MA1000: Calculus

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Evaluation

- Mid Term: 30 Marks
- Assignments: 30 Marks
- End Semester: 40 Marks

Text Books

- Thomas' Calculus, Pearson Education, 4th Edition.
- Piskunov, Differential and Integral Calculus, Vol. I & II, Mir. Publishers.
- Kreyszig, Advanced Engineering Mathematics, Wiley Eastern, 10th Edition.

The Syllabus

- Sequences, Series, Power series.
- Limit and Continuity, Intermediate Value Theorem, Differentiability, Rolle's Theorem, Mean Value Theorem, Taylor's Formula.
- Riemann Integration, Mean value theorem, Fundamental theorem of integral calculus.
- Functions of several variables, Limit and Continuity, Geometric representation of partial and total derivatives, Derivatives of composite functions.
- Directional derivatives, Gradient, Lagrange multipliers- Optimization problems,
- Multiple integrals, Evaluation of line and surface integrals.

Module 1: Sequences, Series, and Power Series

1. Definition of a Sequence, Examples
2. The Definition of Convergence and Divergence of Sequences
3. Testing the Convergence of a Sequence
4. Definition of Series and its convergence
5. Definition of Power Series and its convergence

Sequences

1. Informally, a sequence is a list of objects.
2. We will only consider numerical (i.e., real number) sequences.
3. Our sequences will be infinite, like $1, 3, 5, \ldots, (2n-1), \ldots$.

Thus a sequence is a a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order.

Sequences: More Examples

- $1, 3, 5, \ldots, (2n-1), \ldots$
- $2, 3, 5, 7, 11, \ldots,$
- $1, 1, 1, \ldots, 1, \ldots$
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$
- $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots$

Sequence: Formal Definition

Definition

A *sequence* is a function from the set of positive integers to the set of real numbers:

$$a : \mathbb{Z}^+ \to \mathbb{R}.$$

The images $a(1), a(2), a(3), \ldots, a(n), \ldots$ are called the *terms* of the sequence.

Note:

1. If $a : \mathbb{Z}^+ \to \mathbb{R}$ is a sequence, its terms are rather denoted by $a_1, a_2, a_3, \ldots, a_n, \ldots$.
2. A standard notation for a sequence $a : \mathbb{Z}^+ \to \mathbb{R}$ is $\{a_n\}$:
   That is, we enclose the nth term of the sequence within braces.
3. Sequences are often described by providing formulas for its general (nth) terms.

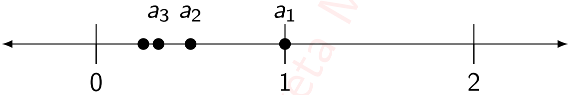
Examples

Sequences are often described by providing formulas for its general (nth) terms

$$
\begin{aligned}
\{2n-1\} &= 1, 3, 5, \ldots, (2n-1), \ldots \\
\left\{\frac{1}{n}\right\} &= 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots \\
\left\{\frac{(n-1)}{n}\right\} &= 0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{(n-1)}{n}, \ldots \\
\left\{\frac{(-1)^{n+1}}{n}\right\} &= 1, -\frac{1}{2}, \frac{1}{3}, \ldots, \frac{(-1)^{n+1}}{n}, \ldots \\
\{(-1)^{n+1}n\} &= 1, -2, 3, -4, \ldots, (-1)^{n+1}n, \ldots \\
\{\sqrt{n}\} & \quad \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots
\end{aligned}
$$

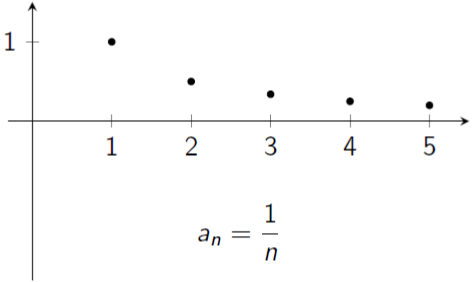
Note: Different sequences behave differently.

Plotting Sequences



$$a_n = \frac{1}{n}$$

Plotting Sequences



Convergence of a Sequence

Definition (Converges, Diverges, Limit)

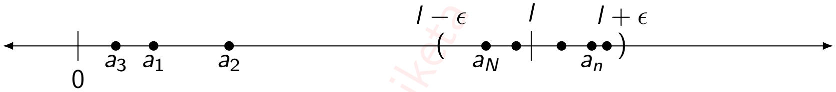
A sequence $\{a_n\}$ *converges* to a number l if to every positive number ϵ, there corresponds an integer N such that for all n,

$$n \geq N \quad \Rightarrow \quad |a_n - l| < \epsilon.$$

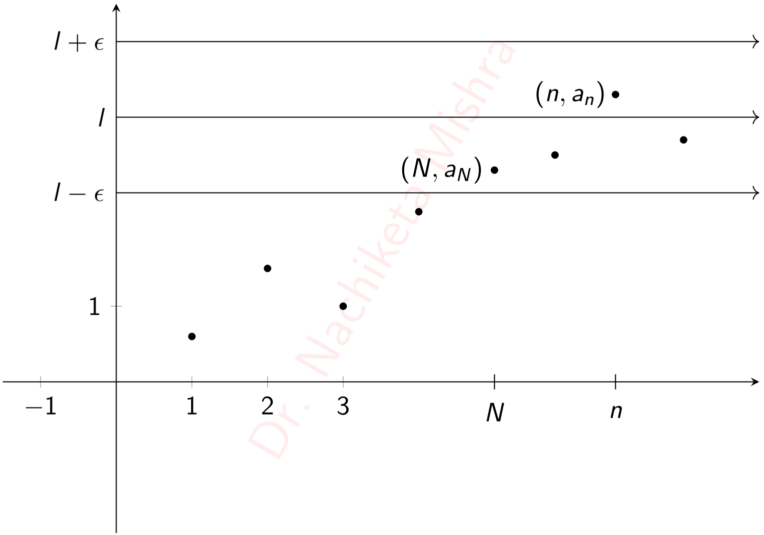
If no such number l exists, we say that $\{a_n\}$ *diverges*.

If $\{a_n\}$ converges to l, we write $\lim_{n\to\infty} a_n = l$ or $a_n \to l$ and call l the *limit* of the sequence.

Convergence, Pictorially I



Convergence, Pictorially II



Applying the Definition

Show using the definition of convergence that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Solution: Here $a_n = \dfrac{1}{n}$ and $l = 0$.
Let $\epsilon > 0$ be given. We must show that there exists a fixed integer N such that for all n,

$$n \geq N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon.$$

We note that $\dfrac{1}{n} < \epsilon \Leftrightarrow n > \dfrac{1}{\epsilon}$.
Thus, if N is any integer greater than $\dfrac{1}{\epsilon}$, then the above implication hold for all integers $n \geq N$.

Homework

Let k be any real constant. Show using the definition of convergence that

$$\lim_{n\to\infty} k = k.$$

Solution: Here $a_n = k$ and $l = k$.
Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n \geq N \quad \Rightarrow \quad |k - k| < \epsilon.$$

Since $|k - k| = 0 < \epsilon$ always, we can choose any positive integer as N and the implication will hold.
This proves that $\lim_{n\to\infty} k = k$ for any constant k.

Divergent Sequences: Example

Show that the sequence $1, -1, 1, -1, 1, \ldots, (-1)^{n+1}, \ldots$ diverges.

