18/1/24

Discrete Structures for Computer Science

→ Discrete : Composed of distant and seperable elements
(Opposite of Continuous)

Ex. Focusing on discrete set ($\mathbb{N}$, $\mathbb{Q}$)

Continuous : ($\mathbb{R}$, $\mathbb{C}$)

→ Structures : Objects that are built from simpler objects according to some definite pattern

→ Discrete Structures for Computer Science :

- Fundamental Subject

→ Computer Science : Science of Data

- Data Representation
- Data Storage
- Data Processing
- Solving Problems using Computers

→ Contents of DSCS :

- Formal Language (Logic)
- Discrete Structures
  - Sets, Relations and functions
  - Graph Theory
  - Proof Techniques
  - Counting

→ Reference :

Discrete Mathematics and Applications – K.H. Rosen

20/1/24

Formal Language (Logic)

→ Zeroth order Logic (Propositional Logic)

↓

First Order Logic (Predicate Logic)

↓

Second Order Logic

→ **Assertion** : A Statement

		Proposition
Ex. Today is Tuesday	(T/F)	No
Ex. It is Raining	(T/F)	No

→ **Proposition**: A declarative statement that takes the truth value i.e. True or False, but not both

→ **Not Part of Logic & Proposition**:

Ex. Imperative Statement (Commanding)

↳ Bring a Cup of Coffee

Ex. Interrogative Statement (Questioning)

↳ Who are you ?

Ex. Exclamatory Statement

↳ Oh! I forgot

Ex. Vague Statement (True or False cannot be determined)

↳ I have done that

$x+5=6$ (Non-constant Variables)

→ **Atomic Proposition**:

A single statement

→ **Compound Proposition**:

Multiple Statements combined together by using logical operators

→ Operations (Logical)

- Unary Operation (Only Negation)
- Binary Operation

① Negation : (¬P)

P	¬P
T	F
F	T

- Only Unary operation available in Formal Language

② Disjunction (or) operator (∨) :

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

③ Conjunction (and) operator (∧) :

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

④ Exclusive OR operation (⊕) :

Either p is true or q is true, but not both

p	q	p ⊕ q
T	T	F
F	T	T
T	F	T
F	F	F

⑤ Implies Operator (⟶)

[¬P ∨ q]

(Implication)

p	q	p ⟶ q
T	T	T
F	T	T
T	F	F
F	F	T

- p implies q

If p, then q

When p, q

q if p

Whenever p, q

q when p

q whenever p

p only if q

p is sufficient for q

q follows from p

q is necessary for p

p is Premise

q is conclusion

(c) Biconditional Operator : ($\leftrightarrow$)

- $(p \to q) \land (q \to p) = p \leftrightarrow q$
- p if and only if q

If ~~th~~ p then q and if q then p

p is necessary and sufficient for q

If p then q and conversely

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(d)

→ Converse, Inverse and Contrapositive :

$p \longrightarrow q$

Converse : $q \to p$

Inverse : $\neg p \to \neg q$

Contrapositive : $\neg q \to \neg p$

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Contrapositive is the same.

$p \to q \cong \neg q \to \neg p$

Converse and Inverse mean the same

29/1/24

Context Sensitive Proposition :

Today is Sunday : Not a Proposition

Today is Holiday : Not a Proposition

If Today is Sunday, then today is a holiday : Proposition

Fixing context ⇒ Conclusion is made

→ Possible values of ~~Propo~~ variables :

p : T, F $= 2^1$

p, q : TT, TF, TT, FF $= 2^2$

k variables : 2^k

→ Tautology :

A propositional expression always evaluates to true for all possible values of propositional variables.

Ex. $P \vee \neg P$

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Ex $(P \to Q) \to (\neg P \vee Q)$

P	Q	$P \to Q$	$\neg P \vee Q$	$(P \to Q) \to (\neg P \vee Q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

→ **Contradictory :**

Always false

Ex. $P \wedge \neg P$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Ex. $(P \to Q) \wedge (P \wedge \neg Q)$

→ **Contingency :**

Neither ~~Tautogy~~ Tautology nor Contradiction

Ex. $P \vee Q$, $P \wedge Q$, $P \to Q$

→ **Implication :**

$A \Rightarrow B$

only when $A \to B$ is a Tautology

→ **Equivalence :**

$A \Leftrightarrow B$

only when $A \leftrightarrow B$ is a Tautology

Ex $(P \to Q) \to (\neg P \vee Q)$

$(P \to Q) \Rightarrow (\neg P \vee Q)$

& $(\neg P \vee Q) \to (P \to Q)$

$(\neg P \vee Q) \Rightarrow (P \to Q)$

$\therefore$ $(P \to Q) \Leftrightarrow (\neg P \vee Q)$

Q) $P \rightarrow (P \vee Q)$

P	Q	$P \vee Q$	$P \rightarrow P \vee Q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$\therefore P \Rightarrow (P \vee Q)$

Q) $((P \rightarrow Q) \rightarrow R) \Rightarrow (P \rightarrow (Q \rightarrow R))$?

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	LHS	RHS	LHS $\rightarrow$ RHS
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	T

True

Case-Study :

① PSCS is a Clasical Subject

② If PSCS is a Classical Subject,
Then BTech (CSE) is a good Program

$\therefore$ BTech (CSE) is a good Program.

~~① $\rightarrow$ Q~~
~~②~~

① : P
② : $P \rightarrow Q$ $\Big\}$ $P \wedge (P \rightarrow Q)$

Conclusion : Q

If $P \wedge (P \to Q) \Rightarrow Q$,

Then way of Arguing is correct.

P	Q	$P \to Q$	$P \wedge (P \to Q)$	LHS $\to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Tautology

$$P \wedge (P \to Q) \to Q$$

$$\therefore \underbrace{P \wedge (P \to Q)}_{\text{Premise}} \Rightarrow \underbrace{Q}_{\text{Conclusion}}$$

Implication ✓

30/1/24 :

Case study 2 :

1. If DSCS is a Classical subject, then B.Tech CSE is a good Program
2. If B Tech (CSE) is a good Program, then M.Tech (CSE)

$\therefore$ If DSCS is a Classical subject, then M.Tech (CSE) is a good Program.

1 : $p \to q$

2 : $q \to R$

$(P \to Q) \wedge (Q \to R)$

Conclusion : $P \to R$

$$(P \to Q) \wedge (Q \to R) \Rightarrow P \to R \ ?$$

P	Q	R	$P \to Q$	$Q \to R$	LHS $\wedge$ RHS	$P \to R$	
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Tautology

$\therefore (P \to Q) \wedge (Q \to R) \Rightarrow P \to R$

Condition follows Premise.

Q) Verify whether

$P \leftrightarrow Q \equiv \neg(P \oplus Q)$

P	Q	$P \leftrightarrow Q$	$P \oplus Q$	$\neg(P \oplus Q)$	$(P \leftrightarrow Q) \leftrightarrow \neg(P \oplus Q)$
T	T	T	F	T	T
T	F	F	T	F	T
F	T	F	T	F	T
F	F	T	F	T	T

Tautology

$\therefore$ True. (Equivalence)

→ Equivalence Laws :

- **Identity :**

$$P \land T \equiv P$$
$$P \lor F \equiv P$$

- **Domination :**

$$P \lor T \equiv T$$
$$P \land F \equiv F$$

- **Idempotent :**

$$P \lor P = P$$
$$P \land P = P$$

- **Double Negation :**

$$\neg(\neg P) = P$$

- **Commutative Law :**

$$P \lor Q \equiv Q \lor P$$
$$P \land Q \equiv Q \land P$$

- **Exclusive OR :**

$$P \oplus Q \equiv (P \land \neg Q) \lor (\neg P \land Q)$$
$$P \oplus Q \equiv (P \lor Q) \land (\neg(P \land Q))$$

- **Implies :**

$$P \rightarrow Q \equiv \neg P \lor Q$$

- **Biconditional :**

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \land (Q \rightarrow P)$$
$$P \leftrightarrow Q \equiv \neg(P \oplus Q)$$

- **Associative Law :**

$$P \lor (Q \lor R) \equiv (P \lor Q) \lor R$$
$$P \land (Q \land R) \equiv (P \land Q) \land R$$

- **Distributive Law :**

$$P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$$
$$P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$$

- **De-Morgan's Law :**

$$\neg(P \land Q) \equiv (\neg P) \lor (\neg Q)$$
$$\neg(P \lor Q) \equiv (\neg P) \land (\neg Q)$$

- **Absorption Law :**

$$P \lor (P \land Q) \equiv P \land (P \lor Q) \equiv P$$
$$P \land (P \lor Q) \equiv P \lor (P \land Q) \equiv P$$

- **Trivial Tautology / Contradiction :**

$$P \lor (\neg P) \equiv T$$
$$P \land (\neg P) \equiv F$$

Q) $\neg(p \to q) \equiv p \land \neg q$?

Sol: As $p \to q \equiv \neg p \lor q$

$\neg(p \to q) \equiv \neg(\neg p \lor q)$

$\neg(\neg p \lor q) \equiv p \land \neg q$

$\downarrow$ LHS $\quad\quad \downarrow$ RHS

Hence Proved

→ **Predicate Logic:**

① Quantifiers

- $\exists x$ (There exists Quantifiers) [for some x] (Existential Quantifiers)
- $\forall x$ (Universal Quantifiers) [for all x]

② Universe of Discourse (UOD)

Set of People
Set of Elements
Set of Members

Ex. Boys are good
Some boys are good
Atleast 3 boys are good
All boys are good

} Proposition according to 0^{th} logic

$B(x)$ = x is a boy
$G(x)$ = x is good

UOD: Set of People

All boys are good: $\forall x\,(B(x) \to G(x))$

$\forall x\,(\neg B(x) \lor G(x))$

If UOD ≡ Set of Boys, Then $\forall x\, G(x)$

31/1/24

→ Some boys are good

UOD : Set of Students

$S = \{S_1, S_2, \dots S_n\}$

$B(x)$: x is a boy

$G(x)$: x is good

ZOL : P

FOL : $\exists x (B(x) \wedge G(x))$

FOL becomes a ZOL
when context is fixed.

