Big-oh Notation

'O' $f(n), g(n)$ Non-negative

$\downarrow$
Step Count

$f(n) = O(g(n))$ iff $\forall_{n \geq n_0} \exists_{c > 0} (f(n) \leq c \cdot g(n))$

$f(n) = O(g(n))$ iff $\exists_{n_0 > 0} \exists_{c > 0} \forall_{n \geq n_0} (f(n) \leq c \cdot g(n))$

$f(n) = 3n+6$ $\quad 3n+6 \leq 5n \quad c=5, \forall n \geq 3, n_0 = 3 \Rightarrow 3n+6 = O(n)$

$3n+6 \leq 15n \quad c=15, n_0 = 1, \forall n \geq 1 \Rightarrow 3n+6 = O(n)$

$f(n) = n^2+5n-2$ $\quad n^2+5n-2 \leq 10n^2 \quad c=10, n_0=1, \forall n \geq 1 \Rightarrow n^2+5n-2 = O(n^2)$

$n^2+5n-2 \leq 3n^2 \quad c=3, n_0=2, \forall n \geq 2 \Rightarrow n^2+5n-2 = O(n^2)$

$n^2+5n-2 \leq 3n^3 \quad c=3, n_0=2, \forall n \geq 2 \Rightarrow O(n^3)$

$n^2+5n-2 \leq 2 \cdot n^{10} \quad c=2, n_0=2, \forall n \geq 2 \Rightarrow O(n^{10})$

$n^2+5n-2 \leq 2^n \quad c=1, n_0=7, \forall n \geq 7 \Rightarrow O(2^n)$

Big-oh Notation

$f(n),\ g(n)$

$f(n) = O(g(n))$ iff $\exists c>0\ \exists n_0>0\ \forall n>n_0\ (f(n) \le c \cdot g(n))$

$3n^2+4n-2 \le 6n^2 \quad c=6,\ n_0=3,\ \forall n \ge 3$

$\le 25n^2 \quad c=25,\ n_0=6,\ \forall n \ge 6$

$n_0=1 \quad c=1$

$3n^2+4n-2 \le 1 \cdot n^2$

$c=1,\ n_0=2, \Rightarrow 3*4+4*2-2 \le 4$ ✗

$n_0=3 \Rightarrow 3*9+4*3-2 \le 9$ ✗

$c=4,\ n_0=1 \Rightarrow 3n^2+4n-2 \le 4 \cdot n^2$

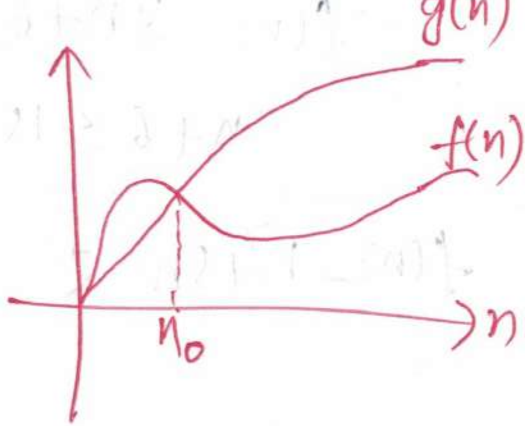
$\hookrightarrow 3+4-2 \le 4$ ✗

$n_0=2 \Rightarrow 3*4+4*2-2 \le 4*4$ ✓

$3n^2+4n-2 \not\le 0.5\,n^2$ for any n_0

$3n^2+4n-2 \not\le 2n^2$

$3n^2+4n-2 \le (3+\epsilon)n^2$ for some $\epsilon>0$ & $n_0>0$



Big-oh Notation

$$3n^2 + 4n - 2 \le 3 \cdot n^3 \quad c=3,\ n_0=4,\ \forall n \ge 4$$

$$\le 10 \cdot 2^n \quad c=10,\ n_0=3,\ \forall n \ge 3$$

$O(n^{2+\epsilon})$, $\epsilon \ge 0$ $= 3n^2+4n-2 = O(n^2)$

$O(c^n), c>1 =$ $\quad = O(n^{2.1})$

$\quad = O(n^3)$

$O(1.1^n) =$ $\quad = O(n^{10})$

$O(1.001^n) =$ $\quad = O(2^n)$

Is $3n^2+4n-2 = O(n)$??

