Electrical Circuits for Engineers (EC1000)

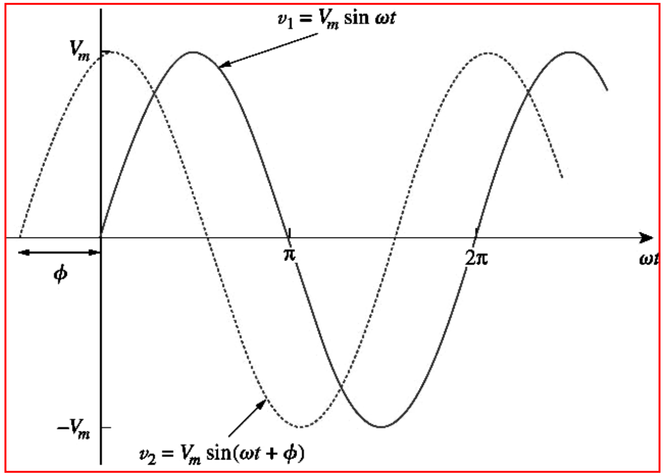
Lecture-7
AC circuits
Sinusoids and Phasor (Ch. 9)

Sinusoids and Phasor

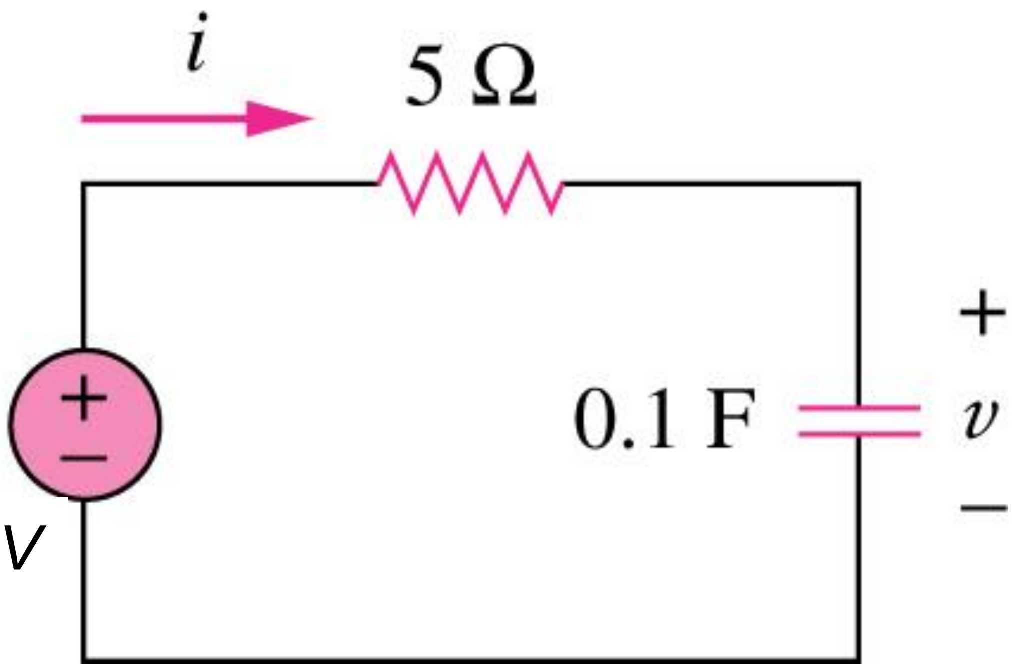
9.1 Motivation
9.2 Sinusoids' features
9.3 Phasors
9.4 Phasor relationships for circuit elements
9.5 Impedance and admittance
9.6 Kirchhoff's laws in the frequency domain
9.7 Impedance combinations

9.1 Motivation (1)

How to determine *v(t)* and *i(t)*?



$v_s = 10 \cos 4t$



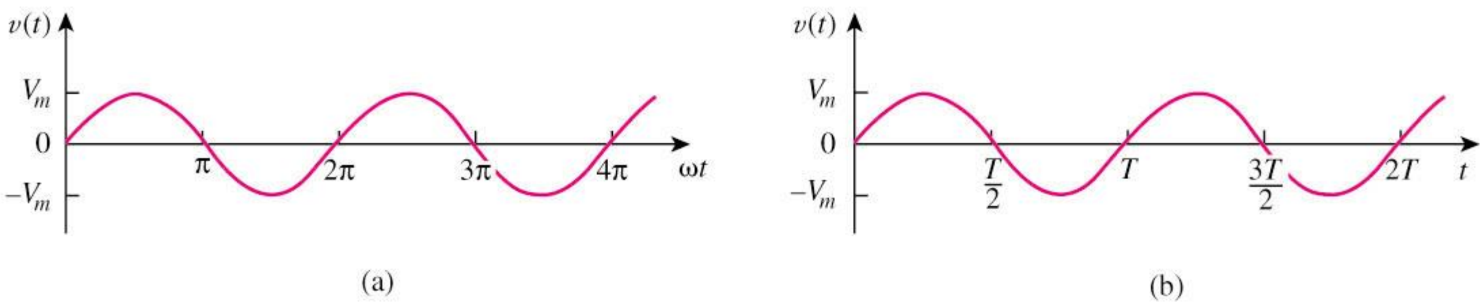
$v_s(t) = 10V$

How can we apply what we have learned before to determine *i(t)* and *v(t)*?

9.2 Sinusoids (1)

A sinusoid is a signal that has the form of the sine or cosine function.
A general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi)$$



Why Sinusoidal signal?

