the Whats (questions)

i) is every continuous fn. differentiable? NO
ii) does every vector space have a basis? YES
iii) will it rain tomorrow? YES/NO (predictions)
iv) is it safe to release a tablet in the market? YES/NO
v) fix integer n. is it prime? YES/NO

PROBABILITY (classical) → number $x \in [0,1]$ assigned to event
→ if x close to 1: likely to happen
else, not.

SAMPLE SPACE → set S containing each possible outcome of an experiment, the elements named as sample point.

lets consider the experiments of:

Tossing a coin, $S = \{Head, Tail\}$
Throwing a dice, $S = \{1,2,3,4,5,6\}$
{ choosing real number b/w -1 & 1, $S = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$

not as part of CLASSICAL PROBABILITY
↳ sample space is finite.

EVENT → Any subset A of a subset S is called an event ($A \subseteq S$)

CLASSICAL PROBABILITY → the probability of an event A occurring is P(A), where:

$$P(A) = \frac{n(A)}{n(S)}$$, $n(A)$ – no. of elements in A.
$n(S)$ – no. of elements in S.

↓

Confine to FINITE SETS.

PROPERTY of CLASSICAL PROBABILITY: its dependent on the number of elements in the event and the sample space.

consider the expt. of tossing a coin, here $S = \{H, T\}$
take $A = \{H\}$ ∴, $P(A) = \frac{n(A)}{n(S)} = 1/2 = 0.5$

→ theoretically there are several other factors that govern S and A, so practically the prob. might be a bit off.

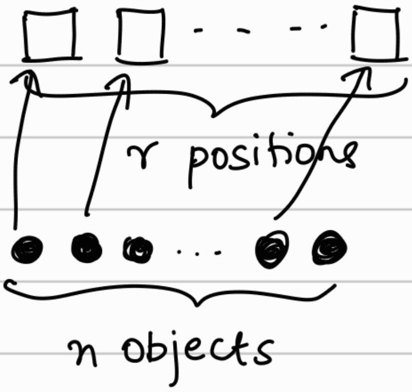
goal → understand & try to reduce the error. form a model that has low error.

consider the expt. of throwing a dice, here $S = \{1,2,3,4,5,6\}$
suppose $A = \{1,2\}$, $P(A) = 2/6 = 1/3$.
now, let $A' = \{1,2,3\}$, $P(A') = 3/6 = 1/2$.
→ obs: $n(A') > n(A)$ and $P(A') > P(A)$.

let $S \rightarrow$ sample space. $P(A) = \frac{n(A)}{n(S)}, n(S) \neq \phi$

REFER to PRIYANK'S page (in WA)

PERMUTATIONS: arrangement of objects in a definite order. the no. of permutations of arranging n obj. taking r at a time is $\longrightarrow$ $^nP_r = \frac{n!}{(n-r)!}$



1st box $\rightarrow$ n choices

2nd $\longrightarrow$ $n-1$

$\vdots$

rth box $\rightarrow$ $n-r+1$ choices

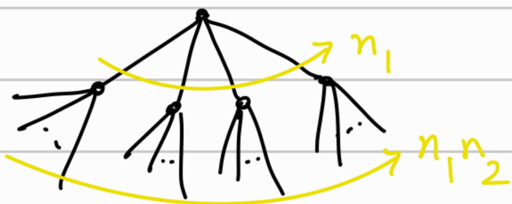
$= n-(r-1)$ choices

$$\therefore, n(\text{ways}) = n \times (n-1) \times \cdots \times (n-(r-1))$$

$$= \frac{n(n-1)\cdots(n-(r-1)) \times (n-r) \times (n-(r+1)) \times \cdots \times 2 \times 1}{(n-r) \times (n-r-1) \times \cdots \times 2 \times 1}$$

$$= \frac{n!}{(n-r)!}$$

MULTIPLICATION PRINCIPLE: consider an expt. having K stages & the i'th stage can occur in n_i ways, then the whole expt can occur in $\prod_{i=1}^{K} n_i$ ways.



(graphs help explain this)

COMBINATIONS: A combination is a selection of r-objects from a given set of n-objects. let this be ${}^nC_r$.

$$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{\left(\frac{n!}{(n-r)!}\right)}{r!} = \frac{n!}{r!(n-r)!} $$

* each combination is repeated $r!$ ways in permutations. ∴, $\frac{{}^nP_r}{r!}$ would ensure that count of each combo. is EXACTLY 1.
(think of it as selecting r objects & then permuting it $r!$ times)

(e.g.) A foundry ships a lot of 20 engine blocks of which 5 contain internal flaws. The purchasor will select 3 blocks at random & test them for hardness. The lot will be accepted if no flaws are found. What is the probability that this lot will be accepted?

(soln.) $P(\text{lot accepted}) = \dfrac{n\left(\text{ways to select 3 engine blocks from 20 \& obtain no flawed engine}\right)}{n\left(\text{ways to select 3 engine blocks from 20}\right)}$

could be $\dfrac{{}^{15}P_3}{{}^{20}P_3}$ also?

$$ = \frac{{}^{15}C_3}{{}^{20}C_3} = \frac{455}{1140} = 0.20 $$

P.T.O.

(e.g.) The distribution of blood types in the United States is roughly 41% type A, 9% type B, 4% type AB and 46% type O. What is the probability that a random person has A, B or AB?

(soln.) let S be sample space & $E_1, E_2, \cdots, E_k$ be events s.t. $\forall_{i \in \{1,2,\ldots,k\}}$ $E_i \subseteq S$ & all E_is → mutually exclusive. ∴,

$$P(E_1 \cup E_2 \cup \cdots \cup E_k) = P(E_1) + P(E_2) + \cdots + P(E_k),$$

here, a person having A, B or AB are mutually exclusive events, hence we can add the probabilities directly.

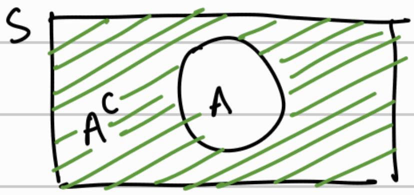
$$\therefore, P(A, B \text{ or } AB) = P(A) + P(B) + P(AB)$$
$$= 0.41 + 0.09 + 0.04$$
$$= 0.54.$$

$\frac{n(A)}{n(S)}$ → relativeness of A wrt S (how much is A deviating from S)

PROPERTIES:

1) $P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$ $\{\text{if } n(S) \neq \phi\}$

ii) let $A \subseteq S$ be an event. Then, $P(A^c) = 1 - P(A)$



$A \cup A^c = S$

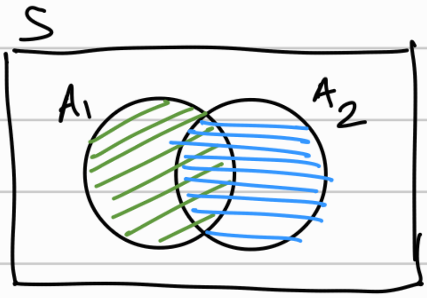
$\Rightarrow P(A \cup A^c) = P(S) = 1$ $\Rightarrow P(A) + P(A^c) = 1$

$\Rightarrow P(A^c) = 1 - P(A)$

$P(A \cup A^c) = P(A) + P(A^c)$, as they are mutually exclusive (nothing common)

ii) let $A_1, A_2 \subseteq S$ be two events. Then,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$



$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$.

we subt. $n(A_1 \cap A_2)$ as it's counted twice.