$\therefore$ ZOL is a special case of FOL &
FOL is a collection of ZOLs

$\because$ x = 1st Roll no. (1)

$B(x)$ = 1 is a boy ← Proposition

$\therefore$ Predicate Logic is collections of propositional logic.

→ Definitions :

① $\exists x \; P(x)$: There exists $x \in$ UOD satisfying Predicate P.

↓ Quantifier ↳ Predicate

Ex. Some boys are good

$\exists x (B(x) \wedge G(x))$

i.e. Atleast one $x \in$ UOD Such that $B(x) \wedge G(x)$ is True.

If $x = S$, then $P(S_1)$ is a Proposition.

$\exists x \; P(x) \equiv P(S_1) \vee P(S_2) \vee \dots P(S_n)$

If atleast, one $P(S_i)$ is true, $\exists x \; P(x)$ is true.

$\equiv P(S_1)$ is true (or)

$P(S_2)$ is true (or) ....

$P(S_n)$ is true

There exists $x \in S$ satisfying Predicate P $\begin{cases} \text{Atleast one } x \in S \\ \text{Some } x \in S \end{cases}$

② Unique x such that $P(x) \equiv \exists! x\, P(x) \equiv$ Exactly one x

or

Unique $x \in$ UOD such that $P(x)$ is true.

$$\exists! x\, P(x) \equiv [P(S_1) \wedge \neg P(S_2) \wedge \neg P(S_3) \ldots \wedge \neg P(S_n)] \vee$$

$$[P(S_2) \wedge \neg P(S_1) \wedge \neg P(S_3) \ldots \wedge \neg P(S_n)] \vee$$

$$\ldots$$

$\exists! x$: Exactly one $x \in$ UOD

Quantifier

③ For All x, $P(x)$

$\forall x\, P(x)$
- For all x, $P(x)$
- For each x, $P(x)$
- For any x, $P(x)$
- For Arbitrary x, $P(x)$

$$\forall x\, P(x) \equiv P(S_1) \wedge P(S_2) \wedge P(S_3) \wedge \ldots P(S_n)$$

Q) Case study:

	UOD : Set of Students	UOD: Set of Boys
(1) Some Boys are good	$\exists x (B(x) \wedge G(x))$	$\exists x\, G(x)$
(2) All boys are good	$\forall x (B(x) \rightarrow G(x))$	$\forall x\, G(x)$
(3) Some boys are not good	$\exists x (B(x) \wedge \neg G(x))$	$\exists x (\neg G(x))$
(4) All boys are not good	~~$\exists x (B(x) \wedge G(x))$~~ $\forall x (B(x) \rightarrow \neg G(x))$	$\forall x (\neg G(x)) \equiv \neg(\exists x\, G(x))$

(3):
- There exists a bad boy
- Some boys are good

(4):
- All boys are ~~bad~~
- No boy is good
- None of the boys are good
- There exists a good boy is false

(5) Not all boys are good. $\exists x(B(x) \land \neg G(x))$

$\neg(\exists x\, G(x))$

$\equiv \exists x(\neg G(x))$

Some boys are bad

There exists a bad boy

Sol: $B(x)$: x is a boy

$G(x)$: x is a good

Q) There Exists atleast 2 good boys

UOD : Set of Boys

$P(x)$: x is good

Sol: $\exists x\, P(x) \land \exists y\, ((x \neq y) \land P(y))$

(OR)

$\exists x\, P(x) \land \neg(\exists! x\, P(x))$

Q) There exists exactly 2 good Boys

UOD : Set of Boys

~~P(x) : x is a boy~~

$P(x)$: x is good

Sol: $\exists! x\, P(x) \land \exists! y\, ((x \neq y) \land P(y))$

(OR)

$\exists x\, P(x) \land \exists y((x \neq y) \land P(y)) \land \forall z\, ((x \neq y \neq z) \land \neg P(z))$

Q) There are Atmost two good boys

0 1 2

$(\neg \forall x\, P(x)) \lor (\exists! x\, P(x)) \lor (\exists! x\, P(x) \land \exists! y\, ((x \neq y) \land P(y)))$

(OR)

Atleast 3

$\neg\left(\exists x\, P(x) \land \exists y\, ((x \neq y) \land P(y)) \land \exists z\, ((x \neq y \neq z) \land P(z))\right)$

31/1/24

1) $\forall x\, p(x) \Rightarrow \exists x\, p(x)$ | $\exists x\, p(x) \not\Rightarrow \forall x\, p(x)$

2) $\exists! x\, p(x) \Rightarrow \exists x\, p(x)$ | $\exists x\, p(x) \not\Rightarrow \exists! p(x)$

$\cancel{\exists x\, p(x) \not\Rightarrow \forall x\, p(x)}$

$\cancel{\exists x\, p(x) \not\Rightarrow \exists!}$

3) $\forall x\, p(x) \not\Rightarrow \exists! x\, p(x)$

$\exists! x\, p(x) \not\Rightarrow \forall x\, p(x)$

Q) Case Study:

Natural Language to FOL

UOD: Residents in Apartments

① Some doctors are men

② All doctors are men

③ All women are Engineers

④ There exists a kid born to man & woman

$$\left.\begin{array}{l} D(x) : x \text{ is a doctor} \\ M(x) : x \text{ is a man} \\ W(x) : x \text{ is a woman} \\ E(x) : x \text{ is an engineer} \end{array}\right\} \text{Unary Predicates}$$

$kid(x, y)$: kid born to x, y } Binary Predicate

Sol:

① $\exists x\, (D(x) \wedge M(x))$

② $\forall x\, (D(x) \rightarrow M(x))$

③ $\forall x\, (W(x) \rightarrow E(x))$

(4) $\exists x \exists y ((M(x) \land W(y)) \to kid(x,y))$ ✓

(or)

$\exists x \exists y (kid(x,y) \to (M(x) \land W(y)))$ ✗

→ Logical Arguements & it's Proofs :

Premise

① Some Doctors are Engineers

UOD: Residents of Apartment

② All Engineers are not Men

Therefore, Some Doctors are not Men.

① $\exists x (D(x) \land E(x))$

② $\forall x (E(x) \to \neg M(x))$

Conclusion $\exists x (D(x) \land \neg M(x))$

$\exists x (D(x) \land E(x)) \land \forall x (E(x) \to \neg M(x)) \Rightarrow \exists x (D(x) \land \neg M(x))$

For Proving something in FOL,

Convert into ZOL through Instantiation

(i) $\exists x\, p(x)$ - Existential Instantiation

$x = a$ for some $a \in$ UOD

on EI, $\exists x\, p(x) \Rightarrow p(a)$, for some a

(ii) $\forall x\, p(x)$ - Universal Instantiation

$x = a$ for any $a \in$ UOD

on UI, $\forall x\, p(x) \Rightarrow p(a)$, for any a

① $D(a) \land E(a)$ for some a

② $E(a) \to \neg M(a)$ for all a

Ⓒ $D(a) \land \neg M(a)$ for some a

(2) ① $D(a) \wedge E(a)$

$\left.\begin{array}{l} D(a) \\ E(a) \end{array}\right\}$ True for some 'a'

$\left[\begin{array}{l} P \wedge Q \Rightarrow P \\ P \wedge Q \Rightarrow Q \end{array}\right]$

② $E(a) \rightarrow \neg M(a)$ } True for any 'a'

$\therefore$ $\neg M(a)$ } True for some 'a'

$[P \wedge (P \rightarrow Q) \Rightarrow Q]$

$\therefore$ $D(a) \wedge \neg M(a)$ } True for some 'a'

On Existential Generalisation:

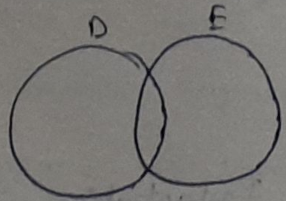
$\exists x\, (D(x) \wedge \neg M(x))$

$\therefore$ Some Doctors are not men.

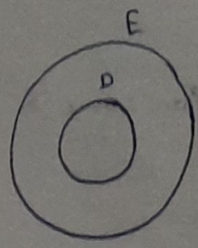
5/2/24

Venn Diagrams:

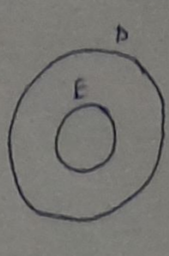
A ① $D \cap E$



② $D \subset E$

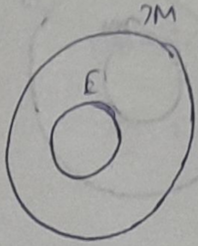


③ $E \subset D$

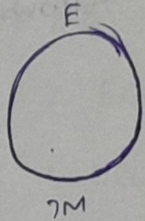


B

① $E \subset \neg M$

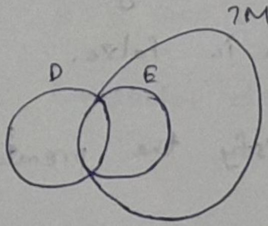


② $E = \neg M$

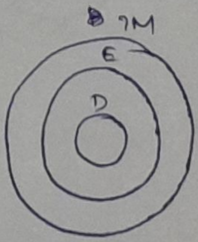


Conclusion

① $D \cap E$ & $E \subset \neg M$

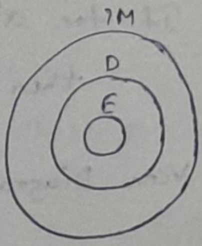


② $D \subset E$ & $E \subset \neg M$

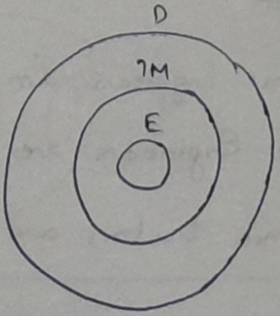


③ $E \subset D$ & $E \subset \neg M$

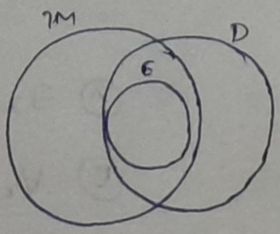
(i)



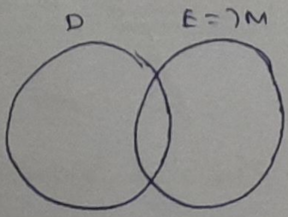
(ii)



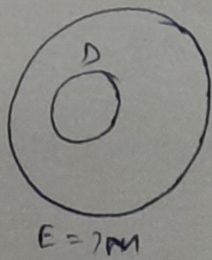
(iii)



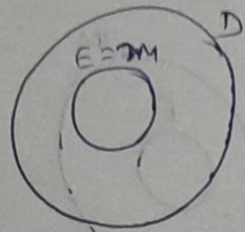
④ $D \cap E$ & $E = \neg M$



⑤ $D < E$ & $E = 7M$



⑥ $E < D$ & $E = 7M$



Therefore, in all 6 cases, Conclusion follows the premises.