Solution:

- Suppose the sequence converges to some number l.
- Let $\epsilon = 1/2$.
- Then there must be an integer N such that each term a_n with index $n \geq N$ lies within $\epsilon = 1/2$ of l: $|a_n - l| < 1/2$.
- The number 1 appears repeatedly as every other term. So, 1 must be within $\epsilon = 1/2$ of l. That is $|l - 1| < 1/2$. This implies that $1/2 < l < 3/2$.
- Similarly, as -1 appears repeatedly as every other term, we also have that $|l - (-1)| < 1/2$. This implies that $-3/2 < l < -1/2$.
- Thus we have that the number l lies in both of the intervals $(1/2, 3/2)$ and $(-3/2, -1/2)$. But this is impossible. So, the sequence diverges.

Divergent Sequences: Example

Show that the sequence $\{\sqrt{n}\}$ diverges.

Solution: The sequence diverges because, as n increases, the terms of the sequence become larger than any fixed number. So, it is not converging to any finite number.

Note: We write $\lim_{n\to\infty} \sqrt{n} = \infty$.

Divergence to Infinity

Definition (Diverges to Infinity)

A sequence $\{a_n\}$ **diverges to infinity** if for every number M, there is an integer N such that for all n with $n \geq N$, $a_n > M$. In this case, we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty.$$

Homework: Provide a definition for the divergence of a sequence to $-\infty$.

Theorem

Let $\{a_n\}$ be a convergent sequence. Then its limit is unique.

Proof:

- Let $\epsilon > 0$ be a positve number.
- Suppose $\{a_n\}$ converges to both l_1 and l_2.
- Then, corresponding to $\epsilon/2 > 0$, we can find integers N_1 and N_2 such that

$$n \geq N_1 \Rightarrow |a_n - l_1| < \frac{\epsilon}{2};$$

$$n \geq N_2 \Rightarrow |a_n - l_2| < \frac{\epsilon}{2}.$$

- Let $N = \max(N_1, N_2)$.
- Then for $n \geq N$, we have

$$|l_1 - l_2| = |(a_n - l_1) - (a_n - l_2)| \leq |a_n - l_1| + |a_n - l_2| < \epsilon.$$

- But ϵ is arbitrary. So, we have that $l_1 = l_2$.

Theorem

Let $\{a_n\}$ *and* $\{b_n\}$ *be convergent sequences with* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then*

1. $\lim_{n\to\infty} (a_n + b_n) = a + b.$
2. $\lim_{n\to\infty} (a_n - b_n) = a - b.$
3. $\lim_{n\to\infty} (ka_n) = ka.$ (*Any number* k).
4. $\lim_{n\to\infty} (a_n b_n) = ab.$
5. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$ *if* $b \neq 0.$

Proof(1):

- Let $\epsilon > 0$ be given.
- Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, corresponding to $\epsilon/2 > 0$, there are integers N_1 and N_2 such that
$$n \geq N_1 \Rightarrow |a_n - a| < \frac{\epsilon}{2};$$
$$n \geq N_2 \Rightarrow |b_n - b| < \frac{\epsilon}{2}.$$
- Let $N = \max(N_1, N_2)$.
- Then for $n \geq N$, we have
$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
- This proves Part 1: $\lim_{n\to\infty} (a_n + b_n) = a + b$.

Proof(4):

Part 2 of the theorem implies the following:

Fact

$$\lim_{n\to\infty} a_n = a \Leftrightarrow \lim_{n\to\infty} (a_n - a) = 0.$$

Consider the following identity:

$$a_n b_n - ab = (a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a).$$

From the fact above and Part 3 of the theorem, we have that

$$\lim_{n\to\infty} [a(b_n - b)] = 0 \quad \text{and} \quad \lim_{n\to\infty} [b(a_n - a)] = 0.$$

We will now prove that

$$\lim_{n\to\infty} [(a_n - a)(b_n - b)] = 0.$$

Proof(4):

Given $\epsilon > 0$, corresponding to $\sqrt{\epsilon} > 0$, there are integers N_1 and N_2 such that

$$n \geq N_1 \Rightarrow |a_n - a| < \sqrt{\epsilon};$$

$$n \geq N_2 \Rightarrow |b_n - b| < \sqrt{\epsilon}.$$

Let $N = \max(N_1, N_2)$. Then

$$n \geq N \Rightarrow |(a_n - a)(b_n - b)| < \epsilon.$$

This proves that

$$\lim_{n\to\infty} [(a_n - a)(b_n - b)] = 0.$$

Thus the fact and the identity in the preceding slide imply that

$$\lim_{n\to\infty} (a_n b_n) = ab.$$

Examples

(a) $\displaystyle \lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1\cdot 0 = 0.$

(b) $\displaystyle \lim_{n\to\infty}\frac{n-1}{n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1-0 = 0.$

(c) $\displaystyle \lim_{n\to\infty}\frac{5}{n^2} = 5\cdot \lim_{n\to\infty}\frac{1}{n}\cdot \lim_{n\to\infty}\frac{1}{n} = 5\cdot 0\cdot 0 = 0.$

Homework

1. Prove the other parts of the preceding theorem.
2. Prove that $\lim_{n\to\infty} \dfrac{4-7n^6}{n^6+3} = -7$.
3. Prove that if the sequence $\{a_n\}$ diverges and c is any nonzero constant, then the sequence $\{ca_n\}$ also diverges.

Theorem (The Sandwich Theorem for Sequences)

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ for all n beyond some index N and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$, then $\lim_{n\to\infty} b_n = l$.

Corollary

If $|b_n| \le c_n$ and $c_n \to 0$, then $b_n \to 0$.

Proof.

$$|b_n| \le c_n \Leftrightarrow -c_n \le b_n \le c_n.$$

$c_n \to 0$. So, by the Sandwich theorem, we have $b_n \to 0$. □

Applying the Sandwich Theorem: Examples

$\frac{1}{n} \to 0$. So, we have the following true:

(a) $\frac{\cos n}{n} \to 0$ since $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$.

(b) $\frac{1}{2^n} \to 0$ since $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$.

(c) $(-1)^n \frac{1}{n} \to 0$ since __________ ?

Theorem (The Continuous Function Theorem for Sequences)

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to l$ and if f is a function that is continuous at l and is defined at all a_n, then $f(a_n) \to f(l)$.

Applying the Continuous Function Theorem: Examples

1. Show that $\sqrt{(n+1)/n} \to 1$.

   Solution: We know that $\dfrac{n+1}{n} \to 1$. Take $a_n = (n+1)/n$, $f(x) = \sqrt{x}$ and $l = 1$. The function $f(x)$ is continuous at $l = 1$. Thus by the theorem, $\sqrt{(n+1)/n} \to \sqrt{1} = 1$.

2. Show that $2^{1/n} \to 1$.

   Solution: The sequence $\left\{\dfrac{1}{n}\right\}$ converges to 0. Take $a_n = 1/n$ and $f(x) = 2^x$ and $l = 0$.
   The function is continuous at $l = 0$.
   Then we have that $2^{1/n} \to 2^0 = 1$.

Using l'Hôpital's Rule

Theorem

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for all $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = l \quad \Rightarrow \quad \lim_{n\to\infty} a_n = l.$$

Applying l'Hôpital's Rule: Examples

Show that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Solution: The function $f(x) = \dfrac{\ln x}{x}$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers.
Thus by the preceding theorem $\lim_{n\to\infty} \frac{\ln n}{n}$ will equal $\lim_{x\to\infty} \frac{\ln x}{x}$ if the latter limit exists.

