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DIFFERENTIAL EQUATIONS

→ **Definition**: An equation involving one dependent variable and its derivatives with respect to one or more independent variable.

→ **Ordinary DE**: A DE involving one independent variable.

Partial DE: A DE involving more than one independent variable

→ Ex $\frac{dy}{dx} = -ky$

$$\left.\begin{aligned} &\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \end{aligned}\right] \text{Ordinary DE}$$

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = 0 \quad - \text{ Partial DE}$$

→ General Ordinary DE of the nth order:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0$$

$$F(x, y, y', y'', \dots y^{(n)}) = 0$$

Q) $y'' - 5y' + 6y = 0$

Check whether e^{2x} is a solution.

Sol:

$$\left.\begin{aligned} y &= e^{2x} \\ y' &= 2e^{2x} \\ y'' &= 4e^{2x} \end{aligned}\right\} \begin{aligned} &\Rightarrow 4e^{2x} - 5(2e^{2x}) + 6e^{2x} = 0 \\ &\Rightarrow 4e^{2x} - 10e^{2x} + 6e^{2x} = 0 \\ &\Rightarrow 0 = 0 \end{aligned}$$

$\therefore e^{2x}$ is a solution

$$\left.\begin{aligned} y &= e^{3x} \\ y' &= 3e^{3x} \\ y'' &= 9e^{3x} \end{aligned}\right\} \begin{aligned} &\Rightarrow 9e^{3x} - 5(3e^{3x}) + 6(e^{3x}) = 0 \\ &\Rightarrow 9e^{3x} - 15e^{3x} + 6e^{3x} = 0 \\ &\Rightarrow 0 = 0 \end{aligned}$$

$\therefore e^{3x}$ is also a solution.

Generally,

$y = c_1e^{2x} + c_2e^{3x}$ is a general equation

For an nth order DE,

There are ∞ solutions.

There are n linearly Independent solutions

There are n arbitrary constants

Q) Prove that

$$xy = \log y + c$$

is a solution of

$$\frac{dy}{dx} = \frac{y^2}{1-xy}$$

$\forall c \in$ Constant

Sol :

$$\frac{dy}{dx} = \frac{y^2}{1-xy}$$

$$xy = \log y + c$$

$$x \cdot \frac{dy}{dx} + y = \frac{1}{y}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{1}{y} - y$$

$$= \frac{1-y^2}{y}$$

$$\therefore \frac{dy}{dx} = \frac{1-y^2}{xy}$$

→ Order : Highest Order Derivative
Always Exists

Degree : Index of the highest order derivative
May or may not exist.

Ex. $\left(\frac{d^2y}{dx^2}\right)^{3/2} + \left(\frac{d^3y}{dx^3}\right)^{2/3} = 0$

Order = 3

$\because \frac{d^3y}{dx^3}$

$\left(\left(\frac{d^2y}{dx^2}\right)^{3/2} + \left(\frac{d^3y}{dx^3}\right)^{2/3}\right)^6 = 0$ (Taking 2nd term to RHS and powering 6 on both sides)

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^9 + \left(\frac{d^3y}{dx^3}\right)^4 = 0$

Degree = 4

Ex. $\sin\left(\frac{d^2y}{dx^2}\right) + y = 0$

Order = 2
Degree D.N.E

If Derivative is inside Transcendental functions, Degree D.N.E

Ex. $\frac{d^2y}{dx^2} = -\sin^{-1}(y)$

Order = 2
Degree = 1

Ex $y'' + e^{y'} + 1 = 0$ (Non-Linear)

Order = 2
Degree = D.N.E

Ex. $y'' + y' + \int y\,dx = 0$ (Linear)

Order = 3 (Differentiating both sides)
Degree = 1

→ Differential Equations

Linear DE

DE whose degree = 1

Non-Linear DE

↳ Ex. $y' \times y'' = x^2$

$(y')^2 + y' = y$

$y^2 + y' = x$

→ Finding Solutions of L.D.E :

① Equation of the form

$$\frac{dy}{dx} = f(x)$$

Sol : Just Integrate

$$y = \int f(x)\, dx + c$$

Ex. $y' = e^{3x} - x$

$$y = \int (e^{3x} - x)\, dx$$

$$= \frac{e^{3x}}{3} - \frac{x^2}{2} + c$$

$$\boxed{\therefore y = \frac{e^{3x}}{3} - \frac{x^2}{2} + c}$$

$[c \in \mathbb{C}]$

Q) (a) $xy' = 1$

(b) $y' = xe^{x^2}$

(c) $y' = \sin^{-1}(x)$

(d) $(1+x^2)y' = x$

(e) $(1+x^2)y' = \tan^{-1} x$

(a) $y' = \frac{1}{x}$

$\boxed{\therefore y = \ln|x| + c}$ $[c \in C]$

(b) $y' = xe^{x^2}$

$y = \int xe^{x^2} dx$ $\quad x^2 = t$

$2x\,dx = dt$

$= \frac{1}{2}\int e^t dt$

$= \frac{e^t}{2} + c$

$\boxed{\therefore y = \frac{e^{x^2}}{2} + c}$ $(c \in C)$

(c) $\boxed{y = x \cdot \sin^{-1}x + \sqrt{1-x^2} + c}$ $[c \in C]$

(d) $y' = \frac{x}{1+x^2}$

$y = \int \frac{x\,dx}{1+x^2}$

$1+x^2 = t$

$2x\,dx = dt$

$\Rightarrow y = \frac{1}{2}\int \frac{dt}{t}$

$= \frac{1}{2}\ln|t| + c$

$\boxed{\therefore y = \frac{1}{2}\ln|1+x^2| + c}$ $[c \in C]$

(e) $y' = \frac{\tan^{-1}x}{1+x^2}$

$t = \tan^{-1}x$

$dt = \frac{1}{1+x^2}dx \Rightarrow y = \int t\,dt = \frac{t^2}{2} + c$

$\boxed{\therefore y = (\tan^{-1}x)^2/2 + c}$ $[c \in C]$

② <u>Seperable Equations</u> :

Equations of the form

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

$$\Rightarrow \frac{dy}{g(y)} = f(x) \cdot dx$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx + c \quad [c \in C]$$

Ex. $xyy' = y - 1$

$\Rightarrow xy \cdot \frac{dy}{dx} = y - 1$

$\Rightarrow \frac{dy}{dx} = \frac{y-1}{y} \cdot \frac{1}{x}$

$\Rightarrow \frac{y}{y-1}\,dy = \frac{1}{x}\,dx$

$$\Rightarrow \int \frac{y}{y-1}\,dy = \int \frac{1}{x}\,dx$$

$$\Rightarrow \int \left(\frac{1}{y-1} + 1\right) dy = \ln|x| + c' \qquad [c' \in C]$$

$$\Rightarrow y + \ln|y-1| = \ln|x| + c$$

$$\boxed{\therefore\ y + \ln|y-1| + \ln|x| + c} \qquad [c \in C]$$

9(b) $xy' = (1-x^2)\tan y$

$$\Rightarrow x\frac{dy}{dx} = (1-x^2)\tan y$$

$$\Rightarrow \frac{dy}{\tan y} = \frac{1-x^2}{x}\,dx$$

$$\Rightarrow \int \frac{dy}{\tan y} = \int \frac{1-x^2}{x}\,dx$$

$$\Rightarrow \int \frac{\cos y}{\sin y}\,dy = \ln|x| - \frac{x^2}{2} + c' \qquad [c' \in C]$$

$t = \sin y$

$dt = \cos y\, dy$

$$\Rightarrow \boxed{\ln|\sin y| = \ln|x| - \frac{x^2}{2} + c} \qquad [c \in C]$$

(c) $y' = 2xy$

$$\Rightarrow \frac{dy}{y} = 2x\,dx$$

$$\Rightarrow \boxed{\ln|y| = x^2 + c} \qquad [c \in C]$$

(a) $x^5 y' + y^5 = 0$

$$\Rightarrow x^5 \frac{dy}{dx} = -y^5$$

$$\Rightarrow -\frac{dy}{y^5} = \frac{dx}{x^5}$$

$$\Rightarrow \frac{4}{y^4} = -\frac{4}{x^4} + c' \qquad [c' \in C]$$

$$\Rightarrow \frac{1}{y^4} + \frac{1}{x^4} = c$$

$$\Rightarrow \frac{x^4+y^4}{x^4y^4} = c \quad [c \in \mathbb{C}]$$

(d) $(1+x^2)\,dy + (1+y^2)\,dx = 0$

$$\Rightarrow (1+x^2)\,dy = -(1+y^2)\,dx$$

$$\Rightarrow \frac{dx}{1+x^2} = -\frac{dy}{1+y^2}$$

$$\Rightarrow \tan^{-1}(x) = -\tan^{-1}(y) + c \quad [c \in \mathbb{C}]$$

$$\Rightarrow \boxed{\tan^{-1}x + \tan^{-1}y = c}$$

(e) $y \log y \, dx - x\,dy = 0$

$$\Rightarrow y \log y \, dx = x\,dy$$

$$\Rightarrow \frac{dy}{y \log y} = \frac{dx}{x}$$

$$\ln y = \cancel{\frac{\log y}{\log 10}} \; \frac{\log y}{\log e}$$

$$[\log y = \ln y \cdot \log e]$$

$$\Rightarrow \frac{dy}{y(\ln y \cdot \log e)} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dt}{t} \cdot \frac{1}{\log e} = \ln|x| + c \qquad \begin{bmatrix} \ln y = t \\ \frac{1}{y}\,dy = dt \end{bmatrix} \quad (c \in \mathbb{C})$$

$$\Rightarrow \ln(10) \cdot \ln|t| = \ln|x| + c$$

$$\Rightarrow \ln(10) \cdot \ln|\ln y| = \ln|x| + c$$

→ <u>Picard's Theorem</u> :

If $f(x,y)$ and $\partial f / \partial y$ are continuous functions on a closed rectangle R, then through each point (x_0, y_0) in the interior of R there passes a unique integral curve of the equation $dy/dx = f(x,y)$

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If we consider a fixed x_0 in this theorem, then the integral curve that passes through

Q) $y' = 2\sin x\cos x$, $y=1$ when $x=0$

$$\frac{dy}{dx} = 2\sin x\cos x = \sin 2x$$

$$\Rightarrow y = \frac{-\cos 2x}{2} + c$$

$$1 = \frac{-\cos 0}{2} + c$$

$$\Rightarrow 1 = c - \tfrac{1}{2}$$

$$\boxed{c = 3/2}$$

$$\Rightarrow y = \frac{3 - \cos 2x}{2}$$

Q) $y' = \log x$, $y=0$ when $x=e$

$$\frac{dy}{dx} = \log x$$

$$\Rightarrow y = x\log x - 1 + c$$

$$0 = e - 1 + c$$

$$\Rightarrow \boxed{c = 1 - e}$$

$$\Rightarrow y = x\log x - e$$

Q) $x(x^2-4)y' = 1$, $y=0$ when $x=1$

$$\frac{dy}{dx} = \frac{1}{x(x^2-4)}$$

$$y = \int \left(\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} \right) dx$$

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→ **Family of Curves** :

[Orthogonal Trajectories]

General Solution of a 1st order DE contains one arbitrary constant, i.e. Parametre

Ex. $y = cx$

$$\frac{dy}{dx} = c$$

$$\Rightarrow x\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

Ex. $x^2+y^2 = 2cx$

$$2x + 2y\frac{dy}{dx} = 2c$$

$$\Rightarrow x + y\frac{dy}{dx} = c$$

$$x^2+y^2 = 2\left(x+y\frac{dy}{dx}\right)x$$

$$\Rightarrow x^2+y^2 = 2x^2 + 2xy\frac{dy}{dx}$$

$$\Rightarrow 2xy\frac{dy}{dx} = y^2 - x^2$$

$$\therefore \frac{dy}{dx} = \frac{y^2-x^2}{2xy}$$

$$\frac{-dx}{dy} = \frac{y^2-x^2}{2xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

Ex $x^2+y^2 = c^2$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$m_1 \cdot m_2 = -1$$

[for Orthogonal]

$$m_1 = \frac{dy}{dx} = \frac{-x}{y}$$

$$\boxed{\therefore m_2 = -\frac{dx}{dy}}$$

$$\cancel{\therefore m_2 = \frac{y}{x}}$$

$$\frac{-dx}{dy} = \frac{-x}{y}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\therefore \ln y = \ln x + c$$

$$\Rightarrow \boxed{y = mx}$$

$[c = \ln(m)]$

→ <u>Homogenous Function</u>:

$f(x,y)$

$$\left.\begin{array}{l} x \longrightarrow tx \\ y \longrightarrow ty \end{array}\right\}$$

$$f(tx, ty) = t^n f(x,y)$$

Then, $f(x,y)$ is called a Homogenous function of ~~[illegible]~~ degree 'n'.

Ex. $f(x,y) = x^2+y^2$

$$f(tx, ty) = t^2(x^2+y^2)$$
$$= t^2 f(x,y)$$

$\therefore$ Degree $= 2$

& Homogenous function

→ <u>Homogenous DE</u>:

$$\frac{dy}{dx} = f(x,y) \neq g(x)\cdot h(y)$$

$$M(x,y)\,dy + N(x,y)\,dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-N(x,y)}{M(x,y)}$$

<u>Note</u>: $N(x,y)$ & $M(x,y)$ are Homogenous ~~[illegible]~~ functions of same degree 'n'.

i.e. $$\frac{dy}{dx} = \frac{-t^n \cdot N(x,y)}{t^n \cdot M(x,y)} = -\frac{N(x,y)}{M(x,y)}$$

$$-\frac{N(tx, ty)}{M(tx, ty)}$$

→ Homogenous function of Degree 0.

Useful for Non-seperable form.

$z = \frac{y}{x}$

$\hookrightarrow y = xz$

$\therefore \frac{dy}{dx} = z + x\frac{dz}{dx}$

$\therefore \frac{dy}{dx} = f(x,y) = g(1, \frac{y}{z})$

$= g(1, z)$

$\therefore g(1, z) = z + x\frac{dz}{dx}$

$\cancel{\Rightarrow h(1,z)}$

$\Rightarrow h(z) = x\frac{dz}{dx}$

$\Rightarrow \frac{dz}{h(z)} = \frac{dx}{x}$ $\quad [h(z) = g(z) - z]$

$\Rightarrow \int \frac{dz}{h(z)} = \ln|(x)| + c$

$\hookrightarrow$ Seperable form

<u>Note</u>: Ex. $\sin(x/y)$ is a homogenous function of degree 0

Ex. $(x+y)\,dx - (x-y)\,dy = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$

$x+y \longrightarrow t(x+y)$ [order = 1]

$x-y \longrightarrow t(x-y)$ [order = 1]

$$\left[\begin{array}{l} z = \frac{y}{x} \\ \Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx} \end{array}\right]$$

$\Rightarrow \frac{dy}{dx} = \frac{1 + y/x}{1 - y/x} = \frac{1+z}{1-z}$

$$\Rightarrow \quad z + x\frac{dz}{dx} = \frac{1+z}{1-z}$$

$$\Rightarrow \quad x\frac{dz}{dx} = \frac{1+z}{1-z} - z$$

$$= \frac{1+z-z+z^2}{1-z}$$

$$= \frac{1+z^2}{1-z}$$

$$\therefore \quad x\frac{dz}{dx} = \frac{1+z^2}{1-z}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dz}{1+z^2}\cdot(1-z)$$

$$\Rightarrow \quad \int \frac{dx}{x} = \int \frac{1-z}{1+z^2}\,dz$$

$$= \int \frac{dz}{1+z^2} - \int \frac{z}{1+z^2}\,dz$$

$$= \tan^{-1}(z) - \frac{1}{2}\int \frac{2z\,dz}{1+z^2} \qquad 1+z^2 = t$$

$$2z\,dz = dt$$

$$\Rightarrow \quad \ln|x| = \tan^{-1}(z) - \frac{1}{2}\ln|1+z^2| + c$$

$$\therefore \quad \ln|x| = \tan^{-1}(z) - \frac{1}{2}\ln|1+z^2| + c$$

$$\Rightarrow \quad \boxed{\therefore \ln|x| = \tan^{-1}\left(y/x\right) - \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right| + c}$$

$$\Rightarrow \quad z + x\frac{dz}{dx} = \frac{1+z}{1-z}$$

$$\Rightarrow \quad x\frac{dz}{dx} = \frac{1+z}{1-z} - z$$

$$= \frac{1+z-z+z^2}{1-z}$$

$$= \frac{1+z^2}{1-z}$$

$$\therefore \quad x\frac{dz}{dx} = \frac{1+z^2}{1-z}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dz}{1+z^2}\cdot(1-z)$$

$$\Rightarrow \quad \int \frac{dx}{x} = \int \frac{1-z}{1+z^2}\,dz$$

$$= \int \frac{dz}{1+z^2} - \int \frac{z}{1+z^2}\,dz$$

$$= \tan^{-1}(z) - \frac{1}{2}\int \frac{2z\,dz}{1+z^2} \qquad 1+z^2 = t$$

$$2z\,dz = dt$$

$$\Rightarrow \quad \ln|x| = \tan^{-1}(z) - \frac{1}{2}\ln|1+z^2| + c$$

$$\therefore \quad \ln|x| = \tan^{-1}(z) - \frac{1}{2}\ln|1+z^2| + c$$

$$\Rightarrow \quad \boxed{\therefore \ln|x| = \tan^{-1}(y/x) - \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right| + c}$$

Homework

(a) $(x^2 - 2y^2)\,dx + xy\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2 - x^2}{xy}$$

$$f(x,y) = \frac{2y^2 - x^2}{xy}$$

$$f(tx, ty) = \frac{2(ty)^2 - (tx)^2}{t(x)\,t(y)}$$

$$= \frac{t^2(2y^2 - x^2)}{t^2(xy)}$$

$$= \frac{2y^2 - x^2}{xy}$$

$$= t^0\left(\frac{2y^2 - x^2}{xy}\right)$$

$$\therefore f(tx, ty) = t^0\,(f(x,y))$$

Hence $\frac{dy}{dx} = \frac{2y^2 - x^2}{xy}$ is a Homogenous DE

$$\frac{dy}{dx} = \frac{2y^2 - x^2/x^2}{xy/x^2} = \frac{2\left(\frac{y}{x}\right)^2 - 1}{\left(y/x\right)}$$

$$\left[\begin{array}{l} \frac{y}{x} = z \Rightarrow y = xz \\ \qquad \frac{dy}{dx} = x\frac{dz}{dx} + z \end{array}\right]$$

$$\Rightarrow x\frac{dz}{dx} + z = \frac{2z^2 - 1}{z}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{z^2 - 1}{z}$$

$$\Rightarrow \frac{dx}{x} = \frac{2}{z^2-1}\, dz$$

$$\Rightarrow \int \frac{dx}{x} = \frac{1}{2}\int \frac{2z}{z^2-1}\, dz$$

$$\Rightarrow \ln|x| + c = \frac{1}{2}\ln|z^2-1| \qquad [c \in \mathbb{R}]$$

$$\Rightarrow \ln|x| - \frac{1}{2}\ln\left|\frac{y^2}{x^2}-1\right| + c = 0$$

$$\Rightarrow \ln x + \ln c' = \frac{1}{2}\ln\left|\frac{y^2}{x^2}-1\right|$$

$$\Rightarrow xc = \sqrt{\frac{y^2}{x^2}-1}$$

(OR)

$$\boxed{x^2 c = \frac{y^2}{x^2} - 1}$$

(b) $x^2y' - 3xy - 2y^2 = 0$

$$\Rightarrow x^2 \frac{dy}{dx} = 3xy + 2y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3xy+2y^2}{x^2}$$

$$f(x,y) = \frac{3xy+2y^2}{x^2}$$

$$f(tx,ty) = \frac{3(tx)(ty) + 2(ty)^2}{(tx)^2}$$

$$= \frac{t^2(3xy+2y^2)}{t^2(x^2)}$$

$$f(tx,ty) = t^0(f(x,y))$$

$\therefore$ Homogenous DE

$$\frac{dy}{dx} = \frac{3xy + 2y^2}{x^2}$$

$$\left[\begin{array}{l} y = zx \\ \frac{dy}{dx} = x\frac{dz}{dx} + z \end{array}\right]$$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{3x(zx) + 2(zx)^2}{x^2}$$

$$= 3z + 2z^2$$

$$\Rightarrow x\frac{dz}{dx} = 2(z + z^2)$$

$$\frac{dz}{dx} = \frac{2z(1+z)}{x}$$

$$\Rightarrow \frac{dx}{x} = \frac{dz}{2z(1+z)}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dz}{2z(1+z)}$$

$$\Rightarrow \ln|x| + c = \int \frac{1+z-z}{2z(1+z)}\, dz$$

$$= \frac{1}{2}\int\left(\frac{1}{z} - \frac{1}{z+1}\right) dz$$

$$= \frac{1}{2}\left(\ln|z| - \ln|z+1|\right)$$

$$\Rightarrow 2\ln|x| + c = \ln|z| - \ln|z+1|$$

$$\Rightarrow 2\ln|x| + \ln|c| + \ln\left|\frac{y}{x}+1\right| = \ln\left|\frac{y}{x}\right|$$

$$\Rightarrow cx^2\left(\frac{y}{x}+1\right) = \frac{y}{x}$$

$$\boxed{cx^2y + cx^3 = y}$$

$$\boxed{x^2y + x^3 = y/c}$$

Q) $x^2y' = 3(x^2+y^2)\tan^{-1}\left(\frac{y}{x}\right) + xy$

$$x^2\frac{dy}{dx} = 3(x^2+y^2)\tan^{-1}\left(\frac{y}{x}\right) + xy$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{\frac{3(x^2+y^2)\tan^{-1}\left(\frac{x}{y}\right) + xy}{x^2}}_{f(x,y)}$$

$$f(tx, ty) = \frac{3(t^2x^2+t^2y^2)\tan^{-1}\left(\frac{ty}{tx}\right) + t^2(xy)}{t^2x^2}$$

$$= t^0 f(x,y)$$

$\therefore$ Homogenous

$$\begin{bmatrix} y = zx \\ \frac{dy}{dx} = x\frac{dz}{dx} + z \end{bmatrix}$$

$$\Rightarrow x\frac{dz}{dx} + z = \frac{3(x^2+z^2x^2)\tan^{-1}(z) + zx^2}{x^2}$$

$$= 3(1+z^2)\tan^{-1}(z) + z$$

$$\Rightarrow x\frac{dz}{dx} = 3(1+z^2)\tan^{-1}(z)$$

$$\Rightarrow \frac{dx}{x} = \frac{dz}{3(1+z^2)\tan^{-1}(z)}$$

$$\Rightarrow \ln|x| + c = \frac{1}{3}\ln|\tan^{-1}(z)|$$

$$\Rightarrow 3\ln|x| + c = \ln|\tan^{-1}(z)|$$

$$\therefore \boxed{cx^3 = \tan^{-1}\left(\frac{y}{x}\right)}$$

9) $x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} = y\sin\left(\frac{y}{x}\right) + x$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) + x}{x\sin\left(\frac{y}{x}\right)} \rightarrow f(x,y)$$

$$f(tx,ty) = \frac{ty\sin\left(\frac{ty}{tx}\right) + tx}{tx\sin\left(\frac{ty}{tx}\right)}$$

$$= f(x,y)\cdot t^0$$

$$\therefore f(tx,ty) = t^0 f(x,y)$$

$\therefore$ Homogenous DE

$$\frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) + x}{x\sin\left(\frac{y}{x}\right)}$$

$$\left[\begin{array}{c} y = zx \\ \frac{dy}{dx} = x\frac{dz}{dx} + z \end{array}\right]$$

$$\frac{dy}{dx} = \frac{z x \sin(z) + x}{x \sin(z)}$$

$$\Rightarrow x \frac{dz}{dx} = \frac{x}{x \sin(z)}$$

$$\Rightarrow \frac{dx}{x} = \sin(z)\, dz$$

$$\Rightarrow \int \frac{dx}{x} = \int \sin(z)\, dz$$

$$\Rightarrow \ln|x| + c = -\cos(z)$$

$$\Rightarrow \boxed{\ln|x| + \cos\left(\frac{y}{x}\right) = c}$$

Q) $xy' = y + 2xe^{-y/x}$

$$x \frac{dy}{dx} = y + 2xye^{-y/x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + 2xe^{-y/x}}{x} \rightarrow f(x,y)$$

$$f(tx, ty) = \frac{ty + 2txe^{-ty/tx}}{tx}$$

$$= \frac{y + 2xe^{-y/x}}{x}$$

$$= t^0 f(x,y)$$

$$\therefore f(tx, ty) = t^0 f(x,y)$$

$\therefore$ Homogenous DE

$$\Rightarrow x\frac{dz}{dx} = 3(1+z^2)\tan^{-1}(z)$$

$$\Rightarrow \frac{dx}{x} = \frac{dz}{3(1+z^2)\tan^{-1}(z)}$$

$$\Rightarrow \ln|x| + c = \frac{1}{3}\ln|\tan^{-1}(z)|$$

$$\Rightarrow 3\ln|x| + c = \ln|\tan^{-1}(z)|$$

$$\therefore \boxed{cx^3 = \tan^{-1}\left(\frac{y}{x}\right)}$$

Q) $x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} = y\sin\left(\frac{y}{x}\right) + x$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right)+x}{x\sin\left(\frac{y}{x}\right)} \rightarrow f(x,y)$$

$$f(tx,ty) = \frac{ty\sin\left(\frac{ty}{tx}\right)+tx}{tx\sin\left(\frac{ty}{tx}\right)}$$

$$= f(x,y)\cdot t^0$$

$$\therefore f(tx,ty) = t^0 f(x,y)$$

$\therefore$ Homogenous DE

$$\frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right)+x}{x\sin\left(\frac{y}{x}\right)}$$

$$\left[\begin{array}{c} y = zx \\ \frac{dy}{dx} = x\frac{dz}{dx} + z \end{array}\right]$$

$$\frac{dy}{dx} = \frac{z x \sin(z) + x}{x \sin(z)}$$

$$\Rightarrow x \frac{dz}{dx} = \frac{x}{x \sin(z)}$$

$$\Rightarrow \frac{dx}{x} = \sin(z)\, dz$$

$$\Rightarrow \int \frac{dx}{x} = \int \sin(z)\, dz$$

$$\Rightarrow \ln|x| + c = -\cos(z)$$

$$\Rightarrow \boxed{\ln|x| + \cos\left(\frac{y}{x}\right) = c}$$

Q) $xy' = y + 2xe^{-y/x}$

$$x \frac{dy}{dx} = y + 2xe^{-y/x}$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{\frac{y + 2xe^{-y/x}}{x}}_{f(x,y)}$$

$$f(tx, ty) = \frac{ty + 2txe^{-ty/tx}}{tx}$$

$$= \frac{y + 2xe^{-y/x}}{x}$$

$$= t^0 f(x,y)$$

$$\therefore f(tx, ty) = t^0 f(x,y)$$

$\therefore$ Homogenous DE

$$\frac{dy}{dx} = \frac{y + 2xe^{-y/x}}{x}$$

$$\left[\begin{array}{l} y = zx \\ \frac{dy}{dx} = x\frac{dz}{dx} + z \end{array}\right]$$

$$\Rightarrow x\frac{dz}{dx} = \frac{zx + 2xe^{-z}}{x} - z$$

$$\Rightarrow x\frac{dz}{dx} = 2e^{-z}$$

$$\Rightarrow \frac{dx}{x} = \frac{dz}{2e^{-z}} = \frac{1}{2}e^{z}\,dz$$

$$\Rightarrow \ln|x| + c = \frac{1}{2}e^{z}$$

$$\Rightarrow e^{y/x} = 2\ln(cx)$$

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$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

Two Cases :

Case (i) $\frac{a}{a'} = \frac{b}{b'} = k$ (let)

$$\frac{dy}{dx} = \frac{k(a'x + b'y) + c}{a'x + b'y + c'}$$

$$\left[\begin{array}{l} a'x + b'y = z \\ \frac{dz}{dx} = a' + b'\frac{dy}{dx} \end{array}\right]$$

$$\left[\frac{dy}{dx} = \frac{1}{b'}\left(\frac{dz}{dx} - a'\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b'}\left(\frac{dz}{dx} - a'\right) = \frac{kz + c}{z + c'}$$

$$\Rightarrow \frac{dz}{dx} - a' = \frac{kb'z + b'c}{z + c'}$$

$$\Rightarrow \frac{dz}{dx} = \frac{kb'z + b'c}{z + c'} + a'$$

$$= \frac{kb'z + b'c + a'z + a'c'}{z + c'}$$

$$= \frac{z(a' + kb') + b'c + a'c'}{z + c'}$$

$$\Rightarrow dx = \frac{z + c'}{z(a' + kb') + b'c + a'c'}\, dz$$

Ex. $\frac{dy}{dx} = \frac{2x + 3y + 4}{6x + 9y + 7}$

$\frac{2}{6} = \frac{3}{9}$ ✓

$\therefore k = 1/3$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{3}(6x + 9y) + 4}{6x + 9y + 7}$$

$$\left[\begin{array}{l} 6x + 9y = z \\ \Rightarrow \frac{dz}{dx} = 6 + 9\frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} = \frac{\frac{dz}{dx} - 6}{9} \end{array}\right]$$

$$\Rightarrow \frac{\frac{dz}{dx} - 6}{9} = \frac{\frac{1}{3}(z) + 4}{z+7}$$

$$\Rightarrow \frac{dz}{dx} - 6 = \frac{3z + 36}{z+7}$$

$$\Rightarrow \frac{dz}{dx} = \frac{3z + 36 + 42 + 6z}{z+7}$$

$$\therefore \frac{dz}{dx} = \frac{9z + 78}{z+7}$$

$$\Rightarrow dx = \frac{z+7}{9z+78}\, dz$$

$$\Rightarrow x + c = \int \frac{z+7}{9z+78}\, dz$$

$$= \int \left(\frac{z}{9z+78} + \frac{7}{9z+78} \right) dz$$

$$= \left(\int \frac{z\, dz}{9z+78} \right) + \frac{7}{9} \ln|9z+78|$$

$$= \frac{1}{9} \int \frac{9z\, dz}{9z+78} + \frac{7}{9} \ln|9z+78|$$

$$= \frac{1}{9} \int \left(1 - \frac{78}{9z+78} \right) dz + \frac{7}{9} \ln|9z+78|$$

$$= \frac{1}{9} \left(z - \frac{78}{9} \ln|9z+78| \right) + \frac{7}{9} \ln|9z+78|$$

$$\Rightarrow x + c = \frac{z}{9} - \frac{15}{81} \ln|9z+78|$$

$$\Rightarrow x + c = \frac{6x+9y}{9} - \frac{15}{81} \ln|54x+81y+78|$$

$$\boxed{\therefore x + c = \frac{2x+3y}{3} - \frac{15}{81} \ln|54x+81y+78|}$$

9) $\frac{dy}{dx} = \frac{x+y+3}{x-3y+7}$

$$\begin{bmatrix} x = X+h \\ y = Y+k \end{bmatrix}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(X+h)+(Y+k)+3}{(X+h)-3(Y+k)+7}$$

$$= \frac{X+Y+h+k+3}{X-3Y+h-3k+7}$$

$$h+k+3=0 \quad -①$$
$$h-3k+7=0 \quad -②$$

$$4k=4 \Rightarrow \boxed{k=1}$$
$$\boxed{h=-4}$$

$$\begin{bmatrix} x = X-4 \\ y = Y+1 \end{bmatrix}$$

$$\therefore \frac{dy}{dx} = \frac{X+Y}{X-3Y} \qquad \begin{bmatrix} dx = dX \\ dy = dY \end{bmatrix}$$

$$\Rightarrow \frac{dY}{dX} = \frac{X+Y}{X-3Y}$$

$$\begin{bmatrix} Y = zX \\ \frac{dY}{dX} = z + X\frac{dz}{dX} \end{bmatrix}$$

$$\Rightarrow z + X\frac{dz}{dX} = \frac{X+zX}{X-3zX}$$

$$\Rightarrow X\frac{dz}{dX} = \frac{1+z}{1-3z} - z$$

$$= \frac{1+z-z+3z^2}{1-3z}$$

$$\Rightarrow X\frac{dz}{dX} = \frac{(1+3z^2)}{(1-3z)}$$

$$\Rightarrow \frac{dX}{X} = \frac{1-3(z)}{z(1+3(z))}\,dz$$

$$\Rightarrow \ln|x+4| + c = \int \frac{1-3(z)}{z(1+3z)}\,dz$$

$$= \int\left(\frac{1}{z(1+3z)} - \frac{3}{1+3z}\right)dz$$

$$= 3\int\frac{1}{3z(1+3z)}\,dz - \ln|1+3z|$$

$$= 3\int\left(\frac{1}{3z} - \frac{1}{1+3z}\right)dz - \ln|1+3z|$$

$$= \ln|z| - \ln|1+3z| - \ln|1+3z|$$

$$= \ln|z|$$

$$= \ln\left|\frac{y}{x}\right|$$

$$= \ln\left|\frac{y-1}{x+4}\right|$$

$$\int\frac{dx}{\sqrt{1-x^2}}$$

$$\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$= \int\frac{dx}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$= \int\frac{d(x/a)}{\sqrt{1-(x/a)^2}}$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \ln|x+4| + \ln c = \ln\left|\frac{y-1}{x+4}\right|$$

$$\boxed{\therefore c(x+4)^2 = y-1}$$

$$\Rightarrow \frac{dX}{X} = \frac{1-3Z}{1+3Z^2}\, dZ$$

$$\Rightarrow \frac{dX}{X} = \left(\frac{1}{1+3Z^2} - \frac{3Z}{1+3Z^2}\right) dZ$$

$$\Rightarrow \ln|X| + c = \int \frac{dZ}{1+3Z^2} - \frac{1}{2}\int \frac{2\times 3Z}{1+3Z^2}\, dZ$$

$$\Rightarrow \ln|x+a| + c = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}Z) - \frac{1}{2}\ln(1+3Z^2) + C$$

<u>Homework:</u>

1. (a) $ae \neq bd$

$$\Rightarrow \frac{a}{d} \neq \frac{b}{e}$$

$$\frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+ey+f}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{ax+by+c}{dx+ey+f}$$

$$\begin{bmatrix} x = X+h \\ y = Y+k \end{bmatrix} \longrightarrow \begin{bmatrix} dx = dX \\ dy = dY \end{bmatrix}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a(X+h)+b(Y+k)+c}{d(X+h)+e(Y+k)+f}$$

$$= \frac{aX+bY+ah+bk+c}{dX+eY+dh+ek+f}$$

$$ah+bk+c = 0 \Rightarrow adh+bdk+cd = 0$$
$$dh+ek+f = 0 \Rightarrow adh+aek+af = 0$$

$$\Rightarrow \boxed{k = \frac{-af-cd}{bd+ac}}$$

$$aeh+bek+ce = 0$$
$$dbh+bek+bf = 0$$

$$\Rightarrow \boxed{h = \frac{-ce-bf}{(ae+bd)}}$$

$$\boxed{\therefore x = X - \frac{ce+bf}{ae+bd}}$$

$$\boxed{y = Y - \frac{af+cd}{bd+ac}}$$

~~(b)~~ ~~i.e. then~~

$\therefore$ DE becomes a Homogenous DE.

