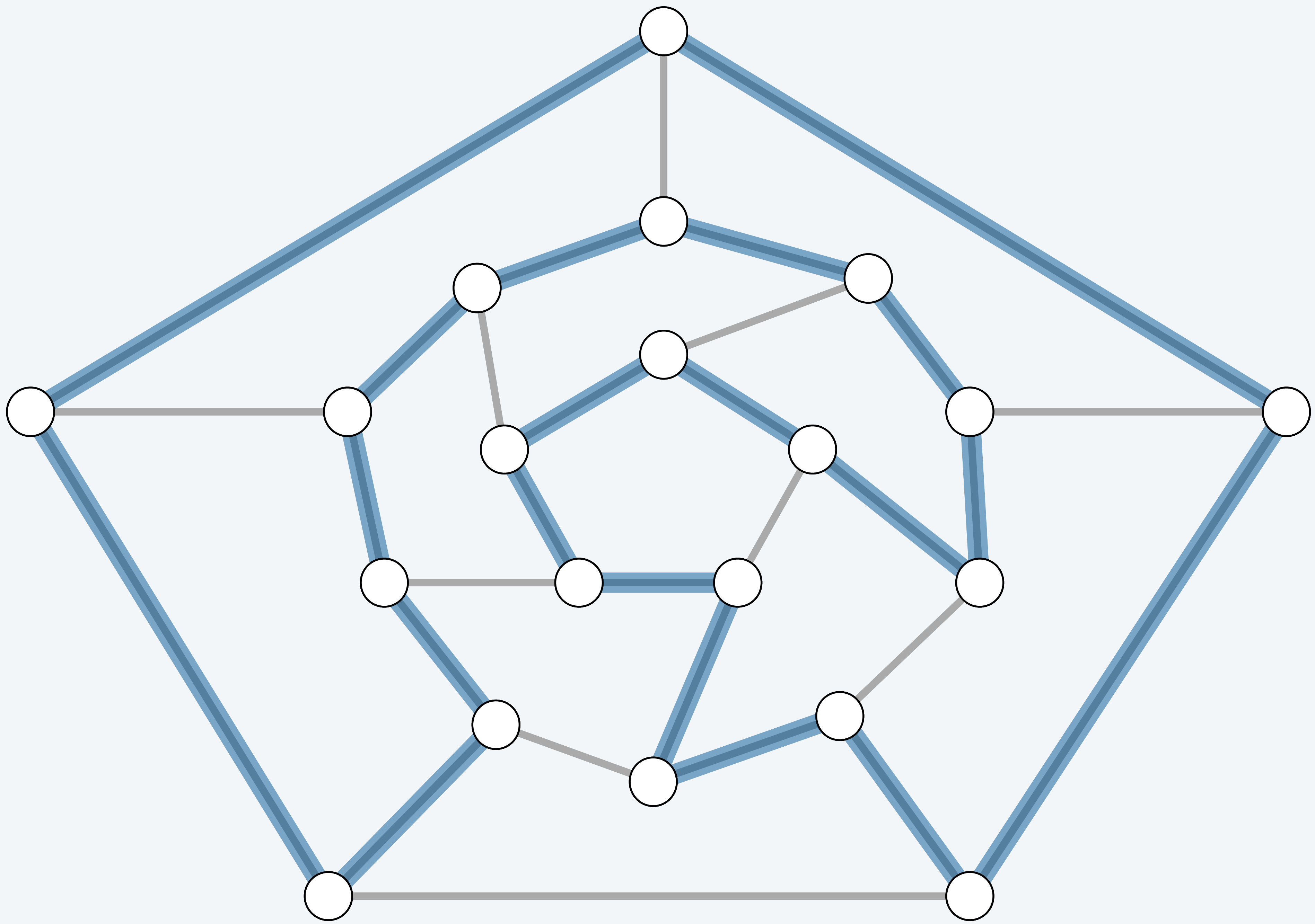
# Hamilton cycle

HAMILTON-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a cycle $\Gamma$ that visits every node exactly once?



yes

# Hamilton cycle

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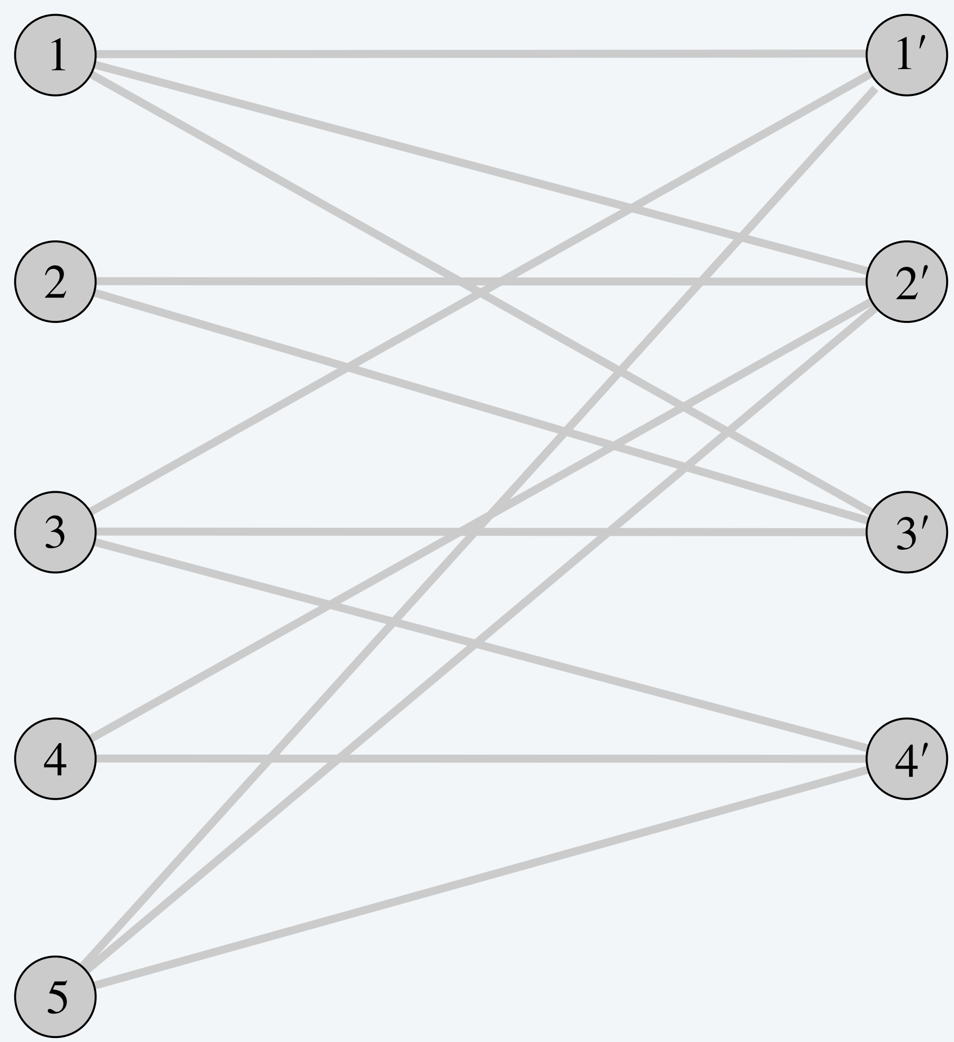


