1/2/23

ASSIGNMENT - 2

Y. HARITH
CS23I1027
(B2)

(PH1000)

Q1) (i) Given:

$\vec{E} = E\hat{z}$

$\vec{B} = B\hat{x}$

$\vec{u} = \frac{E}{\sqrt{2}B}(\hat{y}+\hat{z})$

To Find:

Trajectory of the Particle

Solution:

Assume, mass of Particle $= m$
charge of Particle $= q$

Let's see the forces. By Lorrentz Force Equation:

$$F = q(\vec{v}\times\vec{B} + \vec{E})$$

At $t=0$, $$F_0 = q\left[\left(\frac{E}{\sqrt{2}B}(\hat{y}+\hat{z}) \times B\hat{x}\right) + E\hat{z}\right]$$

$$\Rightarrow F_0 = q\left[\frac{E}{\sqrt{2}B}(\hat{y}-\hat{z}) + E\hat{z}\right]$$

$\therefore$ Force acts only in yz plane and no force exists in x plane and Hence, no velocity in $\hat{x}$.

$\therefore$ Velocity at any time t, $v(t)$ can be expressed as

$$v(t) = v_y(t)\,\hat{y} + v_z(t)\,\hat{z}$$

$dy = v_y \cdot dt$	$dz = v_z \cdot dt$
$\Rightarrow v_y(t) = \frac{dy}{dt}$	$\Rightarrow v_z(t) = \frac{dz}{dt}$

$$\therefore v(t) = \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

Now, Let's use the equation of Lorentz Law to find Force at any time 't':

$$\vec{F} = q((\vec{V} \times \vec{B}) + \vec{E})$$

$$\Rightarrow \vec{F} = q\left[\left(\left(\frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right) \times B\hat{x}\right) + E\hat{z}\right]$$

$$= q\left[\left(\frac{dz}{dt}B\right)\hat{y} - \left(B\frac{dy}{dt}\right)\hat{z} + E\hat{z}\right]$$

$$= q\left[\left(B \cdot \frac{dz}{dt}\right)\hat{y} + \left(E - B \cdot \frac{dy}{dt}\right)\hat{z}\right]$$

$$\therefore \vec{F} = \left(qB \cdot \frac{dz}{dt}\right)\hat{y} + \left[q\left(E - B\frac{dy}{dt}\right)\right]\hat{z} \quad -(1)$$

Also, We know that $F = ma$ [from Newton's Second Law]

$$\Rightarrow \vec{F} = m\left(\frac{dv(t)}{dt}\right) \qquad \left[v(t) = \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right]$$

$$= m \cdot \frac{d}{dt}\left(\frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right)$$

$$= m\left(\frac{d^2y}{dt^2}\hat{y} + \frac{d^2z}{dt^2}\hat{z}\right)$$

$$\therefore \vec{F} = \left(m \cdot \frac{d^2y}{dt^2}\right)\hat{y} + \left(m\frac{d^2z}{dt^2}\right)\hat{z} \quad -(2)$$

Now (1) = (2), Since both forces must be equal.

Let's Equate both forces component by component.

$$qB \cdot \frac{dz}{dt} = m\frac{d^2y}{dt^2}$$

$$\Rightarrow \frac{d^2y}{dt^2} = \left(\frac{qB}{m}\right)\frac{dz}{dt} \quad (3)$$

$$q\left(E - B\frac{dy}{dt}\right) = m\frac{d^2z}{dt^2}$$

$$\Rightarrow \frac{d^2z}{dt^2} = \frac{q}{m}\left(E - B\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{d^2z}{dt^2} = \left(\frac{qB}{m}\right)\left(\frac{E}{B} - \frac{dy}{dt}\right) \quad -(4)$$

om ④,

$$\Rightarrow \left(\frac{m}{qB}\right)\left(\frac{d^2z}{dt^2}\right) = \frac{E}{B} - \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{E}{B} - \frac{m}{qB}\left(\frac{d^2z}{dt^2}\right)$$

Differentiating both w.r.t time,

$$\Rightarrow \frac{d^2y}{dt^2} = 0 - \frac{d}{dt}\left(\frac{m}{qB}\left(\frac{d^2z}{dt^2}\right)\right)$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{m}{qB}\left(\frac{d^3z}{dt^3}\right) \quad - ⑤$$

Put ⑤ in ③ :

$$\frac{-m}{qB}\left(\frac{d^3z}{dt^3}\right) = \frac{qB}{m}\left(\frac{dz}{dt}\right)$$

$$\Rightarrow \frac{d^2}{dt^2}\left(\frac{dz}{dt}\right) = \frac{-q^2B^2}{m^2}\left(\frac{dz}{dt}\right)$$

