Engineering Electromagnetics

Lecture 37

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by

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Maxwell's equations

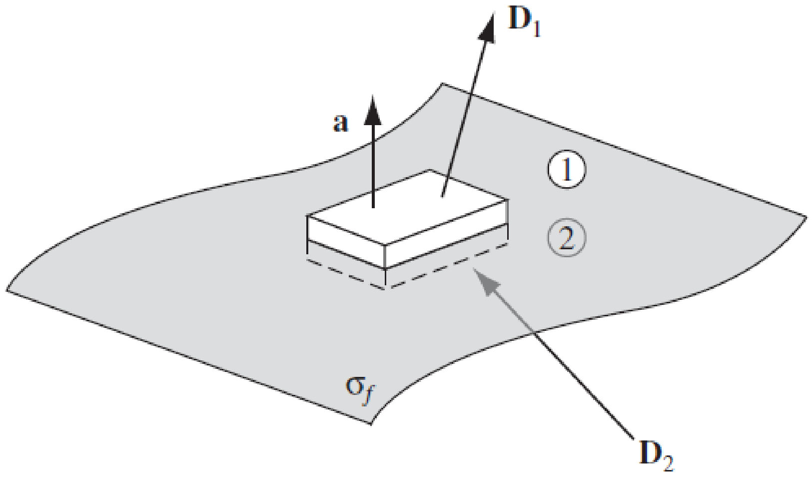
$$
\begin{aligned}
&\text{(i)} \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho && \text{(Gauss's law)},\\
&\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0 && \text{(no name)},\\
&\text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law)},\\
&\text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \text{(Ampère's law with Maxwell's correction)}.
\end{aligned}
$$

$$
\begin{aligned}
&\text{(i)} \quad \nabla \cdot \mathbf{D} = \rho_f, && \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},\\
&\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, && \text{(iv)} \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.
\end{aligned}
$$

$$
\left.
\begin{aligned}
&\text{(i)} \quad \oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}\\
&\text{(ii)} \quad \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0
\end{aligned}
\right\} \text{ over any closed surface } \mathcal{S}.
$$

$$
\left.
\begin{aligned}
&\text{(iii)} \quad \oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a}\\
&\text{(iv)} \quad \oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a}
\end{aligned}
\right\} \text{ for any surface } \mathcal{S} \text{ bounded by the closed loop } \mathcal{P}.
$$

Boundary conditions in Electrodynamics



$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f.}$$

$$\boxed{\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}.}$$

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0.} \quad \boxed{\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.}$$

General boundary conditions in case of *linear* media,

$$\left.\begin{array}{ll} \text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}. \end{array}\right\}$$

In particular, if there is no free charge or free current at the interface, then

$$\begin{array}{ll} \text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{0}. \end{array}$$

Poynting Theorem

In Chapter 2, we found that the work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is (Eq. 2.45)

$$W_e = \frac{\epsilon_0}{2} \int E^2 \, d\tau,$$

where **E** is the resulting electric field. Likewise, the work required to get currents going (against the back emf) is (Eq. 7.35)

$$W_m = \frac{1}{2\mu_0} \int B^2 \, d\tau,$$

where **B** is the resulting magnetic field. This suggests that the total energy stored in electromagnetic fields, per unit volume, is

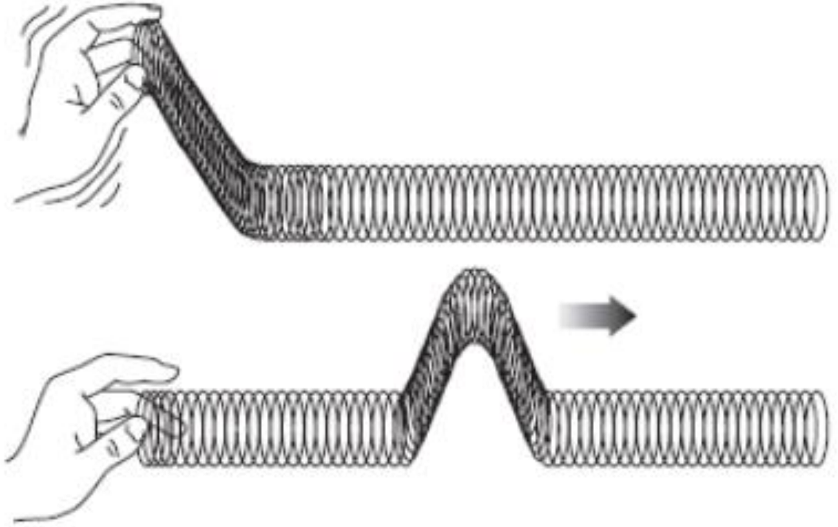
$$\boxed{u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right).}$$

