Vector Spaces

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Mathematical structures in linear algebra

(1) **Field** (See Chapter 1)

(2) **Vector Space**

(3) **...........**

A vector space V over a field F

A vector space $\langle\ V,\ F,\ +,\ .\ \rangle$ consists of the following:

(1) a field F of scalars;

(2) a set V of objects, called vectors;

(3) an operation $+ : V \times V \longrightarrow V$, called vector addition, which satisfies the following axioms:

   (a) addition is commutative

$$\alpha + \beta = \beta + \alpha, \quad \text{for all } \alpha, \beta \in V;$$

   (b) addition is associative

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma, \quad \text{for all} \alpha, \beta, \gamma \in V;$$

(c) there is a unique vector $\mathbf{0} \in V$ called the zero vector such that

$$\alpha + \mathbf{0} = \alpha, \text{ for all } \alpha \in V;$$

(d) for each vector $\alpha \in V$, there is a unique vector $-\alpha \in V$ such that $\alpha + (-\alpha) = \mathbf{0}$;

(4) an operation $. \; : \; F \times V \longrightarrow V$, called scalar multiplication, which satisfies the following axioms.

(e) $1.\alpha = \alpha$, for all $\alpha \in V$;
(f) $(c_1 c_2)\alpha = c_1(c_2\alpha)$, for all $c_1, c_2 \in F, \alpha \in V$;
(g) $c(\alpha + \beta) = c\alpha + c\beta$, for all $\alpha, \beta \in V, c \in F$;
(h) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$, for all $c_1, c_2 \in F, \alpha \in V$.

Example 1 : The Euclidean space $\langle \mathbb{R}^n, \mathbb{R}, +, . \rangle$

$$V = \mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}\} \text{ and } F = \mathbb{R}.$$

Let $\alpha = (x_1, x_2, \dots, x_n)$, $\beta = (y_1, y_2, \dots, y_n) \in V$

Let us define vector addition and scalar multiplication as follows:

Define $+ \ : \ V \times V \longrightarrow V$ **as (vector addition)**

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

and $. \ : \ F \times V \longrightarrow V$ **as**

$$c\alpha = (cx_1, cx_2, \dots, cx_n)$$

Then $\langle \mathbb{R}^n, \mathbb{R}, +, . \rangle = \mathbb{R}^n$ is a vector space.

Verification

(a)

$$\begin{aligned}\alpha+\beta &= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)\\ &= (y_1+x_1, y_2+x_2, \ldots, y_n+x_n)\\ &= \beta+\alpha.\end{aligned}$$

(b) Let $\gamma = (z_1, z_2, \ldots, z_n)$. Then

$$\begin{aligned}\alpha+(\beta+\gamma) &= (x_1, x_2, \ldots, x_n) + (y_1+z_1, y_2+z_2, \ldots, y_n+z_n)\\ &= (x_1+y_1+z_1, x_2+y_2+z_2, \ldots, x_n+y_n+z_n)\\ &= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n) + (z_1, z_2, \ldots, z_n)\\ &= (\alpha+\beta)+\gamma.\end{aligned}$$

(c) Let $\mathbf{0} = (0, 0, \ldots, 0) \in V = \mathbb{R}^n$ such that
$\alpha + \mathbf{0} = (x_1, \ldots, x_n) + (0, \ldots, 0) = (x_1, \ldots, x_n) = \alpha$.

(d) For every $\alpha = (x_1, x_2, \ldots, x_n)$, there exists
$-\alpha = (-x_1, -x_2, \ldots, -x_n) \in V$ such that $\alpha + (-\alpha) = \mathbf{0}$.

(e) $1 \cdot \alpha = (1 \cdot x_1, 1 \cdot x_2, \ldots, 1 \cdot x_n) == (x_1, x_2, \ldots, x_n)\alpha$.

(f) For scalars c_1, c_2 and a vector $\alpha = (x_1, x_2, \ldots, x_n)$, we have

$$\begin{aligned}(c_1 c_2) \cdot \alpha &= (c_1 c_2 x_1, c_1 c_2 x_2, \ldots, c_1 c_2 x_n) \\ &= c_1 \cdot (c_2 x_1, c_2 x_2, \ldots, c_2 x_n) \\ &= c_1 \cdot (c_2 \cdot \alpha).\end{aligned}$$

contd.

(g) Let $\alpha = (x_1, x_2, \ldots, x_n)$ and $\beta = (y_1, y_2, \ldots, y_n)$, we have

$$\begin{aligned} c(\alpha + \beta) &= c(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ &= (c \cdot (x_1 + y_1), c \cdot (x_2 + y_2), \ldots, c \cdot (x_n + y_n)) \\ &= (cx_1 + cy_1, cx_2 + cy_2, \ldots, cx_n + cy_n)) \\ &= c\alpha + c\beta. \end{aligned}$$

(h) For scalars c_1, c_2 and a vector $\alpha = (x_1, x_2, \ldots, x_n)$, we have

$$\begin{aligned} (c_1 + c_2) \cdot \alpha &= ((c_1 + c_2)x_1, (c_1 + c_2)x_2, \ldots, (c_1 + c_2)x_n) \\ &= (c_1x_1 + c_2x_1, c_1x_2 + c_2x_2, \ldots, c_1x_n + c_2x_n) \\ &= (c_1x_1, c_1x_2, \ldots, c_1x_n) + (c_2x_1, c_2x_2, \ldots, c_2x_n) \\ &= c_1 \cdot \alpha + c_2 \cdot \alpha. \end{aligned}$$

Example 2: The n-tuple space $\langle F^n, F, +, .\rangle$

Let F be a field and let

$$V = F^n = \{(x_1, x_2, \ldots, x_n) \ : \ x_i \in F\}.$$

Let $\alpha = (x_1, x_2, \ldots, x_n), \beta = (y_1, y_2, \ldots, y_n) \in V = F^n$.

Define $+ \ : \ V \times V \longrightarrow V$ as (vector addition)

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

and $. \ : \ F \times V \longrightarrow V$ as

$$c\alpha = (cx_1, cx_2, \ldots, cx_n)$$

Show that $\langle F^n, F, +, .\rangle$ is a vector space.

Example 3 : The space of $m \times n$ matrices $\langle F^{m\times n}, F, +, .\rangle$

Let F be a field and

$$V = F^{m\times n} = \{A = [a_{ij}]_{m\times n} \quad : \quad a_{ij} \in F\}.$$

We define vector addition and scalar mulatiplication as follows, where $A = [a_{ij}], B = [b_{ij}] \in V$ and $c \in F$

$$[A + B]_{ij} = [a_{ij} + b_{ij}]$$

and

$$[cA]_{ij} = [ca_{ij}].$$

Show that $F^{m\times n}$ is a vector space over F

Note that $F^{n\times n}$ is not a field

Example 4: The set of all real valued continuous functions defined on $[0,1]$

Let $V = \{f : f : [0,1] \longrightarrow \mathbb{R}$ and f is continuous on [0,1]$\}$
We define

$$+ \; : \; V \times V \longrightarrow V$$

$$\text{as } (f+g)(s) = f(s) + g(s) \text{ for } s \in [0,1].$$

$$\cdot \; : \; \mathbb{R} \times V \longrightarrow V$$

$$\text{as } \; (cf)(s) = cf(s) \text{ for } s \in [0,1]$$

Show that $\langle V, \mathbb{R}, +, . \rangle$ is a a vector space.

