$$\begin{bmatrix} 2 & 3/4 & -1 & -1 & 0 \\ 0 & 7/24 & 1/2 & 1/2 & -1 \\ 0 & 21/8 & 3/2 & 3/2 & -3 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 9R_2$

$$\begin{bmatrix} 2 & 3/4 & -1 & -1 & 0 \\ 0 & 7/24 & 1/2 & 1/2 & -1 \\ 0 & 0 & -3 & -3 & 6 \end{bmatrix}$$

$R_1 \rightarrow R_1/2$

$R_2 \rightarrow R_2/(7/24)$

$R_3 \rightarrow R_3/(-3)$

$$\begin{bmatrix} 1 & 3/8 & -1/2 & -1/2 & 0 \\ 0 & 1 & 12/7 & 12/7 & -24/7 \\ 0 & 0 & 1 & +1 & -2 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - \frac{12}{7} \cdot R_3$

$R_1 \longrightarrow R_1 + \frac{1}{2} \cdot R_3$

$$\begin{bmatrix} 1 & 3/8 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

$R_1 \longrightarrow R_1 - \frac{3}{8} \cdot R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix} \Longleftarrow \text{Row-Reduced Echelon Form of A.}$$

15/11/24

ASSIGNMENT - 1

(MA1002)

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B2 (CSE-DD)

① Given:

$$W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid (2x_1 + \tfrac{3}{4}x_2 - x_3 - x_4 = 0),$$
$$((x_1 + \tfrac{2}{3}x_2 - x_5) = 0) \ \&$$
$$(9x_1 + 6x_2 - 3x_3 - 3x_4 - 3x_5 = 0)$$

To find:

Finite set, 'S', S.T. span (S) = W

Solution:

Given equations are :

$$2x_1 + \tfrac{3}{4}x_2 - x_3 - x_4 + 0.x_5 = 0$$

$$x_1 + \tfrac{2}{3}x_2 + 0.x_3 + 0.x_3 - x_5 = 0$$

$$9x_1 + 6x_2 - 3x_3 - 3x_4 - 3x_5 = 0$$

Writing this in the form $AX = 0$

$$\begin{bmatrix} 2 & 3/4 & -1 & -1 & 0 \\ 1 & 2/3 & 0 & 0 & -1 \\ 9 & 6 & -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Converting $AX = 0$ into its Row reduced Echelon form

$$\begin{bmatrix} 2 & 3/4 & -1 & -1 & 0 \\ 1 & 2/3 & 0 & 0 & -1 \\ 9 & 6 & -3 & -3 & -3 \end{bmatrix}$$

$$R_2 \to R_2 - \tfrac{1}{2}.R_1$$

$$\begin{bmatrix} 2 & 3/4 & -1 & -1 & 0 \\ 0 & 7/24 & 1/2 & 1/2 & -1 \\ 9 & 6 & -3 & -3 & -3 \end{bmatrix}$$

$$R_3 \to R_3 - \tfrac{9}{2}.R_1$$

② Given:

$A, B \in \mathbb{F}^{m \times n}$

R is the row-reduced echelon form of Matrix A

To prove / disprove:

Non-zero vectors of R form a basis for the row space of A.

Proof:

Let $S = \{ r_i \in R \mid r_i \neq 0 , i \in \{1, 2, \dots m\}, r_i \rightarrow \text{row of } R \}$

Let $|S| = n$, i.e. no. of non-zero rows of $R = n$

As we know,

Row space of A (R.S(A)) is the subspace spanned by the rows of A.

For S to be a basis of R.S(A), it must satisfy:

(a) Linear Independent set

(b) Span(S) = R.S(A)

(a) To prove: $S \rightarrow$ Linearly Independent Set

$S = \{ r_1, r_2, \dots, r_n \}$; $n \rightarrow$ no. of non-zero rows in R

As we know, for a row-reduced echelon form:

(i) First non-zero entry of each non-zero row of R is 1.

(ii) Each column of R containing leading non-zero entry of some row has all entries as 0.

Consider $c_1 r_1 + c_2 r_2 + \dots c_n e_n = 0$, $c_i \in \mathbb{R}$

as each c_i has a unique column in which it has a leading 1, the corresponding column in the RHS vector would result in a non-zero ~~vecto~~ value.

$\therefore c_1 = c_2 = c_3 = \dots = c_n = 0$ $[\because \forall r_i \in S]$

$\therefore r_1, r_2, r_3, \dots, r_n$ are linearly Independent.