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}).}$$

$$\frac{dW}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau - \frac{1}{\mu_0} \oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a},$$

where $\mathcal{S}$ is the surface bounding $\mathcal{V}$. This is **Poynting's theorem**; it is

Wave type-2

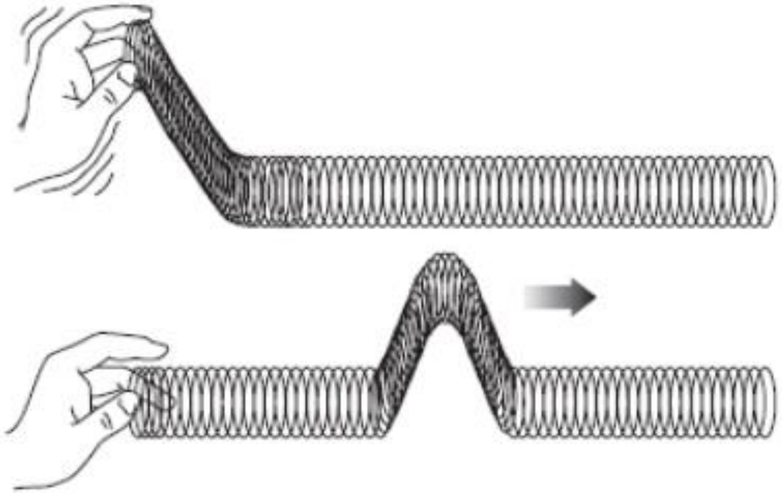


Q: What kind of wave it is?

Q: Medium is displaced in which direction?

Optics, Hecht

Wave type-2



(b) A transverse wave

Medium is displaced in a direction perpendicular to that of the motion of the wave

Optics, Hecht

A few points to note

- Which one moves→ disturbance or atoms/medium?
- Or the disturbance and medium both move in the direction of wave propagation?

A few points to note

- In all cases, the energy-carrying disturbance advances through the medium

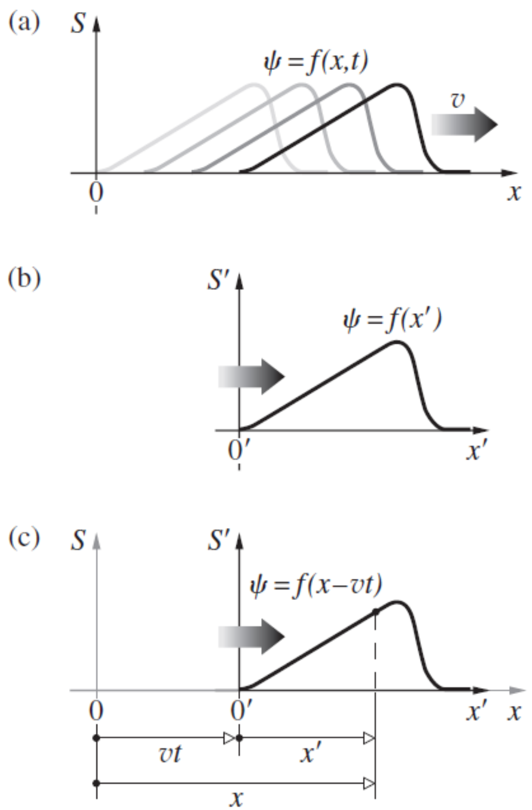
  NOT the individual participating atoms → remain in the vicinity of their equilibrium positions