$$3n^2+4n-2 \le c \cdot n$$

$$3n+4-\frac{2}{n} \le c$$

$\exists c>0\ \exists n_0>0\ \forall n \ge n_0$

$c = 10^9 \quad n_0 = 10^{12} \quad \forall n \ge 10^{12}$

$$3n+4-\frac{2}{n} \not\le 10^9$$

$3n^2+4n-2 \ne O(n)$

$\ne O(n^{1.5})$

$\ne O(n^{2-\epsilon})$, $\epsilon > 0$

Big-oh Notation

$5n^3 + 6n^2 \cdot 2^n - 100 \le 10 \cdot n^2 \cdot 2^n$ $c = 10$ $n_0 = 4$

$= O(n^2 \cdot 2^n)$

$= O(n^3 \cdot 2^n)$

$= O(n^2 \cdot 3^n)$

$= O(n^{2+\epsilon} \cdot (2+\delta)^n)$

$\epsilon \ge 0, \delta \ge 0$

$\ne O(2^n)$

$\ne O(n \cdot 2^n)$

$= O(3^n)$

$= O(2^n)$

$= O(2.001^n)$

$n \log n + 10 = O(n \log n)$

$= O(n^2)$

$= O(n^{1.1})$

$= O(2^n)$

$\ne O(n)$

$= O(n \log n \log n)$

$\ne O(n \log \log n)$

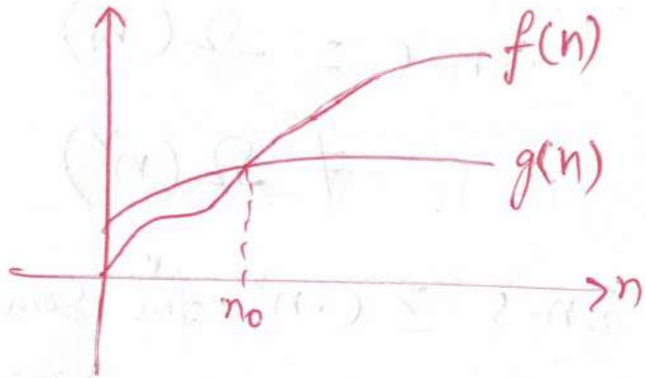
Big-Omega Notation

$$f(n) = \Omega(g(n)) \text{ iff } \exists c>0 \; \exists n_0>0 \; \forall n \geq n_0 \; (f(n) \geq c \cdot g(n))$$

$3n-5 \geq 2 \cdot n \quad c=2, n_0=5 \Rightarrow 3n-5 = \Omega(n)$

$3n-5 \geq 0.5\, n$

$3n-5 \geq 2 \log_2 n \quad \Rightarrow \Omega(\log_2 n)$



$4n^2 - n + 2 \geq 2 \cdot n^2 \quad c=2, n_0=2$

$= \Omega(n^2)$

$\geq 0.001 n^2 \Rightarrow \Omega(n^2)$

$\geq 10 \cdot n \Rightarrow \Omega(n)$

$\geq 5\sqrt{n} \Rightarrow \Omega(\sqrt{n})$

$\geq 6 \cdot n^{1.9} \Rightarrow \Omega(n^{1.9})$

$= \Omega(n^{2-\epsilon})$

$\epsilon \geq 0$

$= \Omega(\log n)$

$= \Omega(n \log n)$

Big-Omega Notation

$$f(n) = \Omega(g(n)) \text{ iff } \exists_{c>0}\ \exists_{n_0>0}\ \forall n \geq n_0\ (f(n) \geq c \cdot g(n))$$

$3n-5 = \Omega(n)$

$\neq \Omega(n^2)$

$3n-5 \geq c \cdot n^2$ for some $c>0$ $\forall n \geq n_0$

$\frac{3}{n} - \frac{5}{n^2} \geq c$ Suppose $c = 10^{-6}$ choose $n_0 = 10^9$ $n \geq n_0$

The above Inequality is False

$3n-5 \neq \Omega(2^n)$

$\neq \Omega(n \log n)$

$6n^2 - n + 100 = \Omega(n^2)$

$= \Omega(n)$

$= \Omega(\log n)$

$= \Omega(n^{2-\epsilon})$ ($\epsilon \geq 0$)

