Laplace Transforms

$$F(p) = \mathcal{L}\{f(x); p\} = \int_a^b \underbrace{K(x,t)}_{\text{Kernel}} f(t)\,dt$$

$f(x) \to f(t)$

Laplace transformation

$$K(x,t) = e^{-pt} \quad [0, \infty)$$

$$\mathcal{L}\{f(x); p\} = \int_0^\infty e^{-pt} f(t)\,dt = F(p) \quad p>0$$

integration possible (exists)

$$\boxed{\mathcal{L}\{1; p\} = \int_0^\infty e^{-pt} \cdot 1\,dt = \left[\frac{e^{-pt}}{-p}\right]_0^\infty = \frac{1}{p}}$$

$$\mathcal{L}\{x, p\} = \frac{1}{p^2} \; ; \; \mathcal{L}\{x^n, p\} = \frac{n!}{p^{n+1}} \quad n \in \mathbb{N} \text{ is } Z^+$$

In general

gamma function

$$\mathcal{L}\{x^n, p\} = \frac{\Gamma(n+1)}{p^{n+1}}$$

$$\mathcal{L}\{\sin x; p\} = \int_0^\infty e^{-px} \sin x\,dx$$

$$\mathcal{L}\{\cos x; p\} = \int_0^\infty e^{-px} \cos x\,dx$$

$$\boxed{\begin{aligned} \mathcal{L}\{e^{2x}; p\} &= \frac{1}{p-2} \\ \mathcal{L}\{e^{ax}; p\} &= \frac{1}{p-a} \end{aligned}}$$

$$\mathcal{L}\{e^x; p\} = \int_0^\infty e^{-px} e^x\,dx$$

$$= \int_0^\infty e^{-(p-1)x}\,dx$$

$$= \left[\frac{-e^{-(p-1)x}}{(p-1)}\right]_0^\infty$$

$$= \frac{1}{p-1}$$

$$e^{ix} = \cos x + i\sin x$$

$$\mathcal{L}\{e^{ix}; p\} = \mathcal{L}\{\cos x; p\} + i\,\mathcal{L}\{\sin x; p\}$$

$$= \frac{1}{p-i} = \frac{p+i}{p^2+1}$$

$$\Rightarrow \boxed{\mathcal{L}\{\cos x; p\} = \frac{p}{p^2+1}} \;\&\; \boxed{\mathcal{L}\{\sin x; p\} = \frac{1}{p^2+1}}$$

$$\mathcal{L}\{e^{iax}; p\} = \frac{1}{p-ia}$$

$$\Rightarrow \boxed{\mathcal{L}\{\cos ax; p\} = \frac{p}{p^2+a^2}} \;\&\; \boxed{\mathcal{L}\{\sin ax; p\} = \frac{a}{p^2+a^2}}$$

Change of scale property

Let $\mathcal{L}\{f(x)\} = F(p)$

$\cancel{\mathcal{L}\{f(x)\}}$ $\boxed{\mathcal{L}\{f(ax)\} = \frac{1}{a} F(p/a)}$

$\boxed{ax = t}$

$$\mathcal{L}\{f(ax)\} = \int_0^\infty f(ax) e^{-px}\,dx = \int_0^\infty f(t)\, e^{-t/a\, p} \frac{dt}{a}$$

$$= \frac{1}{a}\int_0^\infty f(t)\, e^{-(p/a)t}\,dt = \frac{1}{a} F(p/a)$$

$$\mathcal{L}\{\cos x; p\} = \frac{p}{p^2+1} \;;\; \mathcal{L}\{\sin x; p\} = \frac{1}{p^2+1}$$

$$\mathcal{L}\{\cos ax; p\} = \frac{1}{a}\,\frac{p/a}{\frac{p^2}{a^2}+1} = \frac{p}{p^2+a^2}$$

$\mathcal{L}\{\sin ax; p\} = \frac{1}{a}\,\frac{1}{\ldots} = \frac{a}{\ldots}$

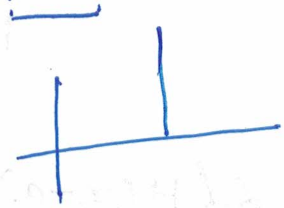
③

$\boxed{\mathcal{L}\{\cosh ax\} = \dfrac{p}{p^2-a^2}}$; $\boxed{\mathcal{L}\{\sinh ax\} = \dfrac{a}{p^2-a^2}}$

which fns have L·T

a) piecewise continuous

b) $|f(x)\,e^{-\alpha x}| < M \quad x \geq x_0$



$\exists M$ such that $|f(x)| < M \cdot e^{\alpha x}$

Sufficient condition for existence of Laplace transform

If ~~Let~~ $f(x)$ is piecewise continous fn & exponential order exists, then laplace transform of $f(x)$ exists.