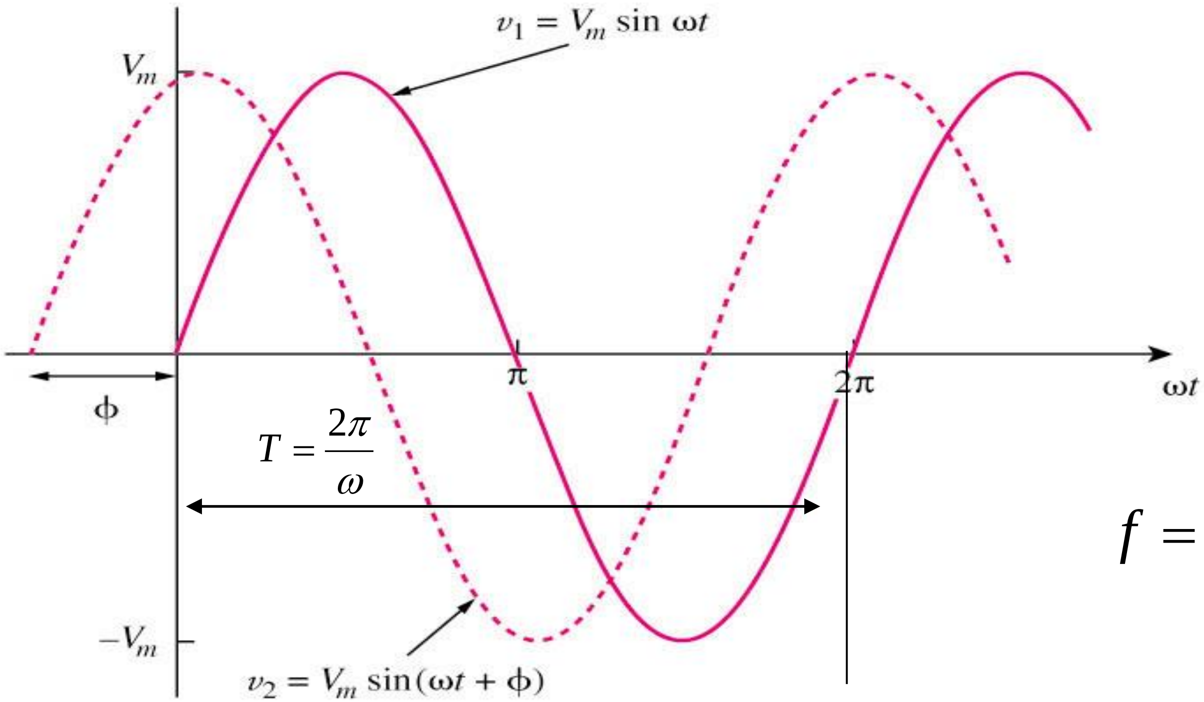
1. Nature itself is sinusoidal
2. AC can be easily generated and transmitted
3. Any periodic signal can be a sum of sinusoids_Fourier Analysis.
4. It can be easily handled mathematically.

where

Vm = the **amplitude** of the sinusoid
ω = the angular frequency in radians/s
Φ = the phase
ωt = the argument of the sinusiod

9.2 Sinusoids (2)

A periodic function is one that satisfies *v(t) = v(t + nT)*, for all t and for all integers n.



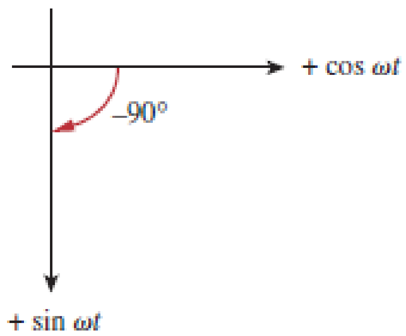
$$f = \frac{1}{T} Hz \quad \omega = 2\pi f$$

- Only two sinusoidal values with the same frequency can be compared by their amplitude and phase difference.
- If phase difference is zero, they are in phase; if phase difference is not zero, they are out of phase.

9.2 Sinusoids (2)

- A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes.
- This is achieved by using the following trigonometric identities:

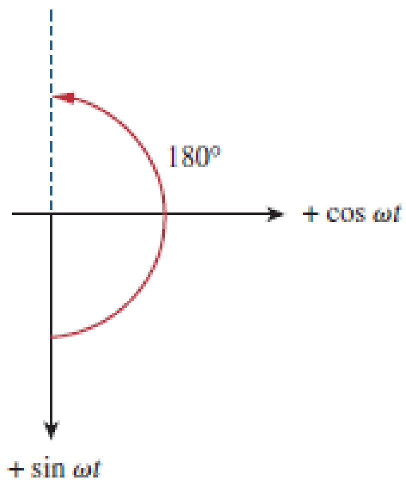
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$



(a)

With these identities, it is easy to show that

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm\cos \omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp\sin \omega t$$



Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

9.2 Sinusoids (3)

Example 1

Given a sinusoid, $5\sin(4\pi t - 60^o)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Solution:

Amplitude = 5, phase = –60^o, angular frequency = 4π rad/s, Period = 0.5 s, frequency = 2 Hz.

Given the sinusoid $30\sin(4\pi t - 75^\circ)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Answer: 30, -75°, 12.57 rad/s, 0.5 s, 2 Hz.

9.2 Sinusoids (4)

$$\sin(\omega t \pm 180^\circ) = -\sin\omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos\omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm\cos\omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp\sin\omega t$$

Example 2

Find the phase angle between $i_1 = -4\sin(377t + 25^o)$ and $i_2 = 5\cos(377t - 40^o)$, does i_1 lead or lag i_2?

Solution:

Since $\sin(\omega t+90^o) = \cos \omega t$

$$i_2 = 5\sin(377t - 40^o + 90^o) = 5\sin(377t + 50^o)$$

$$i_1 = -4\sin(377t + 25^o) = 4\sin(377t + 180^o + 25^o) = 4\sin(377t + 205^o)$$

therefore, i_1 leads i_2 155^o.

