Calculus Midterm :-

Name:- O.V.S. lokesh Reddy
Roll NO:- EC20B1050

1) Given

$$\lim_{n\to\infty}(\sqrt{n^2+n}-n)$$

By rationalizing

$$\lim_{n\to\infty}(\sqrt{n^2+n}-n)\frac{(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} \Rightarrow \lim_{n\to\infty}\frac{(\cancel{n^2}+n-\cancel{n^2})}{\sqrt{n^2+n}+n}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}(\sqrt{1+1/n}+1)} \Rightarrow \lim_{n\to\infty}\frac{1}{1+\sqrt{1+1/n}} \Rightarrow \frac{1}{\sqrt{1+0}+1} \quad [\text{As } \lim_{n\to\infty}\frac{1}{n}=0]$$

$$\Rightarrow 1$$

$$\therefore \lim_{n\to\infty}\sqrt{n^2+n}-n=1$$

By definition of convergence that

$a_n=\sqrt{n^2+n}-n$ and $l=1$

let $\epsilon>0$ be given, we must show that there exists a fixed integer N such that for all n

$$n\geq N \Rightarrow \left|\sqrt{n^2+n}-n-1\right|<\epsilon$$

we note that

$$\sqrt{n^2+n}<\epsilon+n+1$$

squaring

$$n^2+n<\epsilon^2+(n+1)^2+2\epsilon n+2\epsilon$$

$$\cancel{n^2}+n<\epsilon^2+\cancel{n^2}+1+2n+2\epsilon n+2\epsilon$$

$$\epsilon^2+1+n+2\epsilon n+2\epsilon>0$$

$$n>\frac{-(\epsilon^2+1+2\epsilon)}{(1+2\epsilon)} \Rightarrow n>\frac{-(\epsilon+1)^2}{(1+\epsilon 2)}$$

Thus, if N is any integer greater than $\frac{-(\epsilon+1)^2}{1+2\epsilon}$, then above implication hold for all integers $n\geq N$//

2) Given

Name:- O.V.S.Lokesh Reddy
Roll No:- E(20B1050

$a_1 = 10$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{10}{a_n}\right)$ for $n \geq 1$

let The sequence $\{a_n\}$ be converges to l

then $\lim_{n\to\infty} a_n = l$ we can also write $\lim_{n\to\infty} a_{n+1} = l$

so, Given $\lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{10}{a_n}\right) = l$

$$\lim_{n\to\infty} a_n + \frac{10}{\lim_{n\to\infty} a_n} = 2l$$

$\Rightarrow$

$$l + \frac{10}{l} = 2l \Rightarrow l^2 + 10 = 2l^2 \Rightarrow l = \sqrt{10}$$

Here $\{a_n\}$ is decresing sequence (+ve numbers).
As limit exist for a_n, The $\{a_n\}$ sequence converges
limit of $a_n \Rightarrow \lim_{n\to\infty} a_n = \sqrt{10}$//

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3) Given sequence $\{a_n\}$, where $a_1 = 2$ and

$a_{n+1} = 2 + \frac{a_n}{a_{n+1}}$, For $n \geq 1$;

$$\Rightarrow a_{n+1}^2 = 2a_{n+1} + a_n \Rightarrow a_n = (a_{n+1})^2 - 2a_{n+1}$$

let The sequence $\{a_n\}$ be converges to l then $\lim_{n\to\infty} a_n = l$
we can also write $\lim_{n\to\infty} a_{n+1} = l$

so, $\lim_{n\to\infty} a_n \Rightarrow \lim_{n\to\infty} (a_{n+1})^2 - 2a_{n+1} = l$

$$\lim_{n\to\infty} (a_{n+1})^2 - 2\lim_{n\to\infty} a_{n+1} = l \Rightarrow l^2 - 2l = l \Rightarrow l^2 = 3l$$

$\Rightarrow \therefore l = 3$ [As The term of $\{a_n\}$ are positive] $\Rightarrow \lim_{n\to\infty} a_n = 3$//

As limit exist for $\{a_n\}$, Then $\{a_n\}$ sequence is convergence

$$\lim_{n\to\infty} a_n = 3$$//

Name: av.s.lokesh Reddy
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4) Given $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$, where p is a constant

we know $\ln n \le n$

Taking power p on both sides (let p be +ve)

Then $(\ln n)^p < n^p$

$$\frac{1}{(\ln n)^p} > \frac{1}{n^p}$$

Taking ~~limit~~ sigma on both sides

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p} > \sum_{n=2}^{\infty} \frac{1}{n^p}$$

we know $\sum_{n=2}^{\infty} \frac{1}{n^p}$ is Divergent for $0 \le p \le 1$ and convergent for $p > 1$

i) $0 \le p \le 1$

as $\sum \frac{1}{n^p}$ Diverges By comparision test $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ Diverges

ii) $p > 1$

$$\frac{1}{(\ln n)^p} > \frac{1}{n^p} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{(\ln n)^p}}{\frac{1}{n^p}} > 0$$