Applying the l'Hôpital's rule, we see that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

Thus we conclude that $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

Applying l'Hôpital's Rule: Examples

Find

$$\lim_{n\to\infty} \frac{5^n}{7n}.$$

Solution: By l'Hôpital's rule (differentiating with respect to n),

$$\lim_{n\to\infty} \frac{5^n}{7n} = \lim_{n\to\infty} \frac{5^n \ln 5}{7} = \infty.$$

Applying l'Hôpital's Rule: Homework

Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

Note: Here the limit leads to the indeterminate form 1^∞. We can apply l'Hôpital's rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n.

Bounded Sequences

Definition

A sequence $\{a_n\}$ is said to be **bounded above** if there exists a number M such that $a_n \leq M$ for all n. In this case, M is called an **upper bound** for the sequence. If M is an upper bound for $\{a_n\}$ and no number less than M is an upper bound for $\{a_n\}$, then M is called the **least upper bound** for $\{a_n\}$.

Examples:

1. The sequence $1, 3, 5, \ldots$ has no upper bound.
2. The sequence $1, -1, 1, -1, \ldots$ is bounded above by $M = 1$. In fact, $M = 1$ is the least upper bound for this sequence.
3. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is bounded above by $M = 1$. In fact, $M = 1$ is the least upper bound for this sequence.

Homework: Define the following concepts for sequence: **bounded below**, **lower bound** and **greatest lower bound**. Also provide examples.

Monotonic Sequences

Definition

A sequence $\{a_n\}$ is said to be

- **monotonically increasing** or **non-decreasing** if $a_n \leq a_{n+1}$ for all n;
- **monotonically decreasing** or **non-increasing** if $a_n \geq a_{n+1}$ for all n.

Examples of monotonic sequences:

1. The sequence 1,2,3,. . . , n, . . . of positive integers is monotonically increasing.
2. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is monotonically increasing.
3. The constant sequence $3, 3, 3, \ldots$. is both m.i and m.d.
4. The sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}, \ldots$ is a monotonically decreasing sequence.

The Completeness Property of Real Numbers

A monotonically increasing sequence that is bounded above has a least upper bound.

Theorem

Let $\{a_n\}$ *be a monotonically increasing sequence that is bounded above. Then it converges to its least upper bound.*

Theorem

Let $\{a_n\}$ *be a monotonically decreasing sequence that is bounded below. Then it converges to its greatest lower bound.*

Proof:

- Let $\{a_n\}$ be a monotonically increasing sequence that is bounded above.
- Let l be the least upper bound of the sequence.
- Then we have that $a_n \leq l$ for all n.
- Let $\epsilon > 0$ be any real number.
- Then $l - \epsilon$ cannot be an upper bound for the sequence as $l - \epsilon < l$.
- Thus there is an integer N such that $a_N > l - \epsilon$.
- But $\{a_n\}$ is monotonically increasing. So, we have that

$$n \geq N \quad \Rightarrow \quad a_n \geq a_N > l - \epsilon.$$

- This implies that $n \geq N \quad \Rightarrow \quad l - \epsilon < a_n < l + \epsilon \quad$ or $\quad |a_n - l| < \epsilon$.
- Hence a_n converges to l.

Cauchy Sequences

Definition

A sequence $\{a_n\}$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that

$$n \geq N, m \geq N \quad \Rightarrow \quad |a_n - a_m| < \epsilon.$$

Theorem

A sequence converges if and only if is a Cauchy sequence.

Subsequences

Definition

Let $\{a_n\}$ be a sequence. If $\{n_k\}$ is a sequence of positive integers such that $n_1 < n_2 < n_3, \ldots,$ then $\{a_{n_k}\}$ is called a subsequence of $\{a_n\}$.

For example, the sequence of prime numbers is a subsequence of the sequence of positive integers.

Theorem

1. If a sequence $\{a_n\}$ converges to l, then every subsequence of $\{a_n\}$ also converges to l.
2. Every bounded sequence $\{a_n\}$ has a convergent subsequence.

Some Special Sequences

1. If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
2. If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.
3. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
4. If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
5. If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.
7. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

Solution:

(1) To prove: If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Let $\epsilon > 0$ be given. We will find an integer N such that

$$n \geq N \quad \Rightarrow \quad \left|\frac{1}{n^p} - 0\right| < \epsilon.$$

$$\frac{1}{n^p} < \epsilon \;\Leftrightarrow\; \frac{1}{n} < \epsilon^{1/p} \;\Leftrightarrow\; n > (1/\epsilon)^{1/p}.$$

Choose any positive integer $N > (1/\epsilon)^{1/p}$.

Solution:

(2) To prove: If $p > 0$, then $\lim_{n \to \infty} \sqrt[n]{p} = 1$.

If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$, and by binomial theorem,

$$1 + nx_n \leq (1 + x_n)^n = p$$

so that

$$0 < x_n \leq \frac{p-1}{n}.$$

Hence by the Sandwich theorem, $x_n \to 0$. This implies that $\sqrt[n]{p} \to 1$.

If $p = 1$, the result is trivial.

If $0 < p < 1$, the result is obtained by taking reciprocals.

Solution:

(3) To Prove: $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Put $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$, and by the binomial theorem,

$$n = (1 + x_n)^n \leq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \qquad (n \geq 2).$$

Hence $x_n \to 0$ or $\sqrt[n]{n} \to 1$.

Solution:

(4) To prove: If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

Let k be an integer such that $k > \alpha$ and $k > 0$. For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k} \qquad (n > 2k).$$

Since $\alpha - k < 0$, $n^{\alpha-k} \to 0$ by Part 1.

Series

Consider a tiny frog which is initially at the point 0 of the number line. It makes successive rightward jumps along the number line as follows.

It jumps 1 unit in step 1.

It jumps $\frac{1}{2}$ units in step 2.

It jumps $\frac{1}{4}$ units in step 3.

It jumps $\frac{1}{8}$ units in step 4.

It jumps $\frac{1}{16}$ units in step 5.

$$\vdots$$

Thus the total distance travelled by the frog is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots.$$

The total distance travelled by the frog is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots :$$

Partial sum		**Partial sum**	**Value**
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_2 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

The sequence $\{s_n\}$ of partial sums converges to 2.

In this case, we say that the series **converges** to 2.

And say that the **sum** of the series is 2.

Definition (Series, nth Term, Partial Sum, Converges, Sum)

Given a sequence $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

is called a **series**. The number a_n is called the nth term of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &\vdots \end{aligned}$$

is called the **sequence of partial sums** of the series and the number s_n is called the ***n*th partial sum**.

Definition *Contd.*

If the sequence $\{s_n\}$ of partial sums converges to a limit l, we say that the series **converges** and that its **sum** is l. In this case, we also write

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n = l.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Example: Geometric Series

A **geometric series** is a series of the form

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}.$$

Here a and r are fixed real numbers and $a \neq 0$. The **common ratio** r can be positive or negative or even zero.

We will now prove that the geometric series converges for each r with $|r| < 1$ and it diverges otherwise.

Geometric Series

Case 1: $r = 1$**:** The nth partial sum of the geometric series is

$$s_n = a + a(1) + a(1^2) + \ldots + a(1^{n-1}) = na.$$

So, the sequence of partial sums diverges to ∞ or $-\infty$ depending on whether $a > 0$ or $a < 0$. Hence, in this case, the series diverges.

Case 2: $r = -1$**:** The nth partial sum of the geometric series is

$$s_n = a + a(-1) + a(1) + a(-1) + \ldots + a((-1)^{n-1}).$$

So, the sequence of partial sums diverges as it oscillates between a and 0. Hence, in this case too, the series diverges.

