1. If $\lim_{x \to x_0} f(x) = L$, then show that $\lim_{x \to x_0} |f(x)| = |L|$. Is the converse true?

$$|f(x) - L| < \varepsilon$$

there exists a δ for which

$$|x - x_0| < \delta$$

$$|f(x) - L| < \varepsilon$$

$$\Rightarrow |f(x)| - |L| < \varepsilon$$

$$\Rightarrow \big||f(x)| - |L|\big| < \varepsilon$$

$$\Rightarrow \lim_{x \to x_0} |f(x)| = |L|$$

$$(x-2)(x^2+2x+4)$$

$$= x^3 + \cancel{2x^2} + \cancel{4x}$$

$$- \cancel{2x^2} - \cancel{4x} - 8$$

$$= \underline{\underline{x^3 - 8}}$$

POST QUIZ-2

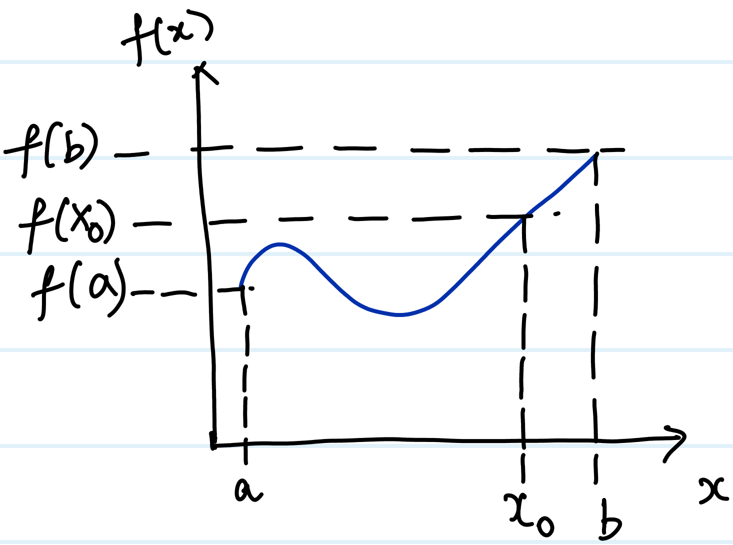
01/11/2023

Intermediate Value Theorem

$y = f(x) \longrightarrow$ continuous in $[a, b]$ takes every value b/w $f(a)$ & $f(b)$.

if y_0 is any value b/w $f(a)$ & $f(b)$,

$y_0 = f(x_0) \ \forall x_0 \in [a, b]$



DIFFERENTIATION

- how a curve bends at a pt. on the curve.
- rate of change of a function at a pt.
- helps define physical concepts: vel, acc, jerk

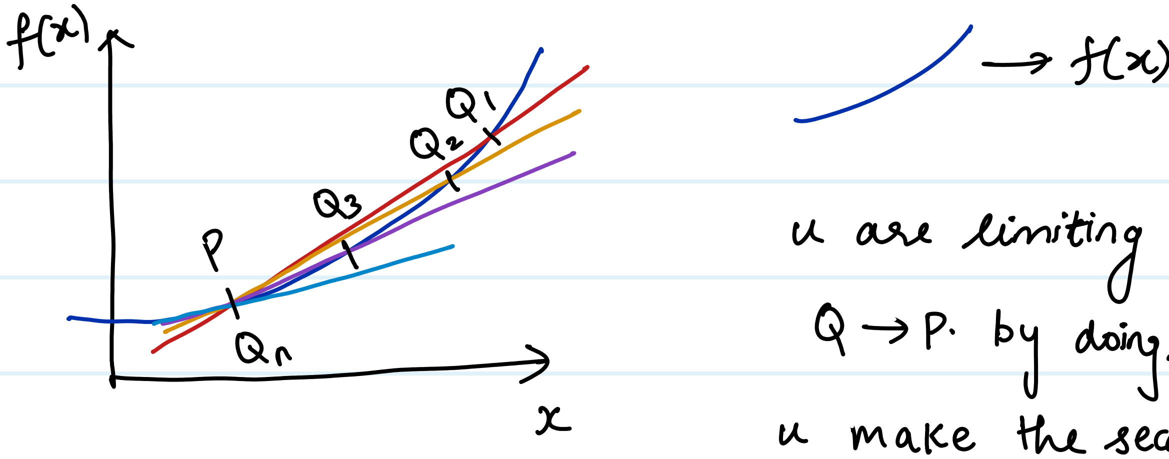
$\left(\frac{ds}{dt}\right) \quad \left(\frac{d^2s}{dt^2}\right) \quad \left(\frac{d^3s}{dt^3}\right)$