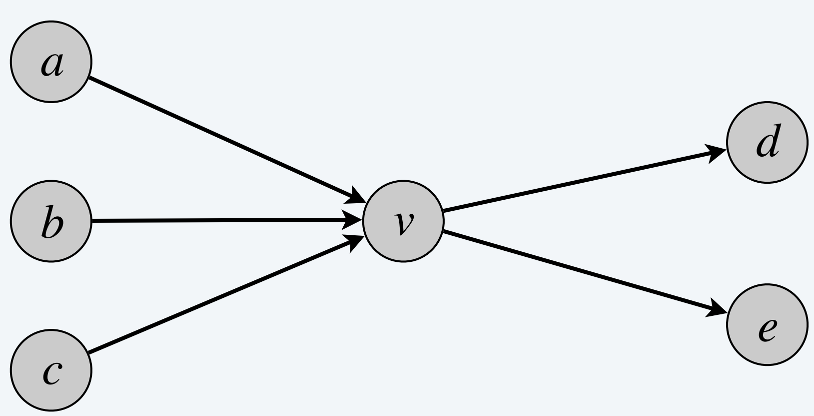

**no**

# Directed Hamilton cycle reduces to Hamilton cycle

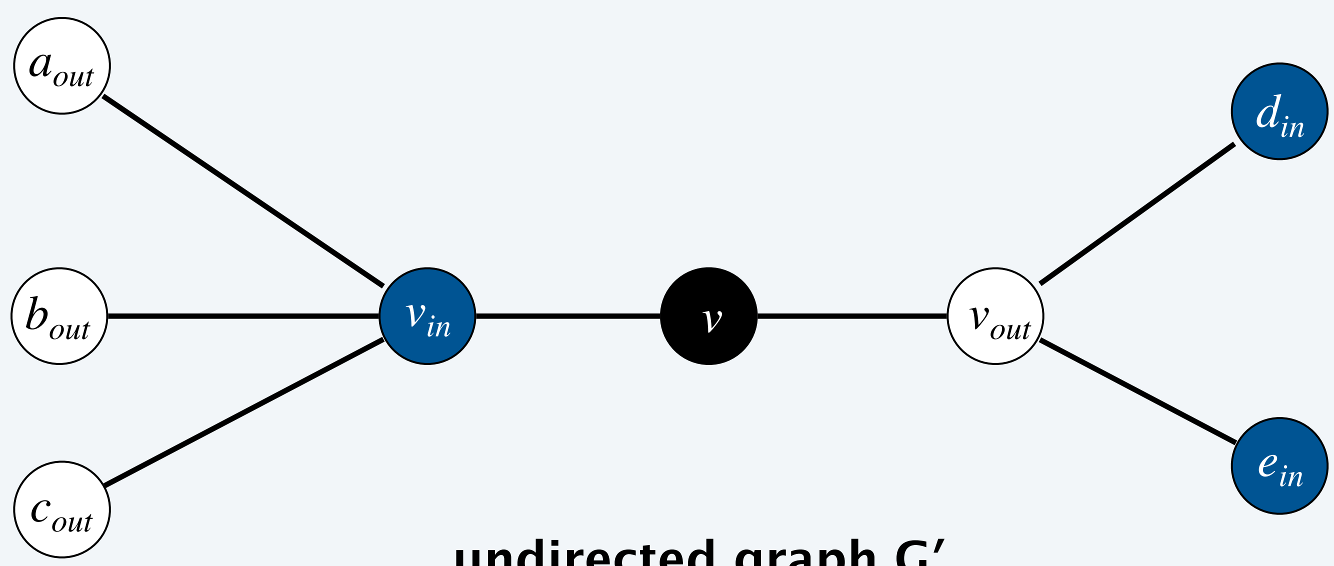
DIRECTED-HAMILTON-CYCLE. Given a directed graph $G = (V, E)$, does there exist a directed cycle $\Gamma$ that visits every node exactly once?

Theorem. DIRECTED-HAMILTON-CYCLE $\le_P$ HAMILTON-CYCLE.

Pf. Given a directed graph $G = (V, E)$, construct a graph $G'$ with $3n$ nodes.




directed graph G




undirected graph G′

# Directed Hamilton cycle reduces to Hamilton cycle

Lemma. $G$ has a directed Hamilton cycle iff $G'$ has a Hamilton cycle.

Pf. $\Rightarrow$

- Suppose $G$ has a directed Hamilton cycle $\Gamma$.
- Then $G'$ has an undirected Hamilton cycle (same order). ▪

Pf. $\Leftarrow$

- Suppose $G'$ has an undirected Hamilton cycle $\Gamma'$.
- $\Gamma'$ must visit nodes in $G'$ using one of following two orders:

  $\ldots, black, white, blue, black, white, blue, black, white, blue, \ldots$

  $\ldots, black, blue, white, black, blue, white, black, blue, white, \ldots$
- Black nodes in $\Gamma'$ comprise either a directed Hamilton cycle $\Gamma$ in $G$, or reverse of one. ▪

# 3-satisfiability reduces to directed Hamilton cycle

Theorem. 3-SAT $\le_P$ DIRECTED-HAMILTON-CYCLE.

