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Linear Algebra Assigment-1

1) Given equations,

$$2x_1 + \frac{3}{4}x_2 - x_3 - x_4 = 0 \quad - (1)$$

$$x_1 + \frac{2}{3}x_2 - x_5 = 0 \quad - (2)$$

$$9x_1 + 6x_2 - 3x_3 - 3x_4 - 3x_5 = 0 \quad - (3)$$

It is of form $AX=0$ where

$$A = \begin{bmatrix} 2 & 3/4 & -1 & -1 & 0 \\ 1 & 2/3 & 0 & 0 & -1 \\ 9 & 6 & -3 & -3 & -3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

row reducing A,

$$\begin{bmatrix} 2 & 3/4 & -1 & -1 & 0 \\ 1 & 2/3 & 0 & 0 & -1 \\ 9 & 6 & -3 & -3 & -3 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1/2} \begin{bmatrix} 1 & 3/8 & -1/2 & -1/2 & 0 \\ 1 & 2/3 & 0 & 0 & -1 \\ 9 & 6 & -3 & -3 & -3 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 9R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3/8 & -1/2 & -1/2 & 0 \\ 0 & 1 & 12/7 & 12/7 & -24/7 \\ 0 & 21/8 & 3/2 & 3/2 & -3 \end{bmatrix} \xleftarrow{R_2 \leftarrow \frac{24R_2}{7}} \begin{bmatrix} 1 & 3/8 & -1/2 & -1/2 & 0 \\ 0 & 7/24 & 1/2 & 1/2 & -1 \\ 0 & 21/8 & 3/2 & 3/2 & -3 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_1 \leftarrow R_1 - \frac{3}{8}R_2 \\ R_3 \leftarrow R_3 - \frac{21}{8}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -8/7 & -8/7 & 9/7 \\ 0 & 1 & 12/7 & 12/7 & -24/7 \\ 0 & 0 & -3 & -3 & 6 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3/-3} \begin{bmatrix} 1 & 0 & -8/7 & -8/7 & 9/7 \\ 0 & 1 & 12/7 & 12/7 & -24/7 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\downarrow R_1 \leftarrow R_1 + \frac{8R_3}{7}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 12/7 & 12/7 & -24/7 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{12R_3}{7}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}.$$

The free variables are x_4, x_5. Let $x_4 = s, x_5 = t$.

$x_1 - x_5 = 0 \Rightarrow x_1 = t$,

$x_2 = 0$

$x_3 + x_4 - 2x_5 = 0 \Rightarrow x_3 = 2t - s$.

solution $W_S = \{ (t, 0, 2t-s, s, t) : t, s \in \mathbb{R} \}$

$W_S = \{ t(1,0,2,0,1) + s(0,0,-1,1,0) : t, s \in \mathbb{R} \}$

$=$ span of $\{ (1,0,2,0,1), (0,0,-1,1,0) \}$.

$\therefore$ Vectors that span W are $(1,0,2,0,1), (0,0,-1,1,0)$.

2) Let R be row reduced echleon matrix row equivalent to A, let $\{P_1, P_2, \ldots, P_r\}$ $(r < n)$ be non zero rows of R.

Row equivalence preserves row space, hence A and R have same row space.

R has its own non zero rows linearly independent and spanning row space of R.

claim 1:- R is ~~linea~~ $P_1, \ldots, P_r$ are linearly independent

Let $c_1P_1 + \cdots + c_rP_r = 0$, $c_i \in F$.

As R is in row reduced echleon form, leading entry in each row occurs in unique column.

Hence $c_1 = c_2 = \cdots = c_n = 0$.

This proves $P_1, P_2, \ldots, P_r$ are linearly independent

<u>claim 2:-</u> $R^n = \text{span } A$.

Let $x = (x_1, x_2, \ldots, x_n)$ be arbitrary vector in R^n.

As R is obtained from A by performing row operation,

$(x_1, x_2, \ldots, x_n)$ can be written as $x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

where e represents operation performed.

This proves that $R^n = \text{span } A$.

$\therefore$ R forms basis for row space A.

3) Let $V_1 = (1,2,3,4)$, $V_2 = (4,3,2,1)$.

Let us check if V_1, V_2 are linearly dependent.

$c_1 V_1 + c_2 V_2 = 0$, $c_i \in R$.

$c_1 (1,2,3,4) + c_2 (4,3,2,1) = 0$.

$\Rightarrow$ $c_1 + 4c_2 = 0$ — 1

$2c_1 + 3c_2 = 0$ — 2

$3c_1 + 2c_2 = 0$ — 3

$4c_1 + c_2 = 0$. — 4

solving 1×2, 2

$2c_1 + 8c_2 = 0$

$2c_1 + 3c_2 = 0$

$\Rightarrow c_2 = 0 \Rightarrow c_1 = 0$ i.e., only solution is $c_1 = c_2 = 0$.

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v_1, v_2 are linearly independent.

