# Engineering Electromagnetics

## Lecture 23

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*by*

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- For a point charge *q* at the origin, calculate the flux of **E** through a spherical surface of radius r.

# Flux

- In the case of a point charge q at the origin, the flux of E through a spherical surface of radius r is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0} q$$

- For any closed surface

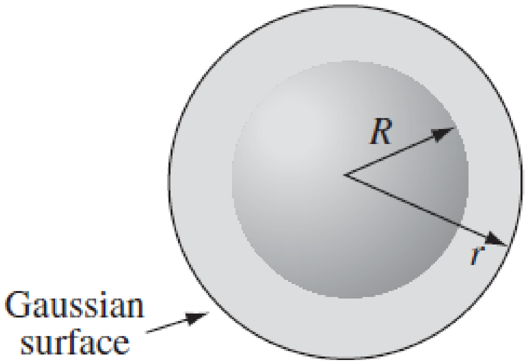
$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left(\oint \mathbf{E}_i \cdot d\mathbf{a}\right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i\right)$$

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}}$$

where $Q_{enc}$ is the total charge enclosed within the surface



From Griffith Book




$$\int_{\mathcal{S}} |\mathbf{E}|\, da = |\mathbf{E}| \int_{\mathcal{S}} da = |\mathbf{E}|\, 4\pi r^2.$$

$$|\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q,$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.$$

Notice a remarkable feature of this result: The field outside the sphere is exactly *the same as it would have been if all the charge had been concentrated at the center.*

# Divergence of E

As it stands, Gauss's law is an *integral* equation, but we can easily turn it into a *differential* one, by applying the divergence theorem:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau.$$

Rewriting $Q_{enc}$ in terms of the charge density $\rho$, we have

$$Q_{enc} = \int_{\mathcal{V}} \rho\, d\tau.$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau.$$

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{enc}}$$

And since this holds for *any* volume, the integrands must be equal:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

**Gauss's law in differential form**

From Griffith Book

# Curl of E

the simplest possible configuration: a point charge at the origin. In this case

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.$$

$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l}.$ In spherical coordinates, $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$, so

$$\mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr.$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}}\frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$



From Griffith Book

# Curl of E

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2} dr = \frac{-1}{4\pi\epsilon_0} \frac{q}{r} \bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{r_a} - \frac{q}{r_b} \right)$$

where $r_a$ is the distance from the origin to the point **a** and $r_b$ is the distance to **b**. The integral around a *closed* path is evidently zero (for then $r_a = r_b$):

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{l} = 0,}$$

and hence, applying Stokes' theorem,

$$\boxed{\nabla \times \mathbf{E} = \mathbf{0}.}$$

From Griffith Book

# Energy stored in Electric field

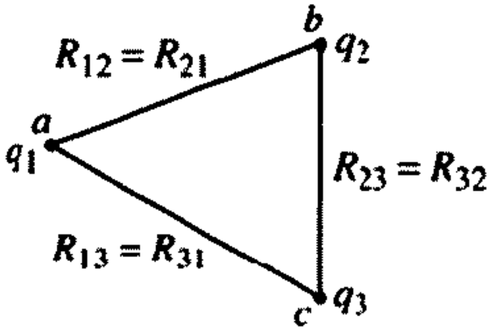



**Figure 3.32** Potential energy in a system of three point charges

$$W = W_3 + W_2 + W_1 = 0 + q_2 V_{b,c} + q_1(V_{a,c} + V_{a,b})$$
$$= \frac{1}{4\pi\epsilon}\left[\frac{q_2 q_3}{R_{23}} + \frac{q_1 q_3}{R_{13}} + \frac{q_1 q_2}{R_{12}}\right]$$

$$W = \frac{1}{2}[q_1(V_{a,c} + V_{a,b}) + q_2(V_{b,a} + V_{b,c}) + q_3(V_{c,a} + V_{c,b})]$$

The total energy can now be written as

$$W = \frac{1}{2}[q_1 V_1 + q_2 V_2 + q_3 V_3] = \frac{1}{2}\sum_{i=1}^{3} q_i V_i$$

We can generalize this equation for a system of $n$ point charges as

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V_i} \quad (3$$

*Equation (3.64) allows us to compute the electrostatic potential energy for a group of point charges in their mutual field.*

If the charges are continuously distributed, (3.64) becomes

$$\boxed{W = \frac{1}{2}\int_v \rho_v V\, dv} \quad (3.65)$$

where $\rho_v$ is the volume charge density within $v$.

or

$$\boxed{W = \frac{1}{2}\int_s \rho_s V\, ds}$$

# Conductors

(ii) $\rho = 0$ inside a conductor.

