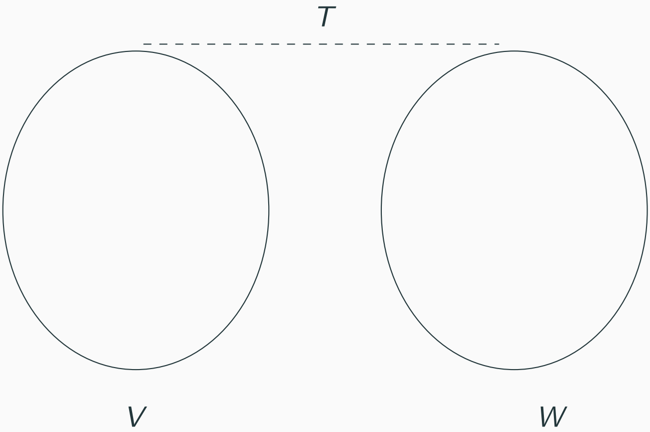
Representation of Transformations by Matrices

Shalu M A
IIITDM Kancheepuram, Chennai

Linear transformations and Matrices



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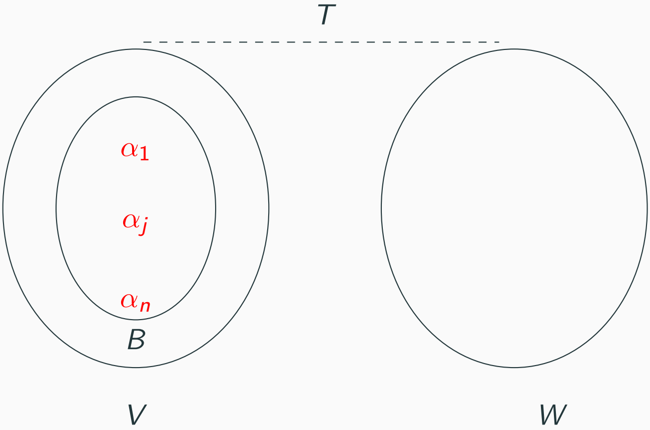

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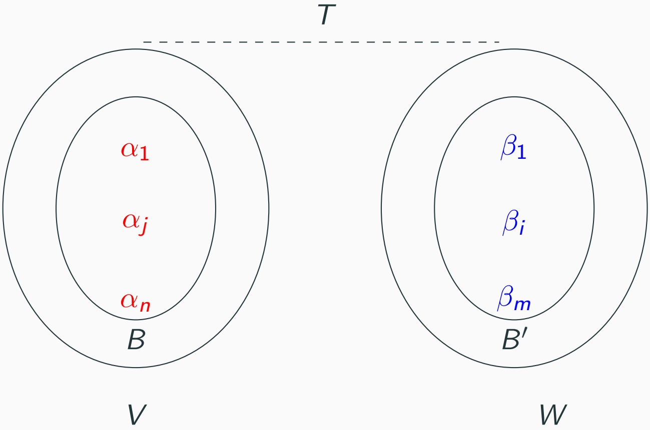

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Ordered basis, $B' = \{\beta_1, \beta_2, \ldots, \beta_m\}$

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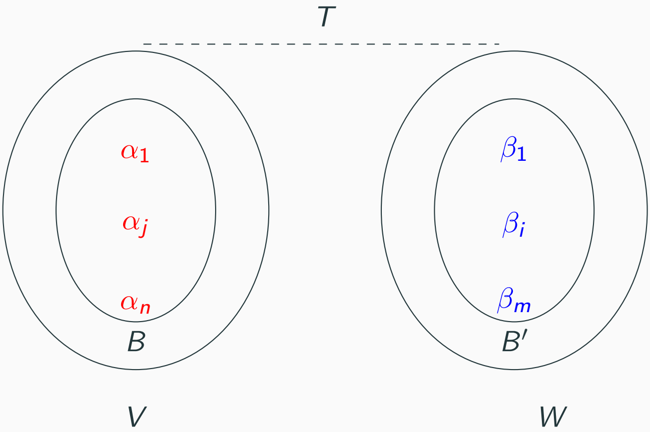

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Ordered basis, $B' = \{\beta_1, \beta_2, \ldots, \beta_m\}$

Note that (i) $T : V \longrightarrow W$ is a L.T.

L.T. and Matrix

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$$[T(\alpha)]_{B'} = \begin{bmatrix} \sum_{j=1}^{n} A_{1j}x_j \\ \sum_{j=1}^{n} A_{2j}x_j \\ \cdots \\ \sum_{j=1}^{n} A_{mj}x_j \end{bmatrix}$$

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$$\implies \quad [T(\alpha)]_{B'} = A\,[\alpha]_B$$

where $A = ([T(\alpha_1)]_{B'}, [T(\alpha_2)]_{B'}, \dots, [T(\alpha_n)]_{B'})$

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for every vector $\alpha \in V$.

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Proof (See the previous slides)

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Note : Let V be a finite dimensional vector space and B an ordered basis for V. If $T : V \longrightarrow V$ a linear operator, then A is denoted as $[T]_B$.