Example 5: The space of polynomial functions over a field

Assignment

Problem 1

Let $V = \{(x, y) : \; x, y \in \mathbb{R}\}$. We define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$c(x, y) = (cx, y)$$

Prove or disprove that $\langle V, \mathbb{R}, +, .\rangle$ is a vector space.

Solution: Verify $(c_1 + c_2)\alpha \neq c_1\alpha + c_2\alpha$.

Note 1

Let V be a vector space over a field F. We have

$$\mathbf{0} = \mathbf{0} + \mathbf{0}, \qquad \text{(additive identity)}$$

$$c\mathbf{0} = c(\mathbf{0} + \mathbf{0}), \qquad c \in F$$

$$c\mathbf{0} = c\mathbf{0} + c\mathbf{0}, \qquad (c(\alpha + \beta) = c\alpha + c\beta)$$

Add $-(c\mathbf{0}) \in V$ on both sides

$$c\mathbf{0} + (-(c\mathbf{0})) = (c\mathbf{0} + c\mathbf{0}) + (-(c\mathbf{0})),$$

$$\mathbf{0} = c\mathbf{0} + (c\mathbf{0} + -(c\mathbf{0}))\,, \quad (\mathit{Associative})$$

Note 1 contd.

$$\mathbf{0} = c\mathbf{0} + \mathbf{0}, \quad (\textit{Existence of inverse})$$

$$\mathbf{0} = c\mathbf{0} \quad (\textit{additive identity})$$

$$c\mathbf{0} = \mathbf{0} \quad \text{for all} \quad c \in F.$$

Qn. Show that $0\alpha = \mathbf{0}$ for all $\alpha \in V$, where 0 is the additive identity in the field F and $\mathbf{0}$ is the zero vector in the vector space V.

Note 2

$$
\begin{aligned}
\mathbf{0} &= 0\alpha, \quad \text{(see last question)} \\
&= (1 + (-1))\,\alpha \\
&= 1.\alpha + (-1)\alpha \qquad (\text{ Reason: } \ (c_1 + c_2)\alpha = c_1\alpha + c_2\alpha) \\
&= \alpha + (-1)\alpha \qquad (\mathit{Reason}: \ 1.\alpha = \alpha)
\end{aligned}
$$

$\Longrightarrow$ additive inverse of α, $\quad -\alpha = (-1)\alpha.$

Note 3

Prove that if $c\alpha = \mathbf{0}$, then $c = 0$ or $\alpha = \mathbf{0}$.

Proof : Suppose that $c \neq 0$ (else $0\alpha = \mathbf{0}$).

Since $0 \neq c \in F$ and F is a field, $\quad c^{-1} \in F$.

$$c\alpha = \mathbf{0}$$

$$\implies c^{-1}(c\alpha) = c^{-1}\mathbf{0} = \mathbf{0}$$

$$\implies \quad (c^{-1}c)\alpha = \mathbf{0} \qquad \text{Reason: } (c_1c_2)\alpha = c_1(c_2\alpha)$$

$$\implies \quad 1.\alpha = \mathbf{0} \qquad \textit{Reason} \quad : \quad c^{-1}c = 1$$

$$\implies \quad \alpha = \mathbf{0} \qquad \textit{Reason} \quad : \quad 1.\alpha = \alpha$$

Definition

Let V be a vector space over a field F. A vector $\beta \in V$ is said to be a linear combination of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in V if there exist scalars $c_1, c_2, \ldots, c_n$ in F such that

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \sum_{i=1}^{n} c_i\alpha_i.$$

Problem 2

Show that $(x, y, z) \in \mathbb{R}^3$ is a linear combination of vectors $\alpha = (1, 1, 1)$, $\beta = (0, 1, 1)$ and $\gamma = (0, 0, 1)$.

Solution : Find scalars (if exist) $a, b, c \in \mathbb{R}$ such that

$$(x, y, z) = a\alpha + b\beta + c\gamma$$

$$(x, y, z) = a(1, 1, 1) + b(0, 1, 1) + c(0, 0, 1)$$

$$(x, y, z) = (a, a + b, a + b + c)$$

$$(x, y, z) = x(1, 1, 1) + (y - x)(0, 1, 1) + (z - y)(0, 0, 1).$$

Problem 3

Prove or disprove that $(1, 2, 3)$ is a linear combination of $\alpha = (1, 1, 1)$ and $\beta = (0, 1, 1)$.
Ans. No.

$$(1, 2, 3) = a(1, 1, 1) + b(0, 1, 1)$$

$\implies a + b = 2$ **and** $a + b = 3$**, lead us to a contradiction.**

Problem 4

Let $\mathbb{R}$ be the real field. Find all vectors in $\mathbb{R}^3$ that are linear combination of $(1,0,-1),(0,1,1)$ and $(1,1,1)$.

Solution: Objective is to find all linear combinations of vectors $(1,0,-1),(0,1,1)$ and $(1,1,1)$.

That is, find all (x,y,z) such that there exist $a,b,c \in \mathbb{R}$ such that

$$a(1,0,-1)+b(0,1,1)+c(1,1,1)=(x,y,z).$$

That is, find a,b,c (if exist) such that

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\implies AX = Y$$

Problem 4 contd.

Find a row-reduced echelon matrix which is row-equivalent to A.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

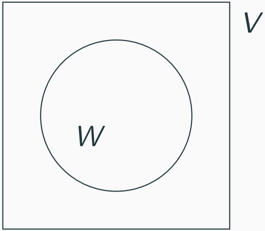
By Theorem 12, A is invertible ($A \sim I$). By Theorem 13, the system $AX = Y$ has a solution X for all Y. Hence for every $Y^t = (x, y, z) \in \mathbb{R}^3$, there exists $X^t = (a, b, c)$ such that

$$a(1, 0, -1) + b(0, 1, 1) + c(1, 1, 1) = (x, y, z).$$

Subspace

Definition

Let V be a vector space over a field F. A subspace of V is a subset W of V which itself a vector space over F with the operations of vector addition and scalar multiplication defined on V.



Examples of subspaces

Example-1. Let V be a vector space over the field F. Then the subset $\{\mathbf{0}\}$ of V is a subspace of V and it is called the zero subspace.