Integrating both w.r.t time,

$$\Rightarrow \int \frac{d}{dt}\left(\frac{d^2z}{dt^2}\right) dt = \frac{-q^2B^2}{m^2}\int \frac{dz}{dt}\,dt$$

$$\Rightarrow \frac{d^2z}{dt^2} + C_1' = \frac{-q^2B^2}{m^2}(z + C_1'')$$

[C_1' and C_1'' are constants of Integration]

$$\Rightarrow \frac{d^2z}{dt^2} = \frac{-q^2B^2}{m^2}(z + c)$$

Let $\frac{qB}{m} = \omega$ & $z + C_1 = z'$

$$\Rightarrow \frac{d^2z'}{dt^2} = -\omega^2(z') \quad - ⑥$$

This is an equation of a SHM

$$\Rightarrow \frac{d^2z'}{dt^2} + \omega^2(z') = 0$$

Let $z'(t) = e^{\lambda t}$

$$\Rightarrow \frac{d^2}{dt^2}(e^{\lambda t}) + \omega^2 e^{\lambda t} = 0$$

$$\Rightarrow \lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0$$

$$\Rightarrow (\omega^2 + \lambda^2)\, e^{\lambda t} = 0$$

As $e^{\lambda t} \neq 0$ $\quad [\because z'(t) \neq 0]$

$$\Rightarrow \lambda = i\omega \quad \text{(OR)} \quad \lambda = -i\omega$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$z_1'(t) = C_1 e^{-it\omega} \qquad z_2'(t) = C_2 e^{+it\omega}$$

$\therefore$ General solution : $z'(t) = z_1'(t) + z_2'(t)$

[Euler's Identity: $e^{i\theta} = \cos\theta + i\sin\theta$]

$$\Rightarrow z'(t) = C_1 e^{-it\omega} + C_2 e^{it\omega}$$

$$= C_1[\cos(\omega t) - i\sin(\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)]$$

$$\Rightarrow z'(t) = (C_1 + C_2)\cos(\omega t) - i(C_1 - C_2)\sin(\omega t) \qquad [C_1 \,\&\, C_2 \in \mathbb{C}]$$

$$\therefore z'(t) = C_1'\cos(\omega t) - C_2'\sin(\omega t)$$

$$z'(t) = z(t) + c$$

$$\boxed{\therefore z(t) = C_1\cos(\omega t) - C_2\sin(\omega t) + C_3} \quad - (7) \qquad [C_3 \in C]$$

Now Putting (6) in (4) :

[
$z(t) = C_1\cos(\omega t) - C_2\sin(\omega t) + C_3$

$z'(t) = C_1\omega(-\sin(\omega t)) - C_2\omega\cos(\omega t)$

$z''(t) = -C_1\omega^2\cos(\omega t) + C_2\omega^2\sin(\omega t)$

$= -\omega^2(C_1\cos(\omega t) - C_2\sin(\omega t))$
]

$$\frac{d^2 z}{dt^2} = (\omega)\left(\frac{E}{B} - \frac{dy}{dt}\right)$$

$$\Rightarrow -\omega^2(z(t) - C_3) = \omega\left(\frac{E}{B} - \frac{dy}{dt}\right)$$

$$\Rightarrow \omega(C_3 - z(t)) = \frac{E}{B} - \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{E}{B} + \omega(z(t) - C_3)$$

Integrating both sides w.r.t time,

$$\Rightarrow \int dy = \int \left[\frac{E}{B} + \omega\left(z(t) - C_3\right)\right] dt$$

[C_4'' is the constant of Integration]

$$\Rightarrow y = \left(\frac{E}{B}\right)t + \omega \int z(t)\,dt - \omega C_3 t + C_4''$$

$$\Rightarrow y(t) = \omega \int \left(C_1 \cos(\omega t) - C_2 \sin(\omega t) + C_3\right) dt - \omega C_3 t + C_4'' + \left(\frac{E}{B}\right)t$$

$$= \omega\left(\frac{C_1 \sin(\omega t)}{\omega} + \frac{C_2 \cos(\omega t)}{\omega} + C_3 t\right) + C_4' - \omega C_3 t + C_4'' + \left(\frac{E}{B}\right)t$$

$$= C_1 \sin(\omega t) + C_2 \cos(\omega t) + \cancel{\omega C_3 t} + \left(\frac{E}{B}\right)t - \cancel{\omega C_3 t} + C_4$$

$$\boxed{\therefore y(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t) + \left(\frac{E}{B}\right)t + C_4} \quad -(8) \qquad [C_4 \in C]$$

Now, As we obtained $y(t)$ & $z(t)$ in terms of arbitary constant.

