Invertible linear transformations

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Invertible function

A function $f : X \longrightarrow Y$ is invertible if there exists a function $g : Y \longrightarrow X$ such that
(i) $gof : X \longrightarrow X$ and
(ii) $fog : Y \longrightarrow Y$ are identity functions.

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$$f(x) = f(y) \Longrightarrow x = y$$

Lemma

A linear transformation : $V \longrightarrow W$ is invertible if and only if

(1) T is $1:1$, that is, $T(\alpha) = T(\beta) \Longrightarrow \alpha = \beta$.

(2) T is onto, that is, $R(T) = W$.

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Proof : Suppose that $T : V \longrightarrow W$ is an invertible linear transformation. Then there exists a function $T^{-1} : W \longrightarrow V$ such that

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(ii) $T^{-1}T : V \longrightarrow V$ are identity functions.

It is enough to prove that

$$T^{-1}(c\beta_1 + \beta_2) = cT^{-1}(\beta_1) + T^{-1}(\beta_2) \text{ for all } \beta_1, \beta_2 \in W, c \in F$$

Theorem 7 contd.

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Let $\alpha_1 = T^{-1}(\beta_1)$ and $\alpha_2 = T^{-1}(\beta_2)$. Since T is invertible, α_1, α_2 are unique vectors in V such that $T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$.

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Let $\alpha_1 = T^{-1}(\beta_1)$ and $\alpha_2 = T^{-1}(\beta_2)$. Since T is invertible, α_1, α_2 are unique vectors in V such that $T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$. Since T is a linear transformation,

$$T(c\alpha_1 + \alpha_2) = cT(\alpha_1) + T(\alpha_2)$$

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$$T(c\alpha_1 + \alpha_2) = cT(\alpha_1) + T(\alpha_2) = c\beta_1 + \beta_2$$

$$\implies T^{-1}(c\beta_1 + \beta_2) = c\alpha_1 + \alpha_2 = cT^{-1}(\beta_1) + T^{-1}(\beta_2)$$

Hence T^{-1} is a linear transformation.

Non-singular linear transformation

A linear transformation $T : V \longrightarrow W$ is non-singular if

$$T(\alpha) = 0 \quad \Longrightarrow \quad \alpha = 0$$

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$\Longrightarrow \alpha \in \{0\}$ and thus $N(T) \subseteq \{0\} - - - -(2)$.
From (1) and (2)

$$N(T) = \{0\}$$

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$$\Longrightarrow T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k) = 0$$

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Hence $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_k)\}$ is a linearly independent subset of W and it completes Case 1.

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$$T^{-1}(Z) = A^{-1}Z = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \Longrightarrow T^{-1}(z_1, z_2) = (z_2, z_1 - z_2)$$

Problem 2

Find the inverse of a linear operator T on R^3 defined as

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$$T(x_1, x_2, x_3) = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Problem 2 contd.

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Problem 2 contd.

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = [I|A^{-1}]$$

Problem 2 contd.

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

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$$T^{-1}(Y) = A^{-1}Y$$

Problem 2 contd.

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$$T^{-1}(Y) = A^{-1}Y$$

$$T^{-1}(y_1, y_2, y_3) = \left(\frac{1}{3}y_1, \frac{1}{3}y_1 - y_2, -y_1 + y_2 + y_3\right)$$

Theorem 9

Let V and W be finite dimensional vector spaces over the field such that dim $V =$ dim W. If $T : V \longrightarrow W$ is a linear transformation, the following are equivalent.

(i) T is invertible.

(ii) T is non-singular.

(iii) T is onto, that is, $R(T) = W$.

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Proof : Let dim $V =$ dim $W = n$.

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Proof : Let dim $V =$ dim $W = n$.
By Rank-Nullity-Dimension Theorem,

$$\text{rank } (T) + \text{ nullity } (T) = n - - - -(1)$$

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

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First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one.

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2).

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1).

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$.

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First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$. $\implies$ rank $(T) = n$, see (1) .

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First we prove that $(i) \implies (ii) \implies (iii)$.

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First we prove that $(i) \implies (ii) \implies (iii)$.

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$\implies R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim W).
$\implies$ T is onto.

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Next we prove that $(iii) \implies (i)$.

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Next we prove that $(iii) \implies (i)$.

T is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.

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Next we prove that $(iii) \implies (i)$.

T is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.
$\implies$ nullity $(T) = 0$, see (1). $\implies N(T) = \{0\}$.

Theorem 9 contd.