Theorem 1

Let V be a vector space over the field F. A non-empty subset W of V is a subspace of V if and only if

$$\forall\ \alpha, \beta \in W, c \in F \Longrightarrow c\alpha + \beta \in W.$$

Proof:

Case 1: Suppose that W is a subspace of V. So, W is a vector space over the field F. Thus, if $c \in F$ and $\alpha \in W$, then $c\alpha \in W$ (closed under scalar multiplication). Again, as W is closed under vector addition, for any vector $\beta \in W$ one has $c\alpha + \beta \in W$. Hence

$$\forall\ \alpha, \beta \in W, c \in F \Longrightarrow c\alpha + \beta \in W.$$

Theorem 1 contd.

Case 2: Suppose that W is a non-empty subset of V with the property that

$$\forall\ \alpha, \beta \in W, c \in F \Longrightarrow c\alpha + \beta \in W. - - - - -(1)$$

Since $W \neq \phi$, there exists $\gamma \in W$ and hence $\mathbf{0} = (-1)\gamma + \gamma \in W$, by equation (1).
For all $\alpha, \beta \in W$, $\alpha + \beta = 1 \cdot \alpha + \beta \in W$ by equation (1).
For all $\alpha \in W$, $c\alpha = c\alpha + 0 \in W$ by equation (1).
Furthermore, $-\alpha = (-1)\alpha + \mathbf{0} \in W$ for all $\alpha \in W$ by (1).
Since $W \subseteq V$, $\langle W, F, +, . \rangle$ satisfies the rest of the axioms (verify!) of a vector space and thus W is a subspace of V.

Examples of subspaces

Example-2. Note that $W = \{(0, x_2 \ldots, x_n) : x_i \in F\}$ is a subspace of F^n.

Proof: Clearly $\mathbf{0} = (0, 0, \ldots, 0) \in W$. So $\phi \neq W \subseteq F^n$. Let $\alpha = (0, x_2, \ldots, x_n)$, $\beta = (0, y_2, \ldots, y_n) \in W$ and $c \in F$.

$$c\alpha + \beta = (0, cx_2 + y_2, \ldots, cx_n + y_n) \in W$$

By Theorem 1, W is a subspace of F^n.

Example-3. Prove that $W = \{(1 + x_2, x_2, x_3, \ldots, x_n) \ : \ x_i \in F\}$ is not a subspace of F^n.

Reason: $\mathbf{0} = (0, 0, \ldots, 0) \notin W$.

Example-4. Prove that the solution set of the homogeneous system $AX = 0$ is subspace of $F^{n\times 1}$ where $A \in F^{m\times n}$.

Let $S = \{X \in F^{n\times 1} \ : \ AX = 0\}$. Clearly $\mathbf{0} \in S \neq \phi$.
Let $X_1, X_2 \in S$. Then $AX_1 = AX_2 = 0$.

$$\implies \ A(cX_1 + X_2) = cAX_1 + AX_2 = 0,$$

This implies

$$\forall\ X_1, X_2 \in S, c \in F \implies cX_1 + X_2 \in S.$$

Hence, by Theorem 1, S is a subspace of $F^{n\times 1}$.

Theorem 2

Let V be a vector space over the field F. Let W_1, W_2 be two subspaces of V. Then $W_1 \cap W_2$ is a subspace of V.

Proof: Since W_1 and W_2 are subspace of V. Since W_1 and W_2 are themselves vector spaces, $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$. Thus, $\mathbf{0} \in W_1 \cap W_2$, that implies $W_1 \cap W_2 \neq \emptyset$.

Now, let $\alpha, \beta \in W_1 \cap W_2$ and $c \in F$. This implies $\alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$. Since W_1 and W_2 are subspaces of V, by Theorem 1, we have $c\alpha + \beta \in W_1$ and $c\alpha + \beta \in W_2$. This implies $c\alpha + \beta \in W_1 \cap W_2$. Hence, by Theorem 1, $W_1 \cap W_2$ is a subspace of V.

Corollary: Intersection of any collection of subspaces of a vector space V is a subspace of V.

The subspace spanned by S

Definition: Let S be a **subset** of a vector space V. The subspace spanned by S is defined as the intersection all subspaces of V which contains S.

Subspace spanned by $S = \bigcap\{W : S \subseteq W, W \text{ is a subspace of } V\}$.

Note-1. Subspace spanned by S is the smallest subspace which contains S.

Note-2. If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we call the subspace spanned by S as the subspace spanned by the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$L(S)$= the set of all linear combinations of vectors in S

Let S be a **non-empty subset** of a vector space V. The set of all linear combinations of vectors in S is denoted by $L(S)$. In other words

$$L(S) = \left\{ \sum_{i=1}^{n} c_i \alpha_i \quad : \quad c_i \in F, \alpha_i \in S, n \in \mathbb{N} \right\}.$$

Example. Let $S = \{(1,0,0),(0,0,1)\}$. Then

$$\begin{aligned} L(S) &= \{a(1,0,0) + b(0,0,1) : \; a, b \in \mathbb{R}\} \\ &= \{(a,0,b) : \; a, b \in \mathbb{R}\}. \end{aligned}$$

Lemma

$S \subseteq L(S)$ and $L(S)$ is a subspace of V.

Proof: Since $S \neq \phi$, let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Observe that $\alpha_1 = 1 \cdot \alpha_1 + 0 \cdot \alpha_2 + \cdots + 0 \cdot \alpha_n$. So, $\alpha_1 \in L(S)$. Similarly, it can be shown that $\alpha_i \in L(S)$. Therefore, $S \subseteq L(S)$ and hence, $L(S)$ is nonempty.

To show $L(S)$ is a subspace of V, by Theorem 1, it is equivalent to check that for all $\alpha, \beta \in L(S)$ and $c \in F$, $c\alpha + \beta \in L(S)$.

Since $\alpha, \beta \in L(S)$, we have $\alpha = \sum_{i=1}^{n} c_i\alpha_i$ and $\beta = \sum_{i=1}^{n} d_i\alpha_i$.

Thus, $c\alpha + \beta = \sum_{i=1}^{n}(cc_i)\alpha_i + \sum_{i=1}^{n} d_i\alpha_i = \sum_{i=1}^{n}(cc_i + d_i)\alpha_i$ is a linear combination of vectors in S. Thus $c\alpha + \beta \in L(S)$. Hence, $L(S)$ is a subspace of V.

Theorem 3

Let S be a non-empty subset of a vector space V over the field F. Then the subspace spanned by the set S is the set of all linear combinations of vectors in S.

Proof.

Let

$$W^* = \bigcap \{W : \ S \subseteq W, \ W \text{ is a subspace of } V\}$$

We want to show that

$$W^* = L(S) - - - (a)$$

By the previous lemma, $S \subseteq L(S)$ and $L(S)$ is a subspace of V, and thus

$$W^* \subseteq L(S) - - - -(i).$$

Theorem 3 contd.