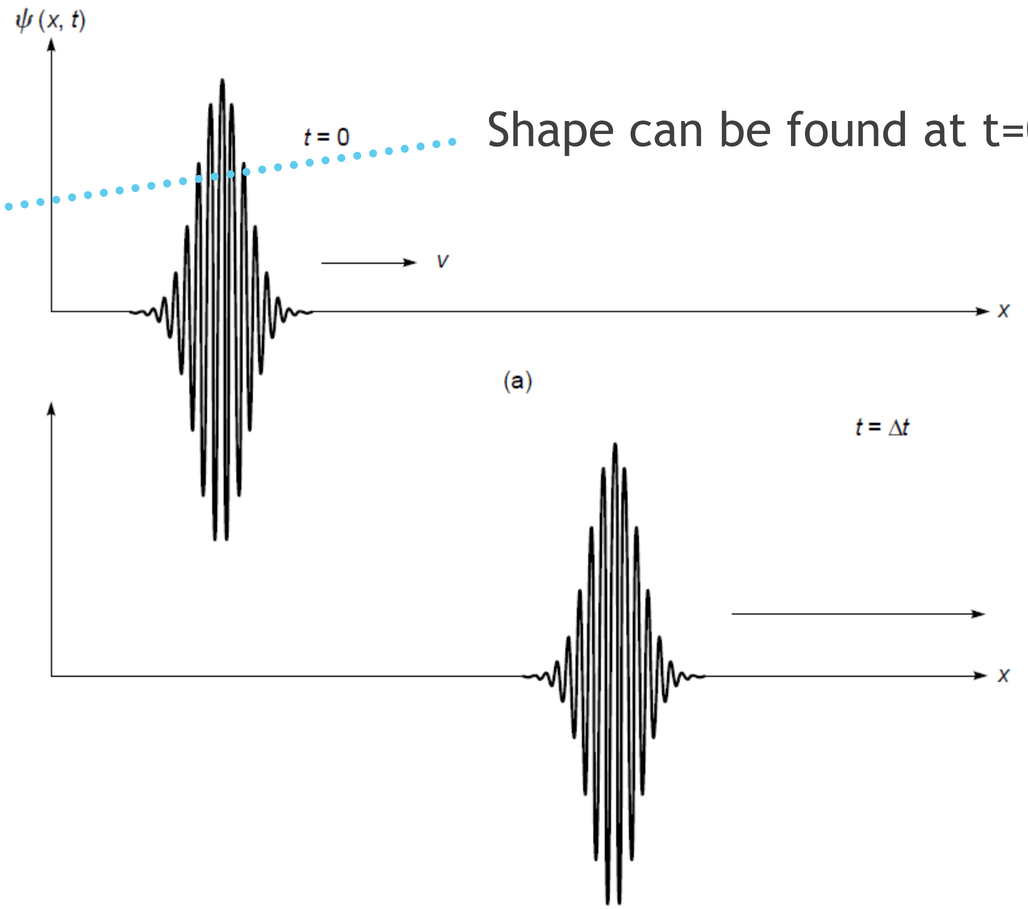
- disturbance advances, not the material medium.

- That’s one of several crucial features of a wave that distinguishes it from a stream of particles.

1D Wave

As disturbance moves: *ψ(x, t)* = *f(x, t)* [*f(x, t)* → *specific function or wave shape*]





Shape can be found at t=0

The disturbance at *t* in *S′* = at *t* = 0 in *S*. *So, x' to be replaced by x*-*vt* → ***ψ(x, t) = f(x-vt)***

Optics, Hecht; Ghatak

SINUSOIDAL WAVES: FREQUENCY AND WAVELENGTH

$$y(x, t) = a \cos k(x - vt)$$

- A wave propagating along _?_ direction.
- Can two points separated by a distance have same displacement?

