Engineering Electromagnetics

Lecture 10

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by

Debolina Misra

Assistant Professor in Physics
IIITDM Kancheepuram, Chennai, India

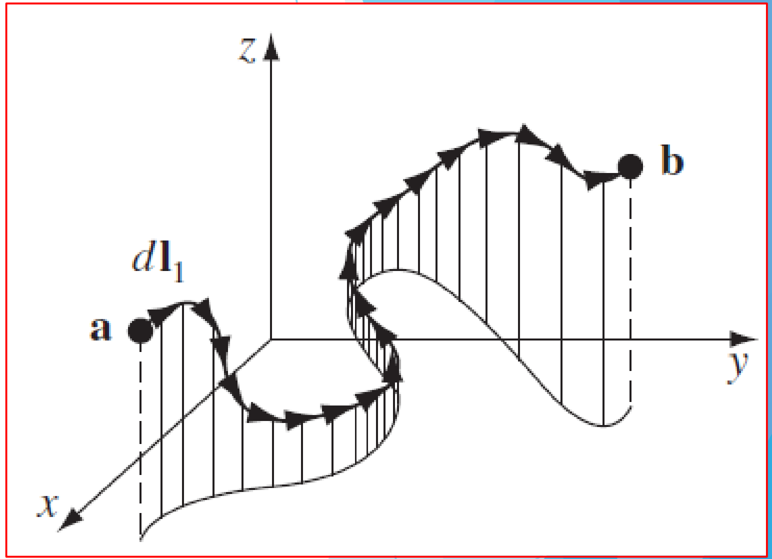
The Fundamental Theorem for Gradients

Now we move a little further, by an additional small displacement $d\mathbf{l}_2$; the incremental change in T will be $(\nabla T).d\mathbf{l}_2$.

In this manner, proceeding by infinitesimal steps, we make the journey to point **b**.

At each step we compute the gradient of T (at that point) and dot it into the displacement $d\mathbf{l}$ →this gives us the change in T .

Evidently the *total* change in T in going from **a** to **b** (along the path selected) is



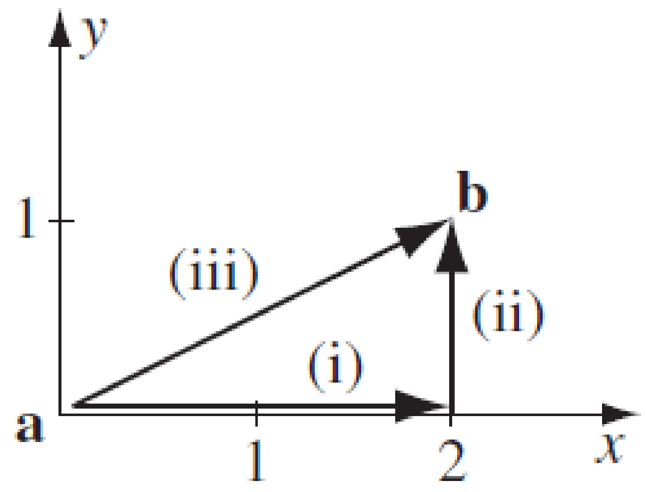
$$\boxed{\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).}$$

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of the path taken from **a** to **b**

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} =$ **?**

Problem-2

Example 1.9. Let $T = xy^2$, and take point **a** to be the origin (0, 0, 0) and **b** the point (2, 1, 0). Check the fundamental theorem for gradients.

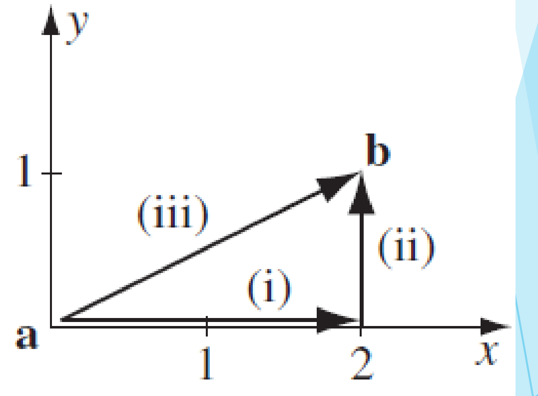


Solution

Although the integral is independent of path, we must *pick* a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii) (Fig. 1.27). As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\nabla T = y^2\,\hat{\mathbf{x}} + 2xy\,\hat{\mathbf{y}}$.

(i) $y = 0$; $d\mathbf{l} = dx\,\hat{\mathbf{x}}$, $\nabla T \cdot d\mathbf{l} = y^2\,dx = 0$, so

$$\int_{\mathrm{i}} \nabla T \cdot d\mathbf{l} = 0.$$



(ii) $x = 2$; $d\mathbf{l} = dy\,\hat{\mathbf{y}}$, $\nabla T \cdot d\mathbf{l} = 2xy\,dy = 4y\,dy$, so

$$\int_{\mathrm{ii}} \nabla T \cdot d\mathbf{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2.$$

The total line integral is 2. Is this consistent with the fundamental theorem? Yes: $T(\mathbf{b}) - T(\mathbf{a}) = 2 - 0 = 2$.

