Quick summary

Exact Differential equations and Integrating Factors

Given first order ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad -(1)$$

if it is not exact

$$\Rightarrow \boxed{\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x} \quad -(2)}$$

Suppose eq. (1) has solution $f(x,y) = c$ and $\mu(x,y)$ is the required integrating factor

$\Rightarrow$ $\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,dy = 0$ is an exact differential equation

$$\Rightarrow \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad -(3)$$

$$\Rightarrow \mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

$$\Rightarrow \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -M\frac{\partial \mu}{\partial y} + N\frac{\partial \mu}{\partial x}$$

Case-1

$$\boxed{\mu = \mu(x)}$$

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial \mu}{\partial x}$$

$$\frac{1}{\mu}\frac{\partial \mu}{\partial x} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

↓ function of ... ↓ $g(x)$ ... le)

$\rightarrow$ case-2

$$\boxed{\mu = \mu(y)}$$

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -M\frac{\partial \mu}{\partial y}$$

$$\frac{1}{\mu}\frac{\partial \mu}{\partial y} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M}$$

↓ function of y-only ↓ $h(y)$

AIM: To find the integrating factor μ (2)

$\mu = \mu(x)$

then

$$\frac{1}{\mu}\frac{d\mu}{dx} = g(x)$$

where $$g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$\Rightarrow \boxed{\mu = e^{\int g(x)\,dx}}$$

$\mu = \mu(y)$

then

$$\frac{1}{\mu}\frac{\partial \mu}{\partial y} = h(y)$$

where

$$h(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-N}$$

$$\Rightarrow \boxed{\mu = e^{\int h(y)\,dy}}$$

required integrating factors in each case.

here $$\mu = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$$

here $$\mu = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy}$$

Look at the symmetry in the expressions for μ.

Find an integrating factor of $y\,dx + (x^2y - x)\,dy = 0$.

Solution: Here,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1-(2xy-1)}{x^2y-x} = \frac{-2(xy-1)}{x(xy-1)}$$

$$= \frac{-2}{x} \quad : \text{function of } x \text{ only} = g(x)$$

hence $\mu = \mu(x)$

$$= e^{+\int g(x)\,dx}$$

$$= e^{\int \frac{-2}{x}dx} = e^{\ln(1/x^2)} = \frac{1}{x^2}$$

is the required integrating factor of the given differential equation.

Other integrating factors.

Integrating factor of homogeneous D.E (in degree)

$$\boxed{I.F = \frac{1}{Mx+Ny}}, \quad \text{such that } Mx+Ny \neq 0.$$

Example: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

here $M = x^2y - 2xy^2$; $N = -x^3 + 3x^2y$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ and $\frac{\partial N}{\partial x} = -3x^2 + 6xy$.

$$\left.\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x^2 - 10xy.\right\} \text{ Not exact}$$

$\mu = \mu(x)$?

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4x^2 - 10xy}{-x^3 + 3x^2y}$$ is not a function of x alone. $\neq g(x)$

$\mu = \mu(y)$?

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{4x^2 - 10xy}{x^2y - 2xy^2}$$ is not $h(y)$ alone.

Then how do we find integrating factor.

For homogeneous D.E $\boxed{I.F \text{ as } \frac{1}{Mx+Ny}.}$

$\boxed{\text{Try}}$

$$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$
$$= x^2y^2$$

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{x^2 \cdot y^2} = \mu(x,y).$$

$\mu M dx + \mu N dy = 0$

$$\Rightarrow \frac{x^2y - 2xy^2}{x^2y^2}\,dx + \frac{-x^3 + 3x^2y}{x^2y^2}\,dy = 0$$

Now $M' = \mu M = \frac{1}{y} - \frac{2}{x}$; $N' = \mu N = -\frac{x}{y^2} + \frac{3}{y}$.

$\frac{\partial M'}{\partial y} = \frac{-1}{y^2}$ and $\frac{\partial N'}{\partial x} = \frac{-1}{y^2}$

$\boxed{\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}}$ $\therefore \mu = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$ is an integrating factor of given differential equation $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.

In general μ is an integrating factor for given D.E $M dx + N \partial y = 0$

if

$$\boxed{\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}}$$

Need to be used for Homework 1 and 2 in slide no - 81 (470/520)

H-W 87-Slide

First order linear differential equation

Standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\Rightarrow \frac{dy}{dx} = Q(x) - P(x)y.$$

$$\Rightarrow dy = (Q(x) - P(x)y)\,dx \quad \text{or}$$

$$\Rightarrow (Q(x) - P(x)y)\,dx - dy = 0$$

$$M\,dx + N\,dy = 0.$$

Here $M = Q(x) - P(x)\cdot y \qquad N = -1.$

$$\frac{\partial M}{\partial y} = -P(x) \quad \text{and} \quad \frac{\partial N}{\partial x} = 0 \quad \left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$$

$\therefore$ Not exact.

$$I.F = e^{\int P(x)\,dx} \qquad \text{why?}$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-P(x) - 0}{-1} = P(x) \rightarrow \text{fn of } x \text{ only.}$$

$$\therefore \boxed{\mu = e^{\int P(x)\,dx}} = \underline{I.F}$$

Now $\mu(M\,dx + N\,dy) = 0$

$$\equiv \underbrace{\left(Q(x)\,e^{\int P(x)\,dx} - P(x)\cdot e^{\int P(x)\,dx}\cdot y\right)}_{\text{New } M} dx - \underbrace{e^{\int P(x)\,dx}}_{\text{New } N}\cdot dy = 0$$

$$f(x,y) = \int m\,dx + \int\left(N - \frac{\partial}{\partial y}\int m\,dx\right)dy$$

$$\int m \cdot dx = \int (I \cdot F)\, Q(x) \cdot dx - y \int P(x) \cdot (I \cdot F)\, dx \quad - (1)$$

$$\frac{\partial}{\partial y} \int m \cdot dx = - \int P(x) \cdot (I \cdot F)\, dx$$

$$\left(N - \frac{\partial}{\partial y} \int m \cdot dx \right) = -(I \cdot F) + \int P(x) \cdot (I \cdot F)\, dx \quad - (2)$$

Now.

$$\int \left(N - \frac{\partial}{\partial y} \int m \cdot dx \right) dy = - \int (I \cdot F)\, dy + y \cdot \int P(x)\, (I \cdot F)\, dx \quad - (3)$$

Now

$$f = \int m\, dx + \int \left(N - \frac{\partial}{\partial y} \int m \cdot dx \right) dy = (1) + (3)$$

$$= \int (I \cdot F)\, Q(x)\, dx - y \int P(x) \cdot \cancel{(I \cdot F)\, dx}$$

$$- y \cdot (I \cdot F) + y \int P(x) \cdot \cancel{(I \cdot F)\, dx}$$

$$= \int (I \cdot F)\, Q(x)\, dx - y \cdot (I \cdot F) = \cancel{A} \; C$$

constant

$$\Rightarrow \quad y \cdot (I \cdot F) = \int (I \cdot F)\, Q(x)\, dx + D$$

or
$$\boxed{y = \frac{1}{(I.F.)} \int (I.F.)\, Q(x)\, dx + D \quad \left(I.F = e^{\int P(x)\, dx} \right)}$$

is the solution of 1^{st} order O.L.D.E.

$$\frac{dy}{dx} + P(x)\, y = Q(x).$$