SLOPE of A CURVE

$y=f(x)$. let $P \rightarrow$ pt on y. Slope at $P = ?$

Slope at $P \equiv$ tangent at P.

you obtain tangent by limiting the secant to the curve.



u are limiting $Q \rightarrow P$. by doing so, u make the secant QP into a tangent at P.

find slope of $y=x^2$ at $P(2,4)$

$$\frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{(2+h)-2} = \frac{h^2+4+4h-4}{h}$$

$$= \frac{h^2+4h}{h} = h+4$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} (h+4) = 4.$$

$\therefore$ slope at $P(2,4) = 4$.

line acting as tangent at $P(2,4)$

$\hookrightarrow \quad m = \dfrac{y-y_0}{x-x_0}$

$\Rightarrow \quad y = m(x-x_0)+y_0$

$\Rightarrow \quad y = 4(x-2)+4$

$= 4x-8+4$

$\Rightarrow \quad \boxed{y = 4x-4}$

DEFN: SLOPE of CURVE

slope of $y=f(x)$ at any pt. $P(x_0, f(x_0))$
is m, where:

$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

EXAMPLE:

for $y=mx+b$, show tangent line
is itself. at any pt. $P(x_0, mx_0+b)$

SOLUTION :

$$f(x) = mx+b$$
$$f(x_0) = mx_0+b$$
$$f(x_0+h) = m(x_0+h)+b$$
$$= mx_0+mh+b$$

$m =$ slope at $(x_0, f(x_0))$

$$\Rightarrow m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{(mx_0+mh+b)-(mx_0+b)}{h}$$

$$m = \frac{y-b}{x-0}$$

$$\boxed{y = mx+b}$$

$$= \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$$

$\therefore$, tgt line : $y = mx+b \Rightarrow$ the line itself!

EXAMPLE : find slope of $y = 1/x$ at $x = a \neq 0$.

What happens to tangent at $(a, 1/a)$ as a changes?

slope at $(a, f(a))$

$$f(x) = 1/x$$

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \frac{a-(a+h)}{a(a+h)h} = \frac{-h}{a(a+h)h}$$

$$\Rightarrow m = \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}$$

$$\Rightarrow \boxed{m = -1/a^2}$$

as a changes, m changes to $-1/a^2$

DEFINITION: DERIVATIVE at a pt

expression $\rightarrow$ $\left.\dfrac{f(x_0+h) - f(x_0)}{h}\right\} \rightarrow$ DIFFERENCE QUOTIENT of f at x_0 with increment h.

- diff. quotient $\rightarrow$ secant slope.
  derivative of diff. quotient $\rightarrow$ tangent
- diff quotient $\rightarrow$ average rate of change. derivative of diff. quotient $\rightarrow$ instantaneous rate of change $\left(\frac{df}{dx} \text{ at } x = x_0\right)$

if the diff. quotient has a limit as $h \to 0$, that limit is called the derivative of f at x_0.

DERIVATIVE: derivative of $f(x)$ wrt. x is the function $f'(x)$ whose value at x is:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

(OR) $$\boxed{f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}}$$

DEFINITION : DIFFERENTIABILITY

if $f'(x)$ exists for $f(x)$ at $x = x_0$,
$f(x) \longrightarrow$ differentiable at $x = x_0$

if $f'(x)$ exists for $f(x)$ $\forall x \in$ domain of $f(x)$, then $f(x) \longrightarrow$ differentiable.

