Engineering Electromagnetics

Lecture 15

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by

Debolina Misra

Assistant Professor of Physics
IIITDM Kancheepuram, Chennai, India

- For a point charge q at the origin, calculate the flux of **E** through a spherical surface of radius r.

Flux

- For a point charge q at the origin, calculate the flux of **E** through a spherical surface of radius r.

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0} q$$

- the flux through any surface enclosing the charge is q/ε_0
- Now suppose that instead of a single charge at the origin, we have a bunch of charges scattered about.

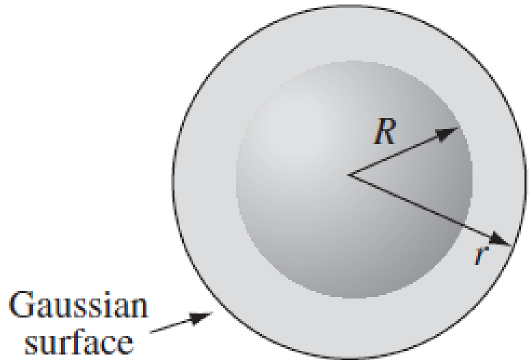
$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left(\oint \mathbf{E}_i \cdot d\mathbf{a}\right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i\right)$$

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}}$$

where Q_{enc} is the total charge enclosed within the surface

From Griffith Book

Example 2.3. Find the field outside a uniformly charged solid sphere of radius R and total charge q.



$$\oint_{\mathcal{S}} |\mathbf{E}|\, da = |\mathbf{E}| \oint_{\mathcal{S}} da = |\mathbf{E}|\, 4\pi r^2.$$

$$|\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q,$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.$$

Notice a remarkable feature of this result: The field outside the sphere is exactly *the same as it would have been if all the charge had been concentrated at the center.*

Divergence of E

As it stands, Gauss's law is an *integral* equation, but we can easily turn it into a *differential* one, by applying the divergence theorem:

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}}$$

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau.$$

Rewriting Q_{enc} in terms of the charge density ρ, we have

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho \, d\tau.$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau.$$

And since this holds for *any* volume, the integrands must be equal:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

Gauss's law in differential form

From Griffith Book

Line integral of E

$\boxed{\text{Let us calculate } \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.}$

the simplest possible configuration: a point charge at the origin. In this case

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.$$

In spherical coordinates, $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$, so

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr.$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2} dr = \frac{-1}{4\pi\epsilon_0} \frac{q}{r} \bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right)$$

For a closed path → a = b → $r_a = r_b$ ⇨ $\oint \mathbf{E} \cdot d\mathbf{l} = 0$

Apply Stokes' theorem to convert line integral to surface integral $\oint \boldsymbol{E}.d\boldsymbol{l} = \iint (\boldsymbol{\nabla} \times \boldsymbol{E}).\boldsymbol{ds}$ ⇨ $\boxed{\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}.}$

From Griffith Book

Curl of E = zero → Electrostatic potential

$$\boxed{\nabla \times \mathbf{E} = \mathbf{0}}$$

- If curl of a vector field is zero→ it can be represented as gradient of scalar field

$$\boxed{\vec{\mathbf{E}} = -\nabla V}$$

The curl of a gradient is always zero

$$\nabla \times (\nabla T) = \mathbf{0}.$$

You can check it just taking any scalar T (x,y,z) in Cartesian coordinate system→ then take the Grad→ and finally curl→ it should give you Zero

- The potential difference between two points a and b is

From fundamental theorem of Gradient

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} \Rightarrow \boxed{V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}}$$

From Griffith Book

Advantage of the potential formulation

$$\nabla \times \mathbf{E} = \mathbf{0}$$

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

- **E** → not just any vector, a special one.

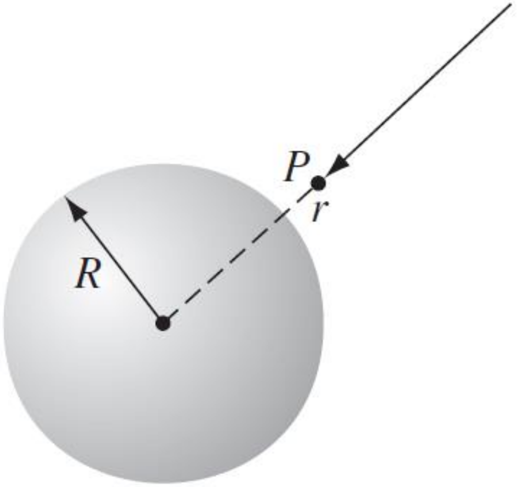
From Griffith Book

Problem-1

- Find the Field and potential inside and outside a spherical shell of radius R that carries a uniform surface charge.
- At any point (r>R and r<R)

- q is the total charge on the sphere (surface charge)

$$\boxed{V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}}$$



Thank You