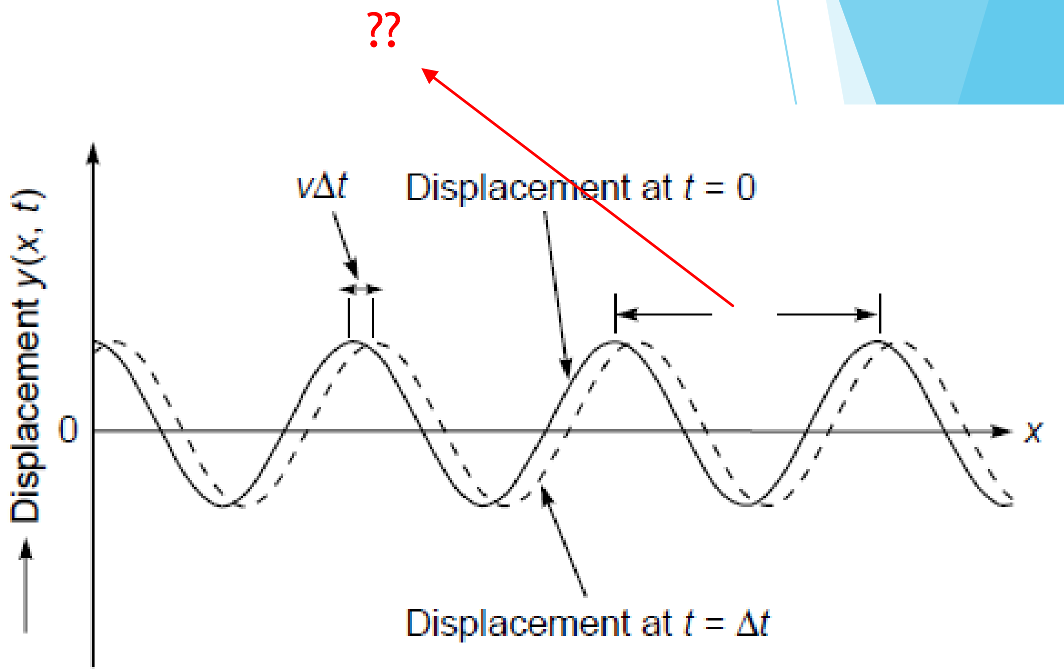


Fig. 11.4 The curves represent the displacement of a string at $t = 0$ and $t = \Delta t$, respectively, when a sinusoidal wave is propagating in the $+x$ direction.

Optics by Ghatak

SINUSOIDAL WAVES: FREQUENCY AND WAVELENGTH

$$y(x, t) = a \cos k(x - vt)$$

- It can be seen from the figure that, at a particular instant, any two points separated by a distance λ → same displacement
- [λ → wavelength]
- maximum displacement of the particle (from its equilibrium position) is ?
- which is known as the ___ of the wave.

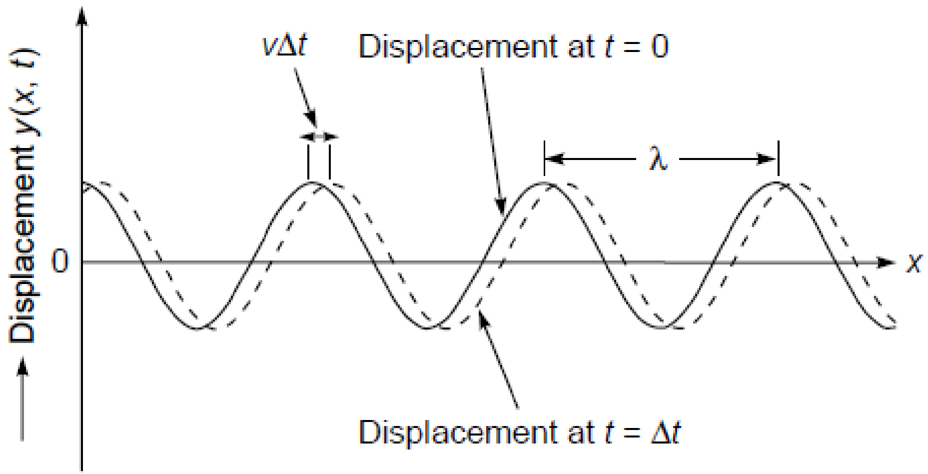


Fig. 11.4 The curves represent the displacement of a string at $t = 0$ and $t = \Delta t$, respectively, when a sinusoidal wave is propagating in the $+x$ direction.

Optics by Ghatak

SINUSOIDAL WAVES: FREQUENCY AND WAVELENGTH

$$y(x, t) = a \cos k(x - vt)$$

- It can be seen from the figure that, at a particular instant, any two points separated by a distance λ → same displacement
- λ → wavelength
- maximum displacement of the particle (from its equilibrium position) is → ‘a’
- which is known as the amplitude of the wave.