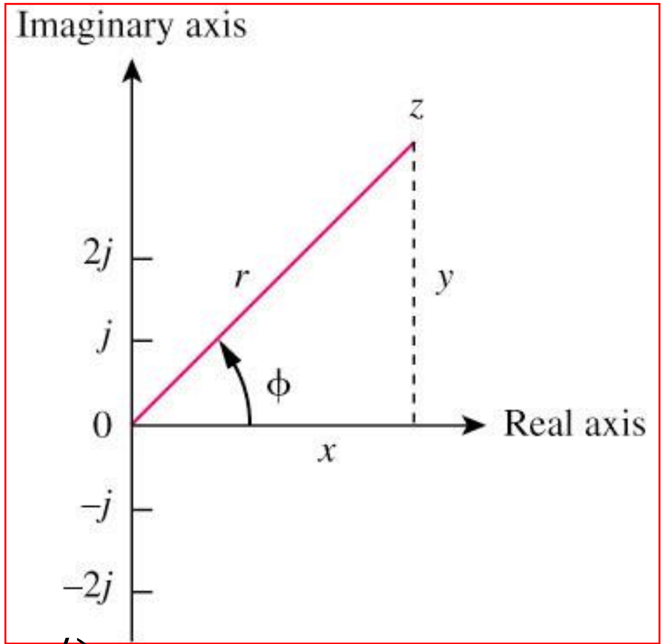
Calculate the phase angle between $v_1 = -10\cos(\omega t + 50^\circ)$ and $v_2 = 12\sin(\omega t - 10^\circ)$. State which sinusoid is leading.

: v_2 leads v_1 by 30°.

9.3 Phasor (1)

Sinusoids are easily expressed in terms of ***phasors***, which are more convenient to work with than sine and cosine functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.



It can be represented in one of the following three forms:

a. Rectangular $z = x + jy = r(\cos\phi + j\sin\phi)$

b. Polar $z = r\angle\phi$

c. Exponential $z = re^{j\phi}$

where $r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\frac{y}{x}$

$$z = x + jy = r\underline{/\phi}, \qquad z_1 = x_1 + jy_1 = r_1\underline{/\phi_1}$$

$$z_2 = x_2 + jy_2 = r_2\underline{/\phi_2}$$

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

9.3 Phasor (3)

Mathematic operation of complex number:

1. Addition
2. Subtraction
3. Multiplication
4. Division
5. Reciprocal
6. Square root
7. Complex conjugate
8. Euler's identity

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

$$z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \phi_1 - \phi_2$$

$$\frac{1}{z} = \frac{1}{r} \angle -\phi$$

$$\sqrt{z} = \sqrt{r} \angle \phi/2$$

$$z^* = x - jy = r \angle -\phi = re^{-j\phi}$$

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

9.3 Phasor (2)

Example 9.3

Evaluate the following complex numbers:

Evaluate these complex numbers:

Example 9.3

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$

$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

Practice Problem

Evaluate the following complex numbers:

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$

Answer: (a) $-15.5 - j13.67$, (b) $8.293 + j7.2$.

9.3 Phasor (4)

Transform a sinusoid to and from the time domain to the phasor domain:

$$v(t) = V_m \cos(\omega t + \phi) \longleftrightarrow V = V_m \angle \phi$$

(time domain) (phasor domain)

- Amplitude and phase difference are two principal concerns in the study of voltage and current sinusoids.
- Phasor will be defined from the cosine function in all our proceeding study. If a voltage or current expression is in the form of a sine, it will be changed to a cosine by subtracting from the phase.

9.3 Phasor (5)

Example 4

Transform the following sinusoids to phasors:

$i = 6\cos(50t - 40^o)$ A

$v = -4\sin(30t + 50^o)$ V

$$\sin(\omega t \pm 180^\circ) = -\sin\omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos\omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm\cos\omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp\sin\omega t$$

Solution:

a. $\mathrm{I} = 6\angle -40^\circ$ A

b. Since $-\sin(A) = \cos(A+90^o)$;

$v(t) = 4\cos(30t+50^o+90^o) = 4\cos(30t+140^o)$ V

Transform to phasor => $\mathrm{V} = 4\angle 140^\circ$ V

9.3 Phasor (6)

Example 5:

Transform the sinusoids corresponding to phasors:

a. $\mathbf{V} = -10\angle 30^\circ \ \text{V}$

b. $\mathbf{I} = j(5 - j12) \ \text{A}$

Solution:

a) $v(t) = 10\cos(\omega t + 210^o)$ V

b) Since $\mathbf{I} = 12 + j5 = \sqrt{12^2 + 5^2} \angle \tan^{-1}(\frac{5}{12}) = 13\angle 22.62^\circ$

$i(t) = 13\cos(\omega t + 22.62^o)\ A$

9.3 Phasor (7)

The differences between v(t) and V:

v(t) is instantaneous or time-domain representation

V is the frequency or phasor-domain representation.

v(t) is time dependent, V is not.

v(t) is always real with no complex term, V is generally complex.

Note: Phasor analysis applies only when frequency is constant; when it is applied to two or more sinusoid signals only if they have the same frequency.

9.3 Phasor (8)

Relationship between differential, integral operation in phasor listed as follow:

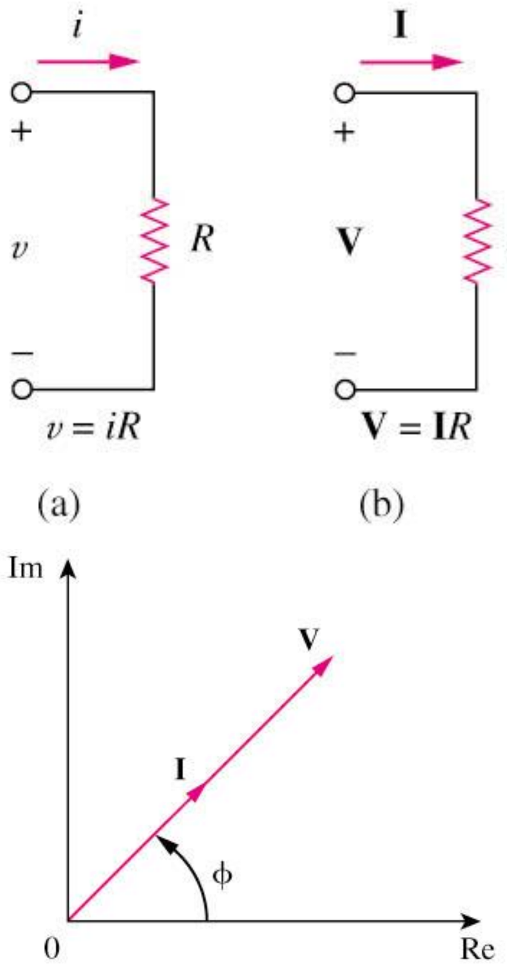
$$v(t) \longleftrightarrow \mathbf{V} = V\angle\phi$$

