Theorem (The Fundamental Theorem of Calculus)

If f is Riemann integrable on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Proof</u>: Let $\epsilon > 0$ be given. given f is R.I

Choose a partition $P = \{x_0, x_1, \cdots x_n\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \epsilon$

The mean value theorem implies that there is a t_i in $[x_{i-1}, x_i]$ such that

$$\underbrace{\frac{F(x_i) - F(x_{i-1})}{\Delta x_i} = F'(t_i)}_{\text{M.V.T}} \underbrace{= f(t_i)}_{\text{given } F' = f} \text{ (given)}$$

$$\Rightarrow \quad F(x_i) - F(x_{i-1}) = f(t_i) \cdot \Delta x_i \quad (1 \le i \le n)$$

Thus $\displaystyle\sum_{i=1}^{n} f(t_i) \cdot \Delta x_i = \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right)$

$$= F(b) - F(a)$$

Thus

$$\underbrace{\sum_{i=1}^{n} f(t_i)\cdot \Delta x_i}_{\text{normal Riemann sum.}} = F(b) - F(a) \quad \text{— (3)}$$

We also note that

$$\underset{\substack{\text{always}\\ \text{lower values}\\ \text{in ith interval}}}{L(P,f)} \le \sum_{i=1}^{n} f(\underset{\substack{t_i \text{ is a}\\ \text{point} \in [x_{i-1}, x_i]}}{t_i})\cdot \Delta x_i \le \underset{\substack{\text{always upper}\\ \text{values.}}}{U(P,f)} \quad (\text{why?})$$

also $$L(P,f) \le \int_a^b f(x)\,dx \le U(P,f)$$

L ——— in between ——— U

$$\int_a^b f(x)\,dx \ \overset{>}{\underset{<}{\text{or}}} \ \sum_i f(t_i)\cdot \Delta x_i$$

So therefore

$$\left| \sum_{i=1}^{n} f(t_i)\cdot \Delta x_i - \int_a^b f(x)\,dx \right| < U(P,f) - L(P,f) < \varepsilon$$

$$\Rightarrow \quad \left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon \quad (\text{from } (3))$$

Since this holds for every $\varepsilon > 0$, we have

$$\boxed{F(b) - F(a) = \int_a^b f(x)\,dx} \rightarrow \text{Hence Proved.}$$

$\boxed{\text{Second part}}$ → Slide 58

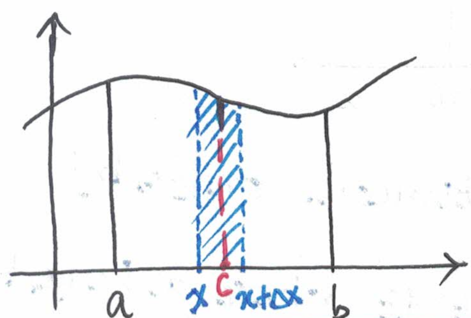
f is R.I on $[a,b]$. For $a \le x \le b$ if $F(x) = \int_a^x f(x)\,dx$ and

Proof:

If f is continuous on $[a, b]$ and given

$$\underbrace{F(x) = \int_a^x f(t)\, dt}_{(1)} \quad \text{where } a \le x \le b$$

then $F'(x) = f(x)$



what $F'(x)$ as per definition of derivative

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\boxed{\int_a^{x+\Delta x} f(t) - \int_a^{x} f(t)\, dt}}{\Delta x} \quad \text{(as per (1), given)}$$

area under curve $y = f(t)$ between x and $x+\Delta x$

(use the concept of Riemann sum)

$$= \lim_{\Delta x \to 0} \underbrace{\frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)\, dt}$$

mean value theorem of definite integrals states that there exists a 'c' ($\boxed{x \le c \le x+\Delta x}$) such that

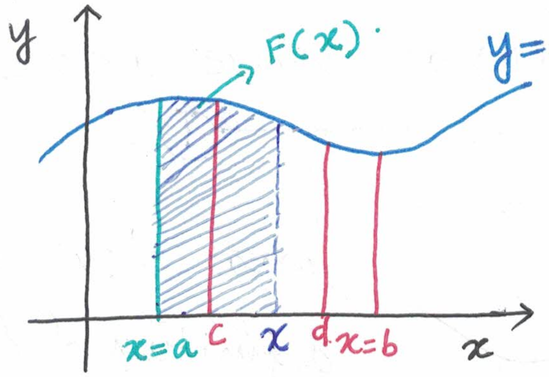
$$f(c) \cdot \Delta x = \int_x^{x+\Delta x} f(t)\, dt$$

$$\lim_{\Delta x \to 0} \frac{f(c) \cdot \Delta x}{\Delta x} = f(x) \quad \text{(why?)}$$

as $\Delta x \to 0 \Rightarrow c \to x$ $\quad \therefore F'(x) = f(x)$

Relevance of Fundamental theorem of Calculus

Part 1:



function $f(x)$ is continuous on $[a, b]$

Suppose $F(x) = \int_a^b f(t)\,dt$, where x in (a, b)

Then, First $\boxed{\text{FTC}}$ Fundamental theorem of calculus states

that $\dfrac{dF}{dx} = F'(x) = \dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$

- Every continuous function has an antiderivative
- connection between differentiation & integration

$$F(x) = \int_a^x f(t)\,dt \Rightarrow \underset{1^{st}\ \text{FTC}}{F'(x) = f(x)}$$

$\boxed{\text{Second FTC}}$

$$F(d) - F(c) = \int_a^d f(t)\,dt - \int_a^c f(t)\,dt = \boxed{\int_c^d f(t)\,dt}$$

Second FTC helps to evaluate definite integrals

difference in anti-derivative values at the limits.