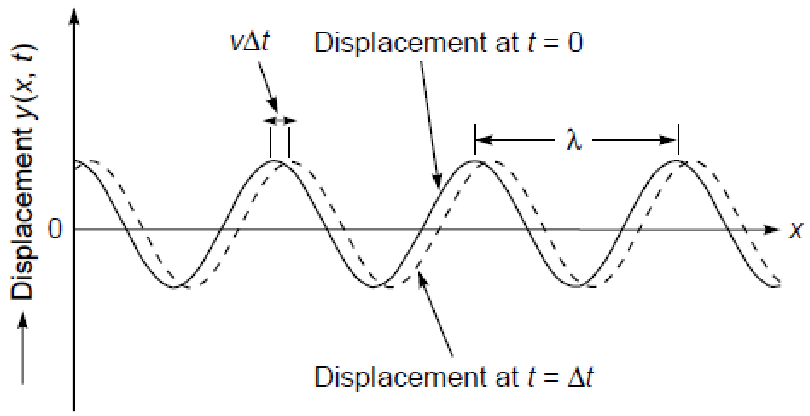


Fig. 11.4 The curves represent the displacement of a string at $t = 0$ and $t = \Delta t$, respectively, when a sinusoidal wave is propagating in the $+x$ direction.

SINUSOIDAL WAVES: Time dependence

$$y(x,t) = a\cos k(x - vt)$$

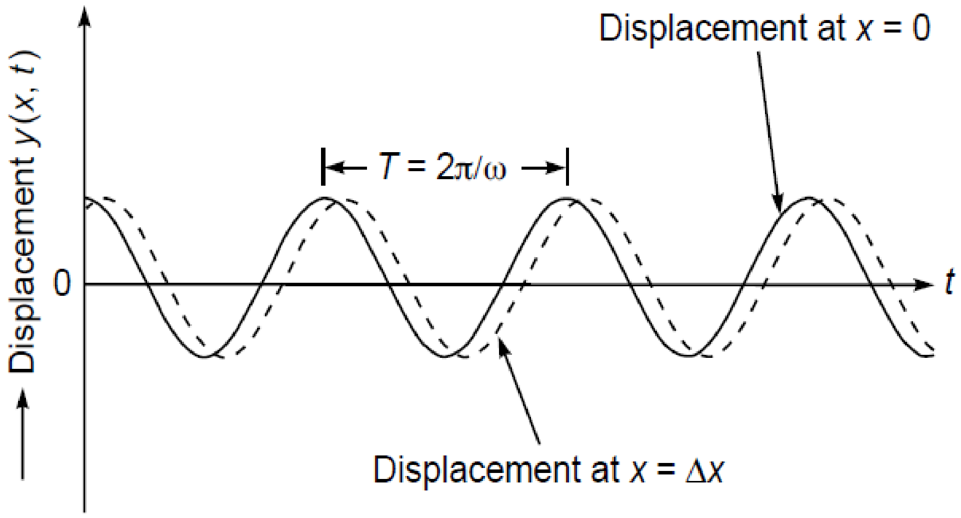


Fig. 11.5 The curves represent the time variation of the displacement of a string at $x = 0$ and $x = \Delta x$, respectively, when a sinusoidal wave is propagating in the $+x$ direction.

Optics by Ghatak

$$y(t) = a\cos\omega t \qquad \text{at } x = 0$$

$$y(t) = a\cos(\omega t - k\Delta x) \qquad \text{at } x = \Delta x$$

where

$$\omega = kv$$

- Corresponding to a particular point, the displacement repeats itself after a time __?
- Called Time period of the wave

$$T = 2\pi/\omega$$

- How is T related to v?
- No. of oscillation a particle carries out in 1s.

1D differential wave Equation

$$\psi(x, t) = f(x')$$
$$x' = x \mp vt,$$

taking the partial derivative w.r.t x, keeping t constant

$$\frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\partial f}{\partial x'} \qquad (1)$$

$$\frac{\partial x'}{\partial x} = \frac{\partial (x \mp vt)}{\partial x} = 1$$

partial derivative w.r.t time and keeping x constant

$$\frac{\partial \psi}{\partial t} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t} = \frac{\partial f}{\partial x'}(\mp v) = \mp v\frac{\partial f}{\partial x'} \qquad (2)$$

combining (1) & (2) $$\frac{\partial \psi}{\partial t} = \mp v\frac{\partial \psi}{\partial x} \qquad (3)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2} \qquad (4)$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\mp v\frac{\partial f}{\partial x'}\right) = \mp v\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial t}\right) \qquad (5)$$

from (2) : $$\frac{\partial \psi}{\partial t} = \frac{\partial f}{\partial t}$$

Hence, (5) becomes $$\frac{\partial^2 \psi}{\partial t^2} = \mp v\frac{\partial}{\partial x'}\left(\frac{\partial \psi}{\partial t}\right)$$