(iii) Any net charge resides on the surface. That's the only place left.

(iv) A conductor is an equipotential.

(v) **E** is perpendicular to the surface, just outside a conductor.

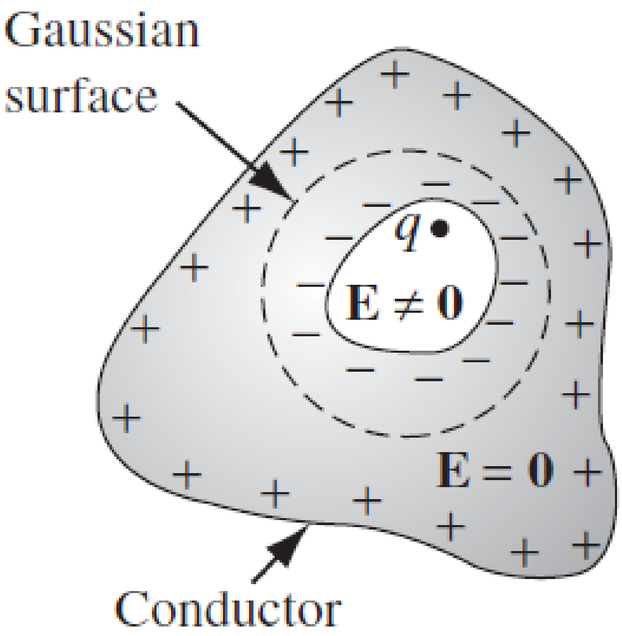


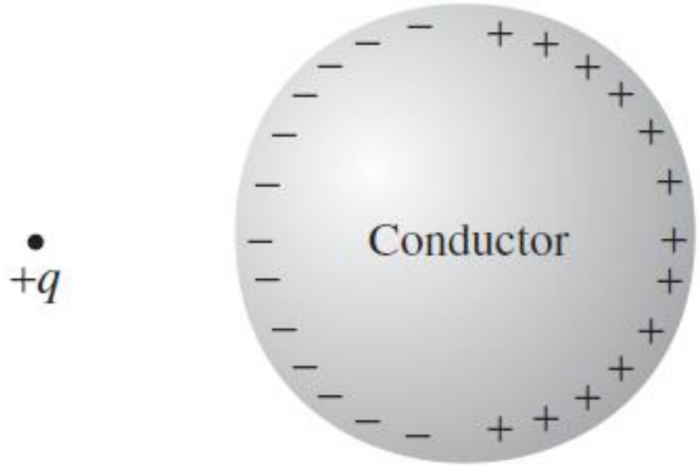
From Griffith Book

# Cavity in a conductor

within that cavity you put some charge, then the field *in the cavity* will *not* be zero.

if we surround the cavity with a Gaussian surface, all points of which are in the conductor (Fig. 2.45), $\oint \mathbf{E} \cdot d\mathbf{a} = 0$, and hence (by Gauss's law) the net enclosed charge must be zero. But $Q_{\text{enc}} = q + q_{\text{induced}}$, so $q_{\text{induced}} = -q$. Then if the conductor as a whole is electrically neutral, there must be a charge $+q$ on its outer surface.



