MA1002: Linear Algebra

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Text books

1. Linear Algebra, Kenneth Hoffman and Ray Kunze, Prentice-Hall, Second Edition (available online).
2. Topics in Algebra, I. N. Herstein, Wiley.
3. Linear Algebra and its Applications, Gilbert Strang, 4th edition.
4. Introduction to Linear Algebra, Krishnamurthi (BITS Pilani).

Evaluation Scheme (Tentative)

Quiz 1	25
Quiz 2	25
End Semester Examination	50

Applications of Linear Algebra

- To summarize and manipulate data (Machine Learning, Image Processing)
- To model satellites, jet engines (Eigen values)
- Many more!
- Prepare a detailed report of an application of linear algebra tools in CSE/ECE/ME and Design

You have first assignment !

Mathematical Structures

1. Field (Members are called scalars)
2. Vector Space (Members are called vectors)

A non empty set F together with operations

(i)$+ : F \times F \to F$ (addition) and

(ii) $\cdot \ : F \times F \to F$ (multiplication)

is said to be a field if the following axioms and properties are satisfied.

Closure axiom: For every $a, b \in F \implies$

$$a + b \in F \quad \text{and} \quad a \cdot b \in F$$

Associative axiom: For all $a, b, c \in F \implies$

$$a + (b + c) = (a + b) + c \text{ and}$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

Identity axiom: There exist elements $0, 1 \in F$ such that

$$a + 0 = 0 + a = a, \qquad \forall a \in F$$
$$a \cdot 1 = 1 \cdot a = a, \quad \forall a \in F$$

Inverse axiom:(i) For every $a \in F$, there exists $b \in F$ such that

$$a + b = b + a = 0$$

(b is called additive inverse of a)

and (ii) for every $a \in F - \{0\}$, there exists $c \in F - \{0\}$ such that

$$a \cdot c = c \cdot a = 1$$

(c is called multiplicative inverse of a)

Commutative Property:

$$a + b = b + a \quad \forall a, b \in F \quad \text{and}$$
$$a \cdot b = b \cdot a \quad \forall a, b \in F$$

Distributive Property: For every $a, b, c \in F$

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and}$$
$$(a + b) \cdot c = a \cdot c + b \cdot c$$

Notation: A field F with respect to operations $+, \cdot$ is usually denoted as $(F, +, \cdot)$

$\mathbb{N} = \{1, 2, 3, \dots, \}$ - set of all natural numbers

Is $(\mathbb{N}, +, \cdot)$ a field?

Closure axiom:

We know that

for all $a, b \in \mathbb{N}, \quad a + b$ and $a \cdot b$ are also elements of $\mathbb{N}$

Associative axiom:

Associative property is true for $\mathbb{N}$ for both $+$ and $\cdot$

Identity axiom:

$0 \notin \mathbb{N}$ and hence identity axiom is not satisfied

Thus $(\mathbb{N}, +, \cdot)$ is not a field.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ - set of all integers

Is $(\mathbb{Z}, +, \cdot)$ a field?

Closure axiom: For all $a, b \in \mathbb{Z}$, $\quad a + b$ and $a \cdot b$ are also elements of $\mathbb{Z}$

Associative axiom: Associative property is true for $\mathbb{Z}$ for both $+$ and $\cdot$

Identity axiom: There exists $0, 1 \in \mathbb{Z}$ such that $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$, $\forall a \in \mathbb{Z}$

Inverse axiom: For every $a \in \mathbb{Z}$, there exists $-a \in \mathbb{Z}$ such that $a + (-a) = -a + a = 0$. But for $2 \in \mathbb{Z}$ there doesnot exist a multiplicative inverse in $\mathbb{Z}$

Thus $(\mathbb{Z}, +, \cdot)$ is not a field

$\mathbb{Z}_2 = \{0, 1\}$ - congruence class modulo 2.

Is $(\mathbb{Z}_2, +, \cdot)$ a field?

Closure axiom:

$$0 + 0 = 0, \quad 1 + 0 = 1, \quad 0 + 1 = 1, \quad 1 + 1 = 0$$
$$0 \cdot 0 = 0, \quad 1 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 1 = 1$$

Thus, the closure axiom is true.

Associative axiom:

From closure axiom we can see that associative axiom also holds true.

Identity axiom:

$0 \in \mathbb{Z}_2$ is the additive identity and $1 \in \mathbb{Z}_2$ is the multiplicative identity.

Inverse axiom:

Additive inverse of 0 is 0 and 1 is 1.

Multiplicative inverse of 1 is 1.

Commutative property:

From closure axiom, we can see that addition and multiplication are commutative in $\mathbb{Z}_2$

Distributive property: For every $a, b \in \mathbb{Z}_2$

$$(a+b) \cdot 0 = 0 = a \cdot 0 + b \cdot 0$$
$$(a+b) \cdot 1 = a + b = a \cdot 1 + b \cdot 1$$

Thus distributive law holds.

Therefore, $(\mathbb{Z}_2, +, \cdot)$ is a field.

$\mathbb{R}$ - Set of real numbers.

Is $(\mathbb{R}, +, \cdot)$ a field?

Closure axiom:

Addition of two real numbers is a real number. Similarly,

Multiplication of two real numbers is a real number.

Thus, the closure axiom is true.

Associative axiom:

Addition and Multiplication are always associative in $\mathbb{R}$.

Identity axiom:

$0 \in \mathbb{R}$ is the additive identity and $1 \in \mathbb{R}$ is the multiplicative identity.

Inverse axiom:

Additive inverse of an element $a \in \mathbb{R}$ is $-a$.

Multiplicative inverse of an element $a \in \mathbb{R} - \{0\}$ is $\frac{1}{a}$.

Commutative property:

It is true that for every $a, b \in \mathbb{R}$

$$a + b = b + a \text{ and } a \cdot b = b \cdot a$$

Distributive property:

Distributive property is true for real numbers with respect to $+$ and $\cdot$

Therefore, $(\mathbb{R}, +, \cdot)$ is a field.

Questions

$\mathbb{Q}$- Set of all rational numbers

$\mathbb{C}$- Set of all complex numbers

1. Is $(\mathbb{Q}, +, \cdot)$ a field?
2. Is $(\mathbb{C}, +, \cdot)$ a field?

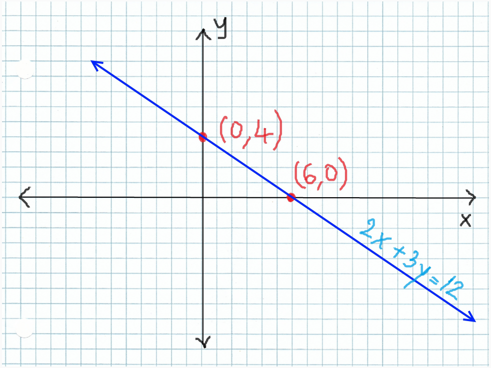
Linear Equations

Equation of a line :

$$y = mx + c$$

$$ax + by = c$$

Example of a line

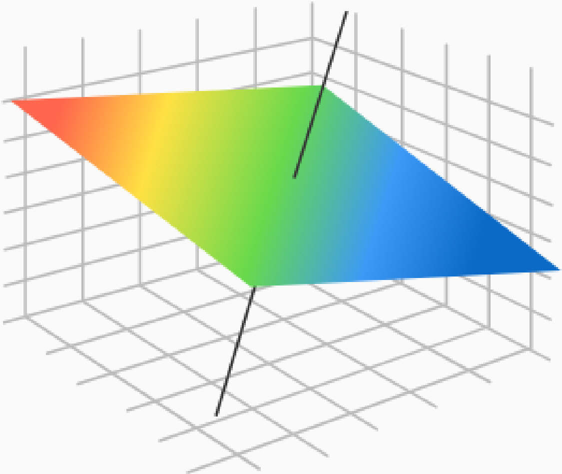


Equation of a plane :

$$z = ax + by + c$$

$$Ax + By + Cz = D$$

Example of a plane



Systems of linear equations

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n &= b_1 \\
A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad\qquad & \vdots \\
A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n &= b_m
\end{aligned}
$$

- m equations
- n varaibles $(x_1, x_2, \ldots, x_n)$
- $A_{ij}, b_i \in F \quad (F$ is a field)

Matrix form

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

$$AX = B$$

$A = [A_{ij}]_{m \times n}$, $1 \leq i \leq m$, $1 \leq j \leq n$

$X = [x_j]_{n \times 1}$, $\qquad B = [b_i]_{m \times 1}$

The solution set of the linear system $AX = B$ is

$$S = \{X \in R^n \ : \ AX = B\}$$

Problem 1

Solve the following system of linear equations

$$4x - y = 5$$

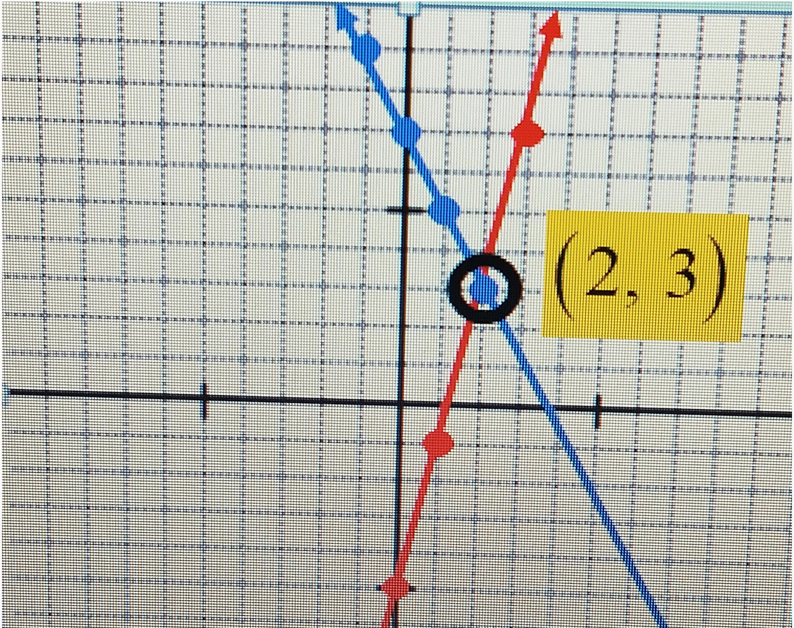
$$2x + y = 7$$

Matrix form

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Solve graphically (sketch it!)

Problem 1 (solution)



Problem 1

Solve the following system of linear equations

$$4x - y = 5$$

$$2x + y = 7$$

Matrix form

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Solve graphically (sketch it!)

Solution $S = \{(2, 3)\}$

Our objective is to design an efficient machinery to solve $AX = B$

Solution of Problem 1 through elementary row operations

$$4x - y = 5 \quad (Eq1)$$

$$2x + y = 7 \quad (Eq2)$$

Let us employ a high school technique

Multiply (Eq2) by 2

$$4x - y = 5 \quad (Eq1)$$

$$4x + 2y = 14 \quad (Eq2)$$

(Eq2)-(Eq1) $\Longrightarrow$

$$4x - y = 5 \quad (Eq1)$$

$$0x + 3y = 9 \quad (Eq2)$$

We have three equivalent systems say red, blue and green

contd.