It remains to show that $L(S) \subset W^*$. It is enough to show that, if W is a subspace containing S, then $L(S) \subseteq W$.

Let $x \in L(S)$. Then x is a linear combination of vectors in S. Since W is a subspace of V and $S \subseteq W$, every linear combination of vectors in S is also a member of W and thus $x \in W$.

$$x \in L(S) \Longrightarrow x \in W. \text{ Thus } L(S) \subseteq W.$$

Hence we proved that

$$L(S) \subseteq \bigcap \{W : S \subseteq W, W \text{ is a subspace of } V\} = W^* - -(ii).$$

By (i) and (ii),

$$W^* = \bigcap \{W : S \subseteq W, W \text{ is a subspace of } V\} = L(S) - - -(a).$$

Note 1: (Visit previous lecture notes)

Find the solution space of the system $RX = 0$

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\}$$

No. of non-zero rows of R, $r = 2$, **No. of variables,** $n = 5$.
$k_1 = 2, k_2 = 4 \Longrightarrow$ Pivot variables $= \{x_{k_1}, x_{k_2}\} = \{x_2, x_4\}$.
No. of free variables $= n - r = 5 - 2 = 3$.
Free variables $= \{x_1, x_3, x_5\}$.

Set the free variables as: $x_1 = a,\ x_3 = b,\ x_5 = c$
$\Longrightarrow \quad x_2 = 3b - \frac{1}{2}c, \quad x_4 = -2c$.

Note 1 contd. (back to chapter one !)

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c) : a, b, c \in \mathbf{R}\right\}$

$$S = \left\{ a(1,0,0,0,0) + b(0,3,1,0,0) + c(0, -\frac{1}{2}, 0, -2, 1) : a, b, c \in \mathbf{R} \right\}$$

$$= \text{Span of } \left\{ (1,0,0,0,0), (0,3,1,0,0), (0, -\frac{1}{2}, 0, -2, 1) \right\}.$$

Problem

Let W be set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfies

$$\begin{aligned} 2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 &= 0 \\ x_1 + \tfrac{2}{3}x_3 - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0 \end{aligned}$$

Find a finite set of vectors which spans W.

Row space and Column space of a matrix

Let $A \in F^{m\times n}$ with rows $\{R_1, R_2, \ldots, R_m\}$ and columns $\{C_1, C_2, \ldots, C_n\}$. Then

Row space of $A =$ The subspace spanned by $R_1, R_2, \ldots, R_m$.

Column space of $A =$ The subspace spanned by $C_1, C_2, \ldots, C_n$.

Note .: Row space of $A \subseteq F^{1\times n}$ and Column space of $A \subseteq F^{m\times 1}$.

Example

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where

$$R_1 = (1,0,0), R_2 = (0,1,0), C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Row space of A

$$= \{a(1,0,0) + b(0,1,0) : \ a, b \in F\} = \{(a,b,0) : \ a, b \in F\}.$$

Column Space of A

$$= \left\{ a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \ a, b, c \in F \right\} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \right\}.$$

Observations

(i) Column space of AB is same as column space of A.

(ii) Row space of AB is same as row space of B.

Observations cont...

$$AX = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \end{bmatrix}$$

$$= x \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} + z \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$$

$$= xC_1 + yC_2 + zC_3 \quad (C_i \text{ is the ith column of } A).$$

(1) AX is a linear combination of columns of the matrix A.

(2) Every column of AB is a linear combination of columns of A.

(3) **Every row of AB is a linear combination of rows of B.**

Linearly Dependent (L.D.) and Linearly Independent (L.I.)

Definition:

Let V be a vector space over the field F. A subset $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of V is said to be linearly dependent if there exist scalars $c_1, c_2, \ldots, c_n \in F$, $c_i \neq 0$ for at least one i, such that

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \mathbf{0}.$$

Definition:

A set which is not linearly dependent is called a linearly independent set.

In other words, if $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \mathbf{0}$, then $c_1 = c_2 = \cdots = c_n = 0$.

Note

1. **Any set which contains a linearly dependent set is linearly dependent.**
2. **Any subset of a linearly independent set is linearly independent.**
3. **Any set which contains the $\mathbf{0}$ vector is linearly dependent. Reason $1 \cdot \mathbf{0} = \mathbf{0}$.**

Problem 1

Show that $\alpha_1 = (3, 0, -3), \alpha_2 = (-1, 1, 2), \alpha_3 = (4, 2, -2)$ and $\alpha_4 = (2, 1, 1)$ are linearly dependent (L.D.) on R^3.

Solution: Find scalars c_1, c_2, c_3, c_4 (at least one $c_i \neq 0$) such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \mathbf{0}$.

$$2\alpha_1 + 2\alpha_2 - \alpha_3 + 0\alpha_4 = \mathbf{0}.$$

Problem 2

Show that $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ is a linearly independent subset of F^3.

Solution: Consider $c_1 e_1 + c_2 e_2 + c_3 e_3 = \mathbf{0}$

$$\implies \quad c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

$$\implies \quad (c_1, c_2, c_3) = (0, 0, 0)$$

$$\implies \quad c_1 = c_2 = c_3 = 0.$$

Hence $\{e_1, e_2, e_3\}$ is a $L.I.$ subset of F^3.

Note

Note:

$\{e_1 = (1, 0, 0, \ldots, 0, 0), e_2 = (0, 1, 0, \ldots, 0, 0) \ \ldots, e_n = (0, 0, 0, \ldots, 0, 1)\}$
is a linearly independent subset of F^n.

Applications

(i) The columns of A forms a linearly independent set if and only if $\mathbf{AX} = \mathbf{0}$ has only trivial solution.

(ii) If A is an invertible matrix if and only if the columns of A forms a linearly independent set (By note (i) and Theorem 13, chapter 1).

Basis

Definition:

Let V be a vector space over the field F. A non-empty set $\mathbb{B} \subseteq V$ is a basis for V if

1. $\mathbb{B}$ is a linearly independent subset of V and
2. $V = \text{ span } \mathbb{B}$ $(= L(\mathbb{B})$).

Definition:

A vector space V is called finite dimensional if it has a finite basis.

Example-1

The set $\mathbb{B} =$
$\{e_1 = (1, 0, 0, \ldots, 0, 0), e_2 = (0, 1, 0, \ldots, 0, 0) \; \ldots, e_n = (0, 0, 0, \ldots, 0, 1)\}$
is a basis of $\mathbb{R}^n$.
Verification:
Claim 1: $\mathbb{B}$ is a linearly independent set in $\mathbb{R}^n$.
Consider

$$c_1(1, 0, \ldots, 0) + c_2(0, 1, 0, \ldots, 0) + \cdots + c_n(0, 0, \ldots, 0, 1) = (0, 0, \ldots, 0)$$

$$\implies (c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$$

$$\implies c_1 = c_2 = \cdots = c_n = 0$$

$\implies \mathbb{B}$ is a *L.I.* set.