# Gauss's Law in the Presence of Dielectrics

Polarization creates → accumulation of **bound charges**

$$\boxed{\rho_b = -\nabla \cdot \mathbf{P} \text{ and } \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}}$$

We are now ready to put it all together: the field attributable to bound charge plus the field due to everything *else* (which, for want of a better term, we call **free charge**, $\rho_f$). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, that is *not* a result of polarization. Within the dielectric, the total charge density can be written:

$$\boxed{\rho = \rho_b + \rho_f}$$

and Gauss's law reads $\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$

# Gauss's Law in the Presence of Dielectrics

$\rho_b = -\nabla \cdot \mathbf{P}$ and $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$

Total charge density:

$$\rho = \rho_b + \rho_f$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

**electric displacement.** $\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}}$

In terms of **D**, Gauss's law reads $\boxed{\nabla \cdot \mathbf{D} = \rho_f}$

or, in integral form, $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$

where $Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume.

# Potential due to Dipole

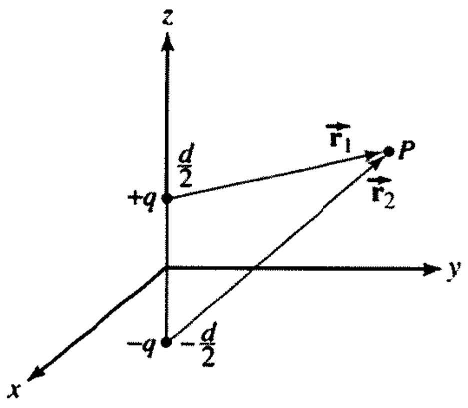



**Figure 3.24** An electric dipole

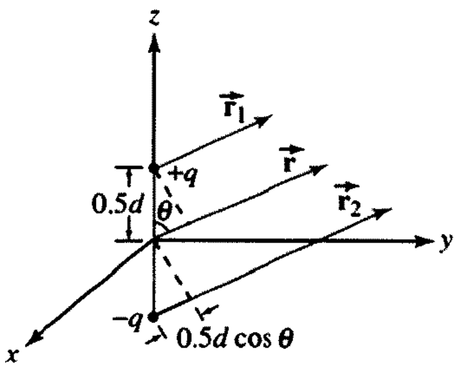



**Figure 3.25** Distance approximations when $P$ is far away from the dipole ($r \gg d$)

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

$$r_1 \approx r - 0.5d\cos\theta, \quad r_2 \approx r + 0.5d\cos\theta$$

and

$$r_1 r_2 = r^2 - (0.5d\cos\theta)^2 \approx r^2$$

The potential at $P$ can now be written as

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{d\cos\theta}{r^2}\right]$$

$$\boxed{\vec{\mathbf{p}} = qd\vec{\mathbf{a}}_z}$$

$$\boxed{V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\vec{\mathbf{p}}\cdot\vec{\mathbf{a}}_r}{4\pi\epsilon_0 r^2}}$$

Note that the potential at a point falls off as the square of the distance for a dipole, whereas it is inversely proportional to distance for a single-point charge.

# Field due to dipole

$$\vec{\mathbf{E}} = \frac{p}{4\pi\epsilon_0 r^3}[2\cos\theta\,\vec{\mathbf{a}}_r + \sin\theta\,\vec{\mathbf{a}}_\theta]$$

However,

$$\begin{aligned} 2\cos\theta\,\vec{\mathbf{a}}_r + \sin\theta\,\vec{\mathbf{a}}_\theta &= 3\cos\theta\,\vec{\mathbf{a}}_r - (\cos\theta\,\vec{\mathbf{a}}_r - \sin\theta\,\vec{\mathbf{a}}_\theta) \\ &= 3\cos\theta\,\vec{\mathbf{a}}_r - \vec{\mathbf{a}}_z \end{aligned}$$

Thus, we can write the electric field intensity at point $P$ as

$$\vec{\mathbf{E}} = \frac{3(\vec{\mathbf{p}}\cdot\vec{\mathbf{r}})\vec{\mathbf{r}} - r^2\vec{\mathbf{p}}}{4\pi\epsilon_0 r^5}$$

*An* **electric dipole** *is defined as two charges of equal strength but of opposite polarity but separated by a small distance.* Associated with each dipole is a vector called the **dipole moment**. If $q$ is the magnitude of each charge and $\vec{\mathbf{d}}$ is the distance vector from the negative to the positive charge, then the dipole moment is $\vec{\mathbf{p}} = q\vec{\mathbf{d}}$.

