Tutorial on Power Series techniques and special functions

Homework.

1. Solve the Bessel's equation of order p by Frobenius method

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

2. Use Frobenius method to solve

$$2x^2y'' + x(2x+1)y' - y = 0.$$

Tutorial

1.) Find ordinary and singular (regular/irregular) points of the following differential equations.

a) $x^2(x^2-1)^2y'' - x(1-x)y' + 2y = 0$

b) $x^4y'' + (\sin x)y = 0.$

c) $x^2y'' + (\sin x)y' + (\cos x)y = 0.$

2.) Find the roots of the indicial equation of the following differential equation about $x = 0$.

a) $x^3y'' + (\cos 2x - 1)y' + 2xy = 0.$

b) $4x^2y'' - 4x \cdot e^x \cdot y' + 3\cos x \cdot y = 0$

3.) Show that $x=0$ is an irregular singular point for the following differential equation

$$y'' + \frac{1}{x^2} y' - \frac{1}{x^3} y = 0,$$ also find the general

Solution

$$\hookrightarrow \equiv x^3 y'' + x y' - y = 0.$$

4.) Find the general solution of the following differential equation about point $x=0$.

$$(1-x^2) y'' - 2x y' + 2y = 0.$$

5.) Find a solution of the following differential equation about point $x=0$.

$$x^2 y'' + x y' + (x^2 - p^2) y = 0,$$ where $p \geq 0$ is a real number.

6.) Show that the indicial equation has only one root for $x^2 y'' + x y' + x^2 y = 0$ and that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$ is a corresponding particular soln

$\uparrow$

$J_0(x)$ - Bessel function of first kind of order 0.

7.) Find the first three terms of the legendre's Series of the function

$$f(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \end{cases}$$

8.) Let $P_n(x)$ be the nth legendre polynomial, where $n \geq 0$ is any integer. Prove the following

a) $P_n(1) = 1$ b) $P_n'(1) = \frac{1}{2} n(n+1)$

9.) Let y be a polynomial solution of the differential equation $(1-x^2)y'' - 2xy' + 12y = 0$. If $y(1) = 2$, then find the value of the integral $\int_{-1}^{1} y^2 \, dx$

10.) Suppose the legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ has an nth degree polynomial solution $y_n(x)$ such that $y_n(1) = 3$. If $\int_{-1}^{+1} \left(y_n^2(x) + y_{n-1}^2(x)\right) dx = \frac{144}{15}$, then find the value of n.