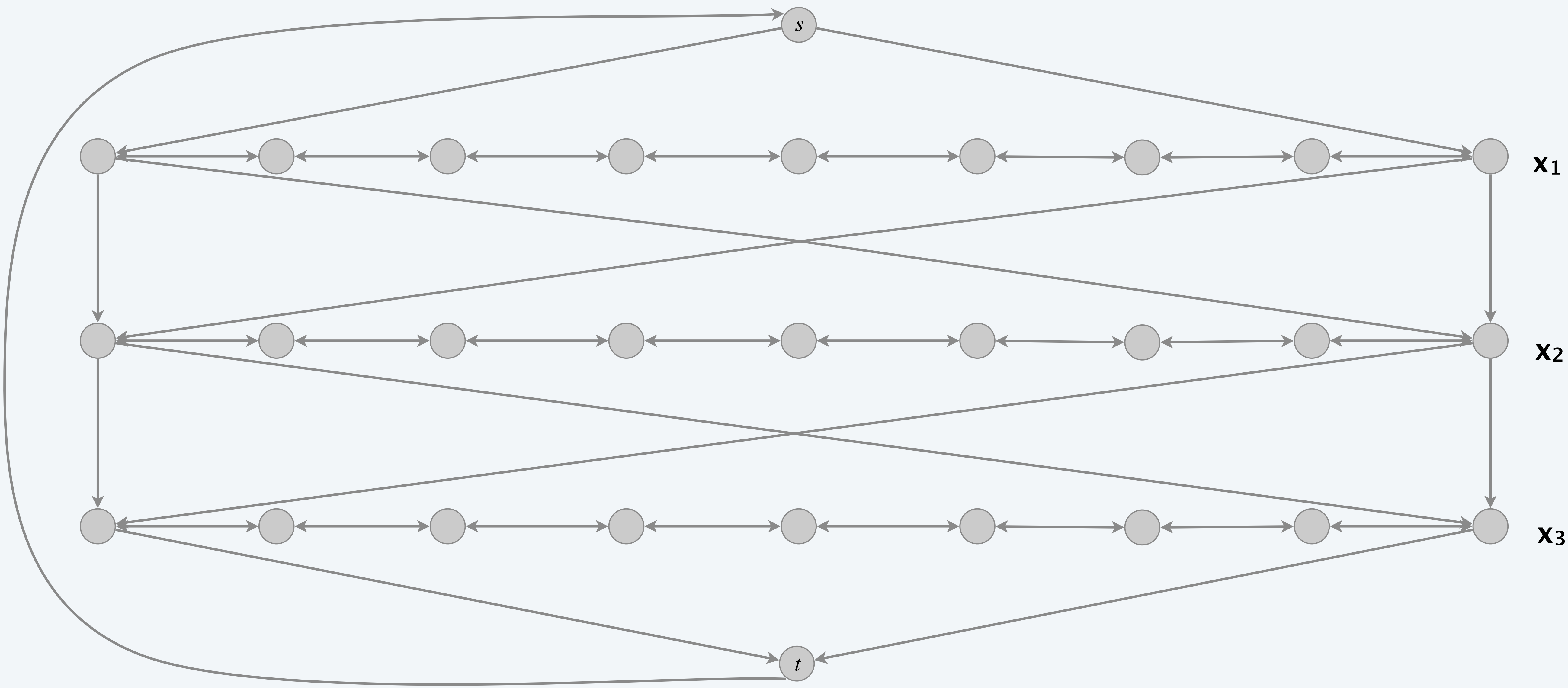
Pf. Given an instance $\Phi$ of 3-SAT, we construct an instance $G$ of DIRECTED-HAMILTON-CYCLE that has a Hamilton cycle iff $\Phi$ is satisfiable.

Construction overview. Let $n$ denote the number of variables in $\Phi$. We will construct a graph $G$ that has $2^n$ Hamilton cycles, with each cycle corresponding to one of the $2^n$ possible truth assignments.

# 3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.

- Construct $G$ to have $2^n$ Hamilton cycles.
- Intuition: traverse path $i$ from left to right $\Leftrightarrow$ set variable $x_i = true$.



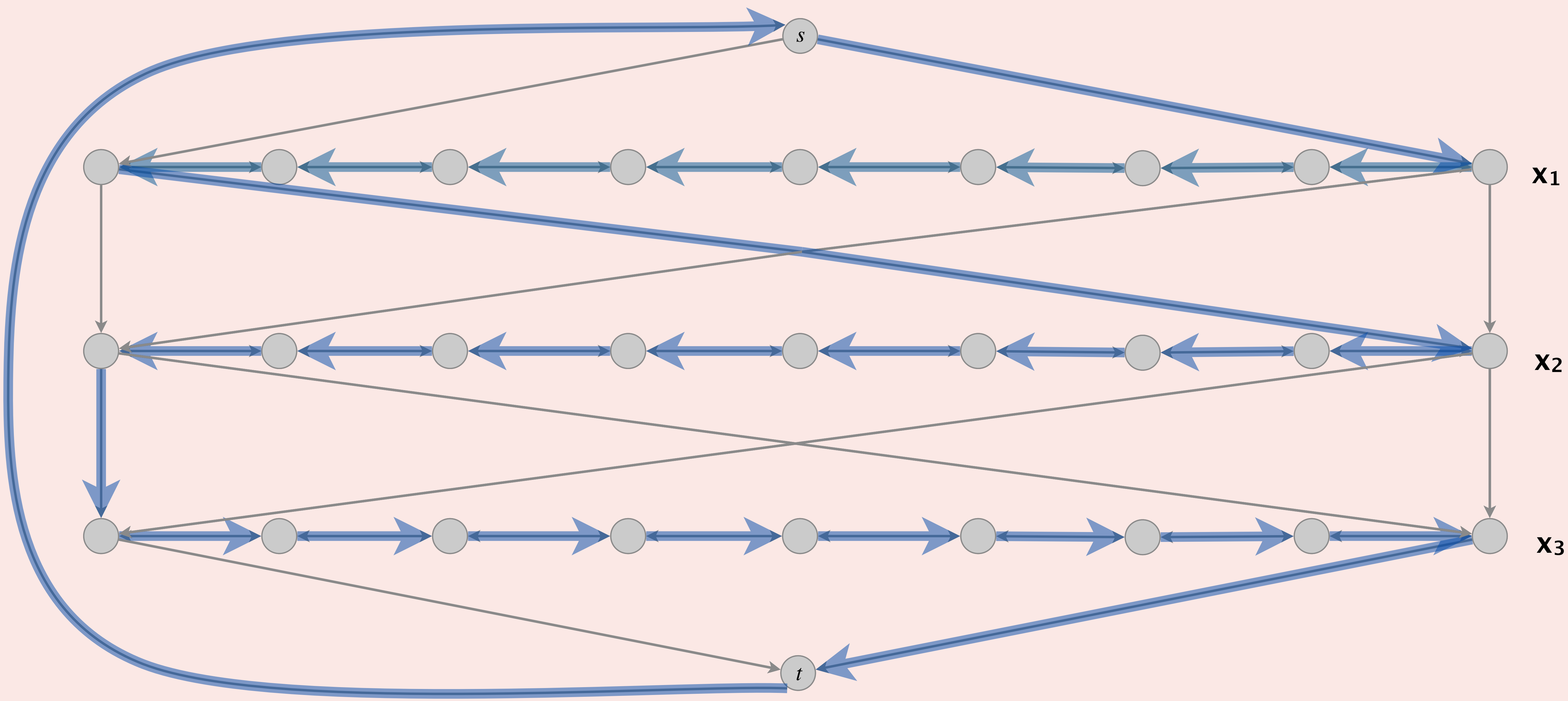
## Which is truth assignment corresponding to Hamilton cycle below?

