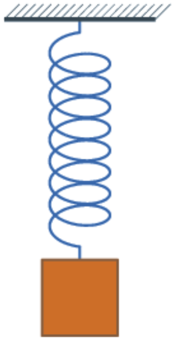
Oscillatory Motion : Damped oscillation



$$x = A \sin(\omega t + \delta)$$

- In real life, non-conservative forces (friction, viscosity..) are present.
- Mechanical energy (amplitude) of the system diminishes with time.

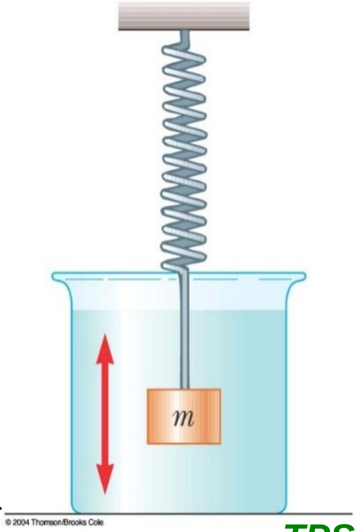
In first order approximation, the damping force is proportional to the velocity $F_d = -b\dot{x}$. , Eqn. of motion,

$$m\ddot{x} = -kx - b\dot{x}, \quad \ddot{x} + 2\nu\dot{x} + \omega_0^2 x = 0; \quad \nu = \frac{b}{2m}$$

Assume the general solution, $x = A\exp(-\nu t)\sin(\omega t + \delta)$

$$\dot{x} = A(-\nu)\exp(-\nu t)\sin(\omega t + \delta) + A\exp(-\nu t)\omega\cos(\omega t + \delta)$$

$$\ddot{x} = A(-\nu)^2\exp(-\nu t)\sin(\omega t + \delta)$$
$$+ 2A(-\nu)\omega\exp(-\nu t)\cos(\omega t + \delta)$$
$$+ A(-\omega^2)\exp(-\nu t)\sin(\omega t + \delta)$$



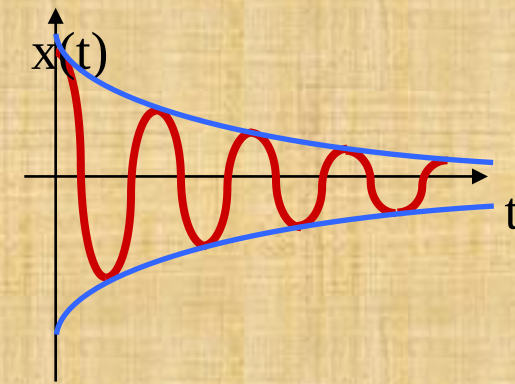
Oscillatory Motion : Damped oscillation

Putting in the eqn. of motion,

$$\nu^2 x - 2A\nu\omega\exp(-\nu t)\cos(\omega t+\delta) - \omega^2 x - 2\nu^2 x$$
$$+2A\nu\omega\exp(-\nu t)\cos(\omega t+\delta) + \omega_0^2 x = 0$$

