① Given Curve:

$$x^2+y^2 = 2cx$$

Solution:

$$x^2+y^2=2cx$$

Differentiating the whole equation,

$$\frac{d}{dx}(x^2+y^2=2cx)$$

$$\Rightarrow \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(2cx)$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 2c$$

$$\Rightarrow x + y\frac{dy}{dx} = c$$

$$\Rightarrow \boxed{c = x + y\frac{dy}{dx}}$$

Substitute this in the original equation,

$$x^2+y^2 = 2cx$$

$$\Rightarrow x^2+y^2 = 2\left(x+y\frac{dy}{dx}\right)x$$

$$\Rightarrow x^2+y^2 = 2x^2 + 2xy\frac{dy}{dx}$$

$$\Rightarrow y^2-x^2 = 2xy\frac{dy}{dx}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{y^2-x^2}{2xy}}$$

Now, To find Orthogonal Trajectory to the given family of curves, we replace $\frac{dy}{dx}$ with $\frac{-dx}{dy}$

$$\frac{-dx}{dy} = \frac{y^2-x^2}{2xy} \qquad [\because m_1 m_2 = -1]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

Let $f(x,y) = \frac{2xy}{x^2-y^2}$

Now $f(tx,ty) = \frac{2(tx)(ty)}{(tx)^2-(ty)^2}$

$$= \frac{2t^{\cancel{2}}xy}{\cancel{t^2}(x^2-y^2)}$$

$$= \frac{2xy}{x^2-y^2}$$

$$= f(x,y)$$

$\therefore f(tx,ty) = t^0 \cdot f(x,y)$

$f(x,y)$ is a Homogenous equation of Degree 0.

And $\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$ is a Homogenous D.E.

$$\left[\begin{array}{l} \therefore y = zx \\ \Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx} \end{array}\right]$$

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{2x(zx)}{x^2-z^2x^2}$$

$$\Rightarrow x\left(\frac{dz}{dx}\right) = \frac{2\cancel{x}(\cancel{x})}{\cancel{x^2}(1-z^2)} - z$$

$$\Rightarrow x\left(\frac{dz}{dx}\right) = \frac{2z}{1-z^2} - \frac{z-z^3}{1-z^2}$$

$$\Rightarrow x\left(\frac{dz}{dx}\right) = \frac{2+z^3}{1-z^2} = \frac{z(1+z^2)}{(1-z^2)}$$

$$\Rightarrow \frac{dx}{x} = \frac{1-z^2}{z(1+z^2)} dz$$

Integrating both sides, We get

$$\Rightarrow \int \frac{1}{x} dx = \int \frac{1-z^2}{z(1+z^2)} dz \quad [z, x \in R]$$

$$\Rightarrow \ln|x| + c = \int \left(\frac{1}{z} - \frac{2z}{1+z^2}\right) dz \quad [c \in R]$$

[from Partial fractions]

$$\Rightarrow \ln|x| + c = \ln|z| - \ln|1+z^2| + c' \quad [c' \in R]$$

$$\Rightarrow \ln|x| + C = \ln|z| - \ln(1+z^2)$$

[As $z = y/x$]

$$\Rightarrow \ln|x| + C = \ln|y/x| - \ln\left(1 + y^2/x^2\right)$$

$$\Rightarrow \boxed{\ln\left(1 + \frac{y^2}{x^2}\right) - \ln|y/x| + \ln|x| = C} \quad [C \in R]$$

is the family of orthogonal Trajectories of the given family of curves.

