Engineering Electromagnetics

Lecture 6

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by

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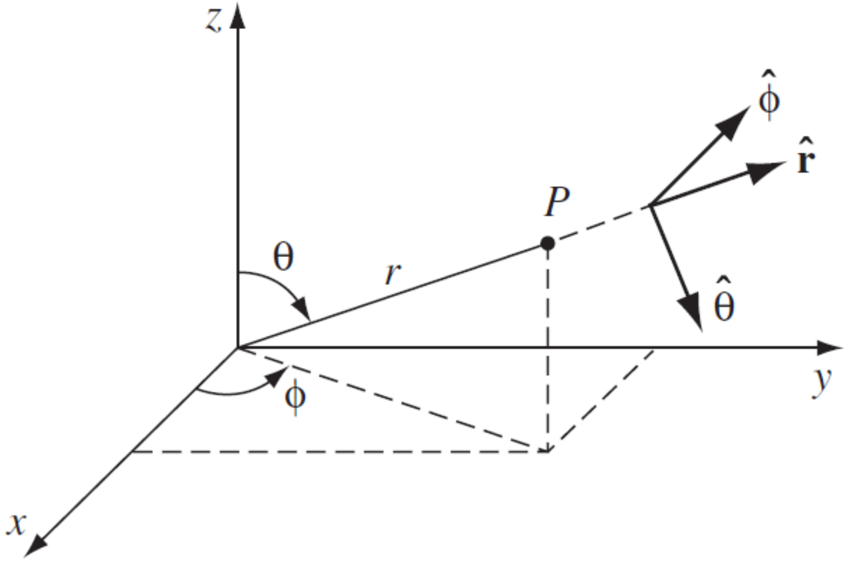
Spherical Coordinates



$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

$\hat{r}.\hat{r} = \hat{\theta}.\hat{\theta} = \hat{\varphi}.\hat{\varphi} =$

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$\hat{r} x \hat{r} = \hat{\theta}\, x\hat{\theta} = \hat{\varphi} x \hat{\varphi} =$

$\hat{r} x \hat{\theta} =;\ \hat{\theta} x \hat{\varphi} =;\ \hat{\varphi} x\ \hat{r} =$

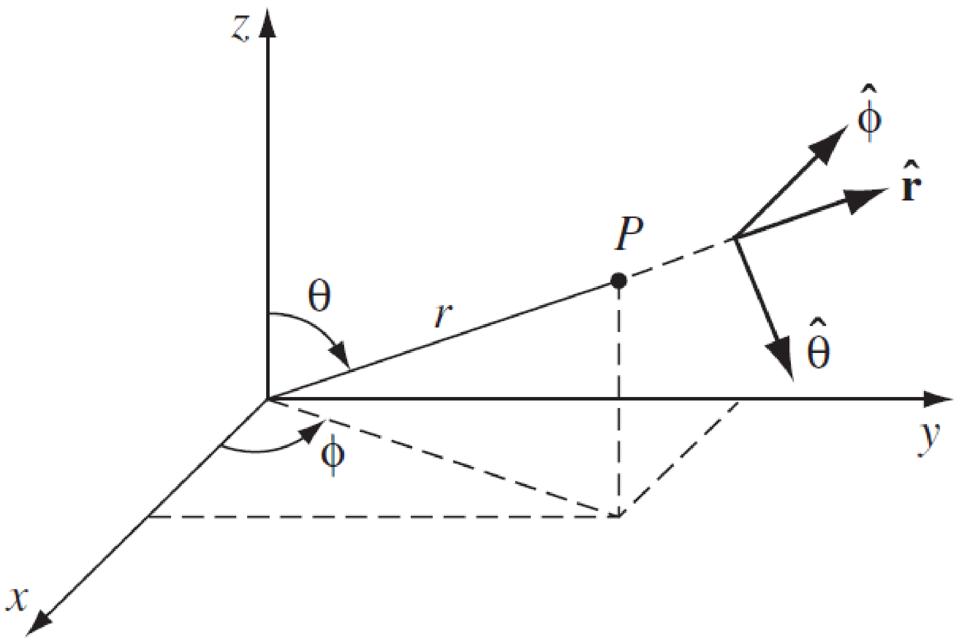
$\hat{\varphi} x\ \hat{\theta} =$

- three unit vectors: $\hat{r}, \hat{\theta}, \hat{\varphi}$ → pointing in the direction of increase of the corresponding coordinates.
- They constitute an orthogonal (mutually perpendicular) basis set
- any vector **A** can be expressed in terms of them, in the usual way:

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left[\frac{z}{r}\right]$$
$$\phi = \tan^{-1}\left[\frac{y}{x}\right]$$

$$\boxed{\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}}$$

Cartesian→ Spherical Coordinates



Cartesian → Spherical

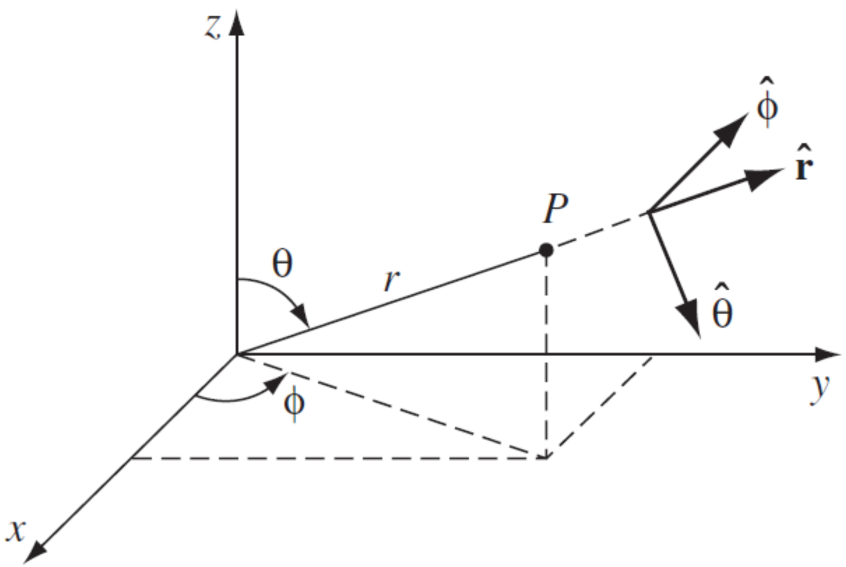
$$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}\end{aligned}$$

$$\begin{bmatrix}\hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}}\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}}\end{bmatrix}$$

For any vector **A**

$$\begin{bmatrix}A_r \\ A_\theta \\ A_\phi\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}A_x \\ A_y \\ A_z\end{bmatrix}$$

Spherical → Cartesian Coordinates



From $A = A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}$ to $A = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$$A_x = \mathrm{A}.\,\hat{x} = (A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}).\,\hat{x} = A_r\hat{r}.\hat{x} + A_\theta\hat{\theta}.\hat{x} + A_\varphi\hat{\varphi}.\,\hat{x}$$

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} &= \sin\theta\cos\phi, & \hat{\mathbf{r}}\cdot\hat{\mathbf{y}} &= \sin\theta\sin\phi, & \hat{\mathbf{r}}\cdot\hat{\mathbf{z}} &= \cos\theta \\
\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{x}} &= \cos\theta\cos\phi, & \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{y}} &= \cos\theta\sin\phi, & \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{z}} &= -\sin\theta \\
\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{x}} &= -\sin\phi, & \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{y}} &= \cos\phi, & \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} &= 0
\end{aligned}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Q: limits?
φ →
θ →?
r → ?