$[\omega_0^2 - \omega^2 - \nu^2]x = 0,$ *for nontrivial solution,*

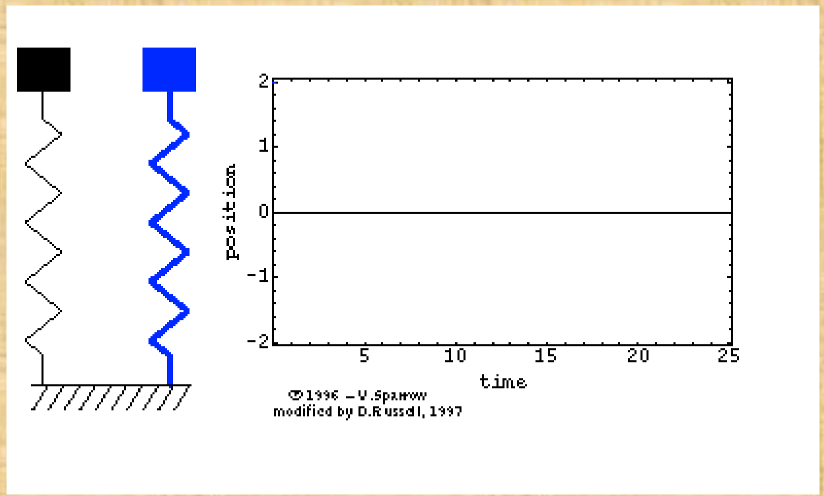
$$\omega = \sqrt{\omega_0^2 - \nu^2}$$



$$x = A\exp(-\nu t)\sin(\omega t+\delta)$$

Amplitude

Damping coefficient

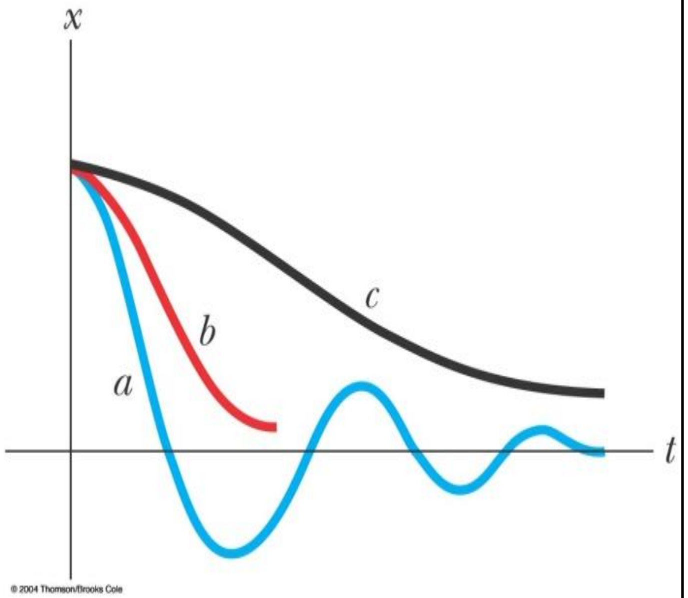


$\omega = \sqrt{\omega_0^2 - \nu^2} \quad \Rightarrow$ *Damping reduces the frequency*

Oscillatory Motion : Damped oscillation

$$\omega = \sqrt{\omega_0^2 - \nu^2} \quad \Rightarrow \textit{Damping reduces the frequency}$$

- a) When the retarding force is small **($\nu<\omega_0$),** the oscillatory character of the motion is preserved, but the amplitude decreases exponentially with time and the motion ultimately ceases– ***under-damped***



- b) When **($\nu=\omega_0$),** the system will not oscillate, the system is said to be ***critically damped***

- c) When **($\nu>\omega_0$),** the system is said to be ***over-damped***

For critically damped and over damped there is no angular frequency

Damped oscillation: Energy

$$x = Ae^{-\nu t}\sin(\omega t + \delta)$$
$$\omega^2 = \omega_0^2 - \nu^2$$

$$KE = \frac{1}{2}m\dot{x}^2 \quad = \frac{1}{2}mA^2 e^{-2\nu t}\{\omega\cos(\omega t + \delta) + \nu\sin(\omega t + \delta)\}^2$$

$$KE = \frac{1}{2}mA^2 e^{-2\nu t}\{\nu^2\sin^2(\omega t + \delta) + 2\nu\cos(\omega t + \delta)\sin(\omega t + \delta) + \omega^2\cos^2(\omega t + \delta\}$$

$$\langle E \rangle = \frac{\int_0^T E\,dt}{\int_0^T dt} = \frac{1}{T}\int_0^T E\,dt$$

$$PE = \int_0^x kx\,dx = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 A^2 e^{-2\nu t}\sin^2(\omega t + \delta)$$