- Venn Diagrams Rules :

→ If Conclusion (claim) is true, then it must be true for all Venn Diagrams.

If for atleast one Venn Diagram is false, then the conclusion is false.

→ Venn Diagrams must satisfy the premises.

Case Study:

Q) Ⓐ Some Engineers are Men

Ⓑ All Engineers are Doctors

Ⓒ Some Doctors are Lawyers

∴ Some Lawyers are Men

Sol:

Ⓐ $\exists x\,(E(x) \land M(x))$

Ⓑ $\forall x\,(E(x) \rightarrow D(x))$

Ⓒ $\exists x\,(D(x) \land L(x))$

Conclusion: $\exists x\,(L(x) \land M(x))$

By Instantiation:

(A) $E(a) \wedge M(a)$ is true for some 'a' (EI)

(B) $E(a) \rightarrow D(a)$ for all a is true (UI)

(C) $D(b) \wedge L(b)$ is true for some 'b' (EI)

$[a \in UOD]$

$(b \in UOD)$

$P \wedge Q \Rightarrow P$

$\therefore E(a) \wedge M(a) \Rightarrow M(a)$ is true for some 'a' $\in$ UOD

$\therefore D(b) \wedge L(b) \Rightarrow L(b)$ is true for some 'b' $\in$ UOD

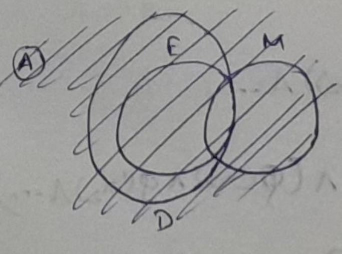
$M(a) \wedge L(b)$ is true for some $a \in$ UOD &
some $b \in$ UOD

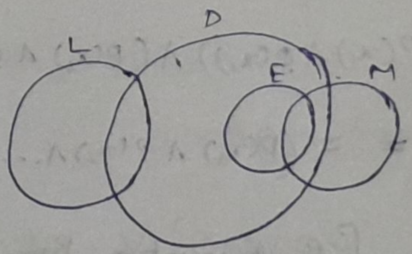
Either
$a = b$ (or) $a \neq b$

$\therefore$ Conclusion does not follow the premises.

~~(OR)~~

(OR)





$L \wedge M$ is not possible in this venn Diagram Case.

→ Scope of Quantifiers:

→ $\exists x \; p(x)$

Scope of x is Restricted to $p(x)$ only

→ $\exists x \; P(x) \wedge Q(x)$

x is bound to P

x is free from Q

→ $\exists x \; (P(x) \wedge Q(x))$

x is bound to P & Q.

→ $\exists x \; P(x) \wedge R$

$\equiv \exists x \; (P(x) \wedge R(x))$

[R : Simple Proposition]

→ $\forall x \; (P(x) \wedge Q(x)) \Leftrightarrow \forall x \; P(x) \wedge \forall x \; Q(x)$

Sol :

$$A \Leftrightarrow B \;\begin{cases} A \Rightarrow B \\ \wedge \\ B \Rightarrow A \end{cases}$$

$A \Rightarrow B$:

$(P(x_1) \wedge Q(x_1)) \wedge (P(x_2) \wedge Q(x_2)) \wedge \ldots$

$= (P(x_1) \wedge P(x_2) \wedge \ldots) \wedge (Q(x_1) \wedge Q(x_2) \wedge \ldots)$

[By Associative Property]

$=$ RHS

$= \forall x \; P(x) \wedge \forall x \; Q(x)$

$\therefore A \Rightarrow B$ is true

$B \Rightarrow A$:

$(p(x_1) \wedge p(x_2) \wedge p(x_3) \wedge \dots) \wedge (q(x_1) \wedge q(x_2) \wedge q(x_3) \wedge \dots)$

$= (p(x_1) \wedge q(x_1)) \wedge (p(x_2) \wedge q(x_2)) \wedge \dots$

[By Associative Property]

$=$ LHS

$= \forall x (p(x) \wedge q(x))$

$\therefore B \Rightarrow A$ is true

$\therefore A \Leftrightarrow B$ [$\because A \Rightarrow B \wedge B \Rightarrow A$]

Hence Proved

6/2/24

Q) $\forall x (p(x) \vee q(x)) \Leftrightarrow \forall x p(x) \vee \forall x q(x)$?

~~$A \Rightarrow B$~~

Sol:

$A \Rightarrow B$:

$(p(x_1) \vee q(x_1)) \wedge (p(x_2) \vee q(x_2)) \wedge \dots$

[Distributive Property]

$= (p(x_1) \vee q(x_1)) \wedge p(x_2) \vee (p(x_1) \vee q(x_1)) \wedge q(x_2) \vee \dots$

$= (p(x_1) \wedge p(x_2)) \vee \underbrace{(q(x_1) \wedge p(x_2)) \vee (p(x_1) \wedge q(x_2))}_{\text{Extra terms}} \vee (q(x_1) \wedge q(x_2)) \vee \dots$

$\not\Rightarrow (p(x_1) \wedge p(x_2)) \vee (q(x_1) \wedge q(x_2))$

$\Rightarrow \forall x p(x) \vee \forall x q(x)$

$\therefore A \not\Rightarrow B$

(OR)

Counter Example:

$x \in$ UOD : Set of Natural Numbers

$p(x)$: x is an even number

$q(x)$: x is an odd number

$\forall x\,(p(x) \vee q(x)) \equiv$ for all Natural numbers 'x', x is odd (or) x is even.

$\forall x\, p(x) \vee \forall x\, q(x) \equiv$ For all natural numbers 'x' are odd (or) all natural numbers 'x' are even.

$\therefore$ False :

$B \Rightarrow A$:

$\forall x\, p(x) \vee \forall x\, q(x)$

$\equiv (p(x_1) \wedge p(x_2) \wedge \dots) \vee (q(x_1) \wedge q(x_2) \dots)$

$\equiv ((p(x_1) \wedge p(x_2)) \vee q(x_1)) \wedge ((p(x_1) \wedge p(x_2)) \vee q(x_2)) \wedge \dots$

[Distributive Property]

$\equiv (p(x_1) \vee q(x_1)) \wedge \underbrace{(p(x_2) \vee q(x_1))}_{} \wedge \underbrace{(p(x_1) \vee q(x_2))}_{\text{Extra terms}}$

$\wedge\ (p(x_2) \vee q(x_2)) \wedge \dots$

$\Rightarrow (p(x_1) \vee q(x_1)) \wedge (p(x_2) \vee q(x_2)) \wedge \dots$

$\Rightarrow \forall x\,(p(x) \vee q(x))$

True

Q) $\exists x\,(p(x) \vee q(x)) \iff \exists x\, p(x) \vee \exists x\, q(x)$

Sol : $A \Leftrightarrow B \begin{cases} A \Rightarrow B \\ B \Rightarrow A \end{cases}$

$A \Rightarrow B$:

$$(p(x_1) \vee q(x_1)) \vee (p(x_2) \vee q(x_2)) \vee \cdots$$

$$\Rightarrow (p(x_1) \vee p(x_2) \vee p(x_3) \vee \cdots) \vee (q(x_1) \vee q(x_2) \vee \cdots)$$

(Associative Property)

$$\Rightarrow \exists x\, p(x) \vee \exists x\, q(x) \Rightarrow A \Rightarrow B$$

$B \Rightarrow A$:

Associative Property

True $\Rightarrow$ $B \Rightarrow A$

$\therefore \exists x\,(p(x) \vee q(x)) \iff \exists x\, p(x) \vee \exists x\, q(x)$

Q) $\exists x\,(p(x) \wedge q(x)) \iff \exists x\, p(x) \wedge \exists x\, q(x)$

Sol : Counter Example :

UOD : Residents

$p(x)$: x is a Doctor

$q(x)$: x is a Lawyer

$\exists x(p(x) \wedge q(x)) \equiv$ Some doctors are Lawyers

$\Downarrow \quad \not\Uparrow$

$\exists x\, p(x) \wedge \exists x\, q(x) \equiv$ There exists a Doctor and There exists a Lawyer

$A \Rightarrow B$ is true

$B \Rightarrow A$ is false

~~Q) $\exists x\, p(x)$ —~~

H.W

Q) $\exists x\,(p(x) \to q(x)) \Leftrightarrow \exists x\, p(x) \to \exists x\, q(x)$ (Ans : $\Rightarrow$)

Q) $\forall x\,(p(x) \to q(x)) \Leftrightarrow \forall x\, p(x) \to \forall x\, q(x)$ (Ans : $\Rightarrow$)

Q) $\forall x\,(p(x) \oplus q(x)) \Leftrightarrow \forall x\, p(x) \oplus \forall x\, q(x)$ (Ans : $\Leftarrow$)

Q) $\exists x\,(p(x) \oplus q(x)) \Leftrightarrow \exists x\, p(x) \oplus \exists x\, q(x)$ (Ans : $\Rightarrow$)

→ **Nested Quantifiers :**

Expressions Including multiple Quantifiers

1. Sum of two +ve Integers is always +ve.
2. If a person is female and parent, then this person is someone's mother
3. Every doctor is richer than some engineears

① $\forall x\, \forall y\, ((x > 0) \wedge (y > 0) \to (x+y) > 0)$

UOD : Set of Integers

② $\forall x\,(Par(x) \wedge Fem(x) \to \exists y\, Mot(x,y))$

③ $\forall x\,(D(x) \longrightarrow \exists y\,(Eng(y) \wedge Rich(x,y)))$

$\exists y$ is in the scope of $\exists x$.

