Laplace Transformation: Motivation

Consider a power series written in the form

$$\sum_{n=0}^{\infty} a(n)x^n.$$

Its *continuous analog* is the improper integral

$$\int_0^{\infty} a(t)x^t dt.$$

For a fixed value of x, if this improper integral exists, this integral associates a number with the number x.

If the integral exists (i.e., converges) for each value in some interval $[a, b]$, then this integral defines a function $F(x)$ of x on this interval.

Suppose we change the parameter x to e^{-p}. Then the integral takes the form

$$\int_0^\infty e^{-pt}a(t)dt.$$

This is called the Laplace transform of the function $a(t)$ and is denoted by

$$L[a(t)] = F(p).$$

Thus

$$L[a(t)] = \int_0^\infty e^{-pt}a(t)dt = F(p).$$

Laplace Transformation

Definition

The **Laplace transformation** of a function $f(x)$ is

$$L[f(x)] = \int_0^\infty e^{-px} f(x) dx \quad = F(p),$$

provided the improper integral exists.

Note

$$\lim_{x \longrightarrow \infty} \frac{x^n}{e^{px}} = 0, \qquad \text{provided} \qquad p > 0$$

.

The Laplace transform of $f(x) = 1$

$$L[1] = \int_0^\infty e^{-px}\,dx = \left[\frac{e^{-px}}{-p}\right]_0^\infty = \frac{-1}{p}[0-1] = \frac{1}{p} \qquad \text{for } p > 0.$$

The Laplace Transform of $f(x) = x^n$, n a Positive Integer

$$
\begin{aligned}
L[x^n] &= \int_0^\infty e^{-px} x^n \, dx \\
&= \int_0^\infty x^n \, d\left(\frac{e^{-px}}{-p}\right) \\
&= \left[-\frac{x^n e^{-px}}{p}\right]_0^\infty + \frac{n}{p}\int_0^\infty e^{-px} x^{n-1} \, dx \\
&= 0 + \frac{n}{p} L[x^{n-1}] \qquad \text{for } p > 0 \\
&= \frac{n(n-1)}{p^2} L[x^{n-2}] \qquad \text{for } p > 0 \\
&\vdots \\
&= \frac{n!}{p^n} \cdot L[x^0] = \frac{n!}{p^n} \cdot L[1] = \frac{n!}{p^{n+1}}
\end{aligned}
$$

The Laplace transform of $f(x) = e^{ax}$, $a \in \mathbb{R}$

$$
\begin{aligned}
L[e^{ax}] &= \int_0^\infty e^{-px} e^{ax}\, dx \\
&= \int_0^\infty e^{-(p-a)x}\, dx \\
&= \frac{-1}{p-a} e^{-(p-a)x}\Big|_0^\infty \\
&= \frac{1}{p-a}, \qquad \text{for } p > a.
\end{aligned}
$$

Note

We similarly have

$$L[e^{iax}] = \frac{1}{p - ia} \qquad \text{for } p > 0$$

and

$$L[e^{-iax}] = \frac{1}{p + ia} \qquad \text{for } p > 0.$$

(Prove!)

Linearity of Laplace transformation

$$
\begin{aligned}
L[\alpha f(x) + \beta g(x)] &= \int_0^\infty e^{-px} \left[\alpha f(x) + \beta g(x)\right] dx \\
&= \alpha \int_0^\infty e^{-px} f(x) dx + \beta \int_0^\infty e^{-px} g(x) dx \\
&= \alpha L[f(x)] + \beta L[g(x)]
\end{aligned}
$$

In particular,

$$L[f(x) + g(x)] = L[f(x)] + L[g(x)]$$

and

$$L[\alpha f(x)] = \alpha L[f(x)].$$

Note

Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Implications:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$L[\cos ax]$

$$
\begin{aligned}
L[\cos ax] &= L\left[\frac{1}{2}\left(e^{iax} + e^{-iax}\right)\right] \\
&= \frac{1}{2}L[e^{iax}] + \frac{1}{2}L[e^{-iax}] \\
&= \frac{1}{2}\left(\frac{1}{p-ia}\right) + \frac{1}{2}\left(\frac{1}{p+ia}\right) \qquad \text{for } p > 0 \\
&= \frac{p}{p^2+a^2}
\end{aligned}
$$