Using (2) again, $$\frac{\partial^2 \psi}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x'^2}$$

Or, $$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

1-D differential wave equation

Optics, Hecht

Wave equation

$$f(z,t) = A\cos[k(z - vt) + \delta]$$

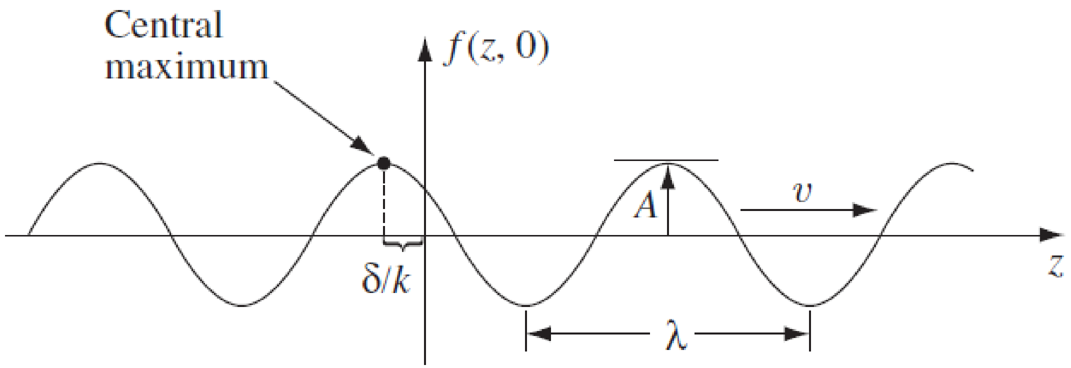
$$\lambda = \frac{2\pi}{k}$$

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2},}$$

$$f(z,t) = A\cos(kz - \omega t + \delta). \qquad (9.12)$$

A sinusoidal oscillation of wave number k and (angular) frequency ω traveling to the *left* would be written

$$f(z,t) = A\cos(kz + \omega t - \delta).$$



E = A cos (x/2-100t) V/m

E = A sin (y/2+100t) V/m

But E is a vector → ?

E = A cos (x/2-100t) $\hat{\mathbf{z}}$

From Griffith book

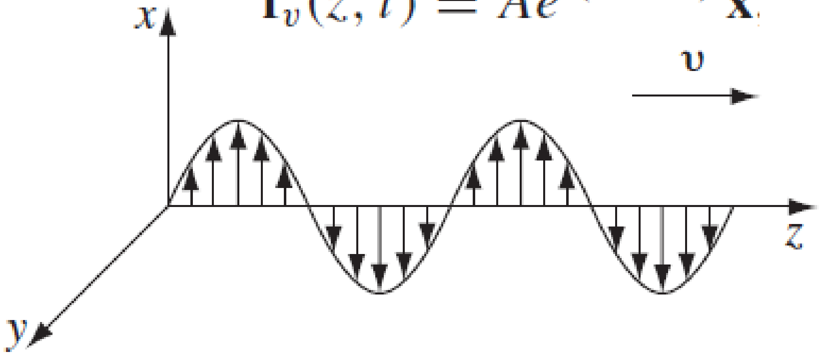
Polarization

- The waves that travel down a string when you shake it are called transverse, because the displacement is perpendicular to the direction of propagation.
- For longitudinal waves, displacement from equilibrium is along the direction of propagation.
- Sound waves, are longitudinal; electromagnetic waves, are transverse.
- Now there are, of course, two dimensions perpendicular to any given line of propagation. Accordingly, transverse waves occur in two independent states of polarization: you can shake the string up-and-down ("vertical" polarization) or left-and-right ("horizontal" polarization)