Proof Techniques

1. Direct Proof $(P \rightarrow Q)$
2. Proof by Contraposition $(\neg Q \rightarrow \neg P)$
3. Proof by Contradiction
4. Proof by Mathematical Induction

Ex. (1) $\forall x (p(x) \wedge q(x)) \Rightarrow \forall x\, p(x) \wedge \forall x\, q(x)$

(2) $\neg(\forall x\, p(x) \wedge \forall x\, q(x)) \Rightarrow \neg(\forall x (p(x) \wedge q(x)))$

$\Rightarrow (\exists x\, \neg p(x) \vee \exists x\, \neg q(x)) \Rightarrow \exists x (\neg p(x) \vee \neg q(x))$

$\Rightarrow \exists x (R(x)) \vee \exists x (S(x)) \Rightarrow \exists x (R(x) \vee S(x))$

$\begin{bmatrix} R(x) \equiv \neg p(x) \\ S(x) \equiv \neg q(x) \end{bmatrix}$

(3) $P \rightarrow Q$

Assume P along with $\neg Q \Longrightarrow \neg P$

$\therefore P \rightarrow Q$ is true

Ex. $\sqrt{2}$ is irrational

$P \rightarrow Q$ form

~~If x is $\sqrt{2}$, then x is irrational~~

$\therefore$ If $\underbrace{x \text{ is } \sqrt{2}}_{P}$, Then $\underbrace{x \text{ is irrational}}_{Q}$

$P \wedge \neg Q \equiv \sqrt{2}$ is rational [Assume]

$\therefore \sqrt{2} = \frac{a}{b}$ S.T $b \neq 0$ & $GCD(a,b) = 1$

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 2b^2$

$\therefore a^2$ is an even number

$\therefore a$ is an even number

Claim: If $\underbrace{a^2 \text{ is even}}_{p}$, then $\underbrace{a \text{ is even}}_{q}$

$p \wedge \neg q \equiv a^2$ is even and a is $\underbrace{\text{not even}}_{\text{odd}}$

$\therefore a = 2k+1$ $[k \in \mathbb{W}]$

$\downarrow$

$a^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2k' + 1$

$(2k^2 + 2k = k')$

$\therefore a^2$ is odd

$\therefore a^2$ is not even

$p \wedge \neg q \Rightarrow \neg p$

$\therefore$ Assumption of $\neg q$ along with p is wrong.

$$\left[\begin{array}{l} p \rightarrow q \equiv \neg p \vee q \\ \quad\quad \neg(\neg p \vee q) = p \wedge \neg q \end{array}\right]$$

If a^2 is even, then a is even

$a^2 = 2b^2$ (let $a = 2\lambda$)

$(2\lambda)^2 = 2b^2$

$4\lambda^2 = 2b^2$

$\Rightarrow b^2 = 2\lambda^2$

If b^2 is even, then b is even

$\therefore$ If both a & b are even,

then $GCD(a,b) \geq 2$

$\therefore$ Assumption of $P \wedge \neg Q$ is wrong

$\therefore P \rightarrow Q$

Claim:

If $3N+2$ is odd, then N is odd

Sol: $3N+2$ is odd $\equiv 3N+2 = 2k+1$ $(k \in W)$

$\Rightarrow 3N = 2k-1$ — ①

Proof by Contradiction:

$3N+2$ is odd and N is $\underbrace{\text{not odd}}_{\text{even}}$

$P \wedge \neg Q$

N is even $\Rightarrow N = 2k'$ $(k' \in W)$

$\Rightarrow 3(2k') = 2k-1$

$\Rightarrow 2(3k') = 2k-1$

$\Rightarrow \underbrace{2(k'')}_{\text{even}} = \underbrace{2k-1}_{\text{odd}}$

$\therefore P \wedge \neg Q \Rightarrow \neg P$

Therefore, Assumption of $P \wedge \neg Q$ is wrong

and Hence $P \rightarrow Q$

(4) Claim:

$$1+2+3+\ldots n = \frac{k(k+1)}{2}, \quad k \geq 1$$

Direct Proof:

~~S = 1 + 2 + 3 + ...~~ ~~KCP~~

~~S = 1 + 2 + 3 + ... k~~

~~S = k + k-1 + k~~

$$S = 1 + 2 + 3 + \ldots k$$

$$S = k + k-1 + k-2 + \ldots 1$$

$$2S = (k+1) + (k+1) + \ldots \text{ k times}$$

$$\therefore S = \frac{k(k+1)}{2}$$

Proof by Mathematical Induction:

1. Base Step
2. Induction hypothesis
3. Induction Step

(1) Base Step:

$k=1$: $1 = \frac{1(1+1)}{2}$

1 - Directly Proved

$k=2$: $1+2 = \frac{2(2+1)}{2}$

$\Rightarrow 3 = 3$

Directly Proved

② Hypothesis

Assume that the claim is true for

$k \geq 0$ (or) $k \geq 1$ (or) $k \geq 2$

[As $k = 3$ has not been proven in base step, cannot be used $(k \geq 3)$ (or) even $(k \geq 4)$, etc]

③ Induction step :

We should prove that for $k+1$ also that claim is true.

$$\underbrace{1+2+3+ \cdots + k + (k+1)}_{RHS} = \underbrace{\frac{(k+1)(k+2)}{2}}_{RHS}$$

$$\frac{k(k+1)}{2} + k+1$$

$$\frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

Steps :

① Base :

$k=1$: Direct Proof with no Assumptions

② Induction Hypothesis :

Assume that claim is true for $k \geq 1$

③ Induction step :

The proof for $k+1$, $k \geq 1$

Assuming $k \geq 1 \rightarrow k+1 = 2$

Assuming $k \geq 2 \rightarrow k+1 = 3$

Assuming $k \geq 3 \rightarrow k+1 = 4$

$\vdots \qquad \vdots$

→ In formal language :

$$\left[\underbrace{P(1)}_{\text{Base}} \wedge \underbrace{\forall k \left((k \geq 1) \wedge P(k)\right.}_{\text{Hypothesis}} \underbrace{\left.\rightarrow P(k+1)\right)}_{\text{Induction}} \right]$$

$$\Rightarrow \forall k \; P(k), \; k \geq 1$$

Q) Claim :

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} < 1 \quad , \; n \geq 1$$

Proof : (by MI)

① Base : $n = 1$

$\frac{1}{2} < 1$ → Direct Proof

② Hypothesis :

Assume that claim is true for $n \geq 1$

③ Induction :

We should prove that claim is true for $n \geq 1$

$$\underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n}}_{<1} + \frac{1}{2^{n+1}} < 1$$

$$\equiv \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2^n}\right) < 1$$

$$= \frac{1}{2} + \frac{1}{2}(<1) < 1$$

∴ The claim is true

9) Claim :

The no. of diagonals in a n-sided polygon is $\frac{n(n-3)}{2}$

Sol : Direct Proof :

$$^{n}C_2 - n$$

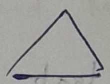
$$\Rightarrow \frac{n(n-1)}{2} - n$$

$$\Rightarrow \frac{n(n-3)}{2}$$

Proof by M.I :

① Base Case :

$n=3$



Diagonals $= 0 = \frac{3(3-3)}{2} = 0$

$n=4$

Diagonals $= 2 = \frac{4(4-3)}{2} = 2$

② Induction Hypothesis :

Assume that claim is true for $n \geq 3$

③ Induction step :

We should prove for $(n+1)$-sided polygon

$$\text{Diagonals} = \frac{(n+1)(n-2)}{2}$$

$$= \frac{n(n-3)}{2} + n-2+1$$

$$= \frac{(n+1)(n-2)}{2}$$

$$[P(3) \land \forall n((n \geq 3) \land P(n) \rightarrow P(n+1))] \; \forall n \; P(n), \; n \geq 3$$

→ Coin change Problem :

Given Integer x, Can we give change by using 2 denomination coins of ₹3 and ₹7

$x=1$ ✗	$x=8$ ✗
$x=2$ ✗	$x=9$ ✓
$x=3$ ✓	$x=10$ ✓
$x=4$ ✗	$x=11$ ✗
$x=5$ ✗	$x=12$ ✓
$x=6$ ✓	$x=13$ ✓
$x=7$ ✓	$x=14$ ✓

① Base Case : $x=12 = 4\times3$ – Direct Proof

② Hypothesis : Assume that change can be given for $x \geq 12$

③ Induction : We should prove that change is possible for $x+1$

Case 1 : There exists atleast 2 – ₹3 coins in x

$$x+1 = x-6+7$$

$$= x-(2\times3)+7$$

Case 2 : There exists atleast 2 – ₹7 coins in x

$$x+1 = x-14+15$$

$$= x-2\times7+3\times5$$

→ Weak Mathematical Induction

Base & Hypothesis – Easy & Induction – hard

14/2/24

→ Strong Mathematical Induction:

- Weak MI:

$$\frac{P(1) \wedge \forall k \geq 1 \, (P(k) \rightarrow P(k+1))}{\forall_{k \geq 1} \, P(k)}$$

1 base & 1 Assumption

Strong MI:

$$\frac{P(1) \wedge \forall_{k \geq 1} \, (P(1) \wedge P(2) \wedge P(3) \ldots P(k) \rightarrow P(k+1))}{\forall_{k \geq 1} \, P(k)}$$

Multiple bases and Multiple Assumptions

- In Weak Induction,
  Base and Hypothesis are simple and Induction step is complex
- In strong Induction,
  Base ~~&~~ and Hypothesis are strong and Induction step is simple.

→ Revisiting coin change Problem:

$x = 12$: 4×3

$x = 13$ $2 \times 3 + 1 \times 7$

$x = 14$ 2×7

$x = 15$ 5×3

$x = 16$ $3 \times 3 + 1 \times 7$

$x = 17$ $1 \times 3 + 2 \times 7$

① Base : $x = 12$, $x = 13$, $x = 14$ ⟹ Direct Proof

$$\begin{array}{ccc} x=12 & x=13 & x=14 \\ \downarrow & \downarrow & \downarrow \\ 4\times 3 & 2\times 3 + 1\times 7 & 2\times 7 \end{array}$$

② Hypothesis :

Assume that change can be given for $x, x-1, x-2$ S.T. $x-2 \geq 12$

③ Induction :

We should Prove that change is possible for $x+1$.

$$(x-2) + \text{₹}3 = \underline{\underline{x+1}}$$

$$\therefore \forall_{x \geq 12} \; P(x)$$

~~Pigeon Pigeon-hole Principle : (PHP)~~

★

→ **Pigeon-hole principle (PHP's) :**

① Given $(n+1)$ pigeons and n pigeon holes, There exists a pigeon hole with atleast 2 pigeons

② Given 'n' pigeons and 'm' pigeon holes.