$L[\sin ax]$

$$\begin{aligned}
L[\sin ax] &= L\left[\frac{1}{2i}\left(e^{iax} - e^{-iax}\right)\right] \\
&= \frac{1}{2i}L[e^{iax}] - \frac{1}{2i}L[e^{-iax}] \\
&= \frac{1}{2i}\left(\frac{1}{p-ia}\right) - \frac{1}{2i}\left(\frac{1}{p+ia}\right) \qquad \text{for } p > 0 \\
&= \frac{a}{p^2+a^2}
\end{aligned}$$

Examples

Find the Laplace transformation of

(a) $f(x) = 2x^2 + 5e^{-2x} - 3\sin(4x)$

(b) $g(x) = \sinh(bx)$

(c) $h(x) = \sin^2 ax$

Solution: (a)

$$\begin{aligned} L\left[2x^2 + 5e^{-2x} - 3\sin(4x)\right] &= 2L[x^2] + 5L[e^{-2x}] - 3L[\sin 4x] \\ &= 2\frac{2!}{p^3} + 5\frac{1}{p+2} - 3\frac{4}{p^2+4^2} \\ &= \frac{4}{p^3} + \frac{5}{p+2} - \frac{12}{p^2+16} \end{aligned}$$

(b)

$$
\begin{aligned}
L[\sinh(bx)] &= L\left[\frac{1}{2}\left(e^{bx} - e^{-bx}\right)\right] \\
&= \frac{1}{2}L[e^{bx}] - \frac{1}{2}L[e^{-bx}] \\
&= \frac{1}{2}\left(\frac{1}{p-b} - \frac{1}{p+b}\right) \qquad \text{provided } p > |a| \\
&= \frac{b}{p^2 - b^2}
\end{aligned}
$$

(c)

$$
\begin{aligned}
L[\sin^2 ax] &= L\left[\frac{1}{2}(1-\cos 2ax)\right] \\
&= \frac{1}{2}L[1] - \frac{1}{2}L[\cos 2ax] \\
&= \frac{1}{2p} - \frac{1}{2}\left(\frac{p}{p^2+(2a)^2}\right) \\
&= \frac{2a^2}{p(p^2+4a^2)}
\end{aligned}
$$

Examples

Find a function $f(x)$ whose Laplace transformation is

(a) $\dfrac{30}{p^4}$

(b) $\dfrac{1}{p^2+p}$

(c) $\dfrac{4}{p^3}+\dfrac{6}{p^2+4}$

Solution:

(a)

$$
\begin{aligned}
\frac{30}{p^4} &= 5 \cdot \frac{3!}{p^{3+1}} \\
&= 5 \cdot L[x^3] \\
&= L[5x^3]
\end{aligned}
$$

(b)

$$\begin{aligned}
\frac{1}{p^2+p} &= \frac{1}{p(p+1)} \\
&= \frac{1}{p} - \frac{1}{p+1} \\
&= L[1] - L[e^{-x}] \\
&= L[1 - e^{-x}]
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{4}{p^3} + \frac{6}{p^2+4} &= 2 \cdot \frac{2!}{p^{2+1}} + 3 \cdot \frac{2}{p^2+2^2} \\
&= 2L[x^2] + 3L[\sin 2x] \\
&= L[2x^2 + 3\sin 2x]
\end{aligned}$$

Existence of Laplace Transform: Sufficient Conditions

Theorem

The Laplace transform of a function $f(x)$ exists if it satisfies the the following two conditions:

(a) *$f(x)$ is* **piecewise continuous** *on the interval $0 \leq x \leq A$ for any positive A.*

(b) *$f(x)$ is of* **exponential order**, *i.e., there exist real constants $M \geq 0$, $K > 0$, and c, such that $|f(x)| \leq Ke^{cx}$, where $x \geq M$.*

Remark : However, the conditions of the theorem are not *necessary*. For example, $f(x) = x^{-1/2}$ has infinite discontinuity at $x = 0$. So this function fails to satisfy the first condition of the theorem but its Laplace transform exists. Indeed

$$L[x^{-1/2}] = \sqrt{\pi/p}.$$

(Prove!)