How do you find a unit vector normal to the surface $x^3+y^3+3xyz=3$ at the point $(1,2,-1)$?

How do you find a unit vector normal to the surface $x^3+y^3+3xyz=3$ ay the point (1,2,-1)?

Calling

$$f(x, y, z) = x^3 + y^3 + 3xyz - 3 = 0$$

The gradient of $f(x, y, z)$ at point x, y, z is a vector normal to the surface at this point.

The gradient is obtained as follows

$\nabla f(x, y, z) = (f_x, f_y, f_z) = 3(x^2 + yz, y^2 + xz, xy)$ at point $(1, 2, -1)$ has the value $3(-1, 3, 2)$ and the unit vector is

$$\frac{\{-1, 3, 2\}}{\sqrt{1 + 3^2 + 2^2}} = \left\{-\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \sqrt{\frac{2}{7}}\right\}$$

https://socratic.org/questions/how-do-you-find-a-unit-vector-normal-to-the-surface-x-3-y-3-3xyz-3-ay-the-point-

Divergence: examples

- If $\boldsymbol{f} = x\hat{x} + y\hat{y} - \hat{z}$, $\nabla . f = ?$
- If $\boldsymbol{f} = 2\hat{y}$, $\nabla . f = ?$
- If $\boldsymbol{f} = x^2\hat{x}, \nabla . f = ?$

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.}$$

Solution

- $\boldsymbol{f} = x\hat{x} + y\hat{y} - \hat{z}$, $\boldsymbol{\nabla}.\boldsymbol{f} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} =$
- $\boldsymbol{f} = 2\hat{y}$, $\boldsymbol{\nabla}.\boldsymbol{f} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} =$
- $\boldsymbol{f} = x^2\hat{x}, \boldsymbol{\nabla}.\boldsymbol{f} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} =$

The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.}$$

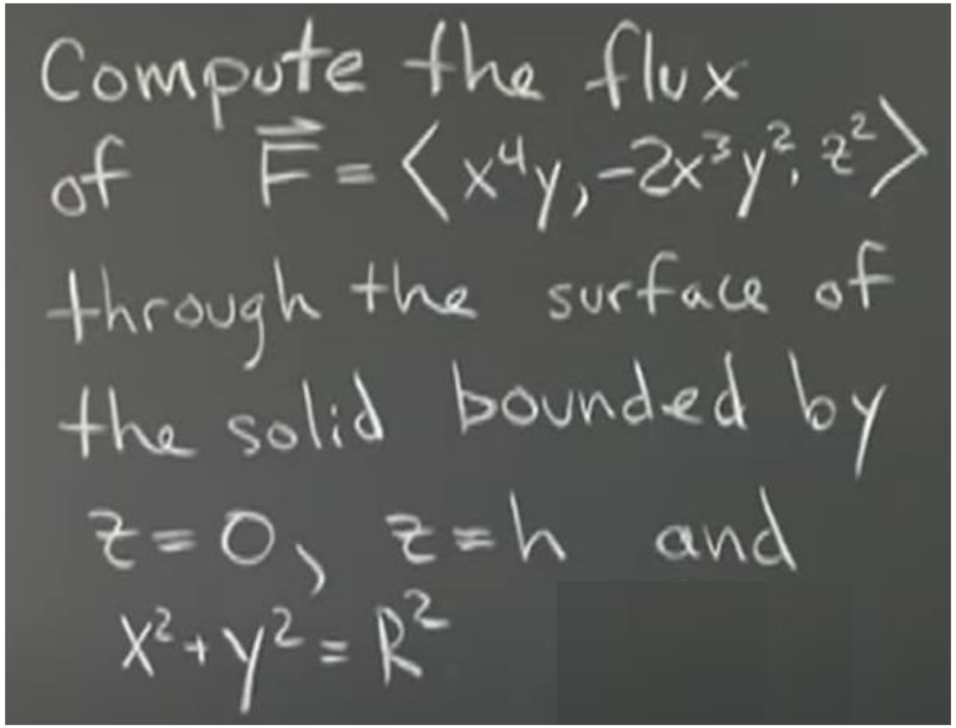
Gauss's theorem/divergence theorem

integral of a *derivative* (in this case the *divergence*) over a *region* (in this case a *volume*, *V*) =value of the function at the *boundary* (in this case the *surface S* that bounds the volume).

Notice that the boundary term is itself an integral (specifically, a surface integral). This is reasonable: the boundary of a *volume* is a (closed) surface.