If $n > m$, then $\exists$ a pigeon hole ≥ 2 pigeons.

$\geq \left\lceil \frac{n}{m} \right\rceil$ pigeons

③ If $n(r-1)+1$ Objects are distributed into n boxes, then there exists a box containing r objects atleast.

Proof:

$\exists$ box containing $\geq r$ elements

$p \wedge \neg q$ $\quad$ $\forall$ boxes containing $\leq r-1$ objects

Total objects $< n(r-1)$ objects

$\therefore$ Our Assumption is wrong.

$\therefore$ q follows p

Q) Claim: ~~The age group~~

In any group of 13 people, then there exists atleast 2 persons having same birth ~~to~~ month.

Sol:

PHP $\quad$ 13 people : 13 pigeons

12 months : 12 pigeon holes

Jan $\quad$ Feb $\quad$ ... $\quad$ Dec

According to PHP 1, '$n+1$' pigeons distributed into n pigeon holes.

$\exists$ pigeon hole with atleast 2 pigeons.

Q) Claim:

In any set of 52 distinct Integers, $\exists$ atleast 2 Integers whose sum (or) difference is divisible by 100

Sol:

PHP $\quad$ 52 distinct Integers : 52 pigeons

51 remainder of 100 : 51 pigeon holes

0 $\quad$ 1, 99 $\quad$ 2, 98 $\quad$ 3, 97 $\quad$ ... $\quad$ 49, 51 $\quad$ 50

'n+1' pigeons are distributed to n pigeon holes

Then there exists a pigeon holes with

≥ 2 ~~holes~~ pigeons.

There exists 2 Integers in the same box

Assume a_i, a_j are 2 pegions. (in same hole)

Case 1

$Rem(a_i) = Rem(a_j)$

$$\frac{a_i - a_j}{100}$$

Case 2

$Rem(a_i) \neq Rem(a_j)$

$$\frac{a_i + a_j}{100}$$

14/2/24

Q) Claim: In any set of 7 distinct Integers, There exists atleast 2 integers whose sum (or) difference is divisible by 10.

Sol: $a_i \bmod 10$

7 distincts Integers - 7 Pegions

6 remainder pairs - 6 pegion holes

0 | 1,9 | 2,8 | 3,7 | 4,6 | 5

By PHP, there exists a pegion hole containing ≥ 2 pigeons

Assume a_i, a_j are 2 pigeons (in same hole)

Case I

$Rem(a_i) \neq Rem(a_j)$

$$\frac{a_i + a_j}{10}$$

Case II

$Rem(a_i) = Rem(a_j)$

$$\frac{a_i - a_j}{10}$$

9) **Claim :**

In any group of 'n' people, there exists atleast 2 persons with equal no. of ~~fr~~ friends

Sol : Note :

Fix a person P_i $\left.\begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \end{matrix}\right\}$ Friends

Observation : ① If $\exists P_i$ with 0 friends, then $\nexists P_j$ with $n-1$ friends

② If $\exists P_i$ with '$n-1$' friends, then $\nexists P_j$ with 0 friends

Case ① : $\{0, 1, \ldots\ n-2\}$

'n' pegions – 'n' people

'$n-1$' pegion holes

↳ 0 to '$n-2$'

$\underset{0}{\cup}\ \underset{1}{\cup}\ \underset{2}{\cup} \ldots \underset{n-2}{\cup}$

$\exists$ 2 persons with equal no. of friends.

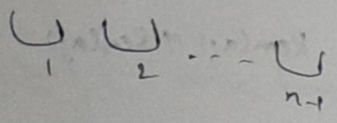
Case ②:

$\{1, 2, 3, \ldots\ n-1\}$

n pegions : n persons

'n-1' pegion holes

$\hookrightarrow$ 1 to n-1



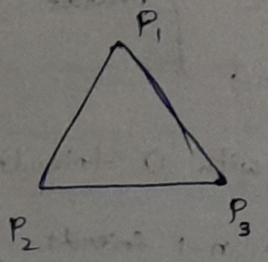
There exists 2 persons having equal no. of friends.

Q) <u>Claim</u>:

In any group of 3 people, there exists ~~2 persons~~ atleast 2 persons with equal no. of friends.

<u>Sol</u>:

Case ①:

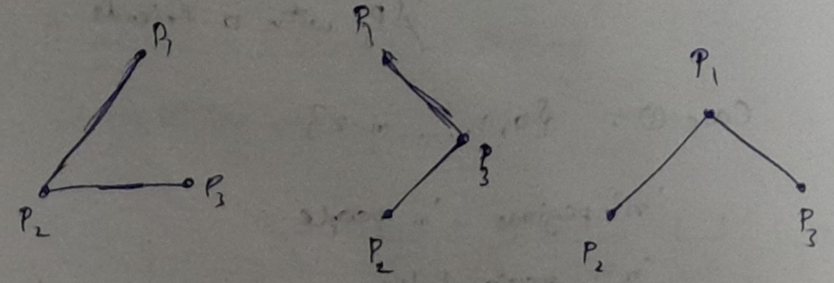


#friends $(P_1) = 2$

#friends $(P_2) = 2$

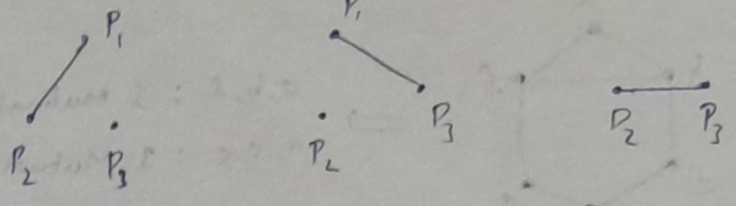
#friends $(P_3) = 2$

Case ②:

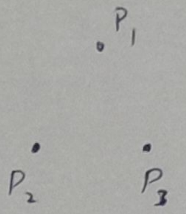


There exists 2 persons with equal no of friends.

Case ③ :



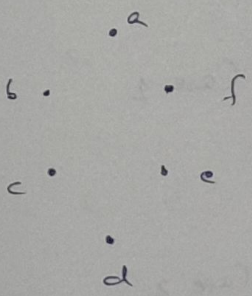
Case ④ :



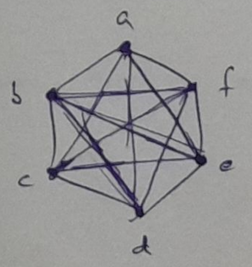
Q) <u>Claim</u> :

In any group of 6 people, there exists atleast 3 mutual friends (or) 3 mutual enemies.

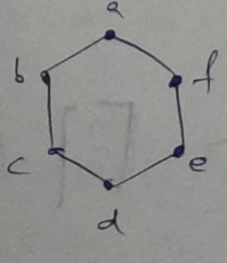
<u>Sol</u> :



$\Longrightarrow$ 6 Mutual Enemies

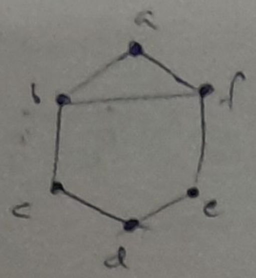


$\Longrightarrow$ 6 Mutual friends



$\Longrightarrow$ a, c, e : Mutual enemies (3)

b, d, f : Mutual enemies (3)



$\Longrightarrow$ a, b, f : 3 Mutual friends

a, c, e : 3 Mutual enemies

19/2/24

Ramsey Theorem:

In a group of 6 people,

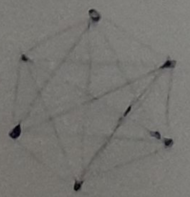
there exists atleast 3 MF (or) Atleast 3 ME

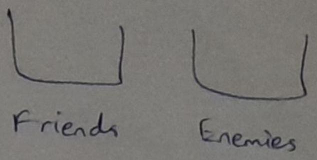
Proof:

Fix person P_i

	Friend	Enemy	
	5	0	Case 1
	4	1	
P_i	3	2	
	2	3	Case 2
	1	4	
	0	5	

P_i 5 people



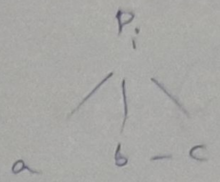


$n = 5$ (pigeons) $m = 2$ pigeon holes $\geq \left\lceil \frac{n}{m} \right\rceil$ pigeons

$\exists$ pigeon hole with atleast 3 pigeons

Case 1:

$P_i \geq 3$ friends



[b, c are friends]

$\exists$ Friend pair in a, b, c – $\{P_i, b, c\}$

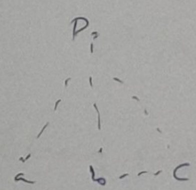
3 M-F

$\nexists$ Friend part in a, b, c – $\{a, b, c\}$

3 M-E

Case 2:

$P_i \geq 3$ Enemies



[b, c are enemies]

$\exists$ Enemy Pair in a, b, c – $\{P_i, b, c\}$ – 3 M-F

$\nexists$ Enemy Pair in a, b, c – $\{a, b, c\}$ – 3 M-F

Claim:

In a group of 10 people, there exists ~~[illegible]~~

($\geq$ 3 MF or $\geq$ 4 ME) and ($\geq$ 4MF or $\geq$ 3 ME)

Sol:

$(P \vee Q) \wedge (R \vee S)$

$\equiv ((P \vee Q) \wedge R) \vee ((P \vee Q) \wedge S)$

$$\equiv (P \wedge R) \vee (Q \wedge R) \vee (P \wedge S) \vee (Q \wedge S)$$

$$\Big(\big((\geq 3\ MF) \vee (\geq 4\ ME) \big) \wedge (\geq 4\ MF) \Big)$$

$$\vee$$

$$\Big(\big((\geq 3\ MF) \vee (\geq 4\ ME) \big) \wedge (\geq 3\ ME) \Big)$$

$$\equiv \big((\geq 3\ MF) \wedge (\geq 4\ MF) \big) \vee \big((\geq 4\ ME) \wedge (\geq 4\ MF) \big) \vee \big((\geq 3\ MF) \wedge (\geq 3\ ME) \big) \vee \big((\geq 4\ ME) \wedge (\geq 3\ ME) \big)$$