To form a basis on R^4, two more linearly independent vectors are required. Consider standard basis vectors ~~v_3~~ $v_3 = (1,0,0,0)$, $v_4 = (0,1,0,0)$ that are linearly indepepen from each other and v_1, v_2.

$\therefore$ Basis of R^4 containing ~~R^4~~ is v_1, v_2 is $\{(1,2,3,4), (4,3,2,1), (1,0,0,0), (0,1,0,0)\}$.

4) $N(T) = \{(4x, 3x, 2x, x) : x \in R\}$ = span of $\{(4,3,2,1)\}$

The set $\{(4,3,2,1)\}$ is linearly independent and spans $N(T)$. Thus forms basis $N(T)$.

nullity $(T) = 1$

$\Rightarrow$ rank $(T) = \dim(R^4) - \dim(N(T)) = 3$.

$\Rightarrow$ Image of T must be R^3 for T to be onto

Basis for R^4 w.r.t $N(T) = \{(4,3,2,1), b_2, b_3, b_4\}$, where b_2, b_3, b_4 are linearly independent and not in $N(T)$. We take $b_2 = \cancel{e_1} = (1,0,0,0)$, $b_3 = \cancel{e_2} = (0,1,0,0)$, $b_4 = \cancel{e_3} = (0,0,1,0)$

For $v = (4,3,2,1) \in N(T)$, $T(v) = 0$.

$T(b_2) = e_1 = (1,0,0)$, $T(b_3) = e_2 = (0,1,0)$, $T(b_4) = e_3 = (0,0,1)$

The columns of T corresponds to images of basis vectors.

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$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rows of T are e_1, e_2, e_3 which span R^3. Hence $T: R^4 \to R^3$

$\therefore$ Onto linear transformation $T: R^4 \to R^3$ with $N(T) = \{(4x, 3x, 2x, x) : x \in R\}$ is given by $T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5) Rank Nullity Theorem:-

Let V and W be vector spaces over field F and T be linear transformation from V into W. suppose that V is finite-dimensional. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

Proof:- Let $\{\alpha_1 \ldots \alpha_k\}$ be basis for N, the null space of T. There are vectors $\alpha_{k+1} \ldots \alpha_n$ in V such that $\{\alpha_1 \ldots \alpha_n\}$ is basis for V. We shall now prove that $\{T\alpha_{k+1} \ldots T\alpha_n\}$ is a basis for range of T. The vectors $T\alpha_1 \ldots T\alpha_n$ certainly span range of T, since $T\alpha_j = 0$ for $j \le k$, we see that $T\alpha_{k+1} \ldots T\alpha_n$ span range. To see that these vectors are independent, suppose we have scalars c_i such that

$$\sum_{i=k+1}^{n} c_i T(\alpha_i) = 0 \Rightarrow T\left(\sum_{i=k+1}^{n} c_i \alpha_i\right) = 0$$

and accordingly the vector $\alpha = \sum_{i=k+1}^{n} c_i \alpha_i$ is the null

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space of T. Since $\alpha_1 \cdots \alpha_k$ form a basis for N, there must be scalars $b_1 \cdots b_k$ such that $\alpha = \sum_{i=1}^{k} b_i \alpha_i$

thus, $\sum_{i=1}^{k} b_i \alpha_i - \sum_{j=k+1}^{n} c_j \alpha_j = 0$

and since $\alpha_1, \cdots, \alpha_n$ are linearly independent,

$b_1 = b_2 = \cdots = b_k = 0 = c_{k+1} = c_{k+2} = \cdots = c_n$

If r is rank of T, $T\alpha_{k+1} \cdots T\alpha_n$ form a basis for range of T i.e., $r = n-k$ (k = nullity (T))

$\Rightarrow r+k = n \Rightarrow \text{rank}(T) + \text{nullity}(T) = \dim(V)$

6) $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$B = \{(1,1), (1,2)\}$.

Let $v_1 = (1,1)$, $v_2 = (1,2)$. Alternate order basis

$v_1' = Pv_1$, $v_2' = Pv_2$.

$v_1' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (2,0)$

$v_2' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (3,1)$

$\Rightarrow B' = \{(2,0), (3,1)\}$

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$B_1 = \{(2,1), (1,2)\}$.

We need to find matrix Q such that $[\alpha]_B = Q[\alpha]_{B_1}$

$\Rightarrow V_1, V_2$ ~~can~~ must be expressed as linear combination of B_1

$\Rightarrow V_1 = a_1 v_1' + b_1 v_2'$, $a_1 b \in \mathbb{R}$

$\Rightarrow 2a_1 + b_1 = 1$
$a_1 + 2b_1 = 1$

solving above eqs, $a_1 = 1/3$, $b_1 = 1/3$.

$V_2 = a_2 V_2' + a_2 V_2'$

$\Rightarrow 2a_2 + b_2 = 1$
$a_2 + 2b_2 = 2$

solving above eqs., $a_2 = 0$, $b_2 = 1$

The transformation matrix Q is $Q = \begin{bmatrix} 1/3 & 0 \\ 1/3 & 1 \end{bmatrix}$

7) $x+y+z=0$, $x,y,z \in V$

$\Rightarrow z = -(x+y)$, $y = -(x+z)$, $x = -(y+z)$

one of x,y,z can be written as linear combination of other two.