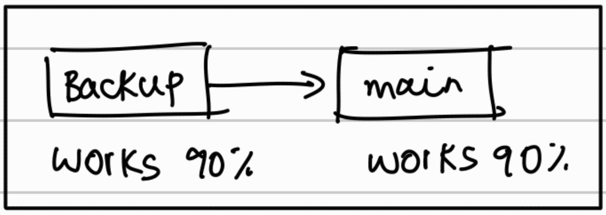
$$\therefore, \quad P(A_1 \cup A_2) = \frac{n(A_1 \cup A_2)}{n(S)} = \frac{n(A_1) + n(A_2) - n(A_1 \cap A_2)}{n(S)}$$

$$= \frac{n(A_1)}{n(S)} + \frac{n(A_2)}{n(S)} - \frac{n(A_1 \cap A_2)}{n(S)}$$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$\therefore, \quad \boxed{P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)}$$

Example:



SYS. } → find prob of sys functioning.

let $A_1 \rightarrow$ main fellow is operable
$A_2 \rightarrow$ backup fellow is operable. } → A_1 & A_2 are indep. events. (what are indep. events?)

$P(\text{working}) = P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$= 0.9 + 0.9 - 0.9 \times 0.9$

$= 1.8 - 0.81 = 0.99 \longrightarrow$ a backup increased working prob. by 9% !

ADDITIVE PROPERTY

let $A_1, A_2, \cdots, A_n$ be n events. Then,

$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \; ? \longrightarrow$ COMPLETE THE PROOF.

Prove : $n(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} n(A_i) - \sum_{i<j} n(A_i \cap A_j) + \sum_{i<j<k} n(A_i \cap A_j \cap A_k) - \cdots + (-1)^{N+1} (A_i \cap A_j \cap \cdots A_N)$ (*)

Proof :

By induction hypothesis -

$\mathbb{N} = \{1, 2, 3 \cdots, \cdots\}$ & $S_1, S_2, \cdots, S_n, \cdots$ be mapped to each $n \in \mathbb{N}$. if S_1 is TRUE and S_n is TRUE, then all $S_1, S_2, \cdots, S_n, S_{n+1}, \cdots$ are all TRUE.

PBC : supp. that some $S_m \rightarrow$ FALSE.

let $F = \{ m \mid S_m \text{ is FALSE} \}$. by assumption, $F \neq \emptyset$ and $F \subseteq \mathbb{N}$. by property, $\exists$ a minimum, say $K \in F$.

So, $K-1 \notin F$. $S_{K-1} \rightarrow$ TRUE. but, wkt.

S_{n+1} is TRUE. $\therefore$, $S_K \rightarrow$ TRUE. BUT, $K \in F$ implies S_K is FALSE! $\rightarrow$ CONTRADICTION.
$\therefore$, $\forall m$, $S_m \rightarrow$ TRUE.

$S_1 = n(A_1) = n(A_1)$
$S_2 = n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$

assume (*) is TRUE,

$S_{n+1} = n\left[(A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}\right]$

$= n(A_1 \cup A_2 \cup \cdots \cup A_n) + n(A_{n+1}) - n\left[(A_1 \cup A_2 \cdots \cup A_n) \cap A_{n+1}\right]$

?

CONDITIONAL PROBABILITY

$$P(A_1 \mid A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)}$$

REFER to HARITH'S | PARTH'S notes for BAYES thm. & total probability.

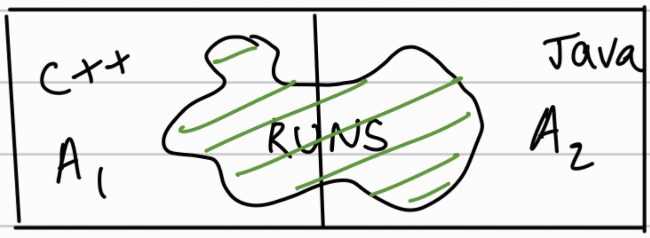
Example:

recall our prev. example -

	compiles (on 1st run)	Doesn't (on 1st run)
C++	72	48
Java	64	16

find prob. of random program compiling on 1st attempt:

$\mathcal{E} = 200$



let A_1 be the event that a random program is written in C++.
let A_2 be the event that a random program is written in Java.
let E be the event that a random program is runs in 1st compilation.

by Total Law of Probability,

$$P(E) = P(A_1) \cdot P(E/A_1) + P(A_2) \cdot P(E/A_2)$$

$$= \frac{\cancel{120}}{200} \times \frac{72}{\cancel{120}} + \frac{\cancel{80}}{200} \times \frac{64}{\cancel{80}}$$

$$= \frac{136}{200} = \frac{68}{100} = 0.68$$

Example: Inquiries to an online computer system arrive on five communication lines. The % of messages received THRU different lines are:

LINE	1	2	3	4	5
%	20	30	10	15	25

% of messages exceeding 100 chars are:

LINE	1	2	3	4	5
%	40	60	20	80	90

find prob. of finding msg. over 100 characters:

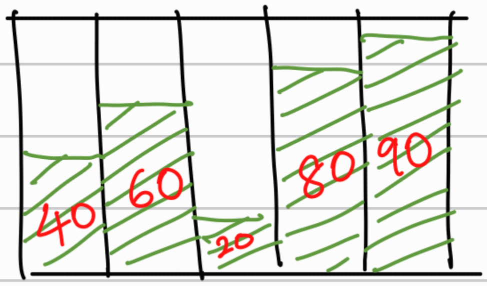
let $A_i \rightarrow$ prob. of message from i'th line.
$E \rightarrow$ prob of message over 100 chars.

$$P(E) = \sum_{i=1}^{5} P(A_i) \cdot P(E/A_i)$$

$$= \frac{20 \times 40 + 30 \times 60 + 10 \times 20 + 15 \times 80 + 90 \times 25}{100 \times 100}$$

$$= \frac{800 + 1800 + 200 + 1200 + 2250}{100 \times 100}$$

$$= \frac{625\cancel{0}}{100 \times 10\cancel{0}} = 0.625$$

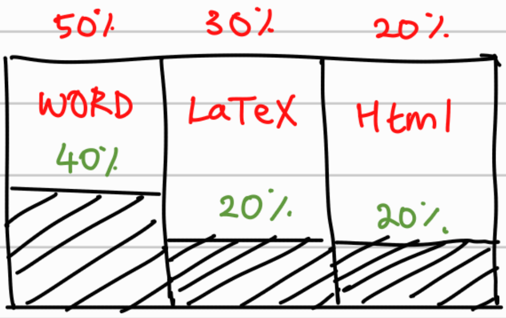


Example:

University dept → 50% docs in WORD, 30% in LaTeX, 20% in HTML → By past data:

$$\left.\begin{array}{ll} 40\% & \text{Word} \to \\ 20\% & \text{LaTex} \to \\ 20\% & \text{HTML} \to \end{array}\right\} \longrightarrow \geq 10 \text{ pgs.}$$

find prob: random doc $\xrightarrow{}$ is in LaTeX → $\geq$ 10 pgs.



$$P(LaTeX/E) = \frac{P(E/LaTeX)\cdot P(LaTeX)}{P(E)}$$

$$P(E) = P(E/LaTex)\cdot P(LaTeX) + P(E/Word)\cdot P(Word) + P(E/Html)\cdot P(Html)$$

$$= \frac{20\times 30 + 40\times 50 + 20\times 20}{100\times 100}$$

$$= \frac{600 + 2000 + 400}{100\times 100}$$

$$= 0.06 + 0.2 + 0.04$$

$$= 0.3$$

$$P(LaTex/E) = \frac{P(E/LaTeX)\cdot P(LaTeX)}{0.3} = \frac{0.2\times 0.3}{\cancel{0.3}}$$

$$= 0.2$$

RANDOM VARIABLES

$X: S \to \mathbb{R}$ is a random variable

(1) Discrete random variable
(2) Continuous random variable

DISCRETE : random variable X is discrete if it takes the values $a_1, a_2, \dots, a_n$ where the set $\{a_1, a_2, \dots, a_n\}$ is discrete.

take $x_1, x_2, \dots, x_n$.
let $r = \min(\{|x_i - x_j| : 1 \leq i, j \leq n\})$
then, $\forall_{1 \leq i, j \leq n}$, $|x_i - x_j| \leq r$. (take $\mathbb{N}$ seq, for example)

look at $1, 1/2, 1/3, 1/4, \dots, 1/n \to$ NOT DISCRETE! $\to$ as u can't find an 'r' s.t. $|x_i - x_j| \leq r$ //

let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence. Then, the set $\{a_1, a_2, \dots, a_n\}$ is not discrete

CONTINUOUS RANDOM VARIABLES : random variable X is continuous if it takes the values $[a,b]$ or (a,b), $[a,b)$ or $(a,b]$.