Any one condition can be proved true & Claim is true

(OR)

Proof:

Fix Person P:

	Friends	Enemies	
	9	0	
1ⓐ	8	1	
	7	2	
2ⓑ	6	3	
P_1	5	4	
	4	5	
	3	6	Case - 1ⓑ
	2	7	
	1	8	Case 2ⓐ
	0	9	

Case - I :

$P_i \geq 6$ friends (or) $P_i \geq 4$ Enemies

(a) (b)

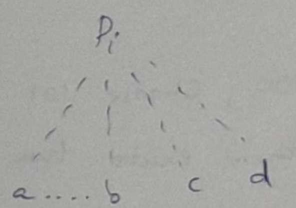
Case (a): $P_i \geq 6$ friends

≥ 3 MF (or) ≥ 3 ME

≥ 4 MF (or) ≥ 3 ME

Hence Proved

Case (b): $P_i \geq 4$ Enemies



$\exists$ enemy pair ~~in~~, Then ~~P_1 along~~ with

(P_1, a, b) - 3 M.E

$\nexists$ enemy pair - $\{a, b, c, d\}$ - 4 M.F

Case 2 :

$P_1 \geq 6$ Enemies (or) $P_i \geq 4$ Friends

(a) (b)

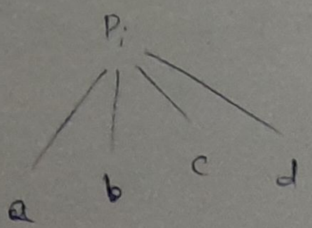
Case (a): $P_i \geq 6$ Enemies

≥ 3 MF (or) ≥ 3 ME

≥ 3 MF (or) ≥ 4 ME

Hence Proved

Case ⑥ : $P_i \geq 4$ friends



$\exists$ Friend pair $\{P_i, \text{pair}\}$ — 3 MF

$\nexists$ Friend pair $\{a, b, c, d\}$ - 4 ME

20/2/24

Claim: From the Integers 1 to 200, Choose 101 distinct Integers Arbitarily, then there exists two such that one divides the other.

Sol: Pigeon hole Principle:

Pigeon holes $\sqcup$ $\sqcup$ $\sqcup$ ------ $\sqcup$
1 3 5 199

[101 pigeons
100 pigeon holes
$\exists$ pigeon hole containing atleast 2 pigeons]

Any odd number = Odd no. $\times 2^0$

Any even number = odd no. $\times 2^i$ $(i \geq 1)$

Any no. = odd no. $\times 2^i$ $(i \geq 0)$

If a_i is odd no., then place a_i in pigeon hole a_i

If a_i is even, then $a_i = x \times 2^j$, $j \geq 1$, Then place it in pigeon hole x.

a_k, a_l

If $a_k > a_l$

$a_k = x \times 2^p$

$a_l = x \times 2^q$

$$\frac{a_k}{a_l} = \frac{x \times 2^p}{x \times 2^q} = 2^{p-q}$$

$\therefore a_k$ is divisible by a_l.

If $a_l > a_k$

$a_k = 2^p$

$a_l = 2^q$

$$\frac{a_l}{a_k} = \frac{x \times 2^q}{x \times 2^p} = 2^{q-p}$$

$\therefore a_l$ is divisible by a_k

$[p \neq q]$

$\because$ Distinct Integers

Claim:

Preparing for ~~a~~ Championship, A chess player is practicing for 77 consecutive days with constraint that he/she plays atleast 1 game a day & the total no. of games played in 77 days does not exceed 132.

Then claim that, there exists a period of consecutive days during which he/she played exactly 21 games.

Sol:

1 Consecutive Day	2 consecutive days		3 consecutive days		
21	20	1	19	1	1
	19	2	18	2	1
	18	3	⋮		
	⋮				
	2	19			
	1	20			

Proof:

Let a_i is the no. of games played until day i

a_1 # games played on day 1

a_2 # games played on day 1 + day 2

$$1 \le \underbrace{a_1, a_2, \ldots\ldots a_{77}}_{\text{distinct Integers}} \le 132 \quad -① $$

$$22 \le a_1, a_2, \ldots a_{77} + 21 \le 153$$

$$\Rightarrow 22 \le \underbrace{a_1+21, a_2+21, \ldots a_{77}+21}_{\text{distinct Integers}} \le 153 \quad -② $$

① & ②:

154 pigeons

153 pigeons holes

$\exists$ pigeon hole containing atleast 2 pigeons

$\hookrightarrow a_i, a_j+21$

$a_i = a_j + 21$

From day j to day i:

Exactly 21 games are played.

$[a_i \ne a_j]$

18/3/24

→ A binary relation R is called as Partial Order (POSET) if

(1) Reflexive

(2) Anti-symmetric

(3) Transitive follow.

Ex. $A = \{1, 2, 3\}$

	R	AS	T	Partial Order
$R_1 = \{(1,1), (2,2), (3,3)\}$	✓	✓	✓	✓
$R_2 = \{(1,1), (2,2), (3,3), (1,2)\}$	✓	✓	✓	✓
$R_3 = \{(1,2), (2,1)\}$	×	×	×	×
~~$R_4 = A \times A$~~				
$R_4 = A \times A$	✓	×	✓	×

Q) $A = \{1, 2, 3\}$

	R	AS	T	PO	
R_1	✓	✓	✓	✓	
R_2	✓	✓	✓	✓	
$R_1 \cup R_2$	✓	×	×	×	Not POSET
$R_1 \cap R_2$	✓	✓	✓	✓	

$$PO \equiv R \rightarrow T$$

$$PO \equiv R \rightarrow AS$$

→ A binary relation 'R' is called as both Equivalent and Partial order if

(1) Reflexive ✓

(2) Antisymmetric ✓

(3) Transitive ✓

(4) Symmetric ✓ follow

Ex. $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1), (2,2), (3,3), (1,2)\}$

$R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$R_4 = A \times A$

	R	S	As	T	Eq & PO
R_1	✓	✓	✓	✓	✓
R_2	✓	×	✓	✓	×
R_3	✓	✓	×	✓	×
R_4	✓	✓	×	✓	×

Q)

	R	S	As	T	Eq	PO	Eq & PO
R_1	✓	✓	✓	✓	✓	✓	✓
R_2	✓	✓	✓	✓	✓	✓	✓
$R_1 \cup R_2$	✓	✓	×	×	×	×	×
$R_1 \cap R_2$	✓	✓	✓	✓	✓	✓	✓

→ Composition of Relations :

A, B, C - 3 sets

$R_1 \subseteq A \times B$

$R_2 \subseteq B \times C$

$$R_1 \circ R_2 = R_1 \cdot R_2 = R_1 R_2 = \left\{ (a,c) / \exists b \text{ in } B, \begin{array}{c} (a,b) \in R_1 \\ \wedge \\ (b,c) \in R_2 \end{array} \right\}$$

$$R \cdot R = R^2 = \left\{ (a,c) / \exists b \text{ in } A, \begin{array}{c} (a,b) \in R \\ \wedge \\ (b,c) \in R \end{array} \right\}$$

$[R \subseteq A \times A]$

$$R^2 \cdot R = R^3 = \left\{ (a,c) / \exists b \text{ in } A, \begin{array}{c} (a,b) \in R^2 \\ \wedge \\ (b,c) \in R \end{array} \right\}$$

$$R^{n-1} \cdot R = R^n = \left\{ (a,c) / \exists b \text{ in } A, \begin{array}{c} (a,b) \in R^{n-1} \\ \wedge \\ (b,c) \in R \end{array} \right\}$$

Q) $A = \{a_1, a_2\}$

$B = \{b_1, b_2, b_3\}$

$C = \{c_1, c_2, c_3\}$

$R_1 = \{(a_1, b_1), (a_2, b_3)\}$

$R_2 = \{(b_1, c_1), (b_1, c_3), (b_3, c_2), (b_2, c_1)\}$

$R_1 \cdot R_2 = \{(a_1, c_1), (a_1, c_3), (a_2, c_2)\}$

$\hookrightarrow R_1 \cdot R_2 \subseteq A \times C$

Ex. $R_1 = \{(a_1, b_1)\}$

$R_2 = \{(b_3, c_2)\}$

$R_1 \circ R_2 = \emptyset$

Ex. $R_1 = A \times B$

$R_2 = B \times C$

$R_1 \cdot R_2 = A \times C$

Claim:

$R_1 \subseteq A \times B$

$R_2 \subseteq B \times C$

$R_3 \subseteq B \times C$

$R_1 \cdot (R_2 \cup R_3) = R_1 \cdot R_2 \cup R_1 \cdot R_3$?

