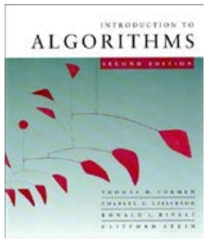
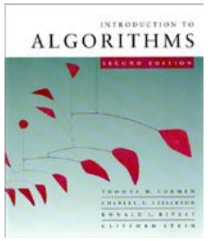


Sorting in linear time

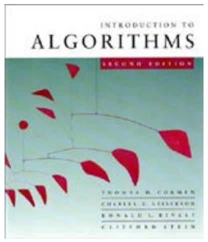
Counting sort: No comparisons between elements.

- ***Input***: $A[1 \,.\,.\, n]$, where $A[j] \in \{1, 2, \ldots, k\}$.
- ***Output***: $B[1 \,.\,.\, n]$, sorted.
- ***Auxiliary storage***: $C[1 \,.\,.\, k]$.



Counting sort

```
for $i \leftarrow 1$ to $k$
    do $C[i] \leftarrow 0$
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$      $\rhd$ $C[i] = |\{\text{key} = i\}|$
for $i \leftarrow 2$ to $k$
    do $C[i] \leftarrow C[i] + C[i-1]$      $\rhd$ $C[i] = |\{\text{key} \le i\}|$
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```

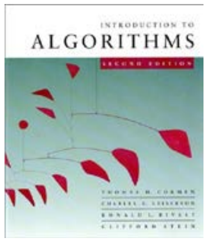


Counting-sort example

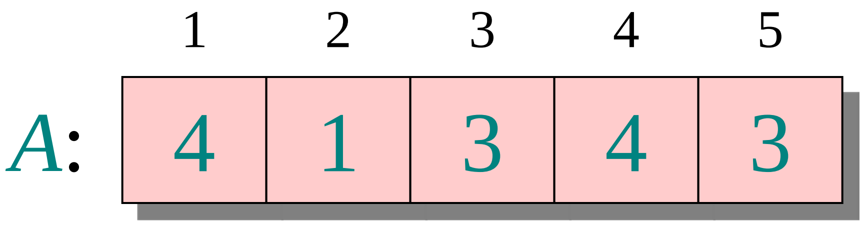
	1	2	3	4	5
A:	4	1	3	4	3

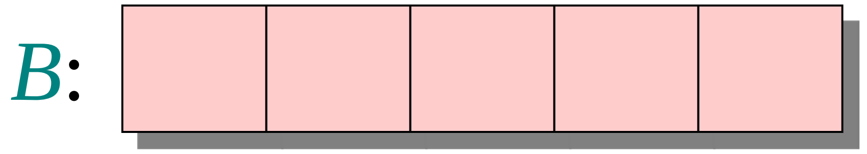
	1	2	3	4
C:				

B:					

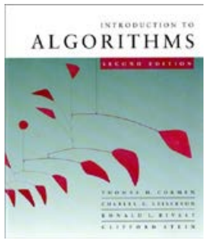


Loop 1

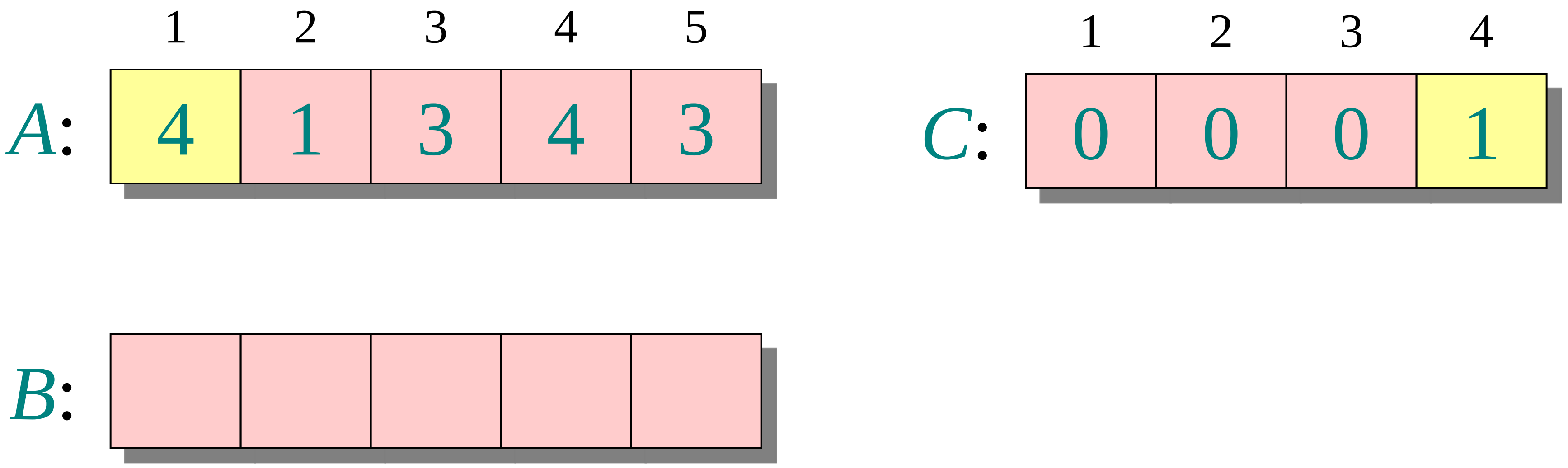




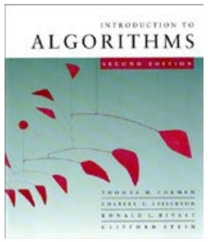
```
for $i \leftarrow 1$ to $k$
    do $C[i] \leftarrow 0$
```



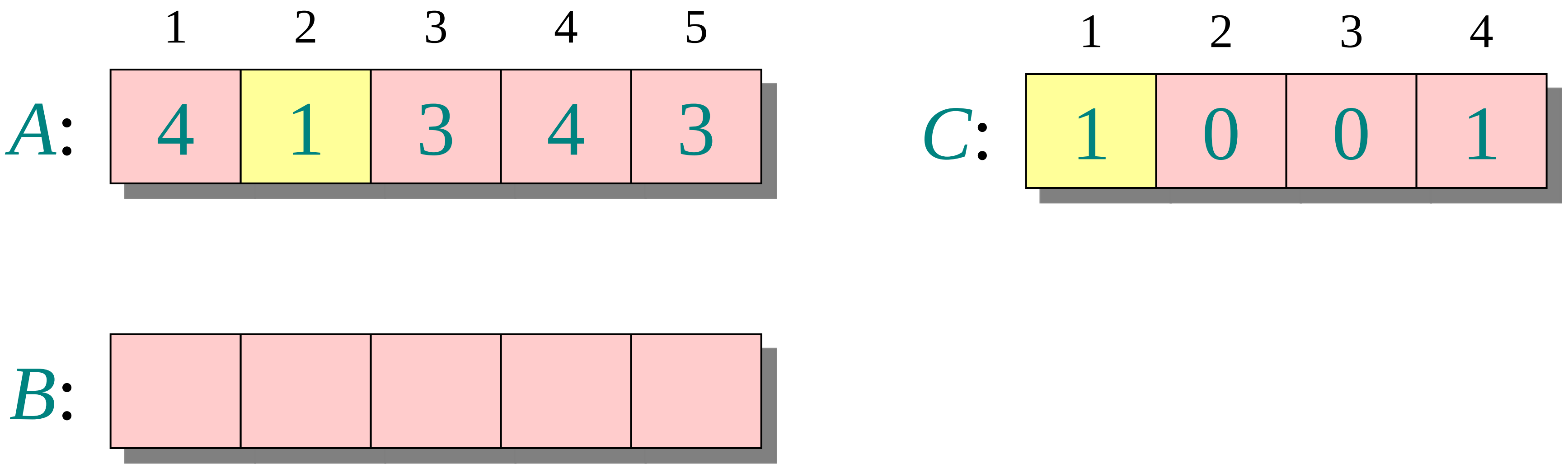
Loop 2



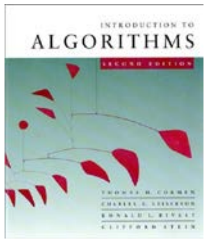
```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$    $\rhd$ $C[i] = |\{\text{key} = i\}|$