Problem -1

Two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are given at a point $P(r, \theta, \phi)$ in space as

$$\vec{\mathbf{A}} = 10\vec{\mathbf{a}}_r + 30\vec{\mathbf{a}}_\theta - 10\vec{\mathbf{a}}_\phi \quad \text{and} \quad \vec{\mathbf{B}} = -3\vec{\mathbf{a}}_r - 10\vec{\mathbf{a}}_\theta + 20\vec{\mathbf{a}}_\phi$$

Determine (a) $2\vec{\mathbf{A}} - 5\vec{\mathbf{B}}$, (b) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$, (c) $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$, (d) the scalar component of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$, (e) the vector projection of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$, and (f) a unit vector perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Solution Both vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are given at the same point P, so the rules of vector operations can be applied directly in the spherical coordinate system.

a) $2\vec{\mathbf{A}} - 5\vec{\mathbf{B}} = (20 + 15)\vec{\mathbf{a}}_r + (60 + 50)\vec{\mathbf{a}}_\theta + (-20 - 100)\vec{\mathbf{a}}_\phi$
$= 35\vec{\mathbf{a}}_r + 110\vec{\mathbf{a}}_\theta - 120\vec{\mathbf{a}}_\phi$

b) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = 10(-3) + 30(-10) + (-10)20 = -530$

c) $$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \vec{\mathbf{a}}_r & \vec{\mathbf{a}}_\theta & \vec{\mathbf{a}}_\phi \\ 10 & 30 & -10 \\ -3 & -10 & 20 \end{vmatrix} = 500\vec{\mathbf{a}}_r - 170\vec{\mathbf{a}}_\theta - 10\vec{\mathbf{a}}_\phi$$

d) The magnitude of $\vec{\mathbf{B}}$: $B = [(-3)^2 + (-10)^2 + (20)^2]^{1/2} = 22.561$

The scalar projection of $\vec{\mathbf{A}}$ onto $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{a}}_B = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{B} = \frac{-530}{22.561} = -23.492$$

Solution -1

e) The vector projection of $\vec{\mathbf{A}}$ onto $\vec{\mathbf{B}}$ is

$$(\vec{\mathbf{A}} \cdot \vec{\mathbf{a}}_B)\vec{\mathbf{a}}_B = \frac{(\vec{\mathbf{A}} \cdot \vec{\mathbf{a}}_B)\vec{\mathbf{B}}}{B} = \frac{-23.492}{22.561}[-3\vec{\mathbf{a}}_r - 10\vec{\mathbf{a}}_\theta + 20\vec{\mathbf{a}}_\phi]$$

$$= 3.123\vec{\mathbf{a}}_r + 10.413\vec{\mathbf{a}}_\theta - 20.825\vec{\mathbf{a}}_\phi$$

f) There are two unit vectors normal to $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. One of the unit vectors is

$$\vec{\mathbf{a}}_{n1} = \frac{\vec{\mathbf{A}} \times \vec{\mathbf{B}}}{|\vec{\mathbf{A}} \times \vec{\mathbf{B}}|} = \frac{500\vec{\mathbf{a}}_r - 170\vec{\mathbf{a}}_\theta - 10\vec{\mathbf{a}}_\phi}{[500^2 + 170 + 10^2]^{1/2}}$$

$$= 0.947\vec{\mathbf{a}}_r - 0.322\vec{\mathbf{a}}_\theta - 0.019\vec{\mathbf{a}}_\phi$$

The other unit vector is

$$\vec{\mathbf{a}}_{n2} = -\vec{\mathbf{a}}_{n1} = -0.947\vec{\mathbf{a}}_r + 0.322\vec{\mathbf{a}}_\theta + 0.019\vec{\mathbf{a}}_\phi$$

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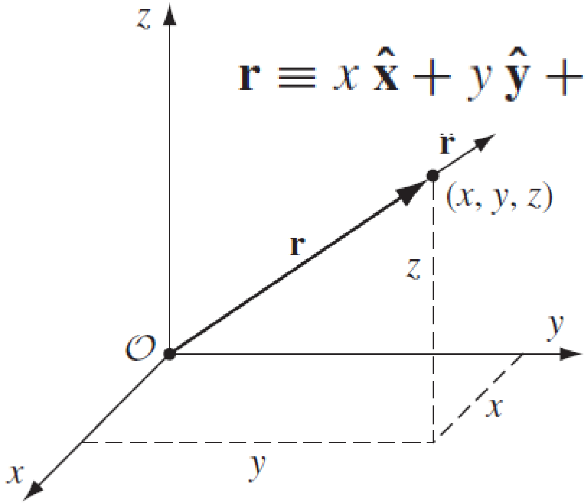
Line, Surface, and Volume elements

- In electrodynamics, we encounter several different kinds of integrals, among which the most important are line (or path) integrals, surface integrals (or flux), and volume integrals.

