Calculus :- [End Semester]

Name :- a.v.s. lokesh Reddy
Roll No :- EC20B1050

1) Given

sequence $\{(1+1/n)^n\}$

Given $a_n = (1+\frac{1}{n})^n$, Now let $y = \lim_{n\to\infty} a_n$

$\Rightarrow \quad y = \lim_{n\to\infty} (1+1/n)^n$

By taking log on both sides ($\log_e$)

$\ln y = \lim_{n\to\infty} n \ln(1+1/n)$ $\quad\quad (\lim_{n\to\infty} 1/n = 0)$

$\ln y = \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n}$

this is of form $0/0$ so, we use 'L' Hospital rule then we know that if $\frac{f(x)}{g(x)}$ is of $0/0, \infty/\infty, -\infty/\infty$ then $h(x) = \frac{f'(x)}{g'(x)}$

$\ln y = \lim_{n\to\infty} \frac{1}{(1+1/n)} \frac{(-1/n^2)}{(-1/n^2)}$

$\ln y = \lim_{n\to\infty} \frac{1}{1+1/n} \Rightarrow \ln y = \frac{1}{1+0} \Rightarrow y = e^1$

$\therefore \lim_{n\to\infty} (1+1/n)^n = e$

as we see this is a decreasing sequence and a_n tends to e

so, ~~upper~~ bounded by e so,

The given sequence $\{(1+1/n)^n\}$ converges to e // as

$\lim_{n\to\infty} a_n = e$ (finite number).

Name:- O.V.S. lokesh Reddy
Roll No:- EC20B1050

3) Given

$$\lim_{x \to 3} (x^2 + 2x) = 15$$

Here $f(x) = x^2 + 2x$, $L = 15$, $x_0 = 3$ then for Given ϵ, there must be a δ for which

$$|x - x_0| < \delta \text{ and } |f(x) - L| < \epsilon \quad -(1)$$

$$\Rightarrow |x - 3| < \delta \text{ and } |x^2 + 2x - 15| < \epsilon$$

$\Rightarrow |x^2 + 2x - 15| < \epsilon$, we have $x^2 + 2x - 15 = (x+5)(x-3)$

let us assume that $\delta = 1$

$\Rightarrow [x-3] < 1$

$x - 3 < 1$ $\quad$ $-(x-3) < 1$

$x < 4$ $\quad$ $x > 2$

$\Rightarrow 2 < x < 4 \longrightarrow (2)$

from (2) $x < 4 \Rightarrow x + 5 < 4 + 5$

$x + 5 < 9$

$$|f(x) - L| = |x-3||x+5| < \delta 9$$

from (1) let $9\delta = \epsilon$

$\delta = \epsilon/9$

$\Rightarrow |x-3||x+5| < 9(\epsilon/9)$

$|x-3||x+5| < \epsilon$

ie $|f(x) - L| < \epsilon$

$\therefore \lim_{x \to 3} (x^2 + 2x) = 15$

Name:- O.V.S. lokesh Reddy
RollNO:- E120B1050

10) Given function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

then we know that $\frac{xy}{\sqrt{x^2+y^2}}$ is continuous every where except at origin $(0,0)$

Now for $f(x,y)$ we should check continuity at $(0,0)$

since $\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{|x^2+y^2|}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$

But $\sqrt{x^2+y^2} \to 0$ as $(x,y) \to (0,0)$

so, $\frac{xy}{\sqrt{x^2+y^2}} \to 0$ at $(0,0)$

$\therefore f(x,y)$ is continuous every where

---

5) Cauchy's mean value theorem proof:-

Here:- $g(a) \neq g(b) \Rightarrow g(b)-g(a) \neq 0$

Now consider a function $F(x)$ defined by $F(x) = f(x) + Ag(x)$ — (1)

where A is some constant to be determine such that $F(a)=F(b)$

$$f(a) + A(g(a)) = f(b) + Ag(b)$$

$$f(b) - f(a) = -A(g(b)-g(a))$$

$$-A = \frac{f(b)-f(a)}{g(b)-g(a)} \quad \text{— (2)}$$

from (1) $f(x)$ & $g(x)$ are continous & differentiable Then $F(x)$ is also continuous & differentiable

by Rolle's Theorem

Name :- O.V.S. lokesh Reddy
Roll NO :- EC20B1050

$F'(c) = 0$, where $c \in [a,b]$

$f'(c) + Ag'(c) = 0$

$$-A = \frac{f'(c)}{g'(c)} \quad -③$$

from ② & ③

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$ Hence proved //

---

8) Given $f(x) = x^2$ on interval $[1,3]$ and is monotonically increasing

Partition $P_n = \{x_0 =$ [redacted]$, x_1 = 2/n + 1 \ldots\ldots x_i = 2i/n + 1, \ldots x_n = 3\}$

let $M_i = \sup(f(x);\ x_{i-1} \le x \le x_i\} = x_i \Rightarrow M_i = (1 + 2i/n)^2$

$m_i = \inf\{f(x),\ x_{i-1} \le x \le x_i\} = x_{i-1} \Rightarrow m_i = (1 + 2/n(i-1))^2$