```



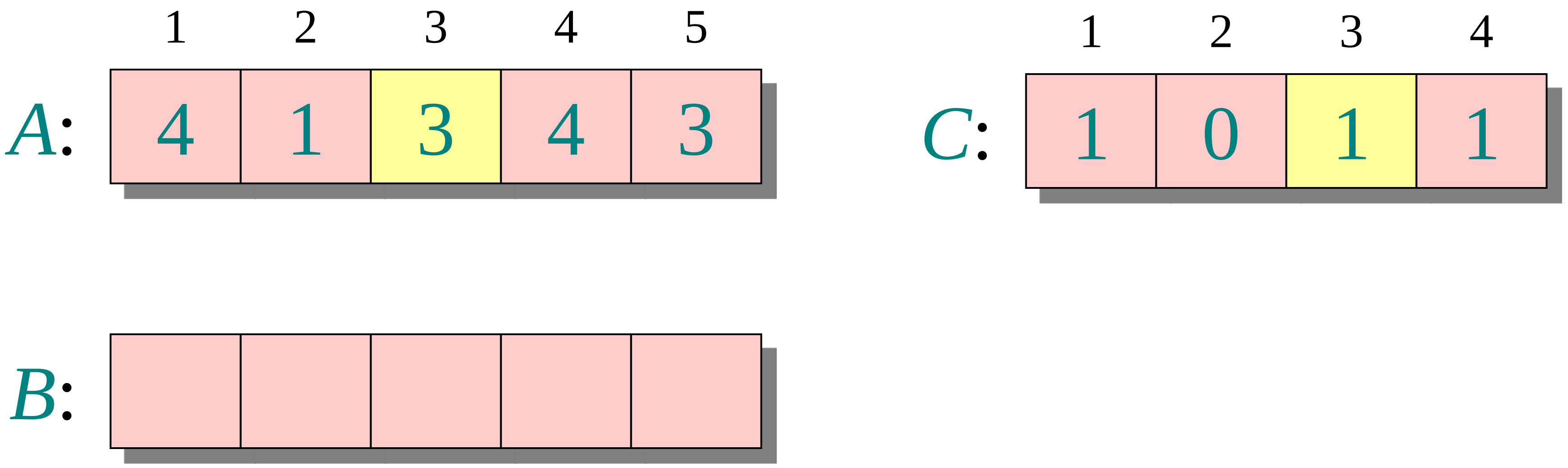
Loop 2



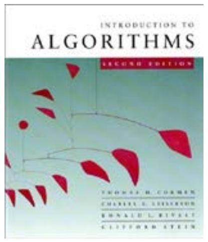
```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$    $\triangleright$ $C[i] = |\{\text{key} = i\}|$
```



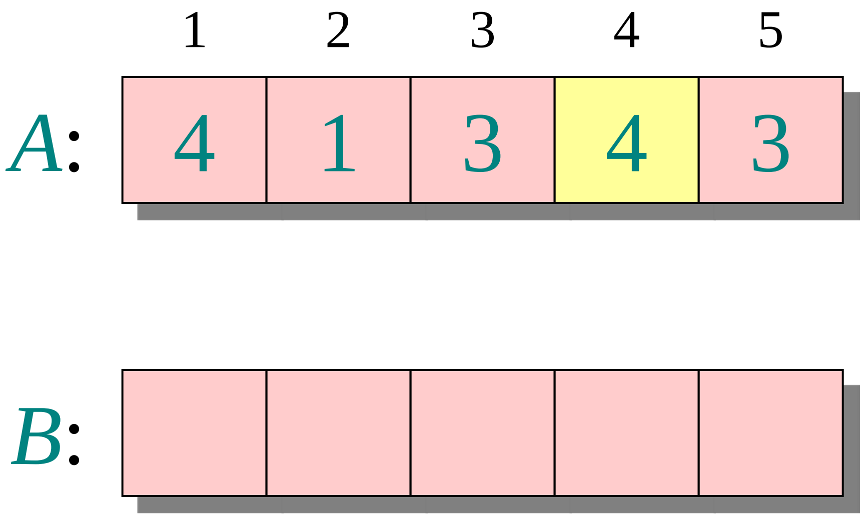
Loop 2



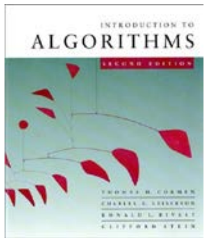
```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$    $\rhd$ $C[i] = |\{\text{key} = i\}|$
```



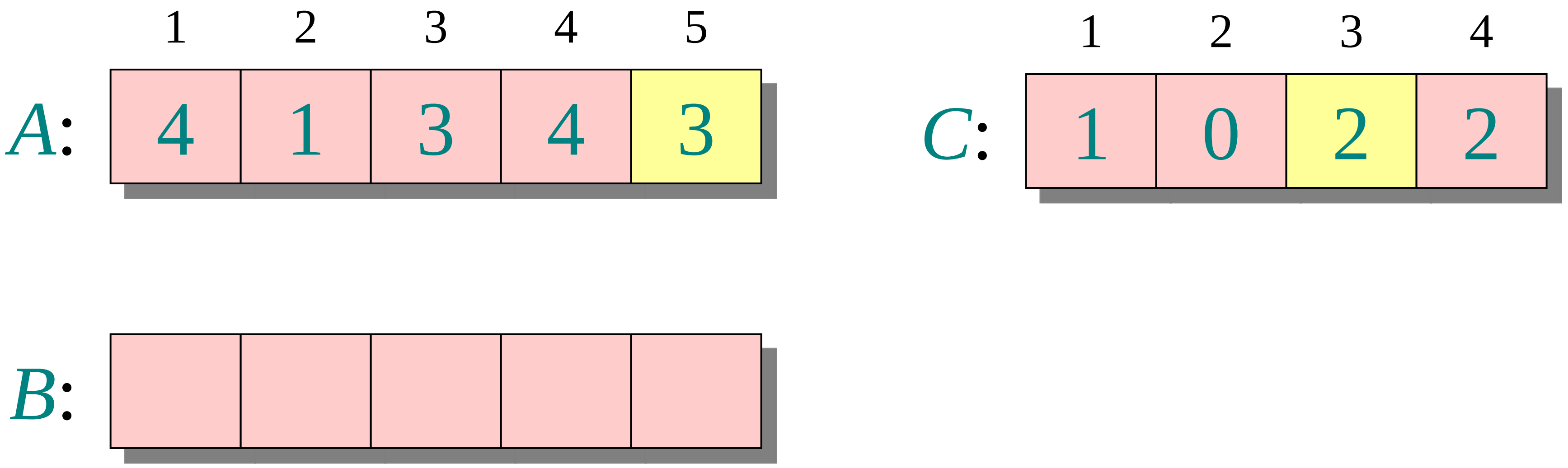
Loop 2



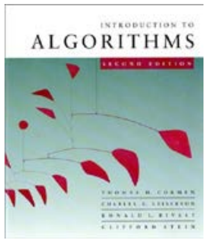
```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$    $\triangleright$ $C[i] = |\{\text{key} = i\}|$
```



Loop 2