* finding derivative $\longrightarrow$ differentiation *

EXAMPLE :

derivative of $y = mx + b$ at any pt $x = m$

$$\Rightarrow \frac{dy}{dx} = \frac{d(mx+b)}{dx} = m$$

find :

derivative of $f(x) = \dfrac{x}{x-1}$. (USE IDEAS TAUGHT)

diff. quotient $\longrightarrow$ $$\frac{f(x+h) - f(x)}{h}$$

$$= \frac{\dfrac{x+h}{(x+h)-1} - \dfrac{x}{x-1}}{h}$$

$$\left[\frac{x-1+1}{x-1} \quad 1 + \frac{1}{x-1} \right]$$

$$= \frac{\left[x(x-1)+h(x-1)\right]-\left[x(x+h)-x\right]}{h(x+h-1)(x-1)}$$

$$= \frac{\left[\cancel{x^2}-\cancel{x}+h(x-1)\right]-\left[\cancel{x^2}+xh-\cancel{x}\right]}{h((x+h)-1)(x-1)}$$

$$= \frac{h(x-1)-hx}{h((x+h)-1)(x-1)}$$

$$= \frac{hx-h-hx}{h((x+h)-1)(x-1)} = \frac{-\cancel{h}}{\cancel{h}((x+h)-1)(x-1)}$$

$$= \frac{-1}{(x+h-1)(x-1)}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

$$\Rightarrow \boxed{f'(x) = \frac{-1}{(x-1)^2}}$$

ONE-SIDED DERIVATIVES

The Right-Hand derivative (RHD) of $f(x)$ at $x = x_0$ is:

$$\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{[Provided limit exists]}$$

The Left-Hand derivative (LHD) of $f(x)$ at $x = x_0$ is:

$$\lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{[provided limit exists]}$$

THEOREM:

$y = f(x)$ has a derivative at $x = x_0$ iff LHD & RHD exist, and LHD = RHD.

DEFINITION: DIFFERENTIABLE ON AN INTERVAL

$y = f(x)$ is differentiable on an open interval (finite/infinite) if derivative exists at all pts. on the interval. Its differentiable in $[a,b]$ if its differentiable on the interior (a,b).

EXAMPLE: Show: $f(x) = |x| \longrightarrow$ differentiable on $(-\infty, 0)$ & $(0, \infty)$ but has no limit at $x = 0$

for $\underline{x > 0}$:

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x)$$

$$= 1.$$

$\underline{x < 0}$:

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x)$$

$$= -1.$$

$\underline{\text{at } x = 0}$:

$$\text{RHD} = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^-} 1 = 1$$

$$\underline{\text{LHD}} : \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$$

$\therefore$, LHD $\neq$ RHD $\Rightarrow$ $f(x) = |x|$ $\to$

not diff. at $x=0$

SHOW: $y = \sqrt{x}$ is not diff.able at $x=0$

THEOREM: DIFFERENTIABLE IMPLIES CONTINUOUS

if $f \to$ diff.able at $x = c$, f is continuous at $x=c$.

PROOF:

let $f \to$ diff.able at $x=c$, i.e, $f'(c)$ exists.

We must show: $\lim\limits_{x \to c} f(x) = f(c)$

(OR)

$$\lim_{h \to 0} f(c+h) = f(c)$$

$$f(c+h) = f(c) + \left(f(c+h) - f(c)\right)$$

$$= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h$$

$$\Rightarrow \lim_{h\to 0} f(c+h) = \lim_{h\to 0} \left[f(c) + \frac{f(c+h)-f(c)}{h} \cdot h\right]$$

$$= \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \cdot h$$

$$= \lim_{h\to 0} f(c) + \lim_{h\to 0} \left[\frac{f(c+h)-f(c)}{h}\right] \lim_{h\to 0} h$$

$$= f(c) + f'(c) \cdot 0$$

$$\Rightarrow \boxed{\lim_{h\to 0} f(c+h) = f(c)}$$

$\therefore$, $\boxed{\text{diff.ability} \Longrightarrow \text{Continuity}}$

THEOREM: INTERMEDIATE VALUE PROPERTY of DERIVATIVES

if $a, b \rightsquigarrow$ 2 pts in an interval on which $f \to$ diff.able, then f' takes all values b/w $f'(a)$ & $f'(b)$

EXAMPLE: unit step fn. CANT be a derivative of any fn. as it doesn't follow intermediate property.