**A.** $x_1 = true, x_2 = true, x_3 = true$

**B.** $x_1 = true, x_2 = true, x_3 = false$

**C.** $x_1 = false, x_2 = false, x_3 = true$

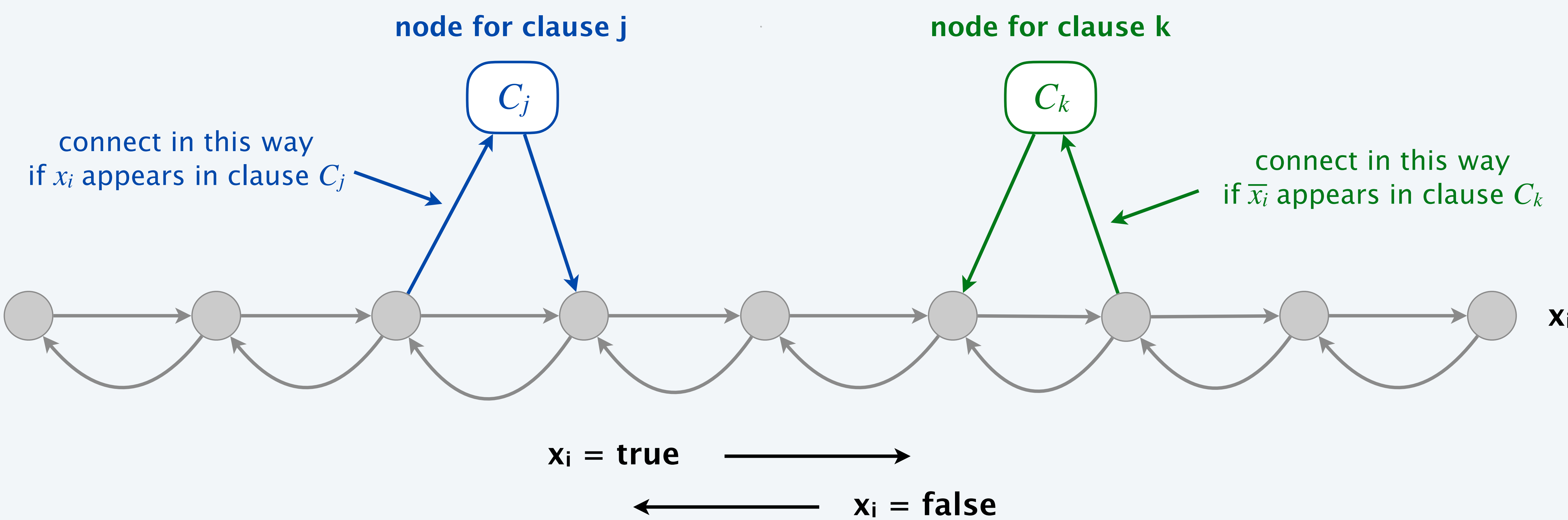
**D.** $x_1 = false, x_2 = false, x_3 = false$



# 3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.

- For each clause: add a node and 2 edges per literal.



# 3-satisfiability reduces to directed Hamilton cycle

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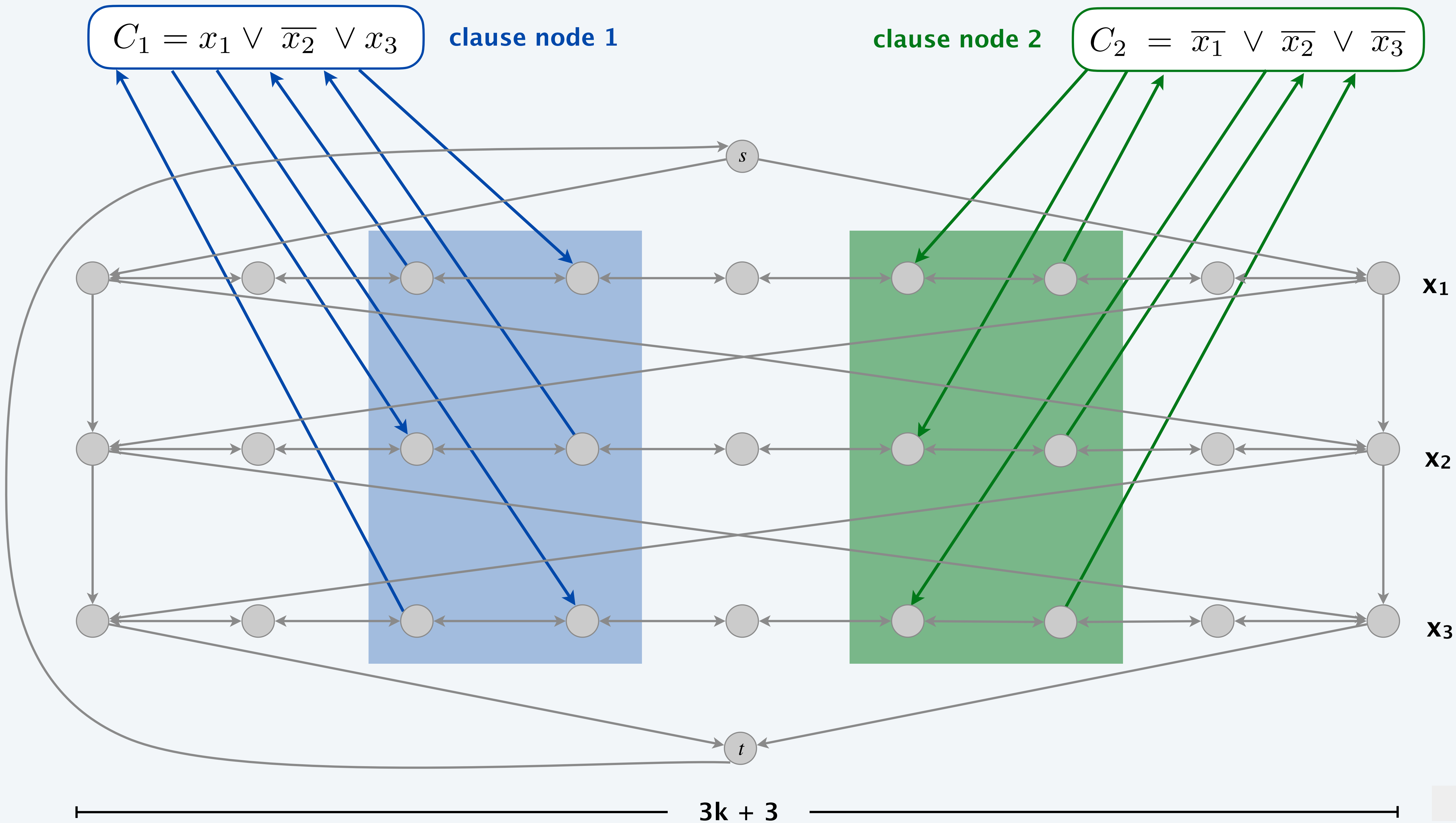


