Indian Institute of Information Technology,
Design and Manufacturing Kancheepuram
MA1002 Linear Algebra (Solution guideline)

Date : 03/10/2024	Mid Semester
Time : 3.30 - 5.00	Marks : 25

1. **Consider the following system of linear equations $kx+y+z=1$, $x+ky+z=1$, and $x+y+kz=1$ where k is a real number. Find the values of k such that the above system has (i) infinite number of solutions, (ii) no solution and (iii) a unique solution. Justify your answer.** (4)

$\begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \implies AX = b$. So we consider the augumented matrix $[A|b]$.

$$(A|b) = \left(\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right) (R_1 \leftrightarrow R_2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right) (R_2 \leftarrow R_2 - kR_1, R_3 \leftarrow R_3 - R_1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1-k^2 & 1-k & 1-k \\ 0 & 1-k & k-1 & 0 \end{array}\right) (R_2 \leftrightarrow R_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1-k & k-1 & 0 \\ 0 & 1-k^2 & 1-k & 1-k \end{array}\right) \qquad (i)$$

When $k=1$, $(A|b) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies x+y+z=1$. The solution set is $S = \{(1-a-b, a, b) : a, b \in R\}$. So the system has infinite number of solutions when $k=1$. ([1]

So $k \neq 1$.

$$(A|b) \sim \left(\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1-k & k-1 & 0 \\ 0 & 1-k^2 & 1-k & 1-k \end{array}\right) (R_2 \leftarrow \frac{1}{1-k}R_2, R_3 \leftarrow \frac{1}{1-k}R_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1+k & 1 & 1 \end{array}\right) (R_1 \leftarrow R_1 - kR_2, R_3 \leftarrow R_3 - (1+k)R_2)$$

$$(A|b) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1+k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k+2 & 1 \end{array}\right) \qquad (ii)$$

When $k = -2$,

$$(A|b) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Due to the last row of $(A|b)$, the system $Ax = b$ has no solution when $k = -2$ [1]
So $k \neq -2, 1$.

$$(A|b) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1+k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k+2 & 1 \end{array}\right) (R_3 \leftarrow \frac{1}{k+2} R_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1+k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{k+2} \end{array}\right) (R_2 \leftarrow R_1 - (1+k)R_3, R_2 \leftarrow R_2 + R_3)$$

$$(A|b) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{k+2} \\ 0 & 1 & 0 & \frac{1}{k+2} \\ 0 & 0 & 1 & \frac{1}{k+2} \end{array}\right)$$

The solution of the system is unique and it is $\{(\frac{1}{k+2}, \frac{1}{k+2}, \frac{1}{k+2})\}$ [2]

2. **Count all possible 3×3 inequivalent classes of row-reduced echelon matrices.** (4)

Let r be the number of non-zero rows.

$r = 0$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$r = 1$: $\begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$r=2$: $\begin{pmatrix}1&0&x\\0&1&y\\0&0&0\end{pmatrix}, \begin{pmatrix}1&x&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$

$r=3$: $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

where x, y are arbitrary scalars. ($\frac{1}{2}$ **mark for each matrix**)

3. **Find all 2×2 elemetary matrices and respective inverses. Justify your answer.** (4)

The elemetary row opertions are

(i) $e_1 : R_1 \leftarrow cR_1$ $(c \neq 0)$, $\quad e_1^{-1} : R_1 \leftarrow \frac{1}{c}R_1$

(ii) $e_2 : R_2 \leftarrow cR_2$ $(c \neq 0)$, $\quad e_2^{-1} : R_2 \leftarrow \frac{1}{c}R_2$

(iii) $e_3 : R_1 \leftarrow R_1 + cR_2$, $\quad e_3^{-1} : R_1 \leftarrow R_1 - cR_2$

(iv) $e_4 : R_2 \leftarrow R_2 + cR_1$, $\quad e_4^{-1} : R_2 \leftarrow R_2 - cR_1$