(b) $ac = bd$

$\Rightarrow \frac{a}{d} = \frac{b}{e}$

$(dx + ey = z)$

2. (a) $\frac{dy}{dx} = \frac{x+y+4}{x-y-6}$

$\frac{1}{1} \neq \frac{1}{-1}$

$$\begin{bmatrix} x = X+h \\ y = Y+k \end{bmatrix} \Rightarrow \begin{bmatrix} dx = dX \\ dy = dY \end{bmatrix}$$

~~$\Rightarrow \frac{dy}{dx} \rightarrow$ $X+h+Y+k$~~

$$\Rightarrow \frac{dY}{dX} = \frac{X+h+Y+k+4}{X+h-Y-k-6} = \frac{X+Y+(h+k+4)}{X-Y+(h-k-6)}$$

$$\begin{aligned} h+k+4 &= 0 \\ h-k-6 &= 0 \end{aligned}$$

$\Rightarrow \boxed{h=1} \Rightarrow \boxed{k=-5}$

$$\begin{bmatrix} x = X+1 \\ y = Y-5 \end{bmatrix}$$

$$\Rightarrow \frac{dY}{dX} = \frac{X+Y}{X-Y}$$

$$\begin{bmatrix} Y = zX \\ \Rightarrow \frac{dY}{dX} = z + X\frac{dz}{dX} \end{bmatrix}$$

$$\Rightarrow z + X\frac{dz}{dX} = \frac{X(1+z)}{X(1-z)}$$

$$\Rightarrow X\frac{dz}{dX} = \frac{1+z}{1-z} - z$$

$$= \frac{1+z-z+z^2}{1-z}$$

$$\Rightarrow X\frac{dz}{dX} = \frac{1+z^2}{1-z}$$

$$\Rightarrow \frac{dX}{X} = \frac{1-z}{1+z^2}\,dz$$

$$\Rightarrow \ln(|x-1|) = \int \frac{1-z}{1+z^2}\,dz$$

$$= \tan^{-1}\left(\frac{Y}{X}\right) - \frac{1}{2}\ln\left|1+\frac{Y^2}{X^2}\right|$$

$$\Rightarrow \boxed{\ln|x-1| = \tan^{-1}\left(\frac{y+5}{x-1}\right) - \frac{1}{2}\ln\left|1+\frac{(y+5)^2}{(x-1)^2}\right|}$$

(c) $\dfrac{dy}{dx} = \dfrac{x+y+4}{x+y-6}$

$x+y = z$

$1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\Rightarrow \dfrac{dz}{dx} = \dfrac{z+4}{z-6} + 1$

$= \dfrac{z+4+z-6}{z-6}$

$= \dfrac{2z-2}{z-6}$

$\Rightarrow dx = \dfrac{z-6}{2z-2}\,dz \Rightarrow 2x = \int \dfrac{z-6}{z-1}\,dz$

$= z - \int \dfrac{4}{z-1}\,dz$

$= z - 5\ln|z-1| + c$

$\Rightarrow 2x = x+y - 5\ln|x+y-1| + c$

$\Rightarrow x = y - 5\ln|x+y-1| + c$

$\Rightarrow \boxed{10x = \dfrac{z^2}{2} - 6z}$

$\boxed{\therefore 10x = \dfrac{z^2}{2} - 6z + c}$

(d) $\dfrac{dy}{dx} = \dfrac{x+y-1}{x+4y+2}$

$\dfrac{1}{1} \neq \dfrac{1}{4}$

$\begin{bmatrix} \therefore x = X+h \\ y = Y+k \end{bmatrix} \Rightarrow \begin{bmatrix} dx = dX \\ dy = dY \end{bmatrix}$

$\dfrac{dY}{dX} = \dfrac{X+h+Y+k-1}{X+h+4(Y+k)+2}$

$$\Rightarrow \frac{dY}{dX} = \frac{X+Y+(h+k-1)}{X+4Y+(h+4k+2)}$$

$$\therefore \begin{aligned} h+k-1 &= 0 \\ h+4k+2 &= 0 \end{aligned}$$

$$k = -1$$

$$h = 2$$

$$\left[\begin{aligned} \therefore\ x &= X+2 \\ y &= Y-1 \end{aligned}\right]$$

$$\Rightarrow \frac{dY}{dX} = \frac{X+Y}{X+4Y}$$

$$\left(\begin{aligned} &Y = ZX \\ &\Rightarrow \frac{dY}{dX} = Z + X\frac{dZ}{dX} \end{aligned}\right)$$

$$\Rightarrow X\frac{dZ}{dX} = \frac{X(1+Z)}{X(1+4Z)} - Z$$

$$\Rightarrow \frac{dX}{X} = \frac{1+4Z}{1-4Z^2}\,dZ$$

$$\Rightarrow \ln|x-2| + c = \int \frac{1}{1-4Z^2}\,dZ + \frac{-1}{2}\int \frac{-8Z}{1-4Z^2}\,dZ$$

$$\Rightarrow \ln|x-2| + c = \int \frac{dZ}{(1)^2-(2Z)^2} - \frac{1}{2}\ln|1-4Z^2|$$

$$= -\int \frac{dZ}{(2Z)^2-(1)^2} - \frac{1}{2}\ln\left|1-4\frac{Y^2}{X^2}\right|$$

$$= -\frac{1}{4}\ln\left(\frac{2Z-1}{2Z+1}\right) - \frac{1}{2}\ln\left|1-\frac{4(y+1)^2}{(x-2)^2}\right|$$

$$\Rightarrow \ln|x-2| + c = \frac{1}{4}\ln\left|\frac{2z+1}{2z-1}\right| - \frac{1}{2}\ln\left|1 - \frac{4(y+1)^2}{(x-2)^2}\right|$$

$$\boxed{\therefore \ln|x-2| + c = \frac{1}{4}\ln\left|\frac{2\left(\frac{y-1}{x+2}\right)+1}{2\left(\frac{y-1}{x+2}\right)-1}\right| - \frac{1}{2}\ln\left|1 - \frac{4(y+1)^2}{(x-2)^2}\right|}$$

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Solving First Order Equations:

Exact Equations

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Ex. $x^2y^3 = c$

$$\Rightarrow 2xy^3dx + 3x^2y^2dy = 0$$

Reverse Scenario:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$\exists$ a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N$$

$$\therefore f(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Definition:

Consider the DE $M(x,y)\,dx + N(x,y)\,dy = 0$.

If there is a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N$$

then $M\,dx + N\,dy$ is called an exact differential and $\underline{M\,dx + N\,dy = 0}$ is called an exact differential equation.

Ex. $y\,dx + x\,dy = 0$

$\Rightarrow xy = c$

$y\,dx + x\,dy = 0$

$\Rightarrow y\,dx = -x\,dy$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

Ex. $\frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0$

$\Rightarrow \frac{x}{y} = c$

→ <u>Necessary</u> <u>Condition</u> <u>for</u> <u>Exactness</u> :

- Suppose that DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact.

- Then $\exists$ a function f such that

$$\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N$$

Then $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial M}{\partial y}$ & $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x}$

But the mixed second order derivatives of a function are equal :

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Q) $y\,dx + (xy - x)\,dy = 0$

Sol: As $M\,dx + N\,dy = 0$

$\Rightarrow M = y \;\&\; N = x^2y - x$

$$\frac{\partial M}{\partial y} = 1 \;\&\; \frac{\partial N}{\partial x} = 2xy - 1$$

$\therefore$ DE is not exact

& Method cannot be used

Note:

Condition is necessary & Sufficient

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\Updownarrow$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

To prove:

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

Then $M(x,y)\,dx + N(x,y)\,dy = 0$

$$\frac{\partial F}{\partial x} = M(x,y)$$

$$\frac{\partial F}{\partial y} = N(x,y)$$

$$\frac{\partial F}{\partial x} = M \Rightarrow \int \frac{\partial F}{\partial x}\,dx = \int M\,dx$$

$$\cancel{\Rightarrow \int \frac{\partial F}{\partial x}\,dx = \int M\,dx}$$

$$\Rightarrow F = \int M\,dx + g(y)$$

$$\therefore \frac{\partial}{\partial y}\left(\int M\,dx\right) + g'(y) = N$$

$$\Rightarrow g'(y) = N - \frac{\partial}{\partial y}\int M\,dx$$

$$0 = \frac{\partial}{\partial x}(g'(y)) = \frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M\,dx\right)$$

$$= \frac{\partial}{\partial x}N - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\int M\,dx\right)$$

$$= \frac{\partial}{\partial x}N - \left(\frac{\partial^2}{\partial x\,\partial y}\int M\,dx\right)$$

$$\left[\frac{\partial^2}{\partial y\,\partial x}\int M\,dx\right]$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= 0$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

[$M(x,y)$ & $N(x,y)$ are continuous, the order of partial differentiation can be varied]

Q) Test for exactness

$$e^y\,dx + (xe^y + 2y)\,dy = 0$$

Sol : $M\,dx + N\,dy = 0$

$M = e^y$	$N = xe^y + 2y$
$\frac{\partial}{\partial y}M = e^y$	$\frac{\partial N}{\partial x} = e^y$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \checkmark$$

$\therefore$ Given equation is exact DE.

$$\frac{\partial f}{\partial x} = M \qquad \Big| \qquad \frac{\partial f}{\partial y} = xe^{y} + 2y$$

$$\Rightarrow \frac{\partial f}{\partial x} = e^{y}$$

$$\Rightarrow f = \left(\int e^{y}\,dx\right) + g(y)$$

$$\Rightarrow g(y) = f - \int e^{y}\,dx$$

$$= f - xe^{y}$$

$$f - xe^{y} = g(y)$$

$$\Rightarrow f = xe^{y} + g(y)$$

$$\left(\frac{\partial f}{\partial y} = xe^{y} + 2y\right)$$

$$\Rightarrow xe^{y} + 2y = xe^{y} + g'(y)$$

$$\therefore g'(y) = 2y$$

$$\boxed{\therefore g'(y) = 2y}$$

$$\therefore g(y) = y^{2} + c$$

$$\boxed{\therefore f = xe^{y} + y^{2} + c}$$

$$f(x,y) = xe^{y} + y^{2} + c$$

Q) $\left(x+\frac{2}{y}\right)dy + y\,dx = 0$

2) $(\sin x \tan y + 1)\,dx + (\cos x \sec^2 y)\,dy = 0$

3) $(y - x^3)\,dx + (x + y^3)\,dy = 0$

4) $(2y^2 - 4x + 5)\,dx = (4 - 2y + 4xy)\,dy$

5) $(y + y\cos xy)\,dx + (x + x\cos xy)\,dy = 0$

Sol :

3) $(y - x^3)\,dx + (x + y^3)\,dy = 0$

$\frac{\partial}{\partial y}(M) = 1 \qquad \frac{\partial}{\partial x}(N) = 1$

Exact

$\frac{\partial F}{\partial x} = 1$

$\Rightarrow F = \int (y - x^3)\,dx + g(y)$

$\Rightarrow F = yx - \frac{x^4}{4} + g(y)$

$g(y) = F - yx + \frac{x^4}{4}$

$F = g(y) + yx - \frac{x^4}{4}$

$\frac{\partial F}{\partial y} = g'(y) + x \quad -(1)$

& $\frac{\partial F}{\partial y} = N = x + y^3 \quad -(2)$

$g'(y) = y^3$

$g(y) = \frac{y^4}{4} + c$

$\cancel{F = \frac{y^4}{4} - \frac{x^4}{4} + yx + c}$

$f(x,y) = c$

$\boxed{xy - \frac{x^4}{4} + \frac{y^4}{4} = c}$

2) $(\sin x \tan y + 1)\, dx + (\cos x \sec^2 y)\, dy = 0$

$$\Rightarrow \frac{\partial F}{\partial x} = \sin x \tan y + 1 = M \qquad \frac{\partial F}{\partial y} = \cos x \sec^2 y = N$$

$$\frac{\partial M}{\partial y} = \sin x \sec^2 y \qquad \frac{\partial N}{\partial x} = -\sin x \sec^2 y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Not Exact

4) $(2y^2 - 4x + 5)\, dx + (2y - 4xy - 4)\, dy = 0$

$$\frac{\partial F}{\partial x} = M = 2y^2 - 4x + 5 \qquad \frac{\partial F}{\partial y} = N = 2y - 4xy - 4$$

$$\frac{\partial M}{\partial y} = 4y \qquad \frac{\partial N}{\partial x} = -4y$$

Not Exact

5) $(y + y\cos xy)\, dx + (x + x\cos xy)\, dy = 0$

$$\frac{\partial F}{\partial x} = M = y + y\cos xy \qquad \frac{\partial F}{\partial y} = N = x + x\cos xy$$

$$\frac{\partial M}{\partial y} = 1 + \cos xy - xy\sin(xy) \qquad \frac{\partial N}{\partial x} = 1 + \cos(xy) - xy\sin(xy)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$f = \int (y + y\cos xy)\, dx + g(y)$$

$$\Rightarrow f = yx + \sin(xy) + g(y)$$

$$\frac{\partial F}{\partial y} = x + x\cos(xy) + g'(y)$$

$$= N$$

$$g'(y) = 0$$

$$\underline{g(y) = c}$$

$$\cancel{\boxed{\therefore F = xy + \sin(xy) + c}}$$

$$f(x,y) = c$$

$$\boxed{xy + \sin(xy) = c}$$

1) $y\,dx + \left(x + \frac{2}{y}\right)dy = 0$

$M = y$ | $N = x + \frac{2}{y}$

$$\frac{\partial M}{\partial y} = 1 \quad \& \quad \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

Exact

$$\frac{\partial F}{\partial x} = M$$

$$F = \int y\,dx + g(y)$$

$$= xy + g(y)$$

$$\frac{\partial F}{\partial y} = x + g'(y) \quad (1)$$

$$\frac{\partial F}{\partial y} = N \quad (2)$$

$$\therefore g'(y) = \frac{2}{y}$$

$$\therefore g(y) = 2\ln y$$

$$f(x,y) = c$$

$$\boxed{xy + 2\ln y = c}$$

$$\cancel{\boxed{F = xy + 2\ln y + c}}$$

$$\cancel{F = xy + \frac{y^2}{4} + c}$$

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Q) $(y)\,dx + (x^2y - x)\,dy = 0$

$M(x,y) = y$ | $N(x,y) = x^2y - x$

$\frac{\partial M}{\partial y} = 1$ | $\frac{\partial N}{\partial x} = 2xy - 1$

$1 \neq 2xy - 1$

Not Exact

$y\,dx + (x^2y - x)\,dy = 0$

$\Rightarrow \left(\frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0$

[Here $\frac{1}{x^2}$ is an Integrating Factor.]

$M(x,y) = \frac{y}{x^2}$ | $N(x,y) = y - \frac{1}{x}$

$\frac{\partial M}{\partial y} = \frac{1}{x^2}$ | $\frac{\partial N}{\partial x} = \frac{1}{x^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Exact

$F = \int \frac{y}{x^2}\,dx + g(y)$

$\Rightarrow F = -y/x + g(y)$

$\Rightarrow \frac{\partial F}{\partial y} = -1/x + g'(y)$ $\qquad \frac{\partial F}{\partial y} = N = y - \frac{1}{x}$

$\boxed{\therefore g'(y) = y}$

$g(y) = \frac{y^2}{2}$

$\therefore F(x,y) = c$

$\Rightarrow \boxed{\frac{y^2}{2} - \frac{y}{x} = c}$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If Solution Exists $\Rightarrow$ If DE is non-exact

$\Uparrow$ $\qquad$ $\Downarrow$

DE can be converted to exact $\Leftarrow$ Integrating factor Exists

$$\mu(x,y)\cdot M(x,y)\,dx + \mu(x,y)\cdot N(x,y)\,dy = 0$$

$$\frac{\partial f}{\partial x} = \mu M \qquad \Bigg| \qquad \frac{\partial f}{\partial y} = \mu N$$

$$\left(\frac{\partial f}{\partial x}\right)\Big/ M = \mu \qquad \Bigg| \qquad \left(\frac{\partial f}{\partial y}\right)\Big/ N = \mu$$

$$\boxed{\left(\frac{\partial f}{\partial x}\right)\Big/ M = \left(\frac{\partial f}{\partial y}\right)\Big/ N = \mu}$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$\Rightarrow \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}}$$

$$\Rightarrow \mu\cdot\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

$$\Rightarrow M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right)$$

Case - I $\qquad\qquad$ **Case - II**

$\mu = \mu(x)$ $\qquad\qquad$ $\mu = \mu(y)$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x}\right)$$

$$\Rightarrow \underbrace{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}_{} = \underbrace{\frac{1}{\mu}\frac{d\mu}{dx}}_{\text{function of } x}$$

$\therefore$ $g(x)$ ← Hence

$$\Downarrow$$

$$g(x) = \frac{1}{\mu}\frac{d\mu}{dx}$$

$$\Rightarrow g(x)\,dx = \frac{1}{\mu}\,d\mu$$

$$\Rightarrow \int g(x)\,dx = \int \frac{1}{\mu}\,d\mu$$

$$\Rightarrow \int g(x)\,dx = \ln|\mu(x)|$$

$$\therefore \mu(x) = e^{\int g(x)\,dx}$$

$$\left[\text{where } g(x) = \frac{1}{N}\underbrace{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}_{\text{Exactness Condition}}\right]$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left(-\frac{M\,\partial \mu}{\partial y}\right)$$

$$\Rightarrow \underbrace{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{} = \underbrace{\frac{1}{\mu}\frac{d\mu}{dy}}_{\text{function of } y}$$

$g(y)$ ← Hence

$$\Downarrow$$

$$g(y) = \frac{1}{\mu}\frac{d\mu}{dy}$$

$$\Rightarrow g(y)\,dy = \frac{1}{\mu}\,d\mu$$

$$\Rightarrow \int g(y)\,dy = \int \frac{1}{\mu}\,d\mu$$

$$\Rightarrow \int g(y)\,dy = \ln|\mu(y)|$$

$$\therefore \mu(y) = e^{\int g(y)\,dy}$$

$$\left[\text{where } g(y) = \frac{1}{M}\underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{Exactness Condition}}\right]$$

$$\cancel{(y)\,dx + (x^2y - x)\,dy = 0}$$

Ex

$$\cancel{\therefore \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x,y)} = \frac{1 - (2xy - 1)}{N(x,y)} = \frac{-2xy}{x^2y - x} = \frac{-2y}{xy - 1}}$$

Ex. $(y)\,dx + (x^2y - x)\,dy = 0$

$M = y$

$N = x^2y - x$

$\frac{\partial M}{\partial y} = 1$

$\frac{\partial N}{\partial x} = 2xy - 1$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y} = 1 - (2xy - 1)$$

$$= 2 - 2xy$$

$$= 2(1 - xy)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x,y)} = \frac{2(1-xy)}{-x(1-xy)} = \frac{-2}{x}$$

$$\mu(x) = e^{\int -\frac{2}{x}\,dx}$$

$$= e^{\ln(x^{-2})}$$

$$= x^{-2}$$

$$\Rightarrow \mu(x) = 1/x^2$$

$\therefore$ Integrating factor $= 1/x^2$

$$F = \int \mu M\,dx + g(y) \quad -(1)$$

$$\& \quad \frac{\partial F}{\partial y} = N \quad -(2)$$

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1) $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ (Ny - Mx) = g(z)$

$z = xy$

P.T:

$\mu = e^{\int g(z)\,dz}$

Sol:

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

$[\mu \to \mu(z)]$

$$\left[\frac{\partial z}{\partial x} = y \;\middle|\; \frac{\partial z}{\partial y} = x\right]$$

$$\Rightarrow \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial z}\cdot\frac{\partial z}{\partial x} - M\cdot\frac{\partial \mu}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{1}{\mu}\left(Ny\frac{\partial \mu}{\partial z} - Mx\frac{\partial \mu}{\partial z}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{1}{\mu}\frac{\partial \mu}{\partial z}(Ny - Mx) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)}{(Ny - Mx)} = \underbrace{\frac{1}{\mu}\frac{d\mu}{dz}}_{\text{function of } z}$$

$$\therefore \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) \Big/ (Ny - Mx) = g(z)$$

$$\Longrightarrow \mu(z) = e^{\int g(z)\,dz}$$

→ **Linear Differential Equation:**

General first order linear Equation is

$$\frac{dy}{dx} = p(x)\,y + q(x)$$

General Second Order Linear Equation is

$$\frac{d^2y}{dx^2} = p(x)\frac{dy}{dx} + q(x)\,y + r(x)$$

General n^{th} order Linear Equation is

$$\frac{d^ny}{dx^n} = a_1(x)\cdot\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots a_{n-1}(x)\cdot\frac{dy}{dx} + a_n(x)\,y + a_{n+1}(x)$$

Note:

$$\frac{d}{dx}\left(e^{\int P dx}\,y\right) = e^{\int P dx}\frac{dy}{dx} + y\,P e^{\int P dx}$$

$$= e^{\int P dx}\left(\frac{dy}{dx} + Py\right)$$

Proof:

$$\frac{dy}{dx} + P(x)\,y = \varphi(x)$$

Convert this into $M\,dx + N\,dy = 0$

$$\Rightarrow dy + P(x)\,y\,dx = \varphi(d(x))$$

$$\Rightarrow \left[P(x)\,y - \varphi(x)\right]dx + dy = 0$$

$$\begin{bmatrix} M(x,y) : P(x)\,y - \varphi(x) \\ N(x,y) : 1 \end{bmatrix}$$

By Comparison

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

$$= P(x) - 0$$

$$= P(x)$$

As $P(x)$ might not always equal to 0,

$\therefore$ Not Exact

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{P(x)}{1} = P(x)$$

$\therefore$ Only function of x

$\therefore$ Integrating factor $= e^{\int P(x)\,dx}$

$[\mu(x)]$

Now $[P(x)\,y - \varphi(x)]\,dx + dy = 0$

$$\Rightarrow e^{\int P(x)\,dx}\,[P(x)\,y - \varphi(x)]\,dx + e^{\int P\,dx}\,dy = 0$$

Now $M'(x,y) : e^{\int P\,dx}\,[P(x)\,y - \varphi(x)]$

$N'(x,y) : e^{\int P\,dx}$

$$\frac{\partial M'}{\partial y} = e^{\int P\,dx} \cdot P(x)$$

$$\frac{\partial N'}{\partial x} = e^{\int P\,dx}\,P(x)$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

$$\therefore f = \int M'\,dx + g(y)$$

$$= \int e^{\int P\,dx}\,[P(x)\,y - \varphi(x)]\,dx + g(y)$$

But $\frac{\partial F}{\partial y} = N = e^{\int P dx}$

$$\Rightarrow \frac{\partial}{\partial y}\left[\int e^{\int P dx}[P(x)y - \varphi(x)]\,dx + g(y)\right] = e^{\int P dx}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\int e^{\int P dx}[P(x)y - \varphi(x)]\,dx\right) + g'(y) = e^{\int P dx}$$

(OR)

$$y \cdot e^{\int P dx} = \int \varphi(x) \cdot e^{\int P dx}\,dx + C$$

$$\therefore y(x) = e^{-\int P dx}\left[\int \varphi(x) \cdot e^{\int P dx}\,dx + c\right]$$

Ex. $\frac{dy}{dx} + \frac{1}{x}y = 3x$

Sol: This equation is a linear equation with $P(x) = 1/x$
$\varphi(x) = 3x$

$$\frac{dy}{dx} + P(x)y = \varphi(x)$$

Integrating Factor : $e^{\int P dx}$

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{\ln x}$$

$$= x$$

Multiplying the given equation by x

$$x\frac{dy}{dx} + y = 3x^2$$

$$\Rightarrow \frac{d}{dx}(xy) = 3x^2$$

$$\therefore xy = x^3 + c$$

$$\left[\because \frac{d}{dx}(I.F)y = \varphi(x)\cdot(I.F)\right]$$

$$\boxed{\therefore\ y = x^2 + c/x}$$

→ **Bernoulli's Equations:**

First order DE of the form

$$\frac{dy}{dx} + P(x)y = \varphi(x)\cdot y^n$$

Bernoulli's Equation is just a linear DE if $n = 0$ (or) 1.

In other cases, it can be reduced to a linear DE by the substitution $z = y^{1-n}$

$$\frac{dy}{dx} + P(x)y = \varphi(x)\cdot y^n$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x)\cdot y^{1-n} = \varphi(x)$$

$$\left[\frac{dz}{dx} = (1-n)\cdot y^{-n}\cdot\frac{dy}{dx}\right]$$

$$\Rightarrow \frac{1}{1-n}\left(\frac{dz}{dx}\right) + P(x)\cdot z = \varphi(x)$$

$$\Rightarrow \frac{dz}{dx} + P(x)(1-n)\cdot z = \varphi(x)\cdot(1-n)$$

$$P(x)\cdot(1-n) = P'(x)$$

$$\varphi(x)\cdot(1-n) = \varphi'(x)$$

$$\Rightarrow \frac{dz}{dx} + z\,P'(x) = \varphi'(x)$$

Integrating factor $= e^{\int P'(x)\,dx}$

$$\therefore z\cdot e^{\int P'dx} = \int \varphi(x)\cdot e^{\int P'dx}\,dx + C$$

$$\Rightarrow y^{1-n} = e^{-\int p'dx}\left[\int q(x)e^{\int p'dx} + c\right]$$

$$\therefore y(x) = \left[e^{-\int(1-n)p(x)dx}\left[\int q(x)e^{\int p(x)(1-n)dx} + c\right]\right]^{1/1-n}$$

Homework

Q) $(e^y - 2xy)y' = y^2$

Sol: $(e^y - 2xy)\dfrac{dy}{dx} = y^2$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{e^y - 2xy}$$

$$\Rightarrow \frac{dx}{dy} = \frac{e^y}{y^2} - \frac{2xy}{y^2}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right)x = e^y\left(\frac{1}{y^2}\right)$$

$$\cancel{z = y^3}$$

$$\cancel{\frac{dz}{dy} = 3y^2\frac{dx}{dy}} \qquad \cancel{\frac{dz}{dx} = 3y^2\frac{dy}{dx}}$$

$$\cancel{\Rightarrow \frac{1}{3y^2}\frac{dz}{dy} +} \qquad \cancel{\Rightarrow \frac{dx}{dy} = 3y^2\frac{dx}{dz}}$$

$$\cancel{\Rightarrow y^2\frac{dx}{dy} + (2y)x = e^y}$$

$$\cancel{\Rightarrow}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right)x = \left(\frac{e^y}{y^2}\right)$$

$$I.F = e^{\int P(y)\,dy}$$

$$= e^{\int 2/y\,dy}$$

$$= e^{2\ln y}$$

$$= y^2$$

$$\therefore x \cdot y^2 = \int \frac{e^y}{y^2} \cdot y^2\,dy + c$$

$$\Rightarrow xy^2 = e^y + c$$

$$\boxed{\therefore x = \frac{e^y}{y^2} + c/y^2}$$

Q) $y - xy' = y'y^2e^y$

$$\Rightarrow y = y'(x + y^2e^y)$$

$$\Rightarrow y' = \frac{y}{x + y^2e^y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + ye^y$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = ye^y$$

$$I.F = e^{\int -1/y\,dy}$$

$$= e^{-\ln y}$$

$$= \frac{1}{y}$$

$$x \cdot \frac{1}{y} = \int ye^y \cdot \frac{1}{y}\,dy + c$$

$$\Rightarrow x/y = e^y + c$$

$$\boxed{x = ye^{y} + cy}$$

Q) $f^2(y)\dfrac{dx}{dy} + 3f(y)\cdot f'(y)\,x = f'(y)$

$$\Rightarrow \frac{dx}{dy} + 3\frac{f'(y)}{f(y)}\,x = \frac{f'(y)}{f^2(y)}$$

$$IF = e^{\int P(y)\,dy}$$

$$= e^{\int \frac{3f'(y)}{f(y)}\,dy}$$

$$= e^{3\ln|f(y)|}$$

$$\therefore IF = f^3(y)$$

$$\therefore x\cdot f^3(y) = \int \frac{f'(y)}{f^2(y)}\cdot f^3(y)\,dy + c$$

$$= \int f'(y)\cdot f(y)\,dy + c$$

$$= \frac{f^2(y)}{2} + c$$

$f(y) = z$

$f'(y)\,dy = dz$

$$\Rightarrow x f^3(y) = \frac{f^2(y)}{2} + c$$

$$\boxed{\therefore x = \frac{1}{2f(y)} + \frac{c}{f^3(y)}}$$

$$\cancel{f(y) = \frac{1}{2x} + \frac{c}{x}}$$

$$\ln\left(\frac{1}{1-4t^2}\right) - \ln\sqrt[3]{t} + \ln\sqrt[6]{1-2t} + \ln\sqrt[6]{1+2t} = \ln x^3$$

$$\Rightarrow \ln\left(\frac{\left(\sqrt[6]{1-2t}\right)\left(\sqrt[6]{1+2t}\right)}{\left(\sqrt[3]{t}\right)\left(1-4t^2\right)}\right) = \ln\left(x^3\right)$$

$$\Rightarrow \frac{\left(1-4t^2\right)^{1/6}}{t^{1/3}\cdot\left(1-4t^2\right)} = x^3$$

$$\Rightarrow \frac{\left(x^2-4y^2\right)^{1/6}\cdot \cancel{x^{1/3}}\cdot x^2}{y^{1/3}\cdot\left(x^2-4y^2\right)\cancel{x^{1/3}}} = x^3$$

$$\Rightarrow \frac{1}{\left(x^2-4y^2\right)^{5/6}\cdot y^{1/3}} = x$$

$$\Rightarrow x\cdot y^{1/3}\cdot\left(x^2-4y^2\right)^{5/6} = 1$$

$$\Rightarrow \boxed{x^6 y^2\left(x^2-4y^2\right)^5 = 1}$$

$$\Rightarrow \cancel{y^2\left(x^2-x^{6/5}-4y^2\right)}$$

$$\Rightarrow \cancel{xy^2\left(x^3-4y^2\right)}$$

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MODULE - 2

The General Second Order Linear DE is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} = R(x) - \varphi(x)\cdot y$$

$$y'' + P(x)\cdot y' = R(x) - \varphi(x)\cdot y$$

Ex. $y'' + y = 0$

Ex. $x^2y'' + 2xy' - 2y = 0$

$$\Rightarrow y'' + \left(\frac{2}{x}\right)y' - \left(\frac{2}{x^2}\right)y = 0$$

$$\left[y'' + P(x)y' + Q(x)y = R(x) \right]$$

H.W

Q) $$\frac{4y^2 - 2x^2}{4xy^2 - x^3}\,dx + \frac{8y^2 - x^2}{4y^3 - x^2y}\,dy = 0$$

(a) As Homogenous eq.

(b) As Exact eq.

Sol:

(a) $$\frac{dy}{dx} = \frac{\dfrac{-4y^2 + 2x^2}{4xy^2 - x^3}}{\dfrac{8y^2 - x^2}{4y^3 - x^2y}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(2x^2 - 4y^2)(4y^3 - x^2y)}{(4xy^2 - x^3)(8y^2 - x^2)}$$

$$= \frac{2y(x^2 - 2y^2)\cancel{(4y^2 - x^2)}}{x\cancel{(4y^2 - x^2)}(8y^2 - x^2)}$$

$$= \underbrace{\left(\frac{2y(x^2 - 2y^2)}{x(8y^2 - x^2)} \right)}_{f(x,y)}$$

$$f(tx, ty) = \frac{2ty(t^2x^2 - 2t^2y^2)}{tx(8t^2y^2 - t^2x^2)}$$

$$= \frac{2y(x^2 - 2y^2)}{x(8y^2 - x^2)}$$

$$= f(x,y) \cdot t^0$$

As $f(tx, ty) = f(x,y)$,

DE is a Homogenous DE.