$$\langle \sin^2() \rangle = \frac{1}{2} = \langle \cos^2() \rangle$$

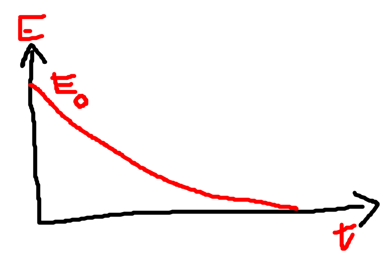
$$\langle \sin(\ ) \rangle = 0 = \langle \cos() \rangle$$

$$E = \frac{1}{2}mA^2 e^{-2\nu t}\{(\nu^2+\omega_0^2)\sin^2(\omega t + \delta) + \nu\sin 2(\omega t + \delta) + \omega^2\cos^2(\omega t + \delta)\}$$

$$\langle E \rangle = \frac{1}{2}mA^2 e^{-2\nu t}\{(\nu^2+\omega_0^2)\langle \sin^2(\omega t + \delta)\rangle + \nu\langle \sin 2(\omega t + \delta)\rangle + \omega^2\langle \cos^2(\omega t + \delta)\rangle\}$$

$$\langle E \rangle = \frac{1}{2}mA^2 e^{-2\nu t}\left\{(\nu^2+\omega_0^2)\frac{1}{2} + \omega^2\frac{1}{2}\right\}$$

Average power dissipation during a time period



$$\langle E \rangle = \frac{1}{2}mA^2\omega_0^2 e^{-2\nu t} = E_0 e^{-2\nu t}$$

$$\langle P \rangle = \frac{d\langle E \rangle}{dt} = \nu\langle E \rangle$$

When damping is small, the amplitude of oscillation does not change much over one oscillation. So we may take the factor exp (- \nu t) as essentially constant.

Oscillatory Motion : Forced oscillation

If an external force, $F_{ext}=F_0\sin\omega t$, (besides damping, $F_d = -bv$, $\nu<\omega_0$) is applied to the system (with restoring force, $F_k = -kx$), the equation of motion can be written as ,

$$ma = F_k + F_d + F_{ext}$$

$$m\ddot{x} = -kx - b\dot{x} + F_0 \sin\omega t$$

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x = \alpha \sin\omega t.$$

$$\nu = \frac{b}{2m} \rightarrow \textit{damping coefficient}$$

$$\alpha = \frac{F_0}{m} \rightarrow \textit{amplitude of driving force}$$

$$\omega_0 = \sqrt{k/m} \rightarrow \textit{natural frequency}$$

$$\text{Assume, } x(t) = A\sin(\omega t + \delta)$$

$$\dot{x} = \omega A\cos(\omega t + \delta),\ \ddot{x} = -\omega^2 \sin(\omega t + \delta)$$

$$(\omega_0^2 - \omega^2)A\sin(\omega t + \delta) + 2\nu\omega A\cos(\omega t + \delta) = \alpha\sin\omega t$$

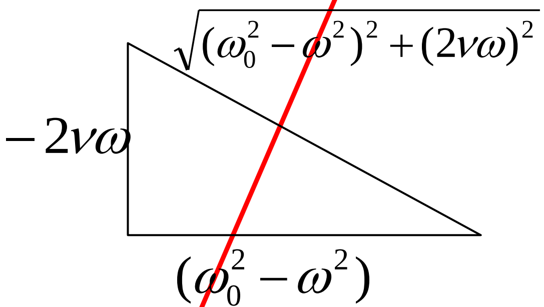
$$\sin(\omega t + \delta) = \sin\omega t\cos\delta + \cos\omega t\sin\delta$$

$$\cos(\omega t + \delta) = \cos\omega t\cos\delta - \sin\omega t\sin\delta$$

$$x(t) = A\sin(\omega t + \delta)$$

Oscillatory Motion : Forced oscillation

$$[\{(\omega_0^2 - \omega^2)\cos\delta - 2\nu\omega\sin\delta\}A - \alpha]\sin\omega t + [(\omega_0^2 - \omega^2)\sin\delta + 2\nu\omega\cos\delta]A\cos\omega t = 0$$



