Hence, $\lim (a_n b_n) = ab = (\lim a_n)(\lim b_n)$.

(*iii*) *Lemma.* To show that if $\lim b_n = b \neq 0$, then $\exists$ a positive number λ and a positive in[teger]
m_3 such that

$$|b_n| > \lambda, \ \forall\, n \geq m_3$$

Let us take $\varepsilon = \frac{1}{2}|b|$, so that there exists a positive integer m_3 such that

$$|b_n - b| < \frac{1}{2}|b|, \ \forall\, n \geq m_3,$$

Thus,
$$|b| - |b_n| \geq |b_n - b| < \frac{1}{2}|b|.$$

$\Rightarrow$
$$|b_n| \geq \frac{1}{2}|b| \text{ (say)}, \ \forall\, n \geq m_3.$$

Let us apply the Lemma to prove the main theorem.

Now

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{ba_n - ab_n}{bb_n}\right| = \left|\frac{b(a_n - a) - a(b_n - b)}{bb_n}\right|$$

$$\leq \frac{|b|\,|a_n - a| + |a|\,|b_n - b|}{|b|\,|b_n|}$$

$$\leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b|, \ \forall\, n \geq m_3$$

Let $\varepsilon > 0$ be given.

Since $\lim a_n = a$, $\lim b_n = b$, therefore, $\exists$ positive integers m_1, m_2 such that

$$|a_n - a| < \frac{1}{4}|b|\varepsilon, \ \forall\, n \geq m_1$$

and
$$|b_n - b| < \frac{1}{4}\frac{|b|^2 \varepsilon}{|a| + 1}, \ \forall\, n \geq m_2.$$

Thus, for $m = \max(m_1, m_2, m_3)$, we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \ \forall\, n \geq m$$

Hence,
$$\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b} = \frac{\lim a_n}{\lim b_n}.$$

...nder certain conditions, of