RESULTS:

if $f(x) = c$, $\dfrac{df(x)}{dx} = \dfrac{dc}{dx} = 0$

if $f(x) = x^n$, $n \in z^+$, $\dfrac{df(x)}{dx} = \dfrac{d}{dx}(x^n) = nx^{n-1}$

$$\lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^n - x^n}{z - x}$$

$$= \lim_{z \to x} \frac{(z-x)\left[z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + x^{n-2}z + x^{n-1}\right]}{(z-x)}$$

$$= \lim_{z \to x} \left(z^{n-1} + z^{n-2}x + \cdots + x^{n-1}\right)$$

$$= \underbrace{\left(x^{n-1} + x^{n-1} + \cdots + x^{n-1}\right)}_{n \text{ times}}$$

$$\Rightarrow \boxed{\lim_{z \to x} \frac{z^n - x^n}{z - x} = nx^{n-1}}$$

EXAMPLE: does $y = x^4 - 2x^2 + 2$ have horizontal tangent?

$$y = x^4 - 2x^2 + 2$$

$$\Rightarrow \frac{dy}{dx} = 4x^3 - 4x$$

$$\Rightarrow 0 = 4x^3 - 4x$$

$$\Rightarrow 4x(x^2 - 1) = 0 \Rightarrow \boxed{x = \pm 1 \text{ (or) } x = 0}$$

pts. where tgt. is horizontal

MORE RULES:

$$\boxed{\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}}$$

DERIVE as HW.

POWER RULE for $n < 0$

$f(x) = x^n \longrightarrow$ diff. able everywhere other than $x = 0$

06/11/2023

$$\frac{d}{dx}\left(f(x)g(x)\right)$$ use diff. quotient

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$\lim_{h\to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$\lim_{h\to 0} f(x+h)\cdot \underbrace{\left[\frac{g(x+h)-g(x)}{h}\right]}_{G(x,h)} + g(x)\underbrace{\left[\frac{f(x+h)-f(x)}{h}\right]}_{F(x,h)}$$

$$\lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0} G(x,h) + \lim_{h\to 0} g(x)\cdot \lim_{h\to 0} F(x,h)$$

$$\boxed{\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + g(x)\cdot f'(x)}$$

Find eqn. of tangent to the curve: $y = x + 2/x$ at $P(1,3)$

$$y = x + 2/x$$

$$m(x) = \frac{dy}{dx} = \frac{d}{dx}\left(x + 2/x\right) = \frac{d}{dx}x + \frac{d}{dx}(2/x)$$

$$m(x) = 1 - 2/x^2$$

$$\Rightarrow m(1) = 1 - 2 = -1$$

tgt: $y = mx + c$

$\Rightarrow y = -x + c$

$x = 1,\ y = 3$

$\Rightarrow 3 = -1 + c \Rightarrow \boxed{c = 4}$

$\therefore \boxed{y = -x + 4}$

SECOND & HIGHER ORDER DERIVATIVES:-

if $y = f(x)$ is a diff.able function, $f'(x)$ is also a function. if $f'(x)$ is also diff.able, we can diff. f' to get $f''(x)$.

$$\Rightarrow \boxed{(f'(x))' = f''(x)}$$

The Chain Rule:

if $f(u)$ is diff·able at the pt. $u = g(x)$ and $g(x)$ is diff·able at x, then the composite function $f(g(x))$ is diff·able at x and:

$$\boxed{f(g(x)) = f'(g(x)) \cdot g'(x)}$$

if $y = f(u)$ & $u = g(x)$, then:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

EXAMPLE: for $t > 0$, $x(t) = \cos(t^2+1)$, $v(t) = ?$

$$v(t) = \frac{d(x(t))}{dt} = \frac{d(\cos(t^2+1))}{dt}$$

$$= \frac{d(\cos(t^2+1))}{d(t^2+1)} \cdot \frac{d(t^2+1)}{dt}$$

$$= -\sin(t^2+1) \cdot 2t$$

$$= -2t\sin(t^2+1)$$

$\therefore$, $\boxed{v(t) = -2t\sin(t^2+1)}$

PARAMETRIC EQNS

if $y = f(t)$ and $x = g(t)$, then:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)$$

$$= \left(\frac{d(f(t))}{dt}\right) \Big/ \left(\frac{d(g(t))}{dt}\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{f'(t)}{g'(t)}}$$

EXAMPLE: $x = a\cos t$; $y = b\sin t$

find eqn. in terms of y, x, a, b.

$$\frac{x}{a} = \cos t \; ; \; \frac{y}{b} = \sin t$$

$$\cos^2 t = \frac{x^2}{a^2} \; ; \; \sin^2 t = \frac{y^2}{b^2}$$

$$\cos^2 t + \sin^2 t = 1 \Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$\Downarrow$$

given parametric form represents an ellipse.