Now,

$$y = zx$$

$$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{2y(x^2-2y^2)}{x(8y^2-x^2)}$$

$$= \frac{2(zx)(x^2-2z^2x^2)}{x(8z^2x^2-x^2)}$$

$$= \frac{2z\cancel{x^2}(1-2z^2)}{\cancel{x^2}(8z^2-1)}$$

$$= \frac{2z(1-2z^2)}{(8z^2-1)}$$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{2z(1-2z^2)}{(8z^2-1)}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{2z-4z^3-8z^3+z}{(8z^2-1)}$$

$$\Rightarrow \frac{dx}{x} = \frac{8z^2-1}{3z-12z^3}\,dz$$

$$\Rightarrow \int\frac{dx}{x} = \frac{1}{3}\int\frac{8z^2-1}{z(1-4z^2)}\,dz$$

$$\cancel{\Rightarrow \ln|x| + c = \frac{1}{3}\left[\int\frac{1-3z^2}{z(1-z^2)}\,dz - \frac{2+5z^2}{z(1-z^2)}\,dz\right]}$$

$$\Rightarrow \ln|x| + c = \frac{1}{3}\int\frac{8z^2-1}{(2z-1)(z)(2z+1)}\,dz$$

$$\ln|x| + c = \frac{1}{3}\left[\int \frac{1}{2x+1}\,dx + \int \frac{1}{2x-1}\,dx + \int \frac{1}{x}\,dx\right]$$

$$= \frac{1}{3}\left[\ln|2x+1| + \ln|2x-1| + \ln|x|\right]$$

$$= \frac{1}{3}\ln\left|\left[x(4x^2-1)\right]\right|$$

$$\ln|xc| = \ln\left(x^{1/3}\cdot(4x^2-1)^{1/3}\right)$$

$$\Rightarrow cx = x^{1/3}\cdot(4x^2-1)^{1/3}$$

$$\Rightarrow \boxed{cx^{2/3} = (4x^2-1)^{1/3}}$$

(b) $$\frac{4y^2-2x^2}{4xy^2-x^3}\,dx + \frac{8y^2-x^2}{4y^3-x^2y}\,dy = 0$$

$$\cancel{\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{4y^2}{4xy^2-x^3} - \frac{2x^2}{4xy^2-x^3}\right) = \frac{(4xy^2-x^3)(8y) - 4y^2(8xy)}{(4xy^2-x^3)^2}}$$

$$\frac{\partial M}{\partial y} = \frac{(4xy^2-x^3)(8y) - (4y^2-2x^2)(8xy)}{x^2(4y^2-x^2)^2} = \frac{x\cdot(16y-8y)}{(4y^2-x^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(4y^3-x^2y)(-2x) + (8y^2-x^2)(-2xy)}{y^2(4y^2-x^2)^2} = \frac{-8xy^3+16xy^3}{y^2(4y^2-x^2)^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$F = \int M\,dx + g(y)$$

$$\rightarrow F = \int \frac{4y^2 - x^2}{4xy^2 - x^3}\,dx + g(y)$$

$$= 2\int \frac{2y^2 - x^2}{x(2y^2 - x^2)}\,dx + g(y)$$

$$= 2\int \frac{2y^2 - x^2}{x(2y-x)(2y+x)}\,dx + g(y)$$

$$= 2\int \left(\frac{(1/4)}{2y+x} + \frac{(-1/4)}{2y-x} + \frac{1}{2x} \right) dx + g(y)$$

$$\Rightarrow F = \frac{1}{2}\ln|2y+x| - \frac{1}{2}\ln|2y-x| + \ln|x| + g(y)$$

$$\left[\frac{\partial f}{\partial y} = N \right]$$

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$$\frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)\cdot y = R(x)$$

- Existence and Uniqueness Theorem:

  For a Second order DE, Given that $P(x)$, $Q(x)$ and $R(x)$ are continuous functions in $[a,b]$.

  Two numbers y_0, y_0', $\exists$ unique solution, which satisfy

  $y(x_0) = y_0$ & $y'(x_0) = y_0'$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$\left[\begin{array}{l} P \to P(x) \\ Q \to Q(x) \\ R \to R(x) \end{array}\right]$$

If $R=0$, Second Order DE is a Homogenous DE

$$\frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)\cdot y = 0$$

Solve the Homogenous DE, We get:

$y_g(x, c_1, c_2)$

Then use y_1, y_2 to get General Equation

$y = y_p + y_g$

General Solution

Particular/specific solution

- $y'' + P(x)y' + Q(x)y = 0$

Source Term is Removed,

Still a 2nd order DE

$\therefore y_g = c_1y_1 + c_2y_2$

$y_p(x)$ is the solution of the original DE

i.e $y'' + P(x)y' + Q(x)y = R(x)$

$\therefore y = y_g(x) + y_p(x)$

$\Rightarrow \underline{\underline{y = c_1y_1 + c_2y_2 + y_p(x)}}$

Note:

$O(y_1+y_2) = O(y_1) + O(y_2)$

Where O is the operator

i.e. $O \equiv \left(\frac{d^2}{dx^2}(\,) + \frac{d}{dx}(\,) + Q(x)\right)y$ here

$$\left[\begin{array}{l} O(y_g) = 0 \\ O(y_p) = R \end{array}\right]$$

→ **Theorem**: (Existance & Uniqueness Theorem)

Let $P(x)$, $Q(x)$, $R(x)$ be continuous functions on a closed Interval $[a,b]$. If x_0 is any point in $[a,b]$ and y_0 and y_0' are any numbers, then

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q(x)\,y = R(x)$$

has one and only solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ & $y'(x_0) = y_0'$.

Ex. $y'' + y = 0$

$y(0) = 0$

$y'(0) = 1$

let $y = \sin x$, Substitute in DE:

$-\sin x + \sin x = 0$ ✓

let $y = \cos x$, Substitute in DE:

$-\cos x + \cos x = 0$ ✓

$\therefore y_g = c_1 \sin x + c_2 \cos x$

$y_g(x) = c_1 \sin x + c_2 \cos x$

$y_g(0) = c_1(0) + c_2(1)$

$\Rightarrow 0 = c_2$

$\boxed{\therefore c_2 = 0}$ $\because y_g(0) = 0$

$y_g'(x) = c_1 \cos x$

$y_g'(0) = c_1$

$\boxed{c_1 = 1}$ $\because y_g'(0) = 1$

$\Rightarrow \boxed{y_g(x) = \sin x}$

Ex. $y'' + y = 0$

$y(0) = 1$

$y'(0) = 0$

$y = \sin x \Rightarrow -\sin x + \sin x = 0$ ✓

$y = \cos x \Rightarrow -\cos x + \cos x = 0$ ✓

$\therefore y_g(x) = c_1 \sin x + c_2 \cos x$

$y_g(0) = c_1(0) + c_2(1)$

$\Rightarrow \boxed{c_2 = 1}$ $\quad \because y(0) = 1$

$y_g'(x) = c_1 \cos x - \sin x$

$y_g'(0) = c_1 - 0$

$\boxed{c_1 = 0}$ $\quad \because y'(0) = 0$

$\therefore \boxed{y_g(x) = \cos x}$

Note:

y_1 & y_2 should be Linearly Independent.

i.e. $y_2 \neq k y_1$

If $y_2 = k y_1$,

$y = c_1 y_1 + c_2 y_2$

$\Rightarrow y = c_1 y_1 + c_2 k y_1$

$= y_1 (c_1 + c_2 k)$

$\therefore y = c_3 y_1$ Not a General Solution

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Theorem:

Consider non-homogenous equation $y'' + P(x)y' + Q(x)\cdot y = R(x)$.

If $y_g \to$ general solution of the corresponding reduced homogenous ~~solution~~ equation $y'' + P(x)y' + Q(x)y = 0$

and $y_p \to$ particular solution of the given corresponding complete given non-homogenous equation,

then $y_g + y_p$ is the complete solution

Theorem:

If $y_1(x)$ and $y_2(x)$ are two solutions of the Homogenous equation.

$$y'' + P(x)y' + Q(x)y = 0$$

Then

$$c_1 y_1(x) + c_2 y_2(x)$$

is also a solution for any constants c_1 & c_2 on $[a,b]$

Proof: $(c_1 y_1)'' + P(x)(c_1 y_1)' + Q(x)(c_1 y_1) = 0$

$$\Rightarrow c_1 [y_1'' + P(x)y_1' + Q(x)y_1] = 0 \quad -(1)$$

$$(c_2 y_2)'' + P(x)(c_2 y_2)' + Q(x)(c_2 y_2) = 0$$

$$\Rightarrow c_2 [y_2'' + P(x)y_2'' + Q(x)y_2] = 0 \quad -(2)$$

(1) + (2):

$$c_1 [y_1'' + P(x)y_1' + Q(x)y_1] + c_2 [y_2'' + P(x)y_2' + Q(x)y_2] = 0$$

$$\Rightarrow (c_1 y_1'' + c_2 y_2'') + P(x)[c_1 y_1' + c_2 y_2'] + Q(x)[c_1 y_1 + c_2 y_2] = 0$$

$$\Rightarrow (c_1 y_1 + c_2 y_2)'' + P(x)(c_1 y_1 + c_2 y_2)' + Q(x)(c_1 y_1 + c_2 y_2) = 0$$

$\therefore c_1 y_1 + c_2 y_2$ is also a solution for the given homogenous equation.

Note :

(1) $c_1 y_1(x) + c_2 y_2(x)$ is AKA linear combination of $y_1(x)$ and $y_2(x)$.

Solve : $y'' + y' = 0$

$$\left.\begin{aligned} y_1 &= 1 \\ y_2 &= e^{-x} \end{aligned}\right\} \text{Solution}$$

$$y = c_1 + c_2 e^{-x}$$

Solve : $x^2 y'' + 2xy' - 2y = 0$

$y_1 = x^n$ & $y_2 = x$

$\hookrightarrow x^2 (n-2) x^{n-2} (n-1) + 2x (n-1) x^{n-1} - 2x^n = 0$

$\Rightarrow x^n (n-1)(n-2) + x^n (2(n-1)) - x^n (2) = 0$

$\Rightarrow (n-1)(n-2) + 2(n-1) - 2 = 0$

$\Rightarrow (n-1)(n) - 2 = 0$

$\Rightarrow n^2 - n - 2 = 0$

~~$\Rightarrow (n+1)(n+2) = 0$~~

$\Rightarrow (n+1)(n-2) = 0$

$n = -1$ (or) $n = 2$

$x^2 (n-1)(n) x^{n-2} + 2x n x^{n-1} - 2x^n = 0$

$\Rightarrow x^n (n(n-1)) + x^n (2n) - x^n (2) = 0$

$\Rightarrow n(n-1) + 2n - 2 = 0$

$\hookrightarrow n = 1$ (or) $n = -2$

Given : y_1, y_2 &

$$y'' + Py' + Qy = R$$

$y(x_0) = y_0$
$y'(x_0) = y_0'$

[y_0 & y_0' are two different constants.]

Finding Sol :

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0)$$

$$y'(x_0) = c_1 \cdot y_1'(x_0) + c_2 y_2'(x_0)$$

To get a Trivial solution,

$$\begin{cases} ax_1 + by_1 = k_1 \\ cx_1 + dy_1 = k_2 \end{cases}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

$$\frac{a}{b} \neq \frac{c}{d}$$

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

$$\Rightarrow y_1(x_0) \cdot y_2'(x_0) - y_2(x_0) \cdot y_1'(x_0) \neq 0$$

Wronskian function

$$W(x) = y_1(x) \cdot y_2'(x) - y_2(x) \cdot y_1'(x)$$

$$[\therefore W(x_0) \neq 0]$$

$$a_1(x), a_2(x), a_3(x) \rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1' & a_2' & a_3' \\ a_1'' & a_2'' & a_3'' \end{vmatrix}$$

Used for Higher order differential eqs

~~functions~~

Lemma 1:

If $y_1(x)$ & $y_2(x)$ $\rightarrow$ two solutions of Homogenous linear equations $y'' + P(x)y' + Q(x)y = 0$ on the interval $[a,b]$ and their Wronskian $W(y_1, y_2)$ is either always 0 or never 0.

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$$W = y_1 y_2' - y_2 y_1'$$

$$W' = y_1 y_2'' + y_2' y_1' - y_2 y_1'' - y_2' y_1'$$

$$\Rightarrow \boxed{W' = y_1 y_2'' - y_2 y_1''}$$

$$y_1'' + P(x) y_1' + Q(x) y_1 = 0 \quad -(1)$$

$$y_2'' + P(x) y_2' + Q(x) y_2 = 0 \quad -(2)$$

$(1) \times y_2 = (3)$

$(2) \times y_1 = (4)$

$$y_2 y_1'' + P(x) y_2 y_1' + Q(x) y_1 y_2 = 0 \quad -(3)$$

$$y_1 y_2'' + P(x) y_1 y_2' + Q(x) y_1 y_2 = 0 \quad -(4)$$

$(4) - (3)$:

$$P(x)\left[y_1 y_2' - y_2 y_1'\right] = y_2 y_1'' - y_1 y_2''$$

$$\Rightarrow P(x) = -\frac{y_1 y_2'' - y_2 y_1''}{y_1 y_2' - y_2 y_1'}$$

$$\boxed{\therefore P(x) = -\frac{W'}{W}}$$

(OR) $W' + P(x) \cdot W = 0$

$$\frac{dW}{dx} + P(x) \cdot W = 0 \quad \cancel{[LDE]}$$

$$\Rightarrow W = c \cdot e^{\int -P \cdot dx} \qquad [c = W(x_0)] \to W_0$$

If $c = 0 \Rightarrow W = 0$

$c \neq 0 \Rightarrow W \neq 0$

$\therefore$ Wronskian's Function can be either

1. Zero (or) never zero.

Lemma 2:

If $y_1(x)$ and $y_2(x)$ are any two solutions of Homogenous linear equation $y'' + P(x)y' + Q(x)y = 0$ on the interval $[a,b]$ ~~and Wronskian~~

$W = 0$ if and only if y_1 & y_2 are linearly dependent.

Sol: (i) L.D $\Rightarrow W = 0$:

y_1 and y_2 are linearly dependent

$$\left.\begin{aligned} y_2 &= ky_1 \\ y_2' &= ky_1' \end{aligned}\right\} \text{ in } (c,d) \in (a,b)$$

By Existential Theorem

$\therefore$ Proved for (a,b)

$$W = y_1 y_2' - y_2 y_1'$$

$$\Rightarrow W = y_1(ky_1' - ky_1 y_1')$$

$$= ky_1 y_1' - ky_1 y_1'$$

$$= 0$$

$$\boxed{\therefore W = 0}$$

(ii) $W = 0 \Rightarrow$ L.D

$$\frac{y_2}{y_1} = k \Rightarrow \left(\frac{y_2}{y_1}\right)' = 0$$

$$\Rightarrow \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0 \qquad y_1 \neq 0$$

$$\Rightarrow y_1 y_2' - y_2 y_1' = 0$$

$$\therefore \underline{\underline{W = 0}}$$

Hence Proved

$$\frac{y_2}{y_1} = k$$

$$\Rightarrow y_2 = k y_1$$

$$y_2' = k y_1'(x_0)$$

$$y_2 = k y_1(x_0)$$

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ASSIGNMENT - I

25/02/24

① Given Curve :

$$x^2+y^2 = 2cx$$

Solution :

$$x^2+y^2=2cx$$

Differentiating the whole equation,

$$\frac{d}{dx}(x^2+y^2=2cx)$$

$$\Rightarrow \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(2cx)$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 2c$$

$$\Rightarrow x + y\frac{dy}{dx} = c$$

$$\Rightarrow \boxed{c = x + y\frac{dy}{dx}}$$

Substitute this in the original equation,

$$x^2+y^2 = 2cx$$

$$\Rightarrow x^2+y^2 = 2\left(x+y\frac{dy}{dx}\right)x$$

$$\Rightarrow x^2+y^2 = 2x^2 + 2xy\frac{dy}{dx}$$

$$\Rightarrow y^2-x^2 = 2xy\frac{dy}{dx}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{y^2-x^2}{2xy}}$$

Now, To find Orthogonal Trajectory to the given family of curves, we replace $\frac{dy}{dx}$ with $\frac{-dx}{dy}$

$$\frac{-dx}{dy} = \frac{y^2-x^2}{2xy} \qquad [\because m_1m_2 = -1]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

let $f(x,y) = \dfrac{2xy}{x^2-y^2}$

Now $f(tx, ty) = \dfrac{2(tx)(ty)}{(tx)^2-(ty)^2}$

$$= \frac{2t^2xy}{t^2(x^2-y^2)}$$

$$= \frac{2xy}{x^2-y^2}$$

$$= f(x,y)$$

$\therefore f(tx, ty) = t^0 \cdot f(x,y)$

$f(x,y)$ is a Homogenous equation of Degree 0.

And $\dfrac{dy}{dx} = \dfrac{2xy}{x^2-y^2}$ is a Homogenous D.E.

$$\left[\begin{array}{l} \therefore y = zx \\ \Rightarrow \dfrac{dy}{dx} = z + x\dfrac{dz}{dx} \end{array}\right]$$

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{2x(zx)}{x^2-z^2x^2}$$

$$\Rightarrow x\left(\frac{dz}{dx}\right) = \frac{2\cancel{x}(\cancel{x^2})}{\cancel{x^2}(1-z^2)} - z$$

$$\Rightarrow x\left(\frac{dz}{dx}\right) = \frac{2z}{1-z^2} - \frac{z-z^3}{1-z^2}$$

$$\Rightarrow x\left(\frac{dz}{dx}\right) = \frac{z+z^3}{1-z^2} = \frac{z(1+z^2)}{(1-z^2)}$$

$$\Rightarrow \frac{dx}{x} = \frac{1-z^2}{z(1+z^2)}\,dz$$

Integrating both sides, we get

$$\Rightarrow \int \frac{1}{x}\,dx = \int \frac{1-z^2}{z(1+z^2)}\,dz \qquad [z, x \in \mathbb{R}]$$

$$\Rightarrow \ln|x| + c = \int \left(\frac{1}{z} - \frac{2z}{1+z^2}\right) dz \qquad [c \in \mathbb{R}]$$

[from Partial fractions]

$$\Rightarrow \ln|x| + c = \ln|z| - \ln|1+z^2| + c' \qquad [c' \in \mathbb{R}]$$

$$\Rightarrow \ln|x| + C = \ln|z| - \ln(1+z^2)$$

[As $z = y/x$]

$$\Rightarrow \ln|x| + C = \ln|y/x| - \ln\left(1 + y^2/x^2\right)$$

$$\Rightarrow \boxed{\ln\left(1+\frac{y^2}{x^2}\right) - \ln|y/x| + \ln|x| = C} \qquad [C \in \mathbb{R}]$$

is the family of orthogonal Trajectories of the given family of curves.

② (a) Given :

$$\frac{dy}{dx} = \frac{x+y-1}{x+4y+2}$$

~~Let~~

As $\frac{1}{1} \neq \frac{1}{4}$ [Coefficients of x and y]

We assume $x = X+h$

$y = Y+k$

to eliminate constant terms from both numerator and denominator

$x = X + h$ | $y = Y + k$

$\Rightarrow dx = dX$ | $\Rightarrow dy = dY$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y-1}{x+4y+2}$$

$$\Rightarrow \frac{dY}{dX} = \frac{X+h+Y+k-1}{X+h+4(Y+k)+2}$$

$$= \frac{X+Y+(h+k-1)}{X+4Y+(h+4k+2)}$$

Now, $h+k-1=0$ &

$h+4k+2=0$

Subtracting both, we get

$3k+3=0$

$\Rightarrow \boxed{k=-1}$

Putting back in any of the equations,

$\boxed{h=2}$

$\therefore x = X+2$

$y = Y-1$

$$\therefore \frac{dY}{dX} = \frac{X+Y}{X+4Y}$$

Let $f(X,Y) = \dfrac{X+Y}{X+4Y}$

Now $f(tX,tY) = \dfrac{tX+tY}{tX+4tY}$

$$= \frac{X+Y}{X+4Y}$$

$$= f(X,Y)$$

As $f(tX,tY) = t^0 \cdot f(X,Y)$

$f(x,y)$ is a Homogenous equation of degree 0.

Hence $\frac{dy}{dx} = \frac{x+y}{x+4y}$ is a Homogenous DE

$$y = zx$$

$$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x+4y}$$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{x + x(z)}{x+4(zx)}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{x(1+z)}{x(1+4(z))} - z$$

$$\Rightarrow x\frac{dz}{dx} = \frac{1+z}{1+4(z)} - \frac{z+4z^2}{1+4(z)}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{1-4z^2}{1+4(z)}$$

$$\Rightarrow \frac{dx}{x} = \frac{1+4(z)}{1-4z^2}\,dz$$

Integrating both sides

$$\Rightarrow \int \frac{dx}{x} = \int \frac{1+4(z)}{1-4(z^2)}\,dz$$

$$\Rightarrow \ln|x| + c = \int \frac{dz}{1-4(z^2)} + 4\int \frac{z}{1-4(z^2)}\,dz$$

$$\Rightarrow \ln|x-2| + c = \int \frac{dz}{(1-2z)(1+2z)} - \frac{1}{2}\int \frac{-8z}{1-4(z^2)}\,dz$$

$[c \in \mathbb{R}]$

$$\Rightarrow \ln|x-2| + c = \frac{1}{2}\int \frac{dz}{2(z)+1} - \frac{1}{2}\int \frac{dz}{2(z)-1} - \frac{1}{2}\ln|1-4(z^2)| + c'$$

$[c' \in \mathbb{R}]$

$$\Rightarrow \ln|x-2| + C = \frac{1}{4}\ln|2(z)+1| - \frac{1}{4}\ln|2(z)-1| - \frac{1}{2}\ln|1-4(z^2)|$$

$$= \frac{1}{4}\ln\left|\frac{2y}{x}+1\right| - \frac{1}{4}\ln\left|2\frac{y}{x}-1\right| - \frac{1}{2}\ln\left|1-4\frac{y^2}{x^2}\right|$$

$$= \frac{1}{4}\ln\left|2\left(\frac{y+1}{x-2}\right)+1\right| - \frac{1}{4}\ln\left|2\left(\frac{y+1}{x-2}\right)-1\right| - \frac{1}{2}\ln\left|1-4\left(\frac{y+1}{x-2}\right)^2\right|$$

$$\therefore \ln|x-2| - \frac{1}{4}\ln\left|2\left(\frac{y+1}{x-2}\right)+1\right| + \frac{1}{4}\ln\left|2\left(\frac{y+1}{x-2}\right)-1\right| + \frac{1}{2}\ln\left|1-4\left(\frac{y+1}{x-2}\right)^2\right| = C$$

is the solution of the given DE.

$[C \in \mathbb{R}]$

(b) Given:

$$\frac{dy}{dx} = \frac{x+y+4}{x+y-6}$$

As $\frac{1}{1} = \frac{1}{1}$, [Coefficients of x and y]

We assume $x+y = z$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y+4}{x+y-6}$$

$$\Rightarrow \frac{dz}{dx} - 1 = \frac{z+4}{z-6}$$

$$\Rightarrow \frac{dz}{dx} = \frac{z+4+z-6}{z-6}$$

$$= \frac{2z-2}{z-6}$$

$$\Rightarrow dx = \frac{z-6}{2(z-1)}\,dz$$

Now, Integrating both sides,

$$\int dx = \int \frac{z-6}{2(z-1)}\, dz$$

$$\Rightarrow x + c = \frac{1}{2}\int \frac{z-6}{z-1}\, dz \qquad [c \in \mathbb{R}]$$

$$= \frac{1}{2}\int \left(1 - \frac{5}{z-1}\right) dz$$

$$= \frac{1}{2}\left[z - 5\ln|z-1| + c'\right] \qquad [c' \in \mathbb{R}]$$

$$\Rightarrow x + C = \frac{1}{2}\left[z - 5\ln|z-1|\right]$$

$$\Rightarrow x + C = \frac{1}{2}\left[x+y - 5\ln|x+y-1|\right]$$

$$\Rightarrow \frac{1}{2}\left[x - y + 5\ln|x+y-1|\right] = c$$

$$\boxed{x - y + 5\ln|x+y-1| = c} \qquad [C \in \mathbb{R}]$$

is the solution of the given DE.

③ Given:

$$M\,dx + N\,dy = 0$$

$$z = x+y$$

Solution:

It is given that the DE '$M\,dx + N\,dy = 0$' has an Integrating factor of function 'z'.

i.e. $\mu(z) = e^{\int g(z)dz}$

Let us assume that the given DE is not exact. Hence we multiply Integrating factor on both sides and make it exact.

Let the Integrating factor be μ.

$M\,dx + N\,dy = 0$

$\Rightarrow \mu M\,dx + \mu N\,dy = 0$

Now, There exists a function f such that

$\frac{\partial f}{\partial x} = \mu M$ & $\frac{\partial f}{\partial y} = \mu N$ as the

DE now is exact.

And $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$ [Condition for exact DE]

$$\Rightarrow \mu \cdot \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \cdot \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

$$\Rightarrow M\left(\frac{\partial \mu}{\partial y}\right) - N\left(\frac{\partial \mu}{\partial x}\right) = \mu\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{\mu}\left[M\left(\frac{\partial \mu}{\partial y}\right) - N\left(\frac{\partial \mu}{\partial x}\right)\right]$$

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left[N\left(\frac{\partial \mu}{\partial x}\right) - M\left(\frac{\partial \mu}{\partial y}\right)\right]$$

We assumed μ to be function of 'z' alone.

As $z = x + y$,

$$\frac{\partial z}{\partial x} = 1 \quad \Big| \quad \frac{\partial z}{\partial y} = 1$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left[N\left(\frac{\partial \mu}{\partial z} \cdot \frac{\partial z}{\partial x}\right) - M\left(\frac{\partial \mu}{\partial z} \cdot \frac{\partial z}{\partial y}\right)\right]$$

$$= \frac{1}{\mu}\left[N\left(\frac{\partial \mu}{\partial z}\right) - M\left(\frac{\partial \mu}{\partial z}\right)\right]$$

$$= \frac{1}{\mu}(N - M)\left(\frac{\partial \mu}{\partial z}\right)$$

$$\Rightarrow \left(\frac{1}{N-M}\right)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\mu}\frac{\partial \mu}{\partial z}$$

As the RHS, ~~Eq.~~ i.e. $\frac{1}{\mu}\frac{\partial \mu}{\partial z}$ is purely a function of 'z',

$\therefore$ The LHS must also be purely a function of 'z'

i.e.,

$$\left(\frac{1}{N-M}\right)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \text{ must be a}$$

function of 'z' alone

$$\therefore g(z) = \frac{\partial M/\partial y - \partial N/\partial x}{N-M}$$

And $\mu(z) = e^{\int g(z)\,dz}$

Hence the condition under which the DE $M\,dx + N\,dy = 0$ has an IF that is a function of $z(x,y) = x+y$ is that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N-M} \text{ must be a function of } z.$$

(4) Given:

$$\frac{dy}{dx} + P(x)\cdot y = \varphi(x)$$

Solution:

$$\frac{dy}{dx} + P(x)\cdot y = \varphi(x)$$

$$\Rightarrow dy + dx\left(P(x)\cdot y\right) = dx\left(\varphi(x)\right)$$

$$\Rightarrow \left[P(x)\cdot y - \varphi(x)\right]dx + [1]\,dy = 0$$

The given DE has been converted to the form $M\,dx + N\,dy = 0$,

where,

$$M = P(x) \cdot y - \varphi(x)$$
$$N = 1$$

Checking whether this DE (converted) is an exact DE or not,

$$\underbrace{\frac{\partial M}{\partial y}} = \underbrace{\frac{\partial N}{\partial x}} \quad \text{(Check)}$$

Exactness Condition

$$\Rightarrow \underbrace{P(x) - 0} = \underbrace{0}$$

The ~~e~~ DE is only exact if $P(x) = 0$. Which is not true generally.

Assuming $P(x) \neq 0$,

We have an Integrating factor with which we multiply and then convert the non-exact DE to an exact DE.

$$M\,dx + N\,dy = 0$$

$$\Rightarrow \left[P(x) \cdot y - \varphi(x)\right] dx + dy = 0$$

$$\Rightarrow \mu\left[P(x) \cdot y - \varphi(x)\right] + \mu\,dy = 0 \qquad [\mu : \text{Integrating factor}]$$

Now, As this is an exact DE,

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\Rightarrow \mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

$$\Rightarrow M \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial x} = \mu \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left(N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y}\right)$$

As we know,

$$\frac{\partial M}{\partial y} = P(x) \quad \& \quad \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right)$$

$$\Rightarrow P(x) = \frac{1}{\mu}\left[\frac{\partial \mu}{\partial x} - \left(P(x)\cdot y - \varphi(x)\right)\frac{\partial \mu}{\partial y}\right]$$

LHS is always purely a function of 'x'.

$\therefore$ RHS must also be a function of x alone.

$\therefore$ μ cannot be a function of y alone.

μ has to be a function of x alone

$$\Rightarrow \underbrace{P(x)}_{\text{function of } x} = \underbrace{\frac{1}{\mu}\left(\frac{\partial \mu}{\partial x}\right)}_{\text{function of } x}$$

$$\boxed{\therefore \mu(x) = e^{\int P(x)\,dx}}$$

$$\left[\begin{array}{l} \because P(x)\,dx = \dfrac{d\mu}{\mu} \\ \Rightarrow \int P(x)\,dx = \ln|\mu(x)| + c \\ \Rightarrow \mu(x) = e^{\int P(x)\,dx} \end{array}\right] [c \in \mathbb{R}]$$

Solving :

$$\mu M\,dx + \mu N\,dy = 0$$

$$F = \int \mu M\,dx + g(y) \qquad \left[\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}\right]$$

$$\Rightarrow F = \int e^{\int P(x)\,dx} \cdot \left[P(x)\cdot y - \varphi(x)\right] dx + g(y)$$

$$\frac{\partial f}{\partial y} = \mu N = e^{\int P(x)\,dx} \quad - (1)$$

$$\& \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int e^{\int P(x)\,dx} \cdot (P(x)\cdot y - \varphi(x))\,dx\right] + g'(y) \quad - (2)$$

As (1) = (2)

$$\frac{\partial}{\partial y}\left[y\int e^{\int P\,dx} \cdot P\,dx - \int e^{\int P\,dx} \cdot \varphi\,dx\right] + g'(y) = e^{\int P(x)\,dx}$$

$$\Rightarrow \int e^{\int P(x)\,dx} \cdot P\,dx + g'(y) = e^{\int P(x)\,dx}$$

$$\left[\begin{array}{l} \text{Let } t = e^{\int P(x)\,dx} \\ dt = e^{\int P(x)\,dx} \cdot P(x)\,dx \end{array}\right]$$

$$\Rightarrow \int dt + g'(y) = e^{\int P(x)\,dx}$$

$$\Rightarrow \cancel{e^{\int P(x)\,dx}} + g'(y) = \cancel{e^{\int P(x)\,dx}}$$

$$\therefore g'(y) = 0$$

$$\Rightarrow g(y) = k \qquad [k \in \mathbb{R}]$$

$$\therefore F = \int e^{\int P(x)\,dx} \cdot [P(x)\cdot y - \varphi(x)]\,dx + k$$

The Solution, $f(x,y) = C$, $C \in \mathbb{R}$

$$\therefore \int e^{\int P(x)\,dx} \cdot [P(x)\cdot y - \varphi(x)]\,dx = C \qquad [C \in \mathbb{R}]$$

$$\Rightarrow \int e^{\int P(x)\,dx} \cdot P(x)\cdot y\,dx - \int e^{\int P(x)\,dx} \cdot \varphi(x)\,dx = C$$

$$\Rightarrow y \cdot e^{\int P(x)\,dx} - \int \varphi(x) \cdot e^{\int P(x)\,dx}\,dx = C$$

$$\boxed{\therefore y(x) = e^{-\int P(x)\,dx}\left[\int \varphi(x) \cdot e^{\int P(x)\,dx}\,dx + c\right]}$$ is the solution

⑤ Given:

$$xy' + y = y^{-2}$$

Solution:

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{x}\right) = \frac{1}{x}(y^{-2})$$

This is a form of Bernoulli's Equation.

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{1}{x}(y^{-2})$$

$$\left[\frac{dy}{dx} + P(x)\,y = \varphi(x)\cdot y^n\right] \quad [z = y^{1-n}]$$

$$z = y^{1-(-2)}$$

$$\Rightarrow z = y^3$$

$$\left[\frac{dz}{dx} = 3y^2\frac{dy}{dx}\right]$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{1}{x}(y^{-2})$$

$$\Rightarrow y^2\frac{dy}{dx} + \frac{1}{x}y^3 = \frac{1}{x}$$

$$\Rightarrow 3y^2\frac{dy}{dx} + \frac{3}{x}y^3 = \frac{3}{x}$$

$$\Rightarrow \frac{dz}{dx} + \left(\frac{3}{x}\right)z = \frac{3}{x}$$

Now $P(x) = \varphi(x) = \frac{3}{x}$

Integrating factor : $e^{\int P(x)\,dx}$

$$= e^{\int \frac{3}{x}dx}$$

$$= e^{3\ln|x|} = x^3$$

Solution:

$$z \cdot x^3 = \int \frac{3}{x} \cdot x^3 \, dx + c \qquad [c \in \mathbb{R}]$$

$$\Rightarrow z \cdot x^3 = \int 3x^2 \, dx + c$$

$$\Rightarrow z \cdot x^3 = x^3 + c$$

$$\Rightarrow y^3 x^3 = x^3 + c$$

$$\boxed{\therefore x^3(y^3 - 1) = c}$$

is the solution of the given DE.