As $x = -(y+z)$, x can be expressed as linear combination of ~~z~~ y i.e., $x \in span\{y,z\} \Rightarrow span\{y,z\} \supseteq span\{x,y\}$ — (1)

As $z = -(x+y)$, z is expressed as linear combination of x,y, i.e., $z \in span\{x,y\} \Rightarrow span\{x,y\} \supseteq span\{y,z\}$ — (2)

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From ①, ②

$\text{span}\{x, y\} = \text{span}\{y, z\}$ — ③

$y = -(x+z)$, y is linear combination of x, z, $y \in \text{span}\{x, z\}$

$\Rightarrow \text{span}\{x, z\} \supseteq \text{span}\{y, z\}$ — ④

similarily, $x \in \text{span}\{z, y\} \Rightarrow \text{span}\{y, z\} \supseteq \text{span}\{x, z\}$ — ⑤.

From ④, ⑤

$\text{span}\{y, z\} = \text{span}\{z, x\}$ — ⑥

From ③, ⑥

$\text{span}\{x, y\} = \text{span}\{y, z\} = \text{span}\{z, x\}$

8) $v_1, v_2 \cdots v_n$ span V. Any vector then can be written as

$v = c_1 v_1 + \cdots + c_n v_n$, $c_i \in F$, $v \in V$

for $i = 2, 3 \cdots n$, v_i can be written as.

$v_i = (v_i - v_1) + v_1$ — ①

$V = \text{span}\{v_1, v_2 \cdots, v_n\}$

From ①, V can be written as

$V = \text{span}\{v_1, v_2 - v_1, v_3 - v_1 \cdots v_n - v_1\}$

$\Rightarrow$ It is proved that $v_1, v_2 - v_1 \cdots v_n - v_i$ span V

If the set $v_1, v_2 \cdots v_n$ is linearly independent, the solution to $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$, $c_i \in F$, is

$c_1 = c_2 = \cdots = c_n = 0$

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consider

$$a_1v_1 + a_2(v_2-v_1) + a_3(v_3-v_1) + \cdots + a_n(v_n-v_1) = 0$$

$$\Rightarrow (a_1 - a_2 - \cdots - a_n)v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n = 0$$

As $v_1, v_2 \cdots v_n$ are linearly independent, their coefficients must be zero.

$$a_1 - a_2 - \cdots - a_n = 0,\ a_2 = 0,\ a_3 = 0 \cdots,\ a_n = 0$$

$$\Rightarrow a_1 = 0.$$

As all coefficients are zero, it is proved that $v_1, v_2 - v_1 \cdots v_n - v_1$ are linearly independent.

9) Row space of AB contains linear combinations of rows of AB. Each row of AB is linear combination of A.

$$AB = A_1\begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} + A_2\begin{bmatrix} \vdots \end{bmatrix} + \cdots + A_n\begin{bmatrix} \vdots \end{bmatrix}$$

Every vector in $R(AB)$ can be expressed as combination of rows of A. i.e., $R(AB) \subseteq R(A)$

Nullspace of AB is,

$$N(AB) = \{x : ABx = 0\},\ x \in N(B)\}.$$

If $x \in N(B)$, $Bx = 0$

$$\Rightarrow ABx = A(Bx) = 0.$$

$$\Rightarrow x \in N(AB) \Rightarrow N(B) \subseteq N(AB) \Rightarrow N(AB) \supseteq N(B)$$

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10) Proof (i ⇒ ii):-

i) $ToT = 0$

If $R(T) = N(T)$, for any $v \in V$, $T(v) \in R(T) = N(T)$

$\Rightarrow T(T(v)) = 0$ showing $ToT = 0$

ii) $T \neq 0$

$R(T) = N(T)$, if $T = 0$, $R(T) = \{0\}$, $N(T) = V$, it contradicts the equality $R(T) = N(T)$, hence $T \neq 0$

$\dim(V) = \text{rank}(T) + \text{nullity}(N(T))$

If $R(T) = N(T)$, $\dim(R(T)) = \dim(N(T))$, let $r = \text{rank}(T)$

$\Rightarrow n = r + r \Rightarrow r = n/2 \Rightarrow n$ must be even

It is proved that if $R(T) = N(T)$, then $ToT = 0$, $T \neq 0$, n is even, $\text{rank}(T) = n/2$.

Proof (ii ⇒ i):-

$ToT = 0$, for any $v \in V$, $T(v) \in N(T)$

$\Rightarrow R(T) \subseteq N(T)$ — ①

$\dim(V) = \text{rank}(T) + \dim(N(T))$

As $\text{rank}(T) = n/2$, $\dim(V) = n/2$

$\Rightarrow \text{nullity}(T) = n/2 \Rightarrow \text{rank}(T) = \text{nullity}(T)$ — ②

From ①, ②

$R(T) = N(T)$

∴ It is proved that above statements are equivalent.