$$(\omega_0^2 - \omega^2)\sin\delta + 2\nu\omega\cos\delta = 0 \quad \rightarrow \tan\delta = -\frac{2\nu\omega}{\omega_0^2 - \omega^2}$$

$$\sin\delta = \frac{-2\nu\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\nu\omega)^2}} \qquad \cos\delta = \frac{(\omega_0^2 - \omega^2)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\nu\omega)^2}}$$

$$\{(\omega_0^2 - \omega^2)\cos\delta - 2\nu\omega\sin\delta\}A - \alpha = 0$$

$$A = \frac{\alpha}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\nu\omega)^2}}$$

Oscillatory Motion : Forced oscillation

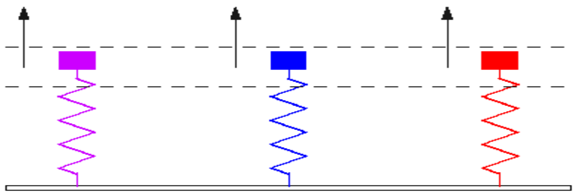
$$x(t)=\frac{\alpha}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}\sin(\omega t+\tan^{-1}\left\{\frac{2\nu\omega}{\omega_0^2-\omega^2}\right\})$$

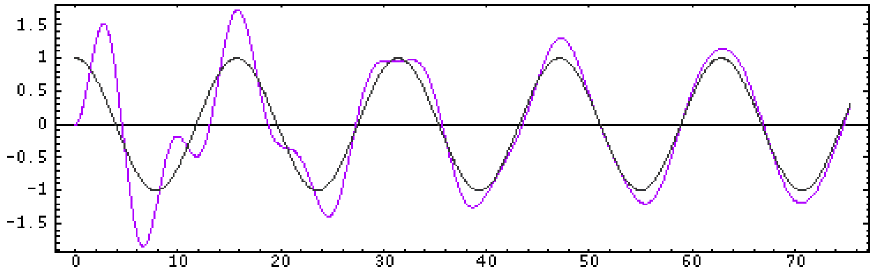
We assume damping is small. To study the response of the system with damping coefficient ν, to the external periodic force (freq.=ω),
We consider cases: 1) $\omega<<\omega_0$, 2) $\omega=\omega_0$ and 3) $\omega>>\omega_0$

1) $\omega<<\omega_0$,

$$\sin\delta=-\frac{2\nu\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}\rightarrow 0$$

$$\cos\delta=\frac{(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}\rightarrow 1$$



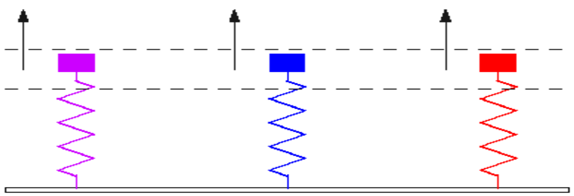


Gives $\delta\rightarrow 0$

$$A=\frac{\alpha}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}=\frac{\alpha}{\omega_0^2}=\frac{F_0}{k}$$

Oscillatory Motion : Forced oscillation

$$x(t)=\frac{\alpha}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}\sin\left(\omega t+\tan^{-1}\left\{\frac{2\nu\omega}{\omega_0^2-\omega^2}\right\}\right)$$