$$\frac{dv}{dt} \longleftrightarrow j\omega\mathbf{V}$$

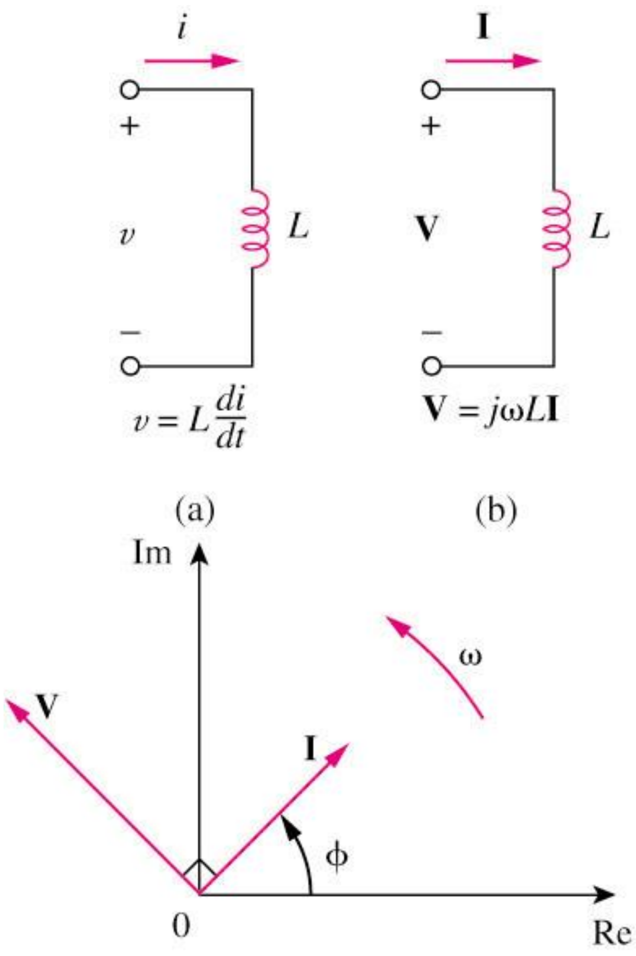
$$\int v\,dt \longleftrightarrow \frac{\mathbf{V}}{j\omega}$$

9.4 Phasor Relationships for Circuit Elements (1)

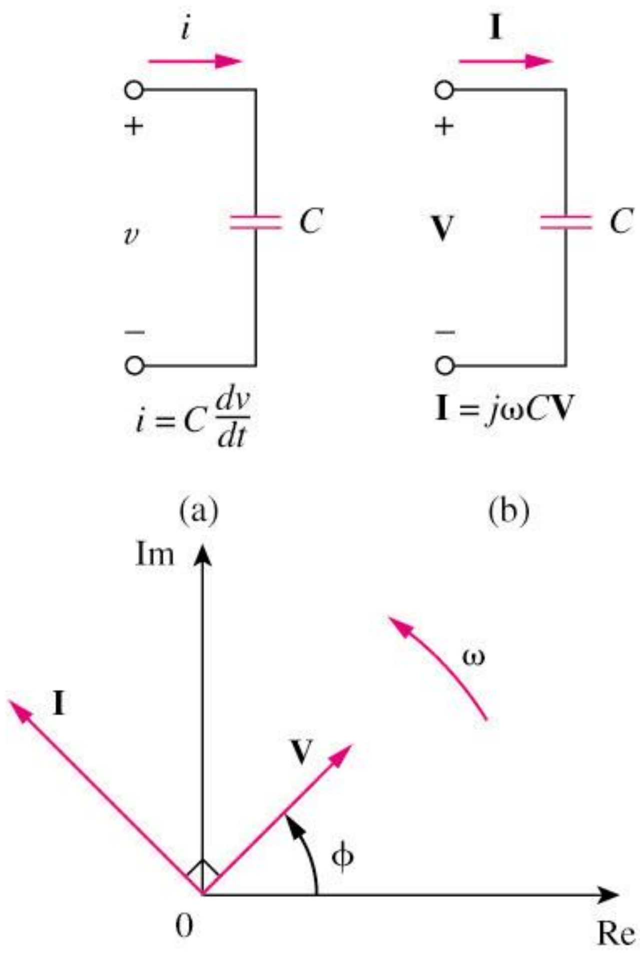
Resistor:



Inductor:



Capacitor:



9.4 Phasor Relationships for Circuit Elements (2)

Summary of voltage-current relationship		
Element	Time domain	Frequency domain
R	$v = Ri$	$V = RI$
L	$v = L\dfrac{di}{dt}$	$V = j\omega LI$
C	$i = C\dfrac{dv}{dt}$	$V = \dfrac{I}{j\omega C}$

Example Problem

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\underline{/45°}$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45°}}{j60 \times 0.1} = \frac{12\underline{/45°}}{6\underline{/90°}} = 2\underline{/-45°}\text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°)\text{ A}$$

9.4 Phasor Relationships for Circuit Elements (3)

Example 7

If voltage $v(t) = 6\cos(100t - 30^o)$ is applied to a 50 μF capacitor, calculate the current, $i(t)$, through the capacitor.

Answer: $i(t)$ = **$30 \cos(100t + 60^o)$ mA**

9.5 Impedance and Admittance (2)

- The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms Ω.

$$Z = \frac{V}{I} = R + jX$$

where R = Re, Z is the resistance and X = Im, Z is the reactance. Positive X is for L and negative X is for C.