```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$    $\rhd$ $C[i] = |\{\text{key} = i\}|$
```

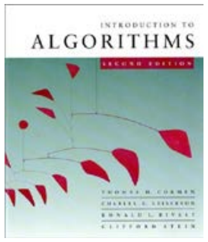


Loop 3

	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	2	3	4
C:	1	0	2	2
C':	1	1	2	2

```
for $i \leftarrow 2$ to $k$
    do $C[i] \leftarrow C[i] + C[i-1]$        ⊳ $C[i] = |\{\text{key} \le i\}|$
```

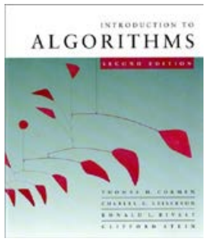


Loop 3

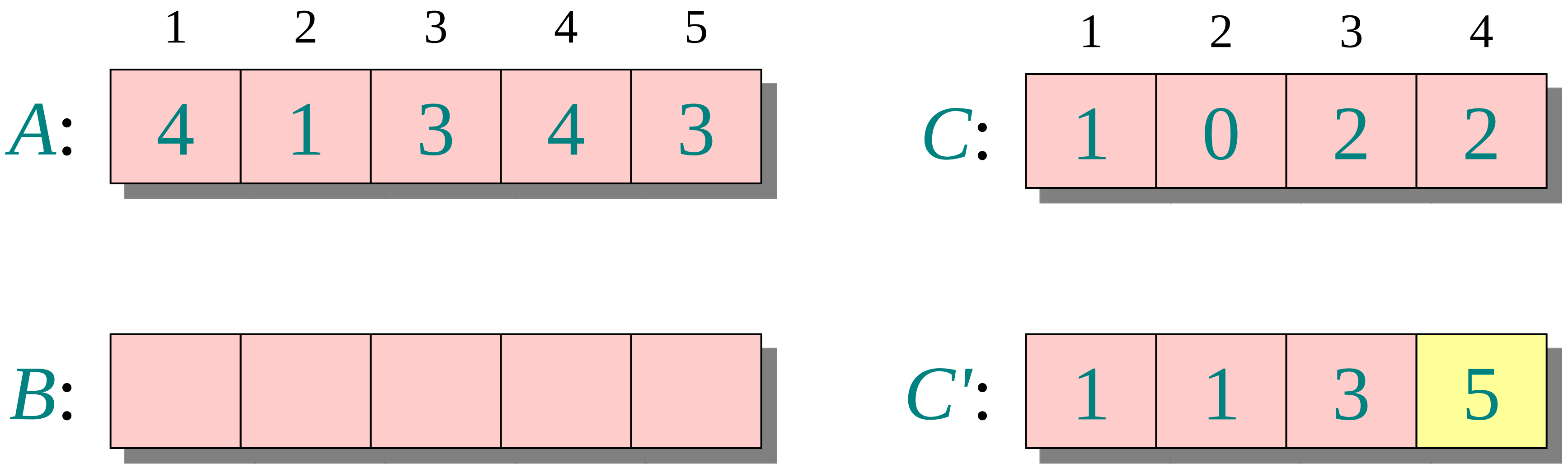
	1	2	3	4	5
A:	4	1	3	4	3
B:					

	1	2	3	4
C:	1	0	2	2
C':	1	1	3	2

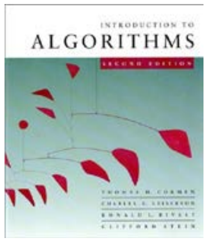
```
for $i \leftarrow 2$ to $k$
    do $C[i] \leftarrow C[i] + C[i–1]$        $\rhd$ $C[i] = |\{\text{key} \le i\}|$
```



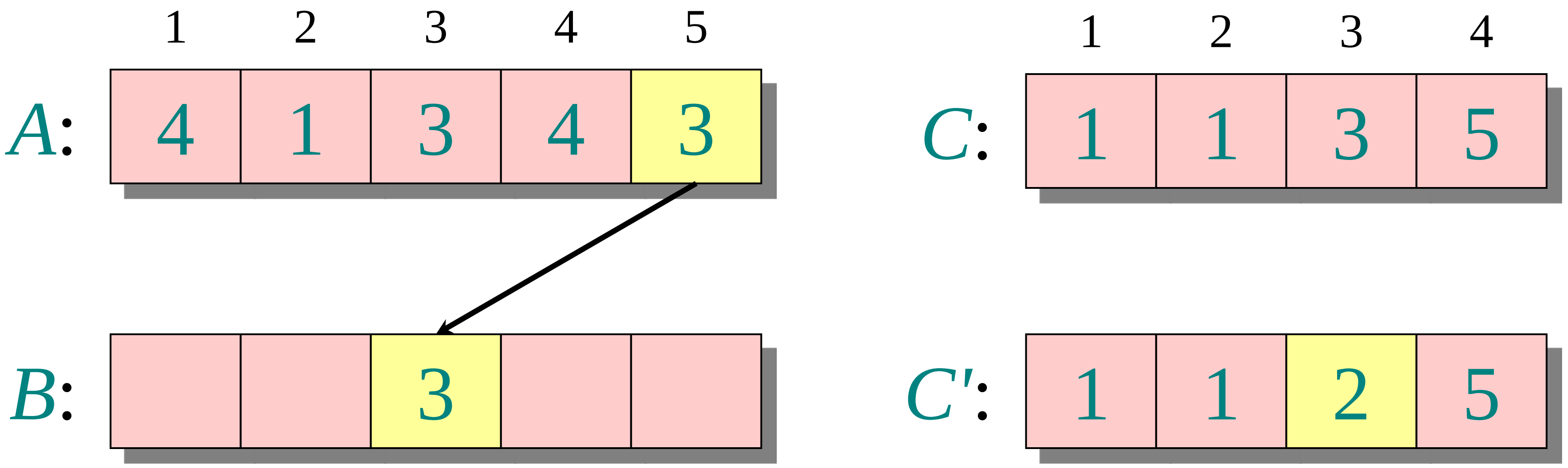
Loop 3



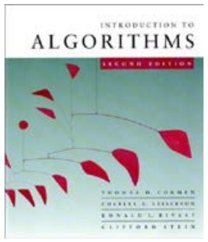
```
for $i \leftarrow 2$ to $k$
    do $C[i] \leftarrow C[i] + C[i–1]$        ⊳ $C[i] = |\{\text{key} \le i\}|$
```



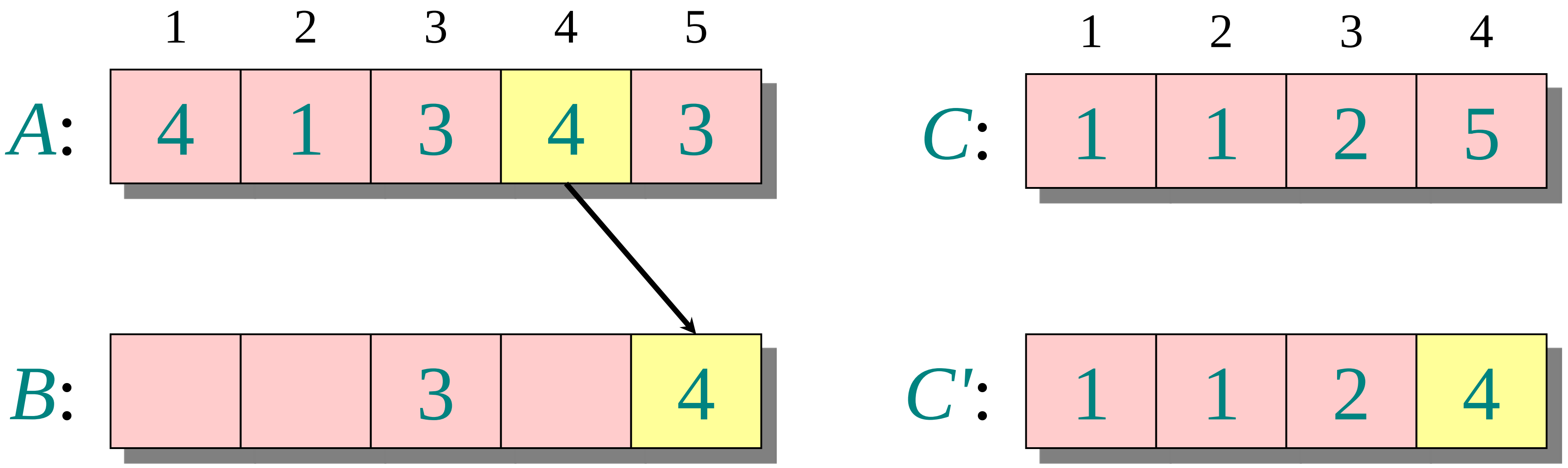
Loop 4



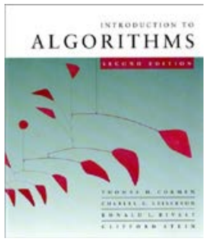
```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```