(aprom pAKalAm)

let X be a discrete random variable. Assume that it takes the values $a_1, a_2, a_3, \cdots a_n$

PROBABILITY DENSITY FUNCTION (Pdf) | PROBABILITY MASS FUNCTION (Pmf) : The Pdf of X is a function -

$P: \mathbb{R} \to \mathbb{R}_{\geq 0}$ s.t.

$$P(x) = \begin{cases} \text{Prob}(X = x) & \text{if } x = a_i \\ & a_i \in \{a_1, a_2, \cdots\} \\ \text{Prob}(X = x) = 0 & \text{if } x \neq a_i \\ & \text{(otherwise)} \end{cases}$$

E.g. Three balls are randomly selected without replacement from an urn containing 20 balls numbered 1 to 20. let X denote the largest number selected.

$$S = \left\{ (a,b,c) \;\middle|\; \begin{array}{l} 1 \leq a,b,c \leq 20 \\ a,b,c \text{ are all different} \end{array} \right\}$$

define $X : S \to \mathbb{R}_{\geq 0}$

$$X\big((a,b,c)\big) = \max\{a,b,c\}$$

clearly, $X \to$ random variables. the min-MAX is 3//

$\hookrightarrow$ possible values $= \{3,4,5,\cdots,20\}$

$\hookrightarrow$ discrete, finite, set.

$\hookrightarrow$ Hence, $X \to$ DRV.

looking at Pdf of X:

$$P(x) = \begin{cases} \text{Prob}(X = x) & \text{if } x \in \{3, 4, \cdots, 20\} \\ \text{Prob}(X = x) = 0 & \text{if } x \notin \{3, 4, \cdots, 20\} \end{cases}$$

$$\text{Prob}(X = a_K) = \frac{\underbrace{{}^{K-1}C_2}_{\text{K will be max only if this is the case}}}{{}^{20}C_3}$$

{considering an ordered set}

$$\therefore \text{ Pdf} = \begin{cases} \left(\dfrac{{}^{x-1}C_2}{{}^{20}C_3}\right) & x \in \{3, 4, \cdots, 20\} \\ 0 & x \notin \{3, 4, \cdots, 20\} \end{cases}$$

CUMULATIVE DISTRIBUTIVE FUNCTION (Cdf):

Cdf of X is a function $F: \mathbb{R} \rightarrow \mathbb{R}_p$ defined as:

$$F(x) = \text{Prob}(X \leq x)$$

EXPECTATION: expectation of X is :

$$E[X] = \sum_x x \, \text{Prob}(x)$$

VARIANCE: Variance of X is :

$$\text{Var}(X) = E\left[(X - E[X])^2\right]$$

let X be a discrete random variable. Then, X^2 is ALSO a discrete random variable $\rightarrow$ if X takes the values $a_1, a_2, a_3, \ldots$, then X^2 takes $a_1^2, a_2^2, \ldots$.

$$f^2(x) = (f \cdot f)(x) = f(x) \cdot f(x) = (f(x))^2$$

In general, for $n \in \mathbb{Z}^+$, $X^n \rightarrow$ discrete random variable.

Further, if $h(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0$, where $a_0, a_1, \ldots, a_m \in \mathbb{R}$. then,

$h(X) = a_m X^m + a_{m-1} X^{m-1} + \cdots + a_1 X + a_0$ is ALSO a discrete random variable.

In general, if $S \xrightarrow{X} \mathbb{R} \xrightarrow{f} \mathbb{R}$, then:
$f \circ X$ is ALSO a random variable

- $\hookrightarrow$ value of composition is DETERMINED by X.
- $\hookrightarrow$ so, still some mapping of $x \in X$ to some $y \in \mathbb{R}$
- $\hookrightarrow$ $\therefore$, DRV.

$$\begin{aligned} Var(X) &= E\left[(X - E[X])^2\right] \\ &= E\left[X^2 + E[X] - 2XE[X]\right] \\ &= E[\underbrace{X^2 + \mu - 2\mu x}] \qquad \mu = E[X] \end{aligned}$$

$\hookrightarrow$ Yet again a DRV

CLASSIFICATION of DISCRETE RANDOM VARIABLES

A random variable X is said to be Bernoulli if it takes the values $\{0, 1\}$ with Pdf:

$$P(0) = p \qquad p \in [0,1]$$
$$P(1) = 1-p$$

(e.g.) tossing a coin $\longrightarrow$ $S = \{H, T\}$; $X : S \to \mathbb{R}$ with values $0, 1$.

$$P(0) = \tfrac{1}{2} \quad (\text{say}, H)$$
$$P(1) = 1 - \tfrac{1}{2} = \tfrac{1}{2} \quad (\text{say}, T)$$

$\therefore$, $X \to$ BERNOULLI RANDOM VARIABLE

Expectation of BRV:

$$E[X] = \sum_x x P(x) = 0 \cdot p(0) + 1 \cdot p(1) = 1 - P(0)$$

VARIANCE of BRV:

$$\text{Var}(X) = E\left[(X-\mu)^2\right] = E\left[X^2 - 2\mu X + \mu^2\right].$$

H.W. p.t. $E[X+Y] = E[X] + E[Y]$ if $X, Y \to$ DRV's

P.T.O.

Prob: let X, Y be 2 DRVs. Then,

$$E[aX+bY] = aE[X] + bE[Y] \quad \forall a,b \in \mathbb{R}$$

$\hookrightarrow$ WRONG statement

Prob: let X be a DRV. Then,

$$E[aX+b] = aE[X]+b, \text{ where } a,b \in \mathbb{R}$$

Soln: $E[aX+b] = \sum_x (ax+b)\, P(ax+b)$

$$= a\sum_x x P(ax+b) + b\sum_x P(ax+b)$$

$$= a \sum_x x P(x) + b \cdot 1$$

* $\sum_x P(ax+b) = 1$, as $P(ax+b)\ \forall x \in X$ represents sum of probabilities of ALL POSSIBLE values the var $aX+b$ can take, hence 1.

* $\sum_x x P_{aX+b}(ax+b) = \sum_x x P_X(x) \longrightarrow$ as Prob of choosing $ax+b$ in $aX+b$ is same as choosing x in X.

$\therefore$, $E[aX+b] = a\sum_x x P(x) + b$

$$= aE[X]+b$$

expectation → like a weighted average.