$$\boxed{\mathcal{L}\{x^{-1/2}\} = \frac{\Gamma(n+1)}{p^{n+1}} = \frac{\Gamma(-\frac{1}{2}+1)}{p^{-1/2+1}} = \frac{\Gamma(\frac{1}{2})}{p^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{p}}}$$

Let $\mathcal{L}\{f(x)\} = F(p)$ then $\mathcal{L}\{e^{-ax} \cdot f(x)\} = F(p+a)$

$$= \int_0^\infty e^{-px} f(x) \cdot e^{-ax}\, dx = \int_0^\infty e^{-(p+a)x} f(x)\, dx$$

$$\mathcal{L}\{f(x); p\} = F(p)$$

$$g(x) = \begin{cases} f(x-a) & x > a \\ 0 & x < 0 \end{cases}$$

$\boxed{\mathcal{L}\{g(x)\} = e^{-pa} \cdot F(p)}$

$$\mathcal{L}\{g(x)\} = \int_0^\infty e^{-px} g(x)\, dx = \int_0^a 0 \cdot e^{-px} dx + \int_a^\infty f(x-a) e^{-px} dx$$

$$x - a = t \quad = \int_0^\infty e^{-p(a+t)} f(t)\, dt = \boxed{e^{-pa} F(p)}$$

Heaviside unit stepfunction

$$f(x-a) = \begin{cases} 1 & x>a \\ 0 & x<a \end{cases}$$

$$H(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

$$\mathcal{L}\{H(x-a) = \int_0^\infty e^{-px} H(x)\, dx$$

$$= \int_0^a 0 \cdot e^{-px} dx + \int_a^\infty 1 \cdot e^{-px} dx = \left[-\frac{e^{-px}}{p}\right]_a^\infty = \frac{e^{-pa}}{p}$$

Laplace transform of the derivative:

$$\mathcal{L}\{f'(x)\} = \int_0^\infty e^{-px} f'(x)\, dx$$

$$= \left[e^{-px} f(x)\right]_0^\infty + \int_0^\infty p \cdot e^{-px} f(x)$$

$$(\mathcal{L}\{f(x)\} = F(p))$$

$$= 0 - f(0) + p \cdot \mathcal{L}\{f(x)\}$$

$$\boxed{\mathcal{L}\{f'(x)\} = pF(p) - f(0)}$$

$$\boxed{\mathcal{L}\{f''(x)\} = p\,\mathcal{L}\{f'(x)\} - f'(0) = p^2 F(p) - p(f(0)) - f'(0)}$$

$$\boxed{\mathcal{L}\{f^{(n)}(x)\} = p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}$$

$$\mathcal{L}\{(\cos x)\} = \mathcal{L}\{(\sin x)'\} = p \cdot \frac{1}{p^2+1} - \sin 0$$

Laplace transform of integration:

$\mathcal{L}\{f(x)\} = F(p)$

$\mathcal{L}\{f(x);p\} = F(p)$

$$\boxed{\mathcal{L}\left\{\int_0^x f(t)\,dt\right\} = \frac{F(p)}{p}} \rightarrow \boxed{\mathcal{L}\left\{\int_0^x \int_0^{t_1} f(u)\,du\right\} = \frac{F(p)}{p^2}}$$

Laplace transformation for multiple of x^n and fn

i.e., $\boxed{\mathcal{L}\{x^n f(x)\} = (-1)^n \frac{d^n}{dp^n} F(p)}$

$$\mathcal{L}\{x \sin x\} = -1 \cdot \frac{d}{dp}\left\{\frac{1}{p^2+1}\right\} \quad \left(\because \mathcal{L}\{\sin x\} = \frac{1}{p^2+1}\right)$$

$$= -\frac{-2p}{(p^2+1)^2} = \frac{2p}{(p^2+1)^2}$$

function divided by x^n then

$$\boxed{\text{Let } \{f(x);p\} = F(p) \text{ then } \mathcal{L}\left\{\frac{f(x)}{x};p\right\} = \int_p^\infty F(p)\,dp}$$

$$\mathcal{L}\left\{\frac{f(x)}{x^n}\right\} = \int_p^\infty \int_{u_1}^\infty \int_{u_2}^\infty \cdots \int_{u_{(n-1)}}^\infty F(u_{n-1})\,du_{n-1}$$

check $\lim_{x\to 0} \frac{f(x)}{x} \rightarrow$ exist

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} \quad \left[\mathcal{L}\{\sin t\} = \frac{1}{p^2+1}\right]$$