② (a) Given :

$$\frac{dy}{dx} = \frac{x+y-1}{x+4y+2}$$

~~Let~~

As $\frac{1}{1} \neq \frac{1}{4}$ [Coeffecients of x and y]

We assume $x = X+h$

$y = Y+k$

to eliminate constant terms from both numerator and denominator

$x = X + h$ | $y = Y + k$

$\Rightarrow dx = dX$ | $\Rightarrow dy = dY$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y-1}{x+4y+2}$$

$$\Rightarrow \frac{dY}{dX} = \frac{X+h+Y+k-1}{X+h+4(Y+k)+2}$$

$$= \frac{X+Y+(h+k-1)}{X+4Y+(h+4k+2)}$$

Now, $h+k-1 = 0$ &

$h+4k+2 = 0$

Subtracting both, we get

$3k+3=0$

$\Rightarrow \boxed{k=-1}$

Putting back in any of the equations,

$\boxed{h=2}$

$\therefore x = X+2$

$y = Y-1$

$$\therefore \frac{dY}{dX} = \frac{X+Y}{X+4Y}$$

Let $f(X,Y) = \dfrac{X+Y}{X+4Y}$

Now $f(tX, tY) = \dfrac{tX+tY}{tX+4tY}$

$$= \frac{X+Y}{X+4Y}$$

$$= f(X,Y)$$

As $f(tX,tY) = t^0 \cdot f(X,Y)$

$f(x,y)$ is a Homogenous equation of degree 0.

Hence $\frac{dy}{dx} = \frac{x+y}{x+4y}$ is a Homogenous DE

$y = zx$

$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x+4y}$

$\Rightarrow z + x\frac{dz}{dx} = \frac{x + x(z)}{x+4(zx)}$

$\Rightarrow x\frac{dz}{dx} = \frac{x(1+z)}{x(1+4(z))} - z$

$\Rightarrow x\frac{dz}{dx} = \frac{1+z}{1+4(z)} - \frac{z+4z^2}{1+4(z)}$

$\Rightarrow x\frac{dz}{dx} = \frac{1-4z^2}{1+4(z)}$

$\Rightarrow \frac{dx}{x} = \frac{1+4(z)}{1-4z^2}dz$

Integrating both sides

$\Rightarrow \int\frac{dx}{x} = \int\frac{1+4(z)}{1-4(z^2)}dz$

$\Rightarrow \ln|x| + c = \int\frac{dz}{1-4(z^2)} + 4\int\frac{z}{1-4(z^2)}dz$

$\Rightarrow \ln|x-2| + c = \int\frac{dz}{(1-2z)(1+2z)} - \frac{1}{2}\int\frac{-8z}{1-4(z^2)}dz$

[CER]

$$\Rightarrow \ln|x-2| + c = \frac{1}{2}\int \frac{dZ}{2(Z)+1} - \frac{1}{2}\int \frac{dZ}{2(Z)-1} - \frac{1}{2}\ln|1-4(Z^2)| + c'$$

$[c' \in \mathbb{R}]$

$$\Rightarrow \ln|x-2| + C = \frac{1}{4}\ln|2(Z)+1| - \frac{1}{4}\ln|2(Z)-1| - \frac{1}{2}\ln|1-4(Z^2)|$$

$$= \frac{1}{4}\ln\left|\frac{2Y}{X}+1\right| - \frac{1}{4}\ln\left|2\frac{Y}{X}-1\right| - \frac{1}{2}\ln\left|1-4\frac{Y^2}{X^2}\right|$$

$$= \frac{1}{4}\ln\left|2\left(\frac{y+1}{x-2}\right)+1\right| - \frac{1}{4}\ln\left|2\left(\frac{y+1}{x-2}\right)-1\right| - \frac{1}{2}\ln\left|1-4\left(\frac{y+1}{x-2}\right)^2\right|$$

$$\therefore \ln|x-2| - \frac{1}{4}\ln\left|2\left(\frac{y+1}{x-2}\right)+1\right| + \frac{1}{4}\ln\left|2\left(\frac{y+1}{x-2}\right)-1\right| + \frac{1}{2}\ln\left|1-4\left(\frac{y+1}{x-2}\right)^2\right| = C$$

is the solution of the given DE.