Piecewise Continuous Functions

Definition

A function $f(x)$ is said to be **piecewise continuous** on an interval $a \leq x \leq b$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \ldots < x_n = b$ so that

1. f is continuous on each subinterval $x_{i-1} < x < x_i$ and
2. f approaches a finite limit as the endpoints are approached from within the subinterval.

Example : Every continuous function is piecewise continuous.

Example : The following is a piecewise continuous function:

$$f(x) = \begin{cases} e^{2x}, & 0 \leq x \leq 1; \\ 4, & x > 1. \end{cases}$$

Functions of Exponential Order

Definition

A function $f(x)$ is of **Exponential Order** if there exist real constants $M \geq 0$, $K > 0$ and c such that

$$|f(x)| \leq Ke^{cx} \qquad \text{when } x \geq M.$$

Examples :

- $f(x) = \cos ax$ is of exponential order. (Take $M = 0$, $K = 1$, and $c = 0$.)
- $f(x) = x^2$ is of exponential order. (Take $c = 1$, $K = 1$, and $M = 4$.)
- $f(x) = e^{x^2}$ is not of exponential order.

Homework: Prove that $x^{-1/2}$ is of exponential order.

A necessary condition for $F(p)$ to be a Laplace Transform

Theorem

If $L[f(x)] = F(p)$, then $F(p) \to 0$ as $p \to \infty$

Proof:

$$
\begin{aligned}
|F(p)| &= \left| \int_0^\infty e^{-px} f(x) dx \right| \\
&\leq \int_0^\infty \left| e^{-px} f(x) \right| dx \\
&\leq \int_0^\infty e^{-px} \ K \ e^{cx} dx, \quad \text{as } f(x) \text{ is of exponential order} \\
&\leq K \int_0^\infty e^{-(p-c)x} dx, \\
&= \frac{K}{p-c} \qquad \text{for} p > c.
\end{aligned}
$$

So, $F(p) \to 0$ as $p \to \infty$.

Note

Thus polynomials in p, $\sin p$, $\cos p$, e^p and $\log p$ *cannot* be Laplace transforms of any functions. On the other hand, a rational function might be a Laplace transform of some function if the degree of the numerator is less than that of the denominator.

Shifiting Formula

Let

$$L[f(x)] = \int_0^\infty e^{-px} f(x)dx = F(p).$$

Then

$$L[e^{ax} f(x)] = \int_0^\infty e^{-px} e^{ax} f(x)dx \quad = \int_0^\infty e^{-(p-a)x} f(x)dx = F(p-a)$$

Examples

Find

(a) $L[x^5 e^{-2x}]$

(b) $L[e^{3x} \cos 2x]$

Solution : (a)

$$L[x^5] = \frac{5!}{p^6} = F(p). \qquad \therefore L[e^{-2x}x^5] = F(p+2) = \frac{5!}{(p+2)^6}$$

(b)

$$L[\cos 2x] = \frac{p}{p^2+4} = F(p). \qquad \therefore L[e^{3x}\cos 2x] = F(p-3) = \frac{p-3}{(p-3)^2+4} = \frac{p-3}{p^2-6p+13}.$$

Laplace Transform of Derivatives

Theorem

If a function $y(x)$ is differentiable, then

$$L[y'] = pL[y] - y(0)$$

and

$$L[y''] = p^2 L[y] - py(0) - y'(0).$$

Proof:

$$\begin{aligned}
L[y'] &= \int_0^\infty e^{-px} y' \, dx \\
&= ye^{-px}\Big|_0^\infty + p\int_0^\infty e^{-px} y dx \\
&= -y(0) + pL[y] \qquad \left(\lim_{x\to\infty} \frac{y}{e^{px}} = 0\right) \\
&= pL[y] - y(0).
\end{aligned}$$