# 3-satisfiability reduces to directed Hamilton cycle

Lemma. $\Phi$ is satisfiable iff $G$ has a Hamilton cycle.

Pf. $\Rightarrow$

- Suppose 3-SAT instance $\Phi$ has satisfying assignment $x^*$.
- Then, define Hamilton cycle $\Gamma$ in $G$ as follows:
  - if $x_i^* = true$, traverse row $i$ from left to right
  - if $x_i^* = false$, traverse row $i$ from right to left
  - for each clause $C_j$, there will be at least one row $i$ in which we are going in "correct" direction to splice clause node $C_j$ into cycle (and we splice in $C_j$ exactly once) ▪

# Traveling salesperson problem

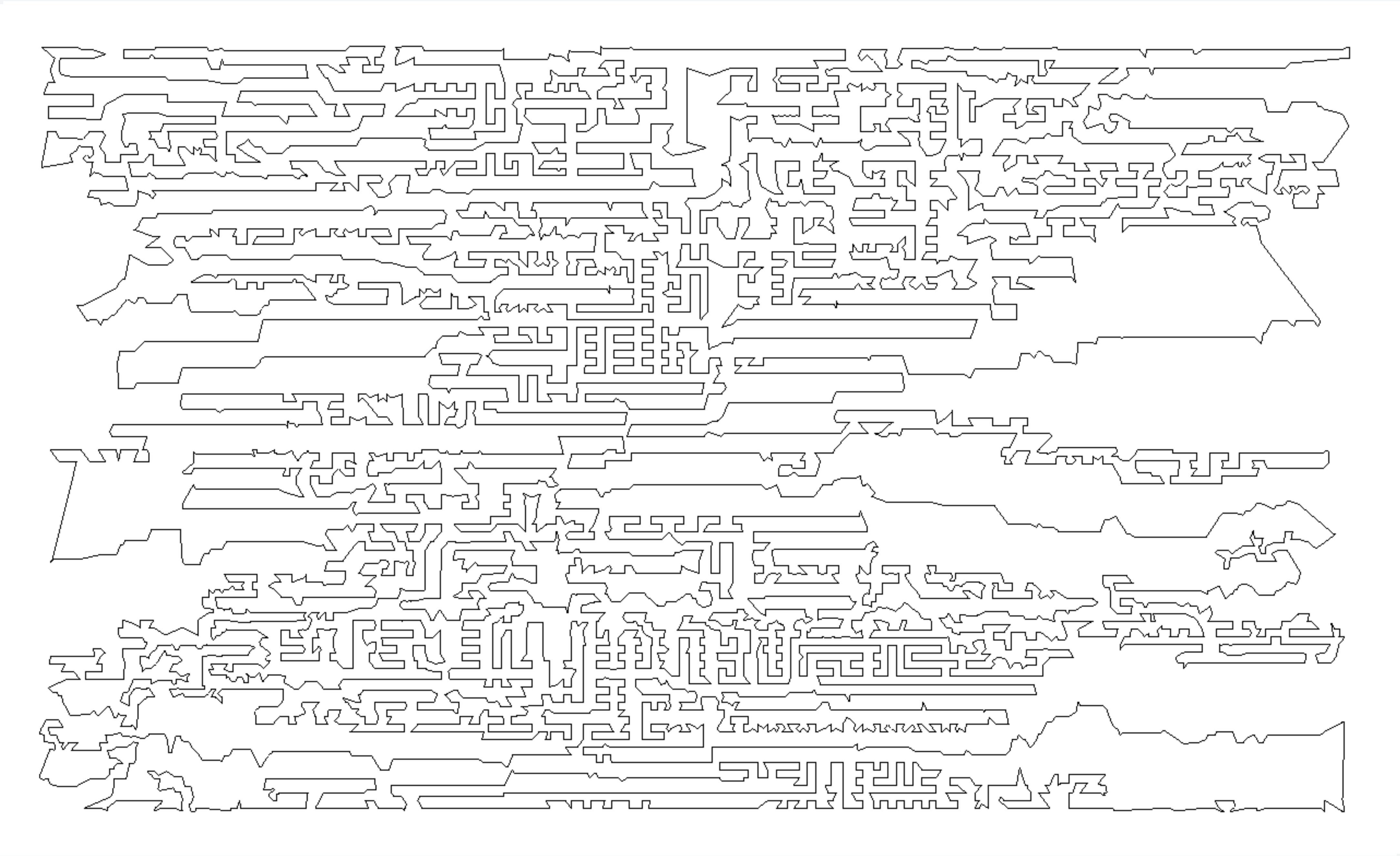
TSP. Given a set of $n$ cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$ ?