# Permittivity

For *linear* dielectrics **P** **E**

$$\vec{\mathbf{P}} = \epsilon_0 \chi \vec{\mathbf{E}} \qquad (3.57)$$

where the proportionality constant $\chi$ is called the *electric susceptibility*, and the factor $\epsilon_0$ is included to make it a dimensionless quantity.

Equation (3.56) can now be expressed as

$$\vec{\mathbf{D}} = \epsilon_0 (1 + \chi) \vec{\mathbf{E}} \qquad (3.58a)$$

The quantity $(1 + \chi)$ is called the *relative permittivity* or the *dielectric constant* of the medium and is symbolized as $\epsilon_r$. Thus, the general expression for the electric flux density finally becomes

$$\vec{\mathbf{D}} = \epsilon_0 \epsilon_r \vec{\mathbf{E}} = \epsilon \vec{\mathbf{E}} \qquad (3.58b)$$

where $\epsilon = \epsilon_0 \epsilon_r$ is the permittivity of the medium.

# Capacitors


**FIGURE 2.51**

Since **E** is proportional to $Q$, so also is $V$. The constant of proportionality is called the **capacitance** of the arrangement:

$$\boxed{C \equiv \frac{Q}{V}}.$$

$$\boxed{\begin{gathered} Q = CV \\ W = \frac{1}{2}CV^2 \end{gathered}}$$

Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation of the two conductors. In SI units, $C$ is measured in **farads** (F); a farad is a coulomb-per-volt. Actually, this turns out to be inconveniently large; more practical units are the microfarad ($10^{-6}$ F) and the picofarad ($10^{-12}$ F).

Griffith book

# Problem-1

Two point charges of 20 nC and $-20$ nC are situated at (1, 0, 0) and (0, 1, 0) in free space. Determine the electric field intensity at (0, 0, 1).

# Solution-1

The two distance vectors are

$$\vec{\mathbf{R}}_1 = \vec{\mathbf{r}} - \vec{\mathbf{r}}_1 = -\vec{\mathbf{a}}_x + \vec{\mathbf{a}}_z, \quad R_1 = |\vec{\mathbf{r}} - \vec{\mathbf{r}}_1| = 1.414\,\text{m}$$

and

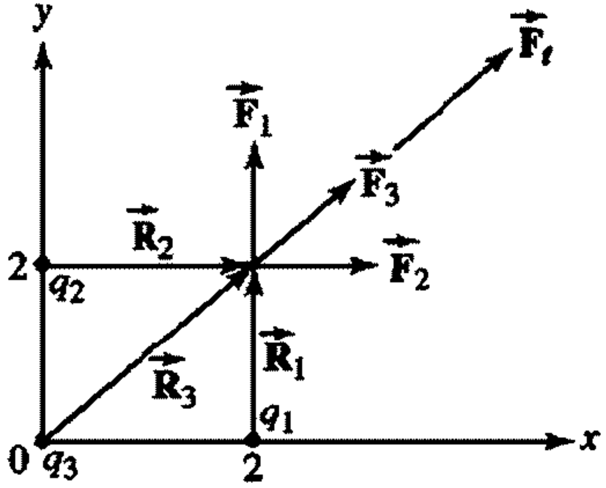
$$\vec{\mathbf{R}}_2 = \vec{\mathbf{r}} - \vec{\mathbf{r}}_2 = -\vec{\mathbf{a}}_y + \vec{\mathbf{a}}_z, \quad R_2 = |\vec{\mathbf{r}} - \vec{\mathbf{r}}_2| = 1.414\,\text{m}$$