We have proved that

$$L[y'] = pL[y] - y(0).$$

Taking y' in the place of y in this formula, we obtain

$$\begin{aligned} L(y'') &= pL[y'] - y'(0) \\ &= p[pL[y] - y(0)] - y'(0) \\ &= p^2 L[y] - py(0) - y'(0). \end{aligned}$$

Example

Find the solution of $y'' + 4y = 4x, \quad y(0) = 1, \quad y'(0) = 5.$
Solution: When L is applied to both sides of the equation, we get

$$L[y''] + 4L[y] = 4L[x].$$

$$L[y'] = pL[y] - y(0) = pL[y] - 1.$$

$$L[y''] = p^2 L[y] - py(0) - y'(0) = p^2 L[y] - p - 5.$$

Hence

$$\begin{aligned} L[y''] + 4L[y] = 4L[x] \quad &\Rightarrow \quad p^2 L[y] - p - 5 + 4L[y] = \frac{4}{p^2} \\ &\Rightarrow \quad (p^2 + 4)L(y) = p + 5 + \frac{4}{p^2} \\ &\Rightarrow \quad L[y] = \frac{p}{p^2 + 4} + \frac{5}{p^2 + 4} + \frac{4}{p^2(p^2 + 4)} \end{aligned}$$

We have

$$
\begin{aligned}
L[y] &= \frac{p}{p^2+4} + \frac{5}{p^2+4} + \frac{4}{p^2(p^2+4)} \\
&= \frac{p}{p^2+4} + \frac{5}{p^2+4} + \frac{1}{p^2} - \frac{1}{p^2+4} \\
&= \frac{p}{p^2+4} + \frac{4}{p^2+4} + \frac{1}{p^2} \\
&= \frac{p}{p^2+2^2} + 2\frac{2}{p^2+2^2} + \frac{1}{p^2} \\
&= L[\cos 2x] + 2L[\sin 2x] + L[x] \\
&= L[\cos 2x + 2\sin 2x + x]
\end{aligned}
$$

Thus the solution is

$$y = \cos 2x + 2\sin 2x + x.$$

Laplace Transform of Integrals

Theorem

Let $L[f(x)] = F[p]$. Then

$$L\left[\int_0^x f(t)dt\right] = \frac{F(p)}{p}.$$

Proof:

- Let $y(x) = \int_0^x f(t)dt$. Then $y(0) = 0$ and $y'(x) = f(x)$.
- Thus $L[y'] = F(p)$. This means that $pL[y] - y(0) = F(p)$.
- Hence $pL[y] = F(p)$ as $y(0) = 0$. Or $L[y] = \dfrac{F(p)}{p}$.
- Hence

$$L\left[\int_0^x f(t)dt\right] = \frac{F(p)}{p}.$$

Laplace Transform of $xf(x)$

Theorem

Let $L[f(x)] = F(p)$. *Then*

$$L[-xf(x)] = F'(p).$$

More generally,

$$L[(-x)^n f(x)] = F^{(n)}(p).$$

Proof: We have

$$F(p) = \int_0^\infty e^{-px} f(x)dx.$$

Differentiating both sides w.r.t p, we get

$$F'(p) = \int_0^\infty e^{-px}(-x)f(x)dx = L[-xf(x)].$$

Differentiating both sides n times w.r.t p, we get

$$F^{(n)}(p) = L[(-x)^n f(x)].$$

Example

Compute $L[xe^{-x}]$, $L[x^2e^{-x}]$ and $L[x^ke^{-x}]$

Solution : Let $f(x) = e^{-x}$. Then $F(p) = L[f(x)] = \frac{1}{p+1}$.