IMPLICIT DIFFERENTIATION

if p/q : rational, then $x^{p/q}$ is diff·able at every pt. where $x^{(p/q)-1}$ is defined and:

$$\frac{d}{dx}\left(x^{(p/q)}\right) = \left(p/q\right)\cdot x^{(p/q)-1}$$

PROOF :

let p and q : $\mathbb{Z}$ and $q>0$. let $y = x^{p/q}$

$$\Rightarrow \quad y^q = x^p$$

$$\Rightarrow q\cdot y^{q-1}\cdot\frac{dy}{dx} = p\cdot x^{p-1} \quad (\text{diff. wrt } x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{p}{q}\cdot\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\cdot\frac{x^{p-1}}{\left(x^{p/q}\right)^{q-1}}$$

$$= \frac{p}{q}\cdot\frac{x^{p-1}}{x^{\frac{pq-p}{q}}}$$

$$= p/q\cdot\frac{x^{p-1}}{x^{p-(p/q)}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = p/q\cdot x^{(p/q)-1}}$$

$y = x^{p/q}$

$\Rightarrow \left(p/q\right)\cdot x^{(p/q)-1}$ is its derivative, where $x^{(p/q)-1}$ is defined...

08/11/2023

LINEARIZATION, STANDARD LINEAR APPROXIMATION :

if f is diff·able at $x = a$, then the approximating function:

$$L(x) = f(a) + f'(a)(x-a)$$

is called the linearization of f at a

$f(x) \simeq L(x)$ of f by L is called the standard linear approximation of f at a

EXAMPLE: find linearization of $f(x) = \sqrt{x+1}$ at $x = 0$

$$L(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt{0+1} + \frac{1}{2\sqrt{0+1}}(x-0)$$

$$\boxed{L(x) = 1 + \frac{x}{2}}$$

DIFFERENTIALS

Let $y = f(x)$ be a diff·able function. The differential dx is an independent variable. The differential dy is:

$$\boxed{dy = f'(x)\,dx}$$

(a) if $y = x^5 + 37x$

$$\Rightarrow \quad dy = (5x^4 + 37)\,dx$$

(b) if $dx = 0.2$ & $x = 1$,

$$dy = (5+37) \times \frac{1}{5} = \frac{42}{5} = 8.4$$

ABSOLUTE EXTREMA

Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if:

$$f(x) \leq f(c) \qquad \forall x \in D$$

similarly for minima,

$$f(x) \geq f(c), \qquad \forall x \in D$$

EXAMPLES: $y = x^2$. extrema depends on D.

1. if $D = (-\infty, \infty)$, then theres no absolute maximum. but, there's an absolute minimum at $x=0$
2. if $D = [0,2] \rightarrow$ max at $x=2$ & min at $x=0$
3. if $D = (0,2] \rightarrow$ no min, but max at $x=2$
4. if $D = [0,2) \rightarrow$ no max, but min at $x=0$
5. if $D = (0,2) \rightarrow$ neither max nor min exists. (no absolute extrema)

EXTREME VALUE THEOREM

if $f \rightarrow$ continuous in $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a,b]$. That is, there are x_1 & x_2 in $[a,b]$ with $f(x_1) = m$ and $f(x_2) = M$.