~~Sol~~ :

Sol : $(a,c) \in R_1 \cdot (R_2 \cup R_3)$

$\Leftrightarrow \exists b$ such that $(a,b) \in R_1 \wedge (b,c) \in R_2 \cup R_3$

$\Leftrightarrow \exists b$ such that $(a,b) \in R_1 \wedge [(b,c) \in R_2 \vee (b,c) \in R_3]$

$\Leftrightarrow \exists b$ such that $\underbrace{[(a,b) \in R_1 \wedge (b,c) \in R_2]}_{P(x)}$

$\vee \underbrace{[(a,b) \in R_1 \wedge (b,c) \in R_3]}_{Q(x)}$

$$\boxed{a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)}$$

$\exists x \, (P(x) \vee Q(x)) \Leftrightarrow \exists x \, P(x) \vee \exists x \, Q(x)$

LHS $\Leftrightarrow$ RHS

LHS $\equiv$ RHS

LHS $\Rightarrow$ RHS

LHS $\Leftarrow$ RHS

→ Composition of Relations:

A, B, C - 3 sets

$R_1 \subseteq A \times B$

$R_2 \subseteq B \times C$

$$R_1 \circ R_2 = R_1 \cdot R_2 = R_1 R_2 = \left\{ (a,c) / \exists b \text{ in } B, \begin{matrix} (a,b) \in R_1 \\ \wedge \\ (b,c) \in R_2 \end{matrix} \right\}$$

$$R \cdot R = R^2 = \left\{ (a,c) / \exists b \text{ in } A, \begin{matrix} (a,b) \in R \\ \cap \\ (b,c) \in R \end{matrix} \right\}$$

$[R \subseteq A \times A]$

$$R^2 \cdot R = R^3 = \left\{ (a,c) / \exists b \text{ in } A, \begin{matrix} (a,b) \in R^2 \\ \cap \\ (b,c) \in R \end{matrix} \right\}$$

$$R^{n-1} \cdot R = R^n = \left\{ (a,c) / \exists b \text{ in } A, \begin{matrix} (a,b) \in R^{n-1} \\ \wedge \\ (b,c) \in R \end{matrix} \right\}$$

Q. $A = \{a_1, a_2\}$

$B = \{b_1, b_2, b_3\}$

$C = \{c_1, c_2, c_3\}$

$R_1 = \{(a_1, b_1), (a_2, b_3)\}$

$R_2 = \{(b_1, c_1), (b_1, c_3), (b_3, c_2), (b_2, c_1)\}$

$R_1 \cdot R_2 = \{(a_1, c_1), (a_1, c_3), (a_2, c_2)\}$

$\hookrightarrow R_1 \cdot R_2 \subseteq A \times C$

Ex. $R_1 = \{(a_1, b_1)\}$

$R_2 = \{(b_3, c_2)\}$

$R_1 \circ R_2 = \phi$

Ex. $R_1 = A \times B$

$R_2 = B \times C$

$R_1 \cdot R_2 = A \times C$

Claim:

$R_1 \subseteq A \times B$

$R_2 \subseteq B \times C$

$R_3 \subseteq B \times C$

$R_1 \cdot (R_2 \cup R_3) = R_1 \cdot R_2 \cup R_1 \cdot R_3$?

~~Sol~~ :

Sol : $(a,c) \in R_1 \cdot (R_2 \cup R_3)$

$\iff \exists b$ such that $(a,b) \in R_1 \land (b,c) \in R_2 \cup R_3$

$\iff \exists b$ such that $(a,b) \in R_1 \land [(b,c) \in R_2 \lor (b,c) \in R_3]$

$\iff \exists b$ such that $\underbrace{[(a,b) \in R_1 \land (b,c) \in R_2]}_{P(x)}$

$\lor \underbrace{[(a,b) \in R_1 \land (b,c) \in R_3]}_{Q(x)}$

$\boxed{a \land (b \lor c) = (a \land b) \lor (a \land c)}$

$\exists x \, (P(x) \lor Q(x)) \iff \exists x \, P(x) \lor \exists x \, Q(x)$

LHS $\iff$ RHS

LHS $\equiv$ RHS

LHS $\Rightarrow$ RHS

LHS $\Leftarrow$ RHS

$\iff \exists b\ ((a,b) \in R_1 \wedge (b,c) \in R_2) \vee$

$\exists b\ ((a,b) \in R_1 \wedge (b,c) \in R_3)$

$\iff (a,c) \in R_1 \cdot R_2 \vee (a,c) \in R_1 \cdot R_3$

$\iff (a,c) \in R_1 \cdot R_2 \cup R_1 \cdot R_3$

$\therefore$ Claim is True.

19/3/24

Claim :

Given :

$$\left.\begin{aligned} R_1 &\subseteq A \times B \\ R_2 &\subseteq B \times C \\ R_3 &\subseteq B \times C \end{aligned}\right\} \text{ Is } R_1(R_2 \cap R_3) \equiv R_1R_2 \cap R_1R_3 \text{ ?}$$

Solution :

Suppose $(a,c) \in R_1 \cdot (R_2 \cap R_3)$

$\iff \exists b$ S.T. $(a,b) \in R_1 \wedge (b,c) \in R_2 \cap R_3$

$\iff \exists b$ S.T. $(a,b) \in R_1 \wedge [(b,c) \in R_2 \wedge (b,c) \in R_3]$

$\iff \exists b\ [(a,b) \in R_1 \wedge (b,c) \in R_2] \wedge [(a,b) \in R_1 \wedge (b,c) \in R_3]$

$[a \wedge (b \wedge c) \equiv (a \wedge b) \wedge (a \wedge c)]$

$\Rightarrow \exists b\ [(a,b) \in R_1 \wedge (b,c) \in R_2] \wedge$

$\exists b\ [(a,b) \in R_1 \wedge (b,c) \in R_3)$

$\Rightarrow (a,c) \in R_1R_2 \wedge (a,c) \in R_1R_3$

$\Rightarrow (a,c) \in R_1R_2 \wedge R_1R_3$

$$\left[\begin{array}{ll} LHS \Leftrightarrow RHS & LHS \Rightarrow RHS \\ \therefore LHS \equiv RHS & \therefore LHS \subseteq RHS \end{array}\right]$$

$$R_1(R_2 \cap R_3) \subseteq R_1R_2 \cap R_1R_3$$

Given claim is not true.

→ ~~counter example~~

Ex. $R_1 = \{(a_1, b_1), (a_1, b_2)\}$

$R_2 = \{(b_2, c_1), (b_1, c_3)\}$

$R_3 = \{(b_1, c_1), (b_1, c_3)\}$

$R_2 \cap R_3 = \{(b_1, c_3)\}$

$R_1 \cdot (R_2 \cap R_3) = \{(a_1, c_3)\}$

$R_1R_2 = \{(a_1, c_3), (a_1, c_1)\}$

$R_1R_3 = \{(a_1, c_1), (a_1, c_3)\}$

Bor

$R_1R_2 \cap R_1R_3 = \{(a_1, c_1), (a_1, c_3)\}$

$\therefore R_1(R_2 \cap R_3) \subseteq R_1R_2 \cap R_1R_3$

Hence proved

$\Longleftrightarrow \exists b\ ((a,b) \in R_1 \wedge (b,c) \in R_2) \vee$

$\exists b\ ((a,b) \in R_1 \wedge (b,c) \in R_3)$

$\Longleftrightarrow (a,c) \in R_1 \cdot R_2 \vee (a,c) \in R_1 \cdot R_3$

$\Longleftrightarrow (a,c) \in R_1 \cdot R_2 \cup R_1 \cdot R_3$

$\therefore$ Claim is True.

19/3/24

Claim:

Given:

$$\left.\begin{aligned} R_1 &\subseteq A \times B \\ R_2 &\subseteq B \times C \\ R_3 &\subseteq B \times C \end{aligned}\right\} \text{ Is } R_1(R_2 \cap R_3) \equiv R_1R_2 \cap R_1R_3 \text{ ?}$$

Solution:

Suppose $(a,c) \in R_1 \cdot (R_2 \cap R_3)$

$\Longleftrightarrow \exists b$ S.T. $(a,b) \in R_1 \wedge (b,c) \in R_2 \cap R_3$

$\Longleftrightarrow \exists b$ S.T. $(a,b) \in R_1 \wedge [(b,c) \in R_2 \wedge (b,c) \in R_3]$

$\Longleftrightarrow \exists b\ [(a,b) \in R_1 \wedge (b,c) \in R_2] \wedge [(a,b) \in R_1 \wedge (b,c) \in R_3]$

$[a \wedge (b \wedge c) \equiv (a \wedge b) \wedge (a \wedge c)]$

$\Rightarrow \exists b\ [(a,b) \in R_1 \wedge (b,c) \in R_2] \wedge$

$\exists b\ [(a,b) \in R_1 \wedge (b,c) \in R_3]$

$\Rightarrow (a,c) \in R_1R_2 \wedge (a,c) \in R_1R_3$

$\Rightarrow (a,c) \in R_1R_2 \wedge R_1R_3$

[LHS $\Leftrightarrow$ RHS   LHS $\Rightarrow$ RHS
$\therefore$ LHS = RHS $\therefore$ LHS $\subseteq$ RHS]

$$R_1(R_2 \cap R_3) \subseteq R_1R_2 \cap R_1R_3$$

Given claim is not true.

~~counter Example~~

Ex. $R_1 = \{(a_1, b_1), (a_1, b_2)\}$

$R_2 = \{(b_2, c_1), (b_1, c_3)\}$

$R_3 = \{(b_1, c_1), (b_1, c_3)\}$

$$R_2 \cap R_3 = \{(b_1, c_3)\}$$

$$R_1 \cdot (R_2 \cap R_3) = \{(a_1, c_3)\}$$

$$R_1R_2 = \{(a_1, c_3), (a_1, c_1)\}$$

$$R_1R_3 = \{(a_1, c_1), (a_1, c_3)\}$$

But

$$R_1R_2 \cap R_1R_3 = \{(a_1, c_1), (a_1, c_3)\}$$

$$\therefore R_1(R_2 \cap R_3) \subseteq R_1R_2 \cap R_1R_3$$

Hence Proved.

⇒ Closure of Relations:

→ Reflexive / Symmetric / Transitive closure

→ $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2)\} \cup \{(2,2), (3,3)\}$

$r(R) = \{(1,1), (2,2), (3,3), (1,2)\}$

$r(R) \longrightarrow$ Reflexive Closure of R

= The minimal superrelation of R which is reflexive.

Definition:

$r(R) = R \cup \{(a,a) \mid (a,a) \notin R \cap a \in A\}$

$r(R)$ should satisfy

(i) $r(R)$ is reflexive

(ii) $R \subseteq r(R)$

(iii) for any relation R', $R \subset R'$

$r(R) \subseteq R'$

Ex. $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}$

$r(R) = R$

$R' = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

→ Symmetric Closure :

~~A~~

$S(R) = R \cup \{(b,a) \mid (a,b) \in R \wedge (b,a) \notin R\}$

S(R) should satisfy

(i) S(R) should be symmetric

(ii) $R \subseteq S(R)$

(iii) For any symmetric relation R'

$R \subset R'$

$S(R) \subseteq R'$

Ex $R = \{(1,2)\} \cup \{(2,1)\}$

$S(R) = \{(1,2),(2,1)\}$

$[A = \{1,2,3\}]$

Ex. If $R = \phi$, $S(R) = R = \phi$

Ex. $R = \{(1,2),(2,3),(3,1)\}$

$S(R) = \{(1,2),(2,3),(3,1),(3,2),(1,3)\}$

i.e The minimal super relation of R which is symmetric.

