$f(x) \rightarrow$ given function

$f'(x) = 0 \leftarrow x_0$

$f''(x_0) \geq 0 \; , \; f''(x_0) \leq 0$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \; \boxed{+ \cancel{\cdots}}$$

$$f(x) - f(x_0) \approx \boxed{\frac{f''(x_0)}{2}} \boxed{(x - x_0)^2}$$

$x_0 \rightarrow$ local minimum.

$$f(x) - f(x_0) > 0$$

Taylor Series for $f(x,y) \to (x_0, y_0)$

X_0

$$f(x,y) = f(x_0,y_0) + \boxed{f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

$$+ \frac{f_{xx}(x_0,y_0)}{2!}(x-x_0)^2 + \boxed{f_{xy}(x_0,y_0)}(x-x_0)(y-y_0) + \boxed{\frac{f_{yy}(x_0,y_0)}{2!}}(y-y_0)^2$$

Let $h = \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$, $H = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}_{(x_0,y_0)}$ 2×2

$$\boxed{f(x,y) - f(x_0,y_0)} \approx \cancel{\boxed{h^T \begin{pmatrix} f_x \\ f_y \end{pmatrix}_{(x_0,y_0)}}} \; (0) + \frac{1}{2} h^T H h$$

$f_x(x_0,y_0) = 0$
$= f_y(x_0,y_0)$

$$\left[f(x,y) - f(x_0,y_0) \right] \approx \boxed{\frac{1}{2} h^T H h}$$

(i) (x_0, y_0) is Local Min $\Rightarrow \frac{1}{2} h^T H h > 0, \forall h \neq 0$

(ii) (x_0, y_0) is Local Max $\Rightarrow \frac{1}{2} h^T H h < 0, \forall h \neq 0$

$$f''(x_0) \gtrless 0 \Rightarrow \boxed{\frac{1}{2} h^T H h} \gtrless 0, \forall h \neq 0$$

$$H = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}_{2\times 2}$$

$$\boxed{f_{xy} = f_{yx}}$$

$$f \in \boxed{C^2} \checkmark$$

$\boxed{\text{Theorem}}$ A is real symmetric matrix $\Rightarrow$ $\Leftarrow$

is positive definite

iff all its eigenvalues are +ve

Def^n → $\boxed{\text{Positive Definite}}$ Matrix

A is +ve definite if $x^T A x > 0$ $\forall\, x \in \mathbb{R}^n$, $x \neq 0$

$A_{n\times n}$, $x_{n\times 1} \in \mathbb{R}^n$

$$x^T_{1\times n} A_{n\times n} x_{n\times 1} \longrightarrow 1\times 1$$

Jacobian:

u & v are functions of two independent variable x and y, then the determinant

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$ is called Jacobian of u, v with respect to x, y

and written as $\dfrac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\dfrac{u,v}{x,y}\right)$

Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{pmatrix} \longrightarrow$$ Extend for Higher Dim.

(1) $J = \partial(u,v)/\partial(x,y)$ & $J' = \partial(x,y)/\partial(u,v)$ then

$$\boxed{JJ' = 1}$$

(2) Chain Rule:-

u, v function of r, s and r, s are function of x, y. then

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \frac{\partial(r,s)}{\partial(x,y)}}$$

Ex:- If $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_3 x_1}{x_2}$ and $y_3 = \dfrac{x_1 x_2}{x_3}$

Show that y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

—x—

Solution:- $\dfrac{\partial y_1}{\partial x_1} = -\dfrac{x_2 x_3}{x_1^2}$, $\dfrac{\partial y_1}{\partial x_2} = \dfrac{x_3}{x_1}$, $\dfrac{\partial y_1}{\partial x_3} = \dfrac{x_2}{x_1}$

$\dfrac{\partial y_2}{\partial x_1} = \dfrac{x_3}{x_2}$, $\dfrac{\partial y_2}{\partial x_2} = -\dfrac{x_3 x_1}{x_2^2}$, $\dfrac{\partial y_2}{\partial x_3} = \dfrac{x_1}{x_2}$

