Determinants

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Definition (Permutation)

Any arrangement $p = (p_1, p_2, \ldots, p_n)$ of the numbers $(1, 2, 3, \ldots, n)$

- Total number of permutations $= n! = n(n-1). \ldots .2.1$
- **Example-1.** Set $n = 2$, so the number permutations of $(1, 2)$ is 2! i.e. 2
$$\Big\{(1,\ 2),\ (2,\ 1)\Big\}$$
- **Example-2.** Set $n = 3$, so the number permutations of $(1, 2, 3)$ is 3! i.e. 6

$$\Big\{(1,2,3),\ (1,3,2),\ (2,1,3),\ (2,3,1),\ (3,1,2),\ (3,2,1)\Big\}$$

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Definition (sign of a permutation)

Permutation p can be restored to natural order by an even/odd number of interchanges. The sign of permutation p is defined as

$$\sigma(p) = \begin{cases} +1, & \text{if number of interchanges} = \text{even} \ , \\ -1, & \text{if number of interchanges} = \text{odd} \ . \end{cases}$$

$(1, 3, 2)$

$(1, 2, 3)$

Number of inter changes $= 1$ (odd)

So, $\sigma(p) = -1$

$$(3, 1, 2)$$

$$(1, 3, 2)$$

$$(1, 2, 3)$$

Number of inter changes $=$ 2 (even)

So, $\sigma(p) = +1$

Definition (Determinant)

For an $n \times n$ matrix $\mathbf{A} = [a_{i,j}]_{n \times n}$, the determinant of $\mathbf{A}$ is defined to be scalar

$$\det(\mathbf{A}) \text{ or } |\mathbf{A}| = \sum_{p}^{n!} \sigma(p)\, a_{1p_1} a_{2p_2} \cdots a_{np_n} \tag{1}$$

- **Note.** here $\{p_1,\, p_2,\, \cdots,\, p_n\}$ are column indices.
- **Example-1.** For $n = 2$, total number of permutations of $(1,\, 2)$ is equals to $n! = 2$

$$\text{permutations} = \Big\{(1,\ 2), (2,\ 1)\Big\} \text{ of } (1,\ 2)$$

Therefore, the determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by the following

$$\begin{aligned}
\det(\mathbf{A}) &= \sum_{p}^{2} \sigma(p)\, a_{1p_1} a_{2p_2} \\
&= \sigma(\underbrace{1}_{p_1}, \underbrace{2}_{p_2})\, a_{11} a_{22} + \sigma(\underbrace{2}_{p_1}, \underbrace{1}_{p_2})\, a_{12} a_{21} \\
&= (+1) a_{11} a_{22} + (-1) a_{12} a_{21} \\
&= a_{11} a_{22} - a_{12} a_{21},
\end{aligned}$$

where $\sigma(1,\ 2) = +1$ and $\sigma(2,\ 1) = -1$

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- **Example-2.** For $n = 3$, $(1, 2, 3)$

$p = (p_1,\ p_2,\ p_3)$	# changes	$\sigma(p)$	$a_{1p_1} a_{2p_2} a_{3p_3}$
$(1,\ 2,\ 3)$	0	$+$	$a_{11} a_{22} a_{33}$
$(1,\ 3,\ 2)$	1	-	$a_{11} a_{23} a_{32}$
$(2,\ 1,\ 3)$	1	-	$a_{12} a_{21} a_{33}$
$(2,\ 3,\ 1)$	2	$+$	$a_{12} a_{23} a_{31}$
$(3,\ 1,\ 2)$	2	$+$	$a_{13} a_{21} a_{32}$
$(3,\ 2,\ 1)$	1	-	$a_{13} a_{22} a_{31}$

$$
\begin{aligned}
\det(\mathbf{A}) &= \sum_{p}^{3!=6} \sigma(p)\, a_{1p_1} a_{2p_2} a_{1p_1} a_{3p_3} \\
&= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} \\
&\quad + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}
\end{aligned}
$$

- **Problem.** Compute $\det(\mathbf{A})$ using definition, where $\mathbf{A}$ is given

