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Differential Equation (MA1001)

Tutorial-3: Series solution for ODE and Legendre polynomial and Bessel Function)

1 Find ordinary and singular (regular and irregular singular) points of the following differential equations.

(a) $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0$

(b) $(x-1)\dfrac{d^2y}{dx^2} + (\cot \pi x)\dfrac{dy}{dx} + (\csc^2 \pi x)y = 0$

(c) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-n^2)y = 0$

(d) $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-n^2)y = 0$

(e) $x^4\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x^{-1}$

2 Find the roots of the indicial equation for following differential equations about $x=0$.

(a) $4x^2\dfrac{d^2y}{dx^2} - 4xe^x\dfrac{dy}{dx} + 3\cos(x)y = 0$

(b) $2x^2\dfrac{d^2y}{dx^2} + x(x+1)\dfrac{dy}{dx} - 3\cos(x)y = 0$

(c)$x^4\dfrac{d^2y}{dx^2} - x^2\sin(x)\dfrac{dy}{dx} + 2(1-\cos(x))y = 0$

3 Find the solution of the following differential equations about point $x=0$.

(a) $2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0$

(b) $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0$

(c) $9x(1-x)\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0$

(d) $x(1-x)\dfrac{d^2y}{dx^2} + (1-5x)\dfrac{dy}{dx} - 4y = 0$

(e) $x\dfrac{d^2y}{dx^2} + (p-x)\dfrac{dy}{dx} - y = 0$

(f) $(2+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (1+x)y = 0$

4 Prove that the solution of the Bessel differential equation is in form $AJ_n(x) + BJ_{-n}(x)$, where

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

is first kind Bessel function, and A and B are constants and n is not an integer.

5 Find the roots of indicial equation for Legendre differential equation:$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ for large value of x, when n is positive integer. (Hint: find indicial equation about point $x=\infty$).

6 Prove the following relations for Bessel function.

(i) $J_n(-x) = (-1)^n J_n(x)$

(ii) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$

(iii) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$

7 Find the polynomials for $P_1(x),\ P_2(x),\ P_3(x),\ P_4(x)$ and $P_5(x)$.

8 Let P_n be a solution of the Legendre differential equation: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ then prove that

(i) $P_n(1) = \dfrac{1}{2}n(n+1)$ (ii) $P_n(1) = (-1)^{n-1}\dfrac{1}{2}n(n+1)$ (Hint: $P_n(1) = (-1)^n$)

9 Let $f(x) = \sum_{r=0}^{\infty} a_n P_n(x)$. Show that coefficients a_n is given by

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x) dx$$

(Hint: Use orthogonal property of Legendre polynomial.)

10 Let $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ and $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$, prove that:
(i) $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

(ii) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

11 Let $y : [-1, 1] \to \mathbb{R}$ with $y(1) = 1$ satisfy the Legendre differential equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 6y = 0 \text{ for } |x| < 1.$$

Find the value of $\int\limits_{-1}^{1} y(x)(x + x^2) dx$

12 Let y be a polynomial solution of the differential equation

$$(1 - x^2) y'' - 2xy' + 6y = 0.$$

If $y(1) = 2$, then find the value of the integral $\int\limits_{-1}^{1} y^2 dx$.

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