As the particle started from Origin,

$y(t) = 0$ @ $t = 0$ &
$z(t) = 0$ @ $t = 0$

$$y(0) = C_1 \sin(0) + C_2 \cos(0) + \frac{E}{B}(0) + C_4$$

$$\Rightarrow 0 = C_2 + C_4$$

$$\boxed{\therefore C_2 = -C_4} \quad -(11)$$

$$z(0) = C_1(\cos(0)) - C_2 \sin(0) + C_3$$

$$\Rightarrow 0 = C_1 + C_3$$

$$\boxed{\therefore C_1 = -C_3} \quad -(12)$$

Let's differentiate $y(t)$ & $z(t)$

$$\boxed{y'(t) = C_1 \omega \cos(\omega t) - C_2 \omega \sin(\omega t) + \frac{E}{B}} \quad -(9)$$

$y'(t)$ Indicates velocity of the particle (y-component) at any time 't'.

$$\boxed{z'(t) = -C_1 \omega \sin(\omega t) - C_2 \omega \cos(\omega t)} \quad - (10)$$

$z'(t)$ Indicates the z-component velocity of the particle at any time t

We know $y'(0) = z'(0) = \frac{E}{\sqrt{2}B}$

$$y'(0) = C_1 \omega \cos(0) - C_2 \omega \sin(0) + \frac{E}{B}$$

$$\Rightarrow \frac{E}{\sqrt{2}B} = C_1 \omega + \frac{E}{B}$$

$$\Rightarrow C_1 \omega = \frac{E}{B}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$\boxed{\therefore C_1 = \frac{E}{B\omega}\left(\frac{1}{\sqrt{2}} - 1\right)} \quad - (13)$$

$$z'(0) = -C_1 \omega \sin(0) - C_2 \omega \cos(0)$$

$$\Rightarrow \frac{E}{\sqrt{2}B} = -C_2 \omega$$

$$\boxed{\therefore C_2 = \frac{-E}{\sqrt{2}B\omega}} \quad - (14)$$

Substituting (11), (12), (13) & (14) in (7), (8), (9), (10)

$$\Rightarrow y(t) = \frac{E}{B\omega}\left(\frac{1}{\sqrt{2}} - 1\right)\sin(\omega t) + \frac{-E}{\sqrt{2}B\omega}\cos(\omega t) + \left(\frac{E}{B}\right)t + \frac{E}{\sqrt{2}B\omega}$$

$$z(t) = \frac{E}{B\omega}\left(\frac{1}{\sqrt{2}} - 1\right)\cos(\omega t) + \frac{E}{\sqrt{2}B\omega}\sin(\omega t) + \frac{E}{B\omega}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$y'(t) = \frac{E}{B}\left(\frac{1}{\sqrt{2}} - 1\right)\cos(\omega t) + \frac{E}{\sqrt{2}B}\sin(\omega t) + \frac{E}{B}$$

$$z'(t) = \frac{E}{B}\left(1 - \frac{1}{\sqrt{2}}\right)\sin(\omega t) + \frac{E}{\sqrt{2}B}\cos(\omega t)$$

Now Substituting $\omega = \frac{qB}{m}$ into these above 4 equations,

$$\therefore\ y(t) = \frac{Em}{qB^2}\left(\frac{1}{\sqrt{2}}-1\right)\sin\left(\frac{qB}{m}t\right) - \frac{Em}{\sqrt{2}qB^2}\cos\left(\frac{qB}{m}t\right) + \left(\frac{E}{B}\right)t + \frac{Em}{\sqrt{2}qB^2}$$

$$z(t) = \frac{Em}{qB^2}\left(\frac{1}{\sqrt{2}}-1\right)\cos\left(\frac{qB}{m}t\right) + \frac{Em}{\sqrt{2}qB^2}\sin\left(\frac{qB}{m}t\right) + \frac{Em}{qB^2}\left(1-\frac{1}{\sqrt{2}}\right)$$