Substituting in equation (3.10), we obtain

$$\vec{\mathbf{E}} = 9 \times 10^9 \left[ \frac{20 \times 10^{-9}}{1.414^3}(-\vec{\mathbf{a}}_x + \vec{\mathbf{a}}_z) - \frac{20 \times 10^{-9}}{1.414^3}(-\vec{\mathbf{a}}_y + \vec{\mathbf{a}}_z) \right]$$

$$= 63.67[-\vec{\mathbf{a}}_x + \vec{\mathbf{a}}_y]\,\text{V/m}$$

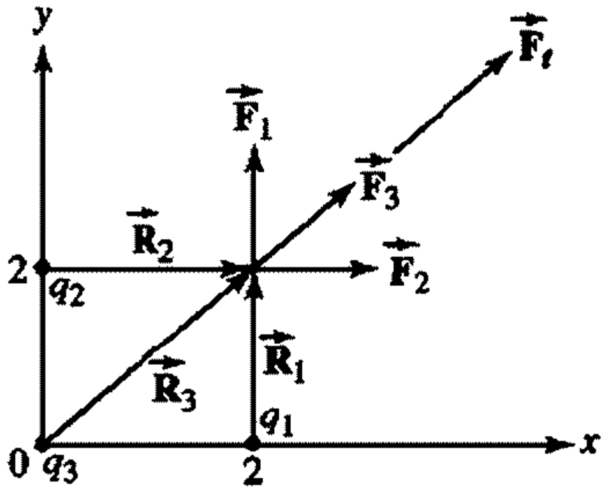
# Problem-2




Three equal charges of 200 nC are placed in free space at (0, 0, 0), (2, 0, 0), and (0, 2, 0). Determine the total force acting on a charge of 500 nC at (2, 2, 0).

# Solution

$$\vec{\mathbf{R}}_1 = \vec{\mathbf{r}} - \vec{\mathbf{r}}_1 = 2\vec{\mathbf{a}}_y \Rightarrow R_1 = 2\,\text{m}$$
$$\vec{\mathbf{R}}_2 = \vec{\mathbf{r}} - \vec{\mathbf{r}}_2 = 2\vec{\mathbf{a}}_x \Rightarrow R_2 = 2\,\text{m}$$
$$\vec{\mathbf{R}}_3 = \vec{\mathbf{r}} - \vec{\mathbf{r}}_3 = 2\vec{\mathbf{a}}_x + 2\vec{\mathbf{a}}_y \Rightarrow R_3 = 2.828\,\text{m}$$




The force on $q$ due to $q_1$ is

$$\vec{\mathbf{F}}_1 = \frac{9 \times 10^9 \times 200 \times 10^{-9} \times 500 \times 10^{-9}}{2^3}[2\vec{\mathbf{a}}_y] = 225\vec{\mathbf{a}}_y\,\mu\text{N}$$

Similarly, we can compute the forces acting on $q$ due to $q_2$ and $q_3$ as

$$\vec{\mathbf{F}}_2 = 225\vec{\mathbf{a}}_x\,\mu\text{N} \quad \text{and} \quad \vec{\mathbf{F}}_3 = 79.6[\vec{\mathbf{a}}_x + \vec{\mathbf{a}}_y]\,\mu\text{N}$$

Thus, the total force experienced by $q$, from (3.6), is

$$\vec{\mathbf{F}}_t = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 = 304.6[\vec{\mathbf{a}}_x + \vec{\mathbf{a}}_y]\,\mu\text{N}$$

# Problem-2

A metallic sphere of radius 10 cm has a surface charge density of 10 nC/$m^2$. Calculate the electric energy stored in the system.

# Solution

The potential on the surface of the sphere is

$$V = \int_s \frac{\rho_s \, ds}{4\pi \epsilon_0 R} = 9 \times 10^9 \times 10 \times 10^{-9} \times 0.1 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$
$$= 113.1 \text{ V}$$

where $Q_t$ is the total charge on the sphere. For uniform charge distribution, the total charge is

$$Q_t = 4\pi R^2 \rho_s = 4\pi (0.1)^2 \, 10 \times 10^{-9} = 1.257 \text{ nC}$$

Thus,

$$W = 0.5 \times 1.257 \times 10^{-9} \times 113.1 = 71.08 \times 10^{-9} \text{ joules (J)}$$ • • •

**Example 4.5.** A metal sphere of radius $a$ carries a charge $Q$ (Fig. 4.20). It is surrounded, out to radius $b$, by linear dielectric material of permittivity $\epsilon$. Find the potential at the center (relative to infinity).