Now,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By observation, (Leading 1)

x_1, x_2 and x_3 are pivot variables &

x_4 and x_5 are free variables

$$\therefore\ x_1 - x_5 = 0 \quad -(1)$$

$$x_2 = 0 \quad -(2)$$

$$x_3 + x_4 - 2x_5 = 0 \quad -(3)$$

By Solving (1), (2) and (3) :

$$\boxed{x_1 = x_5}$$

$$\boxed{x_2 = 0}$$

$$\boxed{x_3 + x_4 = 2x_5}$$

Let $x_4 = a$ and $x_5 = b$

$$\left.\begin{aligned} \therefore\ x_1 &= x_5 = b \\ x_2 &= 0 \\ x_3 &= 2x_5 - x_4 \\ &= 2b - a \\ x_4 &= a \\ x_5 &= b \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 2b-a \\ a \\ b \end{bmatrix}$$

$$\therefore\ W = \{(a, 0, 2a-b, b, a) \mid a, b \in \mathbb{R}\}$$

$$= \{a(1,0,2,0,1) + b(0,0,-1,1,0)\}$$

$$W = \text{span of } \{(1,0,2,0,1), (0,0,-1,1,0)\}$$

$$\boxed{\therefore\ S = \{(1,0,2,0,1), (0,0,-1,1,0)\}}$$

(b) $span(S) = R.S(A)$

(i) $Span(S) \subseteq R.S(A)$

Let $x \in Span(S)$

$\Rightarrow x \in L(S)$, As $span(S) = L(S)$

$\Rightarrow x = c_1 r_1 + c_2 r_2 + \dots + c_n r_n$, $c_i \in F$ & $r_i \in S$

$\Rightarrow x \rightarrow$ Linear combination of rows of A

$\Rightarrow x \in R.S(A)$

$\Rightarrow span(S) \subseteq R.S(A)$

Hence Proved.

(ii) $R.S(A) \subseteq Span(S)$

Let $x \in R.S(A)$

$\Rightarrow x = d_1 A_1 + d_2 A_2 + \dots d_m A_m$, $A_i \in$ row of A, $d_i \in F$

$\Rightarrow x$ is a linear combination of rows of R

$\Rightarrow x \in span(S)$

$\Rightarrow R.S(A) \subseteq span(S)$

from (i) and (ii) $\Rightarrow R.S(A) = sp(S)$

From (a) and (b) $\longrightarrow$ Proved

R forms a basis for $R.S(A)$.

3 Given:

$\{(1,2,3,4),(4,3,2,1)\} \subseteq B_{R^4}$

To find:

Basis of $\mathbb{R}^4$

Solution:

Let $v_1 = (1,2,3,4)$
$v_2 = (4,3,2,1)$

Checking if v_1 and v_2 are Linearly Independent:

$c_1 v_1 + c_2 v_2 = 0$, $c_1, c_2 \in \mathbb{R}$

$\Rightarrow c_1(1,2,3,4) + c_2(4,3,2,1) = 0$

$$\Rightarrow \left.\begin{array}{l} c_1 + 4c_2 = 0 \quad - (1) \\ 2c_1 + 3c_2 = 0 \quad - (2) \end{array}\right\} \quad c_1 = 0 \;\; \& \;\; c_2 = 0$$

$$3c_1 + 2c_2 = 0 \quad - (3)$$

$$4c_1 + c_2 = 0 \quad - (4)$$

$\therefore$ v_1 and v_2 are linearly independent.

For forming a basis on $\mathbb{R}^4$,

Two more linearly independent vectors are required.

Hence, Add Standard Basis vectors that are not in the span of v_1 and v_2

So $v_3 = (0,0,1,0)$

& $v_4 = (0,0,0,1)$

[$(1,0,0,0)$ & $(0,1,0,0)$ would result in Linear Dependence]

Verifying the Linear Independence of all 4 vectors,

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$$

As $c_1v_1 + c_2v_2 = 0$ [from (1) and (2)]

And as v_3 and v_4 are standard Basis,

$$\therefore c_3v_3 + c_4v_4 = 0$$

$\therefore$ Adding the above 2 equations:

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$$

$\therefore$ v_1, v_2, v_3 and v_4 are Linearly independent.