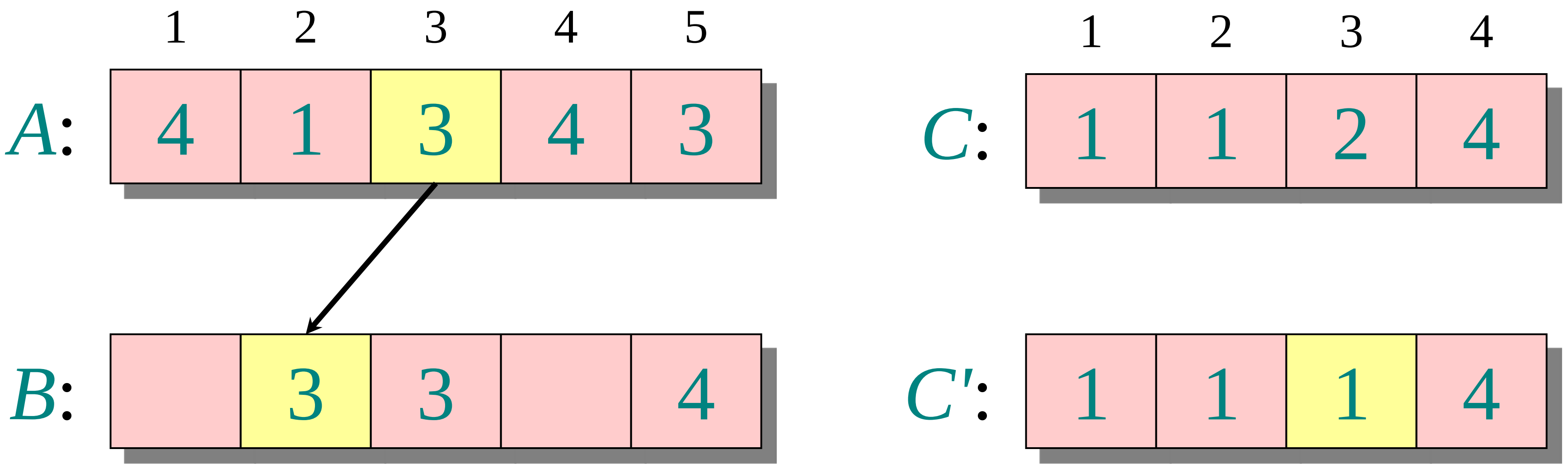
Loop 4



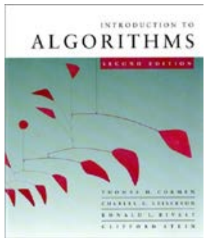
```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```



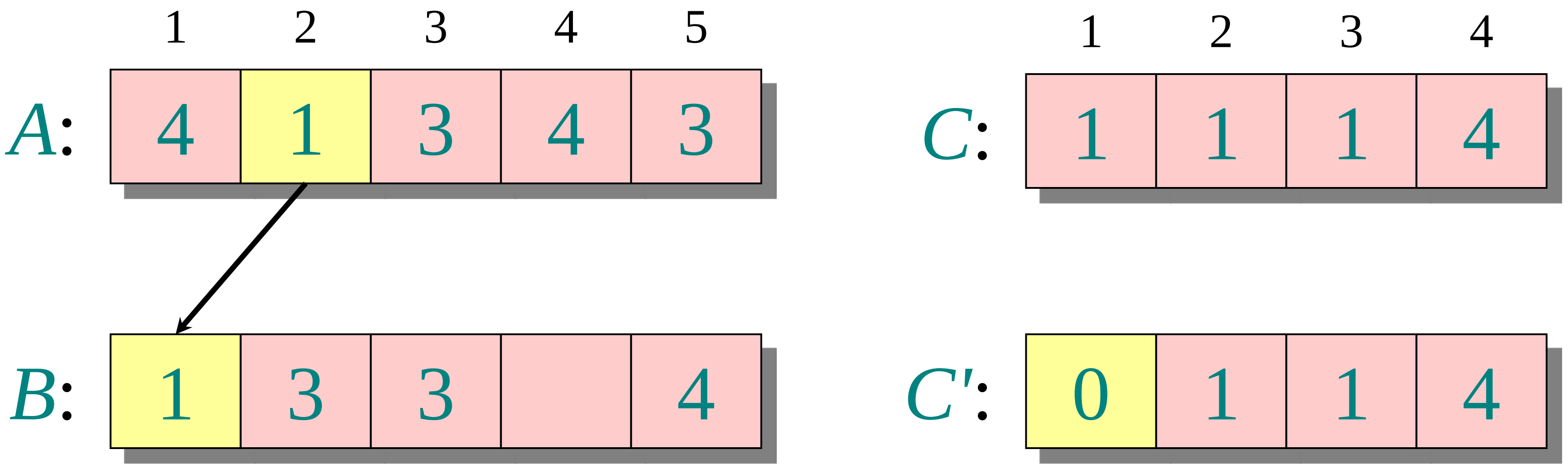
Loop 4



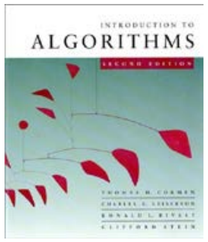
```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```