**Solution**

To compute $V$, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$ (Eq. 4.30). We seem to be in a bind. What we *do* know is the *free* charge $Q$, and fortunately the arrangement is spherically symmetric, so let's begin by calculating $\mathbf{D}$, using Eq. 4.23:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = \mathbf{0}$.) Once we know $\mathbf{D}$, it is a trivial matter to obtain $\mathbf{E}$, using Eq. 4.32:

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

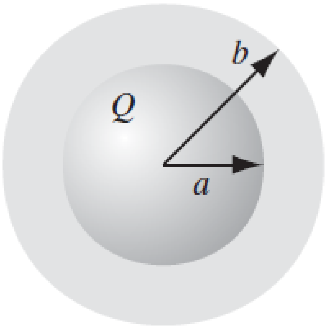



**FIGURE 4.20**

The potential at the center is therefore

$$V = -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l} = -\int_{\infty}^{b} \left( \frac{Q}{4\pi\epsilon_0 r^2} \right) dr - \int_{b}^{a} \left( \frac{Q}{4\pi\epsilon r^2} \right) dr - \int_{a}^{0} (0)\, dr$$

$$= \frac{Q}{4\pi} \left( \frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right).$$

# Problem

An electron and a proton separated by a distance of $10^{-11}$ meter are symmetrically arranged along the $z$ axis with $z = 0$ as its bisecting plane. Determine the potential and $\vec{\mathbf{E}}$ field at $P(3, 4, 12)$.

# Solution

The position vector: $\vec{\mathbf{r}} = 3\vec{\mathbf{a}}_x + 4\vec{\mathbf{a}}_y + 12\vec{\mathbf{a}}_z \quad r = 13$ m

The dipole moment: $\vec{\mathbf{p}} = 1.6 \times 10^{-19} \times 10^{-11}\vec{\mathbf{a}}_z = 1.6 \times 10^{-30}\vec{\mathbf{a}}_z$

From (3.38), the potential at point $P$ is

$$V = \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{r}}}{4\pi\epsilon_0 r^3} = \frac{9 \times 10^9 \times 1.6 \times 10^{-30} \times 12}{13^3} = 7.865 \times 10^{-23}\mathrm{V}$$

The electric field intensity at point $P$, from (3.42), is

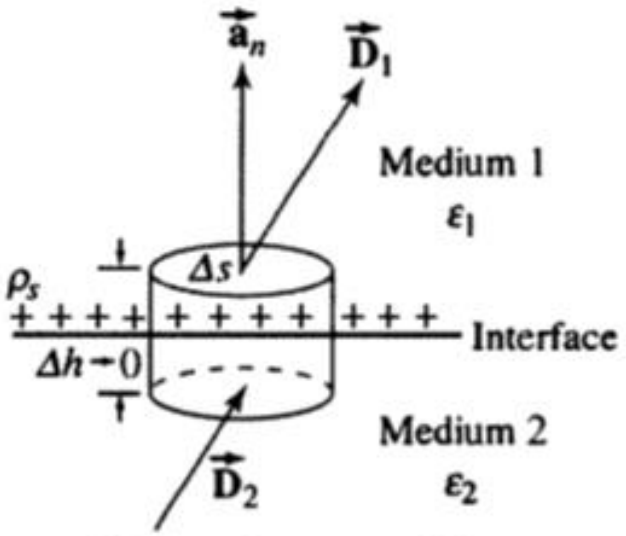
$$\vec{\mathbf{E}} = \frac{9 \times 10^9}{13^5}(1.6 \times 10^{-30})[3 \times 12(3\vec{\mathbf{a}}_x + 4\vec{\mathbf{a}}_y + 12\vec{\mathbf{a}}_z) - 13^2\vec{\mathbf{a}}_z]$$

$$= [4.189\vec{\mathbf{a}}_x + 5.585\vec{\mathbf{a}}_y + 10.2\vec{\mathbf{a}}_z] \times 10^{-24}\ \mathrm{V/m}$$