① Continuing the solution,

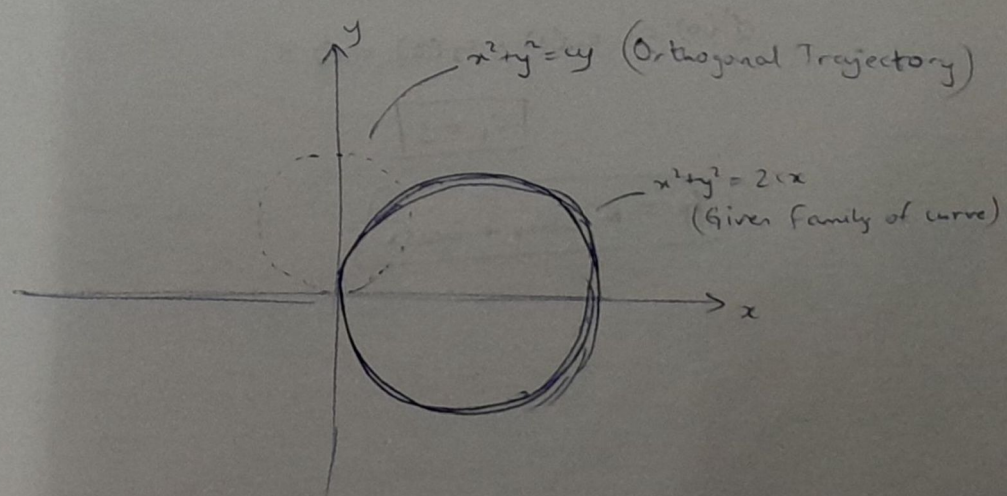
$$\ln\left(1 + \frac{y^2}{x^2}\right) - \ln\left|\frac{y}{x}\right| + \ln|x| = c$$

$$\Rightarrow \ln\left(\frac{x^2+y^2}{x^2}\right) - \ln|y| + \ln|x^2| = c$$

$$\Rightarrow \ln(x^2+y^2) - \ln(y) = \ln(c)$$

$$\Rightarrow \frac{x^2+y^2}{y} = c$$

$$\Rightarrow \boxed{x^2+y^2 = cy}$$



Q) Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any Interval &

Also find y_p if $y(0) = 2$ and $y'(0) = 3$

Sol:

① Check: y_1 & y_2 are solutions

$y_1 = \sin x$	$y_2 = \cos x$
$y_1' = -\sin x$	$y_2'' = -\cos x$
$y_1'' + y_1 = 0$ ✓	$y_2'' + y_2 = 0$ ✓

② $W(y_1, y_2)$

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x$$

$$= -1(\sin^2 x + \cos^2 x)$$

$$= -1 \cdot 1$$

$$= -1$$

Thus, 'sinx' and 'cosx' are Linearly Independent

③ Now, Solve for $y(0)$ and $y'(0)$

$$y(0) = c_1(0) + c_2(1) = 2$$

$$\boxed{c_2 = 2}$$

$$y'(0) = c_1(1) - c_2(0) = 3$$

$$\boxed{c_1 = 3}$$

$$\boxed{y_p = 3\sin x + 2\cos x}$$

Q) Show that $c_1e^{x} + c_2e^{2x}$ is the general solution of

$y'' - 3y' + 2y = 0$ &

Also find y_p if $y(0) = -1$ and $y'(0) = 1$

Sol:

① Check whether y_1 and y_2 are solutions

$y_1 = e^x$	$y_2 = e^{2x}$
$y_1'' = e^x$	$y_2' = 2e^{2x}$
$y_1' = e^x$	$y_2'' = 4e^{2x}$
$y_1'' - 3y_1' + 2y_1 = 0$ ✓	$y_2'' - 3y_2' + 2y_2 = 0$ ✓

② $W(y_1, y_2)$

$$\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^x.e^{2x} - e^{2x}.e^x$$

$$= e^x.e^{2x}$$

$$= e^{3x}$$

Never zero

$\therefore$ 'e^x' and 'e^{2x}' are L. Independent

③ Now solve for $y(0)$ and $y'(0)$

$y(0) = c_1 + c_2 = -1$

$\boxed{c_1 + c_2 = -1}$

$y'(0) = c_1 + 2c_2 = 1$

$\boxed{c_1 + 2c_2 = 1}$

$\rightarrow c_2 = 2$

$c_1 = -3$

$$\boxed{\therefore y_p = -3e^x + 2e^{2x}}$$

→ The Use of known solution to find another;

Let y_1 be solution of $y'' + P(x)y' + Q(x)y = 0$

let $y_2 = Vy_1$,

where $V \to$ non-constant function $\{V(x)\}$

If $y_2 \to$ solution, then:

$$y_2'' + P(x) \cdot y_2' + Q(x) y_2 = 0$$

$$\Rightarrow (Vy_1)'' + P(x) \cdot (Vy_1)' + Q(x) \cdot (Vy_1) = 0$$

$$\Rightarrow (Vy_1' + V'y_1)' + P(x) \cdot [Vy_1' + V'y_1] + Q(x) \cdot Vy_1 = 0$$

$$\Rightarrow (Vy_1'' + V'y_1' + V''y_1 + V'y_1') + P(x)[Vy_1' + V'y_1] + Q(x) \cdot Vy_1 = 0$$

$$\Rightarrow (Vy_1'' + 2V'y_1' + V''y_1) + P(x)[Vy_1' + V'y_1] + Q(x) \cdot (Vy_1) = 0$$

$$\Rightarrow V\underbrace{[y_1'' + P(x)y_1' + Q(x)y]}_{0} + 2V'y_1' + P(x)y_1V' + y_1V'' = 0$$

$$\Rightarrow 2V'y_1' + P(x)(y_1V') + V''y_1 = 0$$

$$\Rightarrow V''y_1 + V'[2y_1' + P(x) \cdot y_1] = 0$$

$$\Rightarrow V'' + V'\left[2\left(\frac{y_1'}{y_1}\right) + P(x)\right] = 0$$

$$\Rightarrow t' + t\left[2\left(\frac{y_1'}{y_1}\right) + P(x)\right] = 0 \qquad \text{Let } t = V'$$

$$\Rightarrow t' = -t\left[P(x) + 2\left(\frac{y_1'}{y_1}\right)\right]$$

$$\Rightarrow dt = -t\left[P(x) + 2\left(\frac{y_1'}{y_1}\right)\right] dx$$

$$\Rightarrow \frac{dt}{t} = -\left(P(x) + 2\left(\frac{y_1'}{y_1}\right)\right) dx$$

Integrating both sides,

$$\Rightarrow \ln|t| = -\int P(x)\,dx - 2\int \frac{dy_1}{y_1} \cdot y_1'$$

$$= -\int P(x)\,dx - 2\ln|y_1|$$

$$\Rightarrow \ln(t) = -\int P(x)\,dx - \ln(y_1^2)$$

$$\therefore t = e^{-\left(\int P(x)\,dx + \ln(y_1^2)\right)}$$

$$\boxed{\therefore t = e^{-\int P(x)\,dx} \cdot \frac{1}{y_1^2}}$$

$$\therefore V = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\,dx$$

Q) Solve :

$$x^2y'' + xy' - y = 0$$

Sol : $\Rightarrow y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$

By Inspection, $y = x$ is a solution

$$V(x) = \int \frac{1}{y_1^2(x)} \cdot e^{-\int P(x)\,dx}\,dx$$

$$= \int \frac{1}{x^2} \cdot e^{-\int \frac{1}{x}dx}\,dx$$

$$= \int \frac{1}{x^2} \cdot \frac{1}{x}\,dx$$

$$= \int \frac{1}{x^3}\,dx$$

$$= \frac{-1}{2x^2}$$

$$\boxed{\therefore V(x) = \frac{-1}{2x^2}}$$

2nd Linearly Independent Solution is $V \cdot y_1(x)$

$$\therefore y_2 = \frac{-1}{2x^2} \cdot x$$

$$\boxed{\therefore y_2 = \frac{-1}{2x}}$$ [$-\frac{1}{2}$ can be the const.]

Solutions : x & $\frac{1}{x}$

$$\boxed{y_p(x) = c_1 x + c_2 x^{-1}}$$

<u>Note</u> :

$$y'' + P(x) \cdot y' + Q(x)\, y = 0 \qquad \{R(x) = 0\}$$

$$\hookrightarrow \quad V(x) = \int \frac{1}{y_1^2(x)} \cdot e^{-\int P(x)\,dx}\, dx$$

$$y_2(x) = V(x) \cdot y_1(x)$$

If $P(x)$ and $Q(x)$ are constants,

$$y'' + py' + qy = 0$$

$\hookrightarrow$ Possible solution : e^{mx}

$$\Rightarrow m^2 e^{mx} + m e^{mx} \cdot p + q e^{mx} = 0$$

$$\Rightarrow (m^2 + mp + q)\, \underbrace{e^{mx}}_{} = 0$$

$\hookrightarrow$ Cannot be 0

$$m^2 + pm + q = 0 \Rightarrow m = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$\therefore y'' + py' + qy = 0$ has solutions $e^{m_1 x}$ and $e^{m_2 x}$

where m_1, m_2 are $\frac{-p \pm \sqrt{p^2 - 4q}}{2}$

Possibilities : ① $p^2-4q>0$: 2 distinct solutions (Linearly Independent)

② $p^2-4q<0$: 2 Imaginary solutions

$$\frac{-p \pm \sqrt{p^2-4q}}{2} = \frac{-p \pm i\sqrt{4q-p^2}}{2}$$

$$= \frac{-p}{2} \pm i\,\frac{\sqrt{4q-p^2}}{2}$$

$$= a \pm ib$$

2 solutions : $e^{(a+ib)x}$ and $e^{(a-ib)x}$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 e^{ax} \cdot e^{ibx} + c_2 \cdot e^{ax} \cdot e^{-ibx}$$

$$y(x) = e^{ax}\left[c_1 e^{ibx} + c_2 e^{-ibx}\right]$$

$$= e^{ax}\left[c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)\right]$$

$$= e^{ax}\left[\cos bx(c_1+c_2) + \sin bx(ic_1 - ic_2)\right]$$

$$= e^{ax}\left[c_1'\cos bx + c_2'\sin bx\right]$$

③ $p^2-4q=0$

$\Rightarrow m_1 = m_2 \Rightarrow$ only 1 Linearly Independent

So, use

$$y_2(x) = y_1(x) \cdot v(x)$$

where $v(x) = \int \frac{1}{y_1^2(x)} \cdot e^{-\int P(x)\,dx}\,dx$

$$= \int \frac{1}{y_1^2(x)} \cdot e^{-px}\,dx$$

$$= \int \frac{1}{e^{2m_1 x}} e^{-px}\,dx$$

$\left[m_1 = \frac{-p}{2}\right]$

from the Quadratic eq. solution

$$= \int e^{-(p+2m_1)x}\,dx$$

$$= \int e^{-(p-p)x}\,dx = x$$

$y_2 = xe^{m_1 x}$ &

$y_1 = e^{m_1 x}$

$\therefore y(x) = c_1 e^{m_1 x} + c_2 e^{m_1 x} \cdot x$

$\therefore y(x) = e^{m_1 x}(c_1 + x c_2)$

~~Auxiallary~~

Auxiliary Equation: $(m^2 + pm + q)e^{mx} = 0$

Equation obtained when solution is substituted.

Ex. $y'' + \underbrace{2}_{p} y' + \underbrace{1}_{q} y = 0$

$m^2 e^{mx} + 2e^{mx} \cdot m + e^{mx} = 0$

$\Rightarrow m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0$

$\underline{\underline{m = -1}} \Longrightarrow e^{-x} = y_1(x)$

$$v(x) = \int \frac{1}{e^{-2x}} \cdot e^{-\int -2\,dx}\, dx$$

$$= \int \frac{1}{e^{-2x}} \cdot e^{-2x}\, dx$$

$$= x$$

$y_2(x) = xe^{-x}$

$$\boxed{y(x) = c_1 e^{-x} + c_2 x e^{-x}}$$

Q) $y'' - 9y' + 20y = 0$

Sol: $m^2 - 9m + 20 = 0$

$\Rightarrow (m-5)(m-4) = 0$

$\therefore m = 5$ (or) $m = 4$

20

-4 -5

$$\boxed{y(x) = c_1 e^{5x} + c_2 e^{4x}}$$

Q) $y'' + y' = 0$

Sol $\Rightarrow m^2 + m = 0$

$\Rightarrow m(m+1) = 0$

$m = 0$ (or) $m = -1$

$$\boxed{y(x) = c_1 + c_2 e^{-x}}$$

Q) $2y'' + 5xy' + 6y = 0$ ~~Can't Talk~~

Sol: Can't talk about Auxiliary equation here as $p(x)$ & $q(x)$ must be constants.

Q) $y'' + 4y' + 4y = 0$

Sol: $m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0$

$\therefore m = -2$

$$V(x) = \int \frac{1}{e^{-4x}} \cdot e^{-\int 4\,dx}\,dx$$

$= x$

$$\boxed{y(x) = c_1 e^{-2x} + c_2 x e^{-2x}}$$

$y'' + P(y') + Q(y) = R(x)$ $\longrightarrow$

- e^{ax} - (1)
- $\sin bx / \cos bx$ - (2)
- $a_0 + a_1 x + \cdots a_n x^n$ - (3)

① $y'' + py' + qy = e^{ax}$

The solution to this is $y_p(x) \longrightarrow$ Particular solution

$y_p(x) = Ae^{ax}$

$\Rightarrow Ae^{ax}(a^2+pa+q) = e^{ax}$

$$\boxed{A = \frac{1}{a^2+pa+q}} \longrightarrow$$ A is the unknown, found by computing this expression.

Denominator $\neq 0$

Denominator $= 0$ when 'a' is the root of the auxiliary equation of $y''+py'+qy=0$

$[m^2+pm+q=0]$

If $y_p(x)$ and $y_g(x)$ have a common solution, then we multiply x with $y_g(x)$.

Q) $y'' + 3y' - 10y = 6e^{4x}$

$m^2+3m-10=0$

$\Rightarrow (m+5)(m-2)=0$

$\therefore m=-5$ (or) $m=2$

-10

$5 \quad -2$

$y_p = Ae^{4x} \longrightarrow$ solution

$$A = \frac{6}{a^2+ap+q} = \frac{6}{16+4(3)+(-10)} = \frac{6}{18} = \frac{1}{3}$$

$$\boxed{y_p = \tfrac{1}{3}e^{4x}}$$

$y_g = y_c(x) + y_p(x)$

① a) $y'' + y' = \sin x$

Sol: Step ①: Compare with $y'' + P(x)y' + Q(x)y = R(x)$

$$P(x) = 1$$
$$Q(x) = 0$$
$$R(x) = \sin x$$

Step ②: Check if we can use method of undetermined coefficients:

Yes, As $P(x)$ & $Q(x)$ are constants

We know, $y_g(x) = y_c(x) + y_p(x)$

Step ③: Find $y_c(x)$

$$y'' + y' = 0$$
$$\Rightarrow m^2 + m = 0$$
$$\Rightarrow m(m+1) = 0$$
$$m = 0 \ \& \ m = -1$$

$$y_c(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$
$$= c_1 e^{0 \cdot x} + c_2 \cdot e^{-1 \cdot x}$$

$$\boxed{\therefore y_c(x) = c_1 + c_2 e^{-x}}$$

Step ④: Find $y_p(x)$

$R(x) = \sin(bx)$ (or) $\cos(bx)$, $b = 1$

We know, $y_p(x) = A\sin bx + B\cos bx$
$= A\sin x + B\cos x$

Substitute $y_p(x)$ in Original Equation,

$$(A\sin x + B\cos x)'' + (A\sin x + B\cos x)' = \sin x$$
$$\Rightarrow (A\cos x - B\sin x)' + A\cos x - B\sin x = \sin x$$

$\Rightarrow -A\sin x - B\cos x + A\cos x - B\sin x = \sin x$

$\Rightarrow -(A+B)\sin x + (A-B)\cos x = 1 \times \sin x + 0 \times \cos x$

$\therefore -(A+B) = 1 \quad \& \quad A-B = 0$

$\boxed{\therefore A = B}$

$-(A+A) = 1$

$\Rightarrow \boxed{A = B = -\frac{1}{2}}$

$y_p(x) = A\sin x + B\cos x$

$\Rightarrow y_p(x) = -\frac{1}{2}(\sin x + \cos x)$

& $y_c(x) = c_1 + c_2 e^{-x}$

As $y_g(x) = y_c(x) + y_p(x)$

$\Rightarrow \boxed{y_g(x) = c_1 + c_2 e^{-x} + -\frac{1}{2}(\sin x + \cos x)}$

(3) $y'' + Py' + Qy = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$

Here, $y_p = A_0 + A_1 x + A_2 x^2 + \dots A_n x_n$

Ex. $y'' - y' - 2y = 4x^2$

Sol: Step (1): Compare

$P(x) = -1$

$Q(x) = -2$

$R(x) = 4x^2$

Step (2):

$P(x)$ & $Q(x)$ are Integers ✓

$\therefore y_g(x) = y_c(x) + y_p(x)$

Step (3):

$y'' - y' - 2y = 0$

$$\Rightarrow m^2 - m - 2 = 0$$

$$\begin{matrix} & -2 & \\ -2 & & 1 \end{matrix}$$

$$\Rightarrow (m-2)(m+1) = 0$$

$$\underline{m = 2} \text{ (or) } \underline{m = -1}$$

$$\boxed{y_c(x) = c_1 e^{2x} + c_2 e^{-x}}$$

Step ④ :

$$R(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$a_0 = a_1 = a_3 = a_4 = \dots = a_n = 0$$

$$\& \quad a_2 = 4$$

We know that,

$$y_p = A_0 + A_1 x + A_2 x^2 + \dots A_n x_n$$

Susstitute y_p in original equation,

$$\Rightarrow (A_0 + A_1 x + A_2 x^2 + \dots A_n x^n)'' - (A_0 + A_1 x + A_2 x^2 + \dots A_n x^n)'$$

$$- 2(A_0 + A_1 x + A_2 x^2 + \dots A_n x^n) = 4x^2$$

$$\Rightarrow (2A_2 + 6A_3 x + 12 A_4 x^2 + \dots) - (A_1 + 2A_2 x + 3A_3 x^2 + \dots)$$

$$- 2(A_0 + A_1 x + A_2 x^2 + \dots) = 4x^2$$

By Comparison,

$$2A_2 - A_1 - 2A_0 = 0$$

$$6A_3 - 2A_2 - 2A_1 = 0$$

$$12 A_4 + 3A_3 - 2A_2 = 4$$

We Assume, $y_p = A_0 + A_1 x + A_2 x^2$ [i.e. $A_3, A_4, \dots A_n = 0$]

$$2A_2 - A_1 - 2A_0 = 0 \quad - ①$$

$$-2A_2 - 2A_1 = 0 \quad - ②$$

$$-2A_2 = 4 \quad - ③$$

$$\boxed{\therefore A_2 = -2} \quad \& \quad \boxed{A_1 = +2} \quad \& \quad \boxed{A_0 = -3}$$

$\therefore y_p = -3 + 2x - 2x^2$

As $y_g(x) = y_c(x) + y_p(x)$

$$\boxed{\therefore y_g(x) = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2}$$

→ <u>Principle of Superposition</u> :

If $y_1(x)$ & $y_2(x)$ are solutions of

$$y'' + P(x) \cdot y' + Q(x) y = R_1(x) \quad \&$$
$$y'' + P(x) \cdot y' + Q(x) \cdot y = R_2(x)$$

respectively,

Then $y(x) = y_1(x) + y_2(x)$ is a solution of

$$y'' + P(x) \cdot y' + Q(x) y = R_1(x) + R_2(x)$$

<u>Note</u> :

Can be Applied for more than 2 D.Es.

→ <u>Variation of Parameters</u> :

In $y'' + P(x) y' + Q(x) \cdot y = R(x)$, Till now we solved only if $P(x)$ and $Q(x)$ are constants. What if they aren't ?

$$y_p(x) = V_1(x)\, y_1(x) + V_2(x) \cdot y_2(x)$$

Substitute y_p in original equation before that :

$$y_p'(x) = V_1' y_1 + y_1' V_1 + V_2' y_2 + y_2' V_2$$

$$\boxed{\text{Let } V_1' y_1 + V_2' y_2 = 0} \quad - (a)$$

$$\Rightarrow y_p' = V_1 y_1' + V_2 y_2' \quad - (1)$$

$$\Rightarrow y_p'' = V_1' y_1' + V_1 y_1'' + V_2 y_2'' + V_2' y_2' \quad - (2)$$

Substitute (1) and (2) in Original Equation :

$$y'' + P(x)y' + Q(x)y = R(x)$$

$$\Rightarrow (v_1'y_1' + v_2'y_2' + v_1y_1'' + v_2y_2'') + P(x)\left[v_1y_1' + v_2y_2'\right] + Q(x)\cdot y = R(x)$$

$$\Rightarrow \left[(v_1y_1'') + P(x)(v_1y_1') + Q(x)\cdot(v_1y_1)\right] + \left[(v_2y_2'') + P(x)(v_2y_2') + Q(x)\cdot(v_2y_2)\right] + v_1'y_1' + v_2'y_2' = R(x)$$

As y_1 & y_2 are solutions of Homogenous equation

$$y'' + P(x)y' + Q(x)y = 0,$$

$$\Rightarrow v_1\left[y_1'' + y_1'P(x) + Q(x)y\right] + v_2\left[y_2'' + P(x)y_2' + Q(x)y_2\right] + v_1'y_1' + v_2'y_2' = R(x)$$

$$\Rightarrow \boxed{v_1'y_1' + v_2'y_2' = R(x)} \quad - (b)$$

Solve (a) and (b) :

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = R(x) \end{cases}$$

$$\left.\begin{aligned} ax + by &= k_1 \\ cx + dy &= k_2 \end{aligned}\right\} \rightarrow x = \frac{\begin{vmatrix} k_1 & b \\ k_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & k_1 \\ c & k_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$v_1' = \frac{-R(x)\cdot y_2}{W} \Rightarrow \boxed{v_1 = \int \frac{-R(x)\cdot y_2}{W}\,dx}$$

$$\because v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \& \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$v_2' = \frac{R(x)\cdot y_1}{W} \Rightarrow \boxed{v_2 = \int \frac{R(x)\cdot y_1}{W}\,dx}$$

$\therefore y_p = -3 + 2x - 2x^2$

As $y_g(x) = y_c(x) + y_p(x)$

$$\boxed{\therefore y_g(x) = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2}$$

→ <u>Principle of Superposition</u> :

If $y_1(x)$ & $y_2(x)$ are solutions of

$$y'' + P(x) \cdot y' + Q(x) y = R_1(x) \quad \&$$
$$y'' + P(x) \cdot y' + Q(x) \cdot y = R_2(x)$$

respectively,

Then $y(x) = y_1(x) + y_2(x)$ is a solution of

$$y'' + P(x) \cdot y' + Q(x) y = R_1(x) + R_2(x)$$

<u>Note</u>:

Can be Applied for more than 2 D.Es.

→ <u>Variation of Parameters</u> :

In $y'' + P(x) y' + Q(x) \cdot y = R(x)$, Till now we solved only if $P(x)$ and $Q(x)$ are constants. What if they aren't ?

$$y_p(x) = V_1(x)\, y_1(x) + V_2(x) \cdot y_2(x)$$

Substitute y_p in original equation before that :

$$y_p'(x) = V_1' y_1 + y_1' V_1 + V_2' y_2 + y_2' V_2$$

$$\boxed{\text{Let } V_1' y_1 + V_2' y_2 = 0} \quad - (a)$$

$$\Rightarrow y_p' = V_1 y_1' + V_2 y_2' \quad - (1)$$

$$\Rightarrow y_p'' = V_1' y_1' + V_1 y_1'' + V_2 y_2'' + V_2' y_2' \quad - (2)$$

Substitute (1) and (2) in Original Equation :

$$y'' + P(x)y' + \varphi(x)y = R(x)$$

$$\Rightarrow (v_1'y_1' + v_2'y_2' + v_1y_1'' + v_2y_2'') + P(x)\left[v_1y_1' + v_2y_2'\right] + \varphi(x)\cdot y$$
$$= R(x)$$

$$\Rightarrow \left[(v_1y_1'') + P(x)\cdot(v_1y_1') + \varphi(x)\cdot(v_1y_1)\right]$$
$$+ \left[(v_2y_2'') + P(x)(v_2y_2') + \varphi(x)\cdot(v_2y_2)\right]$$
$$+ v_1'y_1' + v_2'y_2' = R(x)$$

As y_1 & y_2 are solutions of Homogenous equation
$y'' + P(x)y' + \varphi(x)y = 0$,

$$\Rightarrow v_1\left[\cancelto{0}{y_1'' + y_1'P(x) + \varphi(x)y}\right] + v_2\left[\cancelto{0}{y_2'' + P(x)y_2' + \varphi(x)y_2}\right]$$
$$+ v_1'y_1' + v_2'y_2' = R(x)$$

$$\Rightarrow \boxed{v_1'y_1' + v_2'y_2' = R(x)} \quad - (b)$$

Solve (a) and (b):

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = R(x) \end{cases}$$

$$\left.\begin{aligned} ax + by &= k_1 \\ cx + dy &= k_2 \end{aligned}\right\} \rightarrow x = \frac{\begin{vmatrix} k_1 & b \\ k_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & k_1 \\ c & k_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$v_1' = \frac{-R(x)\cdot y_2}{W} \Rightarrow \boxed{v_1 = \int \frac{-R(x)\cdot y_2}{W}\,dx}$$

$$\because v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \& \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$v_2' = \frac{R(x)\cdot y_1}{W} \Rightarrow \boxed{v_2 = \int \frac{R(x)\cdot y_1}{W}\,dx}$$

$$y_p(x) = V_1(x)\, y_1(x) + V_2(x) \cdot y_2(x)$$

$$\boxed{\therefore\ y_p(x) = -y_1(x) \int \frac{R(x) \cdot y_2(x)}{W(x)}\, dx + y_2(x) \int \frac{R(x) \cdot y_1(x)}{W(x)}\, dx}$$

Ex. $y'' + y = \csc(x)$

Sol: $y_c(x) = c_1 \cos x + c_2 \sin x \longrightarrow$ (for $y'' + y = 0$)

$$y_p(x) = V_1(x)\, y_1(x) + V_2(x)\, y_2(x)$$
$$= (V_1(x))(\cos x) + (V_2(x))(\sin x)$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= \cos^2 x + \sin^2 x$$
$$\therefore\ W = 1$$

$$V_1(x) = -\int \frac{y_2(x) \cdot R(x)}{W}\, dx$$
$$= -\int y_2(x) \cdot R(x)\, dx$$
$$= -\int \sin x \cdot \csc x\, dx$$
$$\therefore\ V_1(x) = -x$$

$$V_2(x) = \int \frac{y_1(x) \cdot R(x)}{W}\, dx$$
$$= \int y_1(x) \cdot R(x)\, dx$$
$$= \int \cos x \cdot \csc x\, dx$$
$$\therefore\ V_2(x) = \ln|\sin(x)|$$

$$y_p(x) = (-x)\, y_1(x) + (\ln|\sin x|)\, y_2(x)$$
$$= -x \cos x + \sin x \cdot \ln(\sin x)$$

$$\boxed{\therefore\ y_p(x) = \sin x \cdot \ln(\sin x) - x \cos x}$$

$$\therefore\ y_g(x) = y_p(x) + y_c(x)$$

$$\boxed{\therefore\ y_g(x) = c_1 \cos x + c_2 \sin x + \sin x \cdot \ln|\sin x| - x \cos x}$$

H.W

Q① Show that e^x and e^{-x} are Linearly Independent solutions of $y''-y=0$ on any Interval

Sol:

According to Lemma 2,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$y_1 = e^x$ | $y_2 = e^{-x}$

$y_1' = e^x$ | $y_2' = -e^{-x}$

$$\Rightarrow W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -e^x \cdot e^{-x} - e^{-x} \cdot e^x$$

$$= -1 - 1 = -2$$

As $W \neq 0$,

$\therefore$ y_1 & y_2 are Linearly Independent

Q② $y = c_1 x + c_1 x^2$

↳ General Solution of $x^2y'' - 2xy' + 2y = 0$ on any interval not containing 0

Also find y_p for which $y(1) = 3$ and $y'(1) = 5$

Sol: $y_1 = x$

$y_2 = x^2$

Step①:

$x^2y_1'' - 2xy_1' + 2y_1 = 0$

$\Rightarrow x^2(0) - 2x(1) + 2x = 0$

$\Rightarrow -2x + 2x = 0$

$\Rightarrow 0 = 0$ ✓

$x^2y_2'' - 2xy_2' + 2y_2 = 0$

$\Rightarrow x^2(2) - 2x(2x) + 2x^2 = 0$

$\Rightarrow 2x^2 - 4x^2 + 2x^2 = 0$

$\Rightarrow 0 = 0$ ✓

(OR)

$$x^2y'' - 2xy' + 2y = 0$$

$$\Rightarrow y'' - \left(\frac{2}{x}\right)y' + \left(\frac{2}{x^2}\right)y = 0$$

By Inspection, $y = x$ is a solution $\Rightarrow \boxed{y_1 = x}$

$$V(x) = \int \frac{1}{y_1^2(x)} e^{-\int P(x)dx} dx$$

$$= \int \frac{1}{x^2} e^{-\int \frac{-2}{x} dx} dx$$

$$= \int \frac{1}{x^2} \cdot x^2 \, dx$$

$$= x$$

2nd linearly Independent Solution is $V(x) \cdot y_1(x)$

$$y_2(x) = x \times x$$

$$\boxed{\therefore y_2 = x^4}$$

Solutions : x and x^2

$$y_p(x) = c_1 x + c_2 x^2$$

(OR)

Step ② :

$$W(y_1, y_2)$$

$$= \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - x^2$$

$$= x^2$$

↳ Never 0 in an interval where $x \neq 0$

$\therefore x$ and x^2 are linearly Independent.

Step ①:

Now solve for $y(1) = 3$ and $y'(1) = 5$

$$y(1) = c_1 + c_2 = 3$$
$$y'(1) = c_1 + 2c_2 = 5$$

$$\boxed{\therefore c_2 = 2} \quad \& \quad \boxed{c_1 = 1}$$

$$\boxed{\therefore y_p(x) = x + 2x^2}$$

~~(OR)~~

$$\cancel{W = x^2}$$

$$\cancel{v_1 = \int \frac{-R(x)\, y_2}{W}\, dx = \int -2/x^2 \cdot x^2}$$

$$\cancel{v_2 = \int \frac{R(x) \cdot y_1}{W}\, dx}$$

Q③ $y = c_1 e^x + c_2 e^{2x}$

Prove $\rightarrow$ $y'' - 3y' + 2y = 0$

Sol: Step①:

$$y_1 = e^x$$
$$y_2 = e^{2x}$$

$y_1'' - 3y_1' + 2y_1 = 0$	$y_2'' - 3y_2' + 2y_2 = 0$
$\Rightarrow e^x - 3e^x + 2e^x = 0$	$\Rightarrow 4e^{2x} - 3(2e^{2x}) + 2e^{2x} = 0$
$\Rightarrow 3e^x - 3e^x = 0$	$\Rightarrow 6e^{2x} - 6e^{2x} = 0$
$\Rightarrow 0 = 0$ ✓	$\Rightarrow 0 = 0$ ✓

(OR)

$$y'' - 3y' + 2y = 0$$
$$\Rightarrow m^2 - 3m + 2 = 0$$

$\Rightarrow (m-2)(m-1) = 0$

2

$-2 \quad -1$

$m = 2$ (or) $m = 1$

$y_p(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow \boxed{y_p(x) = c_1 e^{x} + c_2 e^{2x}}$

Step 2 :

$W(y_1, y_2)$

$y_1 = e^x \quad | \quad y_2 = e^{2x}$

$y_1' = e^x \quad | \quad y_2' = 2e^{2x}$

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$= 2e^{3x} - e^{3x}$

$= e^{3x}$

is never $0 \in \mathbb{R}$

$\therefore$ y_1 & y_2 are linearly Independent

Step 3 :

Find $y_p(x)$

$y_p(0) = c_1 + c_2 = -1$

$y_p'(0) = c_1 + 2c_2 = 1$

$\boxed{\therefore c_2 = 2} \Rightarrow \boxed{c_1 = -3}$

Q4 $y = c_1 e^{2x} + c_2 x e^{2x}$

$\rightarrow y'' - 4y' + 4y = 0$

Sol: Step ①: Check

$y_1 = e^{2x}$ | $y_2 = xe^{2x}$

$y_1' = 2e^{2x}$ | $y_2' = 2xe^{2x} + e^{2x}$

$y_1'' = 4e^{2x}$ | $y_2'' = 4xe^{2x} + 2e^{2x} + 2e^{2x}$

$= 4xe^{2x} + 4e^{2x}$

$y_1'' - 4y_1' + 4y_1 = 0$

$\Rightarrow 4e^{2x} - 8e^{2x} + 4e^{2x} = 0$

$\Rightarrow 0 = 0$ ✓

$y_2'' - 4y_2' + 4y_2 = 0$

$\Rightarrow 4e^{2x}(1+x) - 4e^{2x}(1+2x) + 4xe^{2x} = 0$

$\Rightarrow 0 = 0$ ✓

(OR)

$y'' - 4y' + 4y = 0$

$\Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow \underline{\underline{m = 2}} \Rightarrow y_1 = e^{2x}$

$$V(x) = \int \frac{1}{y_1^2(x)} \cdot e^{-\int P(x)\,dx}\,dx$$

$$= \int \frac{1}{e^{4x}} \cdot e^{-\int -4\,dx}\,dx$$

$$= \int \frac{e^{4x}}{e^{4x}}\,dx$$

$$= x$$

$\therefore y_2 = V(x) \cdot y_1$

$\therefore y_2 = xe^{2x}$

$\therefore y_p(x) = c_1 e^{2x} + c_2 \cdot xe^{2x}$

Step ②:

$W(y_1, y_2)$

$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{vmatrix}$$

$$\Rightarrow W(y_1, y_2) = e^{4x}(1+2x) - 2xe^{4x}$$

$$= e^{4x}$$

$\hookrightarrow$ Never zero

$\therefore$ e^{2x} and xe^{2x} are linearly Independent.