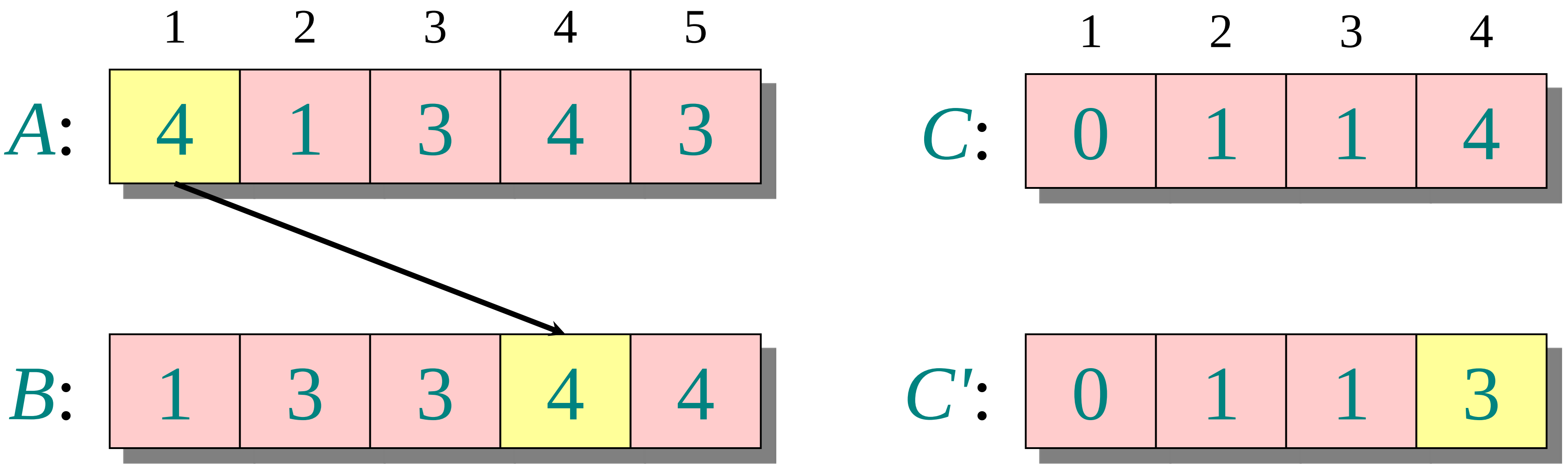
Loop 4



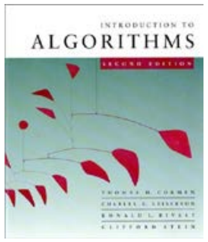
```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```



Loop 4



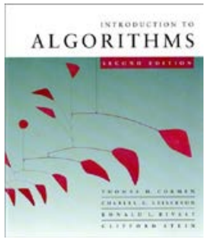
```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```



Analysis

```
Θ(k)        for $i \leftarrow 1$ to $k$
                do $C[i] \leftarrow 0$
Θ(n)        for $j \leftarrow 1$ to $n$
                do $C[A[j]] \leftarrow C[A[j]] + 1$
Θ(k)        for $i \leftarrow 2$ to $k$
                do $C[i] \leftarrow C[i] + C[i-1]$
Θ(n)        for $j \leftarrow n$ downto $1$
                do $B[C[A[j]]] \leftarrow A[j]$
                   $C[A[j]] \leftarrow C[A[j]] - 1$
```

$\Theta(n + k)$



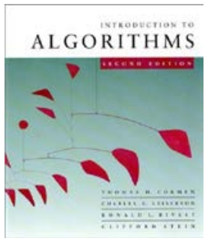
Running time

If $k = O(n)$, then counting sort takes $\Theta(n)$ time.

- But, sorting takes $\Omega(n \lg n)$ time!
- Where's the fallacy?

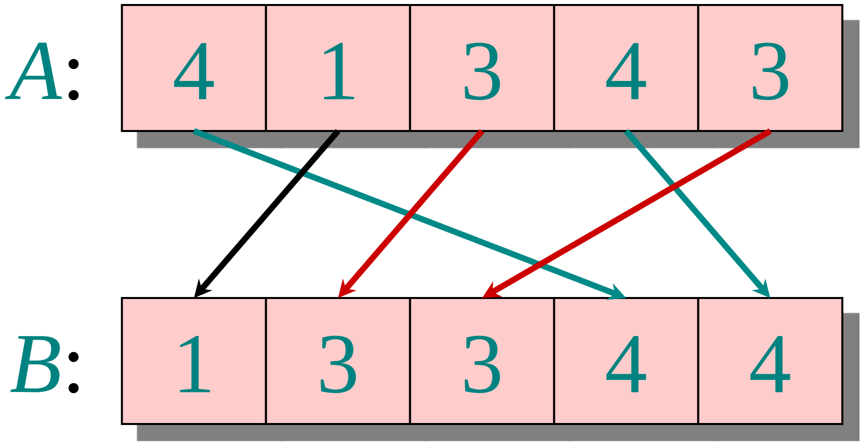
Answer:

- ***Comparison sorting*** takes $\Omega(n \lg n)$ time.
- Counting sort is not a ***comparison sort***.
- In fact, not a single comparison between elements occurs!

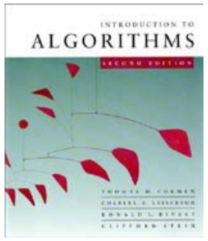


Stable sorting

Counting sort is a ***stable*** sort: it preserves the input order among equal elements.

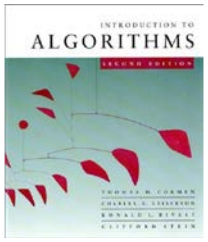


Exercise: What other sorts have this property?