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$p = (p_1,\ p_2,\ p_3)$	# changes	$\sigma(p)$	$a_{1p_1}a_{2p_2}a_{3p_3}$
$(1,\ 2,\ 3)$	0	$+$	$a_{11}a_{22}a_{33} = 1 \times 5 \times 9 = 45$
$(1,\ 3,\ 2)$	1	-	$a_{11}a_{23}a_{32} = 1 \times 6 \times 8 = 48$
$(2,\ 1,\ 3)$	1	-	$a_{12}a_{21}a_{33} = 2 \times 4 \times 9 = 72$
$(2,\ 3,\ 1)$	2	$+$	$a_{12}a_{23}a_{31} = 2 \times 6 \times 7 = 84$
$(3,\ 1,\ 2)$	2	$+$	$a_{13}a_{21}a_{32} = 3 \times 4 \times 8 = 96$
$(3,\ 2,\ 1)$	1	-	$a_{13}a_{22}a_{31} = 3 \times 5 \times 7 = 105$

$$\det(\mathbf{A}) = \sum_{p}^{3!=6} \sigma(p)\, a_{1p_1}a_{2p_2}a_{1p_1}a_{3p_3} = 45 - 48 - 72 + 84 + 96 - 105 = 0$$

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Triangular Determinants.

- $a_{ij} = 0$, when $i > j$, e.g $a_{21} = 0, a_{31=0}, a_{32} = 0$ and so on.

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = a_{11} \times a_{22} \times \cdots \times a_{nn}$$

- From the definition 1, each term $a_{1p_1} a_{2p_2} \cdots a_{np_n}$ contains exactly one entry from each row and each column.
- there is only one term in the expansion of the determinant that does not contain an entry below the diagonal.
- Hence, $\det(\mathbf{U}) = a_{11} \times a_{22} \times \cdots \times a_{nn}$.

Transpose does not alter determinants.

- $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \ \mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- Since,
$$\left\{\sigma(p) a_{1p_1} a_{2p_2} \cdots a_{np_n} \middle| \forall p\right\} = \left\{\sigma(p) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n} \middle| \forall p\right\},$$

- $$\sum_p \sigma(p) a_{1p_1} a_{2p_2} \cdots a_{np_n} = \sum_p \sigma(p) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n}$$

- Hence, $\det(\mathbf{A}) = \det(\mathbf{A}^T)$.

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$\mathbf{A}^T$ $\mathbf{A}$

$a_{p_1 1} a_{p_2 2} a_{p_3 3}$	(p_1, p_2, p_3)	# changes	$\sigma(p)$	$a_{1 p_1} a_{2 p_2} a_{3 p_3}$
$45 = a_{11} a_{22} a_{33}$	(1, 2, 3)	0	+	$a_{11} a_{22} a_{33} = 45$
$48 = a_{11} a_{32} a_{23}$	(1, 3, 2)	1	-	$a_{11} a_{23} a_{32} = 48$
$72 = a_{21} a_{12} a_{33}$	(2, 1, 3)	1	-	$a_{12} a_{21} a_{33} = 72$
$96 = a_{21} a_{32} a_{13}$	(2, 3, 1)	2	+	$a_{12} a_{23} a_{31} = 84$
$84 = a_{31} a_{12} a_{23}$	(3, 1, 2)	2	+	$a_{13} a_{21} a_{32} = 96$
$105 = a_{31} a_{22} a_{13}$	(3, 2, 1)	1	-	$a_{13} a_{22} a_{31} = 105$

Therefore, $\sum_p \sigma(p) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n} = \sum_p \sigma(p) a_{1 p_1} a_{2 p_2} \cdots a_{n p_n}$

Hence, $\det(\mathbf{A}^T) = \det(\mathbf{A})$

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Effects of Row Operations. Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}_{n\times n}$ by one of the 3 elementary row operations:

Type-I Interchange rows i and j i.e. $R_i \leftrightarrow R_j$

Type-II Multiply row i by α i.e $R_i \leftarrow \alpha R_i$, provide $\alpha \neq 0$

Type-III Add α times row j to row i i.e $R_i \leftarrow R_i + \alpha R_j$

Elem. Row Operations Type	The value of $\det(\mathbf{B})$
Type-I	$\det(\mathbf{B}) = -\det(\mathbf{A})$
Type-II	$\det(\mathbf{B}) = \alpha \det(\mathbf{A})$
Type-III	$\det(\mathbf{B}) = \det(\mathbf{A})$