Check for $\mathbb{R}^4 = \text{span}(B)$

Let $x = (x_1, x_2, \ldots x_n)$ be any arbitrary vector in $\mathbb{R}^4$

$$(x_1, x_2, \ldots, x_n) = x_1(1,2,3,4) + x_2(4,3,2,1) + x_3(0,0,1,0) + x_4(0,0,0,1)$$

$\therefore$ $\text{Span}(B) = \mathbb{R}^4$

We can say, $\{(x_1 + 4x_2 + x_3), (2x_1 + 3x_2 + x_4), 3x_1 + 2x_2, 4x_1 + x_2) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\} = \mathbb{R}^4$

$T(b_2) = c_1 = (1, 0, 0)$

$T(b_3) = c_2 = (0, 1, 0)$

$T(b_4) = c_3 = (0, 0, 1)$

The rows of T are c_1, c_2, c_3 which spans $\mathbb{R}^3$.

Hence, $T: \mathbb{R}^4 \to \mathbb{R}^3$

$\therefore$ Onto Linear Transformation, $T: \mathbb{R}^4 \to \mathbb{R}^3$ with $N(T) = \{(4x, 3x, 2x, x) : x \in \mathbb{R}\}$ is given by

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

↳ Zero column

For $T(0,0,0,1) = T(4,3,2,1) - 4T(1,0,0,0) - 3T(0,1,0,0) - 2T(0,0,1,0)$

$= (0,0,0) - (4,0,0) - (0,3,0) - (0,0,2)$

$= (-4, -3, -2)$

$T(0,0,0,1) = (-4, -3, -2)$

$T(X) = AX$, where $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$\therefore T(X) = \{(x_1 - 4x_4,\ x_2 - 3x_4,\ x_3 - 2x_4)\}$

⑤ <u>Rank Nullity Theorem</u>:

Let V and W be vector spaces over field $\mathbb{F}$.

T be Linear Transformation from $V \to W$.

Suppose that V is finite dimensional.

<u>Prove</u>:

$$\text{rank}(T) + \text{Nullity}(T) = \dim(V)$$

) Given:

$$N(T) = \{(4x, 3x, 2x, x) : x \in \mathbb{R}\}$$

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

To find:

Linear Transformation, T

Solution:

$$N(T) = \{(4x, 3x, 2x, x)\}$$

$$= \text{span of } \{(4,3,2,1)\}$$

The set $\{(4,3,2,1)\}$ is linear independent and spans $N(T)$

$\therefore$ Forms Basis $N(T)$

Nullity$(T) = 1$

By Rank-Nullity Theorem, for $T : V \longrightarrow W$

$$\dim(N(T)) + \dim(R(T)) = \dim(V)$$

$$\Rightarrow \dim(R(T)) = \dim(V) - \dim(N(T))$$

$$= 4 - 1$$

$$= 3$$

$$\therefore \dim(R(T)) = 3$$

As $R(T) \subseteq W$, if $\dim(R(T)) = \dim(W)$,

Then, $R(T) = W$

As $\dim(R(T)) = \dim(\mathbb{R}^3) = 3$

$$R(T) = \mathbb{R}^3$$

$\Rightarrow T$ is onto.

Basis for $\mathbb{R}^4$ with respect to $N(T) = \{(4,3,2,1), b_2, b_3, b_4\}$

where b_2, b_3, b_4 are linearly independent and not in $N(T)$.

Let's take $b_2 = (1,0,0,0)$

$b_3 = (0,1,0,0)$

$b_4 = (0,0,1,0)$

for $v = (4,3,2,1) \in N(T)$; $T(v) = 0$

⑥ $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$B = \{(1,1), (1,2)\}$

Let $v_1 = (1,1)$ &

$v_2 = (1,2)$

$v_1' = P v_1$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (2,0)$

$v_2' = P v_2$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (3,1)$

$B' = \{(2,0), (3,1)\}$

$B_1 = \{(2,1), (1,2)\}$

Let's find a Matrix Q s.t. $[\alpha]_B = Q[\alpha]_{B_1}$.

And as v_1 and v_2 must be expressed as a Linear combination of B_1

$\Rightarrow v_1 = a_1 v_1' + b_1 v_2'$, $a, b \in \mathbb{R}$

$\Rightarrow \left.\begin{array}{l} 2a_1 + b_1 = 1 \\ a_1 + 2b_1 = 1 \end{array}\right\} \Rightarrow$ Solving them gives $\boxed{\begin{array}{l} a_1 = 1/3 \\ b_1 = 1/3 \end{array}}$

$v_2 = a_2 v_1' + b_2 v_2'$

$\Rightarrow \left.\begin{array}{l} 2a_2 + b_2 = 1 \\ a_2 + 2b_2 = 2 \end{array}\right\} \Rightarrow$ Solving them gives $\boxed{\begin{array}{l} a_2 = 0 \\ b_2 = 1 \end{array}}$

Proof:

Let $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ be basis for N, the Null space of T.