Geometric Series

Case 3: $|r| \neq 1$**:**

$$\begin{aligned} s_n &= a + ar + ar^2 + \ldots + ar^{n-1} \\ rs_n &= ar + ar^2 + \ldots + ar^{n-1} + ar^n \\ s_n - rs_n &= a - ar^n \\ s_n(1-r) &= a(1-r^n) \\ s_n &= \frac{a(1-r^n)}{1-r} \quad (r \neq 1) \end{aligned}$$

Thus if $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ and so $s_n \to \dfrac{a}{1-r}$;
if $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

Geometric Series: Summary

If $|r| < 1$, the geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ converges to $a/(1-r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example

The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

Homework: Find the sum of the geometric series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2}{5(3^n)} = \frac{2}{5} - \frac{2}{5 \cdot 3} + \frac{2}{5 \cdot 9} - \ldots.$$

Homework

1. A ball is dropped from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is a positive constant that is less than 1. Find the total distance the ball travels up and down.
2. Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.
3. Find the sum of the series
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots.$$
4. Show that the series $\sum_{n=1}^{\infty} n^2$ diverges.
5. Show that the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges.

More Examples

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} + \dots.$$

Solution: Here the nth term $\frac{1}{n(n+1)}$ can be written as

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad (n = 1, 2, 3, \dots).$$

So, the nth partial sum can be written as a telescoping sum:

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Thus $s_n \to 1$. Hence the sum of the series is 1.

Diverging Series

Show that the series $\sum_{n=1}^{\infty} n^2$ diverges.

Solution: Here the nth partial sum is $s_n = \dfrac{n(n+1)(2n+1)}{6}$.
Obviously, $s_n \to \infty$ as $n \to \infty$. Hence the series diverges.

Show that the series $\sum_{n=1}^{\infty} \dfrac{n+1}{n}$ diverges.

Solution: Here each term of the series is greater than 1. So, the nth partial sum $s_n > n$. Thus $s_n \to \infty$ as $n \to \infty$. Hence the series diverges.

An Important Theorem

Theorem

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$.

Proof.

- Let $\sum_{n=1}^{\infty} a_n = l$.
- This means that the sequence $\{s_n\}$ of partial sums converges to l.
- But $a_n = s_n - s_{n-1}$ and $s_n \to l$ and $s_{n-1} \to l$.
- Thus $a_n = s_n - s_{n-1} \to l - l = 0$.

$\square$

The nth Term Test for Divergence

If $\lim\limits_{n\to\infty} a_n$ does not exist or is different from 0, then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Applying the nth Term Test for Divergence

1. $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.
2. $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1 \neq 0$.
3. $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.
4. $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\frac{-n}{2n+5} \to -\frac{1}{2} \neq 0$.

The Converse of the Theorem is not True

$a_n \to 0$ does not imply that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Example: For the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \dots + \frac{1}{2^n} + \frac{1}{2^n} + \dots + \frac{1}{2^n} + \dots,$$

the nth term $a_n \to 0$. But the series diverges as its partial sums s_n increase without bound. Indeed,

$$s_2 > 1, s_4 > 2, s_8 > 3, s_{16} > 4, \dots, s_{2^n} > n, \dots.$$

Note

We will often denote the series $\sum_{n=1}^{\infty} a_n$ simply as $\sum a_n$.

Combining Series

Theorem

Let $\sum_{n=1}^{\infty} a_n = a$ *and* $\sum_{n=1}^{\infty} b_n = b$ *be convergent series. Then*

1. $\sum (a_n + b_n) = a + b.$
2. $\sum (a_n - b_n) = a - b.$
3. $\sum (ka_n) = ka \qquad$ *for any number* k.

Proof (1):

- Let $A_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $B_n = b_1 + b_2 + b_3 + \ldots + b_n$.
- Then the partial sums of the series $\sum(a_n + b_n)$ are

$$\begin{aligned} s_n &= (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n) \\ &= (a_1 + a_2 + a_3 + \ldots + a_n) + (b_1 + b_2 + b_3 + \ldots + b_n) \\ &= A_n + B_n. \end{aligned}$$

- But $A_n \to a$ and $B_n \to b$.
- Hence $s_n = A_n + B_n \to a + b$ by the Addition Rule for sequences.

Corollary

1. *Every nonzero constant multiple of a divergent series diverges.*
2. *If* $\sum a_n$ *converges and* $\sum b_n$ *diverges, then* $\sum (a_n + b_n)$ *and* $\sum (a_n - b_n)$ *both diverge.*

Example

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}.$$

Solution:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \\
&= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} \\
&= 2 - \frac{6}{5} \\
&= \frac{4}{5}.
\end{aligned}$$

Note

Addion or deletion of a finite number of terms does not affect the convergence or divergence of a series. But in the case of convergent series, this may change the sum of the series.

An Important Series

Show that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$$

converges.

Solution: Here the nth partial sum is

$$\begin{aligned}
s_n &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \\
&< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \\
&< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots \\
&= 1 + 2 \\
&= 3
\end{aligned}$$

The sequence of partial sums is bounded above by 3. It is also monotonically increasing. Thus the series converges.

An Important Series

Definition

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots.$$

Note: $e = 2.7182\ldots$

Example: The Harmonic Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

is called the **harmonic series**.

The harmonic series is divergent:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}\right) + \ldots$$

$$> 1 + \frac{1}{2} + + \frac{1}{2} + + \frac{1}{2} + \ldots.$$

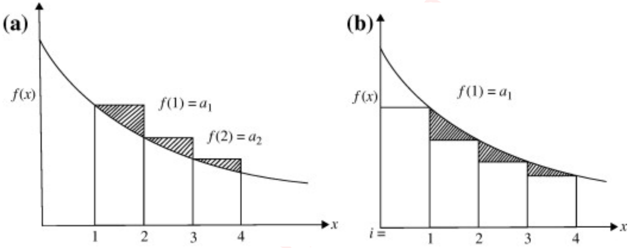
Tests of Convergence for Series: The Integral Test

Theorem (The Integral Test)

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $n \geq N$ (N a positive intger). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_{n=N}^{\infty} f(x)dx$ both converge or both diverge.

Proof

We will prove the theorem for $N = 1$. Let f be a function with $f(n) = a_n$ for all n.

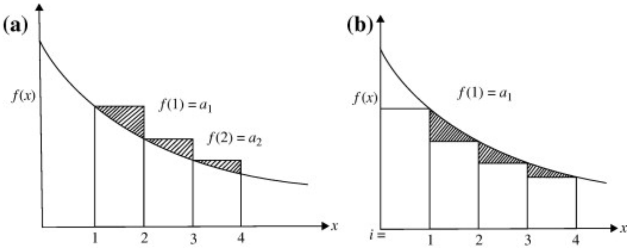


The the rectangles in Figure (a) which have areas $a_1, a_2, \ldots, a_n$ collectively enclose an area $a_1 + a_2 + \ldots + a_n$. This is more than the area under the curve $y = f(x)$ from 1 to $n + 1$. That is,

$$\int_1^{n+1} f(x)dx \leq a_1 + a_2 + \ldots + a_n.$$

Note: The image is from the internet.