$\dfrac{\partial y_3}{\partial x_1} = \dfrac{x_2}{x_3}$, $\dfrac{\partial y_3}{\partial x_2} = \dfrac{x_1}{x_3}$, $\dfrac{\partial y_3}{\partial x_3} = -\dfrac{x_1 x_2}{x_3^2}$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{pmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \partial y_1/\partial x_3 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \partial y_3/\partial x_1 & \partial y_3/\partial x_2 & \partial y_3/\partial x_3 \end{pmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$J = \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_1 x_2 \\ x_2 x_3 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_3 x_1 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4.$$

Ex:- $u = \ln(x + 3y^2 - z^3)$, $v = 4x^2yz$, $w = 2z^2 - xy$

Evaluate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$

Sol^n:- $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = \begin{pmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{pmatrix}$$

at point $(1,-1,0)$,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 4(-1+6) = 20.$$

$\boxed{\text{Prob-01}}$

If $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$

& $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, find $= \dfrac{\partial(u,v)}{\partial(r,\theta)}$

$\boxed{\text{Prob-02}}$

$x = u(1-v)$
$y = uv$, prove that $JJ' = 1$

Lagrange Multiplier Method

Let $u = f(x, y, z)$ ——— (1)

be a function of three variable x, y, z which are connected by the relation

$$\phi(x, y, z) = 0 \quad \text{———} \quad (2)$$

For u to have stationary values

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0 \quad \text{———} \quad (3)$$

Also differentiating (2), we get

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi = 0 \quad \text{--- (4)}$$

Multiplying (4) by parameter λ & add to (3)

$$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

This equation satisfied if

$$\begin{cases} \dfrac{\partial u}{\partial x} + \lambda\dfrac{\partial \phi}{\partial x} = 0 \\ \dfrac{\partial u}{\partial y} + \lambda\dfrac{\partial \phi}{\partial y} = 0 \\ \dfrac{\partial u}{\partial z} + \lambda\dfrac{\partial \phi}{\partial z} = 0 \end{cases}$$

Algorithm

(1) Write $F = f(x,y,z) + \lambda \phi(x,y,z)$

(2) Obtain the equations

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

(3) Solve the above equations together with $\phi(x,y,z) = 0$

The value of x, y, z so obtaind will give the stationary value of $f(x,y,z)$

Example :- 01

A rectangular box open at top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.

Solution :- Let x, y and z ft be the edges & S is its surface

$$S = xy + 2yz + 2zx \quad \text{--- (i)}$$

$$xyz = 32 \quad \text{--- (ii)}$$

Elemenating z from (i)

$$S = xy + 2(y+x)\frac{32}{xy}$$

$$S = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$\frac{\partial S}{\partial x} = y - 64/x^2 = 0$$

and $$\frac{\partial S}{\partial y} = x - 64/y^2 = 0$$

Solving these, we get $x = y = 4$

Now
$$\begin{cases} r = \boxed{\dfrac{\partial^2 S}{\partial x^2}} = 128/x^3 \\ s = \dfrac{\partial^2 S}{\partial x \partial y} = 1 \\ t = \dfrac{\partial^2 S}{\partial y^2} = 128/y^3 \end{cases}$$

$x = y = 4$

$rt - s^2 = 2 \times 2 - 1$

$= +ve$

$\{$ r is also $+ve$

Hence S is minimum for $x = y = 4$

then from (ii) , $\boxed{z = 2}$

Lagrange Method

$$F = xy + 2yz + \lambda(xyz - 32)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= y + 2z + \lambda yz = 0 \\ \frac{\partial F}{\partial y} &= x + 2z + \lambda zx = 0 \\ \partial F/\partial z &= 2y + 2x + \lambda xy = 0 \end{aligned}\right\} \begin{aligned} &\text{(iii)} \\ &\text{(iv)} \\ &\text{(v)} \end{aligned}$$

$$x \times \text{(iii)} - y \times \text{(iv)} = 2zx - 2zy = 0 \Rightarrow \boxed{x = y}$$

Value $z = 0$ neglected as this not satisfy (ii).