• •

# Problem-1

The plane $z = 0$ marks the boundary between free space and a dielectric medium with a dielectric constant of 40. The $\vec{\mathbf{E}}$ field next to the interface in free space is $\vec{\mathbf{E}} = 13\vec{\mathbf{a}}_x + 40\vec{\mathbf{a}}_y + 50\vec{\mathbf{a}}_z$ V/m. Determine the $\vec{\mathbf{E}}$ field on the other side of the interface.



a) Boundary conditions

Let $z > 0$ be the dielectric medium 1 and $z < 0$ be the free space medium 2. Then

$$\vec{\mathbf{E}}_2 = 13\vec{\mathbf{a}}_x + 40\vec{\mathbf{a}}_y + 50\vec{\mathbf{a}}_z$$

The unit vector $\vec{\mathbf{a}}_n$ normal to the interface is $\vec{\mathbf{a}}_z$. Because the tangential components of the $\vec{\mathbf{E}}$ field are continuous, then

$$E_{x1} = E_{x2} = 13 \quad \text{and} \quad E_{y1} = E_{y2} = 40$$

For a dielectric–dielectric interface, the normal components of the field are also continuous. That is,

$$\epsilon_1 E_{z1} = \epsilon_2 E_{z2}$$

However, $\epsilon_2 = \epsilon_0$ and $\epsilon_1 = 40\epsilon_0$. Therefore,

$$E_{z1} = \frac{E_{z2}}{40} = \frac{50}{40} = 1.25$$

Thus, the $\vec{\mathbf{E}}$ field in medium 1 is

$$\vec{\mathbf{E}} = 13\vec{\mathbf{a}}_x + 40\vec{\mathbf{a}}_y + 1.25\vec{\mathbf{a}}_z \text{ V/m}$$

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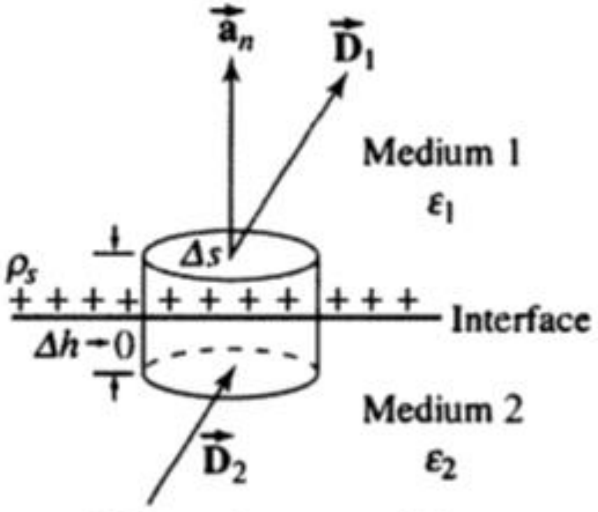


a) Boundary conditions

Calculate the capacitance of a parallel-plate capacitor having a mica dielectric, $\epsilon_R = 6$, a plate area of $10\,\text{in}^2$, and a separation of 0.01 in.

***Solution.*** We may find that

$$S = 10 \times 0.0254^2 = 6.45 \times 10^{-3} \quad \text{m}^2$$

$$d = 0.01 \times 0.0254 = 2.54 \times 10^{-4} \quad \text{m}$$

and therefore

$$C = \frac{6 \times 8.854 \times 10^{-12} \times 6.45 \times 10^{-3}}{2.54 \times 10^{-4}} = 1.349 \quad \text{nF}$$

# Poisson's equation and Laplace's equation

look like, in terms of $V$? Well, $\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V$, so, apart from that persistent minus sign, the divergence of $\mathbf{E}$ is the Laplacian of $V$. Gauss's law, then, says