LOCAL MAXIMUM

A fn. $f \to$ has local maximum at a pt. c in its domain if : $f(x) \leq f(c)$ $\forall x$ in some open interval containing c.

similarly, for local minimum :

$f(x) \geq f(c)$ $\forall x$ in some open interval containing c

LOCAL EXTREME VALUES

if $f \to$ has local maximum / minimum at an interior pt. c of its domain and if f' is defined at c, then:

$$\boxed{f'(c) = 0}$$

PROOF :

suppose $f \to$ local max at $x = c$ (interior)

so that $f(x) - f(c) \leq 0$ $\forall x$ close to c.

let $f'(c) \to$ defined. since $c \to$ interior, then it follows that the derivative is defined by the 2 sided limit :

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

this means that, the Left hand and right hand derivatives exist at c and both equal to $f'(c)$

So, we have :

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

$$\Rightarrow \boxed{f'(c) = 0}$$

CRITICAL POINT

An interior pt of the domain where $f'(x) = 0$ (or) $f'(x)$ is undefined.

EXAMPLE

$f(x) = x^2$ on $[-2, 1]$

$f'(x) = 2x$

$\Rightarrow f'(x) = 0$ only if $\boxed{x = 0}$

absolute maximum (extrema) $\leftarrow f(-2) = 4$ & $f(1)$, $f(0) = 0$ absolute minimum (extrema)

HW:

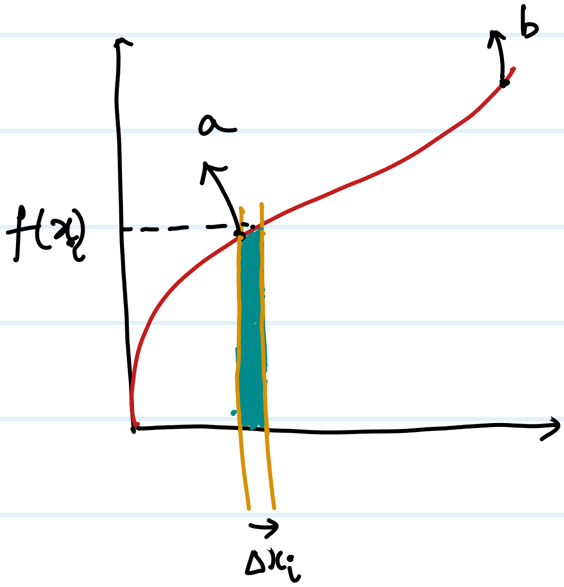
$$g(t) = 8t - t^4 \quad [-2, 1]$$
$$f(x) = x^{2/3} \quad [-2, 1]$$

$\Bigg\}$ → find extrema

ROLLE'S THEOREM : refer slides ... till Riemann Integral.

14/11/2023

RIEMANN INTEGRAL



$$Area_{ab} = \lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\,\Delta x$$

$$= \int_a^b f(x)\,dx$$

$f(x) = 1 - x^2$; $A_{01} = \int_0^1 (1-x^2)dx = \frac{2}{3} = 0.67$

$M_1 = \sup\left(f(x) : x \in [0, 0.5]\right) = 1$

$M_2 = \sup\left(f(x) : x \in [0.5, 1]\right) = 3/4$

Total area $= M_1 \times 0.5 + M_2 \times 0.5$

$= 1 \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2}$

$= \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$

$= 0.875$

estimate 1: size of box: 0.5

upper sum

1

Better estimate → ↓ the size of $\boxed{}$

$M_1 \to 1 - 0 = 1$

$M_2 \to 1 - \frac{1}{16} = \frac{15}{16}$

$M_3 = 1 - \frac{1}{4} = \frac{3}{4}$

$M_4 = 1 - \frac{9}{16} = \frac{7}{16}$.

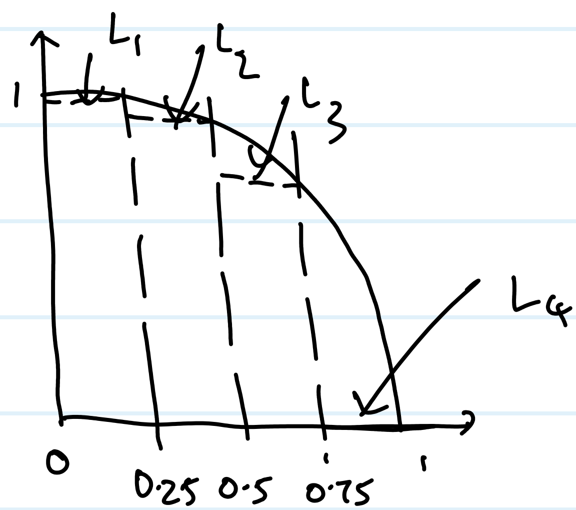
$\text{Area} = (M_1 + M_2 + M_3 + M_4) \times \frac{1}{4}$