By limit comparision test as

$\left[\lim_{n\to\infty} \frac{a_n}{b_n} = c\ (c>0) \right.$ then $\sum b_n$ & $\sum a_n$ both diverges (or) $\left. \text{converges} \right]$

Here $\sum \frac{1}{n^p}$ converges

$\sum \frac{1}{(\ln n)^p}$ converges

iii) $p < 0$, $-p$ is +ve so,

$\sum_{n=2}^{\infty} (\ln n)^{-p} \Rightarrow$ This tends to ∞ so,

$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ Diverges

so, $\frac{1}{(\ln n)^p}$ converges for $p > 1$, Diverges for $p \le 1$

5) Given $a_n = \begin{cases} n/2^n & \text{if } n \text{ is prime} \\ 1/2^n & \text{otherwise} \end{cases}$

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we will try using Root test

let $\sum a_n$ be a series with $a_n \geq 0$, for $n \geq N$ and suppose

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

If $\rho < 1 \rightarrow$ converges
$\rho > 1 \rightarrow$ diverges
$\rho = 1 \rightarrow$ The test is inconclusive.

$$\sqrt[n]{a_n} = \begin{cases} \dfrac{n^{1/n}}{2^{n/n}} & \text{if 'n' is prime} \\ \dfrac{1}{2^{n/n}} & \text{if otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{n^{1/n}}{2} & \text{if 'n' is prime} \\ \dfrac{1}{2} & \text{otherwise} \end{cases}$$

$$\therefore \quad \frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{\sqrt[n]{n}}{2}$$

since $\sqrt[n]{n} \rightarrow 1$, we have $\sqrt[n]{a_n} = \frac{1}{2}$ by sandwich theorum

The limit value of $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2}$, $\frac{1}{2} < 1$ so,

$\sum a_n$ converges By n^{th} root test.

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6) Given

Σa_n diverges

Then for $\Sigma |a_n|$ we know that from inequality

$$x \le |x|$$

$$\Sigma x \le \Sigma |x|$$

So, we can write

$$\Sigma a_n \le \Sigma |a_n|$$ we can write

So, By comparision test as Σa_n diverges

$\Sigma |a_n|$ also diverges.

7) Given series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n+1)(n+2)}{2^n}$

By using ratio test $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right|$

Here

$$\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| \Rightarrow \lim_{n\to\infty} \left| \frac{\cancel{(n+2)}(n+3)}{2^n(2)} \times \frac{\cancel{2^n}}{\cancel{(n+2)}(n+1)} \right| = \frac{1}{2} \; (\rho<1)$$

as $\rho<1$, then $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n+1)(n+2)}{2^n}$ converges.

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3) Given two power series $\Sigma a_n x^n$ and $\Sigma b_n x^n$ are convergent and equal for all values of x in $(-c, c)$

let $f(x) = \Sigma a_n x^n$, $g(x) = \Sigma b_n x^n$

let $h(x) = f(x) - g(x)$

$= \Sigma (a_n - b_n) x^n$

Given both are equal for all x in $(-c, c)$ [then ~~there co-efficients must be equal and~~] $f(x) = g(x)$ so, $h(x) = 0$, for all x in $(-c, c)$

as $h(x) = 0$, we can use the fact that coefficients must be zero,

$a_n - b_n = 0$

$\Rightarrow \boxed{a_n = b_n}$

Since the coefficients are equal we have in fact that power series are equal and radius of converged are also equal.

4) Taylor series for $\sin^2 x$

$$\sum_{n=1}^{\infty} \frac{f^n(0)}{n!} x^n = \frac{f(0)}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \ldots$$

So, we need to find derivates

$f(x) = \sin^2 x$

$f'(x) = 2\sin x \cos x = \sin(2x)$

$f''(x) = 2\cos 2x$ $\qquad f'''(x) = -8\cos 2x$

$f''''(x) = -16\sin 2x$

$$\sum_{n=1}^{\infty} \frac{f^n(a)}{n!} x^n = \frac{\sin^2(a)}{0!} + \frac{\sin(2a)}{1!} x + \frac{2\cos 2(a)}{2!} x^2 + \ldots$$

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9) at $x=0$

$$\sum_{n=1}^{\infty} \frac{f(0)}{n!} x^n = 0 + \frac{0x}{1!} + \frac{2x^2}{2} \cdots$$

$$= 0 + 0 + \frac{2}{2!} x^2 - \frac{f(0)}{3!} x^3 \cdots$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45} x^6 \cdots\cdots$$

for $|\sin^2 x| < 1$

Interval of convergent is $-1 \leq x \leq 1$

$\therefore$ Interval of convergence we get $x \in [-1, 1]$