Example-1 contd.

Claim 2 : $\mathbb{R}^n = \text{span } \mathbb{B}$. That is every vector of $\mathbb{R}^n$ can be written as a linear combination of vectors of $\mathbb{B}$.

Let $x = (x_1, x_2, \ldots, x_n)$ be any arbitrary vector in $\mathbb{R}^n$. As

$$(x_1, x_2, \ldots, x_n) = x_1(1, 0, \ldots, 0) + x_2(0, 1, \ldots, 0) + \cdots + x_n(0, \ldots, 0, 1)$$

$$\implies (x_1, x_2, \ldots, x_n) = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

we get that

$$\text{span } \mathbb{B} = \mathbb{R}^n.$$

Hence, by claims 1 and 2, $\mathbb{B}$ is a basis of $\mathbb{R}^n$.

Note: $\mathbb{B} = \{e_1, e_2, \ldots, e_n\}$ is called the standard basis of $\mathbb{R}^n$.

Example-2

The set $\mathbb{B} = \{e_1 = (1, 0, 0, \ldots, 0, 0), e_2 = (0, 1, 0, \ldots, 0, 0) \ \ldots, e_n = (0, 0, 0, \ldots, 0, 1)\}$ is a basis of the vector space F^n, where F is a field.

Example-3

The set $\mathbb{B} = \{(0,1,1),(1,0,1),(1,1,0)\}$ is a basis for $\mathbb{R}^3$.

Verification:

Claim 1: $\mathbb{B}$ is a linearly independent set in $\mathbb{R}^3$.

Consider

$$c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) = (0,0,0)$$

$$\implies (0, c_1, c_1) + (c_2, 0, c_2) + (c_3, c_3, 0) = (0,0,0)$$

$$\implies (c_2 + c_3, c_1 + c_3, c_1 + c_2) = (0,0,0)$$

$$\implies c_2 + c_3 = 0, \quad c_1 + c_3 = 0, \quad c_1 + c_2 = 0$$

On solving these three equations we obtain $c_1 = 0, \quad c_2 = 0$ and $c_3 = 0$. Thus, $\mathbb{B}$ is a linearly independent set of $\mathbb{R}^3$.

Claim 2: $\mathbb{R}^3 = \text{span } \mathbb{B}$. That is, we have to show that every vector of $\mathbb{R}^3$ is a linear combination of vectors of $\mathbb{B}$.

Let (x, y, z) be an arbitrary vector of $\mathbb{R}^3$. We want to find scalars c_1, c_2, c_3 such that

$$c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) = (x, y, z)$$

$$\implies (0, c_1, c_1) + (c_2, 0, c_2) + (c_3, c_3, 0) = (x, y, z)$$

$$\implies (c_2 + c_3, c_1 + c_3, c_1 + c_2) = (x, y, z)$$

$$\implies c_2 + c_3 = x, \quad c_1 + c_3 = y, \quad c_1 + c_2 = z$$

Example-3 contd.

Upon solving these three equations we obtain
$c_1 = \frac{-x+y+z}{2}$, $c_2 = \frac{x-y+z}{2}$ and $c_3 = \frac{x+y-z}{2}$. Thus,

$$\begin{aligned}&(x, y, z)\\ &= \frac{-x+y+z}{2}(0,1,1) + \frac{x-y+z}{2}(1,0,1) + \frac{x+y-z}{2}(1,1,0).\end{aligned}$$

This implies

$$\text{span } \mathbb{B} = \mathbb{R}^3.$$

Hence, $\mathbb{B}$ is a basis of $\mathbb{R}^3$.

Problem-3:

Find a basis of the solution space of the system $RX = 0$, where

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution.

No. of non-zero rows of R, $r = 2$, **No. of variables,** $n = 5$.

$k_1 = 2, k_2 = 4 \implies$ Pivot variables $= \{x_{k_1}, x_{k_2}\} = \{x_2, x_4\}$.

No. of free variables $= n - r = 5 - 2 = 3$.

Free variables $= \{x_1, x_3, x_5\}$.

Set the free variables as: $x_1 = a,\ x_3 = b,\ x_5 = c$

$\implies \quad x_2 = 3b - \frac{1}{2}c, \quad x_4 = -2c.$

Problem-3 contd...

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c) : a, b, c \in \mathbf{R}\right\}$

$$S = \left\{ a(1,0,0,0,0) + b(0,3,1,0,0) + c(0,-\frac{1}{2},0,-2,1) : a,b,c \in \mathbf{R} \right\}$$

$$= \text{Span of} \left\{ (1,0,0,0,0), (0,3,1,0,0), (0,-\frac{1}{2},0,-2,1) \right\}.$$

Let

$$E_1 = (1,0,0,0,0),\ \ E_3 = (0,3,1,0,0) \text{ and } E_5 = (0,-\frac{1}{2},0,-2,1).$$

We prove that $\{E_1, E_3, E_5\}$ is a linearly independet set.
Let

$$c_1(1,0,0,0,0)+c_2(0,3,1,0,0)+c_3(0,-\frac{1}{2},0,-2,1)=(0,0,0,0,0)$$

$$\Rightarrow (c_1, 3c_2-\frac{1}{2}c_3, c_2, -2c_3, c_3)=(0,0,0,0,0)$$

This implies $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$. Thus, $\{E_1, E_3, E_5\}$ is a linearly independet set.

Hence $\{E_1, E_3, E_5\}$ is a basis of the solution space S.

Problem 4 (assignment)

Find a basis for the solution set S of the system of equations

$$\begin{aligned} 2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \phantom{{}- 3x_5} &= 0 \\ x_1 \phantom{{}- x_2} + \tfrac{2}{3}x_3 \phantom{{}- x_4} - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0 \end{aligned} \quad .$$

Problem-5

Find a basis for the vector space $F^{m\times n}$ of all $m \times n$ matrices over the field F.

Solution: Define the matrices $E_{i,j}$ in the following way:

(i). The ij^{th} entry of matrix $E_{i,j}$ is one
(ii). All other entries are zero.

Verify that the set $B = \{E_{ij} : \; i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n\}$ is a basis for $F^{m\times n}$.

This is an extension of the idea of the standard basis of F^n.

The column vectors of an invertible matrix form a basis

Let $P \in F^{n\times n}$ be an invertible matrix. Let $P_1, P_2, \ldots, P_n$ be the columns of P. Show that $\mathbb{B} = \{P_1, P_2, \ldots, P_n\}$ is a basis of $F^{n\times 1}$.

Claim 1: $\mathbb{B}$ is a L.I. set.