$[C \in \mathbb{R}]$

(b) Given:

$$\frac{dy}{dx} = \frac{x+y+4}{x+y-6}$$

As $\frac{1}{1} = \frac{1}{1}$, [Coefficients of x and y]

We assume $x+y = z$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y+4}{x+y-6}$$

$$\Rightarrow \frac{dz}{dx} - 1 = \frac{z+4}{z-6}$$

$$\Rightarrow \frac{dz}{dx} = \frac{z+4+z-6}{z-6}$$

$$= \frac{2z-2}{z-6}$$

$$\Rightarrow dx = \frac{z-6}{2(z-1)}\,dz$$

Now, Integrating both sides,

$$\int dx = \int \frac{z-6}{2(z-1)} dz$$

$$\Rightarrow x + c = \frac{1}{2} \int \frac{z-6}{z-1} dz \qquad [c \in R]$$

$$= \frac{1}{2} \int \left(1 - \frac{5}{z-1}\right) dz$$

$$= \frac{1}{2} \left[z - 5 \ln|z-1| + c' \right] \qquad [c' \in R].$$

$$\Rightarrow x + C = \frac{1}{2} [z - 5\ln|z-1|]$$

$$\Rightarrow x + C = \frac{1}{2} [x+y - 5\ln|x+y-1|]$$

$$\Rightarrow \frac{1}{2}[x - y + 5\ln|x+y-1|] = c$$

$$\boxed{x - y + 5\ln|x+y-1| = c} \qquad [C \in R]$$

is the solution of the given DE.

③ Given:

$$M\,dx + N\,dy = 0$$

$$z = x + y$$

Solution:

It is given that the DE '$M\,dx + N\,dy = 0$' has an Integrating factor of function 'z'.

i.e. $\mu(z) = e^{\int g(z) dz}$

Let us assume that the given DE is not exact. Hence we multiply Integrating factor on both sides and make it exact.

Let the Integrating factor be μ.

$$M\,dx + N\,dy = 0$$

$$\Rightarrow \mu M\,dx + \mu N\,dy = 0$$

Now, There exists a function f such that

$\frac{\partial f}{\partial x} = \mu M$ & $\frac{\partial f}{\partial y} = \mu N$ as the DE now is exact.

And $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$ [Condition for exact DE]

$$\Rightarrow \mu \cdot \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \cdot \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

$$\Rightarrow M\left(\frac{\partial \mu}{\partial y}\right) - N\left(\frac{\partial \mu}{\partial x}\right) = \mu\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{\mu}\left[M\left(\frac{\partial \mu}{\partial y}\right) - N\left(\frac{\partial \mu}{\partial x}\right)\right]$$

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left[N\left(\frac{\partial \mu}{\partial x}\right) - M\left(\frac{\partial \mu}{\partial y}\right)\right]$$

We assumed μ to be function of 'z' alone

As $z = x + y$,

$$\frac{\partial z}{\partial x} = 1 \quad \Big| \quad \frac{\partial z}{\partial y} = 1$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left[N\left(\frac{\partial \mu}{\partial z} \cdot \frac{\partial z}{\partial x}\right) - M\left(\frac{\partial \mu}{\partial z} \cdot \frac{\partial z}{\partial y}\right)\right]$$

$$= \frac{1}{\mu}\left[N\left(\frac{\partial \mu}{\partial z}\right) - M\left(\frac{\partial \mu}{\partial z}\right)\right]$$

$$= \frac{1}{\mu}(N - M)\left(\frac{\partial \mu}{\partial z}\right)$$

$$\Rightarrow \left(\frac{1}{N - M}\right)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\mu}\frac{d\mu}{dz}$$

As the RHS, i.e. $\frac{1}{\mu}\frac{d\mu}{dz}$ is purely a function of 'z',

$\therefore$ The LHS must also be purely a function of 'z'

i.e.,

$$\left(\frac{1}{N-M}\right)\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \text{ must be a}$$

function of 'z' alone

$$\therefore g(z) = \frac{\partial M/\partial y - \partial N/\partial x}{N-M}$$

$$\text{And } \mu(z) = e^{\int g(z)\,dz}$$

Hence the condition under which the DE $M\,dx + N\,dy = 0$ has an IF that is a function of $z(x,y) = x+y$ is that