Hence

$$L[xe^{-x}] = -F'(p) = -\frac{-1}{(p+1)^2} = \frac{1}{(p+1)^2}.$$

$$L[x^2e^{-x}] = F''(p) = \frac{2}{(p+1)^3}.$$

$$L[x^ke^{-x}] = (-1)^kF^{(k)}(p) = (-1)^k\frac{((-1)^k)k!}{(p+1)^{k+1}} = \frac{k!}{(p+1)^{k+1}}.$$

Laplace Transform of $\dfrac{f(x)}{x}$

Theorem

Let $L[f(x)] = F(p)$. *Then*

$$L\left[\frac{f(x)}{x}\right] = \int_p^{\infty} F(p)dp.$$

Proof: *Homework.*

Example

Find the Laplace transform of $\dfrac{\sin ax}{x}$.

Solution:

$$L\left[\frac{\sin ax}{x}\right] = \int_p^{\infty} \frac{a}{p^2 + a^2}\, dp = \left[\tan^{-1}\left(\frac{p}{a}\right)\right]_p^{\infty} = \frac{\pi}{2} - \tan^{-1}\left(\frac{p}{a}\right).$$

Note

Theorem: If $L[f(x)] = F(p)$, then

$$L\left[\frac{f(x)}{x}\right] = \int_p^\infty F(p)dp.$$

This means:

$$\int_0^\infty e^{-px}\frac{f(x)}{x}dx = \int_p^\infty F(p)dp.$$

Letting $p \to 0$ on both sides, we have

$$\int_0^\infty \frac{f(x)}{x}dx = \int_0^\infty F(p)dp.$$

This identity is useful in computing some integrals that are otherwise impossible.

Example

Find $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

Solution: We know that

$$\int_0^\infty \frac{f(x)}{x}dx = \int_0^\infty F(p)dp,$$

where $F(p) = L[f(x)]$.
Hence

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^\infty \frac{1}{p^2+1}dp = \left[\tan^{-1} p\right]_0^\infty = \pi/2.$$

Convolution of Sequences

Let

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

and

$$b_0, b_1, b_2, b_3, \ldots, b_n, \ldots$$

be two sequences of numbers. Then their **convolution** is the sequence

$$c_0, c_1, c_2, c_3, \ldots, c_n, \ldots,$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots a_n b_0.$$

What is its continuous analog?

Note:

$$c_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \ldots a_0 b_n = b_0 a_n + b_1 a_{n-1} + b_2 a_{n-2} + \ldots b_n a_0.$$

This implies that convolution of sequences is a *commutative* operation.

Convolution of Functions

Definition

Let $f, g : [0, \infty) \to \mathbb{R}$ be two functions. Then their **convolution** is a function $f * g : [0, \infty) \to \mathbb{R}$ given by

$$(f * g)(x) = \int_0^x f(t)g(x-t)dt.$$

Note: Convolution of functions is a *commutative* operation: $f * g = g * f$:

$$\begin{aligned}
(f * g)(x) &= \int_{t=0}^{x} f(t)g(x-t)dt \\
&= \int_{s=x}^{0} f(x-s)g(s)(-1)ds \qquad \text{if we let } s = x - t \\
&= \int_{s=0}^{x} g(s)f(x-s)ds = (g * f)(x)
\end{aligned}$$

Examples

1. Let $f(x) = x$ and g(x)= 1. Then

$$(f * g)(x) = \int_0^x f(t)g(x-t)dt = \int_0^x t dt = \frac{x^2}{2}.$$

2. Let $f(x) = e^x$ and $g(x) = e^{-x}$. Then

$$\begin{aligned}
(f * g)(x) &= \int_0^x f(t)g(x-t)dt \\
&= \int_0^x e^t e^{-(x-t)} dt \\
&= e^{-x} \int_0^x e^{2t} dt \\
&= \frac{e^{-x}}{2}\left(e^{2x} - 1\right) \\
&= \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x.
\end{aligned}$$

Laplace Transform of Convolutions: $L[(f * g)(x)]$

Recall:

$$\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots a_n b_0.$$

Any Guess on $L[(f * g)(x)]$?

Hint: Recall that Laplace transforms are continuous analogs of power series!