Consider $x_1 P_1 + x_2 P_2 + \ldots + x_n P_n = \mathbf{0}$.

$$\implies \begin{bmatrix} P_1 & P_2 & \ldots & P_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \mathbf{0}$$

$$\implies \quad PX = \mathbf{0}$$

$$\implies \quad X = \mathbf{0}, \quad (\text{ P is an invertible matrix})$$

Contd.

$$\implies \quad x_1 = x_2 = \ldots = x_n = 0 \implies \mathbb{B} \text{ is an L.I. set}$$

Claim 2: $F^{n\times 1} = \text{ Span } \mathbb{B}$. That is, every column vector can be expressed as a linear combination of column vectors of matrix P.

Let $Y \in F^{n\times 1}$ be any column vector. Consider the matrix multiplication $P^{-1}Y$, denote this by X. So, $Y = PX$. As the matrix PX can be written as a linear combination of column vectors of P, we have

$$Y = PX = x_1P_1 + x_2P_2 + \ldots + x_nP_n \in L(\{P_1, P_2, \ldots, P_n\})$$

Thus, $F^{n\times 1} = \text{ Span } \mathbb{B}$.

Hence, by claims 1 and 2, $\mathbb{B}$ (the set of all columns of P) is a basis of $F^{n\times 1}$.

A simple way of checking a set of vectors in F^n is a basis or not

Idea: Given any set $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of n vectors in F^n, consider the matrix A where the column vectors are precisely $\alpha_1, \alpha_2, \ldots, \alpha_n$. Show that A is invertible. Then by above observation the set $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for F^n.

Theorem 4

Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$. Then any linearly independent set of vectors in V contains no more than m elements.

Proof: It suffices to show that every subset of V which contains more than m elements is linearly dependent (L.D.). Let S be such a set. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, where $n > m$.

Since span $B = V$, we have

$$\begin{aligned}
\alpha_1 &= A_{11}\beta_1 + A_{21}\beta_2 + \cdots + A_{m1}\beta_m \\
\alpha_2 &= A_{12}\beta_1 + A_{22}\beta_2 + \cdots + A_{m2}\beta_m \\
&\vdots \\
\alpha_n &= A_{1n}\beta_1 + A_{2n}\beta_2 + \cdots + A_{mn}\beta_m.
\end{aligned}$$

Theorem 4 contd.

For each $j = 1, 2, \ldots, n$, we have

$$\alpha_j = A_{1j}\beta_1 + A_{2j}\beta_2 + \cdots + A_{mj}\beta_m = \sum_{i=1}^{m} A_{ij}\beta_i.$$

We want to show that S is L.D. That means we are searching for some $x_1, x_2, \ldots, x_n$, at least one of them is not zero, such that

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = \mathbf{0} - - - - - - -(i)$$

Theorem 4 contd.

Consider

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n$$

$$= \sum_{j=1}^{n} x_j\alpha_j$$

$$= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} A_{ij}\beta_i \right)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}x_j \right) \beta_i \text{ (This is just an rearrangement of terms)}$$

$$= \left(\sum_{j=1}^{n} A_{1j}x_j \right) \beta_1 + \left(\sum_{j=1}^{n} A_{2j}x_j \right) \beta_2 + \cdots + \left(\sum_{j=1}^{n} A_{mj}x_j \right) \beta_m.$$

Theorem 4 contd.

Consider the system of equations

$$\sum_{j=1}^{n} A_{1j}x_j = 0$$

$$\sum_{j=1}^{n} A_{2j}x_j = 0$$

$$\vdots$$

$$\sum_{j=1}^{n} A_{mj}x_j = 0$$

This is a homogeneous linear system with m equations and n variables.

Theorem 4 contd.

Since $m < n$, this system has a non-trivial solution say $x_1^*, x_2^*, \ldots, x_n^*$ (at least one $x_j^* \neq 0$), that is $\sum_{j=1}^{n} A_{ij} x_j^* = 0, \text{ for each } i = 1, 2, \ldots, m - - - (ii)$.

Therefore we got $x_1^*, x_2^*, \ldots, x_n^*$ (at least one $x_j^* \neq 0$) such that

$$x_1^* \alpha_1 + x_2^* \alpha_2 + \ldots + x_n^* \alpha_n = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} x_j^* \right) \beta_i = \mathbf{0} \quad (by \;\; (ii))$$

Hence the set $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a L.D. set.

This completes the proof.

Corollary 1. If V is a finite dimensional vector space, then any two bases of V have the same number of elements.

Proof: $B_1 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ and $B_2 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be two bases of V. Then by definition

(i) B_1 is L.I. and span $B_1 = V$.
(ii) B_2 is L.I. and span $B_2 = V$.

Since span $B_1 = V$ and B_2 is an L.I. set, $n \leq m - - - (a)$ (by Theorem 4).
Similarly, since span $B_2 = V$ and B_1 is an L.I. set, $m \leq n - - - (b)$ (by Theorem 4).
Hence, by (a) and (b), $m = n$.
This completes the proof of the corollary.

Dimension

Definition:

The dimension of a finite-dimensional vector space V is the number of elements in a basis for V.

The dimension of the zero vector space is zero.

If V is not a finite dimensional vector space we say it is an infinite dimensional vector space.

Examples

Example-1.
As $\mathbb{B} = \{e_1 = (1, 0, 0, \ldots, 0, 0), e_2 = (0, 1, 0, \ldots, 0, 0), \ldots,$ $e_n = (0, 0, 0, \ldots, 0, 1)\}$ is a basis of F^n over F,

$$\text{dimension of } F^n = dim(F^n) = n.$$

Example-2. The dimension of $F^{m \times n} = mn$,

as the set $\{E_{ij} : \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n\}$ (see previous lecture) is a basis of $F^{m \times n}$.

Example-3. Let $AX = \mathbf{0}$ be a system of homogeneous equations, where A is an $m \times n$ matrix, X is an $n \times 1$ column vector, and $\mathbf{0}$ is the $m \times 1$ zero vector. Let R be the row-reduced-echelon form of matrix A and r be the number of non-zero rows of R. Then the dimension of the solution space of the homogeneous system of linear equations $AX = \mathbf{0}$ is of dimension $n - r$ (the number of free variables).

Examples

For example consider the system of equations $RX = \mathbf{0}$, where

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution space

$$S = \text{Span of} \left\{ (1,0,0,0,0), (0,3,1,0,0), (0, -\frac{1}{2}, 0, -2, 1) \right\}.$$

We have also verified that the set $\left\{ (1,0,0,0,0), (0,3,1,0,0), (0, -\frac{1}{2}, 0, -2, 1) \right\}$ is a basis of S.

Hence, the dimension of the solution space S is $3 = 5 - 2$.

Examples of infinite dimensional vector spaces

Example-4. Let P be the set of all polynomials with real coefficients. Verify that the set $\{1, x, x^2, x^3, \ldots, x^n, \ldots\}$ is a basis for P. Since $\{1, x, x^2, x^3, \ldots, x^n, \ldots\}$ is an infinite set, the vector space P is an infinite dimensional vector space.