Let us express red system in augmented matrix form

$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right]$$

Multiply second row by $2 \qquad (R_2 \longleftarrow 2 \times R_2)$

$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 4 & 2 & 14 \end{array}\right]$$

Substract first row from second row $\qquad (R_2 \longleftarrow R_2 - R_1)$

$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 4 & 2 & 14 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 0 & 3 & 9 \end{array}\right]$$

Contd.

$$\left[\begin{array}{cc|c} 4 & -1 & 5 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 4 & 2 & 14 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 0 & 3 & 9 \end{array}\right]$$

$R_2 \longleftarrow \frac{1}{3}R_2 \quad \Longrightarrow$

$$\sim \left[\begin{array}{cc|c} 4 & -1 & 5 \\ 0 & 1 & 3 \end{array}\right]$$

$R_1 \longleftarrow \frac{1}{4}R_1 \quad \Longrightarrow$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & 3 \end{array}\right]$$

Contd.

$R_1 \longleftarrow \frac{1}{4}R_1 \qquad \Longrightarrow$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & 3 \end{array}\right]$$

$R_1 \longleftarrow R_1 + \frac{1}{4}R_2 \qquad \Longrightarrow$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Let us write it in the system of equations form

$$x + 0y = 2$$

$$0x + y = 3$$

Salient points

- Multiplying an equation by a non-zero scalar preserves the solution space ($R_i \longleftarrow cR_i, \quad c \neq 0$)
- Replacing i^{th} equation by sum of i^{th} equation and constant multiple of j^{th} equation preserves the solution space. ($R_i \longleftarrow R_i + cR_j$)
- Interchanging two equations preserves the solution space. ($R_i \longleftrightarrow R_j$)

Problem 2

Solve the following system of linear equations.

$$3x - 2y = -6, \quad x + 2y = -10$$

Solve graphically !

Augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -2 & -6 \\ 1 & 2 & -10 \end{array}\right]$$

Interchange first and second rows ($R_1 \longleftrightarrow R_2$)

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -10 \\ 3 & -2 & -6 \end{array}\right]$$

Problem 2 contd.

$R_2 \longleftarrow R_2 - 3R_1 \implies$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -10 \\ 0 & -8 & 24 \end{array}\right]$$

$R_2 \longleftarrow -\frac{1}{8}R_2$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -10 \\ 0 & 1 & -3 \end{array}\right]$$

Problem 2 contd.

$R_1 \longleftarrow R_1 - 2R_2 \implies$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & -3 \end{array}\right]$$

$$x = -4, \quad y = -3$$

Solution $S = \{(-4, -3)\}$

Problem 3

Solve the system of linear equations

$$x + y = 2, \quad 2x + 2y = 5$$

Solve graphically!

Augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 5 \end{array}\right] \quad (R_2 \longleftarrow R_2 - 2R_1)$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]$$

Second row is $0x + 0y = 1$. No solution

Problem 4

Solve the system

$$x + 2y = 5, \quad 2x + 4y = 10$$

Solve graphically! **Augmented matrix is**

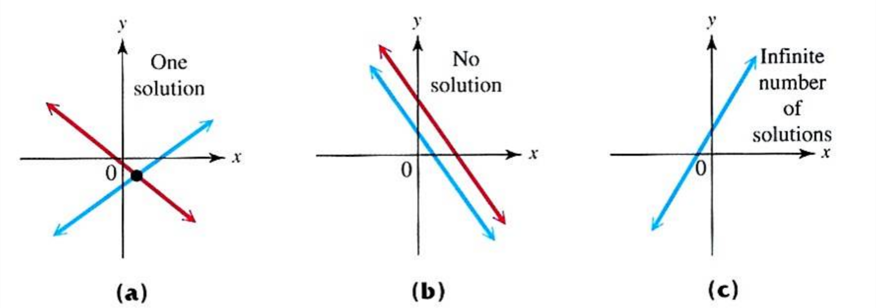
$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 4 & 10 \end{array}\right] \quad (R_2 \longleftarrow R_2 - 2R_1)$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \end{array}\right] \quad \Longrightarrow \quad x + 2y = 5$$

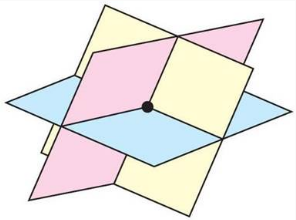
Let $y = c \quad \Longrightarrow \quad x = 5 - 2c$**.**

The solution $S = \{(5 - 2c, c) \ : \ c \in \mathbf{R}\}$

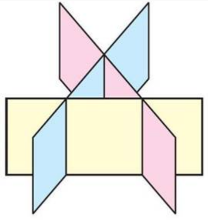
two dimensional problem and possible solutions

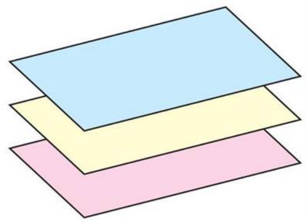


3-dimensional problem with a unique solution

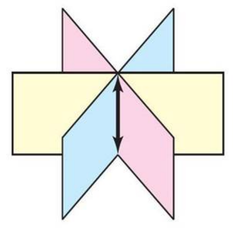


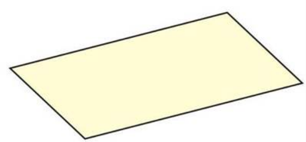
3-dimensional problem with no solutions





3-dimensional problem with infinite number of solutions





Linear combination of equations

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1 \qquad (1)$$

$$A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \qquad (2)$$

Consider $c_1(1) + c_2(2)$(a linear combination) $\implies$

$$c_1\left(A_{11}x_1 + A_{12}x_2 + A_{13}x_3\right) + c_2\left(A_{21}x_1 + A_{22}x_2 + A_{23}x_3\right)$$

$$= c_1 b_1 + c_2 b_2 \qquad (3)$$

Suppose that $x_1 = a, x_2 = b, x_3 = c$ is solution of (1) and (2)

Show that above solution is also a solution of (3).

Consider L.H.S. of (3),

$$c_1 (A_{11}a + A_{12}b + A_{13}c) + c_2 (A_{21}a + A_{22}b + A_{23}c)$$

$$= c_1 b_1 + c_2 b_2$$

So $x_1 = a, x_2 = b, x_3 = c$ is solution of (3)

Converse need not be true (Try !)

Note: If X^* is a solution of k linear equations, then X^* is also a solution of a linear combination of those k equations.

Equivalent systems

We say two sytems are **equivalent** if each equation in each system is a linear combination of equations in the other system.

Why do we focus on equivalent systems?

Theorem 1: Equivalent systems of linear equations have exactly same solutions.

Proof:

let $S_A, S_B \to$ soln. sets of systems. A and B.

Let (A) and (B) be two equivalent systems with solution sets S_A and S_B respectively. Prove that $S_A = S_B$.

Let $X \in S_A$. $\implies X$ satisfies every equation in (A), and every equation in (B) is a linear combination of equations in (A). $\implies X$ satisfies every equation in (B). $\implies X \in S_B$

let $X \in S_A$. $\Rightarrow X$ satisfies every eqn. in A. since every eqn. in B is L.C. of eqn. in A, $\to$ X satisfies every eqn. in B $\to X \in S_B \to S_A \subseteq S_B$

sim. $S_B \subseteq S_A$

$S_A = S_B$

Hence $S_A \subseteq S_B$

Similarly, $S_B \subseteq S_A$

$\implies S_A = S_B$

Problem

Show that the following systems of linear equations are equivalent.

$$\left.\begin{array}{r} x - y = 0 \\ 2x + y = 0 \end{array}\right\} \quad ------ \quad (I)$$

$$\left.\begin{array}{r} 3x + y = 0 \\ x + y = 0 \end{array}\right\} \quad ------ \quad (II)$$

$$3x + y = c_1(x - y) + c_2(2x + y)$$
$$x + y = c_3(x - y) + c_4(2x + y)$$
$$x - y = c_5(3x + y) + c_6(x + y) \quad \Big| \quad 2x + y = c_7(3x + y) + c_8(x + y)$$

Solution

$$
\begin{aligned}
3x + y &= \tfrac{1}{3}(x - y) + \tfrac{4}{3}(2x + y) \\
x + y &= -\tfrac{1}{3}(x - y) + \tfrac{2}{3}(2x + y) \\
x - y &= (3x + y) - 2(x + y) \\
2x + y &= \tfrac{1}{2}(3x + y) + \tfrac{1}{2}(x + y)
\end{aligned}
$$

Note : $AX = 0$ is called a homogeneous system.

Elementary row operations

Consider a matrix $A = [A_{ij}]$, where $A_{ij} \in F$, a field.
The i^{th} row of A is $R_i = [A_{i1}, A_{i2}, \ldots, A_{in}]$
Type 1: Multiplication of one row of A by a non-zero scalar
$c \in F$ $\quad (e \;:\; R_i \longleftarrow cR_i)$ $\quad e_1 : R_i \longrightarrow cR_i$

Type 2: Replacement of i^{th} row by row i plus c times of row j
where $c \in F$ $\quad (e \;:\; R_i \longleftarrow R_i + cR_j)$

Type 3: Interchange of two rows $\quad (e \;:\; R_i \longleftrightarrow R_j)$

$e_2 : R_i \rightarrow R_i + cR_j$

$e_3 : R_i \longleftrightarrow R_j$

Inverse of the Type 1 elementary row operation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Let $e \;:\; R_1 \longleftarrow cR_1, \quad c \neq 0$

$$e(A) = \begin{bmatrix} cA_{11} & cA_{12} & cA_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

We define $e_1 \;:\; R_1 \longleftarrow \frac{1}{c} R_1$

$$e_1(e(A)) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} = A$$

Prove that $e(e_1(A)) = A \implies e_1(e(A)) = A = e(e_1(A))$

Inverse of the Type 1 elementary row operation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Let $e \;:\; R_1 \longleftarrow cR_1, \quad c \neq 0$

$$e(A) = \begin{bmatrix} cA_{11} & cA_{12} & cA_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

We define $e_1 \;:\; R_1 \longleftarrow \frac{1}{c}R_1 \qquad \Longrightarrow$ e_1 **is the inverse elementary operation of** e

$$e_1(e(A)) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} = A$$

Prove that $e(e_1(A)) = A \implies e_1(e(A)) = A = e(e_1(A))$

Inverse of the Type 2 elementary row operation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Let $e \ : \ R_2 \longleftarrow R_2 + cR_1, \quad c \in F$

$$e(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} + cA_{11} & A_{22} + cA_{12} & A_{23} + cA_{13} \end{bmatrix}$$

We define $e_1 \ : \ R_2 \longleftarrow R_2 - cR_1$**.** $\implies \quad e_1(e(A)) = A$ **Similarly,** $e(e_1(A)) = A = e_1(e(A))$

Note: e_1 **is the inverse of** e

Inverse of the Type 3 elementary row operation

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Let $e \ : \ R_1 \longleftrightarrow R_2$.

$$e(A) = \begin{bmatrix} A_{21} & A_{22} & A_{23} \\ A_{11} & A_{12} & A_{13} \end{bmatrix}$$

We define $e_1 \ : \ R_1 \longleftrightarrow R_2$.

$$e_1(e(A)) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} = A$$

Similarly, $e(e_1(A)) = A = e_1(e(A))$

Note: e_1 **is the inverse of** e

Theorem 2

To each elementary row operation e there corresponds an elementary operation e_1, of the same type as e, such that $e(e_1(A)) = A = e_1(e(A))$. In other words, inverse operation of an elementary operation exists and is of an elementary operation of the same type.

Proof :

Type 1	*Inverse of the Type* 1
$e : R_i \longleftarrow cR_i, \; c \neq 0$	$e_1 : R_i \longleftarrow \frac{1}{c} R_i$

Proof of Theorem 2 contd.