Proof:



The rectangles in Figure (b) having areas $a_2, a_3, \ldots, a_n$ collectively enclose an area $a_2 + a_3 + \ldots + a_n$. This is less than the area under the curve $y = f(x)$ from 1 to n. That is,

$$a_2 + a_3 + \ldots + a_n \leq \int_1^n f(x)dx.$$

Adding a_1 to both sides, we get

$$a_1 + a_2 + a_3 + \ldots + a_n \leq a_1 + \int_1^n f(x)dx.$$

Combining the two inequalities gives

$$\int_1^{n+1} f(x)dx \leq a_1 + a_2 + \ldots + a_n \leq a_1 + \int_1^n f(x)dx.$$

The above inequalities hold for each n and continue to hold as $n \to \infty$.

If $\displaystyle\int_1^{\infty} f(x)dx$ is finite, the right-hand inequality implies that $\sum a_n$ is finite.

If $\displaystyle\int_1^{\infty} f(x)dx$ is infinite, the left-hand inequality implies that $\sum a_n$ is infinite.

Hence the series and the integral are both finite or both infinite.

Example: The p-Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

is called the p-series (p any real constant).

The p-series converges if $p > 1$ and diverges if $p \leq 1$.

Example: The p-Series

Show that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

converges if $p > 1$ and diverges if $p \leq 1$.

Solution: If $p > 1$, then $f(x) = \frac{1}{x^p}$ is a positive decreasing function of x (for $x \geq 1$). Now

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \int_1^{\infty} x^{-p} dx \\ &= \left[\frac{x^{-p+1}}{-p+1} \right]_1^{\infty} \\ &= \frac{1}{1-p} \left[\frac{1}{x^{p-1}} \right]_1^{\infty} \\ &= \frac{1}{1-p} [0-1] = \frac{1}{p-1}. \end{aligned}$$

So, the series converges by the integral test in this case.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} x^{-p} dx = \left[\frac{x^{-p+1}}{-p+1} \right]_1^{\infty} = \frac{1}{1-p} \left[x^{1-p} \right]_1^{\infty} = \infty.$$

If $p = 1$, we have the (divergent) harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots.$$

Thus the p-series converges for $p > 1$ and diverges for $p \leq 1$.

Homework

1. Show that the series
$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$
converges using the integral test.
2. Show that the series
$$\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$$
converges if and only if $p > 1$ using the integral test.

Tests of Convergence of Series: The Comparison Test

Theorem (The Comparison Test)

Let $\sum a_n$ be a series of non-negative terms. Then

(a) *$\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n \geq N$, for some integer N.*

(b) *$\sum a_n$ diverges if there is a divergent series of non-negative terms $\sum d_n$ with $a_n \geq d_n$ for all $n \geq N$, for some integer N.*

Proof: In Part (a), the sequence of partial sums of $\sum a_n$ is a monotonically increasing sequence and is bounded above by

$$M = a_1 + a_2 + \ldots + a_{N-1} + \sum_{n=N}^{\infty} c_n.$$

So it converges.

In Part (b), the sequence of partial sums of $\sum a_n$ is not bounded from above. If they were, the partial sums for $\sum d_n$ would be bounded above by

$$M' = d_1 + d_2 + \ldots + d_{N-1} + \sum_{n=N}^{\infty} a_n$$

and $\sum d_n$ would be converging!

Examples

1. The series

$$\sum_{n=1}^{\infty} \frac{7}{7n-2}$$

diverges because its nth term

$$\frac{7}{7n-2} = \frac{1}{n-\frac{2}{7}} > \frac{1}{n}$$

which is the n term of the divergent harmonic series.

2. The series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$$

converges because its terms are all positive and are less than or equal to the corresponding terms of

$$1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$$

which is convergent.

Tests of Convergence of Series: The Limit Comparison Test

Theorem (The Limit Comparison Test)

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. *If* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, *then* $\sum a_n$ *and* $\sum b_n$ *both converge or both diverge.*
2. *If* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
3. *If* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

Proof (1):

Since $\frac{c}{2} > 0$, there exists an integer N such that

$$n \geq N \quad \Rightarrow \quad \left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}.$$

Thus, for $n \geq N$,

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2},$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2},$$

$$\left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n,$$

If $\sum b_n$ converges, then $\sum (3c/2) b_n$ converges and $\sum a_n$ converges by the direct Comparison Test.
If $\sum b_n$ diverges, then $\sum (c/2) b_n$ diverges and $\sum a_n$ diverges by the direct Comparison Test.

Example

Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}.$$

Solution: Let $a_n = \frac{2n+1}{n^2+2n+1}$. For n large, we expect a_n to behave like $2n/n^2 = 2/n$. So, we let $b_n = 1/n$.
Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test.

Example

Does the following series converge or diverge?

$$\sum_{n=2}^{\infty} \frac{1 + n \ln n}{n^2 + 5}.$$

Solution: Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For large n, a_n will behave like $n \ln n/n^2 = \ln n/n$, which is greater than $1/n$ for $n \geq 3$. So, we take $b_n = 1/n$.
Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n + n^2 \ln n}{n^2 + 5},$$

$\sum a_n$ diverges by Part 3 of the Limit Comparison Test.

Homework

Does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

The Ratio Test

Theorem (The Ratio Test)

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

(a) *the series* converges *if $\rho < 1$,*

(b) *the series* diverges *if $\rho > 1$ or is infinite,*

(c) *the test is inconclusive if $\rho = 1$.*

Proof

(a) $\rho < 1$.

Let r be a number between ρ and 1: $\rho < r < 1$.
Then the number $\epsilon = r - \rho$ is positive.
Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho,$$

there is an integer N such that

$$n \geq N \quad \Rightarrow \quad \left| \frac{a_{n+1}}{a_n} - \rho \right| < \epsilon$$

This implies that, for $n \geq N$,

$$-\epsilon < \frac{a_{n+1}}{a_n} - \rho < \epsilon \quad \text{or} \quad \rho - \epsilon < \frac{a_{n+1}}{a_n} < \rho + \epsilon = r.$$

That is,

$$\begin{aligned}
a_{N+1} &< ra_N, \\
a_{N+2} &< ra_{N+1} < r^2 a_N, \\
a_{N+3} &< ra_{N+2} < r^3 a_N, \\
&\vdots \\
a_{N+m} &< ra_{N+m-1} < r^m a_N.
\end{aligned}$$

Consider the series $\sum c_n$, where $c_n = a_n$ for $n = 1, 2, \ldots, N$ and $c_{N+1} = ra_N$, $c_{N+2} = r^2 a_N, \ldots, c_{N+m} = r^m a_N, \ldots$.
Now $a_n \leq c_n$ for all n and

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n &= a_1 + a_2 + \ldots + a_{N-1} + a_N + ra_N + r^2 a_N + \ldots \\
&= a_1 + a_2 + \ldots + a_{N-1} + a_N(1 + r + r^2 + \ldots)
\end{aligned}$$

The geometric series $1 + r + r^2 + \ldots$ converges as $|r| < 1$. So $\sum c_n$ converges.
Since $a_n \leq c_n$, $\sum a_n$ also converges.

(b) $1 < \rho \le \infty$.

From some index M on,

$$\frac{a_{n+1}}{a_n} > 1 \quad \text{and} \quad a_M < a_{M+1} < a_{M+2} < \cdots.$$

So, the terms of the series do not approach zero as n becomes infinite. Hence the series diverges by the nth Term Test.

(c) $\rho = 1$.

Consider the following two series:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$: $\displaystyle\frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1.$

For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\displaystyle\frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1.$

In both cases, $\rho = 1$. But the first series diverges and the second converges.

Example

Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}.$$

Solution: Here

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} \to \frac{2}{3}.$$

The series converges because here $\rho = 2/3 < 1$.