There are vectors $\alpha_{k+1}, \ldots, \alpha_n$ in V such that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in V such that $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is a basis for V.

To prove : $\{T\alpha_{k+1}, \ldots T\alpha_n\}$ is a basis for range of T.

The vectors $T\alpha_1, T\alpha_2, \ldots, T\alpha_n$ certainly span range of T, since $T\alpha_j = 0$ for $j \leq k$, we see that $T\alpha_{k+1}, \ldots, T\alpha_n$ span range.

To see that these vectors are linearly independent, suppose

$$c_1 T(\alpha_1) + c_2 T(\alpha_2) + c_3 T(\alpha_3) + \cdots c_n T(\alpha_n) = 0$$

$$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = 0$$

And accordingly the vector $\alpha = c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n$ is the null space of T.

Since, $\alpha_1, \alpha_2, \ldots \alpha_k$ form a basis for N, there must be scalars $b_1, b_2, \ldots b_k$ S.T. $\alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots b_k\alpha_k$

Hence, $(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_k\alpha_k) - (c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_{k+2} + \cdots + c_n\alpha_n) = 0$

$\because \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent

$$\Rightarrow b_1 = b_2 = b_3 = \cdots = b_k = c_{k+1} = c_{k+2} = \cdots = c_n = 0$$

If $r \to$ rank of T,

$T\alpha_{k+1}, \ldots, T\alpha_n$ form a basis for range of T,

i.e. $r = n - k$ $\boxed{k = \text{nullity}(T)}$

$\Rightarrow r + k = n$

$\Rightarrow \text{rank}(T) + \text{Nullity}(T) = \dim(V)$

Hence Proved

(8) Given:

$V \rightarrow$ Vector space

$v_1, v_2, \dots, v_n$ span V

To prove:

(i) $v_1, v_2-v_1, v_3-v_1, \dots, v_n-v_1$ span V.

(ii) If $v_1, v_2, \dots, v_n$ are Linearly independent,

Then $v_1, v_2-v_1, v_3-v_1, \dots, v_n-v_1$ are linearly independent.

Solution:

Let $\alpha \in V$

$\Rightarrow \exists\, c_1, c_2, \dots, c_n$ S.T. $\alpha = c_1v_1 + c_2v_2 + c_3v_3 + \dots + c_nv_n$

$\alpha = c_1v_1 + c_2v_2 + \dots + c_nv_n$

Add & Subtract $c_jv_1 \ \forall j = 2, 3, \dots, n$

$$\Rightarrow \alpha = (c_1v_1 + c_2v_2 + \dots + c_nv_n) + (c_2v_1 + c_3v_1 + \dots c_nv_1) - (c_2v_1 + c_3v_1 + \dots c_nv_1)$$

$$= (c_1 + c_2 + c_3 + \dots c_n)v_1 + c_2(v_2-v_1) + c_3(v_3-v_1) + \dots + c_n(v_n-v_1)$$

$\therefore$ α is a Linear combination of $(v_1, v_2-v_1, v_3-v_1, \dots, v_n-v_1)$

$\Rightarrow (v_1, v_2-v_1, v_3-v_1, \dots, v_n-v_1)$ also spans V.

Suppose $\{v_1, v_2, \dots, v_n\}$ is Linearly Independent.

$\Rightarrow c_1v_1 + c_2v_2 + \dots + c_nv_n = 0 \rightarrow c_1, c_2, \dots, c_n = 0$

Let $b_1v_1 + b_2(v_2-v_1) + \dots + b_n(v_n-v_1) = 0$

$\Rightarrow (b_1 - b_2 - \dots - b_n)v_1 + b_2v_2 + b_3v_3 + \dots b_nv_n = 0$

$\Rightarrow b_1 = b_2 = b_3 = \dots = b_n = 0$

As $b_2 = 0, b_3 = 0, \dots, b_n = 0 \Rightarrow b_1 = 0$

$\therefore \{v_1, v_2-v_1, v_3-v_1, \dots, v_n-v_1\}$ is also Linearly Independent.