$\downarrow$

$$= \int_p^\infty \frac{1}{p^2+1}\,dp = \left[\tan^{-1} p\right]_p^\infty = \frac{\pi}{2} - \tan^{-1} p$$

$$= \cot^{-1}(p) = \tan^{-1}\left(\frac{1}{p}\right)$$

$$\Rightarrow \mathcal{L}\left\{\int_0^x \frac{\sin t}{t}\,dt\right\} = \frac{1}{p}\tan^{-1}\left(\frac{1}{p}\right)$$

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} \quad \Bigg| \quad \frac{d}{dt}(\sin\sqrt{t}) = \frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}$$

$$\mathcal{L}\left\{\frac{d}{dt}(\sin\sqrt{t})\right\} = \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$$

$$p\,F(P) = \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$$

$$\frac{p\sqrt{\pi}}{\cancel{2}p^{3/2}} e^{-1/4.p} = \frac{1}{\cancel{2}}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$$

$$\therefore \mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\, e^{-\frac{1}{4.p}}$$

$$\mathcal{L}\{J_0(x)\} = \frac{1}{\sqrt{p^2+}}$$

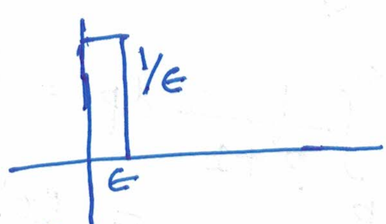
$$J_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{n+2r}}{\Gamma(n+r)-1\ \Gamma r!}$$

$$\sum \ L(x^{2r})$$

$$\downarrow$$

$$\frac{\Gamma 2r+1}{p^{2r+1}}$$

Impulse fn



$$F_\epsilon = \begin{cases} 1/\epsilon & 0<x<\epsilon \\ 0 & x>\epsilon \end{cases} \qquad \delta(x) = \lim_{\epsilon\to 0} F_\epsilon$$

$$\mathcal{L}\{\delta(x)\} = \lim_{\epsilon\to 0} \mathcal{L}\{F_\epsilon\} = \lim_{\epsilon\to 0} \int_0^\infty e^{-px} F_\epsilon \, dx$$

$$= \lim_{\epsilon\to 0} \int_0^\epsilon \frac{1}{\epsilon} e^{-px} dx = \frac{1}{\epsilon}\left[-\frac{e^{-px}}{p}\right]_0^\epsilon$$

$$= \frac{1}{\epsilon}\left[\frac{-e^{-p\epsilon}}{p} + \frac{1}{p}\right] = \lim_{\epsilon\to 0} \frac{1-e^{-p\epsilon}}{p\epsilon} = 1$$

$$\Rightarrow \mathcal{L}\{\delta_0(x)\} = \mathcal{L}\{\lim_{\epsilon\to 0} F_\epsilon(x)\} = 1$$

Inverse Laplace Transformation

$$\Rightarrow \mathcal{L}\{x^n\} = \frac{\Gamma(n+1)}{p^{n+1}} \Rightarrow x^n = \mathcal{L}^{-1}\left\{\frac{\Gamma(n+1)}{p^{n+1}}\right\} \Rightarrow \frac{x^n}{\Gamma(n+1)} = \mathcal{L}^{-1}\left\{\frac{1}{p^{n+1}}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{\sin ax}{a} \; ; \; \mathcal{L}^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \frac{\cos ax}{a}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p-a}\right\} = e^{ax} \; ; \; \mathcal{L}^{-1}\left\{\frac{1}{p+a}\right\} = e^{-ax}$$

$$\mathcal{L}\{x^n f(x) ; p\} = (-1)^n \frac{d^n}{dp^n} F(P) \;\Big|\; F(p) = \mathcal{L}\{f(x) ; p\}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{d^n}{dp^n} F(P)\right\} = (-1)^n x^n f(x)}$$