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$$[T(\alpha)]_B = [T]_B\,[\alpha]_B$$

Problem 1

Let $T : R^2 \longrightarrow R^3$ be a linear transformation defined as

$$T(x_1, x_2) = (x_2, x_1 - x_2, x_1 + x_2)\,.$$

Let $B = \{\alpha_1 = (1,0), \alpha_2 = (0,1)\}$ and
$B' = \{\beta_1 = (1,1,1), \beta_2 = (1,1,0), \beta_3 = (1,0,0)\}$ be respective ordered bases for R^2 and R^3. Find a 3×2 matrix A such that

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Problem 1 contd.

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$$T(\alpha_2) \quad = \quad T(0,1) = (1,-1,1)$$

Problem 1 contd.

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$$\begin{aligned} T(\alpha_2) &= T(0,1) = (1,-1,1) \\ &= (1,1,1) - 2(1,1,0) + 2(1,0,0) \end{aligned}$$

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$$[T(\alpha_2)]_{B'} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Problem 1 contd.

$$A = ([T(\alpha_1)]_{B'}, \quad [T(\alpha_2)]_{B'})$$

Problem 1 contd.

$$A = ([T(\alpha_1)]_{B'},\ \ [T(\alpha_2)]_{B'}) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 2 \end{bmatrix}$$

Problem 1 contd.

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Verification :

$$T(\alpha) = T(x_1, x_2) = (x_2, x_1 - x_2, x_1 + x_2)$$

$$T(\alpha) = (x_1 + x_2)\beta_1 - 2x_2\beta_2 + (-x_1 + 2x_2)\beta_3$$

Problem 1 contd.

$$[T(\alpha)]_{B'} = \begin{bmatrix} x_1 + x_2 \\ -2x_2 \\ -x_1 + 2x_2 \end{bmatrix}, \quad [\alpha]_B$$

Problem 1 contd.

$$[T(\alpha)]_{B'} = \begin{bmatrix} x_1 + x_2 \\ -2x_2 \\ -x_1 + 2x_2 \end{bmatrix}, \quad [\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \alpha = x_1\alpha_1 + x_2\alpha_2$$

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$$A\,[\alpha]_B = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -2x_2 \\ -x_1 + 2x_2 \end{bmatrix} = [T(\alpha)]_{B'}$$

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Note : determinant of T, $det(T) = det(A)$

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Let $T : R^2 \longrightarrow R^2$ be a L.T. defined as $T(x_1, x_2) = (x_1, 0)$. Let $B = \{\alpha_1 = (1, 1), \alpha_2 = (1, 2)\}$ be an ordered basis for R^2. Find $[T]_B$.

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Problem 2 contd

$$[T]_B = ([T(\alpha_1)]_B, [T(\alpha_2)]_B) = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

Problem 2 contd

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Please verify the answer.

Theorem 14

Let V be a finite dimensional vector space over the field F and let

$$B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}. \quad \text{and } B' = \{\beta_1, \beta_2, \dots, \beta_n\}$$

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be two ordered bases for V. Suppose $T : V \longrightarrow V$ is a linear operator. If $P = [P_1, P_2, \ldots, P_n]$ is the $n \times n$ matrix with columns $P_j = [\beta_j]_B$, then

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Proof (Reading assignment)

Similar matrices

Let A and B be $n \times n$ matrices over the field F.

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$\Longrightarrow A = (P^{-1})^{-1}BP^{-1}$.

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Eigen values/ Characteristic values

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Charateristic Polynomial

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So $f(x) = \det\,(xI - A)$ is called the characteristic polynomial of A.

Application (Mechanical Engineering)

Eigenvalues and eigenvectors allow us to "reduce" a linear operation to separate, simpler, problems. For example, if a stress is applied to a "plastic" solid, the deformation can be dissected into "principle directions"- those directions in which the deformation is greatest. Vectors in the principle directions are the eigenvectors and the percentage deformation in each principle direction is the corresponding eigenvalue

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Let us explore a simple case of diagonalization

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$D^2 = P^{-1}A^2P$, $D^k = P^{-1}A^kP$ and $A^k = PD^kP^{-1}$

$A^k \longrightarrow O$ as $k \longrightarrow \infty$ provided $|\lambda_i| < 1$ for $i = 1, 2, \ldots, n$

A few points

1 Find the eign values of the $D, D^2, D^3, \ldots,$ (see last page)

2 Find the eigen values and eigen spaces of $C = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

3 Find the eigen values and eigen spaces of $C^2, C^3, \ldots$

4 Suppose $P^{-1}AP = B$. Show that A and B have same eigen values. If λ is an eigen value of B, find an eigen vector of A corresponding to λ.

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$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

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Hence eigen values of $A = \{1, 2\}$.

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$$A - I = \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - I)X = 0 \Longrightarrow x_1 - x_3 = 0, \ x_2 + \frac{1}{3}x_3 = 0$$

Problem 3 contd.

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Problem 3 contd.

$$\implies \quad E_A(1) = \ \text{span}\left\{(3, -1, 3)\right\}$$

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(b) The eigen space when $\lambda = 2$

Problem 3 contd.

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(b) The eigen space when $\lambda = 2$

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(b) The eigen space when $\lambda = 2$

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$E_A(1) = \{(2a + 2b, a, b) : \ a, b \in R\} \ = \{a(2, 1, 0) + b(2, 0, 1) : a, b \in R\}$

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Problem 3 contd.

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Let us construct a diagonal matrix D with eigen values as diagonal entries.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 3 contd.

Let us construct an inverible matrix P using basis vectors (as columns) of $E_A(1)$ and $E_A(2)$.

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So A is similar to a diagonal matrix D and hence A is diagonalizable.