VARIANCE of BERNOULLI (continued):

$$E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] = E[X^2 - 2\mu X] + E[\mu^2]$$

$$E[X^2 - 2\mu X] = \sum_{x} x^2 p_{X^2}(x^2) - 2\mu \sum_{x} x\, p_X(x)$$

$$= E[X^2] - 2\mu E[X]$$

$$\therefore E[(X-\mu)^2] = E[X^2] - 2\mu E[X] + E[\mu^2]$$

→ μ^2 itself as its const.

$\mu = E[X]$

$$= E[X^2] - 2\mu E[X] + \mu^2$$
$$= E[X^2] - 2\mu^2 + \mu^2$$
$$= E[X^2] - \mu^2$$
$$= E[X^2] - (E(X))^2$$

$$Var(X) = E[X^2] - (E(X))^2.$$

wkt. $E[X] = 1/2$.

$$E[X^2] = \sum_{x} x^2 p_{X^2}(x^2) = 0^2 \cdot p(0^2) + 1^2 \cdot p(1^2)$$
$$= 0 \cdot p(0) + 1 \cdot p(1)$$
$$= 1 \cdot (1 - p(0)) = 1 - p(0)$$
$$= 1 - p$$

$$\text{Var}(X) = (1-p) - (E[X])^2$$
$$= (1-p) - (1\cdot(1-p))^2$$
$$= (1-p)\left[1-(1-p)\right] = \underline{\underline{p(1-p)}}$$

for $p = 1/2$, $\text{Var}(X) = 1/4$ (WHAT DOES THIS PHYSICALLY MEAN ?)

$X \rightsquigarrow 2, 4$ $\quad \begin{cases} P(2) = 1/3, & P(4) = 2/3 \\ P(0) = 1/3, & P(1) = 2/3 \end{cases}$

$Y \rightsquigarrow 0, 1$

BOTH REPRESENT the SAME DISTRIBUTION –
$E[X]$ & $E[Y]$ numerically diff, but $\Delta[E[X], 2]$ & $\Delta[E[Y], 0]$ are SAME. . . . –

$E[X] = \frac{10}{3}$

$E[Y] = 2/3$ $\qquad \frac{10}{3}$ -

BINOMIAL DISTRIBUTIONS

A random variable X is said to be binomial if X takes the values $0, 1, 2, \ldots, n$ for $n \in \mathbb{N}$ with the Pdf

$$p(i) = \text{Prob}(X = i) = {}^nC_i\, p^i (1-p)^{n-i}, \quad \text{where}$$
$$p \in [0, 1].$$

Here, we say that X is Binomial with parameter (n, p).

(e.g.) Lets conduct an experiment with the possible outcomes Loss/Win. Suppose we do the expt. n times and are interested in the no. of Wins. let X be the no. of wins in the procedure (repeat expt. n times). Clearly, X takes the values $0, 1, 2, \cdots, n$, where $n \in \mathbb{N}$.
let Prob(win) $= p$, $P(\text{Loss}) = 1-p$.
(assume outcomes don't affect each other)

$$\therefore, \; p(0) = P(X=0) = {}^nC_0 \, p^0 . (1-p)^n = \underline{\underline{(1-p)^n}}.$$

→ $p(0)$ happens only if we lose ALL n times and thats possible only in 1 outcome
$({}^nC_0 = 1)$

$$p(1) = P(X=1) = \underbrace{{}^nC_1} \cdot p^1 . (1-p)^{n-1}$$
$$= n \cdot p(1-p)^{n-1}$$

→ winning exactly once can happen in n ways $(\{W, L, L, \cdots, L\}, \{L, W, L, \cdots, L\}, \cdots, \{L, L, \cdots, W\})$

→ choose 1 container out of n in which W occurs...

Similarly, $p(i) = P(X=i) = {}^nC_i \, \underbrace{p^i . (1-p)^{n-i}}$

→ winning exactly i times and losing $(n-i)$ times.

↳ can happen in ${}^nC_i$ ways, choose i wins in n expts.

∴, Binomial distribution helps describe such events ......

HOMEWORK
let X be a binomial distribution. (i) find the cdf of X ($Prob(X \leq x)$) (ii) find the expectation of X ($E[X]$) (iii) find the Variance of X. ($Var(X)$)

POISSON DISTRIBUTION

A random variable X is said to be poisson with parameter $\lambda > 0$, if it takes the values $0, 1, 2, \ldots$ with the pdf:

$$p(i) = P(X=i) = \frac{e^{-\lambda} \cdot \lambda^i}{i!}$$

remark: let X be a binomial distribution with parameter (n, p).

$$p(i) = {}^nC_i \, p^i \cdot (1-p)^{n-i}$$

$$= \frac{n!}{i!(n-i)!} p^i \cdot (1-p)^{n-i}$$

let $\lambda = np$

$$\Rightarrow p(i) = \frac{n(n-1)\cdots(n-(i-1))}{i!} \times \left(\frac{\lambda}{n}\right)^i \times \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)\cdots(n-(i-1))}{n^i} \times \frac{\lambda^i}{i!} \times \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

$$= \frac{(1-\lambda/n)^n \cdot \lambda^i}{i!} \times \frac{n(n-1)\times\cdots\times(n-(i-1))}{n^i(1-\lambda/n)^i}$$

$$= \frac{(1-\lambda/n)^n \times \lambda^i}{i!} \times \frac{n \times n(1-1/n) \times n\left(1-\frac{2}{n}\right) \times \cdots \times n\left(1-\frac{(i-1)}{n}\right)}{n^i\left(1-\frac{\lambda}{n}\right)^i}$$

$$= \frac{(1-\lambda/n)^n \times \lambda^i}{i!} \times \frac{\cancel{n^i} \times \left(1-\frac{1}{n}\right)\times\left(1-\frac{2}{n}\right)\times\cdots\times\left(1-\frac{(i-1)}{n}\right)}{\cancel{n^i} \times (1-\lambda/n)^i}$$

$\boxed{\text{let } n \to \infty}$

$$\Rightarrow p(i) = \frac{e^{-\lambda}}{i!} \times \lambda^i \times \frac{1}{1} = \frac{e^{-\lambda} \cdot \lambda^i}{i!} \quad //$$

HOMEWORK (BINOMIAL)

i) $cdf(X) = Prob(X \leq x) = \sum_{i=0}^{x} {}^nC_i \, p^i (1-p)^{n-i}$

ii) $E[X] = \sum_{x \in X} x P_{rob}(x)$

$= \sum_{i=0}^{n} i \cdot {}^nC_i \, p^i (1-p)^{n-i} = np \, (?)$

how?

iii) $Var(X) = E\left[(X - E[X])^2\right]$

$= E\left[X^2 + (E[X])^2 - 2XE[X]\right]$

BINOMIAL

$= E\left[X^2 - 2XE[X]\right] + E\left[(E[X])^2\right]$

$= \sum_x x^2 p_{X^2}(x^2) - 2E[X] \cdot \sum_x x P_X(x) + (E[X])^2$

$= \sum_x x^2 p_X(x) - 2(E[X])^2 + (E[X])^2$

POISSON

$= \sum_{i=0}^{\infty} i^2 \cdot \frac{e^{-\lambda} \cdot \lambda^i}{i!} - (E[X])^2$

$= e^{-\lambda} \cdot \sum_{i=0}^{\infty} \frac{i^2 \cdot \lambda^i}{i!} - \lambda^2$

$= \lambda e^{-\lambda} \cdot \sum_{i=1}^{\infty} \frac{i \cdot \lambda^{i-1}}{(i-1)!} - \lambda^2$

$= \lambda e^{-\lambda} \cdot \sum_{j=0}^{\infty} \frac{(j+1)\lambda^j}{j!} - \lambda^2$