$$\mathcal{L}\left\{\int_0^x f(x)dx ; p\right\} = \frac{F(P)}{p} \Rightarrow \mathcal{L}^{-1}\left\{\frac{F(P)}{P}\right\} = \int_0^x f(x)\,dx$$

$$\mathcal{L}\{f^n(x)\} = p^n F(P) - p^{n-1} f(0) - p^{n-2} f'(0) \cdots f(0)$$

$$\Rightarrow \mathcal{L}^{-1}\{p^n F(P)\} = \frac{d^n}{dx^n} f(x) \quad f(0) = f'(0) \cdots f^{(n-1)}(0) = 0$$

$$\mathcal{L}\left\{\frac{f(x)}{x}\right\} = \int_p^\infty F(u)\,du$$

$$\frac{f(x)}{x} = \mathcal{L}^{-1}\left\{\int_p^\infty F(u)\,du\right\}.$$

$$\mathcal{L}^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{p^2+1-2}{p^2+2p+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{-2}{(p^2+1)^2}\right\}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{x^n}{\Gamma n+1} \hookrightarrow$$

$$\mathcal{L}^{-1}\{F(p+a)\} = e^{-ax} f(x)$$

$$= \frac{(p+3)-3}{(p+3)^{7/2}}$$

$$= e^{-3x}\,\mathcal{L}^{-1}\left\{\frac{1}{p^{5/2}} - \frac{3}{p^{7/2}}\right\}$$

$$\mathcal{L}\{e^{-ax} \cdot f(x) = F(p+a)$$

$$\mathcal{L}^{-1}\{F(p+a) = e^{-ax} \cdot f(x)$$

$$F(p) = \log\left(1+\frac{1}{p^2}\right) = \log\left(\frac{p^2+1}{p^2}\right)$$

$$= \log(p^2+1) - 2\log p$$

$$F'(p) = \frac{2p}{p^2+1} - \frac{2}{p}$$

$$\mathcal{L}^{-1}\{F'(p)\} = \mathcal{L}^{-1}\left\{\frac{2p}{p^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{p}\right\}$$

$$-x f(x) = 2\cos x - 2 \Rightarrow f(x) = \underline{2(1-\cos x)}$$

$$f * g = \mathcal{L}^{-1}\{F(P)\,G(P)\}$$

$$= \int_0^{x} f(u)\,g(x-u)\,du$$

$$\text{or } \int_0^{\infty} g(u)\,f(x-u)\,du$$

CONVOLUTION

1. $\dfrac{1}{p^2(p+1)^2}$ $\quad \mathcal{L}^{-1}\left\{\dfrac{1}{(p+1)^2}\right\} = e^{-x}\mathcal{L}^{-1}\left\{\dfrac{1}{p^2}\right\}$

$$e^{-x}\cdot x = x\cdot e^{-x}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p(p+1)^2}\right\} = \int_0^{x} u\,e^{-u}\,du$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} = \int_0^{x}\int_0^{u_1} u_1\,e^{-u}\,du$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\} = x \qquad g(x) = x\cdot e^{-x}$$

$$\int_0^{x} u(x-u)\,e^{-(x-u)}\,du$$

---

1. $\mathcal{L}\{(D^2+9)\,y = \cos 2x\}$ $\quad y(0)=1;\ y(\pi/2) = -1$

$$\downarrow$$

$$p^2\,\mathcal{L}\{y(x)\} - p\,y(0) - y'(0) + 9\,\mathcal{L}\{y\} = \frac{p}{p^2+4}$$

Let $y'(0) = A$ & let $\mathcal{L}\{y\} = \bar{y}_p$

$$\Rightarrow \quad p^2\bar{y}_p - p + A + 9\,\bar{y}_p = \frac{p}{p^2+4}$$

$$(p^2+9)\,\bar{y}_p = \frac{p}{p^2+4} + p + A$$

$$\bar{y}_p = \frac{p}{(p^2+4)(p^2+9)} + \frac{p}{(p^2+9)} + \frac{A}{p^2+9}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\} + \mathcal{L}^{-1}\left\{\frac{p}{p^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{p^2+9}\right\}$$

$$\cos 3x + \frac{A\sin 3x}{3}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{5}\left[\frac{p}{p^2+4} - \frac{p}{p^2+9}\right]\right\}$$

$$y = \frac{1}{5}\cos 2x - \frac{1}{5}\cos 3x + \cos 3x + \frac{A\sin 3x}{3}$$

$$\mathcal{L}^{-1}\left\{\frac{p}{p^2+4}\right\} = \cos 2x \,; \quad \mathcal{L}^{-1}\left\{\frac{1}{p^2+9}\right\} = \frac{\sin 3x}{3}$$