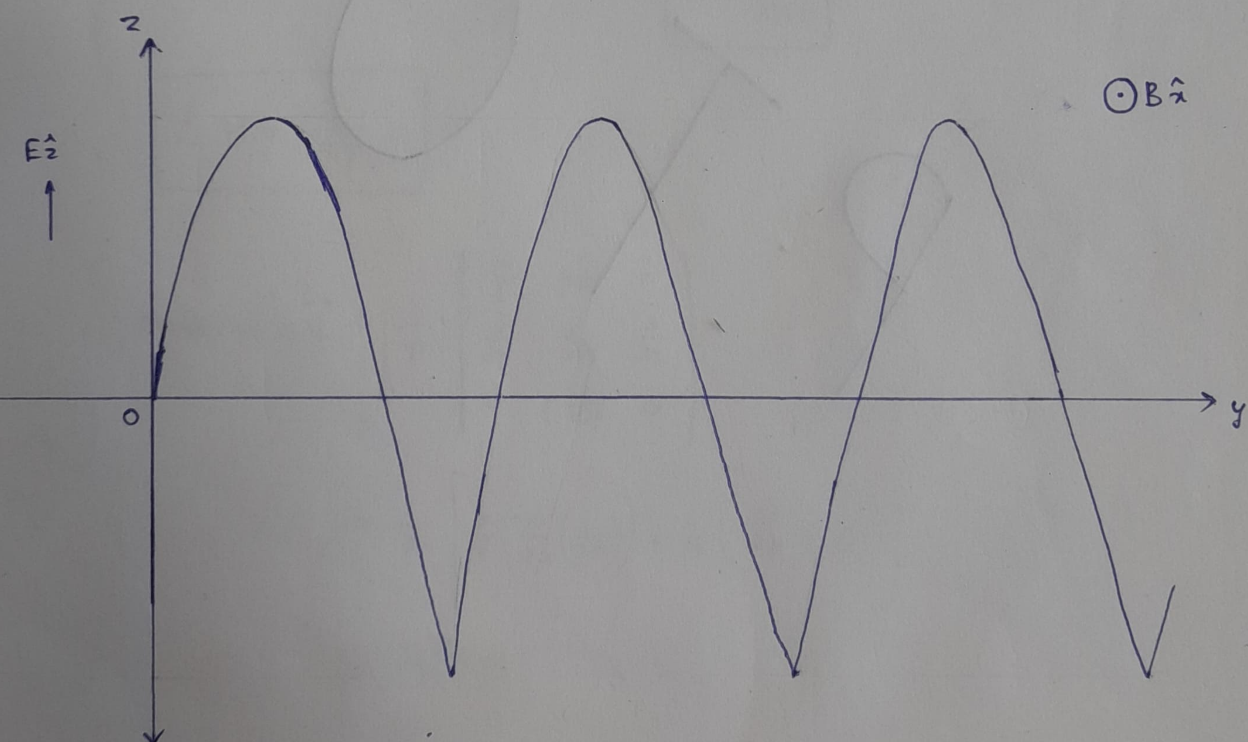
Trajectory of the particle

$$v_y(t) = \frac{E}{B}\left(\frac{1}{\sqrt{2}}-1\right)\cos\left(\frac{qB}{m}t\right) + \frac{E}{\sqrt{2}B}\sin\left(\frac{qB}{m}t\right) + \frac{E}{B}$$

$$v_z(t) = \frac{E}{B}\left(1-\frac{1}{\sqrt{2}}\right)\sin\left(\frac{qB}{m}t\right) + \frac{E}{\sqrt{2}B}\cos\left(\frac{qB}{m}t\right)$$

However, it is not feasible to eliminate the parametre (t) from equations y(t) and z(t) to get a trajectory equation.

The trajectory will look something like this :



The particle's x-coordinate will always be 0, i.e. No motion in x-direction $\because$ No force and Initial velocity in x-direction.

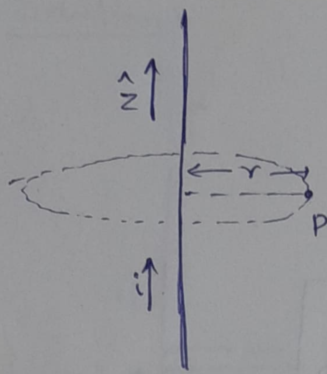
This shape is similar to a Cycloid, but the repeating unit is not exactly a circle, $\therefore$ Trajectory equations don't represent a circle.

1) Given:

$\vec{\nabla} \times \vec{H} = \vec{J}$

$\vec{\nabla} \times \vec{H} = 0$ for Straight Current Carrying ~~conductor~~ Wire

Solution:



Let's Assume an Amperial loop of radius 'r'.

From Ampere's Law,

$$\oint \vec{H} \cdot \vec{dl} = i_{enc}$$

$$\Rightarrow H \cdot 2\pi r = i$$

$$\Rightarrow \boxed{H = \frac{i}{2\pi r}}$$

$$\boxed{\therefore H = \frac{i}{2\pi \rho} \hat{\phi}}$$

Now $\vec{\nabla} \times \vec{H} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \rho \frac{i}{2\pi\rho} & 0 \end{vmatrix}$

$$= \frac{1}{\rho} \left[\hat{\rho}\left(\frac{-\partial}{\partial z}\left(\frac{i}{2\pi}\right)\right) + \hat{z}\left(\frac{\partial}{\partial \rho} \cdot \left(\frac{i}{2\pi}\right)\right) \right]$$

$$= 0$$

As we know,

$$\vec{\nabla} \times \vec{H} = J_{free}.$$

Magnetisation of the wire, $\vec{M} = \chi_m \vec{H}$ [$\chi_m = 0$ if vacuum]

If straight wire is not in vacuum,

$$\boxed{\vec{M} = \frac{\chi_m i}{2\pi\rho} \hat{\phi}}$$

Also, $\underbrace{J}_{\text{Total}} = \underbrace{J_f}_{\text{free}} + \underbrace{J_b}_{\text{bound}}$

$\Rightarrow J = 0 + J_b$

$\Rightarrow J = J_b$

$\underbrace{J_b = \vec{\nabla} \times \vec{M}}$ Volume Bound Current Density

$\therefore J_b = \vec{\nabla} \times \left(\frac{\chi_m i}{2\pi \rho} \hat{\phi} \right)$