Type 2	*Inverse of the Type 2*
$e : R_i \longleftarrow R_i + cR_j$	$e_1 : R_i \longleftarrow R_i - cR_j$

Type 3	*Inverse of the Type 3*
$e : R_i \longleftrightarrow R_j$	$e_1 : R_i \longleftrightarrow R_j$

Note that for an $m \times n$ matrix A, $e(e_1(A)) = A = e_1(e(A))$

Note

$e_1 \;:\; R_2 \longleftarrow 2R_2$ and $e_2 \;:\; R_2 \longleftarrow R_2 - R_1$

$$A = \begin{bmatrix} 4 & -1 & 5 \\ 2 & 1 & 7 \end{bmatrix} \underset{(e_1)}{\longrightarrow} \begin{bmatrix} 4 & -1 & 5 \\ 4 & 2 & 14 \end{bmatrix} \underset{(e_2)}{\longrightarrow} \begin{bmatrix} 4 & -1 & 5 \\ 0 & 3 & 9 \end{bmatrix} = B$$

$$e_2(e_1(A)) = B$$

$e_1^{-1} \;:\; R_2 \longleftarrow \frac{1}{2}R_2$ and $e_2^{-1} \;:\; R_2 \longleftarrow R_2 + R_1$

$$A = \begin{bmatrix} 4 & -1 & 5 \\ 2 & 1 & 7 \end{bmatrix} \underset{(e_1^{-1})}{\longleftarrow} \begin{bmatrix} 4 & -1 & 5 \\ 4 & 2 & 14 \end{bmatrix} \underset{(e_2^{-1})}{\longleftarrow} \begin{bmatrix} 4 & -1 & 5 \\ 0 & 3 & 9 \end{bmatrix} = B$$

$$e_1^{-1}(e_2^{-1}(B)) = A$$

A and B are called row-equivalent matrices.

Row-equivalent matrices

Definition : If A and B are $m \times n$ matrices over the field F, we say B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

Theorem 3

If A and B are row-equivalent $m \times n$ matrices, the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have exactly same solutions.

Proof: Suppose that we pass A to B by a finite sequence of elementary row operations :

$$A = A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow A_k = B$$

Note: If
(1) $A_0X = 0$ and $A_1X = 0$ have same solutions,
(2) $A_1X = 0$ and $A_2X = 0$ have same solutions,
(j) $\dots, \dots,$
(k) $A_{k-1}X = 0$ and $A_kX = 0$ have same solutions,
then $AX = 0$ and $BX = 0$ have same solutions.

<u>T.P.T</u>: for an arbitrary j: $A_jX = 0$ & $A_{j+1}X = 0$ have EXACTLY the same set of solns.

Proof of Theorem 3 contd.

if $A \to B$: single step, $e(A) = B$. $\to$ no matter type, each eqn. in $B \to$ L.C. of eqns. in A $\to$ since e^{-1} is defined, each eqn. in A is L.C. of eqns. in B

It is enough to prove that $A_j X = 0$ and $A_{j+1} X = 0$ have exactly the same solutions(that is one elementary row operation doesn't disturb the set of solutions).

$\Rightarrow AX = 0$, $BX = 0$ $\to$ equivalent

Suppose that B is obtained from A by a single elementary row operation, say e $\quad (i.e., e(A) = B)$. No matter which of the types the operation is : (1) , (2) or (3), each equation in the system $BX = 0$ is a linear combination of the equations in $AX = 0$. Since e^{-1} is an elementary row operation $(i.e., e^{-1}(B) = A)$, each equation in the system $AX = 0$ will also be a linear combination of equations in $BX = 0$. Hence these two systems are equivalent, and by Theorem 1, they have the same solutions.

∴, by T1, same solutions

Problem 1

Show that the following systems are row-equivalent.

$AX = 0$	$BX = 0$
$2x_1 - x_2 + 3x_3 + 2x_4 = 0$ $x_1 + 4x_2 - x_4 = 0$ $2x_1 + 6x_2 - x_3 + 5x_4 = 0$	$x_3 - \frac{11}{3}x_4 = 0$ $x_1 + \frac{17}{3}x_4 = 0$ $x_2 - \frac{5}{3}x_4 = 0$

Solution at page number 8(Hoffman and Kunz)

It's an assignment.

Note that solving the second system is easy !

Note

Let us consider the matrix B from the previous problem.

$$B = \begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \end{bmatrix}$$

- Note that the first non-zero entry of each non-zero row of B is 1.
- Note that each column of B which contains the leading non-zero entry of some row has all its other entries 0.

$$B = \begin{bmatrix} 0 & \color{red}{0} & 1 & -\frac{11}{3} \\ 1 & \color{red}{0} & 0 & \frac{17}{3} \\ 0 & \color{red}{1} & 0 & -\frac{5}{3} \end{bmatrix}$$

Row-reduced matrix

An $m \times n$ matrix R is called row-reduced if :

R.R. if :
a) first non-zero $= 1$

(a) the first non-zero entry of each non-zero row of R is 1;

(b) each column of R which contains the leading non-zero entry of some row has all its other entries 0.

b) other entries in leading non-zero column is 0.

Examples : (i) Identity matrix and (ii)the matrix B (previous problem)

OK. OK.

Examples of non row-reduced matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 1

Find all solutions of the following system of equations by row-reducing the coefficient matrix.

$$\begin{aligned}
\tfrac{1}{3}x_1 + 2x_2 - 6x_3 &= 0\\
-4x_1 \qquad\quad + 5x_3 &= 0\\
-3x_1 + 6x_2 - 13x_3 &= 0\\
-\tfrac{7}{3}x_1 + 2x_2 - \tfrac{8}{3}x_3 &= 0
\end{aligned}$$

Solution: The coefficient matrix of the system is

$$\begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix}$$

Problem 1 contd.

$R_1 \longleftarrow 3R_1,\ R_4 \longleftarrow 3R_4$

$$\sim \begin{bmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -7 & 6 & -8 \end{bmatrix}$$

$R_2 \longleftarrow R_2 + 4R_1,\ R_3 \longleftarrow R_3 + 3R_1,\ R_4 \longleftarrow R_4 + 7R_1$

$$\sim \begin{bmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -67 \\ 0 & 48 & -134 \end{bmatrix}$$

Problem 1 contd.

$R_2 \longleftarrow \frac{1}{24} R_2$

$$\sim \begin{bmatrix} 1 & 6 & -18 \\ 0 & 1 & -\frac{67}{24} \\ 0 & 24 & -67 \\ 0 & 48 & -134 \end{bmatrix}$$

$R_1 \longleftarrow R_1 - 6R_2,\ R_3 \longleftarrow R_3 - 24R_2,\ R_4 \longleftarrow R_4 - 48R_2$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (a\ row - reduced\ matrix)$$

Thus

$$x_1 - \frac{5}{4}x_3 = 0$$
$$x_2 - \frac{67}{24}x_3 = 0$$

Problem 1 contd.

Let $x_3 = a$. $\implies x_1 = \frac{5}{4}a, x_2 = \frac{67}{24}a$

Solution set, $S = \{(\frac{5}{4}a, \frac{67}{24}a, a)\ :\ a \in \mathbf{R}\}$

Note that

(i) x_3 **is a called free variable and**

(ii) x_1, x_2 **are called pivot variables.**

Theorem 4

Every $m \times n$ matrix over the field F is row-equivalent to a row-reduced matrix.

Proof : (Assignment)

THEOREM 4 : Every $m \times n$ matrix over field $\mathbb{F}$ is row equivalent to a row reduced Matrix.

* Take an $m \times n$ matrix A with A_{ij} as its entries ($1 \leq i \leq m$ & $1 \leq j \leq m$).
* Start with the first row and iterate thru it UNTIL you find the first NON-ZERO entry in the row. if $\nexists$ such an element, push the row to the end, and repeat the same step for the next row.
  if $\exists$ such an element, divide row 1 by the element A_{1K} ($K \rightarrow$ col. in which the first leading non-zero occurred in row 1). It's obvious that A_{1j} is now 1, terms left of A_{iK} are 0 (we don't bother abt. the ones to the right now).
* now iterate thru the column K $\forall_i$'s $!= 1$. if you find A_{iK} s.t. $A_{iK} != 0$, perform:
  $R_i \longrightarrow R_i - A_{iK} R_1$ to that row. This ensures that $\forall_i$ $!= 1$, $A_{iK} = 0$
* moving to the next row, there are 2 possibilities:
  I. the leading entry is at A_{2p} s.t. $p < K$.
  II. the leading entry is at A_{2q} s.t. $q > K$.

<u>case I</u> :

* perform $R_2 \rightarrow R_2 / A_{2p}$.
* since w.k.t. $p < K$, $A_{1p} = 0$ {leading at A_{1K}}. $\therefore$, needn't perform $R_1 \rightarrow R_1 - A_{2p} R_2$.

* once iterated & performed $R_i \to R_i - A_{ip}R_2$, we'll have zero-entries in other rows of column p.

Case II :

* perform $R_2 \to R_2 / A_{2q}$

* iterate thru col q. if $\exists_i$ s.t. $A_{ip} != 0$, perform $R_i \to R_i - A_{ip}R_2$. eventually, all other entries in A_{ip} would be 0.
* Notice that, $\because q > K$, even if $A_{1q} \neq 0$, the op. $R_1 \to R_1 - A_{1q}R_2$ would've neither affected the leading zeroes / the leading non-zero terms, as A_{2q}'s are all zero $\forall j<q$ and $K<q$. $\therefore$, the 1st row remains unnaffected.

* In both the cases, we've seen that row 1 & row 2 satisfy the row-reduced condition.
* This procedure can be repeated for all rows, and in the end we will get a Row-Reduced Matrix. (By induction)
* Interchange rows s.t.

$$c_1 < c_2 < c_3 < \cdots < c_n,$$

where $c_i \to$ col. in the i'th row in which leading non-zero occurs. {for ROW REDUCED ECHELON}

We have hence, proved that the matrix A, can be reduced to a row-reduced matrix $\to \therefore$, any matrix. A $\to$ arbitrary.

Problem 2

Find all solutions of the systems of linear equations $AX = 2X$ and $AX = 3X$ where

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$$

$6x - 4y = 2x$
$4x - 2y = 2y$
$-x + 3z = 2z$

$4x - 4y = 0$ — 1
$4x - 4y = 0$ — 2
$-x + z = 0$ — 3

Solution : (i) The system $AX = 2X$ is

$$\begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \quad \left.\begin{aligned} 6x - 4y &= 2x \\ 4x - 2y &= 2y \\ -x + 3z &= 2z \end{aligned}\right\}$$

Problem 2 contd.

$$\Longrightarrow \quad \left.\begin{aligned} 4x - 4y &= 0 \\ 4x - 4y &= 0 \\ -x + z &= 0 \end{aligned}\right\}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The coefficient matrix is

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let us find a row-reduced matrix which is row-equivalent to the above matrix. $R_3 \longleftarrow (-1)R_3$, $R_3 \longleftrightarrow R_1$

$x - z = 0$

$y - z = 0$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 4 & -4 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

Problem 2 contd.

$R_2 \longleftarrow R_2 - 4R_1$ and $R_3 \longleftarrow R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & 4 \\ 0 & -4 & 4 \end{bmatrix}$$

$R_2 \longleftarrow -\frac{1}{4}R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{bmatrix}$$

$R_3 \longleftarrow R_3 + 4R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = z$

$y = z$

let $z = a$

$\therefore S = \{(a, a, a) : a \in \mathbb{R}\}$

Problem 2 contd.