$\neq \Omega(n^3)$

$\neq \Omega(n^{2+\delta})$ $\delta > 0$

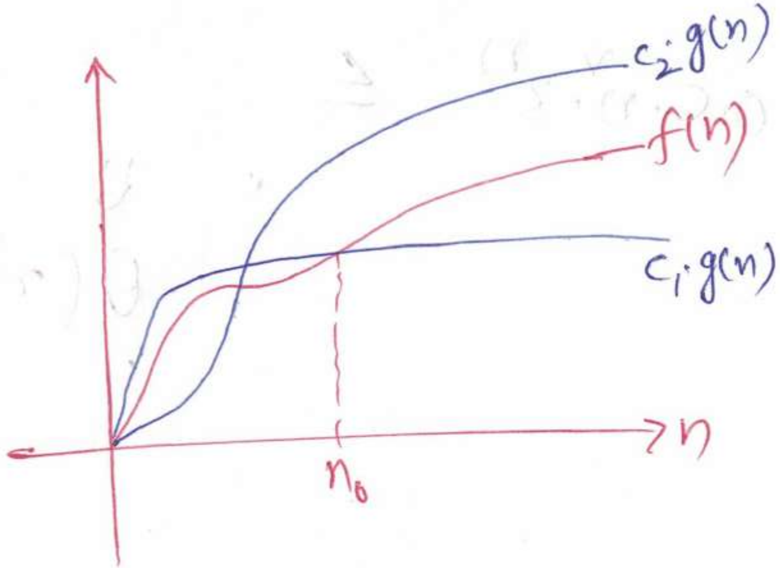
$\neq \Omega(c^n)$ $c > 1$

Tight Bounds (Theta Notation)

$f(n),\ g(n)$

$f(n) = \Theta(g(n))$ iff $\exists c_1 > 0\ \exists c_2 > 0\ \exists n_0 > 0\ \forall n \geq n_0\ \left(c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\right)$

$f(n) = \Theta(g(n))$ iff $f(n) = \Omega(g(n))$
and
$f(n) = O(g(n))$



$2n \leq 3n+2 \leq 4n \quad \forall n \geq 2 \quad c_1 = 2\ c_2 = 4$

$\Rightarrow 3n+2 = \Theta(n)$

$\underbrace{0.5n^2 \leq n^2-3n+100}_{\Omega(n^2)}$, $\underbrace{n^2-3n+100 \leq 4n^2}_{O(n^2)} \quad \forall n \geq 6 \quad c_1 = 0.5\ c_2 = 4$

$\Rightarrow n^2-3n+100 = \Theta(n^2)$

Tight Bounds (Theta Notation)

$$2^n + n^{100} + n^2 \cdot 3^n$$

$$0.5 \cdot n^2 \cdot 3^n \leq \quad \leq 6 \cdot n^2 \cdot 3^n$$

$$\Downarrow$$

$$\Theta(n^2 \cdot 3^n)$$

$2^n = O(3^n)$

$\neq O(1.5^n)$

$2^n = O(n^2 \cdot 3^n)$

$n^{100} = O(3^n)$

$n^{100} = O(n^2 \cdot 3^n)$

Insight into Asymptotic Notation

$$3n+5 = O(n)$$
$$= O(n^2)$$
$$= O(n^3)$$

$$3n+5 = O(n^2)$$

$3n+5 \le c \cdot n^2 \ \exists c>0 \ \exists n_0>0 \ \forall n \ge n_0$

$3n+5 \le c \cdot n \cdot n \ \forall c>0 \ \exists n_0>0 \ \forall n \ge n_0$

$3n+5 \le 0.01 n^2 \quad c=0.01, n_0=1000$

$3n+5 \le 0.0001 n^2 \quad n_0 = 10^6$

work for some $c>0$ $\rightarrow$ $3n+5 = O(n)$

work for any $c>0$ $\rightarrow$ $\left\{ \begin{array}{l} 3n+5 = O(n^{1.1}) \\ \quad = O(n^{1+\epsilon}) \ \epsilon>0 \end{array} \right.$

Little oh [Loose upper bounds]

$f(n) = o(g(n))$ iff $\forall_{c>0} \ \exists_{n_0>0} \ \forall_{n \ge n_0} (f(n) \le c \cdot g(n))$

$$3n+5 \ne o(n)$$
$$= o(n^{1.1})$$
$$= o(n^2)$$
$$= o(n^{1+\epsilon}) \ \epsilon>0$$

$3n+5 \ne o(n)$

For $c=0.001$ $\quad 3n+5 \not\le 0.001 n$

Little oh [Loose upper bounds]