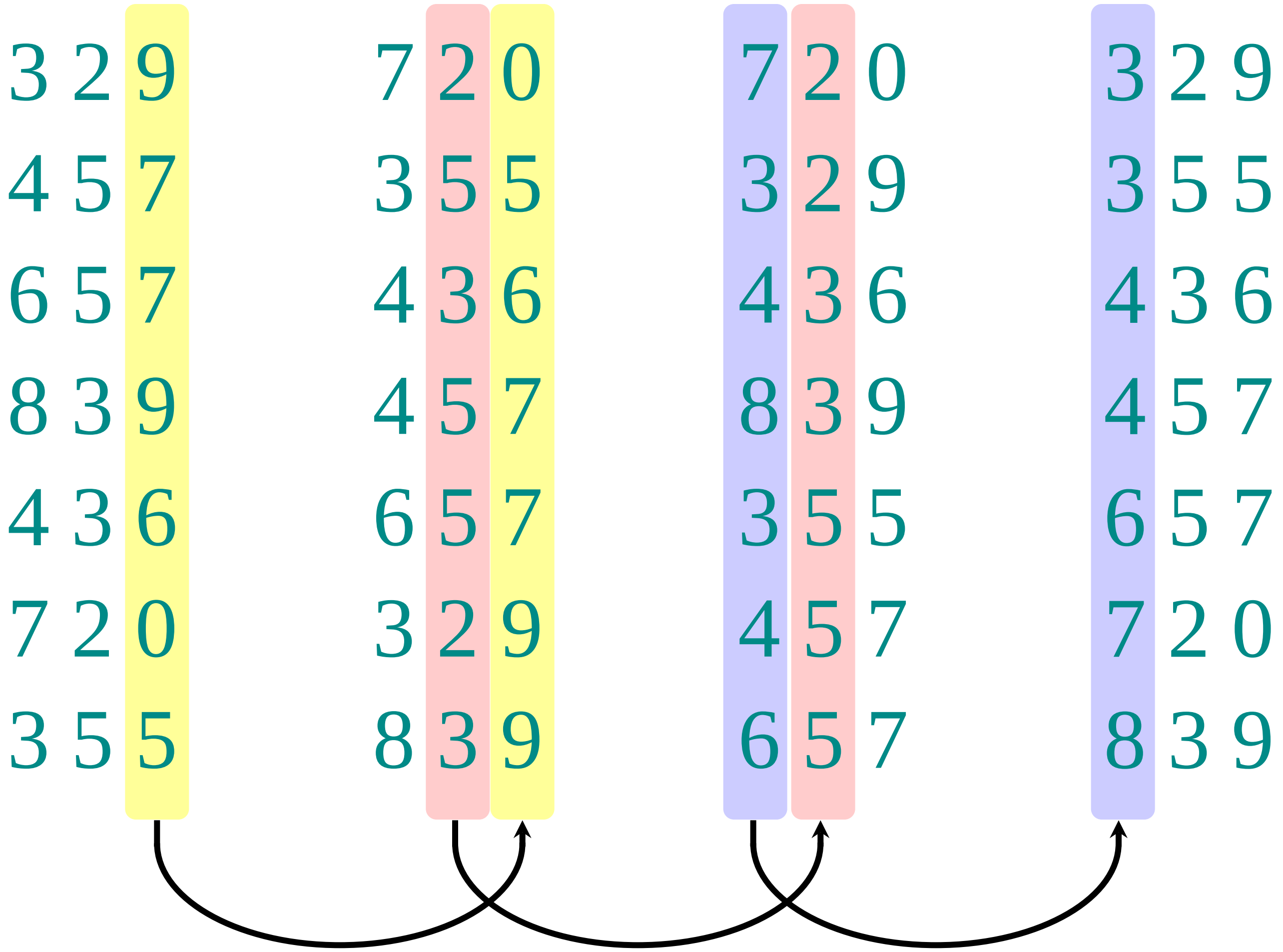


Radix sort

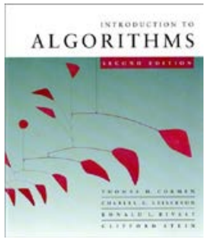
- ***Origin***: Herman Hollerith's card-sorting machine for the 1890 U.S. Census. (See Appendix ⓘ.)
- Digit-by-digit sort.
- Hollerith's original (bad) idea: sort on most-significant digit first.
- Good idea: Sort on ***least-significant digit first*** with auxiliary ***stable*** sort.



Operation of radix sort



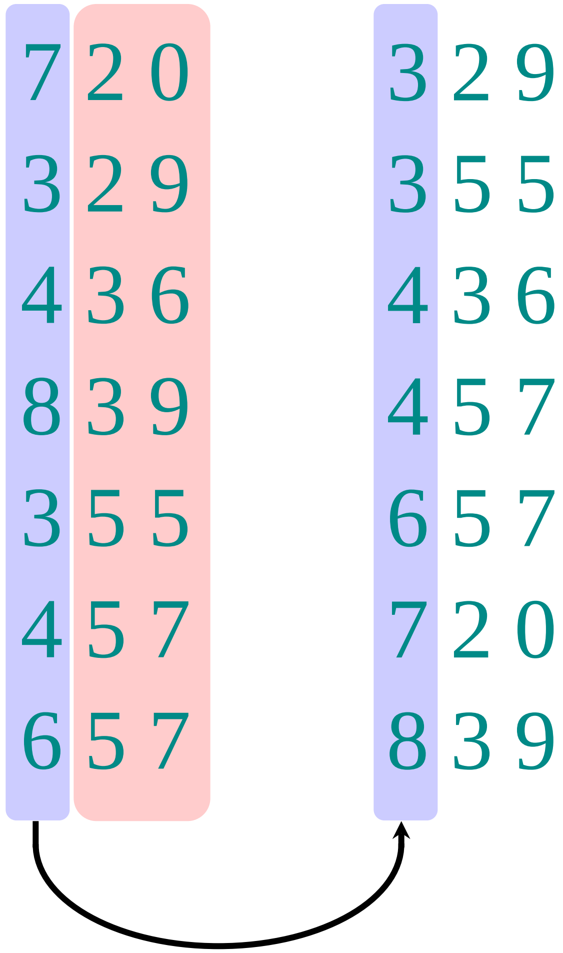
3 2 9	7 2 0	7 2 0	3 2 9
4 5 7	3 5 5	3 2 9	3 5 5
6 5 7	4 3 6	4 3 6	4 3 6
8 3 9	4 5 7	8 3 9	4 5 7
4 3 6	6 5 7	3 5 5	6 5 7
7 2 0	3 2 9	4 5 7	7 2 0
3 5 5	8 3 9	6 5 7	8 3 9

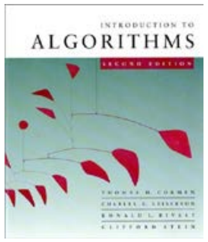


Correctness of radix sort

Induction on digit position

- Assume that the numbers are sorted by their low-order $t-1$ digits.
- Sort on digit t