$= 0$

$\boxed{\therefore J_b = 0}$

Similar to $\vec{\nabla} \times \vec{H} = 0$

$\Rightarrow \vec{\nabla} \times \chi_m \vec{H} = 0$

$\chi_m (\vec{\nabla} \times \vec{H}) = 0$

$\vec{\nabla} \times \vec{H} = 0$

Now, $\underbrace{K_b = \vec{M} \times \hat{n}}$ Surface Bound Current ~~Charge~~ Density

$K_b = \left(\frac{\chi_m i}{2\pi \rho} \hat{\phi} \right) \times \hat{\rho}$

$\Rightarrow \boxed{K_b = \frac{-\chi_m i}{2\pi \rho} \hat{z}}$

Hence, we can say that Curl of H = J and in this case equal to 0 $\because$ No free currents are present.

But there is presence of Bound Currents which gives us a non-zero finite result of Surface Bound Charge Density i.e. K_b.

$\therefore$ Contradiction Resolved.

(a) Given :

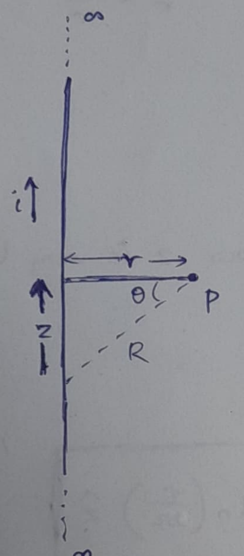
Infinite Straight wire

$i = I$

To find :

$\vec{A}$ at a distance 'r' from the wire

Solution :



As we know that,

$$\vec{A} = \frac{\mu_0}{4\pi} \oint \frac{i\, \vec{dl}}{R}$$

[r is p & i along z]

$$\vec{dl} = |dl| \cdot \hat{dl}$$

$$= |dl| \cdot \hat{z}$$

$$= dz \cdot \hat{z}$$

$$\therefore \vec{A} = \frac{\mu_0 i}{4\pi} \oint \frac{dz}{R} \hat{z}$$

$$= \frac{\mu_0 i}{4\pi} \oint \frac{dz}{\sqrt{r^2+z^2}} \hat{z}$$

$$\Rightarrow \vec{A} = \frac{\mu_0 i}{4\pi} \oint \frac{dz}{\sqrt{r^2+z^2}} \hat{z}$$

And $\tan\theta = \frac{z}{r}$

$$\Rightarrow z = r\tan\theta$$

$$dz = r\sec^2\theta\, d\theta$$

$$\Rightarrow \vec{A} = \frac{\mu_0 i}{4\pi} \oint \frac{r\sec^2\theta\, d\theta}{\sqrt{r^2+r^2\tan^2\theta}} \cdot \hat{z}$$

$$= \frac{\mu_0 i}{4\pi} \oint \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \hat{z}$$

$$= \frac{\mu_0 i}{4\pi} \oint_{-\pi/2}^{\pi/2} \sec\theta\, d\theta\, \hat{z}$$

$$= \frac{\mu_0 i}{4\pi} \Big[\ln|\sec\theta + \tan\theta| \Big]_{-\pi/2}^{\pi/2} \hat{z}$$

$$\Rightarrow \vec{A} = \frac{\mu_0 i}{4\pi} \ln\left(\frac{\sqrt{r^2+z^2}}{r} + \frac{z}{r}\right) \hat{z}$$

$$\sec\theta = \frac{\sqrt{r^2+z^2}}{r}$$

$$\tan\theta = \frac{z}{r}$$

$$= \frac{\mu_0 i}{4\pi} \ln\left(\frac{z}{r} + \sqrt{1+\frac{z^2}{r^2}}\right) \hat{z}$$

As we go to ∞ (i.e. $z \to \infty$ (OR) $z \to -\infty$),

$r << z$

$\Rightarrow \frac{z}{r}$ will become very large and $1 + \frac{z}{r} \approx \frac{z}{r}$

$$\Rightarrow \vec{A} = \frac{\mu_0 i}{4\pi} \ln\left(\frac{z}{r} + \sqrt{\frac{z^2}{r^2}}\right) \hat{z}$$

$$= \frac{\mu_0 i}{4\pi} \ln\left(\frac{2z}{r}\right)\Big]_{-z}^{z} (\hat{z})$$ (Where, Now z is very large)

$$= \frac{\mu_0 i}{2\pi} \ln\left(\frac{2z}{r}\right)\Big]_{0}^{z} (\hat{z})$$

$$\boxed{\therefore \vec{A} = \frac{\mu_0 i}{2\pi} \ln\left(\frac{2z}{r}\right) \hat{z}} \Rightarrow \boxed{\vec{A} = \frac{-\mu_0 i}{2\pi} \ln\left(\frac{r}{c}\right) \hat{z}}$$