$= \lambda e^{-\lambda} \left(\underbrace{\sum_{j=0}^{\infty} j \cdot \frac{\lambda^j}{j!}}_{1} + \underbrace{\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}}_{2} \right) - \lambda^2$

X^2 is not Poisson if X is...

independent of distribution

$$\boxed{P_{X^2}(x^2) = P_X(x)}$$

① $\rightarrow \displaystyle\sum_{j=0}^{\infty} j \cdot \frac{\lambda^j}{j!} = \sum_{j=1}^{\infty} \frac{\lambda^j}{(j-1)!} = \lambda \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!}$

$= \lambda \displaystyle\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$

$= \underline{\underline{\lambda \cdot e^{\lambda}}}$

② $\rightarrow \displaystyle\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda}.$

$\therefore$, $Var(X) = \lambda e^{-\lambda}\left(\lambda e^{\lambda} + e^{\lambda}\right) - \lambda^2$

$= \lambda^2 + \lambda - \lambda^2$

$= \lambda$ //

for Poisson,

$$E[X] = \sum_x x P(x) = \sum_x x \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} = \sum_{i=0}^{\infty} i \cdot \frac{e^{-\lambda} \cdot \lambda^i}{i!}$$

take $i=0$ case out, start from $i=1$ & cancel off i & $i!$ there ...

$$= e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \cdot e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$$

let $j = i-1$.

$$\therefore, E[X] = \lambda e^{-\lambda} \cdot \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda}$$

$$= \lambda.$$

$$Var(X) = E[X] - (E[X])^2$$

$$= \lambda - \lambda^2$$

$$= \underline{\underline{\lambda(1-\lambda)}}$$

X ——————————— X ——————————— X

POISSON $\rightarrow$ $n \rightarrow \infty$ and $p \rightarrow 0$ in **BINOMIAL**.

* identify what R.V. to use .....

contexts for POISSON:

1) no. of misprints in a group of pages of a book.

2) no. of ppl in a community living to 100 years of age.

3) the no. of wrong telephone no.s dialed in a day.

4) the no. of packages of dog biscuits sold in a particular store on each day. (?)

5) the no. of customers entering into a post office on a given day. (?)

6) the no. of vacancies occurring during a yr. in supreme court

7) the no. of α-particles discharged in a fixed period of time from some radioactive material.

Lets assume that the events are occurring at certain pts. of time, assuming the following:

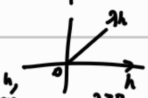
(a) the prob. that exactly 1 event occurs in a time interval of length 'h' is: $\lambda \cdot h + o(h)$, $\lambda > 0$.

(b) the prob. that 2/more events occur in an interval of length 'h' is $o(h)$

(c) for any integer n, $J_1, J_2, \ldots, J_n$, any set of n non-overlapping intervals. let E_i be the event that exactly J_i events occur in the i'th subinterval, then: $E_1, E_2, \ldots, E_n$ are independent.

(e.g.) the no. of earthquakes during a fixed time span.
$h = 21$ days. 0 — 21 day(s)

(a) $\lambda h + o(h)$ makes sense.....
↳ error (not exactly h, but in some form which has worst case as h)



$\lambda > 0$

P.T.O.

let $N(h)$ be the no. of events occurring in the interval $0-h$. $N(h)$ takes the values $\{0, 1, 2, \cdots\}$.

We want to know $P(N(h) = k)$, $k \in \mathbb{N}$

↳ $P($ k out of n sub-intervals contains 1 event & the other $n-k$ contain 0$)$

+

$P($ $N(h) = k$ and at least 1 sub-interval contains 2 / more events$)$

Let B be the event that $N(h) = k$ & at least 1 sub-interval contains 2/more events. find $P(B)$:

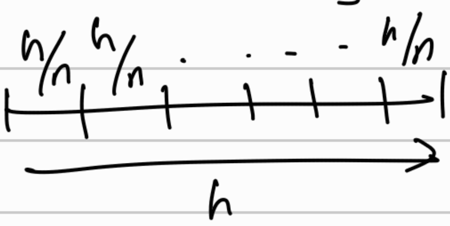
$P(B) \leq P($at least one sub-interval contains ≥ 2 events$)$

→ B is a subset of this

$$\to P\left(\bigcup_{i=1}^{n} \{i\text{'th interval contains} \geq 2 \text{ events}\}\right)$$

$$\leq \sum P(i\text{'th interval contains} \geq 2 \text{ events}\}$$

$$\leq \sum o(h/n)$$



$$= n\, o(h/n)$$

$$= h \cdot \frac{o(h/n)}{(h/n)}$$

as $n \to \infty$, $h/n \to 0$. let $t = h/n$.

$\therefore$, $t \to 0$. $\therefore$, $\dfrac{o(t)}{t} = \dfrac{f(t)}{t} \to 0$.

$\therefore$, $\underline{\underline{P(B) = 0}}$.

$$P(A) = P\left(\begin{array}{l}\text{K sub-intervals contains exactly 1 event \& the} \\ \text{other } n-K \text{ is } 0\end{array}\right)$$

$$= {}^nC_K \left(\lambda \cdot (h/n) + o(h/n)\right)^K \left(1 - \left(\lambda(h/n) + o(h/n)\right)\right)^{n-K}$$

$$= \frac{n!}{K!(n-K)!} \times \frac{1}{n^K} \times \left(\lambda h + n \cdot o\left(\frac{h}{n}\right)\right)^K \cdot \frac{1}{n^{n-K}}\left(n - \left(\lambda h + n\, o(h/n)\right)\right)^{n-K}$$

$$= \frac{n!}{K!(n-K)!} \times \frac{1}{n^n} \times \left(\lambda h + n \cdot o\left(\frac{h}{n}\right)\right)^K \cdot \left(n - (\lambda h + n \cdot o(h/n))\right)^{n-K}$$

$$n \cdot o(h/n) = h \cdot \frac{o(h/n)}{h/n} = h \cdot \frac{f(t)}{t}.$$

if $t \to 0$, $\dfrac{f(t)}{t} \to 0$ //

$$= \frac{n!}{K!(n-K)!} \times \frac{1}{n^n} \times (\lambda h)^K \cdot (n - \lambda h)^{n-K}$$

$$\simeq \frac{n!}{K!(n-K)!} \times \frac{1}{n^n} \cdot \lambda^K \cdot h^K \cdot n^{n-K}$$

$$= \frac{n!}{K!(n-K)!} \times \frac{1}{n^K} \cdot \lambda^K \cdot h^K = \frac{n!}{K!(n-K)!} \times \lambda^K \left(\frac{h}{n}\right)^K$$

$$= \frac{n \times (n-1) \times \cdots \times (n-(k-1)) \times \cancel{(n-k)!}}{k! \times \cancel{(n-k)!}} \times \lambda^k \cdot \left(\frac{h}{n}\right)^k$$

$$= \frac{\cancel{n^k} \times \left(1 - \frac{1}{n}\right) \times \cdots \left(1 - \frac{(k-1)}{n}\right)}{k!} \times \lambda^k \cdot \frac{h^k}{\cancel{n^k}}$$

$$= \frac{1}{k!} \cdot \lambda^k \cdot h^k = \frac{(\lambda h)^k}{k!}$$

its $\frac{(\lambda h)^k}{k!} \cdot e^{-\lambda h}$. check again....

$$\frac{n!}{k!(n-k)!} \times \frac{1}{n^n} \times (\lambda h)^k \times \left(1 - \frac{\lambda h}{n}\right)^{n-k}$$

$$= \frac{n!}{k!(n-k)!} \times \frac{1}{n^n} \times (\lambda h)^k \times \left[\left(1 - \frac{\lambda h}{n-k}\right)^{\frac{n-k}{\lambda h}}\right]^{\lambda h}$$

$$= \frac{n!}{k!(n-k)!} \times \frac{1}{n^k} \times (\lambda h)^k \times e^{-\lambda h}$$

$$\lim_{x \to 0} (1+x)^{1/x} = e$$

$$= \frac{n \times (n-1) \times \cdots \times (n-(k-1))}{k!} \times \frac{1}{n^k} \cdot (\lambda h)^k \times e^{-\lambda h}$$

$$= \frac{\cancel{n^k} \times 1 \times \left(1 - 1/n\right) \times \cdots \times \left(1 - \frac{(k-1)}{n}\right)}{k!} \times \frac{1}{\cancel{n^k}} \times (\lambda h)^k \times e^{-\lambda h}$$

$$= \frac{(\lambda h)^k \cdot e^{-\lambda h}}{k!}$$

(e.g.) Earthquakes happen with assumptions 1, 2 & 3 ...

data : $\lambda = 2/\text{week}$.