can view as a complete graph

13,509 cities in the United States
http://www.math.uwaterloo.ca/tsp

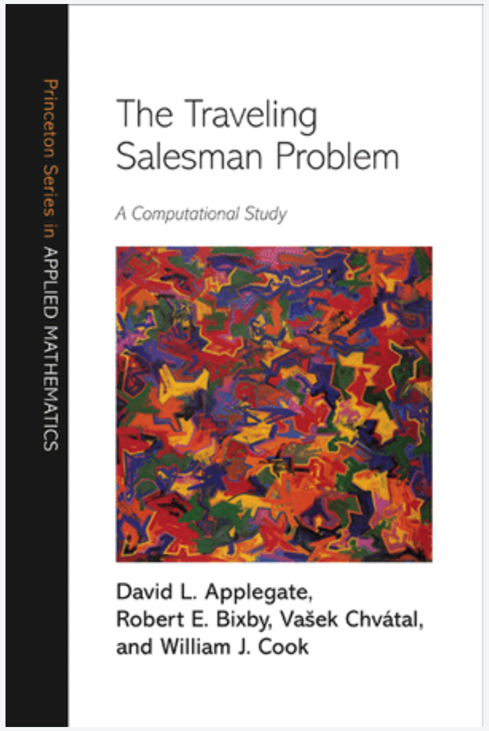
# Traveling salesperson problem

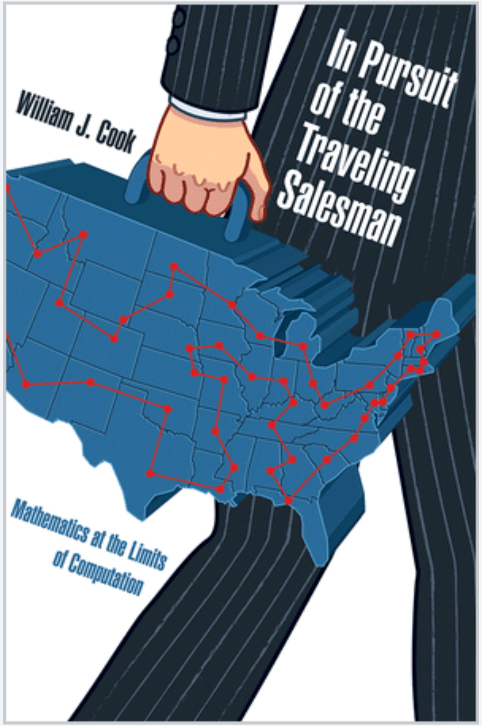
TSP. Given a set of $n$ cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$ ?

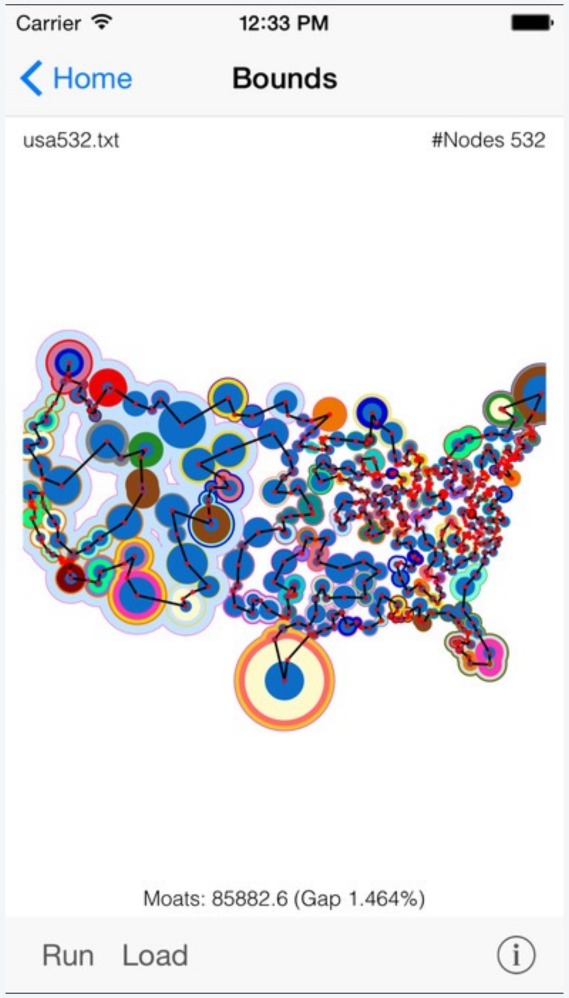


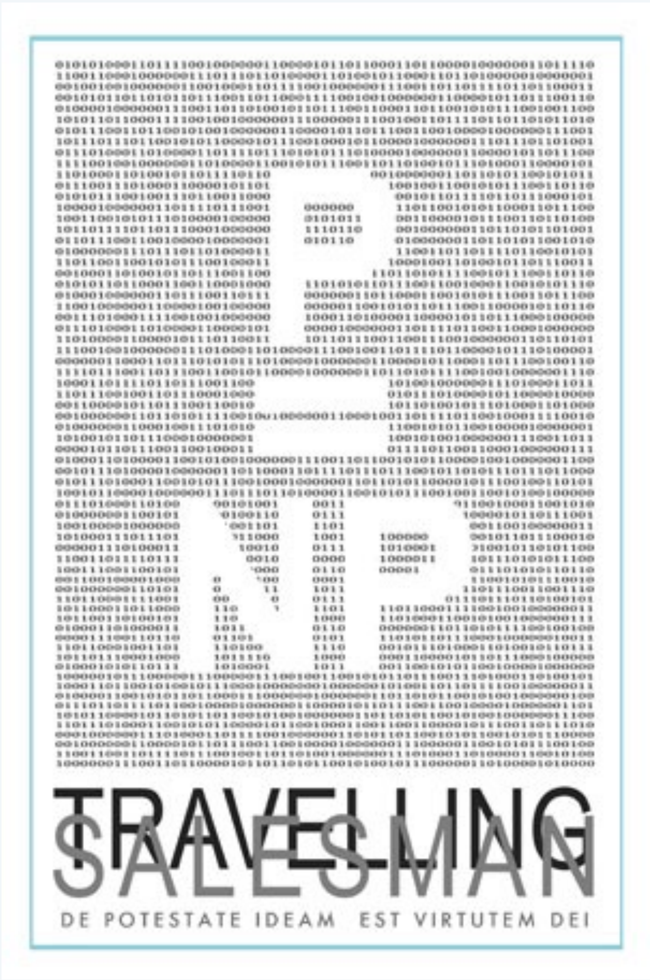
11,849 holes to drill in a programmed logic array
http://www.math.uwaterloo.ca/tsp

# TSP books, apps, and movies



# Hamilton cycle reduces to traveling salesperson problem

TSP. Given a set of $n$ cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$ ?

HAMILTON-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a cycle that visits every node exactly once?

Theorem. HAMILTON-CYCLE $\leq_P$ TSP.

Pf.