Example-5. Let $C[a, b]$ be the set of all continuous functions from $[a, b]$ to $\mathbb{R}$. Then $C[a, b]$ is an infinite dimensional vector space.

Corollary 2. Let V be a finite dimensional vector space and let $n = \dim V$. Then

1. any subset of V which contains more than n vectors is a L.D.;
2. no subset of V which contains fewer than n vectors can span V.

Proof. Let $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be a basis of V. Then
(i) B is L.I. and
(ii) span $B = V$.
Proof of (1). Obvious from Theorem 4.
Proof of (2). Suppose that $V = \text{span}\ \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$. Since B is a linearly independent set and span $B = V$, by Theorem 4 we get $n \leq p$. That means, any set of vectors which spans V contains at least n vectors. This proves (2).

Let S be a linearly independent subset of a vector space V. Suppose that there is a vector $\beta \in V - L(S)$. Then $S \cup \{\beta\}$ is an L.I. subset of V.

Proof. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Claim. The set $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta\}$ is an L.I. set.

Consider

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n + b\beta = \mathbf{0} - - - - - - (i)$$

Then $b = 0$; otherwise $\beta = -\frac{c_1}{b}\alpha_1 - \frac{c_2}{b}\alpha_2 - \cdots - \frac{c_n}{b}\alpha_n \in L(S)$, a contradiction.

Proof continued

From Equation (1) we have

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \mathbf{0}.$$

Since S is an L.I. set, we must have $c_1 = c_2 = \cdots = c_n = 0$. As we already have $c_{n+1} = 0$, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta\}$ is an L.I. subset of V.

Theorem 5

In a finite-dimensional vector space V every non-empty linearly independent set of vectors is part of a basis.

Proof: Let S_0 be a linearly independent subset of V.

Claim: S_0 is a part of a basis for W.

We extend S_0 to a basis for V in the following way.

If span $S_0 = V$, then we are done, as S_0 is L.I. and span $S_0 = V$, hence a basis for V.

If not, that is span $S_0 \neq V$, then there is an element (say) $\beta_1 \in V -$ span S_0. Let $S_1 = S_0 \cup \{\beta_1\}$. By the previous lemma S_1 is an L.I. subset of V.

Theorem 5 contd

If span $S_1 = V$, then we are done, as S_1 is L.I. and span $S_1 = V$, hence a basis for V.

If not, apply the previous lemma to obtain a $\beta_2 \in V - \text{span}\ (S_1)$ such that $S_2 = S_1 \cup \{\beta_2\}$ is an L.I. set.

If we continue in this way, then (in not more than dim V steps) we reach an L.I. set

$$S_m = S_0 \cup \{\beta_1, \beta_2, \ldots, \beta_m\}$$

which is a basis for V.

Example

Let $S_0 = \{(1,1,1)\}$. Find a basis for R^3 which contains S_0.
Solution :

$$L(S_0) = \{a(1,1,1) \ : a \in R\} = \{(a,a,a) \ : \ a \in R\}$$

Clearly, $\beta_1 = (1,1,0) \notin L(S_0)$. By Theorem 5,
$S_1 = S_0 \cup \{\beta_1\} = \{(1,1,1),(1,1,0)\}$ is a L.I. subset of R^3.

$$L(S_1) = \{a(1,1,1) + b(1,1,0) = (a+b, a+b, a) \ : \ a, b \in R\}$$

Clearly $\beta_2 = (1,0,0) \notin L(S_1)$. By Theorem 5,
$S_2 = S_1 \cup \{\beta_2\} = \{(1,1,1),(1,1,0),(1,0,0)\}$ is a L.I. set. Verify that $L(S_2) = R^3$. Hence, S_2 is a basis for R^3.

Definition:

Let V be a finite-dimensional vector space. An ordered basis for V is a finite sequence of vectors which is linearly independent and spans V.

Note: Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let $\alpha \in V = \text{span}\ \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then there exist some scalars $x_1, x_2, \ldots, x_n$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n - - - -(1)$$

This representation is unique.

If not, then there exist scalars $y_1, y_2, \ldots, y_n \in F$ such that

$$\alpha = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n - - - -(2)$$

From (1) and (2), we get

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = \alpha = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n.$$

This implies

$$(x_1 - y_1)\alpha_1 + (x_2 - y_2)\alpha_2 + \cdots + (x_n - y_n)\alpha_n = \mathbf{0}.$$

Since B is linearly independent, we have

$$x_1 - y_1 = x_2 - y_2 = \cdots = x_n - y_n = 0.$$

Thus, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$. Hence the representation is unique.

Coordinate Matrix

Definition:
The coordinate matrix of the vector α relative to the ordered basis B is

$$[\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Example

Example: Find the coordinate matrices of the vector $\alpha = (1, 2, 3)$ with respect to the bases
$B_1 = \{\epsilon_1 = (1, 0, 0), \epsilon_2 = (0, 1, 0), \epsilon_3 = (0, 0, 1)\}$ and
$B_2 = \{\alpha_1 = (1, 1, 1), \alpha_2 = (0, 1, 1), \alpha_3 = (0, 0, 1)\}$.

Solution: Note that

$$\alpha = (1, 2, 3) = 1 \cdot \epsilon_1 + 2 \cdot \epsilon_2 + 3 \cdot \epsilon_3$$

and

$$\alpha = (1, 2, 3) = 1 \cdot \alpha_1 + 1 \cdot \alpha_2 + 1 \cdot \alpha_3.$$

Therefore

$$[\alpha]_{B_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } [\alpha]_{B_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Relation between $[\alpha]_{B_1}$ and $[\alpha]_{B_2}$?

Note that

$$\alpha_1 = \epsilon_1 + \epsilon_2 + \epsilon_3,$$

$$\alpha_2 = 0\epsilon_1 + \epsilon_2 + \epsilon_3$$

and

$$\alpha_3 = 0\epsilon_1 + 0\epsilon_2 + \epsilon_3.$$

So, we have

$$P_1 = [\alpha_1]_{B_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ P_2 = [\alpha_2]_{B_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } P_3 = [\alpha_3]_{B_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Relation between $[\alpha]_{B_1}$ and $[\alpha]_{B_2}$?

Let

$$P = [P_1, P_2, P_3] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Then

$$P \cdot [\alpha]_{B_2} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [\alpha]_{B_1}.$$

Verify that the matrix P is invertible and $[\alpha]_{B_2} = P^{-1}[\alpha]_{B_1}$.

This phenomenon also holds in an arbitrary finite dimensional vector space V.