$= \left(\frac{16}{16} + \frac{15}{16} + \frac{3\times4}{4\times4} + \frac{7}{16}\right) \times \frac{1}{4}$

$= \frac{50}{16} \times \frac{1}{4} = \frac{50}{64}$

$= 0.78125$

upper sum (2)



$L_1 = \frac{15}{16}, \; L_2 = \frac{3}{4}, \; L_3 = \frac{7}{16}$

$L_4 = 0$.

$\left(\frac{15}{16} + \frac{3\times4}{4\times4} + \frac{7}{16}\right) \times \frac{1}{4}$

Lower sum (1)

$= \frac{34}{16} \times \frac{1}{4} = \frac{34}{64} = 0.53125$

15/11/2023

if divisions ↑ | size of intervals ↓, the accuracy ↑

actual sum → sandwiched b/w upper & lower sums.

DEFINITION : $\boxed{\text{PARTITION}}$

Let $[a,b] \to$ closed interval.

then a set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is called a partion of the interval if:

$$\boxed{a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = b}$$

NOTATION :

if $P = \{x_0, x_1, x_2 \ldots, x_n\}$ is a partition of $[a,b]$

length of ith subinterval $[x_{i-1} \cdot x_i]$ is given by:

$$\Delta x_i = x_i - x_{i-1} \quad , \quad i = 1, 2, 3, \ldots, n.$$

$$\sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = b - a \longrightarrow \text{Telescopic Sum}$$

$$\boxed{x_n - x_0} \longleftarrow (\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + (\cancel{x_3} - \cancel{x_2}) + \ldots + (x_n - \cancel{x_{n-1}})$$

UPPER RIEMANN SUM, LOWER RIEMANN SUM

Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

Let $P = \{x_0, x_1, x_2, \ldots x_n\}$ be a partition of $[a,b]$

Let $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ & $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$

$i \in \{1, 2, 3, \ldots, n\}$

Then the upper Reimann sum corresponding to P is:

$$\boxed{U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i}$$

The lower Reimann sum corresponding to P is:

$$\boxed{L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i}$$

Also, $L(P,f) \leq U(P,f)$ → PROVE THIS....

Lemma: Let $f : [a,b] \to R$ be bounde fn. with

$$m \leq f(x) \leq M.$$

Then, for any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$ & $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$
$\forall i \in \{1, 2, \cdots, n\}$

$m \leqslant m_i \leqslant M_i \leqslant M \quad \Rightarrow \quad m\Delta x_i \leqslant m_i \Delta x_i \leqslant M_i \Delta x_i \leqslant M \Delta x_i$

$$\Rightarrow \sum_{i=1}^{n} m\Delta x_i \leqslant \sum_{i=1}^{n} m_i \Delta x_i \leqslant \sum_{i=1}^{n} M_i \Delta x_i \leqslant \sum_{i=1}^{n} M\Delta x_i$$

$$\Rightarrow \sum_{i=1}^{n} m\Delta x_i \leqslant L(P,f) \leqslant U(P,f) \leqslant \sum_{i=1}^{n} M\Delta x_i$$

$$\Rightarrow \boxed{L(P,f) \leqslant U(P,f)}$$

COROLLARY:

Let $f \to$ bounded, real valued for $[a, b]$.

$\Rightarrow$ Set of lower sums is bounded above.