Now $$U(P_n, f) = \sum_{i=0}^{n} M_i \Delta x_i$$

$$= \sum_{i=0}^{n} (1 + 2i/n)^2 (2/n)$$

$$\Rightarrow 2/n \sum_{i=0}^{n} \left(1 + \frac{4i^2}{n^2} + \frac{4i}{n}\right)$$

$$\Rightarrow 2/n \left(n + \frac{4}{n^2} \frac{\cancel{n}(n+1)(2n+1)}{6} + \frac{4}{\cancel{n}} \frac{\cancel{n}(n+1)}{2}\right)$$

$$\Rightarrow 2\left(1 + \frac{4}{6}(1+1/n)(2+1/n)\right) + 2(1+1/n) //$$

Name :- a.v.s. lokesh Reddy

Roll No :- EC20B1050

Now $L(P_n, f) = \sum_{i=0}^{n-1} m_i \Delta x_i$

$= \frac{2}{n}\left(\sum_{i=0}^{n}\left(1+\frac{2(i-1)}{n}\right)^2\right)$

$\Rightarrow 2/n\left(\sum_{i=1}^{n-1} 1 + \frac{4(i)^2}{n^2} + 4i/n\right)$

$\Rightarrow 2/n\left(n + \frac{4}{6}\frac{(n-1)(n)(2n-1)}{n^2} + \frac{4}{n}\frac{(n-1)(n)}{2}\right)$

$\Rightarrow 2\left(1+\frac{2}{3}(1-1/n)(2-1/n) + 2/(1-1/n)\right)$

them $\lim_{n\to\infty} L(f, P_n) = 2(1+1/3(1-0)(2-0)+2)$

$= 2\left(\frac{3+4+6}{3}\right) = 26/3$

$\lim_{n\to\infty} U(f, P_n) = 2(1+2/3(1)(2)+2) \Rightarrow 26/3$

As $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = 26/3$ so it is Riemann integrable.

$\int_1^3 x^2\,dx = \left[x^3/3\right]_1^3 = \frac{27}{3} - 1/3 = 26/3$

so, The common limit equals to the Riemann integral $\int_1^3 x^2\,dx$ //

Given

6) $f(x) = 3x^{2/3}$ in interval $-27 \le x \le 8$

Name:- O.V.S. lokesh Reddy
Roll No:- EC20B1050

For absoulte maximum (or) minimum,

$f'(x) = 0$, $\Rightarrow$ $\cancel{3}\left(\frac{2}{\cancel{3}}\right)x^{-1/3} = 0$

$\Rightarrow \frac{2}{x^{1/3}} = 0 \Rightarrow x = \infty$

and we are given interval $[-27, 8]$ and at end points

$f(-27) = 3 \times 9 = 27$

$f(8) = 3 \times 4 = 12$

Hence absolute maximum value is 27 at $x = -27$
absolute minimum value is 12 at $x = 8$

as x in closed interval we have absolute extremas and we had checked at end points and at $f'(x) = 0$.

---

7) Given a bounded function $f: [a, b] \to R$

we know for any partion $P = (x_0, \ldots, x_n)$ of $[a, b]$

$M_i = \sup\{f(x), x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x), x \in [x_{i-1}, x_i]\}$
$(i = 1, 2, \ldots n)$

Then
we know

$$m_i \le M_i$$

$$m_i \Delta x_i \le M_i \Delta x_i$$

$$\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i$$

$$\Rightarrow L(P, f) \le U(P, f)$$

$\therefore$ For any partion P, The lower Riemann integral is less or equal to upper Riemann integral.

Name :- O.V.S. lokesh Reddy
RollNO :- EC20B1050

4) Given $f(x)$ this has to be continuous at point a then we know that to be continuous at $x=0$ then it limit at a should be equal to $f(a)$ so,

As we for $f(x)$, $\lim_{x \to a} f(x) = f(a)$ then only this is continuous

By limit definition $L = f(a)$, $x_0 = x \to a$

then for given ϵ, there must be a δ for which $|x - a| < \delta$ and $|f(x) - L| < \epsilon$ so,

Now to be continuous

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$ Hence proved.

Name :- O.V.S.lokesh Reddy
Roll NO :- EC20B1050

9) a) Here for $x^2+y^2=1$ and $y\geq 0$ then
this a closed set as it contains it boundary point

b) $\{(x,y) | x^2+y^2=1$ and $y>0$
this a open set as all boundary points are not there as
$y=0$, we get boundary so, This is open set.

Name :- O.V.S. lokesh Reddy
Roll No :- EC20B1050

2) a) Given series

$$\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4n}$$

Here $a_n = \frac{1}{4n}$

then radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(4(n+1))}{4n}\right| \Rightarrow \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)\right|$$

$$R \Rightarrow \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)\right| \Rightarrow R = 1$$

Thus, the given series converge absolutely for $|x-1| < 1$ and diverges for $|x-1| > 1$

At $x=0$, then the series is $\frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{n}$ this diverges

At $x=2$, then the series is $\frac{1}{4}\sum \frac{1}{n}$ this also diverges

Thus the interval of converges of power series is $0 < x < 2$

Name:- O.V.S. lokesh Reddy

Roll NO:- EC20B1050

2) b)

This is a geometric series with common ratio $\ln x$, and first term at $n=0$ so, when we substitute, it we get $a_0 = \ln^0 x = 1$, we know that a geometric series converges if and only if $|r|<1$ so, since $r = \ln x$, this implies

$$|\ln x| < 1$$

$$-1 < \ln x < 1$$

$$1/e < x < e$$

$\therefore$ Interval of convergence is $1/e < x < e$

we know the formula for the sum s of a convergent geometric series as

$$s = \frac{a_0}{1-r} \quad \text{for } 1/e < x < e$$

$$s = \frac{1}{1-\ln x} \quad \text{for } 1/e < x < e$$

$\therefore$ sum of the series in $1/e < x < e$ is $\frac{1}{1-\ln(x)}$