H.W

Q1 (a) $y'' + y = 0$, $y_1 = \sin x$, $y_2 = ?$ & $y_p = ?$

Sol:

$$V(x) = \int \frac{1}{y_1^2(x)} e^{-\int P(x)\,dx}\,dx$$

$$= \int \frac{1}{\sin^2 x} e^{-\int 0\,dx}\,dx$$

$$= \int \frac{-1}{\sin^2 x}\,dx$$

$$= \int -\csc^2 x\,dx$$

$$= \cot x$$

$$y_2(x) = V \times y_1(x)$$

$$= \cot x \times \sin x$$

$$= \cos x$$

$\therefore$ $y_1(x) = \sin x$ & $y_2(x) = \cos x$

& $\boxed{y_p(x) = c_1 \sin x + c_2 \cos x}$

(b) $y'' - y = 0$, $y_1 = e^x$, $y_2 = ?$ & $y_p = ?$

Sol:

$$V(x) = \int \frac{1}{e^{2x}} \cdot e^{-\int 0\,dx}\,dx$$

$$= \int \frac{-1}{e^{2x}}\,dx$$

$$= \int -e^{-2x}\,dx$$

$$= \frac{e^{-2x}}{2}$$

$y_2 = V \cdot y_1$

$\Rightarrow y_2 = \frac{e^{-2x}}{2} \times e^{x}$

$\therefore y_2 = \frac{e^{-x}}{2}$ $\quad \left[\frac{1}{2} \rightarrow c\right]$

$\therefore y_1 = e^{x}$

$y_2 = e^{-x}$

$$\boxed{\therefore y_p(x) = c_1 e^{x} + c_2 e^{-x}}$$

(Q2) $xy'' + 3y' = 0$

$y_1 = 1$

<u>Sol</u>:

$y'' + \left(\frac{3}{x}\right) y' = 0$

$$V(x) = \int \frac{1}{(1)^2} \cdot e^{-\int \frac{3}{x} dx} dx$$

$$= \int e^{-\ln(x^3)} dx$$

$$= \frac{1}{x^3}$$

$\therefore y_2(x) = y_1(x) \times V(x)$

$= \frac{1}{x^3}$

$\therefore y_1(x) = 1$

$y_2(x) = 1/x^3$

$$\boxed{\therefore y_p(x) = c_1 + c_2/x^3}$$

Q3 $y_1 = x^2$

$$x^2y'' + xy' - 4y = 0$$

Sol:

$$y'' + \left(\frac{1}{x}\right)y' - \left(\frac{4}{x^2}\right)y = 0$$

$$V(x) = \int \frac{1}{x^4} \cdot e^{-\int \frac{1}{x}dx} dx$$

$$= \int \frac{1}{x^4} \cdot \frac{1}{x} dx$$

$$= \int \frac{1}{x^5} dx$$

$$= \frac{-1}{4x^4}$$

$$\therefore y_2(x) = y_1(x) \times V(x)$$

$$\Rightarrow y_2(x) = x^2 \times \frac{-1}{4x^4}$$

$$\therefore y_2(x) = \frac{1}{x^2} \qquad \left[-\frac{1}{4} \rightarrow c\right]$$

~~$y_p(x) = c_1$~~

$$y_1(x) = x^2$$

$$y_2(x) = 1/x^2$$

$$\boxed{\therefore y_p(x) = c_1x^2 + c_2/x^2}$$

Q4 $y_1 = x$

$$(1-x^2)y'' - 2xy' + 2y = 0$$

Sol:

$$y'' - \left(\frac{2x}{1-x^2}\right)y' + \left(\frac{2}{1-x^2}\right)y = 0$$

$$V(x) = \int \frac{1}{x^2} \cdot e^{-\int \frac{-2x}{1-x^2}dx} dx$$

$$\Rightarrow V(x) = \int \frac{1}{x^2} \cdot e^{-\int \frac{-2x}{1-x^2} dx} dx$$

$$= \int \frac{1}{x^2} \cdot e^{-\ln|1-x^2|} dx$$

$$= \int \frac{1}{x^2(1-x^2)} dx$$

$$= \int \left(\frac{1}{x^2} + \frac{1}{1-x^2} \right) dx$$

$$= \frac{-1}{x} - \frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x+1|$$

$$y_2(x) = V(x) \times y_1(x)$$

$$\therefore y_2(x) = -1 - \frac{x}{2} \ln|x-1| + \frac{x}{2} \ln|x+1|$$

$$= \frac{x}{2} \ln\left(\frac{x+1}{x-1}\right) - 1$$

$$\therefore y_1(x) = x$$

$$y_2(x) = \frac{x}{2} \ln\left(\frac{x+1}{x-1}\right) - 1$$

$$\boxed{\therefore y_p(x) = c_1 x + c_2 \left(\frac{x}{2} \ln\left(\frac{x+1}{x-1}\right) - 1 \right)}$$

H.W :

① $y'' + y' - 6y = 0$

$$m^2 + m - 6 = 0$$

-6

$3 \quad -2$

$$\Rightarrow (m+3)(m-2) = 0$$

$$m = -3 \text{ (or) } m = 2$$

$$\boxed{\therefore y_p(x) = c_1 e^{-3x} + c_2 e^{2x}}$$

② $y'' + 2y' + y = 0$

$m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0$

$\boxed{m = -1}$

$\boxed{\therefore y_p(x) = c_1 e^{-x} + c_2 x e^{-x}}$

③ $y'' + 8y = 0$

$m^2 + 8 = 0$

$\Rightarrow m^2 = -8$

$\Rightarrow \boxed{m = i(2\sqrt{2})}$

$\boxed{\therefore y_p = c_1 e^{i2\sqrt{2}} + c_2 e^{-i2\sqrt{2}}}$

④ $2y'' - 4y' + 8y = 0$

$2m^2 - 4m + 8 = 0$

$\Rightarrow m^2 - 2m + 4 = 0$

$m = \frac{2 \pm \sqrt{4-16}}{2}$

$= \frac{2 \pm i(2\sqrt{3})}{2}$

$\boxed{\therefore m = 1 \pm i\sqrt{3}}$

$\boxed{y_p = c_1 e^{(1+i\sqrt{3})x} + c_2 e^{(1-i\sqrt{3})x}}$

(OR) $\boxed{y_p = e^x (c_1' \cos\sqrt{3}x + c_2' \sin\sqrt{3}x)}$

⑤ $y'' - 4y' + 4y = 0$

$m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\boxed{\therefore m = 2}$

$$\boxed{\therefore y_p(x) = c_1 e^{2x} + c_2 x e^{2x}}$$

⑥ $y'' - 9y' + 20y = 0$

$\Rightarrow m^2 - 9m + 20 = 0$

$\Rightarrow (m-4)(m-5) = 0$

$\boxed{m=4}$ (or) $\boxed{m=5}$

20

-4 -5

$$\boxed{\therefore y_p(x) = c_1 e^{4x} + c_2 e^{5x}}$$

⑦ $y'' + y' = 0$

$m^2 + m = 0$

$\Rightarrow m(m+1) = 0$

$\boxed{m=0}$ (or) $\boxed{m=-1}$

$$\boxed{\therefore y_p(x) = c_1 + c_2 e^{-x}}$$

H.W:

① $y'' + 3y' - 10y = 6e^{4x}$

Sol:

Step ①:

$P(x) = 3$

$Q(x) = -10$

$R(x) = 6e^{4x}$

Step ②:

$P(x)$ & $Q(x)$: Constants

$y_g(x) = y_p(x) + y_c(x)$

Step ③:

$y'' + 3y' - 10y = 0$

$m^2 + 3m - 10 = 0$

$\Rightarrow (m+5)(m-2) = 0$

$\therefore \boxed{m=-5}$ and $\boxed{m=2}$

-10

5 -2

$$\boxed{y_c(x) = c_1 e^{-5x} + c_2 e^{2x}}$$

Step ④ :

$R(x) = 6e^{4x}$

We know, $y_p(x) = Ae^{ax}$

Substitute in Original equation :

$$(Ae^{ax})'' + 3(Ae^{ax})' - 10(Ae^{ax}) = 6e^{4x}$$

$$\Rightarrow Aa^2e^{ax} + 3Aae^{ax} - 10Ae^{ax} = 6e^{4x}$$

$$\Rightarrow (Aa^2 + 3Aa - 10A)(e^{ax}) = 6(e^{4x})$$

$$\underline{a = 4}$$

$$\Rightarrow (16A + 12A - 10A) = 6$$

$$\Rightarrow 18A = 6$$

$$\Rightarrow \boxed{A = 1/3}$$

$$\boxed{\therefore y_p(x) = 1/3\, e^{4x}}$$

Now, $y_g(x) = y_c(x) + y_p(x)$

$$\boxed{\therefore y_g(x) = c_1 e^{-5x} + c_2 e^{2x} + 1/3\, e^{4x}}$$

Q2 $y'' + 4y = 3\sin x$

Step ① :

$P(x) = 0$

$Q(x) = 4$

$R(x) = 3\sin x$

Step ② ✓

$y_g(x) = y_c(x) + y_p(x)$ $\quad \because P(x), Q(x) \in \mathbb{R}$

step ③ :

$$y'' + 4y = 0$$

$$m^2 + 4 = 0$$

$$\Rightarrow \boxed{m = \pm 2i}$$

$$\boxed{y_c(x) = c_1 e^{2ix} + c_2 e^{-2ix}}$$

$$\hookrightarrow \boxed{y_c(x) = c_1' \cos 2x + c_2' \sin 2x}$$

Step ④ :

We know, $y_p(x) = A\sin x + B\cos x$

$$\left[\begin{array}{l} y_p(x) = A\sin ax + B\cos bx \\ R(x) = 3\sin(bx),\ b = 1 \end{array}\right]$$

$$(A\sin x + B\cos x)'' + 4(A\sin x + B\cos x) = 3\sin x$$

$$\Rightarrow (-A\sin x - B\cos x) + 4(A\sin x + B\cos x) = 3\sin x$$

$$\therefore 3A\sin x + 3B\cos x = 3\sin x$$

$$\boxed{B = 0} \ \& \ \boxed{A = 1}$$

$$\boxed{y_p(x) = \sin x}$$

Now, $y_g(x) = y_c(x) + y_p(x)$

$$\boxed{\therefore y_g(x) = \sin x + c_2 \sin 2x + c_1 \cos 2x}$$

Doubt

Q3 $y'' + 10y' + 25y = 14e^{-5x}$

step ① ✓
step ② ✓

Step ③ :

$$y'' + 10y' + 25y = 0$$

$$\Rightarrow m^2 + 10m + 25 = 0$$

$\Rightarrow \boxed{m=-5}$

$\Rightarrow (m+5)^2 = 0$

$\boxed{\therefore m = -5}$

$$\boxed{\therefore y_c(x) = c_1 e^{-5x} + c_2 x e^{-5x}}$$

Step ④ :

$y_p(x) = Ae^{ax}$

$\underline{\underline{a = -5}}$

Sub in original Eq.

$(Ae^{ax})'' + 10(Ae^{ax})' + 25(Ae^{ax}) = Ae^{ax}$

$Aa^2e^{ax} + 10Aae^{ax} + 25Ae^{ax} = Ae^{ax}$

$\Rightarrow 25Ae^{-5x} - 50Ae^{-5x} + 25Ae^{-5x} = Ae^{-5x}$

$\Rightarrow (25A - 50A + 25A) = A$

$\Rightarrow \boxed{A = 0}$

$\boxed{y_p = 0}$

$$\boxed{\therefore y_g(x) = c_1 e^{-5x} + c_2 x e^{-5x}}$$

Q4 $y'' - 2y' + 5y = 25x^2 + 12$

Step ③ :

$m^2 - 2m + 5 = 0$

$m = \frac{2 \pm \sqrt{4-20}}{2}$

$\boxed{\therefore m = 1 \pm 2i}$

$$\boxed{y_c(x) = e^{x}(c_1' \cos 2x + c_2' \sin 2x)}$$

Step ④ :

$$y_p(x) = A_0 + A_1 x + A_2 x^2$$

Sub in Original eq :

$$(A_0 + A_1 x + A_2 x^2)'' - 2(A_0 + A_1 x + A_2 x^2)' + 5(A_0 + A_1 x + A_2 x^2) = 25x^2 + 12$$

$$\Rightarrow (2A_2) - 2(A_1 + 2A_2 x) + 5(A_0 + A_1 x + A_2 x^2) = 25x^2 + 12$$

$$\Rightarrow (5A_2)x^2 + (5A_1 - 4A_2)x + (5A_0 - 2A_1 + 2A_2) = 25x^2 + 12$$

$$5A_2 = 25 \Rightarrow \boxed{A_2 = 5}$$

$$5A_1 - 4A_2 = 0$$

$$\Rightarrow 5A_1 = 4A_2$$

$$\boxed{\therefore A_1 = 4}$$

$$5A_0 - 2A_1 + 2A_2 = 12$$

$$\Rightarrow 5A_0 = 12 + 8 - 10$$

$$\boxed{\therefore A_0 = 2}$$

$$\therefore y_p(x) = 2 + 4x + 5x^2$$

$$\boxed{\therefore y_g(x) = e^{x}(c_1' \cos 2x + c_2' \sin 2x) + 5x^2 + 4x + 2}$$

Doubt

Q5 $y'' - y' - 6y = 20e^{-2x}$

$$m^2 - m - 6 = 0$$

$$\Rightarrow (m+2)(m-3) = 0$$

-6

$-2 \quad 3$

$$\boxed{y_c(x) = c_1 e^{-2x} + c_2 e^{3x}}$$

$y_p(x) = Ae^{-2x} \rightarrow$ L.D with $y_c(x) \Rightarrow \therefore y_p(x) = xAe^{-2x}$

Sub in Original eq:

$$(xAe^{-2x})'' - (Ae^{-2x})' - 6(xAe^{-2x}) = 20e^{-2x}$$

$$\Rightarrow 4A + 2A - 6A = A$$

$$\boxed{A = 0}$$

$$\boxed{\therefore y_p = 0}$$

$$\boxed{y_g(x) = c_1 e^{-2x} + c_2 e^{5x}}$$

$$A\left[-2xe^{-2x} + e^{-2x}\right]'$$

$$- A\left[-2xe^{-2x} + e^{-2x}\right]$$

$$- 6Axe^{-2x}$$

$$\Rightarrow A\left[-2e^{-2x} + 4e^{-2x} \cdot x - 2e^{-2x}\right]$$

$$- A\left[2xe^{-2x} + e^{-2x}\right]$$

$$- 6Axe^{-2x}$$

(P6) $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$$\boxed{y_c(x) = c_1 e^{x} + c_2 e^{2x}}$$

$$y_p(x) = A\sin(2x) + B\cos(2x) \qquad [b = 2]$$

Sub in original eq:

$$\left(-4A\sin(2x) - 4B\cos(2x)\right) - 3\left(2A\cos(2x) - 2B\sin(2x)\right)$$

$$+ 2\left(A\sin(2x) + B\cos(2x)\right) = 14\sin 2x - 18\cos 2x$$

$$\Rightarrow (\sin 2x)\left[-4A + 6B + 2A\right] = 14(\sin 2x)$$

$$\& \ (\cos 2x)\left[-4B - 6A + 2B\right] = -18(\cos 2x)$$

$$6B - 2A = 14 \ \&$$

$$-2B - 6A = -18$$

$$\therefore \boxed{B = 3} \Rightarrow \boxed{A = 2}$$

$$\boxed{\therefore y_p(x) = 2\sin(2x) + 3\cos(2x)}$$

$$\boxed{\therefore y_g(x) = c_1 e^{x} + c_2 e^{2x} + 2\sin(2x) + 3\cos(2x)}$$

H.W

① $y'' + 4y = 4\cos 2x + 6\cos x + 8x^2 - 4x$

$y'' + 4y = 4\cos 2x$ — ①

$y'' + 4y = 6\cos x$ — ②

$y'' + 4y = 8x^2 - 4x$ — ③

$y_c(x) \Rightarrow y'' + 4y = 0$

$m^2 + 4 = 0$

$\Rightarrow \boxed{m = \pm 2i}$

~~$\therefore y_c(x) = \cos 2x + \sin 2x$~~

$$\boxed{\therefore y_c(x) = c_1'\cos 2x + c_2'\sin 2x}$$

① : $y_p(x) = A\sin 2x + B\cos 2x$

↳ LD with $y_c(x)$

$y_p(x) = x(c_1'\cos 2x + c_2'\sin 2x)$

$\Rightarrow y_p''(x) + 4y_p(x) = 4\cos 2x$

$\boxed{c_1' = 0}$ & $\boxed{c_2' = 1}$

$$\boxed{\therefore y_p(x) = x\sin 2x} \text{ — ①}$$

~~① : $y_p(x) = A\sin 2x + B\cos 2x$~~

~~$(A\sin 2x + B\cos 2x)'' + 4(A\sin 2x + B\cos 2x) = B\cos 2x + A\sin 2x$~~

~~$-4A\sin 2x - 4B\cos 2x + 4A\sin 2x + 4B\cos 2x = B\cos 2x + A\sin 2x$~~

~~$\boxed{A = 0}$ & $\boxed{B = 0}$~~

~~$$\boxed{\therefore y_g(x) = c_1'\cos 2x + c_2'\sin 2x} \text{ — ①}$$~~

$= x(-2c_1'\sin 2x + 2c_2'\cos 2x) + c_1'\cos 2x + c_2'\sin 2x$

$= -2c_1'\sin 2x + 2c_2'\cos 2x$

~~$+ x(-4c_1'\cos 2x - 4c_2'\sin 2x)$~~

$- 2c_1'\sin 2x + 2c_2'\cos 2x$

② : $y_p(x) = A\sin x + B\cos x$

$(A\sin x + B\cos x)'' + 4(A\sin x + B\cos x) = 6\cos x$

$\Rightarrow -A\sin x - B\cos x + 4A\sin x + 4B\cos x = 6\cos x$

$\Rightarrow \boxed{A = 0}$ & $3B = 6 \Rightarrow \boxed{B = 2}$

$$\boxed{\therefore y_p(x) = 2\cos x} \text{ — ②}$$

$\cancel{\therefore y_g(x) = c_1'\cos 2x + c_2'\sin 2x + 2\cos x \quad -②}$

③ : $y_p(x) = A_0 + A_1 x + A_2 x^2$

$$(A_0 + A_1 x + A_2 x^2)'' + 4(A_0 + A_1 x + A_2 x^2) = 8x^2 - 4x$$

$$\Rightarrow (2A_2 + 4A_0) + (4A_1)x + (4A_2)x^2 = 8x^2 - 4x$$

$$2A_2 + 4A_0 = 0$$

$$\boxed{A_1 = -1}$$

$$\boxed{A_2 = 2} \Rightarrow \boxed{A_0 = -1}$$

$$\boxed{\therefore y_p(x) = 2x^2 - x - 1} \quad -③$$

$$\cancel{\therefore y_g(x) = c_1'\cos 2x + c_2'\sin 2x + 2x^2 - x - 1 \quad -③}$$

$$\therefore y_g(x) = y_c(x) + \sum y_p(x)$$

$$\boxed{\therefore y_g(x) = c_1'\cos 2x + c_2'\sin 2x + 2\cos x + 2x^2 - x - 1 + x\sin 2x}$$

Q2 $y'' + 9y = \underbrace{2\sin 3x}_{①} + \underbrace{4\sin x}_{②} \underbrace{- 26e^{-2x}}_{③} + \underbrace{27x^3}_{④}$

$y_c(x) \Rightarrow y'' + 9y = 0$

$$m^2 + 9 = 0$$

$$\Rightarrow \boxed{m = \pm 3i}$$

$$\boxed{y_c(x) = c_1'\cos 3x + c_2'\sin 3x}$$

① $\cancel{y_p(x) = A\sin 3x + B\cos 3x}$

$$\cancel{(A\sin 3x + B\cos 3x)'' + 9(A\sin 3x + B\cos 3x) = A\sin 3x}$$

$$\cancel{-9A\sin 3x - 9B\cos 3x + 9A\sin 3x + 9B\cos 3x = A\sin 3x}$$

① $y_p(x) = A\sin 3x + B\cos 3x$ → Linearly Dependent with $y_c(x)$

$\therefore y_p(x) = x(c_1'\cos 3x + c_2'\sin 3x)$

$y_p''(x) + 9y_p(x) = 2\sin 3x$

$\rightarrow \boxed{c_1' = -\frac{1}{3}}$ & $\boxed{c_2' = 0}$

$$\boxed{\therefore y_p(x) = \frac{-x}{3}\cos 3x} \quad -①$$

② $y_p(x) = A\sin x + B\cos x$

$(-A\sin x - B\cos x) + 9(A\sin x + B\cos x) = 4\sin x$

$\boxed{B = 0}$

$\boxed{A = \frac{4}{2}}$

$$\boxed{\therefore y_p(x) = \frac{1}{2}\sin x} \quad -②$$

③ $y_p(x) = Ae^{-2x}$

$4Ae^{-2x} + 9Ae^{-2x} = -26e^{-2x}$

$\boxed{A = -2}$

$$\boxed{\therefore y_p(x) = -2e^{-2x}} \quad -③$$

④ $y_p(x) = A_0 + A_1x + A_2x^2 + A_3x^3$

$(2A_2 + 6A_3x) + 9(A_0 + A_1x + A_2x^2 + A_3x^3) = 27x^3$

$\boxed{A_3 = 3}$

$\boxed{A_2 = 0}$

$9A_0 + 2A_2 = 0$ & $9A_1 + 6A_3 = 0$

$\boxed{A_0 = 0}$ $\boxed{A_1 = -2}$

$$\boxed{\therefore y_p(x) = 3x^3 - 2x} \quad -④$$

$$y_g(x) = y_c(x) + \sum y_p(x)$$

$$\boxed{\therefore\ y_g(x) = c_1'\cos 3x + c_2'\sin 3x + \frac{1}{8}\sin x - \frac{x}{3}\cos 3x - 2e^{-2x} + 3x^3 - 2x}$$

H.W

① $y'' - 2y' + y = 2x$

$$y'' - 2y' + y = 0 \Rightarrow m^2 - 2m + 1 = 0$$
$$\Rightarrow (m-1)^2 = 0$$
$$\Rightarrow \boxed{m = 1}$$

$$\boxed{y_c(x) = c_1 e^x + c_2 x e^x}$$

& $y_p(x) = A_0 + A_1 x$

$$(A_0 + A_1 x)'' - 2(A_0 + A_1 x)' + (A_0 + A_1 x) = 2x$$

$$(-2A_1 + A_0) + (A_1)x = 2x$$

$$\boxed{A_1 = 2} \quad \& \quad \boxed{A_0 = 4}$$

$$\boxed{\therefore\ y_p(x) = 4 + 2x}$$

$$\boxed{y_g(x) = c_1 e^x + c_2 x e^x + 4 + 2x}$$

(OR)

$$y_p(x) = v_1 y_1 + v_2 y_2$$

$$v_1 = \int \frac{-y_2 R}{W}\,dx \qquad v_2 = \int \frac{y_1 R}{W}\,dx \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \int \frac{-xe^x(2x)}{e^{2x}}\,dx \qquad = \int \frac{e^x \cdot (2x)}{e^{2x}}\,dx \qquad \Rightarrow W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$$= -\int \frac{2x^2}{e^x}\,dx \qquad = \int \frac{2x}{e^x}\,dx \qquad = xe^{2x} + e^{2x} - xe^{2x}$$

$$\boxed{v_1 = 2(x^2 + 2x + 2)e^{-x}} \qquad \boxed{v_2 = -(2)(x+1)e^{-x}} \qquad \boxed{\therefore\ W = e^{2x}}$$

$$y_p(x) = \frac{2(x^2+2x+2)}{e^x} e^x - \frac{2(x+1)}{e^x} \cdot xe^x$$

$$= \cancel{2x^2} + 4x + 4 - \cancel{2x^2} - 2x$$

$$= \underline{2x+4}$$

$$\boxed{\therefore y_g(x) = c_1 e^x + c_2 x e^x + 2x + 4}$$

② $y'' - y' - 6y = e^{-x}$

$$y'' - y' - 6y \Rightarrow m^2 - m - 6 = 0$$

$$\Rightarrow (m+2)(m-3) = 0$$

$$\Rightarrow \boxed{m = -2} \text{ or } \boxed{m = 3}$$

-6

$2 \quad -3$

$$\boxed{\therefore y_c(x) = c_1 e^{-2x} + c_2 e^{3x}}$$

$$y_p(x) = v_1 y_1 + v_2 y_2$$

$$= v_1 e^{-2x} + v_2 e^{3x}$$

$$v_1 = \int \frac{-y_2 R}{W} dx$$

$$= \int \frac{-e^{3x} \cdot e^{-x}}{5e^x} dx$$

$$= -\int \frac{e^x}{5} dx$$

$$\boxed{\therefore v_1 = -\frac{e^x}{5}}$$

$$v_2 = \int \frac{y_1 R}{W} dx$$

$$= \int \frac{e^{-2x} \cdot e^{-x}}{e^{5x}} dx$$

$$= \int e^{-8x} dx$$

$$\boxed{\therefore v_2 = -\frac{e^{-8x}}{8}}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-2x} & e^{3x} \\ -2e^{-2x} & 3e^{3x} \end{vmatrix}$$

$$= 3e^x + 2e^x$$

$$\boxed{\therefore W = 5e^x}$$

$$y_p(x) = \frac{-e^x}{5} e^{-2x} + \frac{-e^{-8x}}{8} e^{3x}$$

$$= \frac{-1}{5e^x} - \frac{1}{8e^{5x}}$$

$$\boxed{\therefore y_p(x) = \frac{-1}{5e^x} - \frac{1}{8e^{5x}}}$$

$$\boxed{\therefore y_g(x) = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{5} e^{-x} - \frac{1}{8} e^{-5x}}$$

③ $y'' + 4y = \tan 2x$

$m^2 + 4 = 0$

$\Rightarrow \boxed{m = \pm 2i}$

$$\boxed{y_c(x) = c_1' \cos 2x + c_2' \sin 2x}$$

$y_p(x) = v_1 y_1 + v_2 y_2$

$= v_1 \cos 2x + v_2 \sin 2x$

$v_1 = \int \frac{-y_2 R}{W} dx$

$= -\int \frac{\sin 2x \tan 2x}{2} dx$

$= -\frac{1}{2} \int \frac{\sin^2(2x)}{\cos(2x)} dx$

$= -\frac{1}{2} \int (\sec(2x) - \cos(2x)) dx$

$= \frac{1}{2} \int (\cos 2x - \sec 2x) dx$

$v_2 = \int \frac{y_1 R}{W} dx$

$= \int \frac{\cos 2x \cdot \tan 2x}{2} dx$

$= \frac{1}{2} \int \sin 2x \, dx$

$$\boxed{v_2 = \frac{-\cos 2x}{4}}$$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$\Rightarrow W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$= 2\cos^2(2x) + 2\sin^2(2x)$

$= 2$

$$\boxed{v_1 = \frac{\sin 2x}{4} - \frac{\ln|\sec(2x) + \tan(2x)|}{4}}$$

$$y_p = \left(\frac{\sin 2x - \ln|\sec(2x) + \tan(2x)|}{4} \right) \cos 2x - \frac{\sin 2x \cos 2x}{4}$$

$$\boxed{y_p = \frac{-(\cos 2x) \ln|\sec(2x) + \tan(2x)|}{4}}$$

$$\boxed{\therefore y_g(x) = c_1' \cos 2x + c_2' \sin 2x - \frac{1}{4} \cos 2x \cdot \ln|\sec 2x + \tan 2x|}$$

④ $y'' + 2y' + 5y = e^{-x}\sec(2x)$

$$y'' + 2y' + 5y = 0 \Rightarrow m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$\Rightarrow \boxed{m = -1 \pm 4i}$$

$$\boxed{y_c(x) = e^{-x}\left(c_1'\cos 4x + c_2'\sin 4x\right)}$$

$$y_p(x) = V_1 y_1 + V_2 y_2$$

$$\Rightarrow y_p(x) = V_1 e^{-x}\cos 4x + V_2 e^{-x}\sin 4x$$

$$V_1 = \int \frac{-y_2 R}{W}\,dx$$

$$\Rightarrow V_1 = \int \frac{-e^{-x}\sin 4x \cdot e^{-x}\sec 2x}{4e^{-2x}}\,dx$$

$$= \frac{1}{4}\int \sin 4x \sec 2x\,dx$$

$$= -\frac{1}{4}\cos 2x$$

$$V_2 = \int \frac{y_1 R}{W}\,dx$$

$$\Rightarrow V_2 = \int \frac{e^{-x}\cos 4x \cdot e^{-x}\sec(2x)}{4e^{-2x}}\,dx$$

$$= \frac{1}{4}\int (\cos 4x)(\sec 2x)\,dx$$

$$= \frac{1}{4}\int (2\cos 2x - \sec 2x)\,dx$$

$$= \frac{\sin 2x}{4} - \frac{\ln|\sec 2x + \tan 2x|}{8}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x}\cos 4x & e^{-x}\sin 4x \\ -4e^{-x}\sin 4x & 4e^{-x}\cos 4x \end{vmatrix}$$

$$= 4e^{-2x}\cos^2 4x - \cancel{e^{-2x}\sin 4x\cos 4x} + \cancel{e^{-2x}\cos 4x\sin 4x} + 4e^{-2x}\sin^2 4x$$

$$= 4e^{-2x}$$

$$\therefore y_p(x) = \left(\frac{-1}{4}\cos 2x\right)\left(e^{-x}\cos 4x\right) + \left(\frac{\sin 2x}{4} - \frac{\ln|\sec 2x + \tan 2x|}{8}\right)\left(e^{-x}\sin 4x\right)$$

$$= \frac{1}{4}e^{-x}\left[\sin 2x\sin 4x - \cos 2x\cos 4x\right] - \frac{e^{-x}\sin 4x \cdot \ln|\sec(2x) + \tan 2x|}{8}$$

$$\boxed{y_p(x) = -\frac{1}{4}e^{-x}\cos 6x - \frac{e^{-x}\sin 4x \cdot \ln|\sec(2x) + \tan(2x)|}{8}}$$

$$\therefore y_g(x) = e^{-x}\left[c_1'\cos(4x) + c_2'\sin(4x) - \frac{\cos 6x}{4} - \frac{\sin 4x \cdot \ln|\sec(2x) + \tan(2x)|}{8}\right]$$

(5) $y'' + 2y' + y = e^{-x}\log x$

$y'' + 2y' + y = 0 \Rightarrow m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0$

$\boxed{m = -1}$

$$\boxed{y_c(x) = c_1 e^{-x} + c_2 x e^{-x}}$$

$y_p(x) = V_1 y_1 + V_2 y_2$

$= V_1 e^{-x} + V_2 x e^{-x}$

$V_1 = \int \frac{-y_2 R}{W} dx$

$= \int \frac{-xe^{-x} \cdot e^{-x}\log x}{e^{-2x}} dx$

$= \int -x\log x \, dx$

$V_2 = \int \frac{y_1 R}{W} dx$

$= \int \frac{e^{-x} \cdot e^{-x}\log x}{e^{-2x}} dx$

$= \int \log x \, dx$

$\therefore V_2 = (x\log x - 1)$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix}$

$= -xe^{-2x} + e^{-2x} + xe^{-2x}$

$\Rightarrow W = e^{-2x}$

$x \quad \log x$

$-1 \quad x\log x - 1$

$= -\left[x^2\log x - x - \int (x\log x - 1)\, dx\right]$

$V_1 = \frac{x^2(1 - 2\ln x)}{4}$

$I = \int x\log x \, dx$

$= x^2\log x - x - \int x\log x \, dx + x$

$2I = x^2\log x$

$I = \frac{x^2}{2}\log x$

$y_p(x) = \frac{x^2}{4}(1 - 2\ln x) \cdot e^{-x} + (x\log x - 1)xe^{-x}$

$= \frac{x^2}{4}e^{-x} - \frac{x^2\ln x}{2}e^{-x} + x^2\ln x \cdot e^{-x} - xe^{-x}$

$= \frac{x^2}{4}e^{-x} + \frac{x^2}{2}\ln x e^{-x} - xe^{-x}$

$$\boxed{y_p(x) = xe^{-x}\left[\frac{x}{4} + \frac{x\ln x}{2} - 1\right]}$$

$$\boxed{y_g(x) = \frac{e^{-x}}{4}\left[c_1 + c_2 x + x^2 + 2x^2\ln x - 4x\right]}$$

(6) $y'' - 3y' + 2y = (1+e^{-x})^{-1}$

$y'' - 3y' + 2y = 0 \rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow \boxed{m=2}$ or $\boxed{m=1}$

$$\boxed{y_c(x) = c_1 e^{+2x} + c_2 e^{x}}$$

$y_p(x) = v_1 y_1 + v_2 y_2$

$= v_1 e^{2x} + v_2 e^{x}$

$v_1 = \int \frac{-y_2 R}{W} dx$

$= \int \frac{-e^{x} \cdot (1+e^{-x})^{-1}}{-e^{3x}} dx$

$= \int \frac{1}{e^{x} + e^{2x}} dx$

$= \int \frac{e^{-x}}{1+e^{x}} dx$

$\therefore v_1 = \ln(1+e^{x}) - e^{-x} - x$

$v_2 = \int \frac{y_1 R}{W} dx$

$= \int \frac{e^{2x}}{-e^{3x}(1+e^{-x})} dx$

$= -\int \frac{1}{1+e^{x}} dx$

$\therefore v_2 = \ln(1+e^{x}) - x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} e^{2x} & e^{x} \\ 2e^{2x} & e^{x} \end{vmatrix}$

$= e^{3x} - e2e^{3x}$

$W = -e^{3x}$

$y_p(x) = e^{2x}(\ln(1+e^{x}) - e^{-x} - x) + (\ln(1+e^{x}) - x)(e^{x})$

$= e^{2x}\ln(1+e^{x}) - e^{x} - xe^{2x} + e^{x}\ln(1+e^{x}) - xe^{x}$

$= e^{x}\left[e^{x}\ln(1+e^{x}) - 1 - xe^{x} + \ln(1+e^{x}) - x\right]$

$= e^{x}\left[(e^{x}+1)\left[\ln(1+e^{x}) - x\right] - 1\right]$

$$\boxed{y_p(x) = e^{x}(e^{x}+1)(\ln(1+e^{x}) - x) - e^{x}}$$

$$\boxed{y_g(x) = c_1 e^{2x} + c_2 e^{x} + e^{x}(e^{x}+1)(\ln(1+e^{x}) - x)}$$

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MODULE-3

(Power Series)

$x^2y'' + bxy' + cy = 0 \longrightarrow y = x^n$

$xy'' + y' + xy = 0 \longrightarrow \sum a_n x^n$

$\downarrow$

Power Series Solution

$xy'' + y' + xy = 0$

Let $a_0 + a_1x + a_2x^2$ is the solution

$\Rightarrow x(2a_2) + (2a_2x + a_1) + x(a_0 + a_1x + a_2x^2) = 0$

$\Rightarrow a_2x^3 + a_1x^2 + (a_0 + 4a_2)x + a_1 = 0$

$a_1 = 0$

$a_2 = 0$

$a_0 + 4a_2 = 0$

$\therefore 0$ is a solution ✓

$y = 0$ ✓

Let $y = \sum a_n x^n$

$\therefore y' = \sum n a_n x^{n-1}$

$\therefore y'' = \sum n(n-1) \cdot a_n \cdot x^{n-2}$

$\therefore xy'' + y' + xy = 0$

$\Rightarrow x \sum n(n-1) \cdot a_n \cdot x^{n-2} + \sum n \cdot a_n x^{n-1} + x \sum a_n x^n = 0$