The equivalent system is

$$\left.\begin{aligned} x - z &= 0 \\ y - z &= 0 \end{aligned}\right\}$$

Let $z = a$ (Note that z is a free variable). Thus $x = a = y$

(ii) Find all solutions of $AX = 3X$

The solution set is

$S = \left\{X \ \in \mathbf{R}^3 \ : \ AX = 3X\right\} = \{(0, 0, a) \ : \ a \in \mathbf{R}\}.$

Note

$$A = \begin{bmatrix} 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. A is a row-reduced matrix. Ye.
2. All non-zero rows are above zero rows. Ye.
3. The k_i denotes the column which contains leading one (called pivot elements) (if exists) of R_i (row i).
   $k_1 = 2$, $k_2 = 4$, and $k_3 = 5$. Ye.
   Note that $k_1 < k_2 < k_3$.

Row-reduced echelon matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

blue zeros forms a staircase (echelon) from right to left.

Row-reduced echelon matrix

An $m \times n$ matrix R is called a row-reduced echelon **matrix if:**

(a) R is row-reduced ; Ye.

(b) every row of R which has all its entries 0 occurs below every row which has a non-zero entry; Ye.

(c) if rows $1, 2, \ldots, r$ are the non-zero rows of R, and if the leading non-zero entry of row i occurs in column k_i, $i = 1, 2, \ldots, r$, then $k_1 < k_2 < \ldots < k_r$. Ye.

B is a row-reduced matrix, but not a row-reduced echelon matrix

not RRE, as

$k_1 > k_2 < k_3$

it shd. be

$k_1 < k_2 < k_3$

$$B = \begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ do $R_1 \longleftrightarrow R_2$ and $R_2 \longleftrightarrow R_3$

Why?

$k_1 = 3, k_2 = 1, k_3 = 2$ **which violates the condition** (c)

Could you find a a row-reduced echelon matrix C **which is row-equivalent to** B**?** $(R_1 \longleftrightarrow R_2, R_2 \longleftrightarrow R_3)$

$$C = \begin{bmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Is it unique?)

Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

Proof.

Assignment. $\square$

$\text{Proof}_{RRE} = \text{proof}_{RR}$ + row interchanges s.t. $c_1 < c_2 < \dots < c_n$

+ push all zero rows below non-zero rows.

$c_1 \rightarrow$ col. in row 1 where leading non-zero occurs

Problem 3

Solve the system of linear equations $AX = b$ **where**

$$A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{bmatrix} \textbf{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 4/3 & 5/3 \\ 0 & 1 & -2/3 & -1/3 & 1/3 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] \longleftarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 1 & -2/3 & -1/3 & 1/3 \\ 0 & 9 & -6 & -3 & 2 \end{array}\right]$$

SYSTEM has NO solutions.

$$S_{Ax=b} = \emptyset$$

Note 1:

Consider a row-reduced echelon matrix R and the system $RX = 0$, where

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\}$$

No. of non-zero rows of R, $r = 2$, No. of variables, $n = 5$

$k_1 = 2, k_2 = 4 \implies$ Pivot variables $= \{x_{k_1}, x_{k_2}\} = \{x_2, x_4\}$.

No. of free variables $= n - r = 5 - 2 = 3$,

Free variables $= \{u_1, u_2, u_3\} = \{x_1, x_3, x_5\}$.

$x_2, x_4 \rightarrow$ leading entries in their rows, $\therefore$ are pivot variables. $\therefore$ free $\rightarrow x_1, x_3, x_5$.

Note 1 contd.

let $x_1 = a,\ x_3 = b,\ x_5 = c.$

$\Rightarrow x_2 = 3b - c/2$

$x_4 = -2c$

$$x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0$$
$$x_4 + 2x_5 = 0$$

Set the free variables as :

$u_1 = x_1 = a,\ u_2 = x_3 = b,\quad u_3 = x_5 = c$

$\Longrightarrow \quad x_2 = 3b - \frac{1}{2}c,\quad x_4 = -2c$

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c)\ :\ a, b, c \in \mathbf{R}\right\}$

$\therefore\ S = \left\{(a, 3b - c/2, b, -2c, c) : a, b, c \in \mathbb{R}\right\}$

if a system has r pivots & n variables in total, then :

$$x_{k_i} + \sum_{j=1}^{n-r} C_{ij} u_j = 0 \Bigg\} \rightarrow \forall_i\text{s from } 1 \text{ to } r.$$

$$\left.\begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j &= 0 \\ x_{k_2} + \sum_{j=1}^{n-r} C_{2j} u_j &= 0 \end{aligned}\right\} \textit{general expression)}$$

here,

$$\begin{bmatrix} 0 & -3 & \tfrac{1}{2} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

$$r \Bigg\downarrow \overbrace{\begin{bmatrix} C_{11} & \cdots & C_{1(n-r)} \\ \vdots & \ddots & \vdots \\ C_{r1} & \cdots & C_{r(n-r)} \end{bmatrix}}^{n-r} \overbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_{n-r} \end{bmatrix}}^{1} \Bigg\downarrow n-r + \begin{bmatrix} x_{k_i} \\ \vdots \\ x_{k_r} \end{bmatrix} \Bigg\downarrow r$$

Note 2

Consider an $m \times n$ row-reduced echelon matrix R with r non-zero rows. Let rows $1, 2, \ldots, r$ be the non-zero rows of R, and suppose that the leading non-zero entry of row i occurs in column k_i. The system $RX = 0$ has r (non-trivial) equations. Let x_{k_i}s are the pivot variables. Let $u_1, u_2, \ldots, u_{n-r}$ denote the $(n-r)$ unknowns which are different from $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$. Then r non-trivial equations of $RX = 0$ are of the form

if $r \rightarrow$ NON-TRIVIAL EQNS $\Rightarrow$ r pivots $\Rightarrow (n-r)$ free vars

$x_{k_i}\ \forall i \in \{1, 2, \ldots, r\}$

$u_i\ \forall i \in \{1, \ldots, n-r\}$

Note 2 contd.

map v_i with corresponding x_j → <u>acc. to eqns.</u>

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0$$

$$x_{k_2} + \sum_{j=1}^{n-r} C_{2j} u_j = 0$$

$$\dots\dots\dots\dots\dots\dots$$

$$x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0$$

All the solutions of the system of equations $RX = 0$ are obtained by assigning any value whatsoever to $u_1, u_2, \dots, u_{n-r}$, and then computing the corresponding values of $x_{k_1}, x_{k_2}, \dots, x_{k_r}$.

Remarks (Note 2 contd.)

if $n > r$: $\exists$ a ZERO ROW $\longrightarrow$ $\exists$ a free var
$\therefore$ a non-trivial solution.

if $n = r$: $\nexists$ a ZERO ROW $\rightarrow$ $\nexists$ a free var, $\therefore$ ONLY trivial solution.

Thus, we have

(i) **If $n > r$, then the system $RX = 0$ has at least one free variable and thus it has a non-trivial solution.**

(ii) **If $n = r$, then the system $RX = 0$ has no free variable and thus it has only trivial solution.**

Theorem 6

If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

Proof: Let R be a row-reduced echelon matrix which is row-equivalent to A. Then the systems $AX = 0$ and $RX = 0$ have same solutions by Theorem 3. If r is the number of non-zero rows of R, then $r \leq m$. Since $m < n$, we have $r < n$. Thus the system $RX = 0$ has $n - r(\geq 1)$ free variables and it admits a non-trivial solution. Hence $AX = 0$ has a non-trivial solution.

THEOREM 6: if A is $m \times n$ s.t. $m < n$, then the homogeneous system $AX = 0$ has a non-trivial solution.

proof: By Theorem 5, w.k.t. every matrix of order $m \times n$ is ROW EQUIVALENT to a ROW REDUCED ECHELON MATRIX, R.
By Theorem 3, the systems $AX = 0$ and $RX = 0$ have the SAME SOLUTION SET (proved using cascading effect).
if $r \to$ no. of NON-ZERO rows, its clear that $r \leq m$. This means $r < n$
$\Rightarrow$ $(n-r)$ rows are ZERO ROWS, meaning that the system has those many free variables. $\therefore$, NON-TRIVIAL soln. exists for $RX = 0$, hence for $AX = 0$ as well.

Note

if R is an $n \times n \to$ square, row reduced echelon matrix, with n - NON ZERO rows, then $R = I$.
!∃ a ZERO row $\Rightarrow$ each row $\to$ leading entry $= 1$ &
$1 = k_1 < 2 = k_2 < 3 = k_3 < \cdots < n = k_n \longrightarrow 1$ only in diag pos.
$\Rightarrow \underline{\underline{R = I}}$.

If R is an $n \times n$ (square) row-reduced echelon matrix with n non-zero rows, then $\color{red}{R = I}$ **(the identity matrix).**

Because : (i) Every row has a leading one and (ii) $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$**.**

Who's your gf?
↙ asked by 21065

Theorem 7

T.P. (i) if $A \to$ equi. to I_n, $AX=0 \to 1$ soln.
(ii) if $AX=0 \to 1$ soln, $A \to$ equi to I_n.

If A is an $n \times n$ (square) matrix, then A is row equivalent to the $n \times n$ identity matrix if and only if the system of equations $AX = 0$ has only the trivial solution.

Proof.

Case 1. Suppose that A is row-equivalent to the $n \times n$ identity matrix I. By Theorem 3, $AX = 0$ and $IX = 0$ have the same solution set. Thus the solution set of $AX = 0$ is

$$S = \{X \ : \ AX = 0\} = \{X \ : \ IX = 0\} = \{X \ : \ X = 0\} = \{0\}$$

Hence the system $AX = 0$ has only the trivial solution.

Proof of Theorem 7 contd.

Case 2. Suppose that the system $AX = 0$ has only the trivial solution. Prove that A is row-equivalent to the $n \times n$ identity matrix.

Let R be an $n \times n$ row-reduced echelon matrix which is row-equivalent to A. By Theorem 3, the systems $AX = 0$ and $RX = 0$ have exactly the same solutions. Since $AX = 0$ has only the trivial solution, $RX = 0$ has only the trivial solution. Hence the system $RX = 0$ has no free variables. Thus the number of free variables (of the system $RX = 0$), $n - r = 0$ where r is the number of non-zero rows of R. So R is an $n \times n$ row-reduced echelon matrix with $n(= r)$ non-zero rows and thus $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$. This proves that $R = I$, an identity matrix.

Hence A is row-equivalent to $R = I$.

THEOREM 7: if $A_{n\times n}$ is row-equivalent to I_n iff. the system $AX=0$ has only the trivial solution.

↑ prep. X

↘ prep. Y

T.P.T. $\rightarrow (X \rightarrow Y) \wedge (Y \rightarrow X)$

Proof:

Case I - $A_{n\times n}$ is ROW EQUIVALENT to I_n

$\Rightarrow$ By Theorem 3, $AX=0$ and $IX=0$ has the SAME solution set (proved using induction / cascading effect)

$\Rightarrow$ I_n is an $n\times n$ matrix with $r=n$, where $r \rightarrow$ no. of NON-ZERO ROWS, I_n doesn't have ANY free variables (no ZERO rows), by Note in Theorem 6.

$\Rightarrow$ $IX=0$ has only the trivial solution

$\Rightarrow$ $AX=0$ has only the trivial solution.

($X \rightarrow Y$ proved)

Case II - $AX=0$ has only the trivial solution

$\Rightarrow$ no. of non-zero rows is n, by note in Theorem 6.

By Theorem 5, ANY $m\times n$ matrix is ROW EQUIVALENT to a ROW REDUCED ECHELON matrix, R

$\Rightarrow$ $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$, since $\nexists$ any ZERO ROW in R, and $k_i \rightarrow$ col. in the i'th row where the leading non-zero (1) occurs.

$\Rightarrow$ $R = I$.

