Rolle's Theorem

If f be a real valued function satisfying:

1. f is continuous on closed interval $[a, b]$
2. f is differentiable on open interval (a, b)
3. $f(a) = f(b)$

then there exists a point $c \in (a, b)$ such that $f'(c) = 0$

Problem-1

It is given that the Rolle's Theorem holds for the function

$$f(x) = x^3 + bx^2 + cx,\ 1 < x < 2 \text{ at the point } x = \frac{4}{3}$$

Find the vaules of b and c.

Solution: Let $a = 1$ and $b = 2$

- If Rolle's Theorem holds, $f(a) = f(b)$ i.e. $f(1) = f(2)$

$$1 + b + c = 8 + 4b + 2c \Rightarrow \boxed{\text{3b+c = -7}} \tag{1}$$

- Since $x = \frac{4}{3} \in (1, 2)$, we obtain using the Rolle's theorem

$$f'\left(\frac{4}{3}\right) = 0 \tag{2}$$

- The derivative $f'(x) = 3x^2 + 2bx + c$. Using the above leads us

$$3\left(\frac{4}{3}\right)^2 + 2b\left(\frac{4}{3}\right) + c = 0 \Rightarrow \boxed{\text{8b+3c = -16}} \tag{3}$$

Problem-1

- Solve (1) and (3) to get the required constants,
- Subtracting (3) from 3 times of (1)

$$9b - 8b = -21 + 16 \Rightarrow \boxed{\text{b = -5}} \tag{4}$$

- Substituting b into (1),

$$\boxed{\text{c} = 8}$$

Cauchy-MVT

If f and g be two real valued functions satisfying:

1. f, g are continuous on closed interval $[a, b]$
2. f, g are differentiable on open interval (a, b)
3. $g'(x) \neq 0$ for all $x \in (a, b)$

then there exists a point $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$

Problem-2

The functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on (a, b) such that

$$f(a) = 4, f(b) = 10, g(a) = 1 \ \& \ g(b) = 3$$

Then show that $f'(c) = 3g'(c)$ where $c \in (a, b)$.

Solution:

- Using the Cauchy-MVT, we obtain

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{10 - 4}{3 - 1} = \frac{f'(c)}{g'(c)}$$

- Direct calculations, we derive

$$\boxed{f'(c) = 3\, g'(c)}$$

Problem-3

Prove the between any two real roots of $e^x \sin(x) = 1$ there exists atleast one root of $e^x \cos(x) + 1 = 0$

Proof. Given $e^x \sin(x) = 1$

- Let $f(x) = e^x \sin(x) - 1$
- Derivetive: $f'(x) = e^x \cos(x) + e^x \sin(x) = \boxed{e^x \cos(x) + 1}$
- Suppose a and b $(a < b)$ are two roots of $f(x)$, then

$$f(a) = 0 = f(b) \Rightarrow \boxed{\text{f(a) = f(b)}}$$

- Also, it is clear that $f(x)$ is $\boxed{\text{continuous}}$ on $[a, b]$ and $\boxed{\text{differentiable}}$ on (a, b)
- By Rolle's theorem, there exists a point $c \in (a, b)$ such that $\boxed{\text{f'(c) = 0}}$
- which implies $\boxed{e^c \cos(c) + 1 = 0}$
- Thus we can say that there exists atleast one root c of $e^x \cos(x) + 1 = 0$

LMVT

If f be a real valued function satisfying:

1. f is continuous on closed interval $[a, b]$
2. f is differentiable on open interval (a, b)

then there exists a point $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$

Problem-4

Using LMVT, show that

$$x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}, \quad 0 < x < 1$$

Solution: Let $f(t) = \sin^{-1} t$ and derivative $f'(t) = \dfrac{1}{\sqrt{1-t^2}}$

- To use LMVT, let $a = 0$ and $b = x$
- there exists $c \in (a, b) = (0, x)$ i.e. $0 < c < x$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \Rightarrow \frac{f(x) - f(0)}{x - 0} = f'(c)$$

- since $f(0) = \sin^{-1} 0 = 0$,

$$\boxed{\mathrm{f(x) = x\, f'(c)}} = \frac{x}{\sqrt{1-c^2}}, \text{ where } 0 < c < x \tag{5}$$