(a) find the probability that at least 3 e.q.s. occur in the next 2 weeks.

$P(N(2) \geq 3)$

$\downarrow$

no. of earthquakes occuring in 2 weeks ≥ 3.

{assume Poisson)}

$$P(N(2) \geq 3) = 1 - \{P(N(2) < 3)\}$$

$$= 1 - \{P(N(2)=0) + P(N(2)=1) + P(N(2)=2)\}$$

$$P = \cancel{\frac{e^{-\lambda} \cdot \lambda^{i}}{i!}}$$

$$= 1 - \left\{\frac{e^{-2} \cdot 2 \times 0}{0!} + \frac{e^{-2} \cdot 2 \times 1}{1!} + \frac{e^{-2} \cdot 2 \times 2}{2!}\right\}$$

GEOMETRIC RANDOM VARIABLE

A r.v. X is a G.r.v. if it takes the values 1, 2, 3, 4, ... with the Pdf :

$$P(X=n) = (1-p)^{n-1} \cdot p \quad , \quad 0 < p < 1.$$
$$n = 1, 2, 3, 4, \ldots$$

(e.g.) suppose we do indep. trials with prob p, $0 < p < 1$, are performed until a success occurs.

let $X \rightarrow$ no. of trials reqd. to get first success. $\therefore$, $X \rightarrow 1, 2, 3, \ldots, n$

Hence, $P(X=n) = p(n) = (1-p)\times(1-p)\times \cdots \times (1-p) \times p$
$$= (1-p)^n \cdot p$$

many Hyper aiaper

H.W: find Pdf, cdf, Var & exp of G.r.v.

THE NEGATIVE BINOMIAL RANDOM VARIABLE

fix r. How many trials reqd. until r successes?

A r.v. X is NBrv if X takes the values $r, r+1, r+2, \ldots$ with pdf:

$$p(n) = P(X=n) = \underbrace{{}^{n-1}C_{r-1} \cdot p^r \cdot (1-p)^{(n-1)-(r-1)}}_{} \cdot p$$

choose how in the first $n-1$ boxes $r-1$ successes are distributed ......

HYPERGEOMETRIC RANDOM VARIABLE

A r.v. X is HG r.v. if X takes the values $0, 1, 2, 3, \ldots$ with the Pdf

$$p(i) = P(X=i) = \frac{{}^{m}C_i \cdot {}^{N-m}C_{n-i}}{{}^{N}C_n}$$

for some $m, n, N \in \mathbb{N}$
$i = 0, 1, 2, \ldots, n$

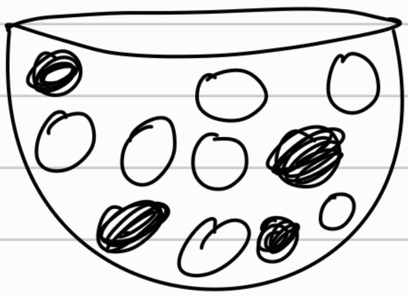
ZETA DISTRIBUTION

A r.v. X is said to have Zeta Distribution if it takes the values 1, 2, 3, ... with Pdf :

$$p(k) = P(X=k) = \frac{C}{k^{\alpha+1}}$$

for some $\alpha > 0$,
$C \in \mathbb{R}$
$k = 1, 2, 3, \cdots$

x ——————————— x ——————————— x



N - white
M - black

select a ball with replacement randomly one at a time until a black one is obtained.

(a) What is the prob. that EXACTLY n-draws are needs.

considers both $M \leq N$ & $M > N$ cases

X takes the values $1, 2, \cdots, \cdots \infty$ (with replacement, no exhaustion)

$$\text{Pdf}: P(X=n) = p(n) = (1-p)^{n-1} \cdot p = \left(1 - \frac{M}{N+M}\right)^{n-1} \cdot \frac{M}{N+M}$$

$$= \left(\frac{N}{N+M}\right)^{n-1} \cdot \frac{M}{N+M}$$

$$= \frac{M \cdot N^{n-1}}{(M+N)^n}$$

THE BANACH match problem

left pckt (L) right pckt (R)

$p(L) = p(R) = 1/2$.

each pckt has N. whats the probability that if one box is empty, the other one has K matches?

Banach would've exhausted $N + (N-K) = 2N-K$ if K in one pckt remains & the other is empty. but, an extra choice is required to know if pckt has been exhausted / not.

∴, in 2N-K selections, N has to be from one pckt & N-K from the other. the problem is to find how this can be done, in 2 cases: one when left is empty & the other when right is.

$$^{2N-K}C_N \cdot (P(L))^N \cdot (P(R))^{N-K}$$

Sim, for right empty. ∴, Prob of such case is:

$$C_N \cdot (P(L))^N \cdot (P(R))^{N-K} + {}^{2N-K}C_N (P(R))^N \cdot (P(L))^{N-K}$$

but, $P(L) = P(R) = 1/2$. ∴, $(1/2)^N \cdot (1/2)^{N-K}$
$= (1/2)^{2N-K}$.

∴, P of case $= 2 \cdot {}^{2N-K}C_N \cdot (1/2)^{2N-K}$

let X be the # of choices until the right pckt is empty, & that the left pckt has K matches.

we want to know:

$$P(X = N+1+N-K) = P(X = 2N-K+1)$$

X takes the values: $N, N+1, N+2, \ldots$

$$^{2N-K+1}C_N \cdot (p(R))^{N+1} \cdot (p(L))^{N-K}$$

$$^{2N-K+1}C_N \cdot \left(\tfrac{1}{2}\right)^{2N-K+1}$$

+1 added coz extra trial reqd. to find if empty.

Sim, for right.

$\therefore$, $\boxed{\text{total prob} = 2 \cdot {}^{2N-K+1}C_N \cdot \left(\tfrac{1}{2}\right)^{2N-K+1}}$

CONTINUOUS RANDOM VARIABLE (C.R.V.)