$$\mathcal{L}^{-1}\left\{\frac{p}{p^2+4}\cdot\frac{1}{p^2+9}\right\} = \int_0^x f(u)\,g(x-u)\,du$$

$$= \frac{1}{3}\int_0^x \cos 2u \,\sin(3x-3u)\,du.$$

---

② $xy'' + y' + 4xy = 0 \qquad y(0) = 3\,; \quad y(0) = 0$

$$L\{xy'' + y' + 4xy\} = 0$$

$$-\frac{d}{dp}\mathcal{L}\{y''\} + L\{y'\} - 4\frac{d}{dp}L\{y\}$$

$$-\frac{d}{dp}\left[p^2 L(y(x)) - p\,y(0) - y'(0)\right] + \left[pL(y(x)) - y(0)\right]$$

$$-4\frac{d}{}\ L\{y(x)\}$$

$$-2p\,\bar{y}_p - p^2\frac{d\bar{y}_p}{dp} + 3 + p\cdot\bar{y}_p - 3 - 4\cdot\frac{d\bar{y}_p}{dp} = 0$$

$$(p^2+4)\cdot\frac{d\bar{y}_p}{dp} + p\,\bar{y}_p = 0.$$

$$\frac{d\bar{y}_p}{dp} + \frac{p}{(p^2+4)}\bar{y}_p = 0$$

$$I.F = e^{\int p/p^2+4\,dp} = e^{\frac{1}{2}\log(p^2+4)}$$

$$\frac{d\bar{y}_p}{\bar{y}_p} + \frac{p}{p^2+4}dp = 0.$$

$$\log\bar{y}_p + \frac{1}{2}\log(p^2+4) = \log c$$

$$\bar{y}_p\cdot\sqrt{p^2+4} = c$$

$$\bar{y}_p = \frac{c}{\sqrt{p^2+4}} \qquad \mathcal{L}\{J_0(x)\} = \frac{1}{\sqrt{p^2+1}}$$

$$\mathcal{L}\{J_0(ax)\} = \frac{a}{\sqrt{p^2+a^2}}.$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{c}{\sqrt{p^2+4}}\right\}$$

$$y(x) = \frac{c}{2}\cdot J_0(2x)$$

CONVOLUTION

$\mathcal{L}^{-1}\{F(p)\} = f(x)$; $\mathcal{L}^{-1}\{G(p)\} = g(x)$

then $\mathcal{L}^{-1}\{F(p)\cdot G(p)\} = f * g = \int_0^x f(u)\, g(x-u)\, du$ To prove

$$\Downarrow \qquad\qquad \Downarrow$$

$$F(p)\, G(p) = \mathcal{L}\left\{ \int_0^x f(u)\, g(x-u)\, du \right.$$

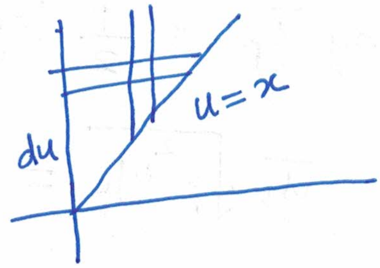
$$= \int_0^\infty e^{-px} \int_0^x f(u)\, g(x-u)\, du$$

$$= \int_{x=0}^{\infty} \int_{u=0}^{x} e^{-px} f(u)\, g(x-u)\, du\, dx$$

$$\int_{u=0}^{\infty} f(u) \int_{x=u}^{\infty} e^{-px} \underbrace{g(x-u)}_{x-u=t}\, dx\, du \quad \Bigg|\ \begin{array}{l} u=0,\ u=x \\ x=0,\ x=\infty \end{array}$$

$$\int_{u=0}^{\infty} f(u) \left[\int_0^\infty e^{-p(u+t)} g(t)\, dt \right] du$$

du



$$= \int_{u=0}^{\infty} f(u) \left[e^{-pu} \int_0^\infty e^{-pt}\, g(t)\, dt \right] du$$

$$= \int_0^\infty f(u)\, e^{-pu} du \cdot \int_0^\infty e^{-pt} \cdot g(t)\, dt$$

$$= \mathcal{L}\{f(x)\} \cdot \mathcal{L}\{g(x)\}$$

$$= \underline{F(p) \cdot G(p)}.$$

<u>Hence proved</u>