$\Rightarrow \sum n(n-1) a_n x^{n-1} + \sum n \cdot a_n \cdot x^{n-1} + \sum a_n x^{n+1} = 0$

$(n \to n+2)$ $\quad$ $(n \to n+2)$

$\Rightarrow \sum (n+2)(n+1) \cdot a_n \cdot x^{n+1} + \sum (n+2) a_n x^{n+1} + \sum a_n x^{n+1} = 0$

$\Rightarrow (n+2)(n+1) a_{n+2} + (n+2) a_{n+2} + a_n = 0$

$$\Rightarrow a_{n+2} \cdot (n+2)\left[(n+1)+1\right] + a_n = 0$$

$$\Rightarrow a_{n+2} \cdot (n+2)^2 + a_n = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{-a_n}{(n+2)^2}}$$

Assume $a_0 = 1 \Rightarrow$ We get $a_2, a_4, a_6, a_8, \ldots$

$a_1 \Longrightarrow$ We get $a_3, a_5, a_7, a_9, \ldots$

$\therefore \sum a_n x^{2n+1}$

$$y = \sum a_n x^n$$

$$a_{n+2} \to a_n$$

is the solution when $\sum a_n x^n$ converges

$$\left[\text{for } \sum a_n \longrightarrow \text{Ratio Test} \longrightarrow \left|\frac{a_{n+1}}{a_n}\right| < 1\right]$$

$$\therefore \left|\frac{a_{n+1} \cdot x^{n+1}}{a_n \cdot x^n}\right| < 1$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \cdot |x| < 1$$

$$\Rightarrow |x| < \left|\frac{a_{n+1}}{a_n}\right| = R$$

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→ Analytical function:

$$f(x) \rightarrow \sum a_n x^n$$

$$\hookrightarrow \sum a_n (x - x_0)^n$$

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

$$\left| \frac{a_{n+1} \cdot x^{n+1}}{a_n x^n} \right| < 1$$

x_0 → Ordinary
x_0 → Single → Regular / Irregular

Ex. $2x^2 y'' + y' + 3xy = 0$

of form $y'' + P(x) y' + Q(x) y = 0$

$P(x)$ & $Q(x)$ → Analytic @ x_0
↳ Ordinary

$$2x^2 y'' + y' + 3xy = 0$$

$$\Rightarrow y'' + \frac{1}{2x^2} y' + \frac{3}{2x} y = 0$$

$$P(x) = 1/2x^2$$

$$Q(x) = 3/2x$$

x_0 → Singular Point
Regular / Irregular

$$y(x) = \sum a_n x^n$$

$$\left[\begin{array}{l} \lim_{x \to x_0} P(x) \cdot (x - x_0) \\ \lim_{x \to x_0} Q(x) \cdot (x - x_0)^2 \end{array} \right]$$

limit finite, ∴ Regular

Singular Point : $P(x)$ & $Q(x)$ are analytic at a point

Ex $y'' + y = 0$

$x_0 = 0$

Sol: $P(x) = 0$, $\varphi(x) = 1$] $y'' + P(x)y' + \varphi(x) = 0$

$\lim_{x \to 0} (0)x = 0$

$\lim_{x \to 0} 1 \cdot x^2 = 0$

} Finite, $\therefore$ Analytic throughout

@ $x_0 = 0$, Both $P(x) = 0$ & $\varphi(x) = 0$

$\therefore x_0 \to$ ordinary Point

$\therefore y = \sum a_n x^n$

$\Rightarrow y' = \sum n a_n x^{n-1}$

$\Rightarrow y'' = \sum n(n-1) \cdot a_n \cdot x^{n-2}$

Now, $y'' - y = 0$:

$\sum n \cdot (n-1) a_n \cdot x^{n-2} + \sum a_n x^n = 0$ $[\because y'' + y = 0]$

$[n \to n+2]$

$\sum (n+2)(n+1) a_{n+2} x^n + \sum a_n x^n = 0$

$\left[\sum a_n x^n + \sum b_n x^n = 0 \quad \therefore a_n + b_n = 0 \ \forall n \right]$

$\therefore a_n + (n+2)(n+1) a_{n+2} = 0 \quad \forall n$

$$\Rightarrow \boxed{a_{n+2} = \frac{(-1)\, a_n}{(n+1)(n+2)}}$$

$$a_2 = \frac{-a_0}{2} = \frac{(-1)a_0}{1 \cdot 2}$$

$$a_4 = \frac{-a_2}{12} = \frac{+a_0}{24} = \frac{(-1)(-1)a_0}{1 \cdot 2 \cdot 3 \cdot 4}$$

Similarly, $a_6 = \frac{(-1)a_0}{6!}$

$$\boxed{\therefore a_{2n} = \frac{(-1)^n a_0}{(2n)!}} \quad -(1)$$

$$a_3 = \frac{(-1)a_1}{2\cdot 3} = \frac{(-1)a_1}{3!}$$

$$a_5 = \frac{(-1)(-1)a_1}{2\cdot 3\cdot 4\cdot 5} = \frac{(-1)(-1)a_1}{5!}$$

Similarly, $a_7 = \frac{(-1)(-1)(-1)a_1}{7!}$

$$\boxed{a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}} \quad -(2)$$

$$\Longrightarrow \frac{a_{2n+1}}{a_{2n}} = \frac{a_1}{a_0}\cdot\frac{1}{(2n+1)}$$

As $y = \sum a_n x^n$

$$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots a_n x^n$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \dots a_{2n} x^{2n}) + (a_1 x + a_3 x^3 + a_5 x^5 + \dots a_{2n+1} x^{2n+1})$$

$$= \left(a_0 - \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 + \dots\right) + \left(a_1 x - \frac{a_1}{3!}x^3 + \frac{a_1}{5!}x^5 + \dots\right)$$

$$= (a_0)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + (a_1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$\therefore y = a_0 \cos x + a_1 \sin x$$

where $\cos x = \sum \frac{x^{2n}\cdot(-1)^n}{(2n)!}$

$\sin x = \sum \frac{x^{2n+1}\cdot(-1)^n}{(2n+1)!}$

$$\left[y = c_1 \cos x + c_2 \sin x \right]$$

Q) $y'' - y = 0$

$x_0 = 0$

$P(x) = 0$

$Q(x) = -1$

$y'' + P(x)y' + Q(x)y = 0$

$$\left.\begin{aligned} \lim_{x \to x_0} P(x) \cdot (x - x_0) &= \lim_{x \to 0} 0 = 0 \\ \lim_{x \to x_0} Q(x) \cdot (x - x_0)^2 &= \lim_{x \to 0} (-1)x = 0 \end{aligned}\right\} \text{Analytic throughout}$$

@ $x_0 = 0$, Both $P(x) = 0$ & $Q(x) = 0$

$\therefore x_0 \to$ Ordinary Point

$\therefore y = \sum a_n x^n$

$\Rightarrow y' = \sum n \cdot a_n \cdot x^{n-1}$

$y'' = \sum (n)(n-1) \cdot a_n \cdot x^{n-2}$

Now,

$y'' - y = 0$

$\Rightarrow \sum (n)(n-1) \cdot a_n \cdot x^{n-2} - \sum x^n \cdot a_n = 0$

$[n \to n+2]$

$\Rightarrow \sum (n+1)(n+2) \cdot a_{n+2} \cdot x^n - \sum a_n \cdot x^n = 0$

$\Rightarrow \therefore (n+1)(n+2) a_{n+2} - a_n = 0 \quad \forall n$

$$\Rightarrow \boxed{a_{n+2} = \frac{a_n}{(n+1)(n+2)}}$$

$\boxed{y = (c_1 + c_2)e^x}$ $\quad \left[\begin{aligned} c_1 &= a_0 \\ c_2 &= a_1 \end{aligned}\right]$

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Q) $y' = y$

$(1-x^2)y'' - 2xy' + P(P+1)y = 0$

Sol: $\Rightarrow y' - y = 0$

$\left[y'' + P(x)y' + Q(x)y = 0\right]$

$P(x) = 1$
$Q(x) = -1$ } Let $x_0 = 0$

$\lim_{x \to x_0} P(x)\cdot(x - x_0) = \lim_{x \to 0} x = 0$

$\lim_{x \to x_0} Q(x)\cdot(x - x_0)^2 = \lim_{x \to 0} (-1)(x)^2 = 0$

$\therefore$ @ $x_0 = 0$, Both $P(x)$ & $Q(x)$ are 0

$\therefore x_0 \rightarrow$ Ordinary Point

$y = \sum a_n x^n$

$y' = \sum n a_n x^{n-1}$

Now

$y' - y = 0$

$\Rightarrow \sum n \cdot a_n \cdot x^{n-1} - \sum a_n x^n = 0$

$[n \rightarrow n+1]$

$\Rightarrow \sum (n+1) a_{n+1} x^n - \sum a_n \cdot x^n = 0$

$\Rightarrow (n+1) a_{n+1} - a_n = 0$

$\Rightarrow \boxed{a_{n+1} = \frac{a_n}{n+1}}$

$a_1 = a_1 = a_0$

$a_2 = \frac{a_1}{2}$

$a_3 = \frac{a_2}{3} = \frac{a_1}{3 \cdot 2}$

$a_4 = \frac{a_3}{4} = \frac{a_1}{4 \cdot 3 \cdot 2}$

$$\boxed{\therefore \quad a_n = \frac{a_1}{n!}}$$

$y = \sum a_n x^n$

$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$= a_0 + a_0 x + \frac{a_1}{2!} x^2 + \frac{a_1}{3!} x^3 + \frac{a_1}{4!} x^4 + \cdots$

$= a_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right]$

$= a_0 e^x$

$$\boxed{y = a_0 e^x}$$

Q) $(1-x^2) y'' - (2x) y' + (p(p+1)) \cdot y = 0$

Sol: $y'' - \left(\frac{2x}{1-x^2}\right) y' + \left(\frac{p(p+1)}{1-x^2}\right) y = 0$

$P(x) = \frac{-2x}{1-x^2}$

$Q(x) = \frac{p(p+1)}{1-x^2}$

let $x_0 = 0$

$\lim_{x \to 0} P(x) \cdot x = \lim_{x \to 0} \frac{-2x^2}{1-x^2} = 0$

$\lim_{x \to 0} Q(x) \cdot x^2 = \lim_{x \to 0} \frac{x^2}{1-x^2} \left[p(p+1) \right] = 0$

@ $x_0 = 0$, $P(x) = 0$ & $Q(x) = 0$

$\therefore x_0 \rightarrow$ Ordinary Point

$$y = \sum a_n x^n$$

$$y' = \sum a_n \cdot n \cdot x^{n-1}$$

$$y'' = \sum n(n-1) \cdot a_n \cdot x^{n-2}$$

$$\cancel{(1-x^2) \sum n(n-1) a_n x^{n-2}}$$

$$y'' - \left(\frac{2x}{1-x^2}\right) y' + \left(\frac{p(p+1)}{1-x^2}\right) y = 0$$

$$\Rightarrow \underset{[n \to n+2]}{\sum n(n-1) a_n \cdot x^{n-2}} - \left(\frac{2x}{1-x^2}\right) \underset{[n \to n+1]}{\sum n \cdot a_n \cdot x^{n-1}} + \frac{p(p+1)}{(1-x^2)} \sum a_n x^n = 0$$

$$\Rightarrow \sum (n+1)(n+2) a_{n+2} x^n - \left(\frac{2x}{1-x^2}\right) \sum (n+1) a_{n+1} x^n + \frac{p(p+1)}{(1-x^2)} \sum a_n x^n = 0$$

$$\Rightarrow (n+1)(n+2) a_{n+2} - \left(\frac{2x}{1-x^2}\right)(n+1)(a_{n+1}) + \frac{p(p+1)}{1-x^2} a_n = 0$$

$$\Rightarrow (1-x^2)(n+1)(n+2) a_{n+2} - (2x)(n+1)(a_{n+1}) + p(p+1) a_n = 0$$

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

$$y = \sum a_n x^n$$

$$1 \cdot y'' = \sum (n+2)(n+1)\, a_{n+2} \cdot x^n$$

$$-x^2 y'' = -\sum n(n-1) \cdot a_n x^n$$

$$-2xy' = -2 \sum n \cdot a_n \cdot x^n$$

$$p(p+1) \cdot y = \sum p(p+1)\, a_n \cdot x^n$$

$$y'' = \sum n(n-1)\, a_n x^{n-2}$$

$$= \sum (n+1)(n+2)\, a_{n+2} \cdot x^n$$

$$-x^2 y'' = -\sum (n+1)(n+2)\, a_{n+2} \cdot x^n$$

$$= -\sum n(n-1)\, a_n \cdot x^n$$

$$-2xy' = -2 \sum a_n \cdot n \cdot x^n$$

$$p(p+1) \cdot y = \sum p(p+1)\, a_n \cdot x^n$$

$$(n+2)(n+1)\, a_{n+2} - n(n-1)\, a_n - 2\, n\, a_n + p(p+1)\, a_n = 0$$

$$\Rightarrow (n+2)(n+1)\, a_{n+2} = a_n \left[n(n-1) + 2n - p(p+1) \right]$$

$$\Rightarrow \boxed{a_{n+2} = \frac{a_n \left(n(n+1) - p(p+1) \right)}{(n+1)(n+2)}}$$

9) $y'' + xy' + y = 0$

Sol: $y'' + P(x) \cdot y' + Q(x) y = 0$

$$P(x) = x$$

$$Q(x) = 1$$

@ $x_0 = 0$, $\lim_{x \to 0} (x \cdot)(x) = 0$

$$\lim_{x \to 0} (1)(x)^2 = 0$$

$\therefore x_0 \rightarrow$ Ordinary Point

$$y = \sum a_n x^n$$

$$y' = \sum n \cdot a_n \cdot x^{n-1}$$

$$y'' = \sum n(n-1) \cdot a_n \cdot x^{n-2}$$

$$y'' + xy' + y = 0$$

$$\Rightarrow \sum n(n-1)\cdot a_n \cdot x^{n-2} + x\sum n\cdot a_n \cdot x^{n-1} + \sum a_n x^n = 0$$

$[n \to n+2]$

$$\Rightarrow \sum (n+1)(n+2)\cdot a_{n+2}\cdot x^n + \sum n\cdot a_n \cdot x^n + \sum a_n x^n = 0$$

$$\therefore \quad (n+1)(n+2)a_{n+2} + na_n + a_n = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+2} + (n+1)a_n = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{-a_n}{n+2}}$$

$$a_2 = \frac{-a_0}{2}$$

$$a_4 = \frac{-a_2}{4} = \frac{+a_0}{2\cdot 4}$$

$$a_6 = \frac{-a_4}{6} = \frac{-a_0}{2\cdot 4\cdot 6}$$

$$\vdots$$

$$a_{2n} = \frac{a_0(-1)^n}{2\cdot 4\cdots 2n} = \frac{(-1)^n a_0}{2^n \cdot n!}$$

$$a_3 = \frac{-a_1}{3}$$

$$a_5 = \frac{-a_3}{5} = \frac{+a_1}{3\cdot 5}$$

$$a_7 = \frac{-a_5}{7} = \frac{-a_1}{3\cdot 5\cdot 7}$$

$$\vdots$$

$$a_{2n+1} = \frac{a_1(-1)^n}{3\cdot 5\cdots(2n+1)} = \frac{a_1(-1)^n \cdot 2^n \cdot n!}{(2n+1)!}$$

As $y = \sum a_n x^n$

$$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \cdots) + (a_1 x + a_3 x^3 + \cdots)$$

$$= \left(a_0 + \frac{-a_0}{2}x^2 + \frac{+a_0}{4\cdot 2}x^4 + \cdots\right) + \left(a_1 x + \frac{-a_1}{3}x^3 + \frac{+a_1}{5\cdot 3}x^5 + \cdots\right)$$

$$= a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 4} - \frac{x^6}{6\cdot 4\cdot 2}\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 3} - \frac{x^7}{7\cdot 5\cdot 3} + \cdots\right)$$

$$= a_0\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n \cdot n!}\right) + a_1\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cdot 2^n \cdot n!}{(2n+1)!}\right)$$

Continuing legendre DE :

$$\boxed{a_{n+2} = \frac{a_n\left(n(n+1) - p(p+1)\right)}{(n+1)(n+2)}}$$

$$\Rightarrow a_{n+2} = \frac{a_n\left(n^2+n-p^2-p\right)}{(n+1)(n+2)}$$

$$= \frac{a_n\left((n-p)(n+p) + (n-p)\right)}{(n+1)(n+2)}$$

$$\boxed{a_{n+2} = \frac{a_n\left((n-p)(n+p+1)\right)}{(n+1)(n+2)}}$$

$$a_2 = \frac{-p(p+1)\,a_0}{2!}$$

$$a_4 = \frac{+p(p+1)(p+2)(p+3)}{4!} \cdot a_0$$

$$a_8 = \frac{-p(p+1)(p-2)(p+3)(p-4)(p+5)(p-6)(p+7)}{8!} \cdot a_0$$

$$a_7 = \frac{(-1)^3 (p-1)(p+2)(p-3)(p+4)(p-5)(p+6)}{7!} \cdot a_1$$

——— × ———

~~Taylor series:~~

~~$f(x) = \sum_{k=0} \frac{f^{k}(0)}{k!} x^k$~~

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{\frac{-x^2}{2}} = 1 - \frac{x^2}{2} + \frac{x^4}{4.2} - \frac{x^6}{6.4.2}$$

$$\therefore y = a_0 e^{-x^2/2} + a_1 y_2$$

$y_2(x) = y_1(x) \cdot V_1(x)$

$$V(x) = \int \frac{1}{y_1^2(x)} \cdot e^{-\int P(x)\,dx}\, dx$$

$$y_1(x) = e^{-x^2/2}$$

$$= \int \frac{1}{e^{-2x^2}} \cdot e^{-\int x\,dx}\, dx$$

$$= \int \frac{1}{e^{-x^2}}\, e^{-x^2/2}\, dx$$

$$= \int e^{x^2/2}\, dx$$

$$y_2(x) = e^{-x^2/2} \cdot \int e^{x^2/2}\, dx$$

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→ **Frobenius Method** :

$P(x)$ is Analytical if $P(x)$ can be expressed as $\sum a_n x^n$.

$$y'' + P(x) \cdot y' + Q(x)\, y = 0$$

If $P(x)$ and $Q(x)$ are analytic at $x = 0$ (or) any x_0, Then x_0 is called Ordinary Point.

If x_0 → Singular Point
- Regular Point [Power series solution Exists]
- Irregular Point [Then, No Power series solution exists]

x_0 is a Regular Point if

$$\lim_{x \to x_0} P(x) \cdot (x - x_0)$$

$$\lim_{x \to x_0} Q(x) \cdot (x - x_0)^2$$ both exist and are finite.

Conventionally, $x_0 = 0$.

If $x_0 \rightarrow$ Regular Point, Then Frobenius Method can be applied,

If $y = x^m \sum a_n x^n$

$$\left[\begin{array}{c} m : \text{exponent} \\ m \in \mathbb{Q} \end{array}\right]$$

Used when $x_0 \rightarrow$ not ordinary

Ex. $y'' + \frac{2}{x} y' - \frac{2}{x^2} y = 0$

$P(x) = 2/x$

$Q(x) = -2/x^2$

$\therefore$ Not Analytical As $P(x)$ & $Q(x)$ cannot be expressed as $\sum a_n x^n$, $n \geq 0$ [for $x_0 = 0$]

$\therefore$ Origin $\rightarrow$ Not Regular Point

$\therefore$ Origin $\rightarrow$ Singular Point

$$\left.\begin{array}{l} x P(x) = 2 \\ x^2 Q(x) = -2 \end{array}\right\} \text{Analytic functions}$$

Ex. legender D.E :

$$(1-x^2) y'' - 2x y' + p(p+1) y = 0$$

$$x = 0$$

$$\Rightarrow y'' - \left(\frac{2x}{1-x^2}\right) y' + \left(\frac{p(p+1)}{1-x^2}\right) y = 0$$

$1 - x^2 = 0 \Rightarrow x$ are singular Points

$x \in [-1, 1]$

$[\because x = \cos\theta]$

legender DE is used in the domain

$x_0 = 1$

$$\lim_{x \to x_0} (x - x_0) P(x)$$

$$= \lim_{x \to 1} (x-1) \frac{-2x}{1-x^2}$$

$$= \lim_{x \to 1} \frac{2x}{1+x}$$

$$= 1$$

$$\lim_{x \to x_0} (x - x_0)^2 Q(x)$$

$$= \lim_{x \to 1} (x-1)^2 \cdot \frac{p(p+1)}{1-x^2}$$

$$= \lim_{x \to 1} \frac{(1-x)}{1+x} p(p+1)$$

$$= 0$$

$x_0 = -1$

$$\lim_{x \to x_0} (x - x_0) P(x)$$

$$= \lim_{x \to -1} (x+1) \frac{-2x}{1-x^2}$$

$$= \lim_{x \to -1} \frac{2x}{x-1}$$

$$= 1$$

$$\lim_{x \to x_0} (x - x_0)^2 Q(x)$$

$$= \lim_{x \to -1} (x+1)^2 \cdot \frac{p(p+1)}{1-x^2}$$

$$= \lim_{x \to -1} \frac{x+1}{1-x} p(p+1)$$

$$= 0$$

$\therefore$ ~~Reg~~ Regular Singular Point.

Ex Bessel's DE of order p $[p \geq 0]$

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

$$\Rightarrow y'' + \frac{1}{x} y' + \left(1 - \frac{p^2}{x^2}\right) y = 0$$

$$x P(x) = 1$$

$$x^2 Q(x) = x^2 - p^2 = -p^2 \; [@ \; x = 0]$$

Q) Determine nature of point $x = 0$ for the following

(i) $y'' + (\sin x) y = 0$

Ordinary $\because P(x) = 0$
$Q(x) = \sin x$

(ii) $x y'' + (\sin x) y = 0$

$$\Rightarrow y'' + \left(\frac{\sin x}{x}\right) y = 0$$

$$P(x) = 0$$

$$Q(x) = \frac{\sin x}{x}$$

$P(x) \rightarrow$ Analytic

$Q(x) \rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1 \rightarrow$ Analytic

$\therefore$ Ordinary [No need for Frobenius Method]

(iii) $x^2 y'' + (\sin x) y = 0$

$$\Rightarrow y'' + \left(\frac{\sin x}{x^2}\right) y = 0$$

$P(x) \rightarrow 0$ (Analytical)

$Q(x) \rightarrow \frac{\sin x}{x^2}$ (Not Analytical)

$$\lim_{x \to 0} x^2 \frac{\sin x}{x^2} = 0$$

$\therefore$ Regular

Singular [$\because Q(x) \rightarrow$ Not-Analytic]

$\therefore$ Regular Singular Point.

Frobenius Method is required

$\rightarrow$ <u>Frobenius Series</u> :

Ex. $x^{1/2}(1 + x + x^2 + \cdots)$

$x^{-1}(1 + x + x^2 + \cdots)$

Q) $(2x^3) y'' + x(2x+1) y' - y = 0$ @ $x_0 = 0$

~~@ x = 0~~

Sol : $y'' + \frac{x(2x+1)}{2x^2} - \frac{1}{2x^2} y = 0$

$$P(x) = \frac{x(2x+1)}{2x^2} \qquad \Big| \qquad q(x) = \frac{-1}{2x^2}$$

$$= \frac{2x^2+1}{2x^2}$$

$$= 1 + \frac{1}{2x^2}$$

↳ Not Analytical

$$\therefore x P(x) = \frac{x^2(2x+1)}{2x^2}$$

$$= x + \frac{1}{2}$$

$\therefore$ Analytical

$\therefore x = 0 \rightarrow$ Regular Singular Point

$$x^2 q(x) = -\frac{1}{2}$$

$\therefore$ Analytical, $\therefore$ Frobenius Method is used

$$2x^2 y'' + x(2x+1)y' - y = 0$$

$$y = x^m \sum a_n x^n$$

[$m \rightarrow$ Not a General Point
↳ Specific Point]

$$\Rightarrow y = \sum a_n x^{m+n} \quad -(1)$$

$$\Rightarrow y' = \sum (m+n) a_n \cdot x^{(m+n-1)} \quad -(2)$$

$$\Rightarrow y'' = \sum (m+n)(m+n-1) a_n \cdot x^{(m+n-2)} \quad -(3)$$

Substitute (1), (2) and (3) in Original DE

$$2x^2 \sum (m+n)(m+n-1) a_n \cdot x^{(m+n-2)} + x(2x+1) \sum (m+n) a_n \cdot x^{(m+n-1)}$$

$$- \sum a_n x^{m+n} = 0$$

$n = n-1$

$$\Rightarrow 2(m+n)(m+n-1) \sum a_n x^{m+n} + 2(m+n) \sum a_n \cdot x^{(m+n+1)}$$

$$+ (m+n) \sum a_n \cdot x^{m+n} - \sum a_n x^{m+n} = 0$$

$$\Rightarrow 2a_n(m+n)(m+n-1) + 2(m+n-1)a_{n-1} + (m+n)a_n - a_n = 0$$

$$\Rightarrow a_{n-1}\left[2(m+n-1)\right] + a_n\left[2(m+n)(m+n-1) + (m+n-1)\right] = 0$$

$$\Rightarrow a_{n-1}\left[2(m+n-1)\right] + a_n\left[(2(m+n)+1)(m+n-1)\right] = 0$$

$$\Rightarrow a_n = \frac{-2(m+n-1)}{(2m+2n+1)(m+n-1)}\, a_{n-1}$$

$$\Rightarrow \boxed{a_n = \left(\frac{-2}{2(m+n)+1}\right) a_{n-1}}$$

Tutorial:

① $x^2y'' + xy' + (x^2-p^2)y = 0$

Sol: $y'' + \left(\frac{1}{x}\right)y' + \left(\frac{x^2-p^2}{x^2}\right)y = 0$

$y'' + \left(\frac{1}{x}\right)y' + \left(1-\frac{p^2}{x^2}\right)y = 0$

$P(x) = \frac{1}{x}$

$Q(x) = 1 - \frac{p^2}{x^2}$ } Not Analytical

$xP(x) = 1$

$x^2 Q(x) = x^2 - p^2$ } Analytical @ $x_0 = 0$

$\therefore x_0 = 0 \rightarrow$ Regular Singular Point

$m(m-1) + P_0 m + Q_0 = 0$

$\Rightarrow m(m-1) + \frac{1}{2}m - \frac{1}{2} = 0$

$\Rightarrow m^2 - \frac{1}{2}m - \frac{1}{2} = 0$

$\Rightarrow 2m^2 - m - 1 = 0 \Rightarrow m = 1, -\frac{1}{2}$

$x^2y'' + xy' + (x^2-p^2)y = 0$

$y = x^m \sum a_n x^n = \sum a_n x^{n+m}$ — ①

$y' = \sum (m+n)a_n \cdot x^{(m+n-1)}$ — ②

$y'' = \sum (m+n)(m+n-1)a_n \cdot x^{(m+n-2)}$ — ③

Sub in Original equation :

$$x^2 \sum (m+n)(m+n-1) a_n \cdot x^{m+n-2}$$
$$+ x \sum (m+n) a_n x^{m+n-1}$$
$$+ (x^2 - p^2) \sum a_n x^{m+n} = 0$$

$$\Rightarrow \sum (m+n)(m+n-1) a_n \cdot x^{m+n}$$
$$+ \sum (m+n) a_n \cdot x^{m+n}$$
$$+ \sum a_n x^{m+n+2} \quad [n \to n-2]$$
$$- p^2 \sum a_n \cdot x^{m+n} = 0$$

$$\Rightarrow \cancel{(m+n)(m+n-1) a_n + (m+n) a_n + a_n = 0}$$

$$\Rightarrow \cancel{a_n \left[(m+n)(m+n) + 1 \right] = 0}$$

$$\Rightarrow \cancel{a_n = \frac{(m+n)^2 + 1}{}}$$

$$\Rightarrow \sum (m+n)(m+n-1) a_n \cdot x^{m+n} + \sum (m+n) a_n \cdot x^{m+n}$$
$$+ \sum a_{n-2} x^{m+n} - p^2 \sum a_n x^{m+n} = 0$$

$$\therefore (m+n)(m+n-1) a_n + (m+n) a_n + a_{n-2} - p^2 a_n = 0$$

$$\Rightarrow a_n \left[(m+n)^2 - p^2 \right] = - a_{n-2}$$

$$\Rightarrow a_n \left[p^2 - (m+n)^2 \right] = a_{n-2}$$

$m^2 - p^2 = 0 \rightarrow$ Indicial Equation

$$\boxed{m = \pm p}$$

$$\boxed{\therefore a_n = \frac{a_{n-2}}{p^2 - (m+n)^2}}$$

Q) $2x^2y'' + 2x^2y' + xy' - y = 0$ $\quad \left[y = \sum a_n x^{n+m}\right]$

$$2x^2y'' = 2\sum (n+m)(n+m-1)\, a_n \cdot x^{n+m}$$

$$2x^2y' = 2\sum (n+m) \cdot a_n \cdot x^{n+m+1}$$

$$= 2\sum (n+m-1)\, a_{n-1} \cdot x^{n+m}$$

$$xy' = \sum (n+m)\, a_n \cdot x^{n+m}$$

$$-y = -\sum a_n x^{n+m}$$

$$\Rightarrow 2(n+m)(n+m-1)\, a_n + 2(n+m-1)\, a_{n-1} + (n+m)a_n - a_n = 0$$

$$\Rightarrow a_n\left[2(n+m)(n+m-1) + (n+m) - 1\right] + 2(n+m-1)\, a_{n-1} = 0$$

$$\Rightarrow a_n\left[2(n+m)(n+m-1) + (n+m-1)\right] + 2(n+m-1)\, a_{n-1} = 0$$

$$\Rightarrow a_n\left[2(n+m)+1\right] + 2a_{n-1} = 0$$

$$\Rightarrow \boxed{a_n = \frac{-2a_{n-1}}{2(n+m)+1}}$$

$n=0$:

$$a_0\left[2m+1\right] = 0$$

$m = 1$

$$a_n = \frac{-2a_{n-1}}{2n+3}$$

$$\boxed{a_n = \frac{-a_{n-1}}{n+3/2}}$$

$$a_1 = \frac{-2a_0}{5}$$

$$a_2 = \frac{-2a_1}{7} = \frac{4a_0}{5 \cdot 7}$$

$m = -1/2$

$$a_n = \frac{-2a_{n-1}}{2n}$$

$$\boxed{\therefore a_n = -\frac{a_{n-1}}{n}}$$

$$a_1 = -a_0$$

$$a_2 = \frac{-a_1}{2} = a_0/2$$

$$a_3 = \frac{-a_2}{3} = \frac{-a_0}{3 \cdot 2}$$

Node — Null — Head

Sub in Original equation :

$$x^2 \sum (m+n)(m+n-1) a_n \cdot x^{m+n-2}$$
$$+ x \sum (m+n) a_n x^{m+n-1}$$
$$+ (x^2 - p^2) \sum a_n x^{m+n} = 0$$

$$\Rightarrow \sum (m+n)(m+n-1) a_n \cdot x^{m+n}$$
$$+ \sum (m+n) a_n \cdot x^{m+n}$$
$$+ \sum a_n x^{m+n+2} \quad [n \to n-2]$$
$$- p^2 \sum a_n \cdot x^{m+n} = 0$$

$$\Rightarrow (m+n)(m+n-1) a_n + (m+n) a_n + a_n = 0$$

$$\Rightarrow a_n \left[(m+n)(m+n) + 1 \right] = 0$$

$$\Rightarrow a_n = \cancel{(m+n)^2 + 1}$$

$$\Rightarrow \sum (m+n)(m+n-1) a_n \cdot x^{m+n} + \sum (m+n) a_n \cdot x^{m+n}$$
$$+ \sum a_{n-2} x^{m+n} - p^2 \sum a_n x^{m+n} = 0$$

$$\Rightarrow \therefore (m+n)(m+n-1) a_n + (m+n) a_n + a_{n-2} - p^2 a_n = 0$$

$$\Rightarrow a_n \left[(m+n)^2 - p^2 \right] = - a_{n-2}$$

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$$= 2\sum (m+n-1)\, a_{n-1} \cdot x^{m+n}$$

$$xy' = \sum (m+n)\, a_n \cdot x^{m+n}$$

$$-y = -\sum a_n x^{m+n}$$

$$\Rightarrow 2(m+n)(m+n-1)\, a_n + 2(m+n-1)\, a_{n-1} + (m+n)a_n - a_n = 0$$

$$\Rightarrow a_n\left[2(m+n)(m+n-1) + (m+n) - 1\right] + 2(m+n-1)\, a_{n-1} = 0$$

$$\Rightarrow a_n\left[2(m+n)(m+n-1) + (m+n-1)\right] + 2(m+n-1)\, a_{n-1} = 0$$

$$\Rightarrow a_n\left[2(m+n)+1\right] + 2a_{n-1} = 0$$

$$\Rightarrow \boxed{a_n = \frac{-2a_{n-1}}{2(m+n)+1}}$$

$n=0$:

$$a_0\left[2m+1\right] = 0$$

$m=1$

$$a_n = \frac{-2a_{n-1}}{2n+3}$$

$$\boxed{a_n = \frac{-a_{n-1}}{n+3/2}}$$

$$a_1 = \frac{-2a_0}{5}$$

$$a_2 = \frac{-2a_1}{7} = \frac{4a_0}{5\cdot 7}$$

$m = -1/2$

$$a_n = \frac{-2a_{n-1}}{2n}$$

$$\boxed{\therefore a_n = -\frac{a_{n-1}}{n}}$$

$$a_1 = -a_0$$

$$a_2 = \frac{-a_1}{2} = a_0/2$$

$$a_3 = \frac{-a_2}{3} = \frac{-a_0}{3\cdot 2}$$

$$y = x^m \sum a_n x^m$$

$$= x \cdot a_0 \left(1 - \frac{2}{5}x + \frac{4}{35} \cdot x^2 \ldots\right)$$

$$y = x^m \sum a_n x^m$$

$$= x^{-\frac{1}{2}} \cdot a_0 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots\right)$$

$$y = c_1 y_1 + c_2 y_2$$

$$\Longrightarrow y = c_1 \left[x a_0 \left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \ldots\right)\right] + c_2 \cdot x^{-\frac{1}{2}} \left[1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots\right]$$

→ **Bessel DE :**

$$a_n = \frac{a_{n-2}}{p^2 - (m+n)^2} \qquad [m = \pm p]$$

$m = p$:

$$a_n = \frac{a_{n-2}}{p^2 - (p+n)^2}$$

$$\therefore a_n = \frac{-a_{n-2}}{(2p+n)(n)}$$

$$a_2 = \frac{-a_0}{4(p+1)}$$

$$a_4 = \frac{-a_2}{8(p+2)} = \frac{a_0}{4 \cdot 8 \cdot (p+1)(p+2)}$$

$$a_3 = \frac{-a_1}{(2p+3)(3)}$$

$$a_5 = \frac{-a_3}{(2p+5)(5)} = \frac{a_1}{3 \cdot 5 \cdot (2p+3)(2p+5)}$$

$$a_{2n} = \frac{(-1)^n \cdot a_0}{4^n \cdot n! \cdot \prod_{m=1}^{n} (p+m)}$$

$m = -p$:

$$a_n = \frac{a_{n-2}}{p^2 - (m-p)^2}$$

$$\therefore a_n = \frac{a_{n-2}}{(n)(2p-n)}$$

$$a_2 = \frac{a_0}{4(p-1)}$$

$$a_4 = \frac{a_2}{8(p-2)} = \frac{a_0}{4 \cdot 8 \cdot (p-1)(p-2)}$$

$$a_3 = \frac{a_1}{3(2p-3)}$$

$$a_5 = \frac{a_3}{5(2p-5)} = \frac{a_1}{5 \cdot 3 \cdot (2p-3)(2p-5)}$$

$$a_{2n} = \frac{a_0}{4^n \cdot n! \prod_{m=n}^{1} (p-m)}$$

$$\left[p^2-(m+n)^2\right] a_n = a_{n-2}$$

$m = p$:

$$\left[(p+1)^2-p^2\right] a_1 = 0$$

$$\therefore a_1 = 0 \Rightarrow a_3 = 0 \Rightarrow a_5 = 0 \ldots$$

$$y(x) = J_p(x)$$

$$= \sum_{n=1}^{\infty} a_{2n} \cdot x^{2n}$$

Bessel function of first kind of order 'p'.

