{3}}$$

$$= \frac{100}{8.854}$$

$$= 11.29 \ F/m$$

$$\boxed{\therefore \varepsilon_r = 11.3 \ F/m}$$