Theorem (The Convolution Theorem)

$$L[(f * g)(x)] = L[f(x)] \times L[g(x)].$$

Proof:

$$\begin{aligned} L[f(x)] \times L[g(x)] &= \left[\int_{s=0}^{\infty} e^{-ps} f(s) ds\right] \times \left[\int_{t=0}^{\infty} e^{-pt} g(t) dt\right] \\ &= \int_{t=0}^{\infty} \int_{s=0}^{\infty} e^{-p(s+t)} f(s) g(t) dsdt \\ &= \int_{t=0}^{\infty} \left[\int_{s=0}^{\infty} e^{-p(s+t)} f(s) ds\right] g(t) dt \end{aligned}$$

Let $x = s + t$. Then $s = x - t$. Also t is fixed during the inner integration. Thus $ds = dx$.

Let $x = s + t$. Then $s = x - t$. Also t is fixed during the inner integration. Thus $ds = dx$. So, we have

$$\begin{aligned} L[f(x)] \times L[g(x)] &= \int_{t=0}^{\infty} \left[\int_{s=0}^{\infty} e^{-p(s+t)} f(s) ds \right] g(t) dt \\ &= \int_{t=0}^{\infty} \left[\int_{x=t}^{\infty} e^{-px} f(x-t) dx \right] g(t) dt \\ &= \int_{t=0}^{\infty} \int_{x=t}^{\infty} e^{-px} f(x-t) g(t) dx dt \end{aligned}$$

$$0 \leq t < \infty, t \leq x < \infty \iff 0 \leq x < \infty, 0 \leq t \leq x$$

(See G.F. Simmons, page 469)

$$
\begin{aligned}
L[f(x)] \times L[g(x)] &= \int_{t=0}^{\infty} \int_{x=t}^{\infty} e^{-px} f(x-t) g(t) dx dt \\
&= \int_{x=0}^{\infty} \int_{t=0}^{x} e^{-px} f(x-t) g(t) dt dx \\
&= \int_{x=0}^{\infty} e^{-px} \left[\int_{t=0}^{x} f(x-t) g(t) dt \right] dx \\
&= \int_{x=0}^{\infty} e^{-px} (f * g)(x) dx \\
&= L\left[(f * g)(x) \right].
\end{aligned}
$$

Hence the theorem is proved.

Example

Using the convolution theorem, find $L^{-1}\left\{\dfrac{1}{p^2(p+1)^2}\right\}$.

Solution: We know that $L[x] = \left\{\dfrac{1}{p^2}\right\}$ and $L[xe^{-x}] = \left\{\dfrac{1}{(p+1)^2}\right\}$.

Hence

$$\frac{1}{p^2(p+1)^2} = L[x] \times L[xe^{-x}] = L\left[(x) * (xe^{-x})\right] = L\left[(xe^{-x}) * (x)\right] = L\left[\int_0^x (te^{-t})(x-t)dt\right]$$

$$= L\left[xe^{-x} + 2e^{-x} + x - 2\right].$$

Example

Using the convolution theorem, find $L^{-1}\left\{\dfrac{p+1}{(p^2+1)^2}\right\}$.

Solution: $\dfrac{p+1}{(p^2+1)} = L[\cos x + \sin x]$ and $\dfrac{1}{(p^2+1)} = L[\sin x]$.
Hence

$$\frac{p+1}{(p^2+1)^2} = L[\cos x + \sin x] \times L[\sin x] = L[(\cos x + \sin x) * (\sin x)].$$

But

$$\begin{aligned}(\cos x + \sin x) * (\sin x) &= \int_0^x \cos(x-t)\sin t dt + \int_0^x \sin(x-t)\sin t dt \\ &= \frac{1}{2}x(\sin x - \cos x) - \frac{1}{2}\sin x.\end{aligned}$$