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N-M} \text{ must be a function of } z.$$

(4) Given:

$$\frac{dy}{dx} + P(x)\cdot y = \varphi(x)$$

Solution:

$$\frac{dy}{dx} + P(x)\cdot y = \varphi(x)$$

$$\Rightarrow dy + dx\left(P(x)\cdot y\right) = dx\left(\varphi(x)\right)$$

$$\Rightarrow \left[P(x)\cdot y - \varphi(x)\right]dx + [1]\,dy = 0$$

The given DE has been converted to the form $M\,dx + N\,dy = 0$, where,

$$M = P(x)\cdot y - \varphi(x)$$
$$N = 1$$

Checking whether this DE (converted) is an exact DE or not,

$$\underbrace{\frac{\partial M}{\partial y}} = \underbrace{\frac{\partial N}{\partial x}} \quad \text{(Check)}$$

Exactness Condition

$$\Rightarrow \underbrace{P(x) - 0} = \underbrace{0}$$

The ~~e~~ DE is only exact if $P(x) = 0$. Which is not true generally.

Assuming $P(x) \neq 0$,

We have an Integrating factor with which we multiply and then convert the non-exact DE to an exact DE.

$$M\,dx + N\,dy = 0$$

$$\Rightarrow [P(x)\cdot y - \varphi(x)]\,dx + dy = 0$$

$$\Rightarrow \mu\,[P(x)\cdot y - \varphi(x)] + \mu\,dy = 0 \qquad [\mu : \text{Integrating factor}]$$

Now, As this is an exact DE,

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\Rightarrow \mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

$$\Rightarrow M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right)$$

As we know,

$$\frac{\partial M}{\partial y} = P(x) \quad \& \quad \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right)$$

$$\Rightarrow P(x) = \frac{1}{\mu}\left[\frac{\partial \mu}{\partial x} - \left(P(x)\cdot y - \varphi(x)\right)\frac{\partial \mu}{\partial y}\right]$$

LHS is always purely a function of 'x'.

$\therefore$ RHS must also be ~~a function of~~ x alone.

$\therefore$ μ cannot be a function of y alone.

μ has to be a function of x alone

$$\Rightarrow \underbrace{P(x)}_{\text{function of } x} = \underbrace{\frac{1}{\mu}\left(\frac{\partial \mu}{\partial x}\right)}_{\text{function of } x}$$

$$\boxed{\therefore \mu(x) = e^{\int P(x)\,dx}}$$

$$\left[\begin{aligned} &\because P(x)\,dx = \frac{d\mu}{\mu} \\ &\Rightarrow \int P(x)\,dx = \ln|\mu(x)| + c \\ &\Rightarrow \mu(x) = e^{\int P(x)\,dx} \end{aligned}\right] \quad [c \in \mathbb{R}]$$

Solving :

$$\mu M\,dx + \mu N\,dy = 0$$

$$F = \int \mu M\,dx + g(y) \qquad \left[\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}\right]$$

$$\Rightarrow F = \int e^{\int P(x)\,dx}\cdot\left[P(x)\cdot y - \varphi(x)\right]dx + g(y)$$

$$\frac{\partial f}{\partial y} = \mu N = e^{\int P(x)\,dx} \quad -①$$

$$\& \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int e^{\int P(x)dx}\cdot\left(P(x)\cdot y - \varphi(x)\right)dx\right] + g'(y) \quad -②$$