A r.v. X is a C.R.V. if $X: S \to \mathbb{R}$ & $\exists$ a function $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$ s.t.

$$P(X \in A) = \int_A f(x)\,dx \quad , \text{ where } A \subseteq \mathbb{R}$$

we confine ourselves to intervals $\{[a,b], (a,b), (a,b], [a,b), (a,\infty), (-\infty, a)\}$

(e.g.) suppose X is a continuous random variable whose Pdf is given by:

$$f(x) = \begin{cases} C(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) find the value of C:

$$P\big(X \in (-\infty, \infty)\big) = \int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx \xrightarrow{} 0 + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx \rightarrow 0 = 1$$

$$\Rightarrow \int_{0}^{2} f(x)\,dx = 1$$

$$\Rightarrow c\int_{0}^{2} (4x - 2x^2)\,dx = 1$$

$$\Rightarrow c\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = 1$$

$$\Rightarrow c\left[8 - \frac{16}{3}\right] = 1$$

$$\Rightarrow \boxed{c = 3/8}$$

(b) find $P(X>1)$: $\displaystyle\int_{1}^{\infty} f(x)\,dx$

$$= \frac{3}{8}\int_{1}^{\infty} (4x - 2x^2)\,dx$$

$$= \frac{3}{8}\left[\int_{1}^{2} (4x - 2x^2)\,dx + \int_{2}^{\infty} (4x - 2x^2)\,dx \rightarrow 0\right]$$

$$= \frac{3}{8}\left[\left(2x^2 - \frac{2}{3}x^3\right)_1^2\right]$$

$$= \frac{3}{8}\left[\left(8 - 16/3\right) - \left(2 - 2/3\right)\right]$$

$$= \frac{3}{8}\left[\frac{8}{3} - \frac{4}{3}\right] = \frac{3}{8} \times \frac{4}{3} = \frac{1}{2}$$

let X be a C.R.V. with the pdf f.

1) $$E[X] = \int_{-\infty}^{+\infty} x f(x)\, dx$$

2) $$Var(X) = E[X^2] - (E(X))^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x^2)\, d(x^2) - \left(\int_{-\infty}^{+\infty} x f(x)\, dx\right)^2 \quad (?)$$

Classification of CRVs :

1) Uniform : A CRV X is uniform if $X \in [\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$ with the pdf -

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha}, & \forall\ \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$

Any two intervals of the same length have the same probability of occurring

let $A = (a, b)$

$$P(X \in A) = \int_a^b f(x)\, dx = \int_a^b \frac{1}{\beta - \alpha}\, dx = \frac{b - a}{\beta - \alpha}$$

$$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx = \cancel{\int_{-\infty}^{\alpha} x f(x)\,dx} \to 0 + \int_{\alpha}^{\beta} x f(x)\,dx + \cancel{\int_{\beta}^{\infty} x f(x)\,dx} \to 0$$

$$= \int_{\alpha}^{\beta} x f(x)\,dx = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha}\,dx$$

$$= \frac{1}{\beta-\alpha} \cdot \frac{\beta^2-\alpha^2}{2}$$

$$= \frac{\beta+\alpha}{2}$$

$X^2 \to$ comp. of X and $f = x^2$

$\to \therefore$, continuous ....

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \int_{-\infty}^{+\infty} x^2 g(x)\,dx = \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{\beta-\alpha}\,dx$$

$$= \frac{1}{3} \cdot \frac{1}{\beta-\alpha} \cdot \beta^3-\alpha^3$$

$(\beta-\alpha)(\beta^2+\alpha^2+\alpha\beta)$

$$= \frac{1}{3} \cdot (\beta^2+\alpha^2+\alpha\beta)$$

CRV

$\uparrow$ uniform

??? ????

$$\boxed{Pdf_X = Pdf_{X^2}}$$

$\downarrow$

Stretch doesn't disturb prob.

$$E[X] = \int_{\alpha}^{\beta} x f(x)\,dx$$

$$= \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha}\,dx = \frac{1}{2}(\alpha+\beta)$$

$$\frac{1}{3}(\alpha^2+\beta^2+\alpha\beta) - \frac{1}{4}(\alpha^2+\beta^2+2\alpha\beta)$$

$$= \frac{4\alpha^2+4\beta^2+4\alpha\beta-3\alpha^2-3\beta^2-6\alpha\beta}{12}$$

$$= \frac{\alpha^2+\beta^2-2\alpha\beta}{12} = \frac{(\alpha-\beta)^2}{12}$$

let X be a CRV on (α, β), i.e., the pdf of X is -

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & x \in (\alpha, \beta) \\ 0, & \text{otheewise} \end{cases}$$

then, X^2 is also a CRV on ( ) ??

↑ if $1<\alpha<\beta$

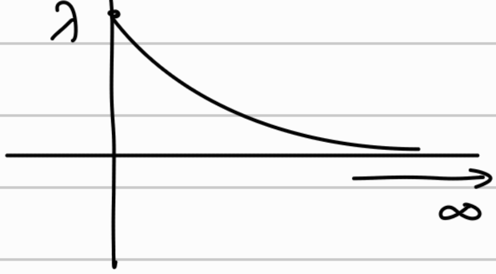
$$X^2 : S \xrightarrow[X]{} (\alpha, \beta) \xrightarrow[x^2]{} (\alpha^2, \beta^2)$$

X ——— X ——— X

EXPONENTIAL RANDOM VARIABLE (ERV)

A continuous r.v. X is said to be ERV with parameter λ if its pdf is following:

$$(\lambda > 0) \quad f(x) = \begin{cases} \lambda e^{-\lambda x}; & x \geq 0 \\ 0; & x < 0 \end{cases}$$



$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\,dx$$

$f(x) = \lambda e^{-\lambda x}$

$e^{-\lambda x}$ → changes rate of change

λ → necessary, as $\int_{-\infty}^{\infty} f(x)\,dx$ SHOULD be 1.

$$= -\int_{0}^{\infty} x e^{-\lambda x}\,d(-\lambda x)$$

$$= -\int_{0}^{\infty} x\,d\left(e^{-\lambda x}\right)$$

$$= -\left[x e^{-\lambda x} + \frac{1}{\lambda}\int e^{-\lambda x}\,d(-\lambda x)\right]$$

if $f(x) = e^{-\lambda x}$ then the integral won't be 1.

$$= -\left[x e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda}\right]_{0}^{\infty}$$

$$= -\left[-\left(\frac{1}{\lambda}\right)\right]$$

$$= \frac{1}{\lambda}$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{\lambda^2}$$

$E[X^2] = \frac{2}{\lambda^2}$ ← using this, find var(x)

prove this also....

if $\lambda >> 0 \to$ graph goes to 0 faster / sharply (?)

(e.g.) Supp. that the length of a phone call is an exp. random var, with par. $\lambda = 1/2$. if someone arrives immediately ahead of you at a public telephone booth find the prob. that you'll have to wait:

(a) more than 10 mins.

let X be the length of the call made by the person in the booth.

$$P(X > 10) = \int_{10}^{\infty} \frac{1}{2} e^{-\frac{1}{2}x} \, dx = 0.365$$

(b) b/w 10 to 20 mins

$$P(10 < X < 20) = \int_{10}^{10} \frac{1}{2} e^{-\frac{1}{2}x} \, dx = \underline{\underline{0.233}}$$

NORMAL RANDOM VARIABLE (NRV)

A r.v. X is said to be NRV if its pdf is:

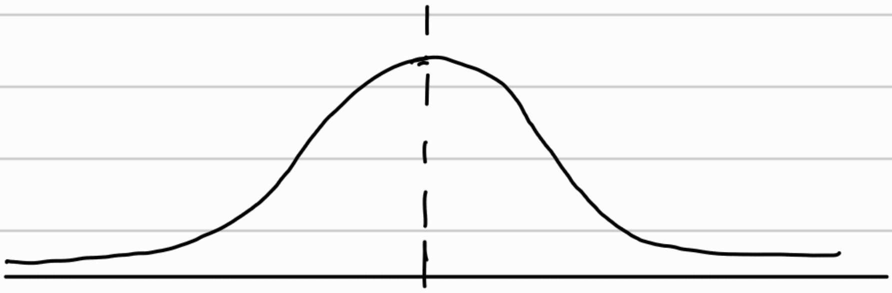
$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-(x-\mu)^2 / 2\sigma^2}$$

defines shape

$-\infty < x < \infty$

adjusts curve

here, μ, σ^2 are the parameters of X.