→ <u>Gamma function</u>:

$$\Gamma(n+1) = n!$$

$$\left[\Gamma(x+1) = \int_0^{\infty} x^t e^{-x}\, dx\right]$$

$$\Gamma(n+1) = n(n-1)! = n\Gamma(n)$$

→ <u>Theorem</u>:

$xP(x)$ & $x^2 Q(x)$ → Valid in $|x| < R$, $R > 0$

Indicial Eq : $m_1 \geq m_2$

∴ Frobenius Eq. exists.

1/4/24

$$(1-x^2)y'' - 2x \cdot y' + p(p+1) = 0$$

$$P_p(x) \leftarrow P_n(x)$$

$$P_n(1) = 1 \longrightarrow P_0(1) = 1$$
$$P_1(1) = 1$$
$$P_2(1) = 1$$

$$\int_{-1}^{+1} P_m(x) \cdot P_n(x)\, dx = \delta_{mn} \cdot \frac{2}{2n+1} \qquad \begin{bmatrix} \delta_{mn} = 1 & \text{if } m = n \\ \delta_{mn} = 0 & \text{if } m \neq n \end{bmatrix}$$

$$f(x) = \sum a_n \cdot P_n(x)$$

Any function

in $[-1, 1]$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Generating Function:

$$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n} (x^2-1)^n$$

Generates Legender ~~Poynt~~ Polynomials

→ Generating functions:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum P_n(x) \cdot t^n \qquad \begin{pmatrix} x = \cos\theta \\ t = \frac{r'}{r} \end{pmatrix}$$

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ 1 & 0 \le x \le 1 \end{cases}$$

$$\sum a_n P_n(x)$$

$\hookrightarrow a_0, a_1, a_2$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\vdots$

Ex. $(1-x^2)y'' - 2xy' + 12y = 0$

$y(1) = 2$ $\qquad \int_{-1}^{+1} y^2 \, dx = 0$

Sol: Given Eq is Legender DE

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Where $n = 3$

$P_n(x) \longrightarrow$ Solution of DE

$\therefore P_3(x)$ is the solution

$\therefore k P_3(x)$ is also a solution

Given $k P_3(1) = 2$

But $P_3(1) = 1$

$\boxed{\therefore k = 2}$

$$P_3(x) = \frac{1}{6\times 8} \frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{3}{6\times 8} \frac{d^2}{dx^2}\left[(x^2-1)^2(2x)\right]$$

$$= \frac{1}{8} \frac{d^2}{dx^2}\left[(x^2-1)^2(x)\right]$$

$$x^5 - 2x^3 + x^2$$

$$= \frac{1}{8} \frac{d}{dx} 5x^4 - 6x^2 + 1$$

$$= \frac{1}{8}(20x^3 - 12x)$$

$$\int_{-1}^{+1} y^2 \, dx = \int_{-1}^{+1} k^2 P_3^2(x) \, dx$$

$$= 4 \int_{-1}^{+1} P_3^2(x) \, dx$$

$$= 4\left(\frac{2}{2(3)+1}\right) = \frac{8}{7}$$

$$\left[\int_{-1}^{+1} P_m(x) \cdot P_n(x) \, dx \quad \text{where } m = n \quad = \frac{2}{2n+1}\right]$$

$$4\int_{-1}^{1} P_3^2(x) \, dx = \frac{1}{16} \times \frac{128}{7} = \frac{8}{7}$$

$$J_p(x) = x^p \sum \frac{1}{n!\,\Pi}$$

$$x^2 J_{m+1} + x^4 J_m(x)$$

$$J_m(x) = x^m \sum \frac{1}{n!(m+m)!} \left(\frac{x}{2}\right)^{2n} \cdot x^2$$

LAPLACE TRANSFORM

$$\rightarrow \boxed{DE} \xrightarrow{L.T} \boxed{A.E} \xrightarrow{\text{Solve}} \boxed{Soln} \longrightarrow \boxed{f}$$

(Differential equation) (Algebraic Equation) $\downarrow$ (p)

$f \rightarrow$ solution of DE

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$$

$$L(y_1 \cdot y_2) = 0$$

$$\Downarrow$$

$$F(p) \cdot L(y) + \varphi(p) \cdot L(y) + R(p) \cdot L(y) - C = 0$$

$$\Rightarrow L(y) = \frac{C}{F(p) + \varphi(p) + R(p)}$$

$$y = L^{-1}\left(\frac{C}{F(p) + \varphi(p) + R(p)}\right)$$

2/4/24

$$L[a(t)] = \int_0^\infty e^{-pt} \cdot a(t)\, dt = F(p)$$

$$L[f(x)] = \int_0^\infty e^{-px} f(x)\, dx = F(p)$$

To prove:

$$L[af(x) + bg(x)] = a \cdot L[f(x)] + b \cdot L[g(x)]$$

Proof: let $af(x) + bg(x) = h(x)$

$$L[h(x)] = \int_0^\infty e^{-px} h(x)\, dx$$

$$= \int_0^\infty e^{-px} (af(x) + bg(x))\, dx$$

$$= \int_0^\infty e^{-px} \cdot a f(x)\, dx + \int_0^\infty e^{-px} \cdot b\, g(x)\, dx$$

$$= a \int_0^\infty e^{-px} f(x)\, dx + b \int_0^\infty e^{-px} g(x)\, dx$$

$$= a \cdot L[f(x)] + b \cdot L[g(x)] = \text{RHS}$$

Hence Proved

$\therefore$ Laplace Transform is a Linear Operator :

$\rightarrow$ $$L[k] = \int_0^\infty e^{-px} \cdot k\, dx$$

$$= k \int_0^\infty e^{-px}\, dx$$

$$= k \cdot \left[\frac{e^{-px}}{-p}\right]_0^\infty$$

$$= -\frac{k}{p}\left(e^{-\infty} - e^{-0}\right)$$

$$= -\frac{k}{p}(0-1)$$

$$\boxed{\therefore L(k) = \frac{k}{p}}$$

$$L[k] = k \cdot L[1]$$

$$= k \cdot \left(\frac{1}{p}\right)$$

$$= k/p$$

$$\boxed{\therefore L[1] = 1/p}$$

$\rightarrow$ $$L[x^n] = \int_0^\infty e^{-px} \cdot x^n\, dx$$

$$= \frac{x^n}{-p} \cdot e^{-px}\Big]_0^\infty - \int_0^\infty \frac{n x^{n-1}}{-p} e^{-px}\, dx$$

$$= -\frac{x^n}{p} e^{-px}\Big]_0^\infty + \frac{n}{p} \int_0^\infty x^{n-1} e^{-px}\, dx$$

e^{-px} x^n

$+p\, e^{-px}$ $\frac{x^{n+1}}{n+1}$

x^n e^{-px}

$-n x^{n-1}$ $\frac{e^{-px}}{-p}$

$$\Rightarrow L[x^n] = \frac{n}{p} \cdot L[x^{n-1}] + \left(\frac{x^n}{p} \cdot e^{-px}\right]_{\infty}^{0}$$

$$= \frac{n}{p} L[x^{n-1}]$$

$$\lim_{x \to 0} \frac{x^n}{p} e^{-px}$$

$$= \frac{1}{p} \lim_{x \to 0} \frac{x^n}{e^{px}}$$

$$= \frac{1}{p} \times 0 = 0$$

$$\boxed{\therefore L[x^n] = \frac{n}{p} \cdot L[x^{n-1}]}$$

$$\frac{1}{p} \lim_{x \to \infty} \frac{x^n}{e^{px}}$$

$$\cancel{\frac{1}{p} \lim_{x \to \infty} \frac{n x^{n-1}}{p e^{px}}}$$

$$= \frac{1}{p} \lim_{x \to \infty} \frac{x^n}{1 + px + \frac{p^2 x^2}{2!} + \dots}$$

$$= \frac{1}{p} \lim_{x \to \infty} \frac{1}{x^{-n} + \underbrace{\frac{p}{x^{n-1}} + \frac{p^2}{2! \cdot x^{n-2}} + \dots}_{0}}$$

$$= 1$$

$$L[x^n] = \frac{n}{p} \cdot \frac{n-1}{p} \cdot L[x^{n-2}]$$

$$= \frac{n}{p} \cdot \frac{n-1}{p} \cdot \frac{n-2}{p} \cdot L[x^{n-3}]$$

$$= \frac{n(n-1)(n-2)\dots 1}{p \cdot p \cdot p \cdots p} \cdot L[x^0]$$

$$= \frac{n!}{p^n} \cdot L[x^0]$$

$$= \frac{n!}{p^n} \cdot \frac{1}{p}$$

$$= \frac{n!}{p^{n+1}}$$

$$\boxed{F(p) = L[x^n] = \frac{n!}{p^{n+1}}}$$

$$\rightarrow L[e^{ax}] = \int_0^{\infty} e^{-px} \cdot e^{ax} \, dx$$

$$= \int_0^{\infty} e^{(a-p)x} \, dx$$

$$= \left. \frac{e^{(a-p)x}}{(a-p)} \right]_0^{\infty}$$

$$= \frac{1}{a-p} \left(e^{(a-p)x} \right]_0^{\infty}$$

$$= \frac{1}{(a-p)} (-1) = \frac{1}{p-a}$$

$$\boxed{L[e^{ax}] = \frac{1}{p-a}}$$

$$\rightarrow \boxed{L[e^{-ax}] = \frac{1}{p+a}}$$

$$\rightarrow L[\cos(ax)] = \int_0^\infty e^{-px} \cdot \cos ax \, dx$$

$\cancel{\cos ax \quad e^{px}}$

$\cancel{\sin ax \quad e^{px}}$

$$\cancel{I = \int_0^\infty e^{-px} \cdot \cos ax \, dx}$$

$x \rightarrow x+\pi$

$$\cancel{I = \int_0^\infty e^{-p}}$$

$$\cos ax = \frac{e^{iax} + e^{-iax}}{2}$$

$$L[\cos(ax)] = \frac{1}{2}\left(L[e^{iax}]\right) + \frac{1}{2}\left(L[e^{-iax}]\right)$$

$$= \frac{1}{2}\left(\frac{1}{p-ia} + \frac{1}{p+ia}\right)$$

$$= \frac{1}{2}\left(\frac{2p}{p^2+a^2}\right) = \frac{p}{p^2+a^2}$$

$$\boxed{L[\cos(ax)] = \frac{p}{p^2+a^2}}$$

$$\rightarrow L[\sin ax]$$

$$\left[\sin(ax) = \frac{e^{iax} - e^{-iax}}{2i}\right]$$

$$\sin ax = \frac{e^{iax} - e^{-iax}}{2i}$$

$$L[\sin(ax)] = \frac{1}{2i}\left(L[e^{iax}] - L[e^{-iax}]\right)$$

$$= \frac{1}{2i}\left(\frac{1}{p-ia} - \frac{1}{p+ia}\right)$$

$$= \frac{a}{p^2+a^2}$$

$$\boxed{\therefore L[\sin(ax)] = \frac{a}{p^2+a^2}}$$

$$\rightarrow L[\cosh x] = L\left[\frac{e^{ax}+e^{-ax}}{2}\right]$$

$$= \frac{1}{2}\left(L[e^{ax}] + L[e^{-ax}]\right)$$

$$= \frac{1}{2}\left(\frac{1}{p-a} + \frac{1}{p+a}\right)$$

$$\boxed{\therefore L[\cosh x] = \frac{p}{p^2-a^2}}$$

$$\rightarrow L[\sinh x] = L\left[\frac{e^{ax}-e^{-ax}}{2}\right]$$

$$= \frac{1}{2}\left(L[e^{ax}] - L[e^{-ax}]\right)$$

$$= \frac{1}{2}\left(\frac{1}{p-a} - \frac{1}{p+a}\right)$$

$$= \frac{a}{p^2-a^2}$$

$$\boxed{\therefore L[\sinh x] = \frac{a}{p^2-a^2}}$$

Q) Find Laplace Transform

$$f(x) = 2x^2 + 5e^{-2x} - 3\sin(4x)$$

$$\Rightarrow L[f(x)] = L[2x^2] + L[5e^{-2x}] - L[3\sin(4x)]$$

$$= 2 \cdot \frac{2!}{p^3} + 5 \cdot \frac{1}{p+2} + (-3)\frac{4}{p^2+16}$$

$$= \frac{4}{p^3} + \frac{5}{p+2} - \frac{12}{p^2+16}$$

Q) Find Inv. Laplace Transform

(a) $\frac{30}{p^4}$

(b) $\frac{1}{p^2+p}$

(c) $\frac{4}{p^3} + \frac{6}{p^2+4}$

Sol:

(a)

$$5 \times \frac{3!}{p^{3+1}}$$

$$= 5 \times L[x^3]$$

$$= L[5x^3]$$

$$\therefore f(x) = 5x^3$$

(c)

$$\left(2 \times \frac{2!}{p^{2+1}}\right) + 3 \times \left(\frac{2}{p^2+2^2}\right)$$

$$= 2L[x^2] + 3L[\sin 2x]$$

$$= L[2x^2] + L[3\sin 2x]$$

$$= L[2x^2 + 3\sin 2x]$$

$$\therefore f(x) = 2x^2 + 3\sin 2x$$

(b)

$$\frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}$$

$$= L[1] - L[e^{-x}]$$

$$= L[1 - e^{-x}]$$

$$\therefore f(x) = 1 - e^{-x}$$

Q) (1) $f(x) = \sin 5x \cos 3x$

(2) $f(x) = \cosh^2 4x$

(3) $f(x) = \cos ax \cdot \sinh(bx)$

Sol: (1) $f(x) = \sin 5x \cdot \cos 3x$

$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cancel{= \sin 4x + \sin x}$

$= \frac{1}{2}(\sin 8x + \sin 2x)$

(2) $f(x) = \cosh^2(4x)$

$= \left(\frac{e^{4x}+e^{-4x}}{2}\right)^2$

(3) $f(x) = \cos(ax) \cdot \sinh(bx)$

$= \left(\frac{e^{iax}+e^{-iax}}{2}\right) \cdot \left(\frac{e^{bx}-e^{-bx}}{2}\right)$

Existence of Laplace Transform:

Piece-wise continuous

Exponential order

As $p \to \infty \implies F(p) \to 0$

$\delta(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$\int_{-\infty}^{\infty} \delta(x) = 1$

3/4/24

→ Shifting:

$L[f(x)] = F(p)$

$L[e^{ax} \cdot f(x)] = F(p-a)$

$f(x)\ \delta(x)$

$-\frac{d}{dx} f(x)\ 1$

$f(x) - f(x) = 0$

$\int_{-\infty}^{\infty} f(x) \cdot \delta(x)$

$= f(x) - f(x)]$

Proof:

$L[f(x)] = \int_0^{\infty} e^{-px} \cdot f(x)\,dx$

$L[e^{ax} f(x)] = \int_0^{\infty} e^{-px} \cdot e^{ax} \cdot f(x)\,dx$

$= \int_0^{\infty} e^{(a-p)x} f(x)\,dx = F(p-a)$

Q) $L[e^{3x} \cdot \sin 4x]$

Sol: $L[\sin ax] = \dfrac{a}{p^2+a^2}$

$\rightarrow L[\sin(4x)] = \dfrac{4}{p^2+16} = F(p)$

$L[e^{3x}\sin 4x] = F(p-3)$

$= \dfrac{4}{(p-3)^2+16}$ $\qquad L[e^{ax}\sin bx] = \dfrac{b}{(p-a)^2+b^2}$

$= \dfrac{4}{p^2-6p+25}$

Q) $L[e^{-4x}\cos 3x]$

Sol: $L[\cos ax] = \dfrac{p}{p^2+a^2}$

$L[\cos 3x] = \dfrac{p}{p^2+9} = F(p)$

$L[e^{-4x} \cdot \cos 3x] = \dfrac{p+4}{p^2+8p+25} = F(p+4)$

Q) $L[x^5 e^{-2x}]$

$L[e^{-2x} \cdot x^5]$

$F(p) = \dfrac{n!}{p^{n+1}} = \dfrac{5!}{p^6}$

$F(p+2) = \dfrac{5!}{(p+2)^6} = L[e^{-2x} \cdot x^5]$

Q) $L[e^{3x} \cdot \cos 2x]$

$F(p) = \dfrac{p}{p^2+4}$

$F(p-3) = \dfrac{p-3}{(p-3)^2+4}$

Q) $L[y^{(n)}] = ?$

$\rightarrow \quad L[y'] = \int_0^\infty e^{-px} \cdot y'(x)\, dx$

$e^{-px} \quad y'(x)$

$-(-pe^{-px}) \quad y(x)$

$= \left(e^{-px} \cdot y(x)\right]_0^\infty + p \int_0^\infty e^{-px} \cdot y(x)\, dx$

$= \left(e^{-px} y(x)\right]_0^\infty + pL[y]$

$\therefore L[y'] = pL[y] - y(0)$

$\rightarrow \quad L[y''] = \int_0^\infty e^{-px} y''(x)\, dx$

$e^{-px} \quad y''(x)$

$= e^{-px} \cdot y'(x)\Big]_0^\infty + p \int_0^\infty e^{-px} y'(x)\, dx \quad -(-pe^{-px}) \quad y'(x)$

$\therefore L[y''] = -y'(0) + pL[y']$

$$\boxed{\therefore L[y^{(n)}] = pL[y^{(n-1)}] - y^{(n-1)}(0)}$$

$\rightarrow L[y''] = p^2 L[y] - p \cdot y(0) - y'(0)$

$$\boxed{\therefore L[y^{(n)}] = p^n L[y] - p^{n-1} \cdot y(0) - p^{n-2} y'(0) - \cdots\cdots y^{(n-1)}(0)}$$

Q) $y'' + 4y = 4x$

$y(0) = 1$

$y'(0) = 5$

$L[y'' + 4y] = L[4x]$

$\Rightarrow L[y''] + L[4y] = L[4x]$

$\Rightarrow p^2 L[y] - p\, y(0) - y'(0) + 4L[y] = 4L[x]$

$\Rightarrow p^2 L[y] - p - 5 + 4L[y] = 4L[x]$

$\Rightarrow L[y](p^2+4) - (p+5) = \frac{4}{p^2}$

$\Rightarrow L[y] = \frac{4/p^2 + p + 5}{p^2+4}$

$$L[y] = \frac{4}{p^2(p^2+1)} + \frac{p}{p^2+4} + \frac{5}{p^2+4}$$

$$= \frac{1}{p^2} - \frac{1}{p^2+4} + \frac{p}{p^2+4} + \frac{5}{p^2+4}$$

$$= \frac{1!}{p^{1+1}} - \frac{1}{2} \cdot \frac{2}{p^2+4} + \frac{p}{p^2+4} + \frac{5}{2} \cdot \frac{2}{p^2+4}$$

Taking Inverse Laplace Transform:

$$\Rightarrow y = x - \frac{1}{2}\sin(2x) + \cos(2x) + \frac{5}{2}\sin(2x)$$

$$= x + \cos(2x) + 2\sin(2x)$$

Q) $L\left[\int_0^x f(t)\,dt\right]$

$F'(x) = f(x)$

$$\Rightarrow L[F(x)] = G(p)$$

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$\mathcal{L}[y'(x)] = ?$

$\mathcal{L}[y(x)] = F(p)$

$\Longrightarrow \mathcal{L}[y'(x)] = p \cdot F(p) - y(0)$

$\Longrightarrow \mathcal{L}[y''(x)] = p^2 \cdot F(p) - p \cdot y(0) - y'(0)$

$\cancel{\mathcal{L}[y^{(n)}(x)] = p^n \cdot F(p) - p^{n-1} \cdot y^{(n-1)}(0) - p^{n-2} \cdot y^{(n-2)}(0) \ldots - y(0)}$

$\mathcal{L}[y^{(n)}(x)] = p^n \cdot F(p) - p^{n-1} y(0) - p^{n-2} y'(0) - \ldots y^{(n-1)}(x)$

Q) $\mathcal{L}[f(x)] = F(p)$

$$y(x) = \int_0^x f(t)\, dt$$

Sol: $\mathcal{L}[y(x)] = \mathcal{L}\left[\int_0^x f(t)\, dt\right] = G(p)$

As $y'(x) = f(x) \Rightarrow \mathcal{L}[y'(x)] = \mathcal{L}[f(x)]$

$\Rightarrow p \cdot G(p) = F(p) \Longrightarrow G(p) = \frac{F(p)}{p}$

As $\mathcal{L}[y'(x)] = p \cdot F(p) - y(0)$

$\Rightarrow \mathcal{L}[f(x)] = p \cdot F(p) - y(0)$

$\Rightarrow F(p) = p\, F(p) - y(0)$

$\Rightarrow F(p) = \frac{y(0)}{p-1}$

$$\left[\begin{array}{l} \mathcal{L}[f(x)] = F(p) \\ \mathcal{L}[f'(x)] = p \cdot F(p) \end{array}\right]$$

Now, $\mathcal{L}\left[\int_0^x f(t)\, dt\right] = \frac{F(p)}{p}$

$$F(p) = \int_0^\infty e^{-px} \cdot f(x)\, dx$$

$$F'(p) = \frac{dF}{dp} = \frac{d}{dp}\left(\int_0^\infty e^{-px} f(x)\, dx\right)$$

$$= \int_0^\infty e^{-px} (-x)\, f(x)\, dx$$

$$= \mathcal{L}[-x f(x)]$$

$$\therefore F'(p) = \mathcal{L}[-x f(x)]$$

Similarly, $F''(p) = \mathcal{L}[(-x)^2 f(x)]$

$$= \mathcal{L}[x^2 f(x)]$$

$$F^n(p) = \mathcal{L}[(-x)^n f(x)]$$

<u>Prove that</u>

If $\mathcal{L}[f(x)] = F(p)$

$$\mathcal{L}\left[\frac{f(x)}{x}\right] = \int_p^\infty F(p)\, dp$$

<u>Sol</u> : $F(p) = \int_0^\infty e^{-px} f(x)\, dx$

$$\text{RHS} = \int_p^\infty F(p)\, dp = \int_p^\infty \left(\int_0^\infty e^{-px} f(x)\, dx\right) dp$$

$$= \int_0^\infty \left(\int_p^\infty e^{-px} f(x)\, dp\right) dx$$

$$= \int_0^\infty \left(f(x) \cdot \int_p^\infty e^{-px}\, dp\right) dx$$

$$= \int_0^\infty \left(f(x) \cdot \left(\frac{e^{-px}}{-x} \right]_p^\infty \right) dx$$

$$= \int_0^\infty \left(f(x) \cdot \frac{1}{-x} \left(e^{-px} \right]_p^\infty \right) dx$$

$$= \int_0^\infty f(x) \cdot \frac{-1}{x} \left[0 - e^{-px} \right] dx$$

$$= \int_0^\infty e^{-px} \cdot \frac{f(x)}{x} \, dx$$

$$= L\left[\frac{f(x)}{x} \right]$$

$$= LHS$$

Hence Proved, RHS = LHS

Note:

$L[f(x)] = F(p)$, [Given $f(0) = 0$]

① $L[f'(x)] = p\,F(p) - f(0) \longrightarrow$ D.E

② $L\left[\int_0^x f(t)\,dt\right] = \frac{F(p)}{p} \longrightarrow$ I.E

③ $L[-x\,f(x)] = F'(p) \Rightarrow L\left[+(x)^n \cdot f(x)\right] = (-1)^n \cdot \frac{d^n F(p)}{dp^n}$

④ $L\left[\frac{f(x)}{x}\right] = \int_p^\infty F(p)\,dp$

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Q) $\mathcal{L}[e^{x} \cdot x] = ?$

$\mathcal{L}[e^{x} \cdot x^2] = ?$

$\mathcal{L}[x^k \cdot e^{-x}] = ?$

Sol:

$$\mathcal{L}[e^{-x}] = \frac{1}{p+1}$$

$$\mathcal{L}[x \cdot e^{-x}] = \frac{-d}{dp}\left(\frac{1}{p+1}\right)$$

$$\therefore \mathcal{L}[x \cdot e^{-x}] = \frac{1}{(p+1)^2}$$

$$\mathcal{L}[x] = \frac{1}{p^2}$$

$$\mathcal{L}[x e^{-x}] = \frac{1}{[p-(-1)]^2} = \frac{1}{(p+1)^2}$$

$$\& \ \mathcal{L}[x^2 \cdot e^{-x}] = \frac{d}{dp}\left(\frac{1}{(p+1)^2}\right)$$

$$\therefore \mathcal{L}[x^2 \cdot e^{-x}] = \frac{-2}{(p+1)^3}$$

$$\& \ \mathcal{L}[x^k \cdot e^{-x}] = (-1)^k \frac{d^k}{dp^k}\left(\frac{1}{p+1}\right)$$

$$= \frac{(-1)^k (-1)^k \cdot k!}{(p+1)^{k+1}}$$

$$\therefore \mathcal{L}[x^k \cdot e^{-x}] = \frac{k!}{(p+1)^{k+1}}$$

Q) $\mathcal{L}\left[x \cdot e^{ax} \cdot \sin bx\right]$

Sol : $\mathcal{L}\left[e^{ax} \cdot (x \sin bx)\right]$

①:

$\mathcal{L}\left[x \sin bx\right]$

$\Rightarrow \mathcal{L}\left[x \cdot (\sin bx)\right]$

②:

$$\mathcal{L}\left[\sin bx\right] = \frac{b}{p^2+b^2}$$

$\cancel{\mathcal{L}\left[e^{ax}(\sin bx)\right]} =$

$$\mathcal{L}\left[x \cdot (\sin bx)\right] = \frac{-d}{dp}\left(\frac{b}{p^2+b^2}\right)$$

$$= -b \cdot \frac{d}{dp}\left(\frac{1}{p^2+b^2}\right)$$

$$= -b \cdot \left(\frac{2p}{(p^2+b^2)^2}\right)$$

$$= \frac{-2bp}{(p^2+b^2)^2}$$

$$\therefore \mathcal{L}\left[e^{ax} \cdot (x \sin bx)\right] = \frac{-2b(p-a)}{\left((p-a)^2+b^2\right)^2}$$

Q) $\mathcal{L}\left[\frac{\sin ax}{x}\right]$

$$\mathcal{L}[\sin ax] = \frac{a}{p^2+a^2}$$

$$\mathcal{L}\left[\frac{\sin ax}{x}\right] = \int_p^\infty \frac{a}{p^2+a^2}\,dp$$

$$= a\int_p^\infty \frac{1}{p^2+a^2}\,dp$$

$$= \frac{a}{a}\cdot\left(\tan^{-1}\left(\frac{p}{a}\right)\right]_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{p}{a}\right)$$

$$\therefore \mathcal{L}\left[\frac{\sin ax}{x}\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{p}{a}\right)$$

<u>Note</u>:

$$\mathcal{L}\left[\frac{f(x)}{x}\right] = \int_p^\infty F(p)\,dp$$

$$\lim_{p\to 0}\int_0^\infty e^{-px}\cdot\frac{f(x)}{x}\,dx = \lim_{p\to 0}\int_p^\infty F(p)\,dp$$

$$\Rightarrow \int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty F(p)\,dp$$

Ex. $$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{p\to 0}\int_0^\infty e^{-px}\cdot\frac{\sin x}{x}\,dx$$

$$\cancel{= \int_0^\infty \left[\frac{\pi}{2} - \tan^{-1}(p)\right] dp}$$

$$\cancel{\Rightarrow \frac{\pi}{2}\int_0^\infty = \int_0^\infty \tan^{-1}(p)\,dp}$$

$$= \int_0^\infty \frac{a}{p^2+a^2}\,dp$$

$$= \tan^{-1}\left(\frac{p}{a}\right)\Big]_0^\infty$$

$$= \frac{\pi}{2} - 0$$

$$= \pi/2$$

8) $\mathcal{L}\left[\int_0^x \frac{1-e^{-u}}{u}\,du\right]$

$$\cancel{\int_0^x \frac{1-e^{-u}}{u}\,du = f(x)}$$

$$f(u) = 1-e^{-u}$$

$$\frac{f(u)}{u} = g(u)$$

$$\cancel{\mathcal{L}[f(x)]}$$

$$\mathcal{L}[f(u)] = \mathcal{L}[1-e^{-u}]$$

$$= \mathcal{L}[1] - \mathcal{L}[e^{-u}]$$

$$= \frac{1}{p} - \frac{1}{p+1}$$

$$= \frac{p+1-p}{p(p+1)}$$

$$\Rightarrow \mathcal{L}[f(u)] = \frac{1}{p(p+1)} = F(p)$$

$$\mathcal{L}\left[\frac{f(u)}{u}\right] = \int_p^\infty \frac{1}{p(p+1)}\,dp$$

$$= \int_p^\infty \left(\frac{1}{p} - \frac{1}{p+1}\right)dp$$

$$= \ln|p| - \ln|p+1| \Big]_p$$

$$= \ln\left|\frac{p}{p+1}\right|\Big]_p^{\infty}$$

$$= \left[\lim_{p\to\infty} \ln\left(\frac{p}{p+1}\right)\right] - \ln\left|\frac{p}{p+1}\right|$$

$$= -\ln\left|\frac{p}{p+1}\right|$$

$$= \ln\left|\frac{p+1}{p}\right|$$

$$\therefore \mathcal{L}\left[\frac{1-e^{-u}}{u}\right] = \ln\left|\frac{p+1}{p}\right|$$

$$\boxed{\mathcal{L}\left[\int_0^x \frac{1-e^{-u}}{u}\right] = \frac{1}{p}\ln\left|\frac{p+1}{p}\right|}$$

Q) Find Inverse Laplace Transform of

$$F(p) = \log\left[\frac{p^2+1}{p(p+1)}\right]$$

Sol:

$$F(p) = \log(p^2+1) - \log(p) - \log(p+1)$$

$$F'(p) = \frac{2p}{1+p^2} - \frac{1}{p} - \frac{1}{p+1}$$

Let Inverse Laplace Transform of $F(p)$ be $f(x)$ &
Let Inverse Laplace Transform of $F'(p)$ be $g(x)$.

$$g(x) = 2\cos x - 1 - e^{-x}$$

$$\therefore -x \cdot f(x) = 2\cos x - 1 - e^{-x}$$

$$\boxed{f(x) = \frac{1+e^{-x} - 2\cos x}{x}}$$

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$$A_n(x) = \sum a_n x^n = \sum a(n) \cdot x^n$$

$$B_n(x) = \sum b_n x^n = \sum b(n) \cdot x^n$$

$$A_n(x) \cdot B_n(x) = C_n(x) \cdot x^n$$

$$C_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots$$

$$C_n = \sum_{k=0}^{n} a(k) \cdot b(n-k)$$

$\rightarrow$ $f, g : [0, \infty) \longrightarrow \mathbb{R}$

$$f * g(x) = \int_0^x f(t) \cdot g(x-t)\, dt$$

[$*$ $\longrightarrow$ ~~Convol~~ Convolution]

$$f * g(x) = g * f(x)$$

<u>Proof</u>:

$$LHS = \int_0^x f(t)\, g(x-t)\, dt$$

$u = x - t$

$du = -dt$

$$\Rightarrow \int_x^0 f(x-u) \cdot g(u)(-du)$$

$$\Rightarrow -\int_x^0 g(u) \cdot f(x-u)\, du$$

$$\Rightarrow \int_0^x g(u) \cdot f(x-u)\, du = \int_0^x g(t) \cdot f(x-t)\, dt = RHS$$