$\Rightarrow$ $\therefore$, A is Row Equivalent to $R = I$.

($Y \rightarrow X$ proved)

Theorem 8

Let $AX = b$ be a given system of equations, where A is an $m \times n$ matrix, X is an $n \times 1$ vector and b is an $m \times 1$ vector. Let $RX = f$ be the row-reduced-echelon form of the system $AX = b$. Let there are r non-zero rows in the augmented matrix $[R|f]$. Then the following are true.

1. No solution. If $r < m$ (meaning that R actually has at least one row of all 0s) and at least one of the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ is not zero, then the system $AX = b$ is inconsistent, i.e, no solution is possible.
2. Unique solution. If the system $AX = b$ is consistent and $r = n$, then the system $AX = b$ has a unique solution.
3. Infinitely many solutions. If the system $AX = b$ is consistent and $r < n$, then the system $AX = b$ has infinitely many solutions.

THEOREM 8 :

Let $AX = b$, where $A \to m \times n$, $X \to n \times 1$ and $b \to m \times 1$. Let $RX = f \to$ R.R.E. form of $AX = b$. Let $r \to$ no. of NON-ZERO rows in $[R|f]$. Then:

① if $r < m$ ($\exists$ a ZERO ROW in R) AND $\exists_i$ s.t. $f_{r+i} \ != 0$, then $AX = b$ is inconsistent, i.e. no soln. exists for $AX = b$.

Since $R \to$ RRE, $\forall_{k > r}$, R_k is ZERO ROW.
$\Rightarrow$ if $\exists_i$ s.t. f_{r+i} is NOT 0, then LHS $= 0$ and RHS $!= 0$, which is impossible. Hence, the system has NO solution / system is inconsistent.

② if $AX = b$ is consistent and $r = n$, then the system $AX = b$ has a unique solution.

Since $Ax = b$ is consistent, the system's solutions exists. Since $r = n$, the system $RX = f$ has ONLY the unique solution, by note 2 in Theorem 5. as the system $RX = f$ is row-equivalent to the row-reduced echelon form of the system $AX = B$, they have the same set of solutions, by Theorem 3.
$\therefore$, $AX = b$ has only the unique solution.

③ if $AX = b$ is consistent and $r < n$, then the system $AX = b$ has infinitely many solutions.

Since $AX = b$ is consistent, solutions exist for the system. if $r \to$ no. of NON-ZERO ROWS (hence, pivots), $(n - r) \to$ no. of free variables. These free variables have to be chosen from $\mathbb{R}$, whose cardinality is uncountable. hence, system has infinite solutions.

Reading assignment

To do

Section 1.5 Matrix multiplication

Assignment

To do.

Solve all exercise problems in section 1.4 (pages 15-16)

Elementary matrices

$A_{m\times m}$: elementary if, $e(I_{m\times m}) \to A_{m\times m}$ single step

An $m \times m$ matrix is said to be an **elementary matrix** if it can be obtained from the $m \times m$ **identity matrix** by means of **a single elementary row operation**.

Example :

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (e : R_1 \longleftarrow cR_1, \quad c \neq 0)$$

$$E = e(I) = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \quad (\text{ E is an elementary matrix})$$

Find all 2×2 elementary matrices

$R_1 \to cR_1$ $\Downarrow$ $\quad$ $R_2 \to cR_2$ $\Downarrow$

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \quad (\textit{Using} \quad \textit{Type} \ \ 1, \quad c \neq 0)$$

$$R_1 \to R_1 + cR_2 \Longrightarrow \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \Longleftarrow R_2 \to R_2 + cR_1 \quad (\textit{Using} \quad \textit{Type} \ \ 2)$$

$$R_i \leftrightarrow R_j \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (\textit{Using} \quad \textit{Type} \ \ 3)$$

Find all 3×3 elementary matrices. **(Assignment)** $\longrightarrow$ doable much?

homework to the reader (LOL!)

ASSIGNMENT: find all 3X3 elementary matrices

SOLUTION :

Type 1 - $e: R_i \to c \cdot R_i$

a) $R_1 \to c \cdot R_1$

b) $R_2 \to c \cdot R_2$

c) $R_3 \to c \cdot R_3$

Type 2 - $e: R_i \to R_i + c \cdot R_j$

a) $R_1 \to R_1 + cR_2$

b) $R_1 \to R_1 + cR_3$

c) $R_2 \to R_2 + cR_1$

d) $R_2 \to R_2 + cR_3$

e) $R_3 \to R_3 + cR_1$

f) $R_3 \to R_3 + cR_2$

Type 3 - $e: R_i \Leftrightarrow R_j$

a) $R_1 \Leftrightarrow R_2$

b) $R_2 \Leftrightarrow R_1$

c) $R_2 \Leftrightarrow R_3$

d) $R_3 \Leftrightarrow R_2$

e) $R_3 \Leftrightarrow R_1$

f) $R_1 \Leftrightarrow R_3$

$(c_1,\ c_1+c_2,\ c_1+c_2+c_3)$

if $\alpha = 1 \cdot \hat{n}_1 + 1 \cdot \hat{n}_2 + 1 \cdot \hat{n}_3$

$\beta = 0 \cdot \hat{n}_1 + 1 \cdot \hat{n}_2 + 1 \cdot \hat{n}_3$

$\gamma = 0 \cdot \hat{n}_1 + 0 \cdot \hat{n}_2 + 1 \cdot \hat{n}_3$

Properties of elementary matrices

Type 1

$e(I) = R_1 \to cR_1$; $e_1(I) = R_1 \to \frac{1}{c}R_1$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \left(e : R_1 \longleftarrow cR_1,\ c \neq 0, e_1 : R_1 \longleftarrow \frac{1}{c}R_1\right)$$

E $\qquad$ E_1

$$E = e(I) = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = e_1(I) = \begin{bmatrix} \frac{1}{c} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EE_1 = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{c} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly (verify), $E_1E = I = EE_1$

$$\boxed{I \xrightarrow[e(I)]{} E \xrightarrow[e_1(E)]{} I \iff e(I) \cdot e_1(I) = I.}$$

Properties of elementary matrices

$$\text{Let} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad (e : R_1 \longleftarrow cR_1, \; c \neq 0)$$

$$\boxed{e(A) = e(I) \cdot A} \quad *$$

$$e(A) = \begin{bmatrix} cA_{11} & cA_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} cA_{11} & cA_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = e(A)$$

$$e(I)A = e(A)$$

Properties of elementary matrices

Type 2

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (e : R_1 \longleftarrow R_1 + cR_2, e_1 : R_1 \longleftarrow R_1 - cR_2)$$

$$E = e(I) = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = e_1(I) = \begin{bmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EE_1 = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly (verify), $E_1 E = I = EE_1$

Properties of elementary matrices

$$\text{Let} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad (e : R_1 \longleftarrow R_1 + cR_2)$$

$$e(A) = \begin{bmatrix} A_{11} + cA_{21} & A_{12} + cA_{22} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} A_{11} + cA_{21} & A_{12} + cA_{22} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = e(A)$$

Properties of elementary matrices

Type 3

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (e : R_1 \longleftrightarrow R_2, e_1 : R_1 \longleftrightarrow R_2)$$

$$E = e(I) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = e_1(I) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EE_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly (verify), $E_1 E = I = EE_1$

Properties of elementary matrices

$$\text{Let} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad (e : R_1 \longleftrightarrow R_2)$$

$$e(A) = \begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix}$$

$$e(I)A = e(A)$$

Theorem 9

Let e **be an elementary row operation and** I **be the** $m \times m$ **identity matrix. Then for every** $m \times n$ **matrix** A**,**

$$e(I)A = e(A)$$

Proof: Assignment

Note: For every elementary row operation e**, there exists an inverse elementary operation of the same type** e_1 **such that**

$$e(I)e_1(I) = I = e_1(I)e(I) \quad (EE_1 = I = E_1E)$$

$e(I) \cdot e_1(I)$

$= e(e_1(I)) = I$ (as e reverses whatever e_1 does)

$(e(I) \cdot A = e(A))$

$\therefore$, $\boxed{e(I) \cdot e_1(I) = I}$

rough work:

case 1 → $e : R_r \to cR_r$

$$\begin{array}{c} m \end{array}\begin{bmatrix} 1 & & \\ & \ddots c \ddots & \\ & & 1 \end{bmatrix}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} m$$

let E be the resultant matrix.
So, the entries are of the form:

$$E_{ij} = \sum_{k=1}^{m} e(I)_{ik} A_{kj}$$

$\forall_{k!=i}$, $I_{ik} = 0$

$\Rightarrow E_{ij} = e(I)_{ii} A_{ij}$

now, if $i = r$, $e(I)_{rr} = c$; $i != r$, $e(I)_{ii} = 1$

$\therefore$ $E_{rj} = e(I)_{rr} \cdot A_{rj} = c \cdot A_{rj} \,\|\, E_{ij} = A_{ij}$

hence, E is of the form:

$$E_{ij} = \begin{cases} c \cdot A_{ij} & ; i = r \\ A_{ij} & ; i != r. \end{cases} \rightarrow r\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ cA_{r1} & cA_{r2} \cdots & cA_{rn} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

SAME as, $e(A) : R_r \to cR_r$

$$e : R_x \to R_x + f \cdot R_y$$

$$e(I)_{ij} = \begin{cases} f & ; i = x \text{ ff } j = y \\ 1 & ; i = j \\ 0 & ; i != j \wedge \sim(i = x \wedge j = y) \end{cases}$$

$$e(I) \cdot A = E \equiv E_{ij} = \sum_{k=1}^{n} e(I)_{ik} A_{kj}$$

if $i = x$, $E_{xj} = \sum_{k=1}^{n} e(I)_{xk} A_{kj}$

$$= e(I)_{xx} A_{xj} + e(I)_{xy} \cdot A_{yj}$$

$$= A_{xj} + f \cdot A_{yj}$$

for other i's,

$$E_{ij} = A_{ij}$$

$$\therefore E_{ij} = \begin{cases} A_{xj} + f \cdot A_{yj} & ; i = x \\ A_{ij} & ; i != x \end{cases}$$

$R_x \to R_x + f \cdot R_y$

$$e(A)_{ij} = \begin{cases} A_{xj} + f \cdot A_{yj} & ; i = x \\ A_{ij} & ; i != x \end{cases}$$

↓ BY OBS.

$$\therefore e(A) = e(I) \cdot A = E$$

THEOREM-9: if $e \to$ elementary row operation and $I \to m \times m$ identity matrix, then $\forall_{m\times n}$ matrices A, $e(A) = e(I)A$

<u>proof</u>: TYPE 1 - $e: R_r \to C * R_r$

BY DOING MANUALLY

LHS: $e(A)$

$$e(A)_{ij} = \begin{cases} C\cdot A_{rj} & ; i=r \\ A_{ij} & ; i!=r \end{cases} \quad - (1)$$

RHS: $e(I)\cdot A = E$.

$$e(I)_{ij} = \begin{cases} c & ; i=j=r \\ 1 & ; i=j!=r \\ 0 & ; i!=j \end{cases} \quad - (2)$$

w.k.t. $\quad E_{ij} = \sum_{k=1}^{n} e(I)_{ik} A_{kj}$

from (2), $e(I) = 0 \ \forall\ i != k$ & for each i, $i=k$ happens only once. $\therefore$, $E_{ij} = e(I)_{ii} A_{ij}$
$e(I)_{ii} = c$ for $i=r$ and $e(I)_{ii} = 1$ for $i!=r$.

$$\Rightarrow \quad E_{ij} = \begin{cases} c\cdot A_{rj} & ; i=r \\ A_{ij} & ; i!=r \end{cases} \quad - (3)$$

comparing (1) and (3), $\boxed{e(A) = e(I)\cdot A = E}$

hence, TYPE 1 proved.