Example

Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Solution: For the series $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n!}$,

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}/(n+1)!}{a^n/n!} = \frac{a}{n+1} \to 0.$$

The series converges because here $\rho = 0 < 1$.

Example

Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}.$$

Solution: Here

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{4^n n! n!} = \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{2(n+1)}{2n+1} \to 1.$$

Because the limit $\rho = 1$, we cannot decide whether the series converges from the ratio test.

But

$$\frac{a_{n+1}}{a_n} = \frac{2(n+1)}{2n+1} > 1 \quad \text{for all} \quad n.$$

Thus $a_{n+1} > a_n$ for all n.
So, $a_1 < a_2 < a_3 < \dots$.
Also $a_1 = 2$. Thus a_n does not converge to 0.

Hence the series diverges.

The Root Test

Theorem (The Root Test)

Let $\sum a_n$ *be a series with* $a_n \geq 0$ *for* $n \geq N$ *(*N *an integer) and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

The Root Test

Theorem (The Root Test)

Let $\sum a_n$ *be a series with* $a_n \geq 0$ *for* $n \geq N$ *(*N *an integer) and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) *the series converges if* $\rho < 1$,

The Root Test

Theorem (The Root Test)

Let $\sum a_n$ *be a series with* $a_n \geq 0$ *for* $n \geq N$ *(N an integer) and suppose that*

$$\lim_{n \to \infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) *the series converges if* $\rho < 1$,

(b) *the series diverges if* $\rho > 1$ *or* ρ *is infinite,*

The Root Test

Theorem (The Root Test)

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$ (N an integer) and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) *the series converges if $\rho < 1$,*

(b) *the series diverges if $\rho > 1$ or ρ is infinite,*

(c) *the test is inconclusive if $\rho = 1$.*

Examples

Which of the following series converges and which diverges?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n^2} \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \to \frac{1}{2} < 1.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ diverges because $\displaystyle\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{(\sqrt[n]{n})^2} \to \frac{2}{1} > 1.$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\displaystyle\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$

Example

Let $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$ Does $\sum a_n$ converge?

Solution: Here

$$\sqrt[n]{a_n} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even.} \end{cases}$$

Therefore,

$$\frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{\sqrt[n]{n}}{2}.$$

We know that $\sqrt[n]{n} \to 1$. So, $\lim_{n\to\infty} \sqrt[n]{a_n} = 1/2$ by the Sandwich Theorem.

Thus here the limit is $\rho < 1$. Hence the series converges by the Root Test.

Alternating Series

Definition

A series in which the terms are alternatively positive and negative is called an **alternating series**.

Examples

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \frac{(-1)^n 4}{2^n} + \ldots$$

$$1 - 2 + 3 - 4 + 5 - 6 + \ldots + (-1)^{n+1} n + \ldots$$

The first series, called the **alternating harmonic series**, converges.

The second series, a geometric series with common ratio $r = -1/2$, converges.

The third series diverges because the nth term does not approach zero.

Homework

Let $\{a_n\}$ be a sequence such that the subsequences $\{a_{2m}\}$ and $\{a_{2m+1}\}$ both converge to the same limit l. Then show that $a_n \to l$.

The Alternating Series Test

Theorem (The Alternating Series Test (Leibniz's Theorem))

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \ldots$$

converges if all three of the following conditions are satisfied:

1. *The u_n's are all positive.*
2. $u_n \geq u_{n+1}$ *for all* $n \geq N$, *for some integer* N.
3. $u_n \to 0$.

Proof:

Let us assume that $N = 1$.
If n is an even integer, say $n = 2m$, then the sum of the first n terms is

$$
\begin{aligned}
s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2m-1} - u_{2m}) \\
&= u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - (u_{2m-2} - u_{2m-1}) - u_{2m}
\end{aligned}
$$

The first equality shows that the s_{2m} is the sum of m non-negative terms. Hence $s_{2m+2} \geq s_{2m}$.

The second equality shows that $s_{2m} \leq u_1$.

So, $\{s_{2m}\}$ is monotonically increasing and bounded above. So, it converges, say

$$\lim_{n \to \infty} s_{2m} = l.$$

Also $u_{2m+1} \to 0$. Hence

$$s_{2m+1} = s_{2m} + u_{2m+1} \to l + 0 = l.$$

Thus we have that $s_{2m} \to l$ and $s_{2m+1} \to l$.

Hence $s_n \to l$. This means that the alternating series converges.

Example

The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

satisfies the three requirements of the alternating series theorem with $N = 1$. So, it converges.

Absolute and Conditional Convergence

Definition (Absolute Convergence)

A series $\sum a_n$ **converges absolutely** if the corresponding series of absolute values $\sum |a_n|$ converges.

Example:
The geometric series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$$

converges absolutely because the corresponding series of absolute values

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

converges.

In contrast, the alternating harmonic series does not converge absolutely: The corresponding series of absolute values is the divergent harmonic series.

Definition (Conditional Convergence)

A series that converges but does not converge absolutely is said to **converge conditionally**.

Example: The alternating harmonic series converges conditionally.

The Absolute Convergence Test

Theorem

If $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof:

For each n,

$$-|a_n| \leq a_n \leq |a_n| \qquad \text{so} \qquad 0 \leq a_n + |a_n| \leq 2|a_n|.$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges.

So, by the Comparison Test, $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges.

But $a_n = (a_n + |a_n|) - |a_n|$. So, $\sum_{n=1}^{\infty} a_n$ can be expressed as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

Examples

(a) For $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \ldots$, the corresponding series of absolute values is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \ldots .$$

The latter converges. Thus the original series converges absolutely. Hence it converges.

(b) For $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \ldots$, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| = \left|\frac{\sin 1}{1}\right| + \left|\frac{\sin 2}{4}\right| + \left|\frac{\sin 3}{9}\right| + \ldots .$$

Since $|\sin n| \leq 1$, the latter converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Thus the original series converges absolutely. Hence it converges.

Power Series

Definition (Power Series, Center, Coefficients)

A **power series about** $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots.$$

A **power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots.$$

Here a is the **center** and $c_0, c_1, c_2, \ldots, c_n, \ldots$ are the **coefficients** of the power series. These are constants.

Example: A geometric series

Taking all the coefficients to be 1 gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots.$$

This the geometric series with first term 1 and common ratio x.
It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots, \qquad -1 < x < 1.$$

Note

The power series

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots, \qquad -1 < x < 1,$$

gives the following polynomial approximations for the non-polynomial function $\dfrac{1}{1-x}$ for values of x near 0:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1 + x \\ P_2(x) &= 1 + x + x^2 \\ &\vdots \end{aligned}$$

Example: Another geometric series

The power series

$$1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\ldots+\left(-\frac{1}{2}\right)^n(x-2)^n+\ldots$$

has center $a=2$ and coefficients $c_0=1, c_1=-1/2, c_2=1/4, \ldots, c_n=(-1/2)^n, \ldots$.

This is a geometric series with first term 1 and common ratio $r=-\dfrac{x-2}{2}$.

The series converges for $|r|=\left|-\dfrac{x-2}{2}\right|<1$ or $0<x<4$.