Problem-4

We have

$$
\begin{aligned}
&0 < c < x && \text{from the last result}\\
&0^2 < c^2 < x^2 && \text{squaring}\\
&0 > -c^2 > -x^2 && \text{multiply by -1}\\
&1 + 0 > 1 - c^2 > 1 - x^2 && \text{adding 1}\\
&\sqrt{1} > \sqrt{1 - c^2} > \sqrt{1 - x^2} && \text{taking square root}\\
&\frac{1}{1} < \frac{1}{\sqrt{1 - c^2}} < \frac{1}{\sqrt{1 - x^2}} && \text{taking reciprocal}\\
&x < \frac{x}{\sqrt{1 - c^2}} < \frac{x}{\sqrt{1 - x^2}} && \text{multiplying by x}\\
&x < xf'(c) < \frac{x}{\sqrt{1 - x^2}} && \text{using (5)}\\
&x < f(x) < \frac{x}{\sqrt{1 - x^2}} && \text{using (5)}\\
&x < \sin^{-1}(x) < \frac{x}{\sqrt{1 - x^2}} && \text{by assumption}
\end{aligned}
$$

Problem-5

Suppose that $f(x)$ is differentiable for all values of x such that

$$f(a) = a,\ f(-a) = -a \text{ and } |f'(x)| \leq 1 \text{ for all } x.$$

Show that $f(x) = x$, in particular for $x = 0$, we have $f(0) = 0$.

Proof. Consider $g(x) = f(x) - x$

- It gives $g(-a) = f(-a) - (-a) = 0$ and $g(a) = f(a) - a = 0$
- which also implies $\boxed{g(-a) = g(a)}$
- By Rolle's theorem, there $x \in (-a, a)$ such that

$$g'(x) = 0 \Rightarrow f'(x) - 1 = 0 \Rightarrow \boxed{f'(x) = 1}$$

- Integrating, $f(x) = x + c$, $c \leftarrow$ integrating constant
- Putting $x = -a$, we can find c by using $f(-a) = -a + c \Rightarrow -a = -a + c = 0 \Rightarrow \boxed{c = 0}$
- Hence $\boxed{f(x) = x}$
- For particular $x = 0$, we have $\boxed{f(0) = 0}$.

Second Derivative Test

Let x_0 satisfies $\boxed{f'(x_0) = 0}$ and let $f(x)$ be differentiable at $x = x_0$, where $a \leq x_0 \leq b$. Then

- $f(x)$ has $\boxed{\text{maximum}}$ at $x = x_0$ if $\boxed{f''(x_0) < 0}$, and
- $f(x)$ has $\boxed{\text{minimum}}$ at $x = x_0$ if $\boxed{f''(x_0) > 0}$

Problem-6

Find the extreme values of the given function $f(x) = \sin(x)^{\sin(x)}$

Solution:

Step-1. Find x_0 such that $\boxed{f'(x_0) = 0}$

$$\log(f(x)) = \sin(x)\log(\sin(x))$$

- Differentiating w.r.t 'x':

$$\frac{1}{f(x)}f'(x) = \cos(x)\log(\sin(x)) + \sin(x)\frac{1}{\sin(x)}\cos(x)$$
$$= \cos(x)\big(\log(\sin(x)) + 1\big)$$
$$f'(x) = f(x)\cos(x)\big(\log(\sin(x)) + 1\big)$$

- Now $\boxed{f'(x) = 0}$ gives

$$0 = f(x)\cos(x)\big(\log(\sin(x)) + 1\big)$$

- $\boxed{f(x) = 0}$ is not possible else there won't be anything to show extreme value of $f(x)$

Problem-6

- So,

$$0 = \cos(x)\big(\log(\sin(x)) + 1\big)$$
$$\Rightarrow \cos(x) = 0 \text{ or } \log(\sin(x)) + 1 = 0$$

- If $\cos(x) = 0$, then $\boxed{x = (2n+1)\frac{\pi}{2}}$, where n is integer
- If $\log(\sin(x)) + 1 = 0$, then $x \Rightarrow \log(\sin(x)) = -1 \Rightarrow \boxed{x = \sin^{-1}(\frac{1}{e})}$, where n is integer
- Finally, we have $x_0 = (2n+1)\frac{\pi}{2}$ and $x_0 = \sin^{-1}(\frac{1}{e})$