TYPE 2 - $e: R_x \rightarrow R_x + f \cdot R_y$

BY DOING MANUALLY

$LHS = e(A)$

$$e(A)_{ij} = \begin{cases} A_{xj} + f \cdot A_{yj} & ; \ i = x \\ A_{ij} & ; \ i != x \end{cases} \quad - (1)$$

$RHS = e(I) \cdot A = E$

$$e(I)_{ij} = \begin{cases} 1 & ; \ i = j \\ 0 & ; \ i != j \\ f & ; \ i = x \text{ and } j = y \end{cases} \quad - (2)$$

w.k.t. $\quad E_{ij} = \sum_{i=1}^{n} e(I)_{ik} \cdot A_{kj}$

if $i = x$, $\quad E_{xj} = \sum_{i=1}^{n} e(I)_{xk} \, A_{kj}$

by (1), $\quad E_{xj} = e(I)_{xx} \cdot A_{xj} + e(I)_{xy} \cdot A_{yj} + 0$

$$E_{xj} = A_{xj} + f \cdot A_{yj}$$

if $i != x$, $\quad E_{ij} = e(I)_{ii} \cdot A_{ij} + 0$

$$= A_{ij}$$

$$\Rightarrow \quad E_{ij} = \begin{cases} A_{xj} + f \cdot A_{yj} & ; \ i = x \\ A_{ij} & ; \ i != x \end{cases} \quad - (3)$$

comparing (1) and (3), $\boxed{e(A) = e(I) \cdot A = E}$

Hence proved Type-2

TYPE 3 - $e: R_x \leftrightarrow R_y$

BY DOING MANUALLY

$LHS = e(A)$

$$e(A)_{ij} = \begin{cases} A_{xj} & ; \; i=y \\ A_{yj} & ; \; i=x \\ A_{ij} & ; \; i!=x \;\&\; y \end{cases} \quad \text{— (1)}$$

$RHS = e(I) \cdot A$

$$e(I)_{ij} = \begin{cases} 1 & ; \; i=x, \, j=y \\ 1 & ; \; i=y, \, j=x \\ 0 & ; \; i,j=x \\ 0 & ; \; i,j=y \\ 1 & ; \; i=j \end{cases} \quad \text{— (2)}$$

w.k.t. $E_{ij} = \sum_{k=1}^{n} e(I)_{ik} A_{kj}$

if $i=x$, $E_{xj} = \sum_{k=1}^{n} e(I)_{xk} A_{kj}$

$= e(I)_{xy} A_{yj} + 0 = A_{yj}$

if $i=y$, $E_{yj} = \sum_{k=1}^{n} e(I)_{yk} A_{kj}$

$= e(I)_{yx} A_{xj} = A_{xj}$

ACC. to (2)

if $i!=x$ && $i!=y$,

$E_{ij} = \sum_{k=1}^{n} e(I)_{ik} A_{kj} = A_{ij}$

$$\Rightarrow \quad E_{ij} = \begin{cases} A_{xj} & ; \; i=y \\ A_{yj} & ; \; i=x \\ A_{ij} & ; \; i!=x \;\&\; y \end{cases} \quad \text{— (3)}$$

Hence proved Type-3

comparing (1) and (3), $\boxed{e(A) = e(I) \cdot A = E}$

Corollary (to Theorem 9)

Let A and B be $m \times n$ matrices over the field F. Then B is row-equivalent to A if and only if $B = PA$ where P is a product of $m \times m$ elementary matrices.

Proof:

Case 1: Suppose that B is row-equivalent to A.

Then B can be obtained from A by a finite sequence of elementary row operations, say $A_i = e_i(A_{i-1})$ $A_k = e_k(A_{k-1})$

$$A = A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow A_{k-1} \longrightarrow A_k = B$$

where $e_i(A_{i-1}) = A_i$, e_i is an elementary row operation for $1 \le i \le k$.

Note that $e_i(A_{i-1}) = e_i(I)A_{i-1}$**, by Theorem 9 and** $e_i(I)$ **is an** $m \times m$ **elementary matrix.**

Corollary contd.

Clearly, $A_1 = e_1(A) = e_1(I)A$ **,** $A_2 = e_2(A_1) = e_2(I)A_1$
$\implies A_2 = e_2(I)e_1(I)A$
Using similar arguments,

$$B = A_k = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(I)A = PA$$

where $P = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(I)$ is a product of $m \times m$ elementary matrices.
Case 2 : Suppose that $B = PA$**, where** P **is a product of** $m \times m$ **elementary matrices.**
Let $P = E_k E_{k-1} \dots E_2 E_1$ **where** E_i **is an** $m \times m$ **elementary matrix for** $1 \leq i \leq k$**. Since** E_i **is an elementary matrix, there exists an elementary row operation** e_i **such that** $E_i = e_i(I)$**.**

$$B = PA = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(I)A$$

Corollary contd.

$$B = PA = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(I)A$$

$$B = PA = e_k(I)e_{k-1}(I)\dots e_2(I)e_1(A)$$

$$B = PA = e_k(I)e_{k-1}(I)\dots e_2(e_1(A))$$

$$\dots\dots\dots\dots\dots\dots$$

$$B = PA = e_k(e_{k-1}(\dots e_2(e_1(A))))$$

Hence B can be obtained from A by a finite sequence of elementary row operations $e_1, e_2, \dots, e_k$. Then B is row-equivalent to A.

THEOREM 9 : let $A, B \to m \times n$ matrices over field $\mathbb{F}$. B is row-equivalent to A iff. $B = PA$, where P is a pdt. of $m \times m$ elementary matrices.

COROLLARY

prep X

prep Y.

<u>proof</u> : case I − $X \to Y$

* it's given that B is row-equivalent to A

$\Rightarrow$ B is obtained from A through a finite set of elementary row-operations

$$A \xrightarrow[e_1]{} A_1 \xrightarrow[e_2]{} A_2 \longrightarrow \cdots\cdots \xrightarrow[e_K]{} A_K = B$$

where K is a finite number, and by observation − $e_i(A_{i-1}) = A_i$, where $i \in \{1, 2, \cdots, K\}$. $e_i(I) \to m \times m$ elementary matrix.

By Theorem 9, w.k.t.

$$e_i(A_{i-1}) = e_i(I) \cdot A_{i-1} = A_i$$

$$A_1 = e_1(A) = e_1(I)A$$

$$A_2 = e_2(A_1) = e_2(I)A_1 = e_2(I) \cdot e_1(I)A$$

$$\vdots$$

$$B = A_K = e_K(A_{K-1}) = e_K(I)A_{K-1}$$

similar extrapolation

$$= e_K(I) \cdot e_{K-1}(I) \cdots e_2(I) \cdot e_1(I)A$$

let $P = e_K(I) \cdot e_{K-1}(I) \cdots e_2(I) \cdot e_1(I)$

$\Rightarrow P \to$ pdt. of $m \times m$ elementary matrices.

$\Rightarrow B = PA$, where $P \to$ pdt. of $m \times m$ elementary matrices.

Hence proved case I − $X \to Y$

case II - $Y \to X$

* it's given that $B = PA$, where $P \to$ pdt. of elementary matrices.

$\Rightarrow P = E_K \cdot E_{K-1} \cdot E_{K-2} \cdots E_2 \cdot E_1$

$\Rightarrow B = E_K \cdot E_{K-1} \cdot E_{K-2} \cdots E_2 \cdot E_1 \cdot A$

$\because E_i \to m \times m$ elementary matrix,

$E_i = e_i(I)$

$\Rightarrow B = e_K(I) \cdot e_{K-1}(I) \cdots e_2(I) \cdot e_1(I) \cdot A$

$K \to$ finite number.

By theorem 9, w.k.t.

$e(I) \cdot A = e(A)$

$\Rightarrow B = e_K(I) \cdot e_{K-1}(I) \cdots e_2(I) \cdot e_1(A)$

$= e_K(I) \cdot e_{K-1}(I) \cdots e_2(e_1(A))$

$= e_K(I) \cdot e_{K-1}(I) \cdots e_3(e_2(e_1(A)))$

$\vdots$

$B = e_K(e_{K-1}(\cdots e_2(e_1(A)) \cdots))$

$\Rightarrow$ B is obtained through a finite no. of elementary row operations from A.

$\Rightarrow$ B is row-equivalent to A.

Hence proved case II - $Y \to X$

since $X \to Y$ and $Y \to X$, $X \Longleftrightarrow Y$

HENCE PROVED COROLLARY

Problem

$B = \left(e_2\left(e_1(A)\right)\right) = e_2(I) \cdot e_1(A) = e_2(I) \cdot e_1(I) \cdot A$

Show that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 8 & 10 \end{bmatrix}$ **are row-equivalent. Find a** 3×3 **matrix** P **such that** $B = PA$.

Solution : Let $e_1 \ : \ R_1 \longleftrightarrow R_2$ **and** $e_2 \ : \ R_3 \longleftarrow R_3 + R_1$

Clearly, $B = e_2(e_1(A)) = e_2(e_1(I)A) = e_2(I)e_1(I)A = PA$

$$P = e_2(I)e_1(I) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{e_2(I)} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{e_1(I)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{e_2(I) \cdot e_1(I) = P.}$$

$e_1 : R_1 \longleftrightarrow R_2$

$e_2 : R_3 \longrightarrow R_3 + R_1$

$B = PA.$

Definition

$A \to n \times n$ matrix over $\mathbb{F}$

if $BA = I$, then $B \to$ left inverse of A.

$AB = I$, then $B \to$ right inverse of A.

if $AB = BA = I$, then $B \to$ 2-sided inverse & $A \to$ invertible

Let A be an $n \times n$ matrix over the field F. An $n \times n$ matrix B such that $BA = I$ is called a left inverse of A; an $n \times n$ matrix B such that $AB = I$ is called a right inverse of A.

If $AB = I = BA$, then B is called a two-sided inverse of A or simply the inverse of A and A is said to be invertible.

Note: If A is an invertible matrix, then A has no zero row. ???

$A \cdot A^{-1} = I$

$I_{ij} = \sum_{k=1}^{n} A_{ik} A^{-1}_{kj}$

$I_{ii} = 1$, $I_{ij} = 0$ $(i != j)$

$I_{ii} = \sum_{k=1}^{n} A_{ik} A^{-1}_{ki}$

$\downarrow$

$1 = (A_{i1} A^{-1}_{1i} + A_{i2} A^{-1}_{2i} + \cdots + A_{in} A^{-1}_{ni})$

if $\exists$ a zero row i in A, $\forall_{k}s$, $A_{ik} = 0$

$\Rightarrow 1 = 0 \cdot A^{-1}_{1i} + 0 \cdot A^{-1}_{2i} + \cdots + 0 \cdot A^{-1}_{ni} \Rightarrow \boxed{1 = 0}$

IMPOSSIBLE

$\therefore$, if A is invertible, A has no zero row.

Lemma

If A has a left inverse B and a right inverse C, then $B = C$.

Proof Suppose that $BA = I$ and $AC = I$.

$$B = BI = B(AC) = (BA)C = IC = C$$

$BA = I$ — ①

$AC = I$ — ②

$B = BI$

$B = B(AC)$

$= (BA)C$

$= IC$

$= C$

$\Rightarrow \boxed{B = C}$

$C = IC$

$= (BA)C$

$= B \cdot (AC)$

$= B \cdot I$

$\Rightarrow \boxed{C = B}$

Note: If A has a left inverse and a right inverse, then A is invertible and the inverse of A is denoted by A^{-1}.