As ① = ②

$$\frac{\partial}{\partial y}\left[y\int e^{\int P dx}\cdot P\,dx - \int e^{\int P dx}\cdot \varphi\,dx\right] + g'(y) = e^{\int P(x)\,dx}$$

$$\Rightarrow \int e^{\int P(x)dx}\cdot P\,dx + g'(y) = e^{\int P(x)\,dx}$$

$$\left[\begin{array}{l} \text{Let } t = e^{\int P(x)dx} \\ dt = e^{\int P(x)dx}\cdot P(x)\,dx \end{array}\right]$$

$$\Rightarrow \int dt + g'(y) = e^{\int P(x)\,dx}$$

$$\Rightarrow \cancel{e^{\int P(x)dx}} + g'(y) = \cancel{e^{\int P(x)dx}}$$

$$\therefore g'(y) = 0$$

$$\Rightarrow g(y) = k \qquad [k \in \mathbb{R}]$$

$$\therefore f = \int e^{\int P(x)dx}\cdot\left[P(x)\cdot y - \varphi(x)\right]dx + k$$

The solution, $f(x,y) = c$, $c \in \mathbb{R}$

$$\therefore \int e^{\int P(x)dx}\cdot\left[P(x)\cdot y - \varphi(x)\right]dx = c \qquad [c \in \mathbb{R}]$$

$$\Rightarrow \int e^{\int P(x)dx}\cdot P(x)\cdot y\,dx - \int e^{\int P(x)dx}\cdot\varphi(x)\,dx = c$$

$$\Rightarrow y\cdot e^{\int P(x)dx} - \int \varphi(x)\cdot e^{\int P(x)dx}\,dx = c$$

$$\boxed{\therefore y(x) = e^{-\int P(x)dx}\left[\int \varphi(x)\cdot e^{\int P(x)dx}\,dx + c\right]}$$ is the solution

⑤ Given:

$$xy' + y = y^{-2}$$

Solution:

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{x}\right) = \frac{1}{x}(y^{-2})$$

This is a form of Bernoulli's Equation.

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{1}{x}(y^{-2})$$

$$\left[\frac{dy}{dx} + P(x)\,y = \varphi(x).\,y^{n}\right] \text{——} \left[z = y^{1-n}\right]$$

$$z = y^{1-(-2)}$$

$$\Rightarrow z = y^3$$

$$\left[\frac{dz}{dx} = 3y^2\frac{dy}{dx}\right]$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{1}{x}(y^{-2})$$

$$\Rightarrow y^2\frac{dy}{dx} + \frac{1}{x}y^3 = \frac{1}{x}$$

$$\Rightarrow 3y^2\frac{dy}{dx} + \frac{3}{x}y^3 = \frac{3}{x}$$

$$\Rightarrow \frac{dz}{dx} + \left(\frac{3}{x}\right)z = \frac{3}{x}$$

Now $P(x) = \varphi(x) = \frac{3}{x}$

Integrating factor : $e^{\int P(x)\,dx}$

$$= e^{\int \frac{3}{x}dx}$$

$$= e^{3\ln|x|} = x^3$$

Solution :

$$z \cdot x^3 = \int \frac{3}{x} \cdot x^3 \, dx + C \qquad [C \in \mathbb{R}]$$

$$\Rightarrow z \cdot x^3 = \int 3 \cdot x^2 \, dx + C$$

$$\Rightarrow z \cdot x^3 = x^3 + c$$

$$\Rightarrow y^3 x^3 = x^3 + c$$

$$\boxed{\therefore x^3 (y^3 - 1) = c}$$

is the solution of the given DE.

① Continuing the solution,

$$\ln\left(1 + \frac{y^2}{x^2}\right) - \ln\left|\frac{y}{x}\right| + \ln|x| = c$$

$$\Rightarrow \ln\left(\frac{x^2+y^2}{x^2}\right) - \ln|y| + \ln|x^2| = C$$

$$\Rightarrow \ln(x^2+y^2) - \ln(y) = \ln(C)$$

$$\Rightarrow \frac{x^2+y^2}{y} = c$$

$$\Rightarrow \boxed{x^2+y^2 = cy}$$