Replacement of Basic Solution

Theorem

Let LPP has a BFS.

If we drop one of the basic vectors and introduce a non-basic vector in the basis set, then the new solution obtained is also a BFS.

Proof:-

$$\text{Max } z = cX$$

$$\text{S.T.C } AX = b, \quad X \geq 0$$

$$X^T \in \mathbb{R}^n, \quad c \in \mathbb{R}^n$$

A & b are real $m \times n$ & $m \times 1$ real matrices respectively.

Let $\rho(A) = m$

& X_B is BFS, then

$$B X_B = b, \quad X_B \geq 0$$

B forms a basis set for the column vectors of A.

$\Rightarrow$ for any $a_j \in A$ (column)

$$a_j = y_{1j} \cdot b_1 + y_{2j} b_2 + \cdots + y_{mj} \cdot b_m$$

$\Rightarrow a_j = B\, y_j$, where $b_i \in B$

& y_{ij} are suitable scalars,

$$b_r = \frac{a_j}{y_{rj}} - \sum_{\substack{i=1 \\ i \neq r}}^{m} \frac{y_{ij}}{y_{rj}} b_i \quad \text{for } y_{rj} \neq 0$$

& then

$$b = \sum_{i=1\,(i\neq r)}^{m} x_{B_i} b_i + x_{B_r}\left[\frac{a_j}{y_{rj}} - \sum_{\substack{i=1 \\ i \neq r}}^{m} \frac{y_{ij}}{y_{rj}} b_i\right]$$

$$= \sum_{\substack{i=1 \\ i \neq r}}^{m} \left[x_{B_i} - x_{B_r}\frac{y_{ij}}{y_{rj}}\right] b_i + \frac{x_{B_r}}{y_{rj}} a_j$$

Now New Basic Solution is $\hat{X}_B$

$$\hat{X}_{B_i} = X_{B_i} - X_{B_r}\frac{y_{ij}}{y_{rj}}, \quad i = 1, 2, \ldots, m, \; i \neq r$$

and $$\hat{X}_{B_r} = \frac{X_{B_r}}{y_{rj}} \quad \text{———} \; (*)$$

We shall now show that $\hat{X}_B$ is feasible also

i.e. Now $\hat{X}_{B_i} \geq 0$ for all 'i'.

$\boxed{\text{Case-I}} \rightarrow \boxed{X_{B_r} = 0}$ result is obvious from $X_{B_i} \geq 0$.

$\boxed{\text{Case II}} \rightarrow \boxed{X_{B_r} \neq 0} \Rightarrow$ only option $\boxed{x_{B_r} > 0}$

So from Eq^n $(*)$ $\hat{x}_{B_r} \geq 0 \Rightarrow y_{rj} > 0$

Remaining 3- alternat sub-cases,

(a) $\left\{ \begin{array}{l} \text{If } y_{ij} = 0 \text{ for } i \neq r \\ \qquad \text{then } \hat{x}_{Bi} \geq 0 \text{ for all } i \neq r \end{array} \right.$

(b) ifelse $y_{ij} < 0$ then $\hat{x}_{Bi} \geq 0$ for all $i \neq r$

(c) else $y_{ij} > 0$ then $\hat{x}_{Bi} \geq 0$ for all $i \neq r$

give rise the condition $\dfrac{x_{Bi}}{y_{ij}} \geq \dfrac{x_{Br}}{y_{rj}}$ $\forall\, i \neq r$.

So if we select 'r' such 'a' way that

$$\frac{x_{Br}}{y_{rj}} = \min_{r} \left\{ \frac{x_{Br}}{y_{ij}} \; ; \; y_{ij} \geq 0 \;\; i \neq r \right\}$$

then the new BS is BFS. $\square$

Formula:- $\hat{B} = (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_m)$

$\hat{b}_i = b_i$ for $i \neq r$

$\hat{b}_r = a_j$

$$\boxed{\hat{X}_B = \hat{B}^{-1} b}$$

Where $\hat{X}_{B_i} = X_{B_i} - X_{B_r} \dfrac{y_{ij}}{y_{rj}}, \ i \neq r$

$\hat{X}_{B_r} = \dfrac{X_{B_r}}{y_{rj}}$ are the basic variables.

Definition (Net Evaluation)

Let X_B be BFS to the LPP

$$\max \quad z = cX$$

s.t.c $\quad Ax = b \;\&\; X \geq 0$

Let c_B be the cost vector corresponding to X_B for each column vectors a_j in A, which is not a column vector of B.

Let $a_j = \sum_{i=1}^{m} y_{ij} \, b_i$

then the number $z_j = \sum_{i=1}^{m} c_{Bi} \, y_{ij}$ is called the Evaluation corresponding to

a_j and the number $z_j - c_j$ is called the net evaluation corresponding to a_j.

Theorem:- 4.3

(Improved BFS)

Let X_B is a BFS to LPP

$\hat{X}_B$ is another BFS (by replacing b_r by a_j in the basis) for which a_j the net evaluation '$z_j - c_j$' is $-ve$

Then $\hat{X}_B$ is improved solution

i.e. $\boxed{\hat{c}_B \hat{X}_B > c_B X_B}$ — (1)

Porof :-

Let LPP is

$\max\ Z = cx, \quad c, x^T \in \mathbb{R}^n$

S.T.C $\quad Ax = b, \ x \geq 0, \ b^T \in \mathbb{R}^m$

x_B is BFS $\quad z_0 = c_B x_B$

Let a_j is introduced in $\hat{x}_B$

such that $z_j - c_j < 0$

$$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}} \quad \& \quad \hat{x}_{Br} = \frac{x_{Br}}{y_{rj}}$$

Hence New value of objective funⁿ is

$$\hat{z} = \sum_{i=1}^{m} \hat{c}_{Bi} \hat{x}_{Bi}$$

$$= \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{Bi} \left(x_{Bi} - x_{Br} \frac{y_{ij}}{y_{rj}} \right) + \hat{c}_{Br} \frac{x_{Br}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{Bi} \left(x_{Bi} - x_{Br} \frac{y_{ij}}{y_{rj}} \right) + c_j \frac{x_{Br}}{y_{rj}}$$

$$\left(\because \hat{c}_{Br} = c_j \right)$$

$$= z_0 - (z_j - c_j) \frac{x_{Br}}{y_{rj}} > z_0$$

Hence the New BFS $\hat{X}_B$ is improved solⁿ. $\square$