Step-2. Need to check the sign of $f''(x_0)$ for both x_0

- We find f''

$$f''(x) = f'(x)\cos(x)\big(\log(\sin(x)) + 1\big) + f(x)\Big[-\sin(x)\big(\log(\sin(x)) + 1\big) + \cos(x)\frac{1}{\sin(x)}\cos(x)\Big]$$

- Substituting $f(x)$ and $f'(x)$

$$f''(x) = \sin(x)^{\sin(x)}\big[\cos(x)\big(\log(\sin(x)) + 1\big)\big]^2 + \sin(x)^{\sin(x)}\Big[-\sin(x)(\log(\sin(x)) + 1) + \frac{\cos^2(x)}{\sin(x)}\Big]$$

- For $x_0 = (2n+1)\frac{\pi}{2}$ i.e $\cos(x_0) = 0$

$$f''(x_0) = 0 + (1)[(-1) + 0] = -1 < 0$$

Hence $x_0 = (2n+1)\frac{\pi}{2}$ has $\boxed{\text{maximum value}}$

- $f_{max} = 1$, since $\sin(x_0) = 1$
- For $x_0 = \sin^{-1}(\frac{1}{e})$ i.e $\log(\sin(x_0)) + 1 = 0$

$$f''(x_0) = 0 + \left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)} \frac{e^2 - 1}{e^2} e = \left(e - \frac{1}{e}\right)\left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)} > 0$$

Hence $x_0 = \sin^{-1}(\frac{1}{e})$ has $\boxed{\text{minimum}}$ value

- $f_{min} = (\frac{1}{e})^{\left(\frac{1}{e}\right)}$, since $\sin(x_0) = \frac{1}{e}$

Problem-7

Find the extreme values of the given function

$$f(x) = \sin^2(x)\sin(2x) + \cos^2(x)\cos(2x), \text{ where } 0 < x < \pi$$

Solution: Find first derivative to get the critical points:

$$\begin{aligned} f'(x) &= 2\sin(x)\cos(x)\sin(2x) + 2\sin^2(x)\cos(2x) - 2\cos(x)\sin(x)\cos(2x) \\ &\quad - 2\cos^2(x)\sin(2x) \\ &= 2\big(\sin(x) - \cos(x)\big)\big(\cos(x)\sin(2x) + \sin(x)\cos(2x)\big) \end{aligned}$$

Therefore $f'(x) = 0$ gives

$$\sin(x) - \cos(x) = 0 \text{ and } \cos(x)\sin(2x) + \sin(x)\cos(2x) = 0$$

- If $\sin(x) - \cos(x) = 0$, $\tan(x) = 1 \Rightarrow \boxed{x = \frac{\pi}{4}}$ on $0 < x < \pi$

Problem-7

- If $\cos(x)\sin(2x)+\sin(x)\cos(2x)=0$, then $\tan(x)=\tan(-2x)$ which gives $x=n\pi+(-2x)$, $n=0,\pm1,\pm2,\cdots$
- $3x=n\pi \Rightarrow x=\frac{n\pi}{3}$
- In the range $0<x<\pi$: put $n=1,2$, we have $\boxed{x=\frac{\pi}{3},\frac{2\pi}{3}}$
- Finally, $\boxed{x_0=\frac{\pi}{4},\frac{\pi}{3},\frac{2\pi}{3}}$

Now, find $f''(x)$:

$$f''(x)=2(\sin(x)-\cos(x))\left(3\cos(x)\cos(2x)-3\sin(x)\sin(2x)\right)$$
$$+2(\sin(x)+\cos(x))(\cos(x)\sin(2x)+\sin(x)\cos(2x))$$

- At $x_0=\frac{\pi}{4}$ i.e $\sin(x_0)-\cos(x_0)=0$

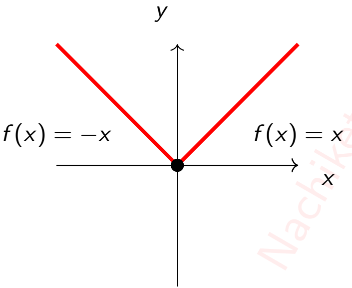
$$f''(x_0)=0+2\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\cdot1+\frac{1}{\sqrt{2}}\cdot0\right)=2>0$$

So, $\boxed{\text{minimum}}$ and its value is $\boxed{\frac{1}{2}}$.