(v) $e_5 : R_1 \leftrightarrow R_2$, $\quad e_5^{-1} : R_1 \leftrightarrow R_2$ [1]

Let $E_i = e_i(I)$ and $E_i^{-1} = e_i^{-1}(I)$.

$$E_1 = \begin{pmatrix}c&0\\0&1\end{pmatrix} \qquad E_1^{-1} = \begin{pmatrix}\frac{1}{c}&0\\0&1\end{pmatrix}$$

$$E_2 = \begin{pmatrix}0&1\\c&0\end{pmatrix} \qquad E_2^{-1} = \begin{pmatrix}0&1\\\frac{1}{c}&0\end{pmatrix} \qquad [1]$$

$$E_3 = \begin{pmatrix}1&c\\0&1\end{pmatrix} \qquad E_3^{-1} = \begin{pmatrix}1&-c\\0&1\end{pmatrix}$$

$$E_4 = \begin{pmatrix}1&0\\c&1\end{pmatrix} \qquad E_4^{-1} = \begin{pmatrix}1&0\\-c&1\end{pmatrix} \qquad [1]$$

$$E_5 = \begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad E_5^{-1} = \begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad [1]$$

Absence of details in your answerbook lead to at least 50 percentage deduction in marks.

4. **Let $A \in F^{n\times n}$. Prove that if $AX = Y$ has a solution $X \in F^{n\times 1}$ for each $Y \in F^{n\times 1}$, then A is invertible.** (3)

Suppose that the system of equations $AX = Y$ has a solution X for each $n\times 1$ matrix Y. Let R be a row-reduced echelon matrix which is row-equivalent to A. By Corollary 12.2, $R = PA$ where P is an $n \times n$ invertible matrix. It is enough to prove that $R = I$

Consider $RX = E \iff (PA)X = E \iff AX = P^{-1}E \qquad (1)$

So $AX = Y$ has a solution X for each $Y \in F^{n\times 1} \implies AX = P^{-1}E$ has a solution

X for E where

$$E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\implies$ $RX = E$ has a solution X for E (by (1)). $\implies$ The last row of R is non-zero. $\implies$ R is an $n \times n$ row-reduced echelon matrix with no zero rows. $\implies$ $R = I$.
Hence A is row-equivalent to $R = I$. By Theorem 12, A is invertible.

5. **Let $V = \{x \in R : x \geq 0\}$ and $F = R$. For $x, y \in V, \alpha \in R$, we define $x + y := xy$, (vector addition) and $\alpha x := |\alpha| x$. (scalar multiplication). Verify, which are the list of axioms for a vector space are satisfied by $(V, R, +, .)$.** (5)

   (a) Closure: For any $x, y \geq 0$ in R, $x + y = xy \in R$ and it is non-negative. (0.5 mark)

   (b) Associativite law: The associativity law is true as it holds for R. (0.5 mark)

   (c) Additive identity: The number 1 is the additive identity. As $x+1 = x \cdot 1 = x$ for all $x \in R$. (2 marks)

   (d) Commutativity law: As the multiplication is commutative in R. (0.5 mark)

   (e) Scalar multiplication: For all non-negative $x \in R$ and $\alpha \in R$, $\alpha x = |\alpha| x$ is also non-negative in R. (0.5 mark)

   (f) We have $1x = |1|x = x$. (.5 mark)

   (g) We also have $(\alpha_1\alpha_2)x = |\alpha_1\alpha_2|x = |\alpha_1||\alpha_2|x$.
   Also, $\alpha_1(\alpha_2 x) = |\alpha_1|(\alpha_2 x) = |\alpha_1|(|\alpha_2|x) = |\alpha_1\alpha_2|x$. Thus, $(\alpha_1\alpha_2)x = \alpha_1(\alpha_2 x)$ (0.5 mark)

6. **Show that $0\alpha = 0$ for all $\alpha \in V$, where 0 is the additive identity in the field F and $\mathbf{0}$ is the zero vector in the vector space V (2)**
   There can be multiple ways to prove this.
   Let V be a vector space over the field F, and let 0 represent both the additive identity in F and $\mathbf{0}$(zero vector) in V.