Theorem 10

(i). If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(ii). If both A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

$$(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB).$$

Note: Product of invertible matrices is invertible.

THEOREM 10:

(i) if A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(ii) if A & B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

proof: (i) if A is invertible, $\exists$ matrices B and C s.t. $AB = I$ and $CA = I$. By the lemma, $B = C$. let $B = C = A^{-1}$.

$\Rightarrow \quad A \cdot A^{-1} = I$ and $A^{-1} \cdot A = I$

$\Rightarrow$ A is left & right inverse of A^{-1}

$\Rightarrow$ A^{-1} is invertible, AND A is the inverse of A^{-1}.

$\Rightarrow (A^{-1})^{-1} = A$.

(ii) let A and B be invertible.

$\Rightarrow \quad A \cdot A^{-1} = I$ and $B \cdot B^{-1} = I$.

we need to prove that:

$$\underset{①}{M \cdot (AB)} = \underset{②}{(AB) \cdot M} = I$$

choose $M = B^{-1}A^{-1}$.

① $M \cdot (AB) = (B^{-1}A^{-1}) \cdot (AB)$

$= B^{-1} \cdot (A^{-1}A) B$

$= B^{-1} \cdot IB = B^{-1}B = I$.

② $(AB)M = (AB)(B^{-1}A^{-1})$

$= A \cdot (BB^{-1}) A^{-1}$

$= A \cdot I \cdot A^{-1}$

$= AA^{-1} = I$.

$\Rightarrow$ ① & ② $= I$.

$\therefore$, if $A, B \rightarrow$ invertible, $AB \rightarrow$ invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 11: An elementary matrix is invertible.

Proof : Let E be an $m \times m$ elementary matrix corresponding to the elementary row operation e. Thus $E = e(I)$. By Theorem 2, there exists an elementary row operation e_1, same type as e, such that

$$e(e_1(A)) = A = e_1(e(A)) \ \text{ for every matrix } \ A.$$

Let $E_1 = e_1(I)$ where I is the $m \times m$ identity matrix. Then

$$EE_1 = e(I)E_1 = e(E_1) = e(e_1(I)) = I$$

and

$$E_1E = e_1(I)E = e_1(E) = e_1(e(I)) = I.$$

$EE_1 = I = E_1E$. Hence E is an invertible matrix.

Thus an elementary matrix is invertible.

THEOREM 11: An elementary matrix is invertible.

proof: let E be an elementary matrix of order $m \times m$.

$\Rightarrow E = e(I)$; where $I \to m \times m$ Identity Matrix

by Theorem 2, $\forall_e$ s $\exists_{e_1}$ s.t.

$$e(e_1(A)) = I = e_1(e(A))$$

let $E_1 = e_1(I)$; where $I \to m \times m$ Identity Matrix

$$\Rightarrow EE_1 = e(I) \cdot E_1 = e(E_1) = e(e_1(I)) = I.$$

&

AND $$E_1E = e_1(I)E = e_1(E) = e_1(e(I)) = I.$$

$\{e(e_1(A)) = I$ & $e_1(e(A)) = I$, by properties of elementary row operations$\}$

$\{e(I) \cdot A = e(A) \to$ By Theorem 9$\}$

$$\Rightarrow EE_1 = I = E_1E.$$

$\therefore$, $E \to$ invertible matrix

Hence, an elementary matrix is invertible

Find inverses of all 2×2 elementary matrices

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{c} \end{bmatrix}$$

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find inverses of all 3×3 elementary matrices. (Assignment)

ASSIGNMENT: find inverses $\forall$ 3X3 elementary matrices.

SOLUTION :

Type 1 –

$e: R_i \to c \cdot R_i$	$e_1: R_i \to \frac{1}{c} R_i$
a) $R_1 \to c \cdot R_1$	a) $R_1 \to \frac{1}{c} R_1$
b) $R_2 \to c \cdot R_2$	b) $R_2 \to \frac{1}{c} R_2$
c) $R_3 \to c \cdot R_3$	c) $R_3 \to \frac{1}{c} R_3$

Type 2 –

$e: R_i \to R_i + c \cdot R_j$	$e_1: R_i \to R_i - c \cdot R_j$
a) $R_1 \to R_1 + cR_2$	a) $R_1 \to R_1 - cR_2$
b) $R_1 \to R_1 + cR_3$	b) $R_1 \to R_1 - cR_3$
c) $R_2 \to R_2 + cR_1$	c) $R_2 \to R_2 - cR_1$
d) $R_2 \to R_2 + cR_3$	d) $R_2 \to R_2 - cR_3$
e) $R_3 \to R_3 + cR_1$	e) $R_3 \to R_3 - cR_1$
f) $R_3 \to R_3 + cR_2$	f) $R_3 \to R_3 - cR_2$

Type 3 –

$e: R_i \Leftrightarrow R_j$	$e_1: R_i \Leftrightarrow R_j$
a) $R_1 \Leftrightarrow R_2$	a) $R_1 \Leftrightarrow R_2$
b) $R_2 \Leftrightarrow R_1$	b) $R_2 \Leftrightarrow R_1$
c) $R_2 \Leftrightarrow R_3$	c) $R_2 \Leftrightarrow R_3$
d) $R_3 \Leftrightarrow R_2$	d) $R_3 \Leftrightarrow R_2$
e) $R_3 \Leftrightarrow R_1$	e) $R_3 \Leftrightarrow R_1$
f) $R_1 \Leftrightarrow R_3$	f) $R_1 \Leftrightarrow R_3$

$$\left.\begin{aligned} E &= e(A) \\ e_1(E) &= e_1(e(A)) \end{aligned}\right\} \to \text{Property of elementary matrices. (proved in Theorem 11)}$$

Theorem 12

If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

Proof: Let R be a row-reduced echelon $n \times n$ matrix which is row-equivalent to A. By Corollary to Theorem 9,

$$R = E_k E_{k-1} \dots E_2 E_1 A \qquad ------ \quad (a)$$

where E_i is an elementary matrix. Note that the inverse of E_i is also an elementary matrix. Since $E_i's$ are invertible,

$$A = E_1^{-1} E_2^{-1} \dots E_k^{-1} R \quad ---- \; (b)$$

Theorem 12 contd.

$(i) \implies (ii)$ Suppose that A is invertible. Using (a), R is a product of invertible matrices and by corolllary to Theorem 10, R is invertible. Note that an invertible matrix has no zero-row. So R is an $n \times n$ row-reduced echelon matrix with no zero row and $k_1 = 1 < k_2 = 2 < \ldots < k_n = n$. Hence R is the $n \times n$ identity matrix. A is row-equivalent to $R = I$.
$(ii) \implies (iii)$ Suppose that A is row-equivalent to $R = I$. By (b)

$$A = E_1^{-1} E_2^{-1} \ldots E_k^{-1} R = E_1^{-1} E_2^{-1} \ldots E_k^{-1} I$$

$$A = E_1^{-1} E_2^{-1} \ldots E_k^{-1}, \text{a product of elementary matrices.}$$

Theorem 12 contd.

$(iii) \implies (i)$ **Suppose that A is a product of elementary matrices.** By Theorem 11, an elementary matrix is invertible. By Corollary to Theorem 10, a product of invertible matrices is invertible. Hence A is invertible.

THEOREM 12: if $A \to n \times n$ matrix, the following are equivalent:

(i) A is invertible

(ii) A is row-equivalent to $n \times n$ Identity matrix

(iii) A is a product of elementary matrices.

<u>proof</u>: w.k.t. $\forall$ matrices A $\exists$ a row-reduced echelon matrix R s.t. R is row equivalent to A (by theorem 5). By corollary in theorem 9,

$$R = E_K \cdot E_{K-1} \cdots\cdots E_2 \cdot E_1 \cdot A \quad -(a)$$

where $E_i \to$ elementary matrix. if E_i is an elementary matrix, E_i^{-1} is also an elementary matrix. Note that E_i's are invertible (elementary matrices are invertible by Theorem 11)

So, $R = E_K \cdot E_{K-1} \cdots\cdots E_2 \cdot E_1 \cdot A$

$$\Rightarrow E_K^{-1} \cdot R = (E_K^{-1} E_K) \cdot E_{K-1} \cdots\cdots E_2 \cdot E_1 A$$

$$= E_{K-1} \cdots\cdots E_2 \cdot E_1 A$$

$$\Rightarrow E_1^{-1} \cdot E_2^{-1} \cdots\cdots E_{K-1}^{-1} \cdot E_K^{-1} \cdot R = A$$

$$\Rightarrow A = E_1^{-1} \cdot E_2^{-1} \cdots\cdots E_{K-1}^{-1} \cdot E_K^{-1} \cdot R \quad -(b)$$

(i) $\longrightarrow$ (ii)

it's given that A is invertible.

from (a), its clear that R is invertible as R is a product of invertible matrices (by Theorem 10). if R is invertible, then R has no zero rows.

$\Rightarrow$ $R \to n \times n$ Row Reduced Echelon matrix with no zero row, and

$k_1 = 1 < k_2 = 2 < \cdots < k_n = n$, where k_i is the column in i'th row where leading non-zero a.k.a 1 occurs.

$\Rightarrow$ $R = I$. (by note in Theorem 6)

$\Rightarrow$ A is row-equivalent to I.

(ii) $\longrightarrow$ (iii)

it's given that R is row-equivalent to I

$\Rightarrow$ $A = E_1^{-1} \cdot E_2^{-1} \cdots E_{k-1}^{-1} \cdot E_k^{-1} \cdot I$

$\Rightarrow$ $A = E_1^{-1} \cdot E_2^{-1} \cdots E_{k-1}^{-1} \cdot E_k^{-1}$

$\Rightarrow$ A is a product of elementary matrices.

(iii) $\longrightarrow$ (i)

it's given that A is a product of elementary matrices.

$\Rightarrow$ A is a product of invertible matrices (elementary matrices are invertible by Theorem 11)

By corollary of theorem 10, product of invertible matrices is invertible.

$\Rightarrow$ A is invertible.

Corollaries (to Theorem 12)

Let A be an $n \times n$ matrix. Consider the augmented matrix $[A|I]$. Suppose that

$$[A|I] \sim [I|B].$$

Note that A is row equivalent to I (thus A is invertible) and I is row equivalent to B. By Corollary to Theorem 9, there exists an $n \times n$ matrix P such that $I = PA \;\; and \;\; B = PI \implies B = P \;\; and \;\; I = BA$ $\implies$ **A is invertible and $B = A^{-1}$.**

Corollary 12.1.

If A is an $n \times n$ matrix and if a sequence of elementary row operations reduces A to the indentity matrix, then that same sequence of operations when applied to I yields A^{-1}.

ATTEMPT TO UNDERSTAND COROLLARY 12-1

$A \to E_1 \to E_2 \to E_3 \to \cdots \to E_K = I.$
$I \to E_1 \to E_2 \to E_3 \to \cdots \to E_K = A^{-1}.$

$E_1 = e_1(A)$
$E_2 = e_2(e_1(A))$

$\downarrow$ By extension,

$E_K = I = e_K(e_{K-1}(\cdots(e_2(e_1(A)))\cdots))$

$$\left.\begin{aligned} e_1(A) &= E_1 = e_1(I)\,A \\ e_2(E_1) &= E_2 = e_2(I)\cdot E_1 = e_2(I)\cdot e_1(I)\cdot A \end{aligned}\right\}\text{ BY THEOREM 9.}$$

$\downarrow$ By extension,

$e_K(E_{K-1}) = E_K = I = e_K(I)\cdot e_{K-1}(I)\cdots e_2(I)\cdot e_1(I)\cdot A$

$\downarrow$ if $XY = I$, X is inverse of Y

$\Rightarrow e_K(I)\cdot e_{K-1}(I)\cdots e_2(I)\cdot e_1(I) = A^{-1}$

$\Rightarrow e_K(e_{K-1}(\cdots(e_2(e_1(I)))\cdots)) = A^{-1}$

$\downarrow$

$\therefore$, if a sequence of elementary row operations form I from A, the same set of operations form A^{-1} from I.