From Griffith book

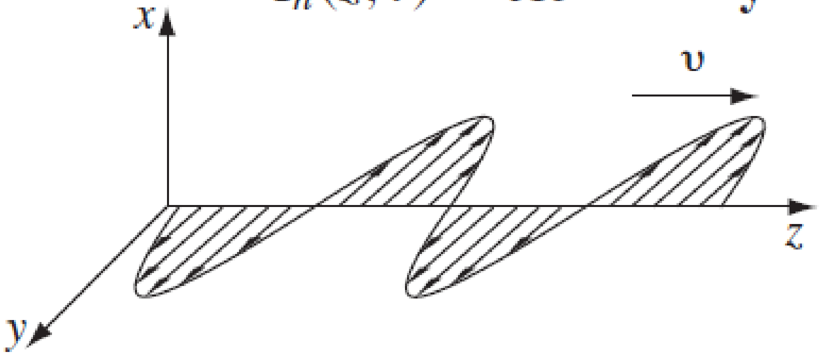
Polarization for transverse waves

$$\tilde{\mathbf{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{x}}$$



(a) Vertical polarization

$$\tilde{\mathbf{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$$



(b) Horizontal polarization

or along any other direction in the xy plane $\tilde{\mathbf{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{n}}$.

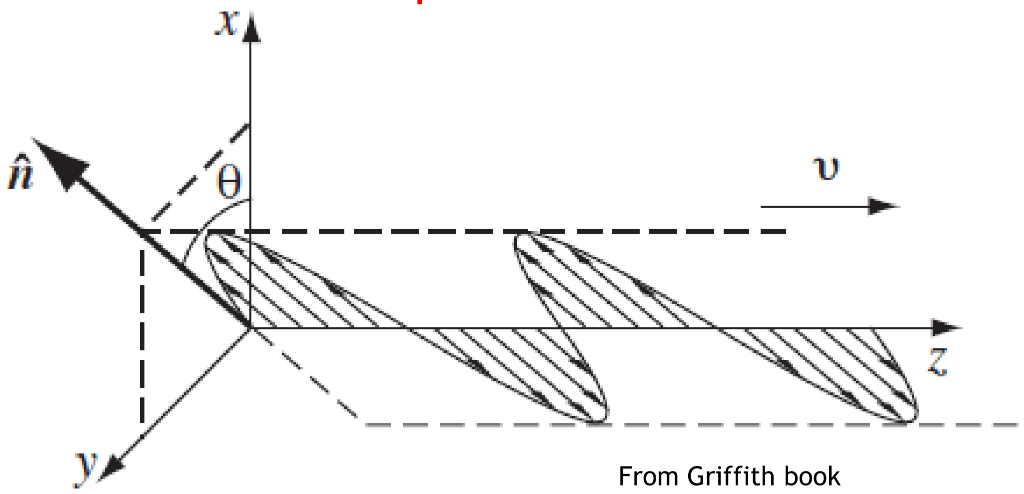
Linearly/plane polarized

The **polarization vector $\hat{\mathbf{n}}$** defines the plane of vibration.

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = 0.$$

(i) $\vec{E} = A\cos(x/2 - 10^8 t)\,\hat{z}$

(ii) $\vec{E} = B\sin(y/2 + 10^6 t)\,\hat{z}$



From Griffith book

EM waves

In regions of space where there is no charge or current, Maxwell's equations read

$$\left.\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{E} = 0, & \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\} \qquad (9.40)$$

They constitute a set of coupled, first-order, partial differential equations for **E** and **B**. They can be *de*coupled by applying the curl to (iii) and (iv):

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

EM wave equations

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Or, since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$,

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.}$$

In vacuum, then, each Cartesian component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the **three-dimensional wave equation**,

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}. \qquad v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s},$$

Plane waves

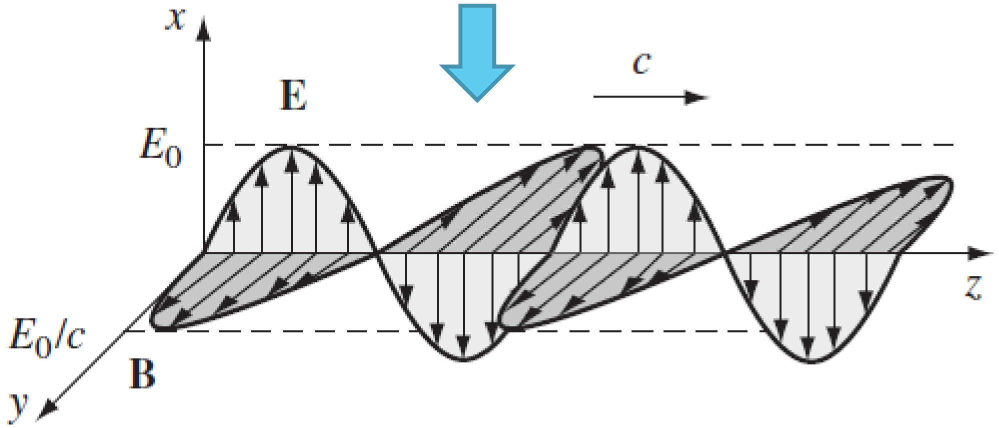
$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