(a) Consider the scalar multiplication of the zero scalar with any vector $\alpha \in V$. We want to show that multiplying the scalar 0 by any vector α results in the zero vector 0_V in the vector space V.
We will use the distributive property of scalar multiplication over addition in vector spaces:

$$0\alpha = (0 + 0)\alpha$$

(b) Apply the distributive property of scalar multiplication:
By the distributive property of scalar multiplication over addition in F, we can expand the right-hand side as follows:

$$(0 + 0)\alpha = 0\alpha + 0\alpha$$

So, we now have the equation:

$$0\alpha = 0\alpha + 0\alpha$$

(c) Subtract 0α from both sides of the equation to eliminate one of the terms on the right-hand side. Since scalar multiplication follows the usual rules of addition and subtraction in V, we can do this as follows:

$$(0\alpha) - (0\alpha) = (0\alpha + 0\alpha) - (0\alpha)$$

This simplifies to:

$$\mathbf{0} = 0\alpha$$

Thus, we have shown that multiplying the scalar 0 by any vector $\alpha \in V$ results in the zero vector 0_V, i.e.,

$$0\alpha = \mathbf{0}$$

This holds for all $\alpha \in V$, as required.

7. **Prove that the inverse of a lower triangular matrix is a lower triangular matrix.** (3)
Let L be an $n \times n$ lower triangular matrix, meaning that all entries above the main diagonal are zero, i.e., for $i < j$, we have $L_{ij} = 0$. We need to show that the inverse L^{-1}, if it exists, is also a lower triangular matrix.

(a) **Diagonals $L_{ii} \neq 0$:** A lower triangular matrix is invertible iff all diagonal entries are non-zero.

(b) **Form of the Matrix Equation:**

Since L is invertible, there exists a matrix L^{-1} such that:

$$LL^{-1} = I_n$$

where I_n is the $n \times n$ identity matrix.

(c) **Consider the Entries of L^{-1}:**

Let $L^{-1} = (b_{ij})$ be the inverse matrix of L. We want to show that $b_{ij} = 0$ for $i < j$, meaning L^{-1} has zero entries above the main diagonal.

(d) **Matrix Multiplication:**

For the product $LL^{-1} = I_n$, consider the elements of the identity matrix on the diagonal and above the diagonal. The element in the (i, j)-th position of the product LL^{-1} is given by:

$$(LL^{-1})_{ij} = \sum_{k=1}^{n} L_{ik} b_{kj}$$

Since L is lower triangular, for $i < k$, $L_{ik} = 0$. Hence, the sum can only include terms with $k \leq i$. This implies:

$$(LL^{-1})_{ij} = \sum_{k=1}^{i} L_{ik} b_{kj}$$

Now consider two cases:

- **Case 1:** $i = j$**:** On the diagonal of I_n, the entries are 1. Hence, for $i = j$, we have:

$$\sum_{k=1}^{n} L_{ik} b_{ki} = \sum_{k=1}^{i} L_{ik} b_{ki} = 1, \quad \text{for } i = 1, 2, 3, \cdots, n$$

  For $i = 1$ and from $L_{11} \neq 0$, we have $b_{ii} \neq 0$. Similarly, for other values of i and $L_{ii} \neq 0$, we can ensure that the inverse matrix L^{-1} must have non-zero diagonal elements.

- **Case 2:** $i < j$**:** For $i < j$, the entry in position (i, j) in I_n is 0. Thus, we have the equation:

$$\sum_{k=1}^{n} L_{ik} b_{kj} = \sum_{k=1}^{i} L_{ik} b_{kj} = 0$$

  Since $L_{ik} = 0$ for $k > i$, and for $i < j$, we conclude that:

$$b_{ij} = 0$$

(e) Since the entries of L^{-1} above the diagonal are all zero, L^{-1} is a lower triangular matrix.