Solution of Integral Equations

Solve the integral equation

$$y(x) = x^2 + \int_0^x y(t)\sin(x-t)dt.$$

Solution : The given integral equation can be written as

$$y(x) = x^2 + y(x) * \sin x.$$

Applying the Laplace transformation on both sides, we have

$$L[y(x)] = L[x^2 + y(x) * \sin x] = L[x^2] + L[y(x) * \sin x] = L[x^2] + L[y(x)] \times L[\sin x].$$

Thus we have

$$L[y(x)] = L[x^2] + L[y(x)] \times L[\sin x] \qquad \text{or} \qquad L[y(x)] = \frac{2}{p^3} + L[y(x)] \times \frac{1}{p^2+1}.$$

Hence

$$L[y(x)]\left(1-\frac{1}{p^2+1}\right)=\frac{2}{p^3} \qquad \text{or} \qquad L[y(x)]=\frac{2}{p^3}\times\frac{p^2+1}{p^2}=\frac{2}{p^3}+\frac{2}{p^5}.$$

So we have

$$L[y(x)]=\frac{2}{p^3}+\frac{2}{p^5}=L[x^2]+\frac{1}{12}L[x^4]=L\left[x^2+\frac{1}{12}x^4\right].$$

Thus the solution is

$$y(x)=x^2+\frac{1}{12}x^4.$$

Homeworks

1. Solve the intial value problem $y'(x) + 5\int_0^x \cos 2(x-t)\, y(t)\, dt = 10, \quad y(0) = 2.$
2. Solve the intial value problem $y'(x) + y(x) - 2\int_0^x y(t)\, dt = x, \quad y(0) = 0.$

Definition (Fourier Series)

Let $a_0, a_1, b_1, a_2, b_2, \ldots a_n, b_n, \ldots$ be any sequence of real numbers. Then the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called a Fourier Series.

Applications: Solving important partial differential equations arising in the theory of sound, heat conduction, electromagnetic waves, and mechanical vibrations.

It is More Powerful than Power Series: Can represent very general functions with many discontinuities, like the impulse function.

Fourier Series: Its Relation to Power Series

Recall: A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots.$$

A Generalization of Power Series:

$$\sum_{n=-\infty}^{\infty} a_n x^n = \ldots + a_{-n} x^{-n} + \ldots + a_{-2} x^{-2} + a_{-1} x^{-1} + a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots.$$

A Further Generalization to Complex Numbers:

$$\sum_{n=-\infty}^{\infty} c_n z^n = \ldots + c_{-n} z^{-n} + \ldots + c_{-2} z^{-2} + c_{-1} z^{-1} + c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots.$$

A Further Generalization to Complex Numbers:

$$\sum_{n=-\infty}^{\infty} c_n z^n = \ldots + c_{-n} z^{-n} + \ldots + c_{-2} z^{-2} + c_{-1} z^{-1} + c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots.$$

Taking $z = e^{ix}$, we obtain

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = \ldots + c_{-n} e^{-inx} + \ldots + c_{-2} e^{-i2x} + c_{-1} e^{-ix} + c_0 + c_1 e^{ix} + c_2 e^{i2x} + \ldots + c_n e^{inx} + \ldots.$$

This is **Fourier Series**! Why? (**Our Fourier series is only a *special case* of this series!**)

A Bit of Complex Analysis

Let a and b be any real numbers. Consider the linear combination

$$a \cos x + b \sin x.$$

There is a unique pair of complex numbers c and d such that the linear combination

$$ce^{ix} + de^{-ix} = a \cos x + b \sin x.$$

(Prove!)

Let n be any integer. For any pair of real numbers a and b, there is a unique choice of complex numbers c and d such that

$$ce^{inx} + de^{-inx} = a \cos nx + b \sin nx.$$

Consider a Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $a_0, a_1, b_1, a_2, b_2, \ldots a_n, b_n, \ldots$ is a sequence of real numbers.

Then, for each $n \geq 1$, we can find complex numbers c_n and c_{-n} such that

$$a_n \cos nx + b_n \sin nx = c_n e^{inx} + c_{-n} e^{-inx}.$$

Hence we have

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} c_n \left(e^{ix}\right)^n.$$