Cartesian



$$\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}.$$

$$r = \sqrt{x^2 + y^2 + z^2},$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

The **infinitesimal displacement vector**, from *(x, y, z)* to *(x + dx, y + dy, z + dz)*

$$\boxed{d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}}$$

Differential volume element

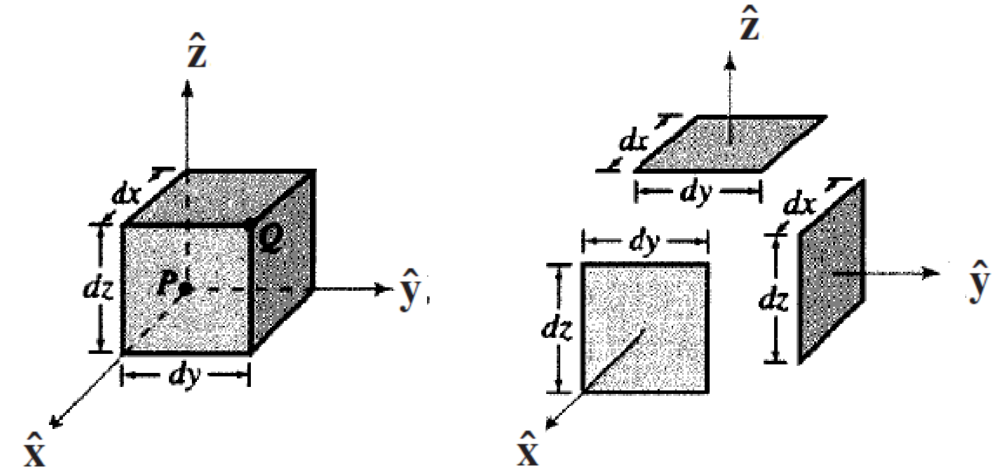
$$dv = dx\,dy\,dz$$

This volume is surrounded by six differential surfaces each expressed in the direction of unit vectors

$$\overrightarrow{ds}_x = dy\,dz\,\hat{\mathbf{x}}$$

$$\overrightarrow{ds}_y = dx\,dz\,\hat{\mathbf{y}}$$

$$\overrightarrow{ds}_z = dx\,dy\,\hat{\mathbf{z}}$$



a) Differential volume   b) Exploded view

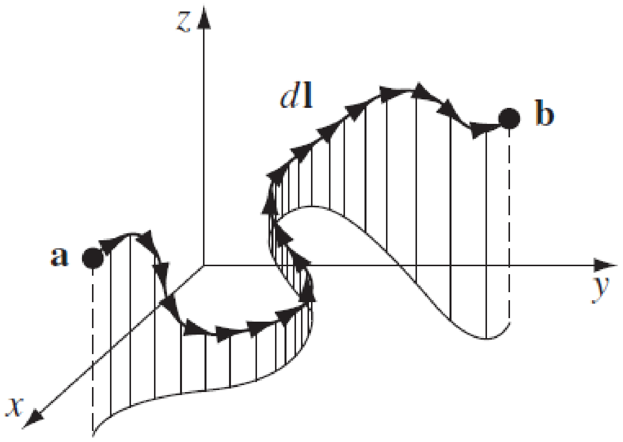
Line Integrals

(a) **Line Integrals.** A line integral is an expression of the form

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l},$$

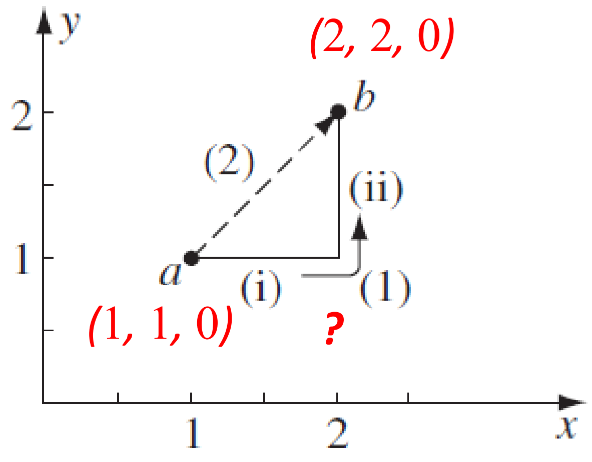
where $\mathbf{v}$ is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point $\mathbf{a}$ to point $\mathbf{b}$. If the path in question forms a closed loop (that is, if $\mathbf{b} = \mathbf{a}$), I shall put a circle on the integral sign:

$$\oint \mathbf{v} \cdot d\mathbf{l}.$$



At each point on the path, we take the dot product of $\mathbf{v}$ (evaluated at that point) with the displacement $d\mathbf{l}$ to the next point on the path. To a physicist, the most familiar example of a line integral is the work done by a force $\mathbf{F}$: $W = \int \mathbf{F} \cdot d\mathbf{l}$.

Problem-2



Example 1.6. Calculate the line integral of the function $\mathbf{v} = y^2\,\hat{\mathbf{x}} + 2x(y+1)\,\hat{\mathbf{y}}$ from the point $\mathbf{a} = (1, 1, 0)$ to the point $\mathbf{b} = (2, 2, 0)$, along the paths (1) and (2) in Fig. 1.21. What is $\oint \mathbf{v} \cdot d\mathbf{l}$ for the loop that goes from $\mathbf{a}$ to $\mathbf{b}$ along (1) and returns to $\mathbf{a}$ along (2)?

Solution

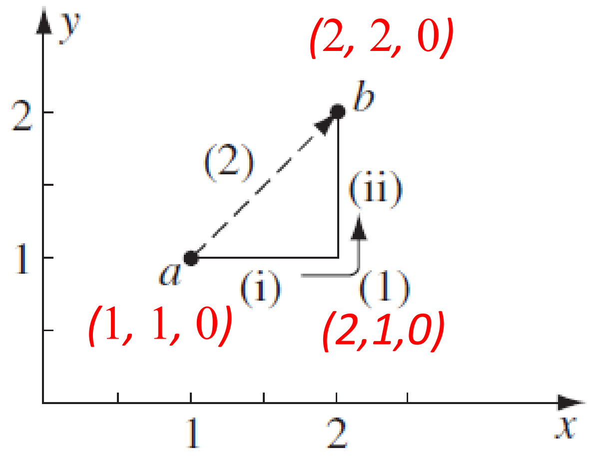
As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$. Path (1) consists of two parts. Along the "horizontal" segment, $dy = dz = 0$, so

(i) $d\mathbf{l} = dx\,\hat{\mathbf{x}},\ y = 1,\ \mathbf{v}\cdot d\mathbf{l} = y^2\,dx = dx$, so $\int \mathbf{v}\cdot d\mathbf{l} = \int_1^2 dx = 1$.

On the "vertical" stretch, $dx = dz = 0$, so

(ii) $d\mathbf{l} = dy\,\hat{\mathbf{y}},\ x = 2,\ \mathbf{v}\cdot d\mathbf{l} = 2x(y+1)\,dy = 4(y+1)\,dy$, so

$$\int \mathbf{v}\cdot d\mathbf{l} = 4\int_1^2 (y+1)\,dy = 10.$$



By path (1), then,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v}\cdot d\mathbf{l} = 1 + 10 = 11.$$

Meanwhile, on path (2) $x = y,\ dx = dy$, and $dz = 0$, so $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dx\,\hat{\mathbf{y}},\ \mathbf{v}\cdot d\mathbf{l} = x^2\,dx + 2x(x+1)\,dx = (3x^2+2x)\,dx$, and

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v}\cdot d\mathbf{l} = \int_1^2 (3x^2+2x)\,dx = (x^3+x^2)\Big|_1^2 = 10.$$

(The strategy here is to get everything in terms of one variable; I could just as well have eliminated x in favor of y.)

For the loop that goes *out* (1) and *back* (2), then,

$$\oint \mathbf{v}\cdot d\mathbf{l} = 11 - 10 = 1.$$

Thank You