$n^2 + 6n + 5 = O(n^2)$

$= O(n^{2+\epsilon}), \epsilon \geq 0$

$= O(c^n), c > 1$

$n^2 + 6n + 5 \leq c \cdot n^{2.1}, \forall c > 0 \ \exists n_0 > 0 \ \forall n \geq n_0$

$< 0.01 \ n^{2.1} \quad n_0 = 10^6$

$n^2 + 6n + 5 \neq o(n^2)$ — Little oh

$= o(n^{2+\epsilon}) \ \epsilon > 0$

$= o(2^n)$

$= o(n!)$

$= o(n^n)$

Little-Omega (Loose lower bounds)

$f(n) = \omega(g(n))$ iff $\forall_{c>0}\ \exists_{n_0>0}\ \forall_{n \geq n_0} (f(n) \geq c \cdot g(n))$

$n^2 + 2^n - n = \Omega(n) = \omega(n)$

$= \Omega(n^2) = \omega(n^2)$

$= \Omega(n^3) = \omega(n^3)$

$= \Omega(2^n) \neq \omega(2^n)$

$\neq \omega(2^n)$

$c = 0.01$ ✓

$c = 10^4$ ✗

$n^3 + 4n^2 - n = \Omega(n^3)$

$= \Omega(n^{3-\epsilon}),\ \epsilon \geq 0$

$= \omega(n^{3-\epsilon}),\ \epsilon > 0$

Remarks

$$n^2+2n+5 = O(n^2)$$
$$= O(n^3)$$
$$= O(n^{2+\epsilon}),\ \epsilon \geq 0$$

$$n^2+2n+5 = \Omega(n^2)$$
$$= \Omega(n)$$
$$= \Omega(n^{2-\epsilon}),\ \epsilon \geq 0$$

$$\underset{n\to\infty}{Lt} \frac{f(n)}{g(n)} \leq c, \quad c: \text{Constant}$$
$$\Rightarrow f(n) = O(g(n))$$

$$\underset{n\to\infty}{Lt} \frac{f(n)}{g(n)} \geq c, \quad c: \text{Constant}$$
$$\Rightarrow f(n) = \Omega(g(n))$$

$$\underset{n\to\infty}{Lt} \frac{f(n)}{g(n)} = c, \quad c: \text{Constant}$$
$$\Rightarrow f(n) = \Theta(g(n))$$

$$\underset{n\to\infty}{Lt} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) = o(g(n))$$ small oh.

$$\underset{n\to\infty}{Lt} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) = \omega(g(n))$$

1. **Reflexivity** — reflexive property

$f(n) = O(f(n))$

$f(n) = \Omega(f(n))$

$f(n) = \Theta(f(n))$

$n = \Theta(n)$

$n = O(n)$

$n = \Omega(n)$

$n \log n = \Theta(n \log n)$

$= O(n \log n)$

$= \Omega(n \log n)$

2. **Symmetric**

$f(n) = O(g(n))$

$\not\Rightarrow g(n) = O(f(n))$

$n = O(n^2)$ $n^2 \neq O(n)$

$f(n) = \Omega(g(n))$

$\not\Rightarrow g(n) = \Omega(f(n))$

$n = \Omega(1)$ $1 \neq \Omega(n)$

Θ is symmetric

$f(n) = \Theta(g(n)) \Rightarrow g(n) = \Theta(f(n))$

③ **Transitive**

$f(n)=O(g(n))$, $g(n)=O(h(n))$ $\Rightarrow f(n)=O(h(n))$

$n^2=O(n^4)$, $n^4=O(2^n) \Rightarrow n^2=O(2^n)$

$f(n)=\Omega(g(n))$, $g(n)=\Omega(h(n))$ $\Rightarrow f(n)=\Omega(h(n))$

$n^2=\Omega(n \log n)$, $n \log n=\Omega(\frac{1}{n}) \Rightarrow n^2=\Omega(\frac{1}{n})$

$f(n)=\Theta(g(n))$, $g(n)=\Theta(h(n))$ $\Rightarrow f(n)=\Theta(h(n))$

④ **Transpose Symmetry**

$f(n)=O(g(n)) \Rightarrow g(n)=\Omega(f(n))$

$n^3=O(2^n) \Rightarrow 2^n=\Omega(n^3)$

$f(n)=\Omega(g(n)) \Rightarrow g(n)=O(f(n))$

$n^3=\Omega(n^2) \Rightarrow n^2=O(n^3)$