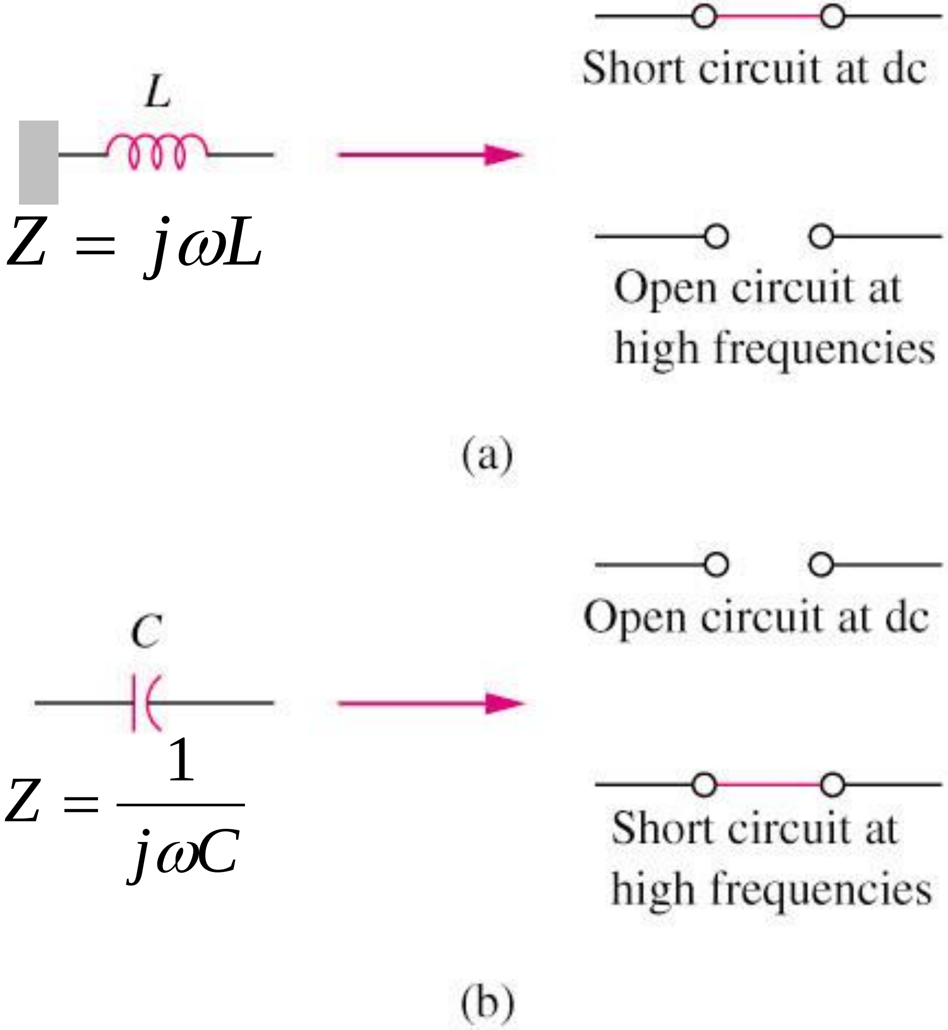
- The admittance Y is the reciprocal of impedance, measured in siemens (S).

$$Y = \frac{1}{Z} = \frac{I}{V}$$

9.5 Impedance and Admittance (2)

Impedances and admittances of passive elements		
Element	**Impedance**	**Admittance**
R	$Z = R$	$Y = \frac{1}{R}$
L	$Z = j\omega L$	$Y = \frac{1}{j\omega L}$
C	$Z = \frac{1}{j\omega C}$	$Y = j\omega C$

9.5 Impedance and Admittance (3)



$$\omega = 0; Z = 0$$

$$\omega \to \infty; Z \to \infty$$

$$\omega = 0; Z \to \infty$$

$$\omega \to \infty; Z = 0$$

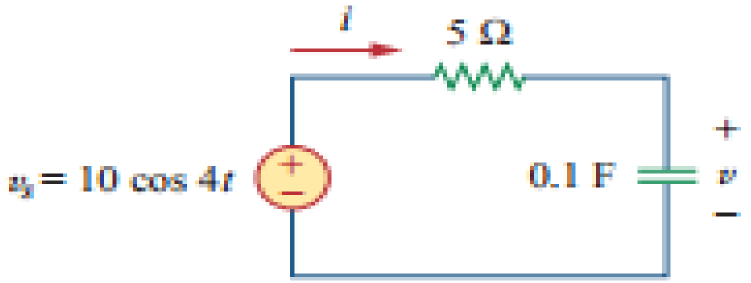
9.5 Impedance and Admittance (4)

After we know how to convert RLC components from time to phasor domain, we can transform a time domain circuit into a phasor/frequency domain circuit.

Hence, we can apply the KCL laws and other theorems to directly set up phasor equations involving our target variable(s) for solving.

Impedance and Admittance

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.



Solution:

From the voltage source $10\cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\underline{/0^\circ}\text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\underline{/26.57^\circ}\text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\underline{/26.57^\circ}}{j4 \times 0.1}$$
$$= \frac{1.789\underline{/26.57^\circ}}{0.4\underline{/90^\circ}} = 4.47\underline{/-63.43^\circ}\text{ V}$$

$$i(t) = 1.789\cos(4t + 26.57^\circ)\text{ A}$$
$$v(t) = 4.47\cos(4t - 63.43^\circ)\text{ V}$$