The sum is

$$\frac{1}{1-r}=\frac{1}{1+\frac{x-2}{2}}=\frac{2}{x}.$$

So,

$$\frac{2}{x}=1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\ldots+\left(-\frac{1}{2}\right)^n(x-2)^n+\ldots, \qquad 0<x<4.$$

Note

The power series

$$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \ldots + \left(-\frac{1}{2}\right)^n (x-2)^n + \ldots, \qquad 0 < x < 4,$$

gives the following polynomial approximations for the non-polynomial function $\frac{2}{x}$ for values of x near 2:

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= 1 - \frac{1}{2}(x-2) \\
P_2(x) &= 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 \\
&\vdots
\end{aligned}$$

Testing Power Series for Convergence Using the Ratio Test

For what values of x does the following series converge?

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$

Solution: We apply the Ratio Test to the series $\sum_{n=1}^{\infty} |u_n|$, where u_n is the nth term of the given power series:

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{n+1} \frac{n}{x^n} \right| = \frac{n}{n+1} |x| \to |x|.$$

Thus the given power series converges absolutely for $|x| < 1$.
It diverges if $|x| > 1$ since the nth term does not converge to zero (?).
At $x = 1$, it becomes the alternating harmonic series. So, it converges (conditionally).
At $x = -1$, we get the negative of the harmonic series. So, it diverges.
Summary: the series converges for $-1 < x \leq 1$ and diverges for other values of x.

Testing Power Series for Convergence Using the Ratio Test

For what values of x does the following series converge?

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots .$$

Solution: We apply the Ratio Test to the series $\sum_{n=1}^{\infty}|u_n|$, where u_n is the nth term of the given power series:

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\,\frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$$

Thus the given power series converges absolutely for $x^2 < 1$.
It diverges for $x^2 > 1$ since the nth term does not converge to zero.
At $x = 1$, the series becomes the alternating series $1 - 1/3 + 1/5 - 1/7 + \dots$, which converges by the Alternating Series Test.
The value at $x = -1$ is the negative of the value at $x = 1$. So, it converges for $x = -1$ as well.
Summary: The series converges for $|x| \le 1$ and diverges otherwise.

Testing Power Series for Convergence Using the Ratio Test

For what values of x does the following series converge?

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

Solution: We apply the Ratio Test to the series $\sum_{n=1}^{\infty} |u_n|$, where u_n is the nth term of the given power series:

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 \quad \text{for every } x.$$

So, the series converges absolutely for all values of x.

Testing Power Series for Convergence Using the Ratio Test

For what values of x does the following series converge?

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \dots.$$

Solution: We apply the Ratio Test to the series $\sum_{n=0}^{\infty} |u_n|$, where u_n is the nth term of the given power series:

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty \quad \text{for every } x \neq 0.$$

So, the nth term of the series does not converge to zero for $x \neq 0$. Hence the series diverges for all values of x except $x = 0$.

Theorem (The Convergence Theorem for Power Series)

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ *converges for* $x = c \neq 0$*, then it converges absolutely for all* x *with* $|x| < |c|$*. If the series diverges for* $x = d$*, then it diverges for all* x *with* $|x| > |d|$*.*

Proof:

Suppose the series $\sum_{n=0}^{\infty} a_n c^n$ converges.

Then $a_n c^n \to 0$. Therefore, corresponding to $\epsilon = 1$, there is an integer N such that, for $n \geq N$,

$$|a_n c^n| < 1 \quad \text{or} \quad |a_n| < \frac{1}{|c|^n}.$$

Now take any x such that $|x| < |c|$ and consider

$$|a_0| + |a_1 x| + \ldots + |a_{N-1} x^{N-1}| + |a_N x^N| + |a_{N+1} x^{N+1}| + \ldots.$$

There are only a finite number of terms prior to $|a_N x^N|$ and so their sum is finite.

Starting from $|a_N x^N|$, the sum of the terms is less than

$$\left|\frac{x}{c}\right|^N + \left|\frac{x}{c}\right|^{N+1} + \left|\frac{x}{c}\right|^{N+2} + \ldots.$$

But the above series is a geometric series with common ratio less than 1 since $|x| < |c|$.

So, it converges. Thus it follows that the given power series converges absolutely for $|x| < |c|$.

To prove the second half of the theorem, we use the first half.

Suppose, for contradiction, that the power series diverges at $x = d$ and converges at a value x_0 with $|x_0| > |d|$.

Then by taking $c = x_0$, we can conclude by the first half of the theorem that the power series converges at $x = d$, which is a contradiction.

Thus, it follows that if the power series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

Note

We prove a similar theorem for power series of the form

$$\sum_{n=0}^{\infty}(x-a)^n$$

by considering the power series $\sum_{n=0}^{\infty} y^n$:

From the theorem, the latter series converges for $y = c \neq 0$ implies that it converges for $|y| < |c|$.

This means that the given power series converges for x with $|x - a| < |c|$.

And so on.

The Radius of Convergence

Theorem

The convergence of the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ *is described by one of the following three possibilities:*

1. *There is a positive number* R *such that the series diverges for* x *with* $|x-a| > R$ *but converges absolutely for* x *with* $|x-a| < R$ *(i.e., for* $a - R < x < a + R$*). The series may or may not converge at either of the end points* $x = a - R$ *and* $x = a + R$*.*
2. *The series converges absolutely for every* x *(*$R = \infty$*).*
3. *The series converges at* $x = a$ *and diverges for all* $x \neq a$ *(*$R = 0$*).*

Proof

- We will prove the theorem for power series of the form $\sum_{n=0}^{\infty} c_n x^n$.
- If the series converges for every x, then we are in Case 2.
- If the series converges only at $x = 0$, then we are in Case 3.
- Otherwise there is a nonzero number d such that $\sum c_n d^n$ diverges.
- Let S denote the set of all numbers x such that $\sum c_n x^n$ converges.
- Then S is nonempty as it contains 0 and a positive number p.
- By the preceding theorem, the power series diverges for all x with $|x| > |d|$. So, S is a bounded set.
- Then by the Completeness Property of the real numbers, S has a least upper bound R.
- If $|x| > R \geq p$, then $x \notin S$ and so $\sum c_n x^n$ diverges.
- If $|x| < R$, then there is a number $b \in S$ such that $b > |x|$. So, $\sum c_n b^n$ converges. Therefore $\sum c_n |x|^n$ converges by the Comparison Test.

The Radius of Convergence

The power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$.

converges if

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x|\cdot\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$$

That is, if

$$|x| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

Definition

The **radius of convergence** of the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

Example: Computing the Radius and Interval of Convergence

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$

Solution: Here $a_n = \dfrac{(-1)^{n-1}}{n}$.
So, the radius of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{1/n}{1/(n+1)} = \lim_{n\to\infty}\frac{n+1}{n} = 1.$$

Thus the given power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

At $x = 1$, it becomes the alternating harmonic series. So, it converges.
At $x = -1$, we get the negative of the harmonic series. So, it diverges.

Thus the interval of converge of the power series is $-1 < x \leq 1$.

Example: Computing the Radius and Interval of Convergence

Find the radius and interval of convergence of

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \ldots + \left(-\frac{1}{2}\right)^n (x-2)^n + \ldots.$$

Solution: Here $a_n = \left(-\frac{1}{2}\right)^n$.

So, the radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{(1/2)^n}{(1/2)^{n+1}} = 2.$$

Thus the given power series converges absolutely for $|x-2| < 2$ and diverges for $|x-2| > 2$.

As we know that it is a geometric ratio $r = -\frac{x-2}{2}$, it diverges for x with $|x-2| = 2$.

Thus the interval of convergence is $|x-2| < 2$ or $0 < x < 4$.