Again -

$y \times (iv) - z \times (v) \Rightarrow \boxed{y = 2z}$

Hence dimension of the box $x = y = 2z = 4$ —— (vi)

$S = xy + 2yz + 2zx$

$(x = 4, y = 4)$

$= 16 + 8z + 8z = 16 + 16z = 16(1+z)$

$S = 16 \times 3 = \boxed{48}$ → Minimum value

2, 4, 4 also satisfy $\phi(x, y, z)$

But $S = 4 + 8z + 4z = 4 + 12z = 4(1+3z)$

for $z = 4$, $S = 4 \times 13 = 52$

Ex:- Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Solution:-

$2x$, $2y$, $2z$ are length, breadth & height of rectangle then

volume $V = 8xyz$

R is radius of sphere $\rightarrow x^2+y^2+z^2=R^2$

$$F = 8xyz + \lambda\left(x^2+y^2+z^2-R^2\right)$$

$F_x = 0$, $F_y = 0$ & $F_z = 0$

Then

$$\begin{cases} 8yz + 2x\lambda = 0 \\ 8zx + 2y\lambda = 0 \\ 8xy + 2z\lambda = 0 \end{cases}$$

$$2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$$

Thus for a max vol $x = y = z$

the rectangular solid is a cube $\square$

Ex!- Find the maximum & minimum distances of the point $(3,4,12)$ from the sphere

$$\boxed{x^2+y^2+z^2=4}$$

Example. Find the maximum & minimum distance of the point $(3,4,12)$ from the sphere $x^2+y^2+z^2=4$

Sol^n:- $P(x,y,z)$ is a point on $A(3,4,12)$ the given point so that

$$AP^2 = (x-3)^2+(y-4)^2+(z-12)^2 = f(x,y,z), \text{ say}$$

find the max / min value of 'f' subject to the condition

$$\phi(x,y,z) = x^2+y^2+z^2$$

$$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$$

$$F(x,y,z) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda\,(x^2+y^2+z^2)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2(x-3) + 2\lambda x \\ \frac{\partial F}{\partial y} &= 2(y-4) + 2\lambda y \\ \frac{\partial F}{\partial z} &= 2(z-12) + 2\lambda z \end{aligned}\right\} = 0$$

$$\lambda = -\frac{x-3}{x} = -\frac{y-4}{y} = -\frac{z-12}{z}$$

$$= \pm \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2} \Big/ \sqrt{x^2+y^2+z^2}$$

$\lambda = \pm \sqrt{f}/1$

Substituting for λ in (iii), we get

$$\begin{cases} x = \dfrac{3}{1+\lambda} = \dfrac{3}{1 \pm \sqrt{f}} \\ y = \dfrac{4}{1 \pm \sqrt{f}} \\ z = \dfrac{12}{1 \pm \sqrt{f}} \end{cases}$$

$$x^2+y^2+z^2 = \frac{9+16+144}{1 \pm \sqrt{f}} = \frac{169}{(1 \pm \sqrt{f})^2}$$

$$\Rightarrow \quad 4 = \frac{169}{(1 \pm \sqrt{f})^2} \quad \Rightarrow (1 \pm f)^2 = \frac{169}{4}$$

$$\Rightarrow \left(1 \pm \sqrt{f}\right)^2 = \frac{169}{4}$$

$$\Rightarrow 1 \pm \sqrt{f} = \pm \frac{13}{2}$$

$$\Rightarrow \sqrt{f} = \frac{13}{2} - 1 \;/\; \frac{13}{2} + 1$$

$$= 6.5 - 1 \;/\; 6.5 + 1$$

$$= 5.5 \;/\; 7.5$$

Max Ap = 7.5

Min Ap = 5.5