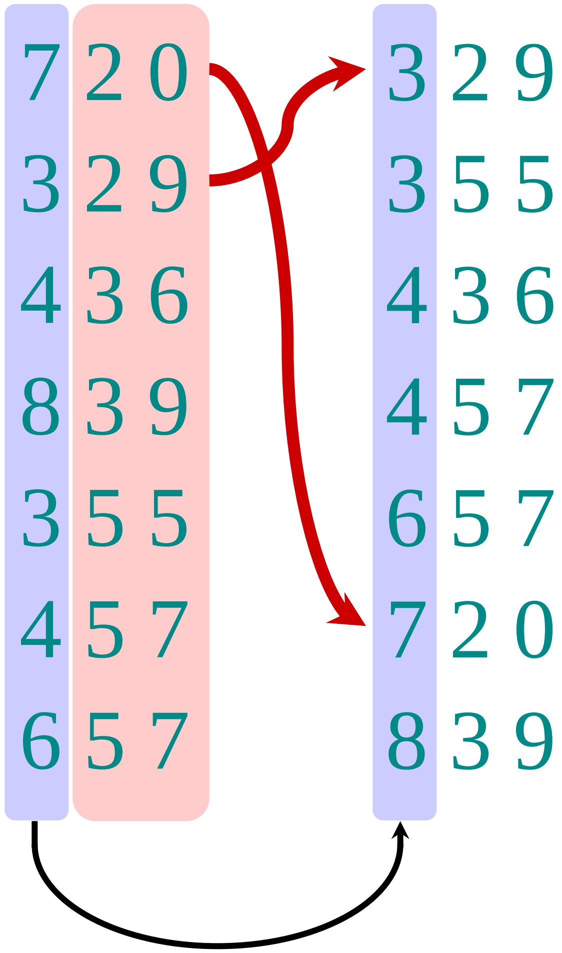


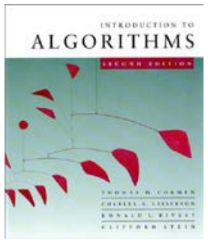


Correctness of radix sort

Induction on digit position

- Assume that the numbers are sorted by their low-order $t - 1$ digits.
- Sort on digit t
  - Two numbers that differ in digit t are correctly sorted.

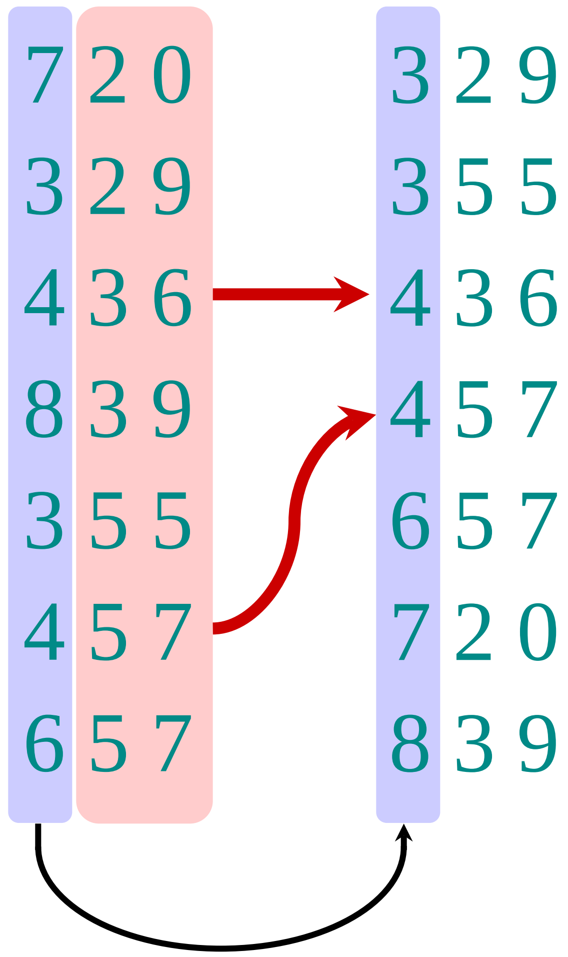


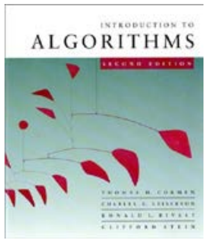


Correctness of radix sort

Induction on digit position

- Assume that the numbers are sorted by their low-order $t - 1$ digits.
- Sort on digit t
  - Two numbers that differ in digit t are correctly sorted.
  - Two numbers equal in digit t are put in the same order as the input ⇒ correct order.

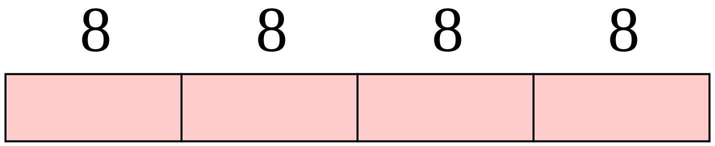




Analysis of radix sort

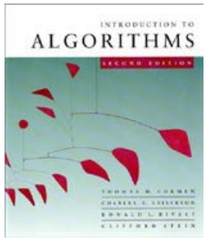
- Assume counting sort is the auxiliary stable sort.
- Sort n computer words of b bits each.
- Each word can be viewed as having b/r base-2^r digits.

Example: 32-bit word



$r = 8 \Rightarrow b/r = 4$ passes of counting sort on base-2^8 digits; or $r = 16 \Rightarrow b/r = 2$ passes of counting sort on base-2^{16} digits.

How many passes should we make?



Analysis (continued)

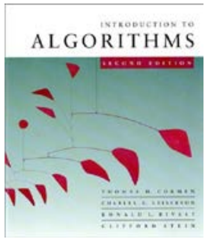
Recall: Counting sort takes $\Theta(n + k)$ time to sort n numbers in the range from 0 to $k - 1$.

If each b-bit word is broken into r-bit pieces, each pass of counting sort takes $\Theta(n + 2^r)$ time. Since there are b/r passes, we have

$$T(n,b) = \Theta\left(\frac{b}{r}\left(n + 2^r\right)\right).$$

Choose r to minimize $T(n, b)$:

- Increasing r means fewer passes, but as $r \gg \lg n$, the time grows exponentially.



Choosing *r*

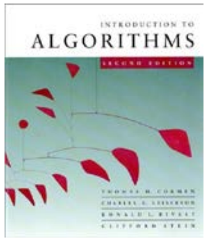
$$T(n,b) = \Theta\left(\frac{b}{r}\left(n + 2^r\right)\right)$$

Minimize $T(n, b)$ by differentiating and setting to 0.

Or, just observe that we don't want $2^r \gg n$, and there's no harm asymptotically in choosing r as large as possible subject to this constraint.

Choosing $r = \lg n$ implies $T(n, b) = \Theta(bn/\lg n)$.

- For numbers in the range from 0 to $n^d - 1$, we have $b = d \lg n \Rightarrow$ radix sort runs in $\Theta(d\,n)$ time.



Conclusions

In practice, radix sort is fast for large inputs, as well as simple to code and maintain.

Example (32-bit numbers):

- At most 3 passes when sorting ≥ 2000 numbers.
- Merge sort and quicksort do at least $\lceil \lg 2000 \rceil = 11$ passes.

Downside: Unlike quicksort, radix sort displays little locality of reference, and thus a well-tuned quicksort fares better on modern processors, which feature steep memory hierarchies.