Hence, Proved

The transformation Matrix ϕ is,

$$\phi = \begin{bmatrix} 1/3 & 0 \\ 1/3 & 1 \end{bmatrix}$$

7 Given:

$$x + y + z = 0,$$

$$x, y, z \in V$$

$$x \neq 0, \; y \neq 0, \; z \neq 0$$

To show:

$$\text{span}\{x, y\} = \text{span}\{y, z\} = \text{span}\{z, x\}$$

Proof:

$$x = -(y+z)$$

$$y = -(x+z)$$

$$z = -(x+y)$$

As any one variable x, y or z, can be expressed as a linear combination of the other two.

i.e. $x \in \text{span}\{y, z\} \Rightarrow \text{span}\{y, z\} \supseteq \text{span}\{x, y\}$ - 1

$z \in \text{span}\{x, y\} \Rightarrow \text{span}\{x, y\} \supseteq \text{span}\{y, z\}$ - 2

From 1 and 2:

$\text{span}\{x, y\} = \text{span}\{y, z\}$ - 3

$y \in \text{span}\{x, z\} \Rightarrow \text{span}\{x, z\} \supseteq \text{span}\{y, z\}$ - 4

Similarly,

$x \in \text{span}\{z, y\} \Rightarrow \text{span}\{y, z\} \supseteq \text{span}\{x, z\}$ - 5

From 4 and 5:

$\text{span}\{y, z\} = \text{span}\{z, x\}$ - 6

from 3 & 6:

$\text{span}\{x, y\} = \text{span}\{y, z\} = \text{span}\{x, z\}$

Hence Proved.

Solution :

(i) LHS $\Rightarrow$ RHS

(a) $T \circ T = 0$,

If $R(T) = N(T)$, for any $v \in V$, $T(v) \in R(T)$

$\Rightarrow T(v) \in N(T)$

$\Rightarrow T(T(v)) = 0$ showing $T \circ T = 0$

(b) $T \neq 0$

$R(T) = N(T)$,

if $T = 0$, $R(T) = \{0\}$

$N(T) = V$

This contradicts the equality $R(T) = N(T)$,

Hence $T \neq 0$.

$\dim(V) = \text{Rank}(T) + \text{Nullity}(T))$

If $R(T) = N(T)$, $\dim(R(T)) = \dim(N(T))$

Let $r \rightarrow$ rank of T

$\Rightarrow n = r + r$

$\Rightarrow r = n/2 \Rightarrow \therefore n$ must be even

$\therefore$ LHS $\Rightarrow$ RHS

(ii) RHS $\Rightarrow$ LHS

$T \circ T = 0$, for any $v \in V$, $T(v) \in N(T)$

$\Rightarrow R(T) \subseteq N(T)$ - ①

$\dim(v) = \text{Rank}(T) + \text{Nullity}(N(T))$

As $\text{rank}(T) = n/2$ & $\dim(v) = n/2$

$\Rightarrow \text{Nullity}(T) = n/2 \Rightarrow \text{rank}(T) = \text{Nullity}(T)$ - ②

from ① & ② :

$R(T) = N(T)$

$\therefore$ LHS $\Rightarrow$ RHS and RHS $\Rightarrow$ LHS, This tells LHS $\equiv$ RHS.

$\therefore$ Both the above statements are equivalent.

——— X ———

9) Given:

$AB \rightarrow$ defined

$R(A) \rightarrow$ Row space of A

$N(B) \rightarrow$ solution set of $BX = 0$

To prove:

(a) $R(AB) \subseteq R(A)$

(b) $N(AB) \supseteq N(B)$

Solution:

(a) Row space of AB contains linear combination of rows of AB. Each row in AB is a linear combination of A.

$$AB = A_1 \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} + A_2 \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} + \dots\dots + A_m \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}$$

Every vector in $R(AB)$ can be expressed as combination of rows of A,

i.e. $R(AB) \subseteq R(A)$

(b) Null space of AB is

$$\cancel{N(AB) = \{x, ABx = 0, x \in N(B)\}}$$

$$N(AB) = \{x, ABx = 0, x \in N(B)\}$$

If $x \in N(B)$, $Bx = 0 \Rightarrow ABx = A(Bx) = 0$

$\Rightarrow x \in N(AB)$

$\Rightarrow N(B) \subseteq N(AB)$

$\Rightarrow N(AB) \supseteq N(B)$

(10) Given:

V: Vector space of dimension n over field F.

$T: V \rightarrow V$: Linear

To prove:

$[R(T) = N(T)] \equiv [T \circ T = 0, T \neq 0, n \text{ is even}, \text{Rank}(T) = n/2]$