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$$\begin{bmatrix} --- & A1* & --- \\ --- & --- & --- \\ --- & Ai* & --- \\ --- & --- & --- \\ --- & Aj* & --- \\ --- & --- & --- \\ --- & An* & --- \end{bmatrix} \xrightarrow{R_i \leftrightarrow R_j} \begin{bmatrix} --- & A1* & --- \\ --- & --- & --- \\ --- & Aj* & --- \\ --- & --- & --- \\ --- & Ai* & --- \\ --- & --- & --- \\ --- & An* & --- \end{bmatrix} = \begin{bmatrix} --- & B1* & --- \\ --- & --- & --- \\ --- & Bi* & --- \\ --- & --- & --- \\ --- & Bj* & --- \\ --- & --- & --- \\ --- & Bn* & --- \end{bmatrix}$$

Proof for Type-I.

It is clear from the above $\mathbf{B}$ agrees with $\mathbf{A}$ except that $\mathbf{B}_{i*} = \mathbf{A}_{j*}$ and $\mathbf{B}_{j*} = \mathbf{A}_{i*}$, then for each permutation $p = (p_1, p_2, \cdots, p_n)$ of $(1, 2, \cdots, n)$

$$\begin{aligned} b_{1p_1} \cdots b_{ip_i} \cdots b_{jp_j} \cdots b_{np_n} &= a_{1p_1} \cdots a_{jp_i} \cdots a_{ip_j} \cdots a_{np_n} \\ &= a_{1p_1} \cdots a_{ip_j} \cdots a_{jp_i} \cdots a_{np_n} \\ \text{since, } \sigma(\underbrace{p_1, \cdots p_i, \cdots p_j \cdots, p_n}_{p}) &= -\sigma(\underbrace{p_1, \cdots p_j, \cdots p_i \cdots, p_n}_{q}) \\ \sum_p \sigma(p) b_{1p_1} \cdots b_{ip_i} \cdots b_{jp_j} \cdots b_{np_n} &= \sum_q \sigma(p) a_{1p_1} \cdots a_{ip_j} \cdots a_{jp_i} \cdots a_{np_n} \\ &= -\sum_p \sigma(p) a_{1p_1} \cdots a_{ip_i} \cdots a_{jp_j} \cdots a_{np_n} \end{aligned}$$

from the previous slide, which is our desired result

$$\det(\mathbf{B}) = -\det(\mathbf{A}).$$

Proof for Type-II and Type-III.

See, C. D. Meyer, Matrix Anal. and App. Linear Algeb., p.463-464. $\square$

Corollary. Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}_{n\times n}$ by one of the 3 elementary row operations:

Type	$\mathbf{A}$	$\mathbf{B}$	$\det(\mathbf{B})$
Type-I	$\mathbf{I}$	$\mathbf{E}$	$\det(\mathbf{E}) = -1$, since $\det(\mathbf{A}) = 1$
Type-II	$\mathbf{I}$	$\mathbf{F}$	$\det(\mathbf{F}) = \alpha$
Type-III	$\mathbf{I}$	$\mathbf{G}$	$\det(\mathbf{G}) = 1$

$$\left.\begin{cases} \det(\mathbf{EA}) &= -\det(\mathbf{A}) = \det(\mathbf{E})\det(\mathbf{A}) \\ \det(\mathbf{FA}) &= \alpha\det(\mathbf{A}) = \det(\mathbf{F})\det(\mathbf{A}) \\ \det(\mathbf{GA}) &= \det(\mathbf{A}) = \det(\mathbf{G})\det(\mathbf{A}) \end{cases}\right\} \Rightarrow \det(\mathbf{PA}) = \det(\mathbf{P})\det(\mathbf{A})$$

where $\mathbf{P}$ is an elementary matrix.

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- For $n = 3$,
- $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}$ (Type-I)
- $\mathbf{EA} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Observation.

For any number of these elementary matrices, $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k$

$$\begin{aligned}
\det(\mathbf{P}_1\mathbf{P}_2\cdots\mathbf{P}_k\mathbf{A}) &= \det(\mathbf{P}_1)\det(\mathbf{P}_2\cdots\mathbf{P}_k\mathbf{A}) \\
&= \det(\mathbf{P}_1)\det(\mathbf{P}_2)\det(\mathbf{P}_3\cdots\mathbf{P}_k\mathbf{A}) \\
&\vdots \\
&= \det(\mathbf{P}_1)\det(\mathbf{P}_2)\cdots\det(\mathbf{P}_k)\det(\mathbf{A})
\end{aligned}$$

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Invertibility and Determinants.