$\left[\begin{array}{l} 2z = c \\ c : \text{some constant} \end{array}\right]$

$$\boxed{\therefore \vec{A} = \left[\frac{\mu_0 i}{2\pi}(\ln(2z)) - \frac{\mu_0 i}{2\pi}\ln(r)\right] \hat{z}}$$

Assume that $\frac{\mu_0 i}{2\pi} \ln(2z) = k$ [$\because$ All are constants]

$$\Rightarrow \boxed{\vec{A} = \left(k - \frac{\mu_0 i}{2\pi} \ln(r)\right) \hat{z}}$$

Divergence of A : $\vec{\nabla} \cdot \vec{A}$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial z}\left(k - \frac{\mu_0 i}{2\pi} \ln(\rho)\right)$$

$$= \frac{\partial}{\partial z}\left(\frac{\mu_0 i}{2\pi} \ln\left(\frac{2z}{r}\right)\right)$$

$$= 0$$

$$\boxed{\therefore \vec{\nabla} \cdot \vec{A} = 0}$$

Curl of A : $\vec{\nabla} \times \vec{A}$

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(k - \frac{\mu_0 i}{2\pi} \ln(r) \right) \hat{z}$$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{A}_\rho & \rho \hat{A}_\phi & \hat{A}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix}$$

$$= \frac{1}{\rho} \left[\hat{\rho} \left(\cancelto{0}{\frac{\partial}{\partial \phi} A_z} \right) - \rho \hat{\phi} \left(\frac{\partial}{\partial \rho} A_z \right) \right]$$

$$= - \left(\frac{\partial}{\partial \rho} A_z \right) \hat{\phi}$$

$$= - \frac{\partial}{\partial \rho} \left(k - \frac{\mu_0 i}{2\pi} \ln(\rho) \right) \hat{\phi}$$

$$= \frac{\mu_0 i}{2\pi\rho} \hat{\phi}$$

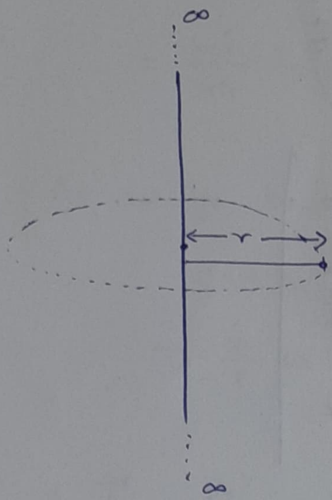
$$\therefore \vec{\nabla} \times \vec{A} = \frac{\mu_0 i}{2\pi\rho} \hat{\phi}$$

And as we know,

$$\vec{\nabla} \times \vec{A} = \vec{B}$$

$$\boxed{\therefore \vec{B} = \frac{\mu_0 i}{2\pi\rho} \hat{\phi}}$$

To check & confirm that this is correct,



Let's draw an Amperial loop of radius r

Now, As we know,

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 i_{enc}$$

$$\Rightarrow B \cdot 2\pi r = \mu_0 i$$

$$\Rightarrow B = \frac{\mu_0 i}{2\pi r}$$

$$\boxed{\therefore B = \frac{\mu_0 i}{2\pi r}\hat{\phi}}$$ [where $r=\rho$]

$\therefore \vec{\nabla}\times\vec{A} = \vec{B}$ Proved

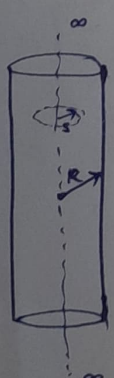
(b) Given:

Radius of wire = R

To find:

Magnetic Potential Inside the wire

Solution:



Now, According to Ampere's Law,

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 i_{enc}$$

Let's draw an Amperian loop of Radius 's',

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 i_{enc}$$

$$\Rightarrow B.(2\pi s) = \mu_0\left(\frac{I}{\pi R^2}\right)(\pi s^2)$$

$$\Rightarrow B.\ 2\pi s = \frac{\mu_0 I s^2}{R^2}$$

$$\boxed{\therefore B = \frac{\mu_0 I s}{2\pi R^2}} \quad \forall s < R$$

And as $\vec{dl}$ is along $\hat{\phi}$,

$$\boxed{\therefore \vec{B} = \frac{\mu_0 I s}{2\pi R^2}\hat{\phi}} \quad \forall s < R$$