Theorem (The Term-by-Term Differentiation Theorem)

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges for* $a - R < x < a + R$ *for some* $R > 0$, *it defines a function* f:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad a - R < x < a + R.$$

This function f *has derivatives all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:*

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}, \end{aligned}$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

Example

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots, \qquad -1 < x < 1.$$

Solution:

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} + \ldots, \qquad -1 < x < 1.$$

$$f'(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \ldots + n(n-1)x^{n-2} + \ldots, \qquad -1 < x < 1.$$

Theorem (The Term-by-Term Integration Theorem)

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R$ $(R > 0)$*. Then*

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ *and*

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$*.*

Example: A series for $\tan^{-1} x$

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, \qquad -1 \leq x \leq 1.$$

Solution: Differentiating the given series, we get

$$f'(x) = 1 - x^2 + x^3 - x^4 + \dots, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and common ratio $-x^2$. So,

$$f'(x) = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2}.$$

Now integration gives

$$f(x) = \int f'(x) dx = \int \frac{dx}{1 + x^2} = \tan^{-1} x + C$$

The series for $f(x)$ is zero when $x = 0$; so $C = 0$.
Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \tan^{-1} x, \qquad -1 < x < 1.$$

Example: A series for $\ln(1+x)$

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots$$

converges for $-1 < t < 1$. Therefore

$$\begin{aligned}
\ln(1+x) &= \int_0^x \frac{1}{1+t}dt \\
&= \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots\right]_0^x \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \qquad -1 < x < 1.
\end{aligned}$$

Theorem (The Multiplication Theorem for Power Series)

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$ *(*$R > 0$*) and*

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely to* $A(x)B(x)$ *for* $|x| < R$*:*

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Example

Multiply the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x} \quad (|x| < 1)$$

by itself to get a power series for $\dfrac{1}{(1-x)^2}$ for $|x| < 1$.

Solution: Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

and

$$\begin{aligned} c_n &= c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 + a_n b_0 \\ &= 1 + 1 + \ldots + 1 = n + 1 \end{aligned}$$

Then, by the Multiplication Theorem, the power series for $\dfrac{1}{(1-x)^2}$ is

$$\begin{aligned} A(x)\cdot B(x) &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty}(n+1)x^n \\ &= 1 + 2x + 3x^2 + 4x^3 + \dots, \quad |x| < 1. \end{aligned}$$

Taylor and Maclaurin Series

Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n + \ldots$$

converges for $a - R < x < a + R$ $(R > 0)$.

Then the Term-by-Term Differentiation Theorem tells us that the sum function has derivatives of all orders within the interval of convergence $a - R < x < a + R$ and

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \ldots \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \ldots \\ f'''(x) &= 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \ldots \end{aligned}$$

and so on. The above equations all hold, in particular, at $x = a$.

So,

$$\begin{aligned} f(a) &= a_0 \\ f'(a) &= a_1 \\ f''(a) &= 1 \cdot 2a_2 \\ f'''(a) &= 1 \cdot 2 \cdot 3a_3 \\ &\vdots \\ f^{(n)}(a) &= n!a_n. \end{aligned}$$

Thus

$$a_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n = 0, 1, 2, \ldots .$$

Hence

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots .$$

This implies that the sum function $f(x)$ has a unique power series expansion. (Why?)

Definition (Taylor Series, Maclaurin Series)

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by** f **at** $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots.$$

The **Maclaurin series generated by** f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots,$$

which is the Taylor series generated by f at $x = 0$.

Note: The Maclaurin series generated by f is often called the Taylor series of f.

Example

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $f(x)$?

Solution: We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives, we get

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \quad f'''(x) = -3!x^{-4}, \ldots, f^{(n)}(x) = (-1)^n n! x^{-(n+1)}, \ldots.$$

So,

$$f(2) = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad f''(2) = 2!\frac{1}{2^3}, \quad f'''(2) = -3!\frac{1}{2^4}, \ldots f^{(n)}(2) = (-1)^n n! \frac{1}{2^{n+1}}, \ldots.$$

Hence the Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \ldots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \ldots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \ldots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \ldots.$$

This is a geometric series with first term $1/2$ and common ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$. The sum is

$$\frac{1/2}{1+(x-2)/2} = \frac{1}{x}.$$

Thus the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$.

Example

The function

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

has derivatives of all orders at $x = 0$. Indeed, $f^{(n)}(0) = 0$ for all n.

Thus the Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

$$= 0 + 0 \cdot x + 0 \cdot x^2 + \ldots 0 \cdot x^n + \ldots$$

$$= 0 + 0 + \ldots + 0 + \ldots.$$

The series converges for every x (the sum is 0) but converges to $f(x)$ only at $x = 0$.

Homework

Find the Taylor series generated by the following functions at $x = 0$:

1. $f(x) = e^x$
2. $f(x) = \sin x$
3. $f(x) = \cos x$

Also prove that, in each case, the Taylor series converges to the corresponding function for all x.

Limit: Motivation

A rock breaks loose from the top of a cliff. What is its average speed

1. during the first two seconds of fall?
2. during the 1-second interval between second 1 and second 2?

Solution: Galileo's law: The distance fallen is proportional to the square of the time it has been falling. Indeed if y denotes the distance fallen in feet in t seconds, then

$$y = 16t^2.$$

1. The average speed during the first two seconds is

$$\frac{\Delta y}{\Delta t} = \frac{16(2^2) - 16(0^2)}{2 - 0} = 32 \text{ ft/sec.}$$

2. The average speed during the 1-second interval between second 1 and second 2 is

$$\frac{\Delta y}{\Delta t} = \frac{16(2^2) - 16(1^2)}{2 - 1} = 48 \text{ ft/sec.}$$

Limit: Motivation

Find the instantaneous speed of the rock at $t = 1$ and $t = 2$ seconds.

Solution: The average speed of the rock over a time interval $[t_0, t_0 + h]$ having length h is

$$\frac{\Delta y}{\Delta t} = \frac{16((t_0 + h)^2) - 16(t_0^2)}{h} \text{ ft/sec.}$$

To calculate the speed at t_0, we cannot simply substitute $h = 0$ in the above formula as we cannot divide by zero.
But we can use this formula to compute the the average speed over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. For $h \neq 0$, the above formula simplifies as follows:

$$\frac{\Delta y}{\Delta t} = \frac{16((t_0 + h)^2) - 16(t_0^2)}{h} = \frac{16(t_0^2 + 2t_0 h + h^2) - 16t_0^2}{h} = \frac{32t_0 h + 16h^2}{h} = 32t_0 + 16h.$$

Thus the instantaneous speed of the rock at $t_0 = 1$ second is 32 ft/sec
and the instantaneous speed of the rock at $t_0 = 2$ second is 32 ft/sec.

Limit: Motivation

Definition

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ of length $h \neq 0$ is

$$\frac{\Delta y}{\Delta t} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

Note: Geometrically, the rate of change of f over $[x_1, x_2]$ is the *slope* of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$.
What does happen when $x_2 = x_1$?
We rather see what happens when x_2 approaches x_1.
When x_2 approaches x_1 $\frac{\Delta y}{\Delta t} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ could be approaching a finite value.

Limit: Motivation

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution: The give formula defines f for all real numbers x except $x = 1$. For $x \neq 1$, the formula simplifies as follows:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1, \qquad \text{for } x \neq 1.$$

For values of x close to 1, $f(x)$ is close to 2.
In this case, we write

$$\lim_{x \to 1} 2.$$

Thus the graph of f is the line $y = x + 1$ with the point $(1, 2)$ removed.

Nonexistence of Limit

Discuss the behavior of the following functions as $x \to 0$:

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$