1. $\mathbf{A}_{n\times n}$ is nonsingular **iff** $\det(\mathbf{A}) \neq 0$
2. $\mathbf{A}_{n\times n}$ is singular **iff** $\det(\mathbf{A}) = 0$

Proof.

Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k$ be a sequence of elementary matrices of Type I, II, or III such that $\mathbf{P}_1\mathbf{P}_2\cdots\mathbf{P}_k\mathbf{A} = \mathbf{E}_A$
From the above observation,

$$\det(\mathbf{E}_A) = \det(\mathbf{P}_1\mathbf{P}_2\cdots\mathbf{P}_k\mathbf{A}) = \det(\mathbf{P}_1)\det(\mathbf{P}_2)\cdots\det(\mathbf{P}_k)\det(\mathbf{A})$$

Since elementary matrices have nonzero determinants i.e. $\det(\mathbf{P}_i) \neq 0$ for all $i = 1, 2, \ldots, k$,

$$\begin{aligned}\det(\mathbf{A}) \neq 0 \Leftrightarrow \det(\mathbf{E}_A) \neq 0 &\Leftrightarrow \text{there are no zero pivots}\\ &\Leftrightarrow \text{every column of } \det(\mathbf{E}_A) \text{(and in } \mathbf{A}\text{) is basic}\\ &\Leftrightarrow \mathbf{A} \text{ is nonsingular.}\end{aligned}$$

$\square$

Minors and Co-factors.

- Simply, minors of $\mathbf{A}_{m\times n}$ are the determinant of any $k \times k$ sub-matrix, where $k \leq \min\{m, n\}$ i.e. $k \leq n$ if $m = n$.

Let M_{ij} be the ij-th minor of $\mathbf{A}_{n\times n}$, then M_{ij} the determinant of the sub-matrix remains when the i-th row and j-th column of $\mathbf{A}$ are deleted and the ij-th co-factor is defined by $C_{ij} = (-1)^{i+j} M_{ij}$

- **Example.** Let, $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16, \quad C_{11} = (-1)^{1+1} M_{11} = 16$$

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26, \quad C_{32} = (-1)^{3+2} M_{32} = -26$$

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Determinant with Co-factors expansions.

$$\begin{aligned}
\det(\mathbf{A}) &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\
&\quad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13} \\
&= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \leftarrow \text{ along the first row}
\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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Similar way,

$\det(\mathbf{A}) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \leftarrow$ along the second row and so on.

- Along the first row :

$$\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix}$$

$$= 3(40 - 24) - 1(16 - 6) - 4(8 - 5)$$

$$= 48 - 10 - 12 = 26.$$

Rank and Determinants.

- rank($\mathbf{A}$) $=$ the size of the largest non-zero minor of $\mathbf{A}_{m\times n}$.

  (Note. *rank(A) = r, at least one minor of size r that not vanishes and every minor of size r+1 and higher vanishes*)

  For, $n = m$

- **Example.** $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3$ of the matrx $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, since

  $\det(\mathbf{A}) = 0 \implies \text{rank}(\mathbf{A}) = 2$.

- **Full rank.** rank($\mathbf{A}$) $= n = size(\mathbf{A})$.

- If det($\mathbf{A}$)$\neq 0$, matrix is full rank.

- If det($\mathbf{A}$)$= 0$, rank($\mathbf{A}$) $< n$ always.

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Some properties.

1. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$, (p.467, C.D. Meyer)
2. $\det(\mathbf{A}^k) = \det(\mathbf{A})^k$, where k is an integer,(use the property-1)
3. $\det(k\mathbf{B}) = k^n\det(\mathbf{B})$, where $k \in \mathbb{R}\backslash\{0\}$, (similar way as Type-II)
4. $\det(\mathbf{A} + \mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$,(e.g. $\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$)
5. $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$, provided $\det(\mathbf{A})$ is invertible, (use definition of inverse then property-1)
6. If $\mathbf{A}$ has two identical rows or columns, then $\det(\mathbf{A}) = 0$,
7. If $\mathbf{A}$ has two prportional rows or columns, then $\det(\mathbf{A}) = 0$.

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THANK YOU