Proof: Let $B_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $B_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be two ordered bases of a finite-dimensional vector space V. Let $\alpha \in V$. Let

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \quad \text{and} \quad \alpha = \sum_{j=1}^{n} y_j \beta_j$$

be the unique representations of α with respect to the bases B_1 and B_2, respectively. Thus,

$$[\alpha]_{B_1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } [\alpha]_{B_2} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

contd..

Since $\beta_j \in V = \text{span } \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for each $j = 1, 2, \ldots, n$, there exist unique scalars $P_{1j}, P_{2j}, \ldots, P_{nj}$ such that

$$\beta_j = P_{1j}\alpha_1 + P_{2j}\alpha_2 + \cdots + P_{nj}\alpha_n.$$

That is

$$\begin{aligned}
\beta_1 &= P_{11}\alpha_1 + P_{21}\alpha_2 + \cdots + P_{n1}\alpha_n \\
\beta_2 &= P_{12}\alpha_1 + P_{22}\alpha_2 + \cdots + P_{n2}\alpha_n \\
&\vdots \\
\beta_n &= P_{1n}\alpha_1 + P_{2n}\alpha_2 + \cdots + P_{nn}\alpha_n.
\end{aligned}$$

contd.

Now,

$$
\begin{aligned}
\alpha &= y_1\beta_1 + y_2\beta_2 + \cdots + y_n\beta_n \\
&= y_1(P_{11}\alpha_1 + P_{21}\alpha_2 + \cdots + P_{n1}\alpha_n) \\
&+ y_2(P_{12}\alpha_1 + P_{22}\alpha_2 + \cdots + P_{n2}\alpha_n) \\
&\vdots \\
&+ y_n(P_{1n}\alpha_1 + P_{2n}\alpha_2 + \cdots + P_{nn}\alpha_n) \\
&= (P_{11}y_1 + P_{12}y_2 + \cdots + P_{1n}y_n)\alpha_1 \\
&+ (P_{21}y_1 + P_{22}y_2 + \cdots + P_{2n}y_n)\alpha_2 \\
&\vdots \\
&+ (P_{n1}y_1 + P_{n2}y_2 + \cdots + P_{nn}y_n)\alpha_n.
\end{aligned}
$$

contd.

But, we already have $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$, and since the representation of α relative to the basis B_1 is unique, we have

$$\begin{aligned} x_1 &= P_{11}y_1 + P_{12}y_2 + \cdots + P_{1n}y_n \\ x_2 &= P_{21}y_1 + P_{22}y_2 + \cdots + P_{2n}y_n \\ &\vdots \\ x_n &= P_{n1}y_1 + P_{n2}y_2 + \cdots + P_{nn}y_n. \end{aligned}$$

Writing these equations in matrix form we get

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

$$\Longrightarrow X = PY, - - - - - (1)$$

where $X = [\alpha]_{B_1}$ and $Y = [\alpha]_{B_2}$.

Next, we claim that the matrix P is invertible, that we show in the following two steps:

Claim (1): $X = \mathbf{0} \iff Y = \mathbf{0}$.

Proof: Let $X = \mathbf{0}$. This implies $x_1 = x_2 = \cdots = x_n = 0$, that further implies $\alpha = \mathbf{0}$.

As $\alpha = y_1\beta_1 + y_2\beta_2 + \cdots + y_n\beta_n = 0$ and $B_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ is linearly independent, we get $y_1 = y_2 = \cdots = y_n = 0$. Thus, $Y = \mathbf{0}$.

By the same argument we can prove that if $Y = \mathbf{0}$ then $X = \mathbf{0}$.

Claim (2): P **is an invertible matrix.**

Proof: Consider the system of equations $PY = \mathbf{0}$. As $X = PY$, we get $X = 0$. Then from Claim 1 it follows that $Y = 0$. This means the sytem $PY = 0$ has only the trivial solution. Hence, by Theorem 7 (of chapter 1) we get that P is invertible.

We just proved the following theorem:

Theorem 7.
Let V be a n-dimensional vector space over the field F and let $B_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $B_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be two ordered bases of V. Then there is a unique, necessarily invertible, $n \times n$ matrix P with entries in F such that (i) $[\alpha]_{B_1} = P[\alpha]_{B_2}$ and (ii) $[\alpha]_{B_2} = P^{-1}[\alpha]_{B_1}$ for every $\alpha \in V$. The columns of P are given by $P_j = [\beta_j]_{B_1}$ for $j = 1, 2, \ldots, n$.

Note: For a given ordered basis B_1 of a finite-dimensional vector space V and a given invertible matrix P, it is possible to construct another ordered basis B_2 of V.

Example: Let $B_1 =$
$\{\alpha_1 = (0,1,1,1), \alpha_2 = (1,0,1,1), \alpha_3 = (1,1,0,1), \alpha_4 = (1,1,1,0)\}$
be an ordered basis for R^4 and let

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}$$

be an invertible matrix. Find an ordered basis for R^4 different from the basis B_1.

Solution

We want to find $\beta_1, \beta_2, \beta_3,$ and β_4 such that

$$[\beta_1]_{B_1} = P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\beta_2]_{B_1} = P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

$$[\beta_3]_{B_1} = P_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \quad [\beta_4]_{B_1} = P_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

solution contd.

From the definition of coordinate matrices, it follows that

$$\beta_1 = 1\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\alpha_4 = (1, 1, 2, 2)$$

$$\beta_2 = 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 1\alpha_4 = (2, 2, 1, 1)$$

$$\beta_3 = 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 4\alpha_4 = (4, 5, 5, 1)$$

$$\beta_4 = 0\alpha_1 + 0\alpha_2 + 0\alpha_3 + 2\alpha_4 = (2, 2, 2, 0).$$

Hence, we got another ordered basis

$$B_2 = \{(1, 1, 2, 2), (2, 2, 1, 1), (4, 5, 5, 1), (2, 2, 2, 0)\}$$

different from the given ordered basis B_1.

Row rank / Column rank

Let A be an $m \times n$ matrix over a field F. Let $\{R_1, R_2, \ldots, R_m\}$ be the rows of A and $\{C_1, C_2, \ldots, C_n\}$ be the columns of A.

Row space of $A =$ span $\{R_1, R_2, \ldots, R_m\}$.

Column space of $A =$ span $\{C_1, C_2, \ldots, C_n\}$.

Then the row space of A is a subspace of $F^{1\times n}$ and the column space of A is a subspace of $F^{m\times 1}$.

Definitions:

Row rank of $A =$ dimension of row space of A.

Column rank of $A =$ dimension of column space of A.

Basis of a row-reduced echelon matrix

Let A be an $m \times n$ matrix over the field F. Let R be the row-reduced echelon form of A. The non-zero rows of R forms a basis for the row space of R. Since two row-equivalent matrices have same row space, thus a basis for the row space of R is also a basis for the row space of A.

Hence

$$\text{Row rank of } A = \text{Row rank of } \ R = \ \text{No. of non-zero rows of } R.$$