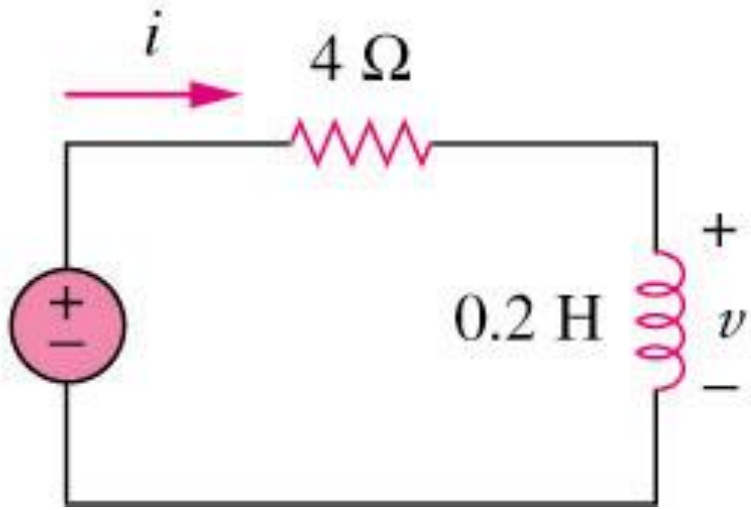
Notice that $i(t)$ leads $v(t)$ by 90° as expected.

9.5 Impedance and Admittance (5)

<u>Example 9.9 and practice problem 9.9</u>

Refer to Figure below, determine *v(t)* and *i(t).*

$$v_s = 5\cos(10t)$$



<u>Answers</u>: *i(t) = 1.118cos(10t – 26.56*

9.6 Kirchhoff's Laws in the Frequency Domain (1)

- Both KVL and KCL are hold in the phasor domain or more commonly called frequency domain.

- Moreover, the variables to be handled are phasors, which are complex numbers.

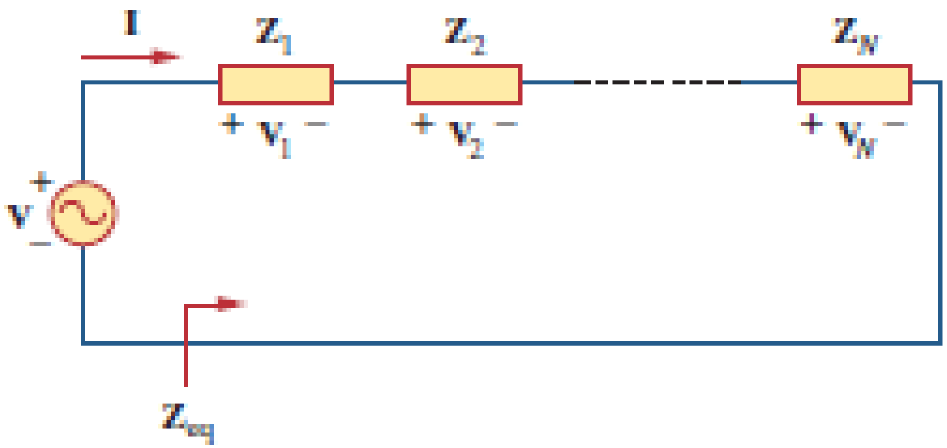
- All the mathematical operations involved are now in complex domain.

9.7 Impedance Combinations (1)

- The following principles used for DC circuit analysis all apply to AC circuit.

- For example:
  a. voltage division
  b. current division
  c. circuit reduction
  d. impedance equivalence
  e. Y-Δ transformation

9.7. Impedance Combinations – Series



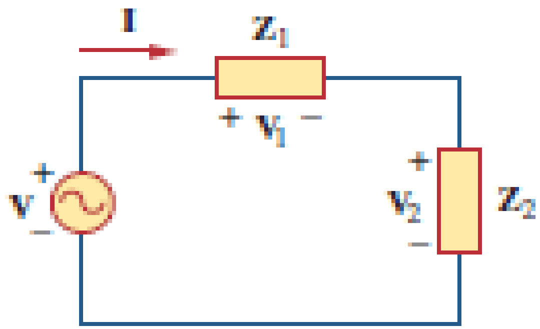


Figure 9.19
Voltage division.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

Since $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$, then

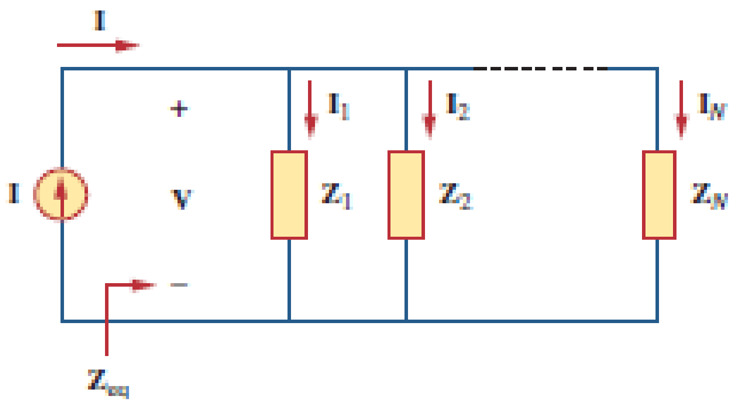
$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}}$$

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N}$$

Impedance Combinations-Parallel

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$



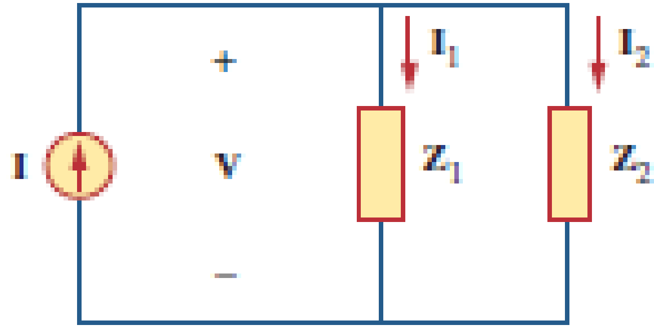


Figure 9.21
Current division.