We know that,

$$\vec{\nabla} \times \vec{A} = \vec{B}$$

And $\vec{A}$ is always along the direction of i, i.e. $\hat{z}$

$$\therefore \vec{A} = A_z \hat{z}$$

$$\Rightarrow \vec{\nabla} \times (A_z)\hat{z} = \vec{B}$$

$$\Rightarrow \left(\frac{-\partial}{\partial\rho} A_z\right)\hat{\phi} = \frac{\mu_0 I s}{2\pi R^2}\hat{\phi}$$

$$\Rightarrow -\frac{\partial}{\partial\rho} A_z = \frac{\mu_0 I s}{2\pi R^2}$$

$$\Rightarrow A_z = -\int \frac{\mu_0 I s}{2\pi R^2} d\rho$$

$$= -\frac{\mu_0 I}{2\pi R^2}\int s \, d\rho$$

$$= -\frac{\mu_0 I}{2\pi R^2}\left(\frac{s^2}{2} + c\right)$$

Let constant of Integration be $-\frac{c^2}{2}$ for simplicity.

$$= -\frac{\mu_0 I}{4\pi R^2}(s^2 - c^2)$$

$$\boxed{\therefore \vec{A} = -\frac{\mu_0 I}{4\pi R^2}(s^2 - c^2)\hat{z}} \quad \forall \rho \leq R \quad -①$$

Let's find out A outside the wire at any distance, say 's'.

From Ampere's Law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 i_{enc}$$

$$\Rightarrow B \cdot 2\pi s = \mu_0 i$$

$$\boxed{\therefore B = \frac{\mu_0 i}{2\pi s}}$$

And $\boxed{\vec{B} = \frac{\mu_0 i}{2\pi s}\hat{\phi}}$

As $\vec{\nabla}\times\vec{A} = \vec{B}$

$$\Rightarrow \left(-\frac{\partial}{\partial\rho}A_z\right)\hat{\phi} = \left(\frac{\mu_0 i}{2\pi s}\right)\hat{\phi}$$

$$\Rightarrow A_z = -\int \frac{\mu_0 i}{2\pi s}\, d\rho$$

$$= -\frac{\mu_0 i}{2\pi}\int s^{-1}\, d\rho$$

$$= -\frac{\mu_0 i}{2\pi}(\ln\rho + c)$$

Assume constant of Integration to be $-\ln c'$ for simplicity

$$\Rightarrow A_z = -\frac{\mu_0 i}{2\pi}\ln(\rho/c')$$

$$\boxed{\therefore A = -\frac{\mu_0 i}{2\pi}\ln(\rho/c')\,\hat{z}} \quad \forall \rho \geq R \quad -②$$

① should be equal to ② at $\rho = R$

$$\Rightarrow -\frac{\mu_0 i}{4\pi R^2}(R^2 - c^2) = -\frac{\mu_0 i}{2\pi}\ln(R/c')$$

$$\Rightarrow 1 - \left(\frac{c}{R}\right)^2 = 2\ln\left(\frac{R}{c'}\right)$$

$1 - \left(\frac{c}{R}\right)^2 = 0$ & $2\ln\left(\frac{R}{c'}\right) = 0$ is a solution

i.e. $c = c' = R$

$$\therefore \vec{A} = -\frac{\mu_0 i}{2\pi}\ln(\rho/R)\,\hat{z} \quad \forall \rho \geq R$$

$$\vec{A} = -\frac{\mu_0 i}{4\pi R^2}(\rho^2 - R^2)\,\hat{z} \quad \forall \rho \leq R$$

$\therefore$ Inside, $\vec{A} = -\frac{\mu_0 i}{4\pi R^2}(s^2 - R^2)\,\hat{z} \quad \forall s \leq R$

Given :

Inner Radius $= a$

Outer Radius $= b$

Current $= I$

To find :

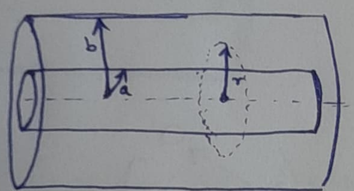
(i) Magnetic field between the tubes

(ii) Magnetisation

(iii) Bound Currents

(iv) Prove that results are correct

Solution :



Let's draw an Amperial loop of radius r such that $a<r<b$.

(i) Now, $\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enc}$

$$\Rightarrow \oint \bar{H} \cdot d\bar{l} = \underbrace{I_{enc}}_{(I_{free})_{enc}}$$

$$\Rightarrow H \cdot 2\pi r = I$$

$$\boxed{H = \frac{I}{2\pi r}}$$

$$\Rightarrow \boxed{\vec{H} = \frac{I}{2\pi r}\hat{\phi}}$$

And, $\bar{B} = \mu_0 \vec{H}(1+\chi_m)$

$$\Rightarrow \boxed{\vec{B} = \mu_0 (1+\chi_m) \frac{I}{2\pi r} \hat{\phi}}$$

(ii) Also, $M = \chi_m \vec{H}$

$$\boxed{\therefore \vec{M} = \frac{\chi_m I}{2\pi r} \hat{\phi}}$$

(iii) $\vec{J_b} = \vec{\nabla} \times \vec{M}$

$$\Rightarrow \vec{J_B} = \vec{\nabla} \times \left(\frac{\chi_m I}{2\pi \rho}\right)\hat{\phi}$$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \rho \chi_m \vec{H} & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\left(-\cancelto{0}{\frac{\partial}{\partial z}(\chi_m \vec{H})} \hat{\rho}\right) + \left(\cancelto{0}{\frac{\partial}{\partial \rho}(\rho \chi_m \vec{H})}\right)\hat{z}\right] = 0$$