$f(-x) = f(x)$

fn. symm. about the line $x = \mu$ → $\mu \rightarrow$ mean/average.

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-(x-\mu)^2/2\sigma^2} dx$$

$t = \dfrac{x-\mu}{\sigma}$

$dt = \dfrac{1}{\sigma} dx$

$\sigma dt = dx$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2} dt \rightarrow = 1$$

let $I = \int_{}^{\infty} e^{-t^2/2} dt \rightarrow$ find integral.

take $dy\,dx = r\,dr\,d\theta$

$$I^2 = \int_0^{\infty} \int_0^{2\pi} e^{-r^2/2} \cdot r\,dr \cdot d\theta$$

$$= \int_0^{\infty} r e^{-r^2/2} dr \cdot \int_0^{2\pi} d\theta = -2\pi(0-1) = 2\pi$$

$\therefore$, $I = \sqrt{2\pi}$.

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \cdot I = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = 1$ //

hence proved //

for a N.R.V X,

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot \int_{-\infty}^{\infty} (x-\mu+\mu) \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\left[\int_{-\infty}^{\infty} (x-\mu) \cdot e^{-(x-\mu)^2/2\sigma^2}\,dx + \int_{-\infty}^{\infty} \mu \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx\right]$$

$x-\mu = t$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot \left[\int_{-\infty}^{\infty} (x-\mu) \cdot e^{-(x-\mu)^2/2\sigma^2}\,dx + \underbrace{\mu \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx}_{I}\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\Bigg[\mu \cdot I + \underbrace{\int_{-\infty}^{\infty} t \cdot e^{-t^2/2\sigma^2}\,dt}_{0}\Bigg]$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot \left[\mu I - 2\sigma^2 \int_{-\infty}^{\infty} e^{-t^2/2\sigma^2} \cdot d\left(-t^2/2\sigma^2\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot \left[\mu I - 2\sigma^2 \cdot 0 \right]$$

$$= \mu \cdot \frac{\cancel{I}^{\,1}}{\cancel{\sqrt{2\pi}\sigma}} = \mu \;//$$

find $\mathrm{Var}(X) = E\left[(X-\mu)^2 \right]$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \cdot (x-\mu)^2 \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}} \cdot dx$$

$t = \frac{x-\mu}{\sigma}$

$\sigma dt = dx$

$$= \frac{-\sigma^2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} t^2 \cdot e^{-t^2/2} dt$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t \cdot (-2t) \cdot e^{-t^2/2} dt$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} t \cdot e^{-t^2/2} \cdot d(-t^2/2)$$

$$= \frac{-1}{\sqrt{2\pi}} \sigma^2 \int_{-\infty}^{\infty} t \cdot d\left(e^{-t^2/2}\right)$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}} \left[t \cdot e^{-t^2/2} - \int_{-\infty}^{\infty} e^{-t^2/2} \cdot dt \right]$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}} \left[\cancel{t \cdot e^{-t^2/2}}^{\,0} \Big|_{-\infty}^{\infty} - \sqrt{2\pi} \right]$$

from earlier results

$$= -\frac{\sigma^2}{\sqrt{2\pi}} \left(-\sqrt{2\pi} \right)$$

$$= \sigma^2$$

let $X \to$ normal distribution with param μ, σ^2,
i.e. pdf -

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad , -\infty < x < \infty$$

lets define $Y = \alpha X + \beta$; $\alpha, \beta \in \mathbb{R}$
$X \to$ cont. $\therefore$, $Y \to$ cont.
is $Y \to$ normal ?

$\underline{\underline{\alpha > 0}}$

$$F_Y(a) = P(Y \leq a) \quad ; \; a \in \mathbb{R}$$

$$= P(\alpha X + \beta \leq a)$$

$$= P\left(X \leq \frac{a-\beta}{\alpha}\right) = F_X\left(\frac{a-\beta}{\alpha}\right)$$

$$F_y(a) = F_x\left(\frac{a-\beta}{\alpha}\right)$$

diff wrt. a (a be var)

$$\Rightarrow f_y(a) = f_x\left(\frac{a-\beta}{\alpha}\right) \cdot \frac{1}{\alpha}$$

$$\Rightarrow f_y(a) = \frac{1}{\alpha} \cdot \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\left(\left(\frac{a-\beta}{\alpha}\right)-\mu\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\alpha} \cdot \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-1/\alpha^2 \left((a-\beta)-\alpha\mu\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\alpha\sigma} \cdot e^{-\frac{(a-(\alpha\mu+\beta))^2}{2(\sigma\alpha)^2}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma'} \cdot e^{-\frac{(a-\mu')^2}{2\sigma'^2}}$$

$\sigma' = \alpha\sigma$

$\mu' = \alpha\mu + \beta$ ||

$\therefore$, y is normal with param μ' & σ', where

$\mu' = \alpha\mu + \beta$

$\sigma' = \alpha\sigma$ ||

if $X \to$ normal, then $Y = \alpha X + \beta$ is ALSO normal.

$X \to \mu, \sigma$, $Y \to \alpha\mu + \beta, \alpha\sigma$ (do for $\alpha < 0$ too – H.W.)

let $Z = \frac{X-\mu}{\sigma} = \frac{X}{\sigma} - \frac{\mu}{\sigma} \equiv \alpha X + \beta$, where

$\alpha = 1/\sigma$ and

$\beta = -\mu/\sigma$

$\therefore$, $Z \to \alpha\mu + \beta, \sigma\alpha$

$= \left(\mu/\sigma - \frac{\mu}{\sigma}\right), 1$ parameters of Z.

$Z \to 0, 1$ ||

standard normal variable.

1) pdf of $Z = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

2) cdf of $Z = \Phi(x) = F_Z(x)$

3) $\forall x \in \mathbb{R}^+ \cup \{0\}$, $\Phi(-x) = 1 - \Phi(x)$

$$P(Z \leq -x) = P(Z \geq x) \quad \left\{ \text{graph symm. about } x=0 \right\}$$

$$\Rightarrow \Phi(-x) = P(Z \geq x)$$

$$\Rightarrow \Phi(-x) = 1 - P(Z \leq x)$$

$$\Rightarrow \Phi(-x) = 1 - \Phi(x) \,//$$

Observation :

* let X be NRV with (μ, σ):

$$F_X(a) = P(X \leq a) = P\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{a-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{a-\mu}{\sigma}\right)$$

(e.g.) if X is normal with parameters $\mu = 3$ & $\sigma^2 = 9$ find

(a) $P(2 < X < 5) = P\left(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right)$

$$= P\left(-\tfrac{1}{3} < Z < \tfrac{2}{3}\right)$$

$$= P\left(Z < \tfrac{2}{3}\right) - P\left(Z < -\tfrac{1}{3}\right)$$

$$= \Phi(2/3) - \Phi(-1/3)$$

$$= \Phi(2/3) - (1 - \Phi(1/3))$$

$$= \Phi(2/3) + \Phi(1/3) - 1$$

$$= 0.74537 + 0.62930 - 1$$

$$= 0.37467$$

$$\simeq 0.3747$$

$$\begin{array}{r} 0.74537 \\ 0.62930 \\ \hline 1.37467 \end{array}$$

Theorem: let X be a binomial distribution with parameters (n, p). then,

$$P\left(a \leq \frac{X - np}{\sqrt{np(1-p)}} \leq b\right) \simeq \Phi(b) - \Phi(a)$$

if $n >>> 0$