- Similarly, max at $x_0=\frac{\pi}{3}$ with $\frac{3\sqrt{3}-1}{8}$; min at $x_0=\frac{2\pi}{3}$ with $-\frac{3\sqrt{3}+1}{8}$

If a function is continuous, then it is not necessarily differentiable

Example-1.



Given function $f(x) = |x|$

- Here at $x = 0$. $LHD = -1$ and $RHD = 1$, $LHD \neq RHS$, which implies $f(x)$ is not differentiable at $x = 0$
- But! Here at $x = 0$. $LHL = 0$ and $RHL = 0$, $LHL = RHL$, which implies $f(x)$ is continuous at $x = 0$

If a function is continuous, then it is not necessarily differentiable

Example-2.

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

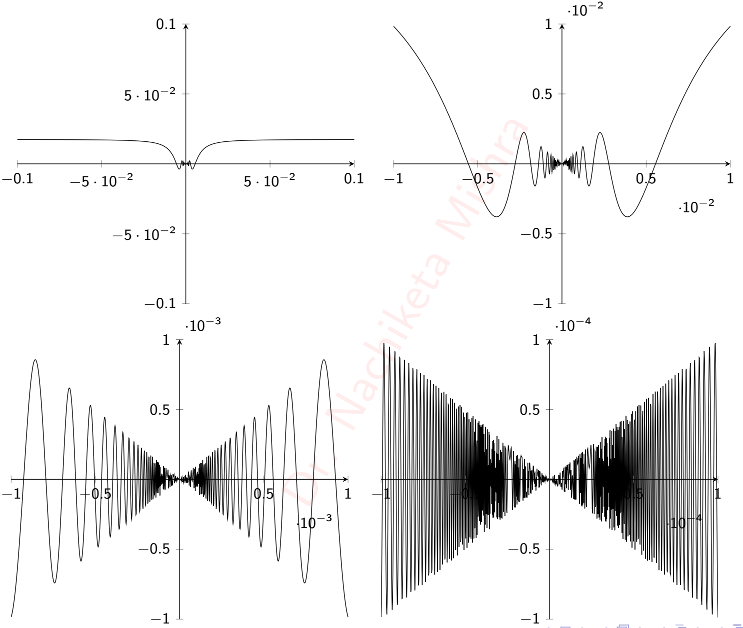
Solution.

$$LHL(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right) = 0; \quad \text{since } 0 < \sin\left(\frac{1}{x}\right) < 1$$

$$RHL(0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0; \quad \text{since } 0 < \sin\left(\frac{1}{x}\right) < 1$$

Since $LHL = RHL$ at $x = 0$, $f(x)$ is continuous at $x = 0$.

Example-2



0.1
5 · 10^−2
−5 · 10^−2
−0.1
−0.1
−5 · 10^−2
5 · 10^−2
0.1
·10^−2
1
0.5
−0.5
−1
−1
−0.5
0.5
1
·10^−2
·10^−3
1
0.5
−0.5
−1
−1
−0.5
0.5
1
·10^−3
·10^−4
1
0.5
−0.5
−1
−1
−0.5
0.5
1
·10^−4

Example-2

- We have $f(0) = 0$
- Now, we show that $f(x)$ is not differentiable at $x = 0$

$$RHD(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h \to 0^+} \sin\left(\frac{1}{h}\right) \leftarrow \text{does not exists}$$

- Which implies $f(x)$ is not differentiable at $x = 0$.
- But! it is continuous at $x = 0$

Home-Work

1. Show that the function $f(x) = \frac{ax+b}{cx+d}$ has no extreme value regardless of the values of a, b, c, d.
   Hint: If $f^{(n)} \neq 0$ but $f^{(i)} = 0, i = 1, 2, .., (n-1)$ with odd n, it has neither max. nor min.
2. Using the function
$$f(x) = \begin{cases} x^2 - \cos(\ln x)) - \sin(\ln x), & \text{if } x \in (0,1] \\ 0, & \text{if } x = 0 \end{cases}$$
   prove that converse of the following statement " $f' > 0$ *in* $(a, b) \Rightarrow f$ *strictly increasing in* (a, b)" is not true.
3. Show that the following function is differentiable at $x = 0$:
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$
4. Answer the following
   4.1 Use a graphing device to estimate the absolute minimum and maximum values of the function $f(x) = x - 2\sin(x)$ on $[0, 2\pi]$.
   4.2 Use calculus to find the exact minimum and maximum values.