$$\boxed{\therefore \vec{J_B} = 0}$$

$$K_b = \vec{M} \times \hat{n}$$

$$= \left(\frac{\chi_m I}{2\pi \rho} \hat{\phi}\right) \times (-\hat{\rho})$$

$$\Rightarrow \boxed{K_b = \frac{\chi_m I}{2\pi \rho} \hat{z}}$$

Check :

$$\oint B \cdot dl = \mu_0 (I_f + I_b) = \mu_0 (I_f + I_{surface} + \cancelto{0}{I_{volume}}) \quad \because \vec{J_b} = 0$$

$$I_{surface} \rightarrow \frac{\chi_m I}{2\pi a} \times 2\pi a = \chi_m I$$

$$= \mu_0 (I)(1+\chi_m)$$

$$\Rightarrow B \cdot 2\pi r = \mu_0 (1+\chi_m) I$$

$$\boxed{\therefore \vec{B} = \frac{\mu_0 (1+\chi_m) I}{2\pi r} \hat{\phi}}$$

$\therefore$ Produce the correct field

Given:

Inner Radius = a

Outer Radius = b

Current = I

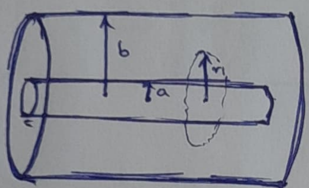
To Find:

(i) Magnetic field between the tubes

(ii) Magnetization

(iii) Bound Currents

Solution:



Let's Consider an Amperial loop of radius $a<r<b$

(i) Now $\oint \vec{H}\cdot d\vec{l} = I$

$$\Rightarrow H \cdot 2\pi r = I$$

$$\boxed{\therefore \vec{H} = \frac{I}{2\pi r}\hat{\phi}}$$

As $\vec{B} = \mu_0(\vec{H} + \chi_m \vec{H})$

$$= \mu_0 \vec{H}(1+\chi_m)$$

$$\boxed{\therefore \vec{B} = \mu_0(1+\chi_m)\frac{I}{2\pi r}\hat{\phi}}$$

(ii) And $\vec{M} = \chi_m \vec{H}$

$$\Rightarrow \boxed{\vec{M} = \frac{\chi_m I}{2\pi r}\hat{\phi}}$$

(iii) $\vec{J}_b = \vec{\nabla} \times \vec{M}$

$$\Rightarrow \vec{J}_B = \vec{\nabla} \times \left(\frac{\chi_m I}{2\pi r} \hat{\phi} \right) = \vec{\nabla} \times \left(\frac{\chi_m I}{2\pi \rho} \hat{\phi} \right)$$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \rho \chi_m \vec{H} & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}(0) = 0$$

$$\boxed{\therefore \vec{J}_B = 0}$$

$$\vec{K}_b = \vec{M} \times \hat{n}$$

$$= \left(\frac{\chi_m I}{2\pi r} \hat{\phi} \right) \times (-\hat{\rho})$$

$$= \frac{\chi_m I}{2\pi r} \hat{z}$$

$$\boxed{\therefore \vec{K}_b = \frac{\chi_m I}{2\pi r} \hat{z}}$$

$\cancel{r=a}$

Q5) Given:

$r = a$

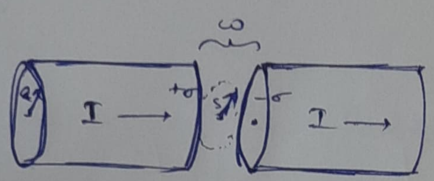
$i = I$ (constant)

gap width, $w << a$

To find:

B at any distance, $s < a$ (in the gap)

Solution:



Let us assume an ~~Am~~ Amperian Loop in the gap of radius s

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 i_{enc}$$

But $i_{enc} = \int \vec{J} \cdot \vec{ds}$

$$\Rightarrow \oint \vec{B} \cdot \vec{dl} = \mu_0 \int \vec{J} \cdot \vec{ds}$$

$$\Rightarrow B \cdot (2\pi s) = \mu_0 \int \vec{J} \cdot \vec{ds}$$

$$J = \frac{I}{2\pi a^2}$$

$$= \mu_0 \times \frac{I}{2\pi a^2} \times \pi s^2$$

$$\Rightarrow B = \mu_0 \times \frac{I}{2\cancel{\pi} a^2} \times \frac{\cancel{\pi} s^2}{2\pi \cancel{s}}$$

$$\boxed{\therefore B = \frac{\mu_0 I s}{2\pi a^2}}$$

& $\boxed{\vec{B} = \frac{\mu_0 I s}{2\pi a^2} \hat{\phi}}$