$$\boxed{\mathbf{E}(z, t) = E_0 \cos(kz - \omega t + \delta)\, \hat{\mathbf{x}}, \quad \mathbf{B}(z, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\, \hat{\mathbf{y}}}$$

E and B in general

If **E** points in the x direction, then **B** points in the y direction

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{x}}, \quad \mathbf{B}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{y}}.$$



The scalar product $\mathbf{k} \cdot \mathbf{r}$ is the appropriate generalization of kz (Fig. 9.11), so

$$\mathbf{E}(\mathbf{r},t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)\,\hat{\mathbf{n}},$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c} E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)(\hat{\mathbf{k}} \times \hat{\mathbf{n}}).$$

where $\hat{\mathbf{n}}$ is the polarization vector. Because **E** is transverse,

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = 0.$$

$$\boxed{\begin{aligned}\mathbf{E}(\mathbf{r},t) &= E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)\,\hat{\mathbf{n}},\\ \mathbf{B}(\mathbf{r},t) &= \frac{1}{c} E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)(\hat{\mathbf{k}} \times \hat{\mathbf{n}}).\end{aligned}}$$

Energy

the energy per unit volume in electromagnetic fields

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right).$$

$$B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2,$$

so the *electric and magnetic contributions are equal*:

$$u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2 (kz - \omega t + \delta)$$

EM Waves in vacuum

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

$$B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2,$$

$$\frac{E}{H} = \sqrt{(\mu_0/\varepsilon_0)}$$

(1) $\mathbf{E} = E_0 \sin(\omega t - ky)\, \hat{\mathbf{z}}$

What is **B**?

If $B = B_0 \sin(\omega t - ky)$
how is B_0 related to E_0?

(2) Our planet is receiving 10 cal/(m^2) per sec energy from the Sun, calculate the amplitudes of the electric field and magnetic fields.

Electromagnetic waves in a matter

Inside matter, but in regions where there is no *free* charge or *free* current, Maxwell's equations become

$$\left.\begin{array}{llll}\text{(i)} & \nabla\cdot\mathbf{D}=0, & \text{(iii)} & \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}, \\[2ex] \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} & \nabla\times\mathbf{H}=\dfrac{\partial\mathbf{D}}{\partial t}.\end{array}\right\} \tag{9.65}$$

If the medium is *linear*,

$$\mathbf{D}=\epsilon\mathbf{E},\quad \mathbf{H}=\frac{1}{\mu}\mathbf{B}, \tag{9.66}$$

and *homogeneous* (so ϵ and μ do not vary from point to point), they reduce to

$$\left.\begin{array}{llll}\text{(i)} & \nabla\cdot\mathbf{E}=0, & \text{(iii)} & \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}, \\[2ex] \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} & \nabla\times\mathbf{B}=\mu\epsilon\dfrac{\partial\mathbf{E}}{\partial t},\end{array}\right\} \tag{9.67}$$

which differ from the vacuum analogs (Eqs. 9.40) only in the replacement of $\mu_0\epsilon_0$ by $\mu\epsilon$.[10] Evidently electromagnetic waves propagate through a linear homogeneous medium at a speed

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}, \tag{9.68}$$

where

$$n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \tag{9.69}$$

is the **index of refraction** of the substance. For most materials, μ is very close to μ_0, so

$$n \cong \sqrt{\epsilon_r}, \tag{9.70}$$

Boundary conditions for waves in a matter: No free charge or current

$$\text{(i)}\ \ \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, \qquad \text{(iii)}\ \ \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel},$$

$$\text{(ii)}\ \ B_1^{\perp} = B_2^{\perp}, \qquad \text{(iv)}\ \ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}.$$

Thank You