LB $\Rightarrow t = 0$

$\Rightarrow u = x - 0$

$\Rightarrow \underline{u = x}$

UB $\Rightarrow t = x$

$\Rightarrow u = x - x$

$\Rightarrow \underline{u = 0}$

(OR) ~~[illegible]~~

$t \longrightarrow a+b-t$

a: LB
b: UB

$\therefore t \rightarrow 0+x-t$

$\underline{\therefore t \rightarrow x-t}$

$$\int_0^x f(t)\cdot g(x-t)\, dt$$

$$= \int_0^x f(x-t)\cdot g(x-x+t)\, dt$$

$$= \int_0^x g(t)\cdot f(x-t)\, dt$$

$$= g * f(x)$$

Ex. $f(x) = x$

$g(x) = 1$

$$f * g(x) = \int_0^x f(t)\cdot g(x-t)\, dt$$

$$= \int_0^x t \cdot 1\, dt$$

$$= \left.\frac{t^2}{2}\right]_0^x = \frac{x^2}{2}$$

Ex. $f(x) = e^x$

$g(x) = e^{-x}$

$$f * g(x) = \int_0^x f(t) \cdot g(x-t)\, dt$$

$$= \int_0^x e^{t} \cdot \cancel{g(x)}\, e^{-(x-t)}\, dt$$

$$= \int_0^x e^{2t} \cdot e^{-x}\, dt$$

$$= e^{-x} \cdot \int_0^x e^{2t}\, dt$$

$$= e^{-x} \cdot \left(\frac{e^{2t}}{2}\right]_0^x = \frac{e^{x} - e^{-x}}{2}$$

$$= \sinh(x)$$

→ <u>Convolution Theorem for Laplace Transform</u>:

$$\mathcal{L}[f * g(x)] = \mathcal{L}[f(x)] \cdot \mathcal{L}[g(x)]$$

<u>Proof</u>:

$$\mathcal{L}[f * g(x)]$$

$$= \int_0^\infty e^{-px} \cdot (f * g(x))\, dx$$

$$\text{RHS} = \mathcal{L}[f(x)] \cdot \mathcal{L}[g(x)]$$

$$= \left[\int_0^\infty e^{-px} \cdot f(x)\, dx\right] \times \left[\int_0^\infty e^{-px} g(x)\, dx\right]$$

(OR) ~~a+b-t → t~~

$t \longrightarrow a+b-t$

a: LB
b: UB

$\therefore t \rightarrow 0+x-t$

$\underline{\therefore t \rightarrow x-t}$

$$\int_0^x f(t) \cdot g(x-t)\, dt$$

$$= \int_0^x f(x-t) \cdot g(x-x+t)\, dt$$

$$= \int_0^x g(t) \cdot f(x-t)\, dt$$

$$= g * f(x)$$

Ex. $f(x) = x$

$g(x) = 1$

$$f * g(x) = \int_0^x f(t) \cdot g(x-t)\, dt$$

$$= \int_0^x t \cdot 1\, dt$$

$$= \left.\frac{t^2}{2}\right]_0^x = \frac{x^2}{2}$$

Ex. $f(x) = e^x$

$g(x) = e^{-x}$

$$f * g(x) = \int_0^x f(t) \cdot g(x-t)\, dt$$

$$= \int_0^x e^{t} \cdot \cancel{g(x)}\, e^{-(x-t)}\, dt$$

$$= \int_0^x e^{2t} \cdot e^{-x}\, dt$$

$$= e^{-x} \cdot \int_0^x e^{2t}\, dt$$

$$= e^{-x} \cdot \left[\frac{e^{2t}}{2}\right]_0^x = \frac{e^{x} - e^{-x}}{2}$$

$$= \sinh(x)$$

→ <u>Convolution Theorem for Laplace Transform</u>:

$$\mathcal{L}[f * g(x)] = \mathcal{L}[f(x)] \cdot \mathcal{L}[g(x)]$$

<u>Proof</u>:

$$\mathcal{L}[f * g(x)]$$

$$= \int_0^\infty e^{-px} \cdot (f * g(x))\, dx$$

$$\text{RHS} = \mathcal{L}[f(x)] \cdot \mathcal{L}[g(x)]$$

$$= \left[\int_0^\infty e^{-px} \cdot f(x)\, dx\right] \times \left[\int_0^\infty e^{-px} g(x)\, dx\right]$$

$$= \int_0^\infty e^{-ps} f(s)\, ds \cdot \int_0^\infty e^{-pt} \cdot f(t)\, dt$$

$$= \int_0^\infty \int_0^\infty e^{-p(s+t)} \cdot f(s) \cdot g(t)\, ds \cdot dt$$

$$s \longrightarrow x - t$$
$$ds = dx$$

$$= \int_0^\infty \int_{s=0}^{s=\infty} e^{-p(x-t+t)} f(x-t)\, g(t)\, d(x-t)\, dt$$

$$= \int_0^\infty \int_t^\infty e^{-px} \cdot f(x-t)\, g(t)\, dx\, dt$$

$x = s + t$
$s = 0 \Rightarrow x = t$
$s = \infty \Rightarrow x = \infty$

15/4/24

Q) $F(p) = \dfrac{1}{p^2(p^2+1)}$, Find Inverse Laplace Transform

$$= \frac{1}{p^2} \times \frac{1}{p^2+1}$$

$$= \mathcal{L}[x] \cdot \mathcal{L}[xe^{-x}]$$

$$= \mathcal{L}[x * xe^{-x}]$$

$$= \int_0^x (x-t)\, t e^{-t}\, dt$$

$$= x \int_0^x t e^{-t} dt - \int_0^x t^2 e^{-t} \cdot dt$$

Th:

$$\mathcal{L}[f * g(x)] = \mathcal{L}[f(x)] \times \mathcal{L}[g(x)]$$

Proof:

$$RHS = \mathcal{L}[f(x)] \times \mathcal{L}[g(x)]$$

$$= \int_0^\infty e^{-px} f(x)\, dx \times \int_0^\infty e^{-px} g(x)\, dx$$

[$x \rightarrow$ dummy variable in both cases]

$$= \int_0^\infty e^{-ps} f(s)\, ds \times \int_0^\infty e^{-pt} g(t)\, dt$$

$$= \int_0^\infty \int_0^\infty e^{-ps} \cdot e^{-pt} \cdot f(s) \cdot g(t)\, ds\, dt$$

$$= \int_{s=0}^{s=\infty} \int_{t=0}^{t=\infty} e^{-p(s+t)} \cdot f(s) \cdot g(t)\, ds\, dt$$

$$s + t = x$$

$$\Rightarrow ds = dx$$

when $s=0 \Rightarrow x=t$

when $s=\infty \Rightarrow x=\infty$

$$= \int_{x=t}^{x=\infty} \int_{t=0}^{\infty} e^{-px} f(x-t)\, g(t)\, dx\, dt$$

As $f * g(x) = \int_0^x f(t) \cdot g(x-t)\, dt$ [i.e. LHS]

$$\mathcal{L}[f * g(x)] = \int_0^\infty e^{-px} (f * g(x))\, dx$$

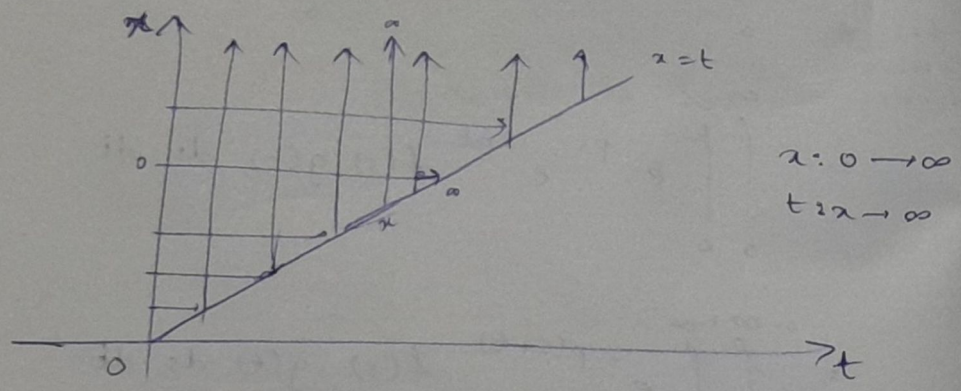
$$= \int_0^\infty e^{-px} \int_0^x f(t) \cdot g(x-t)\, dt\, dx$$

$$LHS = \int_0^\infty e^{-px} \int_0^x g(t) \cdot f(x-t)\, dt\, dx$$

$$\therefore LHS = \int_0^\infty \int_0^x e^{-px} g(t) \cdot f(x-t)\, dt\, dx$$

$$\& \quad RHS = \int_0^\infty \int_t^\infty e^{-px} g(t) \cdot f(x-t)\, dx\, dt$$

$$\therefore RHS = \int_t^\infty \int_0^\infty e^{-px} g(t) \cdot f(x-t)\, dt\, dx$$



$$\therefore RHS = \int_0^\infty \int_0^x e^{-px} g(t) f(x-t)\, dt\, dx$$

$$= LHS$$

Hence <u>Proved</u>

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$$y(x) = x^2 + \int_0^x y(t) \cdot \sin t \, dt$$

$$L[y] = L[x^2] + L\left[\int_0^x y(t) \sin t \, dt\right]$$

$$\left[f * g(x) = \int_0^x f(x-t) g(t) \, dt \right]$$

$$\Rightarrow L[y] = L[x^2] + L[y] \cdot L[\sin x]$$

$$\Rightarrow L[y] = \frac{2}{p^3} + L[y] \cdot \frac{1}{p^2+1}$$

$$\Rightarrow L[y] \cdot \left(1 - \frac{1}{p^2+1}\right) = \frac{2}{p^3}$$

$$\Rightarrow L[y] = \frac{2}{p^3} \times \frac{p^2+1}{p^2}$$

$$= \frac{2(p^2+1)}{p^5}$$

$$y(x) = L^{-1}\left[\frac{2(p^2+1)}{p^5}\right]$$

$$= L^{-1}\left[\frac{2}{p^3}\right] + L^{-1}\left[\frac{2}{p^5}\right]$$

$$= x^2 + \frac{x^4}{12}$$

$$\boxed{\therefore y(x) = x^2 + \frac{x^4}{12}}$$

Q) $y'(x) + 5\int_0^x \cos 2(x-t)\cdot y(t)\,dt = 10$

$y(0) = 2$

~~$y(x) = 10 - 5$~~

Sol: $\mathcal{L}[y'(x)] + 5\,\mathcal{L}\left[\int_0^x \cos(2(x-t))\cdot y(t)\,dt\right] = \mathcal{L}[10]$

$\Rightarrow p\,\mathcal{L}[y(x)] - y(0) + 5\,\mathcal{L}\left[\int_0^x \cos 2(x-t)\cdot y(t)\,dt\right] = \frac{10}{p}$

$\Rightarrow p\,\mathcal{L}[y(x)] - 2 + 5\,\mathcal{L}[\cos(2(x)]\cdot\mathcal{L}[y(x)] = \frac{10}{p}$

$\Rightarrow p\,\mathcal{L}[y] - 2 + 5\,\mathcal{L}[\cos 2x]\cdot\mathcal{L}[y] = \frac{10}{p}$

$\left[\begin{array}{l} f(x) = \cos 2x \\ f(x-t) = \cos 2(x-t) \\ \int_0^x f(x-t)\cdot g(t)\,dt \end{array}\right]$

$\Rightarrow p\,\mathcal{L}[y] - 2 + 5\cdot\left(\frac{p}{p^2+4}\right)\cdot\mathcal{L}[y] = \frac{10}{p}$

$\Rightarrow \mathcal{L}[y]\left(p + \frac{5p}{p^2+4}\right) = \frac{10}{p} + 2$

$\Rightarrow \mathcal{L}[y] = \left(\frac{2p+10}{p}\right) \times \left(\frac{p^2+4}{p^2+9}\right) \times \frac{1}{p}$

$= \frac{(2p+10)(p^3+9p)}{5p^2}$

$= \frac{2(p+5)\cdot p(p^2+9)}{5p^2}$

$= \frac{2(p+5)(p^2+9)}{5p}$

$= \frac{2(p^3+5p^2+9p+45)}{5p}$

$= \frac{2}{5}p^2 + 2p + \frac{18}{5} + \frac{18}{p}$

$= \frac{(2p+10)(p^2+4)}{p^2(p^2+9)}$

$= \frac{2p^3+10p^2+8p+40}{p^2(p^2+9)}$

$= \frac{2p+10}{p^2+9} + \frac{8}{p(p^2+9)} + \frac{40}{p^2(p^2+9)}$

$$\cancel{y = \mathcal{L}^{-1}\left[\frac{2}{5}p^2 + 2p + \frac{18}{5} + \frac{18}{p}\right]}$$

$$\cancel{= \mathcal{L}^{-1}\left[\frac{2p^2}{5}\right] + \mathcal{L}^{-1}[2p] + \mathcal{L}^{-1}\left[\frac{18}{5}\right] + \underbrace{\mathcal{L}^{-1}\left[\frac{18}{p}\right]}_{\Downarrow\ 18}}$$

$$y = \mathcal{L}^{-1}\left[\frac{10}{p^2} - \frac{50}{p^2(p^2+9)} + \frac{2}{p} - \frac{10}{p(p^2+9)}\right]$$

$$= \underbrace{\mathcal{L}^{-1}\left[\frac{10}{p^2}\right]}_{\Downarrow\ 10x} - \underbrace{\mathcal{L}^{-1}\left[\frac{50}{p^2(p^2+9)}\right]}_{\Downarrow} + \underbrace{\mathcal{L}\left[\frac{2}{p}\right]}_{\Downarrow\ 2} - \underbrace{\mathcal{L}\left[\frac{10}{p(p^2+9)}\right]}_{\Downarrow}$$

$$\left(\frac{1}{p^2} - \frac{1}{p^2+9}\right) \times \frac{50}{9}$$

$$\frac{50}{9}\mathcal{L}^{-1}\left[\frac{1}{p^2}\right] - \frac{50}{9}\mathcal{L}^{-1}\left[\frac{1}{p^2+9}\right]$$

$$\frac{50}{9}x - \frac{50}{27}\sin 3x$$

$$\frac{10}{p(p^2+9)} = \frac{1}{p} \times \frac{10}{3} \times \frac{3}{p^2+9}$$

$$\frac{F(p)}{p} = \frac{10}{3} \times \frac{3}{p^2+9}$$

$$\Downarrow$$

$$\int_0^x \frac{10}{3}\sin 3t \, dt$$

$$= \frac{10}{3}\int_0^x \sin 3t \, dt$$

$$= -\frac{10}{9}\left(\cos 3t\right]_0^x$$

$$= -\frac{10}{9}\cos 3x + \frac{10}{9}$$

$$y = 10x + \frac{50}{9}x + \frac{50}{27}\sin 3x + 2 - \frac{10}{9}\cos 3x + \frac{10}{9}$$

Q) $y'(x) + y(x) - 2\int_0^x y(t)\,dt = x$

$y(0) = 0$

<u>Sol</u>: $\mathcal{L}\left[y'(x) + y(x) - 2\int_0^x y(t)\,dt\right] = \mathcal{L}[x]$

$$\Rightarrow \mathcal{L}[y'(x)] + \mathcal{L}[y(x)] - 2\mathcal{L}\left[\int_0^x y(t)\,dt\right] = \mathcal{L}[x]$$

$$\Rightarrow p\mathcal{L}[y(x)] - y(0) + \mathcal{L}[y(x)] - 2\mathcal{L}\left[\int_0^x y(t)\,dt\right] = \mathcal{L}[x]$$

$$\Rightarrow \mathcal{L}[y](p+1) - 2\,\mathcal{L}\left[\int_0^x y(t)\,dt\right] = \mathcal{L}[x]$$

$$\Rightarrow \mathcal{L}[y](p+1) - 2\,\frac{1}{p}\cdot\mathcal{L}[y] = \mathcal{L}[x]$$

$$\Rightarrow \mathcal{L}[y]\cdot\left(p+1-\frac{2}{p}\right) = \mathcal{L}[x]$$

$$\Rightarrow \mathcal{L}[y] = \mathcal{L}[x]\times\left(\frac{p}{p^2+p-2}\right)$$

$$= \frac{1}{p^2}\left(\frac{p}{p^2+p-2}\right)$$

$$\Rightarrow \mathcal{L}[y] = \frac{1}{(p)(p^2+p-2)}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left[\frac{1}{p(p^2+p-2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{(p)(p-1)(p+2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{-1/2}{p} + \frac{1/3}{p-1} + \frac{1/6}{p+2}\right]$$

$A(p-1)(p+2)$
$+ B(p)(p+2)$
$+ C(p)(p-1) = 1$

$p=1 \Rightarrow 3B = 1$

$p=0 \Rightarrow -2A = 1$

$p=-2 \Rightarrow 6C = 1$

$$= -\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{p}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{p-1}\right] + \frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{p+2}\right]$$

$$\boxed{\therefore y(x) = \frac{-1}{2} + \frac{1}{3}e^{x} + \frac{1}{6}e^{-2x}}$$

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Q) $\mathcal{L}[f(x)] = F(p)$

$\mathcal{L}^{-1}[F(ap)] = ?$ Sol : $\frac{1}{a} f(x/a)$

Sol :

$\mathcal{L}[f(x)] = F(p)$

$$\Rightarrow \mathcal{L}[f(x)] = \int_0^\infty e^{-px} \cdot f(x)\,dx$$

$$F(p) = \int_0^\infty e^{-px} f(x)\,dx$$

$$F(ap) = \int_0^\infty e^{-apx} \cdot f(x)\,dx = \int_0^\infty e^{-p(ax)} f(x)\,dx$$

$$\cancel{= \int_0^\infty (e^{a})^{-px} f(x)}$$

Q) $\mathcal{L}^{-1}\left[\frac{p^2-1}{(p^2+1)^2}\right] =$

Sol: $\cancel{\mathcal{L}^{-1}\left[\frac{(p-1)(p+1)}{(p^2+1)(p^2+1)}\right]}$

$$\frac{p^2-1}{(p^2+1)^2} \longrightarrow f'(p)$$

$$F(p) = \frac{p}{p^2+1}$$

$$F'(p) = \frac{p^2-1}{(p^2+1)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{-d}{dp}\left(\frac{-p}{p^2+1}\right)\right]$$

$$F(p) = \mathcal{L}^{-1}\left[\frac{p}{p^2+1}\right] = \cos x$$

$$= x\cos x$$

$$-F'(p) = x\cos x$$

Q) $\mathcal{L}^{-1}\left[\frac{9p^2-1}{(9p^2+1)^2}\right] \Rightarrow \frac{1}{3} f(x/3)$ $\quad [f(x) = x\cos x]$

$$\Rightarrow \frac{1}{3}\left(\frac{x}{3}\cos\frac{x}{3}\right)$$

$$\Rightarrow \frac{x}{9}\cos\left(\frac{x}{3}\right)$$

Q) $\mathcal{L}^{-1}\left[\frac{p}{(p+3)^{7/2}}\right]$

$$= \mathcal{L}^{-1}\left[\frac{p+3}{(p+3)^{7/2}}\right] - \mathcal{L}^{-1}\left[\frac{3}{(p+3)^{7/2}}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{(p+3)^{5/2}}\right] - \mathcal{L}^{-1}\left[\frac{3}{(p+3)^{7/2}}\right]$$

$$= e^{-3x} \cdot L^{-1}\left[\frac{1}{p^{5/2}} - \frac{3}{p^{7/2}}\right]$$

$$\left[\begin{array}{l} \Gamma(n+1) = n \cdot \Gamma(n) = n(n-1)! \\ \Gamma(n+1) = n! \end{array}\right]$$

$$\left[\begin{array}{l} f(x) = x^n, \ n \text{ is not an Integer} \\ L[f(x)] = \dfrac{\Gamma(n+1)}{p^{n+1}} \end{array}\right]$$

$$= e^{-3x} \cdot L^{-1}\left[\frac{1}{p^{5/2}} - \frac{3}{p^{7/2}}\right]$$

$$\left[\begin{array}{l} \Gamma(5/2) = 3/2 \cdot \Gamma(3/2) \\ \quad = 3/2 \cdot 1/2 \cdot \Gamma(1/2) \\ \quad = 3/4 \cdot \sqrt{\pi} \end{array}\right]$$

$$\left[\int_0^\infty t^{-1/2} e^{-t}\, dt = \sqrt{\pi}\right]$$

$$= e^{-3x} \cdot L^{-1}\left[\frac{1}{p^{5/2}}\right] - e^{-3x} \cdot 3 L^{-1}\left[\frac{1}{p^{7/2}}\right]$$

$$= e^{-3x} \times \frac{4}{3\sqrt{\pi}} x^{5/2} - e^{-3x} \times 3 \times \frac{8}{15\sqrt{\pi}} x^{7/2}$$

Q) $L^{-1}\left[\dfrac{6p-4}{p^2-4p+20}\right]$

Sol: $L^{-1}\left[\dfrac{6(p-2)+2\cdot 4}{(p-2)^2+4^2}\right]$

$$= L^{-1}\left[\frac{6(p-2)}{(p-2)^2+4^2} + \frac{2(4)}{(p-2)^2+4^2}\right]$$

$$= 6L^{-1}\left[\frac{p-2}{(p-2)^2+4^2}\right] + 2L^{-1}\left[\frac{4}{(p-2)^2+4^2}\right]$$

$$= 6\cos(2x)\cdot e^{2x} + 2\sin(2x)\cdot e^{2x}$$

Q) $L^{-1}\left[\frac{1}{p}\sin\left(\frac{1}{p}\right)\right]$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Sol:

$$L^{-1}\left[\frac{1}{p}\left(\frac{1}{p} - \frac{1}{3!\,p^3} + \frac{1}{5!\,p^5} - \cdots\right)\right]$$

$$\Rightarrow L^{-1}\left[\frac{1}{p^2} - \frac{1/3!}{p^4} + \frac{1/5!}{p^6} - \cdots\right]$$

Q) $L^{-1}\left[\frac{1}{\sqrt{p^2+a^2}}\right] = J_0(ax)$

Sol: $F(p) = \log\left(1+\frac{1}{p^2}\right)$

Q) $L^{-1}\left[\log\left(\frac{p^2+a^2}{p^2+b^2}\right)\right]$

$$= L^{-1}\left[\underbrace{\log(p^2+a^2) - \log(p^2+b^2)}_{F(p)}\right]$$

$$\cancel{= L^{-1}\left[\log(p^2+a^2)\right] - L^{-1}\left[\log(p^2+b^2)\right]}$$

$$\cancel{f(x) = L^{-1}\left[\log(p^2+a^2)\right] - L^{-1}\left[\log(p^2+b^2)\right]}$$

$$\cancel{f'(x) =}$$

$$F(p) = \log(p^2+a^2) - \log(p^2+b^2)$$

$$F'(p) = \frac{2p}{p^2+a^2} - \frac{2p}{p^2+b^2}$$

$$g(x) = 2\cos(ax) - 2\cos(bx)$$

$$L^{-1}\left[F'(p)\right] = -x f(x)$$

$$\therefore f(x) = \frac{2}{x}\cos\left(\frac{ax}{x}\right) - \frac{2}{x}\cos\left(\frac{bx}{x}\right)$$

Q) Prove that if

$$L^{-1}\left[F(p)\right] = f(x)$$

Then

$$L^{-1}\left[\frac{F(p)}{p^2}\right] = \int_0^x \int_0^u f(t)\, dt\, ds$$

→ **Introduction to Fourier Series:**

$$f(x) = F(k)$$

$$f(t) \rightarrow W$$

$$f(x) = \sum a_n \cos(nx) + \sum b_n \sin(nx)$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \sqrt{\frac{\pi}{2}}\, \delta_{mn}$$

$$\int P_m(x) \cdot P_n(x) = \frac{2}{2n+1} \delta_{mn}$$

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Power Series:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Generalisation of Power Series:

$$\sum_{n=-\infty}^{\infty} a_n x^n = \cdots + a_{-n} x^{-n} + \cdots a_{-2} x^{-2} + a_{-1} x^{-1} + a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$$

Using Complex Numbers,

$$\sum_{n=-\infty}^{\infty} c_n z^n = \cdots + c_{-n} z^{-n} + \cdots c_{-2} z^{-2} + c_{-1} z^{-1} + c_0 + c_1 z + c_2 z^2 + \cdots c_n z^n + \cdots$$

Taking $z = e^{ix}$

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = \cdots + c_{-n} e^{-inx} + \cdots + c_{-2} e^{-i2x} + c_{-1} e^{-ix} + c_0 + c_1 e^{ix} + c_2 e^{i2x} + \cdots c_n e^{inx} + \cdots$$

This is Fourier Series

$$\left.\begin{aligned}\int_{-\pi}^{\pi} \sin x &= 0 \\ \int_{-\pi}^{\pi} \cos x &= 0\end{aligned}\right\} \Rightarrow \begin{aligned}&\int_{-\pi}^{\pi} \sin(mx)\cdot\cos(nx) = 0 \\ &\int_{-\pi}^{\pi} \cos(mx)\cos(nx) = 0 \quad [m \neq n] \\ &\int_{-\pi}^{\pi} \sin(mx)\sin(nx) = 0 \quad [m \neq n]\end{aligned}$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\int_{-\pi}^{\pi} f(x)\cdot \cos nx \, dx = \int_{-\pi}^{\pi} a_n \cos^2(nx)$$

$$= \frac{1}{2}\cdot \int_{-\pi}^{\pi} a_n x$$

$$= a_n \cdot \pi$$

$$\boxed{\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx \, dx} \quad [n \geq 1]$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin(nx)\, dx} \quad [n \geq 1]$$

H.W

$f(x) = x$

$-\pi \leq x \leq \pi$

① $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \, dx = \pi$

② $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\, dx$

$\Rightarrow a_n = 0$

③ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\, dx$

$= \frac{2}{\pi^2}\left[\sin(n\pi) - n\pi\cos(n\pi)\right]$

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$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx$$

for $f(x) = x$;

$$x = \frac{1}{2}(\pi) + \sum_{n=1}^{\infty}\left(\frac{2}{\pi^2}\left[\sin(n\pi) - n\pi\cos(n\pi)\right]\sin(nx)\right)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi^2}\sin(n\pi)\cdot\sin(nx) - \frac{2n}{\pi}\cos(n\pi)\cdot\sin(nx)\right)$$

If n : odd,

$$x = \frac{\pi}{2} + \left(\sum_{n=1}^{\infty}\frac{2n}{\pi}\sin(nx)\right)$$

If n is even,

$$x = \frac{\pi}{2} - \left(\sum_{n=1}^{\infty}\frac{2n}{\pi}\sin(nx)\right)$$

<u>Dirichlet Conditions</u> :

(1) $f(x)$ should be defined & bounded

(2) $f(x)$ should have finite no. of discontinuities and finite no. of maximas and minimas

(3) $f(x)$ is defined & periodic

Ex. $f(x) = x$

$f(x+2\pi) = f(x)$

Note: Fourier series is generally defined for periodic functions.

Ex. $f(x) = \tan x$

Is it satisfied?

① ✗

② ✓

③ ✓

$\tan x \longrightarrow$ Not bounded in $[-\pi, \pi]$

Ex. $f(x) = \sin(1/x)$

Is it satisfied?

① ✓

② ✗

③ ✓

$\sin(1/x) \longrightarrow \infty$ no. of maximas & minimas in $[-\pi, \pi]$

Note:

@ Discontinuity,

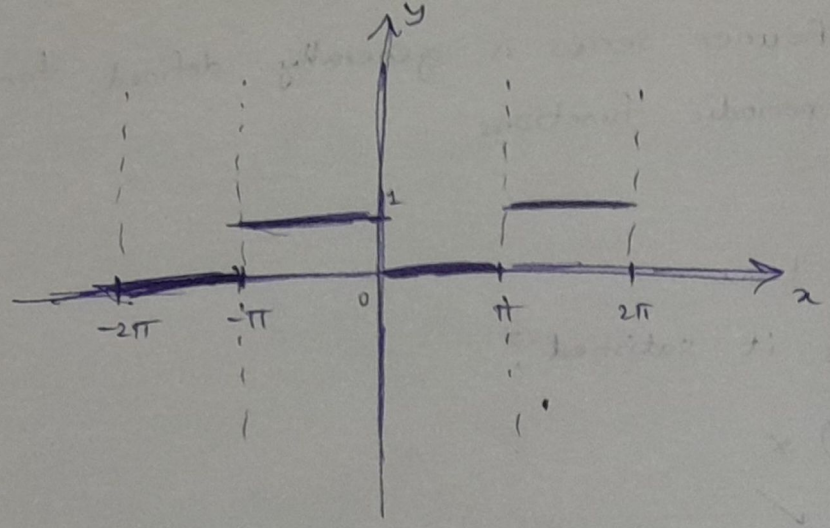
$\frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right] \longrightarrow$ Converges at this point

Ex.

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 \le x < \pi \end{cases}$$

Period $= 2\pi$

Is it following the Dirichlet conditions?



$f(x)$ in the interval $-2\pi < x < 2\pi$.

Finding Fourier series:

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^{0} 1\, dx + \int_{0}^{\pi} 0\, dx \right)$$

$$= \frac{1}{\pi} (\pi)$$

$$= 1$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^{0} \cos nx\, dx + 0 \right)$$

$$\left(\frac{\sin(nx)}{n} \right]_{-\pi}^{0}$$

$$= \frac{1}{\pi n} \sin(n\pi) = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\, dx$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^{0} \sin(nx)\, dx + 0 \right)$$

$$= \frac{1}{\pi n}\left(\cos(n\pi) - 1\right) \qquad \left(\frac{\cos nx}{n}\right]_0^{-\pi}$$

$$= \frac{1}{n\pi}\left[(-1)^n - 1\right]$$

$$\boxed{\therefore\ b_n = \frac{-1}{n\pi}\left[1 - (-1)^n\right]}$$

$$\therefore\ f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{n\pi}\left[1-(-1)^n\right]\sin(nx)$$

$$\therefore\ f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

$$\cancel{f\left(\frac{\pi}{4}\right) = 0 = \frac{1}{2} - \frac{2}{\pi}\left[\frac{1}{\sqrt{2}}\right.}$$

<u>Application of Fourier Series</u>

When $x = \pi/2$,

$$f(\pi/2) = \frac{1}{2} - \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right]$$

Converges to 0

$$\Rightarrow 0 = \frac{1}{2} - \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right]$$

$$\Rightarrow \frac{1}{2} = -\frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right]$$

$$\Rightarrow \frac{\pi}{4} = \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right]$$

Hence, It can be approximated to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

When $x = 0$, $\left.\begin{array}{l} RHL = 0 \\ LHL = 1 \end{array}\right\} \Rightarrow Avg = 1/2$

$\hookrightarrow$ Fourier series $= 1/2$ ✓

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$f(x) = x \quad (0 \le x \le \pi)$

Even extension :

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ -x, & -\pi \le x \le 0 \end{cases}$$

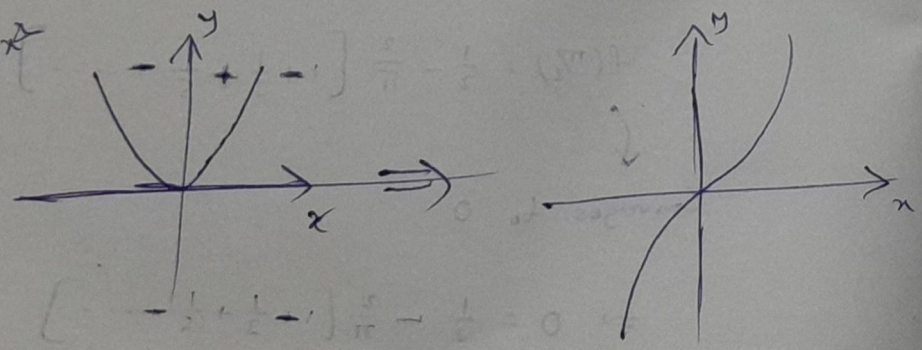
Gives $f(x) = |x|$

Odd extension :

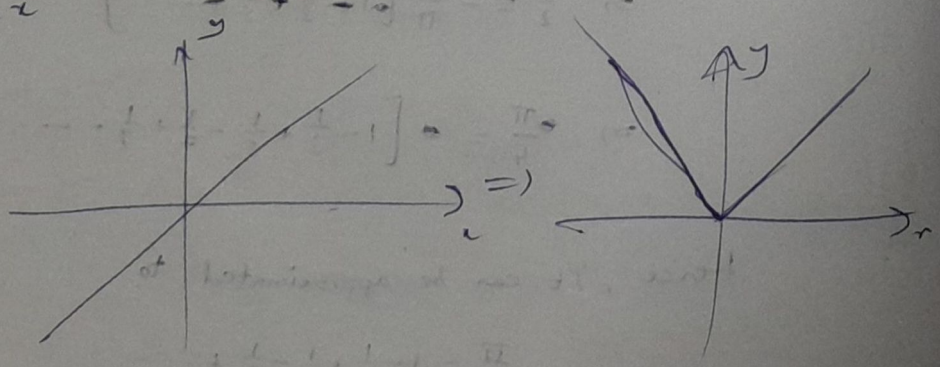
$f(x) = x^2$

$$f(x) = \begin{cases} x^2, & 0 \le x \le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases}$$

$f(x) = x^2$



$f(x) = x$



for $f(x)$, Odd extension

$$f(-x) = -f(x)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$= 0$$

for $f(x)$, Even extension

$$f(-x) = f(x)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cdot \sin(nx)\, dx$$

For Odd extension,

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cdot \cos(nx)\, dx$$

$$b_n = 0$$

→ <u>Shifting of Axes</u>

$f(x) \longrightarrow -\pi$ to $+\pi$

$f(x) \longrightarrow -L$ to $+L \Rightarrow$?

$x : -\pi$ to $+\pi \Rightarrow 2\pi$

$x : -L$ to $+L \Rightarrow 2L$

$$\left[\frac{x}{2L} = \frac{t}{2\pi}\right]$$

$$t = \frac{\pi x}{L}$$

$\hookrightarrow g(t)$

$$a_n = \frac{1}{L}\int_{-L}^{+L} f\left(\frac{\pi x}{L}\right) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

$x : a$ to $b \Rightarrow b - a = 2L$