- Given an instance $G = (V, E)$ of HAMILTON-CYCLE, create $n = |V|$ cities with distance function

$$d(u,v) \;=\; \begin{cases} 1 & \text{if } (u,v) \in E \\ 2 & \text{if } (u,v) \notin E \end{cases}$$

- TSP instance has tour of length $\leq n$ iff $G$ has a Hamilton cycle. ▪

# Exponential algorithm for TSP: dynamic programming

**Theorem.** [Held–Karp, Bellman 1962] TSP can be solved in $O(n^2\, 2^n)$ time.

HAMILTON-CYCLE is a special case

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## A DYNAMIC PROGRAMMING APPROACH TO SEQUENCING PROBLEMS*

MICHAEL HELD† AND RICHARD M. KARP†

### INTRODUCTION

Many interesting and important optimization problems require the determination of a best order of performing a given set of operations. This paper is concerned with the solution of three such *sequencing problems*: a scheduling problem involving arbitrary cost functions, the traveling-salesman problem, and an assembly-line balancing problem. Each of these problems has a structure permitting solution by means of recursion schemes of the type associated with dynamic programming. In essence, these recursion schemes permit the problems to be treated in terms of *combinations*, rather than *permutations*, of the operations to be performed. The dynamic programming formulations are given in §1, together with a discussion of various extensions such as the inclusion of precedence constraints. In each case the proposed method of solution is computationally effective for problems in a certain limited range. Approximate solutions to larger problems may be obtained by solving sequences of small derived problems, each having the same structure as the original one. This procedure of successive approximations is developed in detail in §2 with specific reference to the traveling-salesman problem, and §3 summarizes computational experience with an IBM 7090 program using the procedure.

## Dynamic Programming Treatment of the Travelling Salesman Problem*

RICHARD BELLMAN

*RAND Corporation, Santa Monica, California*

*Introduction*

The well-known travelling salesman problem is the following: "A salesman is required to visit once and only once each of $n$ different cities starting from a base city, and returning to this city. What path minimizes the total distance travelled by the salesman?"

The problem has been treated by a number of different people using a variety of techniques; cf. Dantzig, Fulkerson, Johnson [1], where a combination of ingenuity and linear programming is used, and Miller, Tucker and Zemlin [2], whose experiments using an all-integer program of Gomory did not produce results in cases with ten cities although some success was achieved in cases of simply four cities. The purpose of this note is to show that this problem can easily be formulated in dynamic programming terms [3], and resolved computationally for up to 17 cities. For larger numbers, the method presented below, combined with various simple manipulations, may be used to obtain quick approximate solutions. Results of this nature were independently obtained by M. Held and R. M. Karp, who are in the process of publishing some extensions and computational results.

# Exponential algorithm for TSP: dynamic programming

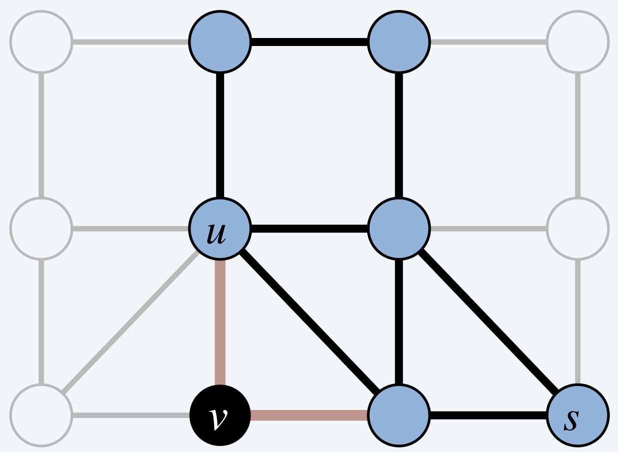
**Theorem.** [Held–Karp, Bellman 1962] TSP can be solved in $O(n^2\, 2^n)$ time.

**Pf.** [dynamic programming]

pick node $s$ arbitrarily

- Subproblems: $c(s, v, X)$ = cost of cheapest path between $s$ and $v \neq s$ that visits every node in $X$ exactly once (and uses only nodes in $X$).
- Goal: $\min_{v \in V} \; c(s, v, V) + c(v, s)$
- There are $\leq n\, 2^n$ subproblems and they satisfy the recurrence:

$$c(s, v, X) = \begin{cases} c(s, v) & \text{if } |X| = 2 \\ \min\limits_{u \in X \setminus \{s, v\}} c(s, u, X \setminus \{v\}) + c(u, v) & \text{if } |X| > 2. \end{cases}$$




- The values $c(s, v, X)$ can be computed in increasing order of the cardinality of $X$. ▪

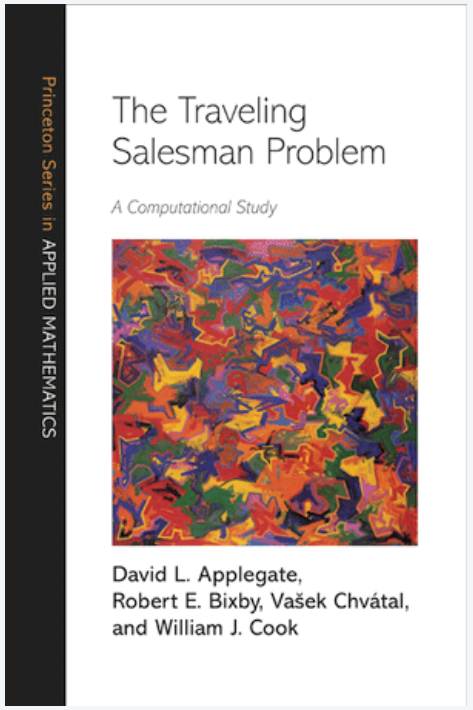
# Concorde TSP solver

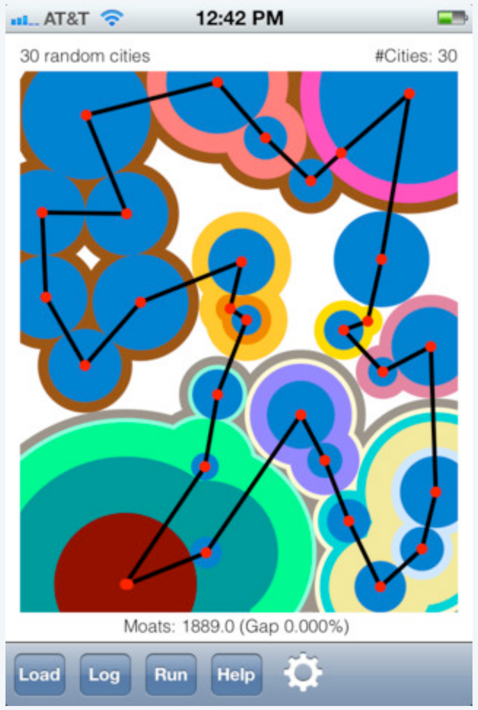
Concorde TSP solver. [Applegate–Bixby–Chvátal–Cook]

- Linear programming + branch-and-bound + polyhedral combinatorics.
- Greedy heuristics, including Lin–Kernighan.
- MST, Delaunay triangulations, fractional $b$-matchings, ...