First we prove that $(i) \Longrightarrow (ii) \Longrightarrow (iii)$.

T is invertible. $\Longrightarrow$ T is one to one. $\Longrightarrow$ T is non-singular (by Lemma 2). $\Longrightarrow N(T) = \{0\}$ (by Lemma 1). $\Longrightarrow$ nullity $(T) = 0$.
$\Longrightarrow$ rank $(T) = n$, see (1) . $\Longrightarrow$ dim $R(T) =$ dim W.
$\Longrightarrow R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim W).
$\Longrightarrow$ T is onto.

Next we prove that $(iii) \Longrightarrow (i)$.

T is onto. $\Longrightarrow R(T) = W$. $\Longrightarrow$ rank $(T) =$ dim $W = n$.
$\Longrightarrow$ nullity $(T) = 0$, see (1). $\Longrightarrow N(T) = \{0\}$.

Claim : T is one to one.

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$.
$\implies$ rank $(T) = n$, see (1) . $\implies$ dim $R(T) =$ dim W.
$\implies R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim W).
$\implies$ T is onto.

Next we prove that $(iii) \implies (i)$.

T is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.
$\implies$ nullity $(T) = 0$, see (1). $\implies N(T) = \{0\}$.

Claim : T is one to one.

Let $T(\alpha) = T(\beta)$.

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$.

$\implies$ rank $(T) = n$, see (1) . $\implies$ dim $R(T) =$ dim W.

$\implies R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim W).

$\implies$ T is onto.

Next we prove that $(iii) \implies (i)$.

T is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.

$\implies$ nullity $(T) = 0$, see (1). $\implies N(T) = \{0\}$.

Claim : T is one to one.

Let $T(\alpha) = T(\beta)$. $\implies T(\alpha - \beta) = 0$.

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$.

$\implies$ rank $(T) = n$, see (1) . $\implies$ dim $R(T) =$ dim W.

$\implies R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim W).

$\implies$ T is onto.

Next we prove that $(iii) \implies (i)$.

T is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.

$\implies$ nullity $(T) = 0$, see (1). $\implies N(T) = \{0\}$.

Claim : T is one to one.

Let $T(\alpha) = T(\beta)$. $\implies T(\alpha - \beta) = 0$. $\alpha - \beta \in N(T) = \{0\}$.

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$.

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$\implies R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim W).

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Next we prove that $(iii) \implies (i)$.

T is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.

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Claim : T is one to one.

Let $T(\alpha) = T(\beta)$. $\implies T(\alpha - \beta) = 0$. $\alpha - \beta \in N(T) = \{0\}$.

$\implies \alpha = \beta$.

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$. $\implies$ rank $(T) = n$, see (1) . $\implies$ dim $R(T) =$ dim W.

$\implies R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim W).

$\implies$ T is onto.

Next we prove that $(iii) \implies (i)$.

T is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.

$\implies$ nullity $(T) = 0$, see (1). $\implies N(T) = \{0\}$.

Claim : T is one to one.

Let $T(\alpha) = T(\beta)$. $\implies T(\alpha - \beta) = 0$. $\alpha - \beta \in N(T) = \{0\}$.

$\implies \alpha = \beta$. $\implies$ T is one to one.

Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

T is invertible. $\implies$ T is one to one. $\implies$ T is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$.
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$\implies R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim W).
$\implies$ T is onto.

Next we prove that $(iii) \implies (i)$.

T is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.
$\implies$ nullity $(T) = 0$, see (1). $\implies N(T) = \{0\}$.

Claim : T is one to one.

Let $T(\alpha) = T(\beta)$. $\implies T(\alpha - \beta) = 0$. $\alpha - \beta \in N(T) = \{0\}$.
$\implies \alpha = \beta$. $\implies$ T is one to one. Note that T is onto (assumption) and T is one to one (above Claim). Hence T is invertible.

Theorem 9A (Assignment)

Let V and W be finite dimensional vector spaces over the field such that dim $V =$ dim W. If $T : V \longrightarrow W$ is a linear transformation, the following are equivalent.

(i) T is invertible.

(ii) T is non-singular.

(iii) T is onto, that is, $R(T) = W$.

(iv) If $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V, then $\{T(\alpha_1), \ldots, T(\alpha_n)\}$ is a basis for W.

(v) There is some basis $\{\alpha_1, \ldots, \alpha_n\}$ for V such that $\{T(\alpha_1), \ldots, T(\alpha_n)\}$ is a basis for W.