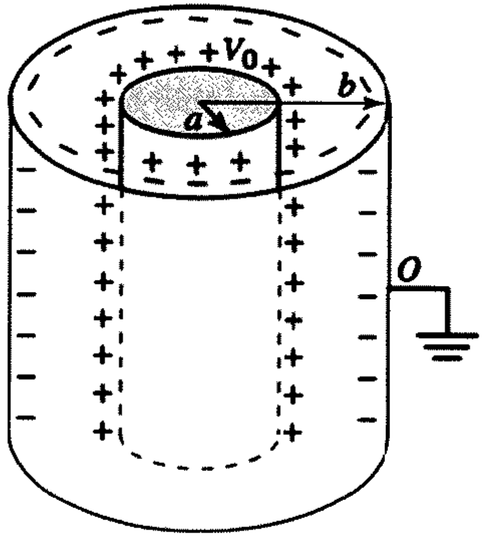
$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}.} \tag{2.24}$$

This is known as **Poisson's equation**. In regions where there is no charge, so $\rho = 0$, Poisson's equation reduces to **Laplace's equation**,

$$\nabla^2 V = 0. \tag{2.25}$$

# Problem 1

The inner conductor of radius $a$ of a coaxial cable (see Figure 3.41) is held at a potential of $V_0$ while the outer conductor of radius $b$ is grounded.




Determine (a) the potential distribution between the conductors, (b) the surface charge density on the inner conductor, and (c) the capacitance per unit length.

Since the two conductors of radii $a$ and $b$ form equipotential surfaces, we expect the potential $V$ to be a function of $\rho$ only. Thus, Laplace's equation reduces to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

Integrating twice, we obtain

$$V = c\ln\rho + d$$

where $c$ and $d$ are constants of integration.

At $\rho = b$, $V = 0 \Rightarrow d = -c\ln b$. Thus,

$$V = c\ln(\rho/b)$$

At $\rho = a$, $V = V_0 \Rightarrow c = V_0/\ln(a/b)$. Hence, the potential distribution within the region $a \le \rho \le b$ is

$$V = V_0\frac{\ln(\rho/b)}{\ln(a/b)}$$

The electric field intensity is

$$\vec{\mathbf{E}} = -\nabla V = -\frac{\partial V}{\partial\rho}\vec{\mathbf{a}}_\rho = \frac{V_0\vec{\mathbf{a}}_\rho}{\rho\ln(b/a)}$$

and the electric flux density is

$$\vec{\mathbf{D}} = \epsilon\vec{\mathbf{E}} = \frac{\epsilon V_0\vec{\mathbf{a}}_\rho}{\rho\ln(b/a)}$$

The normal component of $\mathbf{D}$ at $\rho = a$ yields the surface charge density on the inner conductor as

$$\rho_s = \frac{\epsilon V_0}{a\ln(b/a)}$$

The charge per unit length on the inner conductor is

$$Q = \frac{2\pi\epsilon V_0}{\ln(b/a)}$$

Finally, we obtain the capacitance per unit length as

$$C = \frac{2\pi\epsilon}{\ln(b/a)}$$

From (3.78) and (3.80), we obtain

$$D_{r1} = \frac{Q\epsilon_1}{2\pi r^2(\epsilon_1 + \epsilon_2)} \quad \text{and} \quad E_{r1} = \frac{Q}{2\pi r^2(\epsilon_1 + \epsilon_2)}$$

The potential of the inner sphere with respect to the outer sphere is

$$\begin{aligned} V_{ab} &= -\frac{Q}{2\pi(\epsilon_1 + \epsilon_2)} \int_b^a \frac{1}{r^2}\, dr \\ &= \frac{Q}{2\pi(\epsilon_1 + \epsilon_2)} \left[\frac{b-a}{ab}\right] \end{aligned}$$

Hence, the capacitance of the system is

$$C = 2\pi(\epsilon_1 + \epsilon_2)\left[\frac{ab}{b-a}\right] = C_1 + C_2$$

where

$$C_1 = 2\pi\epsilon_1\left[\frac{ab}{b-a}\right] \quad \text{and} \quad C_2 = 2\pi\epsilon_2\left[\frac{ab}{b-a}\right]$$

Note that $C_1$ and $C_2$ are the capacitances of medium 1 and medium 2, respectively. Thus, the capacitance of the system is equivalent to the parallel combination of the two capacitances. You may have already used this result in the analysis of electrical circuits. • • •

# Thank You