The equivalent impedance is

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

and the equivalent admittance is

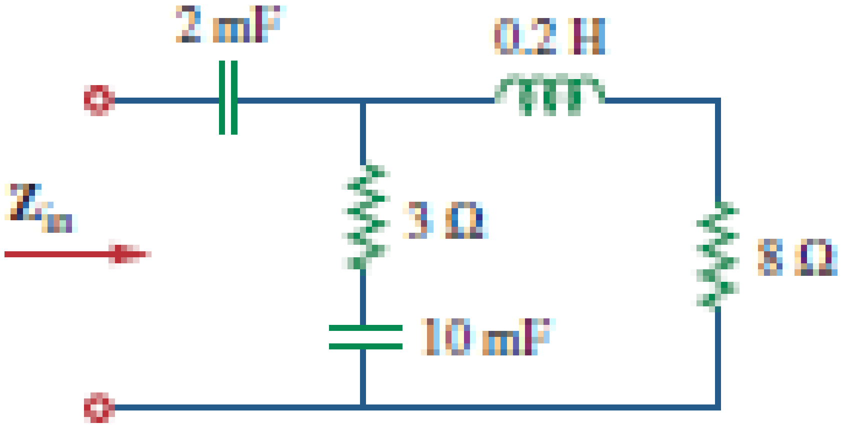
$$\boxed{\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N}$$

$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

9.7 Impedance Combinations (2)

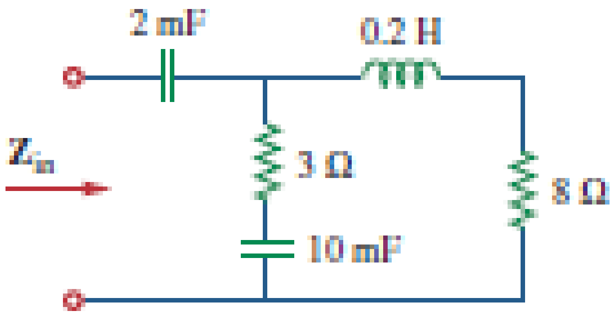
Example 9.10

Determine the input impedance of the circuit in figure below at ω =50 rad/s.



Answer: Z_{in} = 3.22 – j11.07 Ohm

Problems



Solution:

Let

Z_1 = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ohm resistor in series with the 10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ohm resistor

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \,\|\, \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

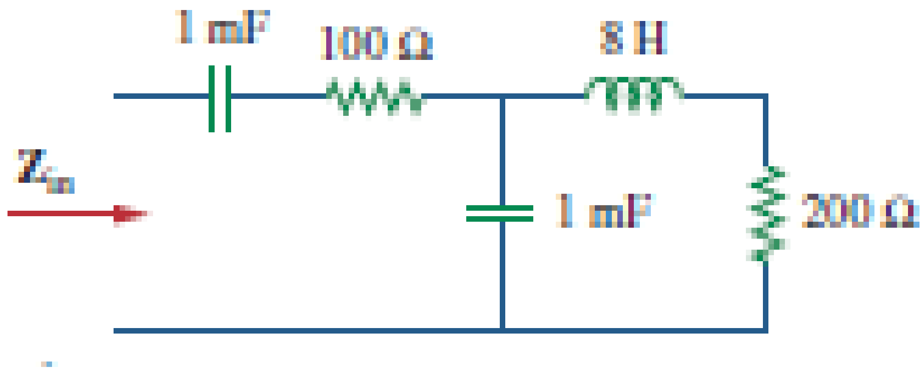
$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

$$\boxed{\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega}$$

Problems

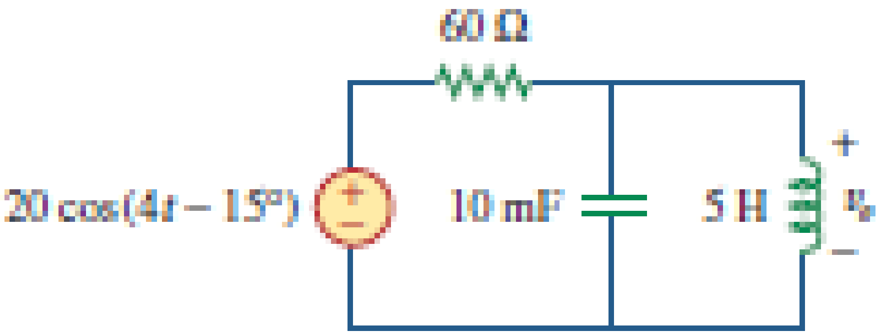
Determine the input impedance of the circuit in Figure at *W* = 10 rad/s.

(**<u>Ans: </u>**149.52-j195)



Example Problem

1. Determine v_o (t) in the circuit of Figure.

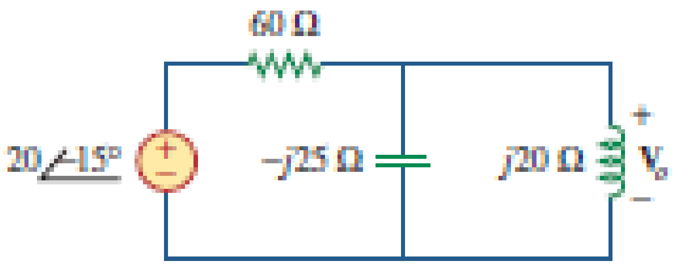


Convert parameters in Phasor Domain

$$v_s = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad \mathbf{V}_s = 20\angle -15^\circ \text{ V}, \quad \omega = 4$$

$$10 \text{ mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5 \text{ H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$