$\Rightarrow$ set of upper sums is bounded below

$L(P,f) \leqslant M(b-a)$ and $m(b-a) \leqslant U(P,f)$

used for better estimates
↑ (Lower → bigger & upper → smaller)

REFINEMENT of a PARTITION & COMMON REFINEMENT

1. Let P_1 be a partition of $[a,b]$. Then a partition P_2 of $[a,b]$ is called a refinement of P_1 if $P_1 \subseteq P_2$

2. Let P_1 and P_2 be partitions of $[a,b]$. Then $P_1 \cup P_2$ is called a common refinement of P_1 and P_2.

Lemma: Let $f:[a,b] \to \mathbb{R}$ be a bounded fn. Let P_1 & P_2 be partitions of $[a,b]$ such that P_2 is a refinement of P_1. Then,

PROVE

$$L(P_1,f) \leq L(P_2,f) \leq U(P_2,f) \leq U(P_1,f)$$

Theorem: Let $f:[a,b] \to \mathbb{R}$ be a bounded function. Let P_1 and P_2 be any partitions of $[a,b]$.

Then,

$$L(P_1,f) \leq U(P_2,f)$$

Let $Q = P_1 \cup P_2$ be the common refinement of P_1 and P_2. Then by above Lemma:

$$L(P_1,f) \leq L(Q,f) \leq U(Q,f) \leq U(P_2,f).$$

Let f: bounded real-valued fn. on $[a,b]$. Then the set of lower sums $\rightarrow$ $\{L(P,f) : P \text{ is partition of } [a,b]\}$ is bounded above by $M(b-a)$.

And the set of upper sums $\rightarrow$ $\{U(P,f) : P \text{ is a partition of } [a,b]\}$ is bounded below by $m(b-a)$.

DEFINITION: Let $f : [a,b] \to \mathbb{R}$ be a bounded function. The upper Riemann Integral of f over $[a,b]$ is

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \text{ is partition of } [a,b]\}$$
$$= \inf U(P,f)$$

The Lower Riemann Integral of f over $[a,b]$ is

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \text{ is partition of } [a,b]\}$$
$$= \sup L(P,f)$$

PROVE: $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$

DEFINITION: Let $f:(a,b)\to\mathbb{R}$ be a bounded function. We say $f \to$ Reimann Integrable on $[a,b]$ if the upper and lower Reimann integrals are equal.

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

PROOF: Show $f(x)=K$ on $[a,b]$ is reimann integrable

Let $P=\{a=x_0, x_1, x_2, \ldots, x_n=b\}$ be any partition of $[a,b]$. Then,

$$M_i = \sup\{f(x): x_{i-1}\leq x\leq x_i\} = \sup\{K: x_{i-1}\leq x\leq x_i\} = K$$

$$m_i = \inf\{f(x): x_{i-1}\leq x\leq x_i\} = \inf\{K: x_{i-1}\leq x\leq x_i\} = K$$

Upper Riemann sum $U(P,f) = \sum_{i=1}^{n} M_i\Delta x_i = K(b-a)$

Lower Riemann sum $L(P,f) = \sum_{i=1}^{n} m_i\Delta x_i = K(b-a)$

Upper Riemann Integral $= \overline{\int_a^b} f(x)\,dx = K(b-a)$

Lower Riemann Integral $= \underline{\int_a^b} f(x)\,dx = K(b-a)$.

upper R.I = Lower R.I.

$\therefore$ $f(x) = K$ is Riemann Integrable.

Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Show $f(x) \to$ not Riemann Integrable.

PROOF : $0 \leq f(x) \leq 1 \quad \forall x \in [0,1]$ so, $f(x) \to$ bounded on $[0,1]$.

Let $P = \{0 = x_0, x_1, x_2, \cdots, x_n = 1\}$ be any partition of $[0,1]$.

$$0 = x_0 \leq x_1 \leq x_2 \cdots \leq x_n = 1.$$

$$M_i = \sup \{ f(x) : x_{i-1} \leq x \leq x_i \} = \sup \{0,1\} = 1.$$
$$m_i = \inf \{ \qquad \} = \inf \{0,1\} = 0.$$

UPPER $= \sum_{i=1}^{n} M_i \Delta x_i = 1$ UPPER.R.I $\neq$ LOWER.R.I

LOWER $= \sum_{i=1}^{n} m_i \Delta x_i = 0$ $\therefore$ NOT . R.Integrable.

IMPORTANT:

$f:[a,b]\to R$ be bounded fn. Then $f \to$ Riemann Integrable iff $\forall\, \varepsilon>0$ there exists a P s.t.

$$U(P,f) - L(P,f) < \varepsilon .$$