From Griffith Book

Div. theorem: example (MIT open course)



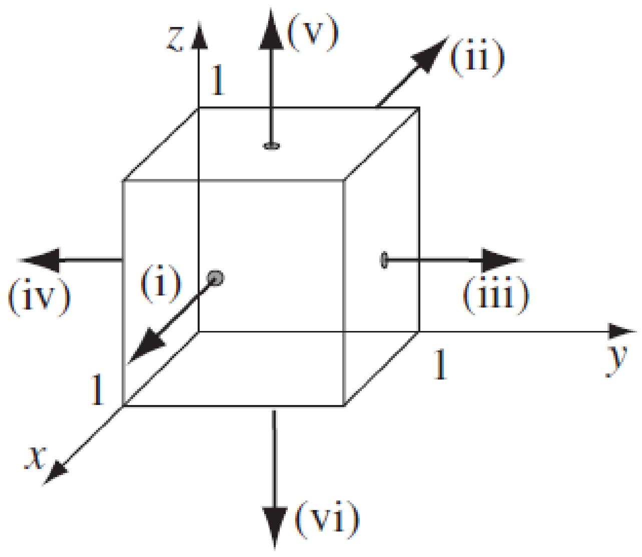
- https://www.youtube.com/watch?v=CCoTAyZ14XM

Problem-3

Check the divergence theorem using the function

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

and a unit cube at the origin.



$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.}$$

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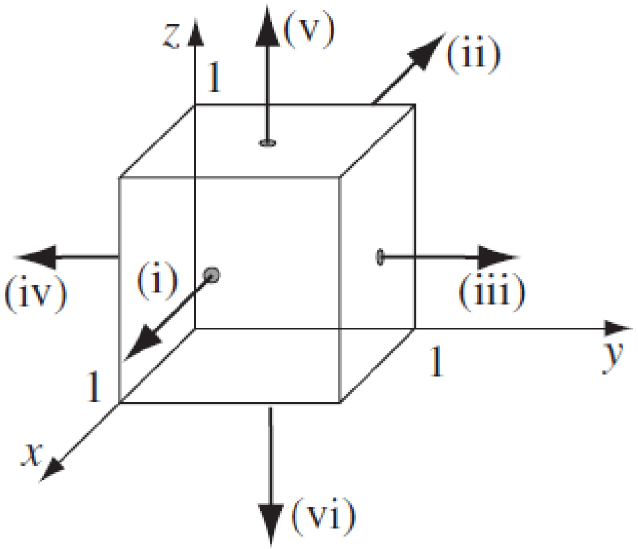
Solution

For the LHS

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$



$$\nabla\cdot\mathbf{v} = 2(x+y)$$

$$\int_{\mathcal{V}} 2(x+y)\,d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz,$$

$$\int_0^1 (x+y)\,dx = \tfrac{1}{2} + y, \quad \int_0^1 (\tfrac{1}{2}+y)\,dy = 1, \quad \int_0^1 1\,dz = 1$$

$$\int_{\mathcal{V}} \nabla\cdot\mathbf{v}\,d\tau = 2$$

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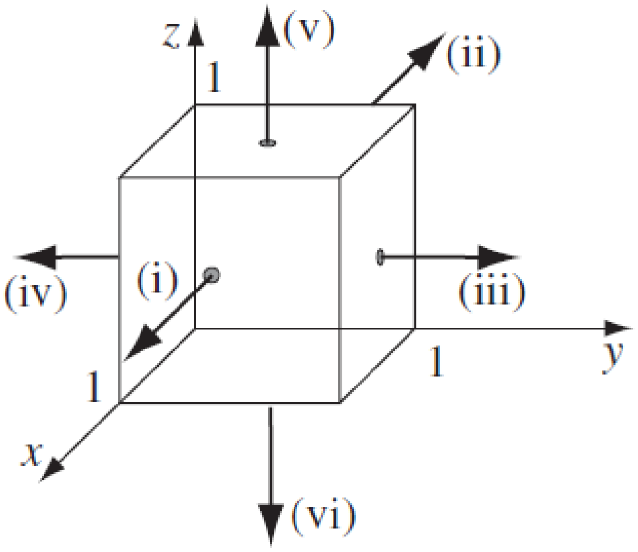
For the RHS

So much for the left side of the divergence theorem. To evaluate the surface integral we must consider separately the six faces of the cube:



(i) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 y^2 dy\, dz = \tfrac{1}{3}.$$

(ii) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1 \int_0^1 y^2\, dy\, dz = -\tfrac{1}{3}.$$

(iii) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 (2x + z^2)\, dx\, dz = \tfrac{4}{3}.$$

Note that for this surface y=1

(iv) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1 \int_0^1 z^2\, dx\, dz = -\tfrac{1}{3}.$$

Note that for this surface y=0 so it is not same as Surface (iii)

(v) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 2y\, dx\, dy = 1.$$

(vi) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1 \int_0^1 0\, dx\, dy = 0.$$

So the total flux is:

$$\oint \mathbf{v} \cdot d\mathbf{a} = \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{4}{3} - \tfrac{1}{3} + 1 + 0 = 2,$$

From Griffith Book

Thank You