Let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

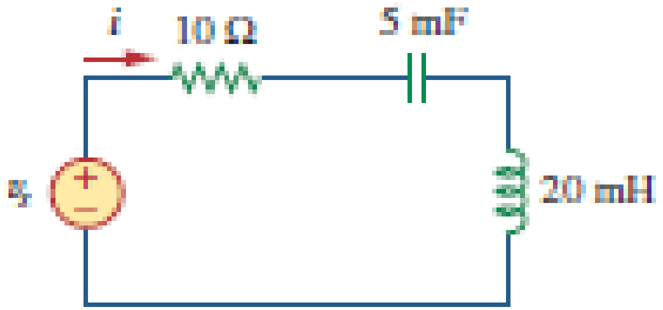
$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\angle -15^\circ)$$

$$= (0.8575\angle 30.96^\circ)(20\angle -15^\circ) = 17.15\angle 15.96^\circ \text{ V}$$

$$\boxed{v_o(t) = 17.15\cos(4t + 15.96^\circ) \text{ V}}$$

Problems

2. Find current i in the circuit of Figure, when vs(t)= 50 cos 200t V.



Solution

$$v_s(t) = 50\cos 200t \longrightarrow V_s = 50 < 0^o, \omega = 200$$

$$5mF \longrightarrow \frac{1}{j\omega C} = \frac{1}{j200x5x10^{-3}} = -j$$

$$20mH \longrightarrow j\omega L = j20x10^{-3}x200 = j4$$

$$Z_{in} = 10 - j + j4 = 10 + j3$$

$$I = \frac{V_s}{Z_{in}} = \frac{50 < 0^o}{10 + j3} = 4.789 < -16.7^o$$

i(t) = **4.789cos(200t–16.7°) A**

Problems

9.41 Find $v(t)$ in the *RLC* circuit of Fig. 9.48.

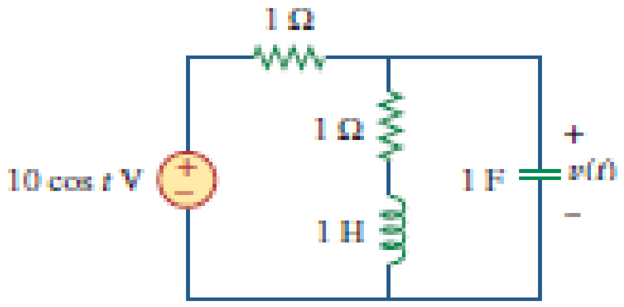


Figure 9.48
For Prob. 9.41.

$\omega = 1$,

$$1\text{ H} \longrightarrow j\omega L = j(1)(1) = j$$

$$1\text{ F} \longrightarrow \frac{1}{j\omega C} = \frac{1}{j(1)(1)} = -j$$

$$\mathbf{Z} = 1 + (1+j)\,||\,(-j) = 1 + \frac{-j+1}{1} = 2 - j$$

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10}{2-j}, \qquad \mathbf{I}_c = (1+j)\mathbf{I}$$

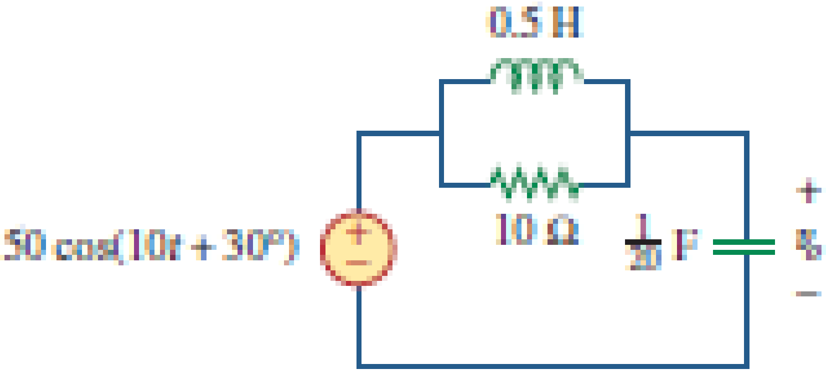
$$\mathbf{V} = (-j)(1+j)\mathbf{I} = (1-j)\mathbf{I} = \frac{(1-j)(10)}{2-j} = 6.325\angle -18.43°$$

Thus,

$$v(t) = \mathbf{6.325cos(t - 18.43°)\ V}$$

Problems

Calculate v_o in the circuit



Answer: $v_o(t) = 35.36 \cos(10t - 105°)$ V.

9.47 In the circuit of Fig. 9.54, determine the value of $i_s(t)$.

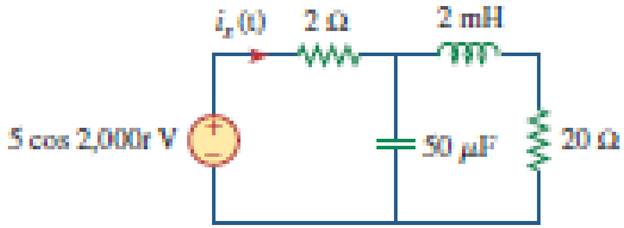


Figure 9.54
For Prob. 9.47.

9.48 Given that $v_s(t) = 20 \sin(100t - 40°)$ in Fig. 9.55, determine $i_x(t)$.

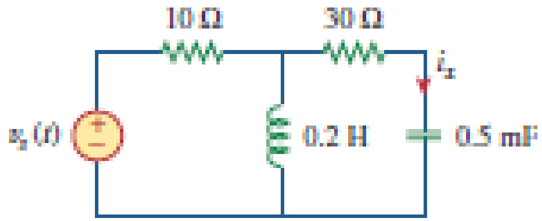


Figure 9.55
For Prob. 9.48.

9.49 Find $v_s(t)$ in the circuit of Fig. 9.56 if the current i_x through the 1-Ω resistor is 0.5 sin 200t A.

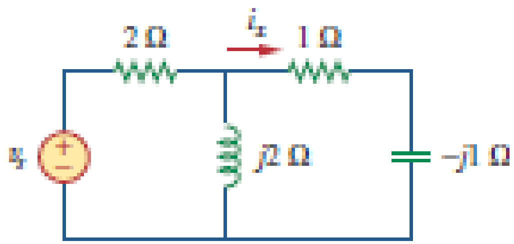


Figure 9.56
For Prob. 9.49.

All the materials extracted from Fundamentals of Electric Circuits by Charles K. Alexander, Matthew N.O. Sadiku, 5th Edition, McGraw Hill, for the purpose of Teaching and Learning only.