HENCE PROVED.

Problem

Find the inverse of

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

Solution : Consider

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \longleftarrow R_2 - \frac{1}{2}R_1, \quad R_3 \longleftarrow R_3 - \frac{1}{3}R_1$$

Solution contd.

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right]$$

$$R_2 \longleftarrow 12R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array}\right]$$

$$R_1 \longleftarrow R_1 - \frac{1}{2}R_2, \quad R_3 \longleftarrow R_3 - \frac{1}{12}R_2$$

Solution contd.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right]$$

$$R_3 \longleftarrow 180R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$$

$$R_2 \longleftarrow R_2 - R_3, \quad R_1 \longleftarrow R_1 + \frac{1}{6}R_3$$

Solution contd.

$$[A|I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right] = [I|B]$$

By Corollary 12.1,

$$A^{-1} = B = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

Problem 2

Find the inverse of

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix}$$

Solution :

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 6 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

Take $R_3 \longleftarrow R_3 - R_1$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 4 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

Solution contd.

Now, take $R_3 \longleftarrow R_3 - R_2$. Then

$$[A|I] \sim \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

Thanks to the last zero row, A is not row-equivalent to I and A is not invertible.

Theorem 13

For an $n \times n$ matrix A the following are equivalent.

(i) A is invertible.

(ii) The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.

(iii) The system of equations $AX = B$ has a solution X for each $n \times 1$ matrix B.

Proof: $(i) \Longrightarrow (ii)$ Suppose that A is invertible. By Theorem 12, A is row-equivalent to I. By Theorem 3, $AX = 0$ and $IX = 0$ have exactly same solutions. The solution set of $AX = 0$ is

$$S = \{X \ : \ AX = 0\} = \{X \ : \ IX = 0\} = \{X \ : \ X = 0\} = \{0\}.$$

Hence the system $AX = 0$ has only the trivial solution $X = 0$.

Theorem 13 contd.

$(ii) \implies (i)$ **Suppose that $AX = 0$ has only the trivial solution $X = 0$.** By Theorem 7, A is row-equivalent to I. By Theorem 12, A is invertible.

$(i) \implies (iii)$ Suppose that A is invertible. That is A^{-1} exists. Consider the system $AX = B$. This implies that $X = A^{-1}B$ is a solution for the system $AX = B$ for each B.

Theorem 13 contd.

$(iii) \implies (i)$ Suppose that the system of equations $AX = B$ has a solution X for each $n \times 1$ matrix B. Let R be a row-reduced echelon matrix which is row-equivalent to A. By an corollary of Theorem 9, $R = PA$, where P is a product of elementary matrices. Since elementary matrices are invertible, so is their product P.

$AX = B$ has a solution X for each B.
$\iff$ $P(AX) = PB$ has a solution X for each B.
$\iff$ $RX = PB$ has a solution X for each B (Note that $R = PA$).
$\iff$ $RX = E$ has a solution X for each E $(= PB)$.

Theoem 13 contd.

Now, take

$$E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$RX = E$ has a solution X.

$\implies$ The last row of R is non-zero.

$\implies$ R is an $n \times n$ row-reduced echelon matrix with no zero rows.

$\implies$ $R = I$.

Hence A is row-equivalent to $R = I$.

By Theorem 12, A is invertible.

THEOREM 13 : for an $n \times n$ matrix A, the following are equivalent:

(i) A is invertible.

(ii) The homogeneous system $AX=0$ has only the trivial solution $X=0$.

(iii) The system of eqns. $AX=B$ has a soln. X for each $n \times 1$ matrix B.

proof: (i) $\longrightarrow$ (ii)

its given that A is invertible.

$\Rightarrow$ By theorem 12, A is row-equivalent to $n \times n$ Identity matrix, I.

$\Rightarrow$ By theorem 3, $AX=0$ and $IX=0$ have exactly the same set of solns. as they are row-equivalent.

$\Rightarrow$ if $S_I \rightarrow$ soln. set for $IX=0$, then:

$$S_I = \{X : X=0\} = \{0\}$$

$\Rightarrow$ The system $AX=0$ has only the trivial soln. $X=0$.

(ii) $\longrightarrow$ (i)

it's given that system $AX=0$ has only $X=0$ as the trivial soln.

$\Rightarrow$ By theorem 7, A is row-equivalent to I.

$\Rightarrow$ $A = E_k \cdot E_{k-1} \cdots E_2 \cdot E_1 \cdot I$, by theorem 9.

Elementary matrices are invertible, by Theorem 11. Product of invertible matrices is invertible, by Theorem 10.

$\Rightarrow$ A is invertible.

$(i) \longrightarrow (iii)$

it's given that A is invertible

let A^{-1} be the inverse.

consider system $AX = B$

$\Rightarrow \quad A^{-1}.(AX) = A^{-1}B$

$\Rightarrow \quad (A^{-1}A)X = A^{-1}B$

$\Rightarrow \quad X = A^{-1}B.$

Hence, X is a soln. for $AX=B$ given by $X = A^{-1}B$, which exists for each B.

$(iii) \longrightarrow (i)$

The system of eqns. $AX=B$ has a soln. X for each $n \times 1$ matrix B.

consider RRE matrix R s.t. R is row-equivalent to A. By corollary in Theorem 9, $R = PA$, where $P \rightarrow$ pdt. of elementary matrices. P is invertible as elementary matrices are invertible, & so is their pdt. (theorems 10 and 11).

$AX = B$ has soln. for each B.

$\Rightarrow \quad P(AX) = PB$ does too.

$\Rightarrow \quad (PA)X = PB$ does too.

$\Rightarrow \quad RX = E$ does too $(E = PB)$

take, $E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$

w.k.t. soln. exists for $RX = E$.

$\Rightarrow$ last row of $R \rightarrow$ NON-ZERO

$\Rightarrow$ $R \rightarrow$ $n \times n$ row reduced echelon matrix with no ZERO ROWS

$\Rightarrow$ $R = I$. Hence, A is row-equivalent to $R = I$.

By Theorem 12, A is invertible.

Corollary 13.1

A square matrix with either a left or right inverse is invertible.

Proof: Let A be an $n \times n$ matrix.

Case 1. Suppose A has a left inverse, say B. That is $BA = I$.

Consider the system $AX = 0$**. That implies** $B(AX) = B0 = 0$**.**

$\implies (BA)X = 0. \implies \quad IX = 0. \implies \ X = 0.$

$$\textbf{Thus} \quad AX = 0 \implies X = 0$$

By Theorem 13, A **is invertible.**

Case 2 : Suppose that A **has a right inverse say** C**. i.e.,** $AC = I$**. So** A **is a left inverse of** C**. By Case 1,** C **is invertible and** $C^{-1} = A$**. Hence** A **is invertible and** $A^{-1} = C$**.**

Corollary 13.2

Let $A = A_1A_2 \dots A_k$ where $A_1, A_2, \dots, A_k$ are $n \times n$ (square) matrices. If A is invertible then each A_i is invertible.

Proof : $A = A_1A_2 \dots A_k \quad - - - (a)$

Suppose that A is invertible. By Theorem 13, $AX = 0 \Longrightarrow X = 0$. We want to show that each A_i is invertible. First, we prove that A_k is invertible. Consider the system $A_kX = 0$.

$\Longrightarrow \quad A_1A_2 \dots A_{k-1}(A_kX) = 0. \Longrightarrow \quad AX = 0. \Longrightarrow X = 0.$

$$\text{Thus } A_kX = 0 \Longrightarrow X = 0$$

By Theorem 13, A_k is invertible. Since A and A_k are invertible, $AA_k^{-1} = A_1A_2 \dots A_{k-1}$ is invertible. By preceeding argument, A_{k-1} is invertible. Continuing in this way, we conclude that each A_i is invertible.

COROLLARY 13.1 : A square matrix with either left (or) right inverse is invertible.

proof : let A be an $n \times n$ square matrix.

Case 1 – A has a left inverse

$\Rightarrow BA = I$.

consider $AX = 0$

$\Rightarrow B\cdot(AX) = B\cdot 0 = 0$

$\Rightarrow (BA)X = 0$

$\Rightarrow IX = 0$

$\Rightarrow X = 0$

So, $AX = 0 \Rightarrow X = 0$

$\therefore$, By Theorem 13, A is invertible.

Case 1 – A has a right inverse

$\Rightarrow AC = I$

$\Rightarrow A \rightarrow$ left inverse of C.

$\Rightarrow$ By case 1, C is invertible.

and $C^{-1} = A$.

$\therefore$, A is invertible with $A^{-1} = C$

COROLLARY 13.1: let $A = A_1 A_2 \cdots A_k$ where $A_1, A_2, \cdots A_k$ are $n \times n$ square matrices. if A is invertible, then each of A_i is invertible.

proof: given that A is invertible, and that

$$A = A_1 A_2 A_3 \cdots A_k \quad - (a)$$

if $A \rightarrow$ invertible, by theorem 13,

$$AX = 0 \Rightarrow X = 0$$

consider $AX = 0$

$$\Rightarrow (A_1 A_2 \cdots A_{k-1} A_k) X = 0$$

$$\Rightarrow A_1 A_2 \cdots A_{k-1} (A_k X) = 0$$

$$\Rightarrow A_k X = 0 \Rightarrow A_k \rightarrow \text{invertible, by Theorem 13.}$$

Since A and $A_k \rightarrow$ invertible,

$AA_k^{-1} = A_1 A_2 \cdots A_{k-1}$ is also invertible, by Theorem 10.

Similarly, A_{k-1} is invertible and so on.

proceeding this way, we get that each A_i is invertible.

Problem 1

Question. Prove or disprove that if A is an $m \times n$ matrix, B is an $n \times m$ matrix and $n < m$, then AB is not invertible.

Solution: Since B is an $n \times m$ matrix and $n < m$, by Theorem 6, the homogeneous system $BX = 0$ has a non-trivial solution, say $X^* \neq 0$.
i.e.

$$BX^* = 0.$$

Consider the system $(AB)X = 0$.

$$(AB)X^* = A(BX^*) = A0 = 0.$$

$\implies X^*$ is a non-trivial solution of the homogeneous system $(AB)X = 0$. By Theorem 13, AB is not invertible.

Problem 2

$$\text{Let} \quad A = \begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix}$$

Does there exist a 3×4 matrix B such that (i) $AB = 0$ and (ii) $B \neq 0$?

Solution of Problem 2

Find a non-trivial solution of the sytem $AX = 0$. Solution set,
$S = \{(\frac{5}{4}a, \frac{67}{24}a, a) \ : \ a \in \mathbf{R}\}$ (Visit previous lecture notes.)
Choose $a = 24 \implies (30, 67, 24)$ is a solution.

$$\implies B = \begin{bmatrix} 30 & 30 & 30 & 30 \\ 67 & 67 & 67 & 67 \\ 24 & 24 & 24 & 24 \end{bmatrix}$$

Verify that $AB = 0, \quad and \quad B \neq 0$

Problem 3

Prove of disprove that A is invertible and find A^{-1} if it exists where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Solution to Problem 3

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$