Remarkable fact. Concorde has solved all 110 TSPLIB instances.

largest instance has 85,900 cities!



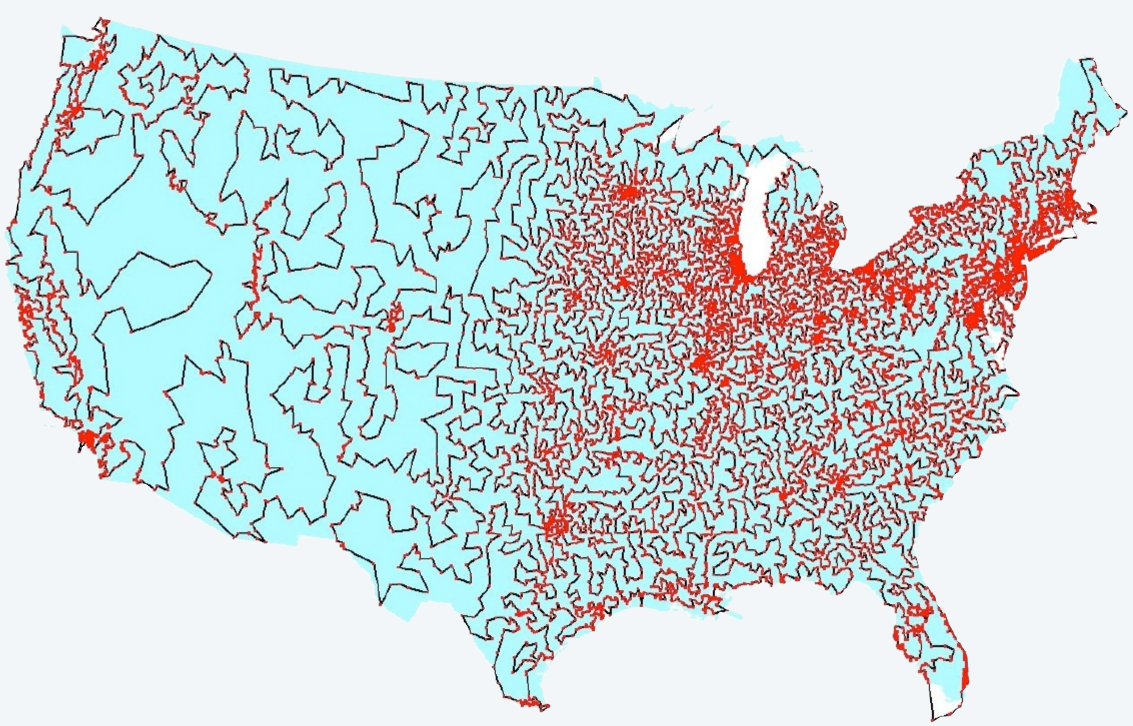



# Euclidean traveling salesperson problem

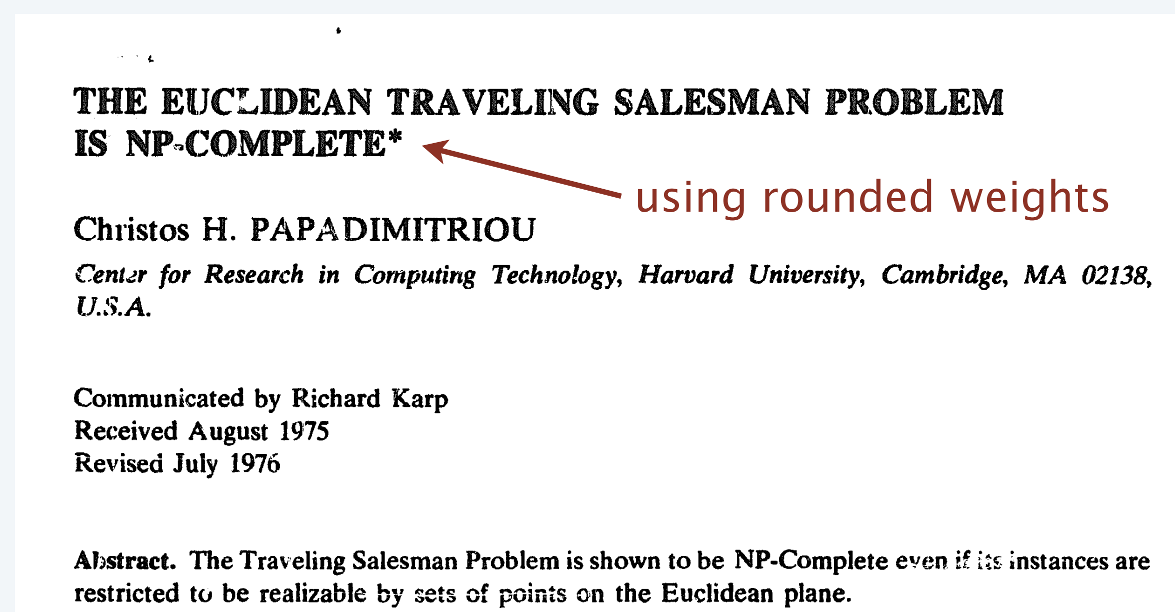
**Euclidean TSP.** Given $n$ points in the plane and a real number $L$, is there a tour that visit every city exactly once that has distance $\leq L$?

**Fact.** 3-SAT $\leq_P$ EUCLIDEAN-TSP.

**Remark.** Not known to be in **NP**.



13509 cities in the USA and an optimal tour



**THE EUCLIDEAN TRAVELING SALESMAN PROBLEM IS NP-COMPLETE***

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Communicated by Richard Karp
Received August 1975
Revised July 1976

**Abstract.** The Traveling Salesman Problem is shown to be NP-Complete even if its instances are restricted to be realizable by sets of points on the Euclidean plane.

using rounded weights