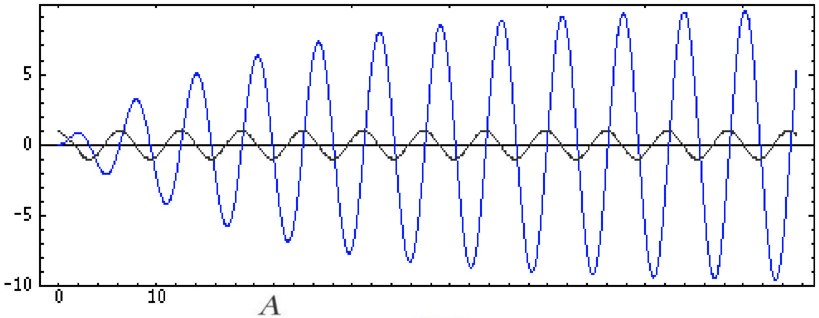


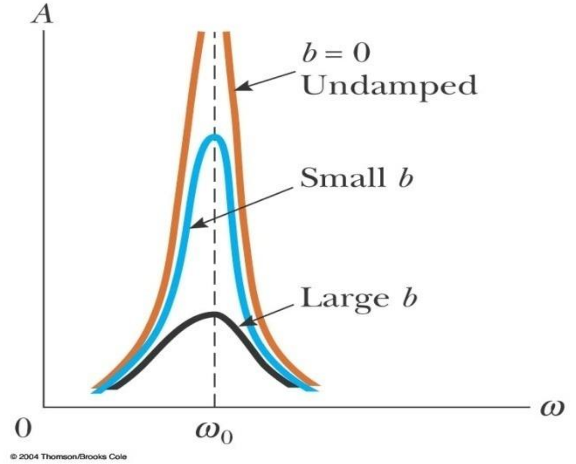
2) $\omega=\omega_0$, RESONANCE

$$\sin\delta=-\frac{2\nu\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}\rightarrow -1$$

$$\cos\delta=\frac{(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}\rightarrow 0$$

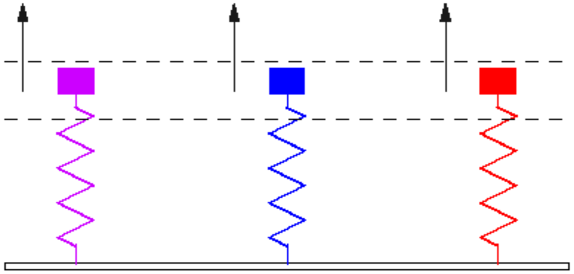
Gives $\delta\rightarrow -\pi/2$ $\quad A=\frac{\alpha}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}=\frac{\alpha}{2\nu\omega}$





Oscillatory Motion : Forced oscillation

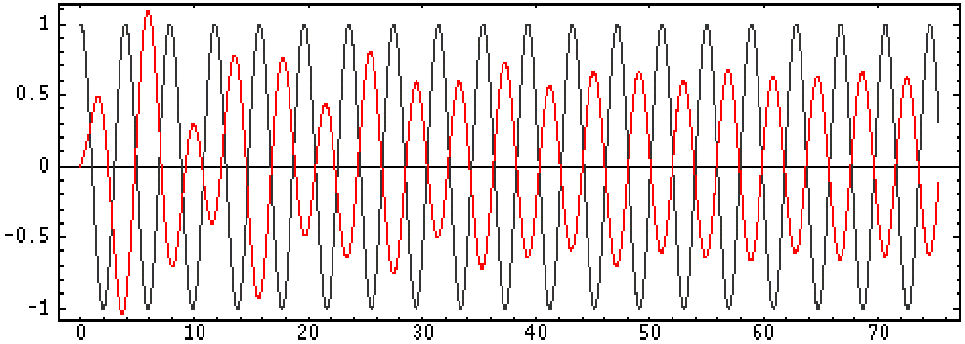
$$x(t) = \frac{\alpha}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}\sin(\omega t + \tan^{-1}\left\{\frac{2\nu\omega}{\omega_0^2-\omega^2}\right\})$$



3) $\omega >> \omega_0$,

$$\sin\delta = -\frac{2\nu\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}}$$

$$= -\frac{2\nu\omega}{\sqrt{(\omega^2)^2+(2\nu\omega)^2}} \approx \frac{2\nu}{\omega} \to 0$$



$$\cos\delta = \frac{(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}} \to -1$$

Gives $\delta \to -\pi$

$$A = \frac{\alpha}{\sqrt{(\omega_0^2-\omega^2)^2+(2\nu\omega)^2}} = \frac{\alpha}{\omega^2}$$