→ Transitive Closure :

$$t(R) = \{R \cup \{(a,c) \mid (a,b) \in R \wedge (b,c) \in R \cap (a,c) \notin R\}\}$$

t(R) should satisfy

(i) t(R) is Transitive

(ii) $R \subseteq t(R)$

(iii) for any transitive relation R'

$R \subset R'$

$t(R) \subseteq R'$

Ex. $A = \{1,2,3\}$

$R = \{(1,2),(2,3),(3,1)\}$

$$t(R) = \{(1,2),(2,3),(3,1),(1,3),(2,1),(3,2),(1,1),(2,2),(3,3)\}$$

$$= A \times A$$

i.e. The minimum superrelation of R which is transitive

Check :

Is R transitive $\longrightarrow$ Yes , $t(R) = R \cup R^2$

$\downarrow$ No

If $R \cup R^2$ transitive $\longrightarrow$ Yes , $t(R) = R \cup R^2$

$\downarrow$ No

If $R \cup R^3$ transitive $\longrightarrow$ Yes , $t(R) = R \cup R^3$

$\downarrow$

......

$\downarrow$

$A \times A$

~~t(R)~~

$t(R)$ = Minimum 'i' S.T.

$$R \cup \bigcup_{j=2}^{i} R^{j}$$ is the transitive

We find Upper bound 'i'.

20/3/24

Note:

Closures do not exist for Asymmetric & Antisymmetric.

→ Relation as a Graph:

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,2)\}$

A Graph containing set of Vertices and set of Edges

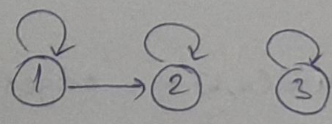
$V(G) = A$

$E(G) = R$

$A = \{1, 2, 3\}$

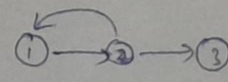
1 2 3

$R = \{(1,1), (2,2), (3,3), (1,2)\}$

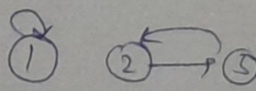


2, $A = \{1, 2, 3\}$,

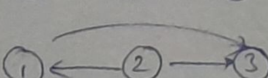
$R_1 = \{(1,2), (2,1), (2,3)\}$



$R_2 = \{(1,1), (2,3), (3,2)\}$



$R_3 = \{(1,3), (2,1), (2,3)\}$



~~$t(R)$ = Minimum 'i' S.T.~~

Jaggu I love you

- Haritha

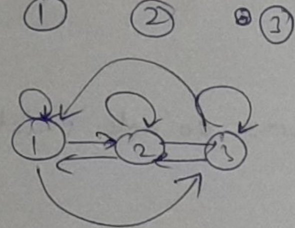
~~$R \cup \bigcup_{j=2}^{i} R^j$ is the transitive~~

~~We need to find Upper bound 'i'.~~

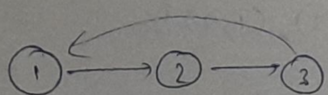
$R_4 = \phi \Rightarrow E(G) = \phi$

Null Graph

$R_5 = A \times A$



Ex. $R = \{(1,2), (2,3), (3,1)\}$

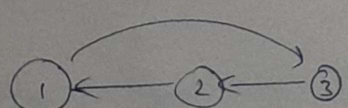


dist $(1,3) = 2$

dist $(2,1) = 2$

dist $(3,2) = 2$

$R^2 = R \cdot R = \{(1,3), (2,1), (3,2)\}$

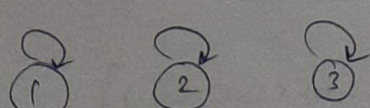


dist $(1,1) = 3$

dist $(2,2) = 3$

dist $(3,3) = 3$

$R^3 = R^2 \cdot R = \{(1,1), (2,2), (3,3)\}$



If $(a,b) \in R^n \longrightarrow$ dist $(a,b) = n$ edges

Equivalence Relations :

Given, $A = \{1,2,3\}$

$R_1 = \{(1,1),(2,2),(3,3)\}$

$R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

$R_3 = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$

$R_4 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

$R_5 = A \times A$

$[1]_{R_i}$	$[2]_{R_i}$	$[3]_{R_i}$
$\{1\}$	$\{2\}$	$\{3\}$
$\{1,2\}$	$\{1,2\}$	$\{3\}$
$\{1,3\}$	$\{2\}$	$\{1,3\}$
$\{1\}$	$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

Equivalence Class :

- Given set A, binary relation $R \subseteq A \times A$, Then

$$[a]_R = \{x \mid (x,a) \in R\}$$

$a \in A$

① $A = \{1,2,3\}$

$R_1 = \{(1,1),(2,2),(3,3)\}$

$[1]_{R_1} = \{1\}$

$[2]_{R_1} = \{2\}$

$[3]_{R_1} = \{3\}$

② $A = \{1,2,3\}$

$R_2 = \{(1,3),(3,1),(1,1),(3,3)\}$

$[1]_{R_2} = \{1,3\}$

$[2]_{R_2} = \emptyset$

$[3]_{R_3} = \{1,3\}$

Equivalence classes for Equivalence Relations

- II Partitions of Given element

II Equivalence Relations

→ Observations on eq. classes for eq. relations:

① $\bigcup_{x \in A} [x]_R = A$ - Reflexivity

② for $\forall x \in A$, $[x]_R \neq \emptyset$ - Reflexivity

③ If $b \in [a]_R$, then $a \in [b]_R$ - Symmetric

④ For $a, b \in A$

(i) $[a]_R = [b]_R$

(ii) $[a]_R \cap [b]_R = \emptyset$

Claim: $\forall a, b \in A$

$[a]_R = [b]_R$ (or) $[a]_R \cap [b]_R = \emptyset$

Proof: Suppose $[a]_R \wedge [b]_R \neq \emptyset$

$x \in [a]_R \wedge x \in [b]_R$

$(x, a) \in R \wedge (x, b) \in R$

Since symmetric: $(a, x) \in R$

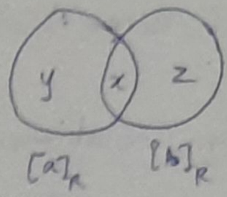
Since Transitive: $(a, b) \in R$

Suppose $y \in [a]_R \to (y, a) \in R$

Transitive $\to (y, b) \in R \to y \in [b]_R$

$[a]_R \subseteq [b]_R$ - ①

~~Maggi Yum Yum~~
~~. Yum Yum Yum~~
~~∴ Maggi Yum Yum~~



Suppose $z \in [b]_R$

$(z,b) \in R \wedge (b,a) \in R$

Transitive $(z,a) \in R \longrightarrow z \in [a]_R$

$\cancel{[b]}_R$

$[b]_R \subseteq [a]_R$ — ②

$\therefore [a]_R = [b]_R$

→ **Counting no. of Equivalent relations:**

Partioning given set $A = \{1, 2, \dots n\}$

~~No.~~

No. of Eq. relations = No. of Partitions of A

$A = \{A_1, A_2, \dots A_k\}$ s.t $A_i \cap A_j = \emptyset$ &

$\bigcup_{i=1}^{k} A_i = A$.

~~$A = \{\} \{A\}$~~ $\{$

Ex. $A - \{1\} - R = \{(1,1)\} \longrightarrow B_1 = 1$

Ex. $A - \{1,2\} \rightarrow R - \{(1,1),(2,2)\}$

$R_2 = A \times A \longrightarrow B_2 = 2$

$A = \{1,2,3\}$

$B_3 = 5$

$B_4 = ?$

$B_n = ??$

No. of Given $A = \{1, 2, \dots n\}$

Eq. relations = No. of Partitions of A

$A = \{A_1, A_2, \dots A_r\}$ $[1 \le r \le n]$

Find A_i containing n^{th} element

$A_1, A_2, \dots \{\dots, n\}, A_{i+1} - A_r$

No. of ~~the~~ elements in $|A_i \setminus n| = k$

$k = 0, 1, 2, \dots n-1$

$\therefore$ Bells Number, $B_n = \sum_{k=0}^{n-1} {}^{(n-1)}C_k \, B_{n-k-1}$

[Recurrence Relation]

~~$B_3 = \{1, 2, 3\}$~~

Ex. $A = \{1, 2, 3\}$

$B_3 = ?$

$k = 0 \Rightarrow$ $\left\{ \begin{array}{l} \{1,2\} \; \{3\} \\ \{1\}\{2\} \; \{3\} \end{array} \right\} = {}^2C_0 \cdot B_2 = 1 \cdot 2$

$k = 1 \Rightarrow$ $\left\{ \begin{array}{l} \{2\} \; \{1,3\} \\ \{1\} \; \{2,3\} \end{array} \right\} = {}^2C_1 \times B_1 = 2 \cdot 1$

$k = 2 \Rightarrow$ $\{1, 2, 3\}\} = {}^2C_2 \times B_0 = 1 \cdot 1$

$B_3 = 5$

Ex. $A = \{1,2,3,4\}$

$B_4 = ?$

$$\begin{array}{ccccc} & & & & 1 \\ & & & 1 & & 1 \\ & & 1 & & 2 & & 1 \\ & 1 & & 3 & & 3 & & 1 \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$B_4 = \sum_{k=0}^{3} {}^3C_k \, B_{3-k}$$

$$= {}^3C_0 B_3 + {}^3C_1 B_2 + {}^3C_2 B_1 + {}^3C_3 B_0$$

$$= B_3 + 3B_2 + 3B_1 + B_0$$

$k=0$ $\quad \left.\{1,2,3\} \ \{4\}\right\} \rightarrow {}^3C_0 \cdot B_3 = 5$

$k=1$ $\quad \left.\begin{matrix}\{2,3\}, \{1,4\} \\ \{1,3\}, \{2,4\} \\ \{1,2\}, \{3,4\}\end{matrix}\right\} \rightarrow {}^3C_1 \cdot B_2 = 6$

$k=2$ $\quad \left.\begin{matrix}\{3\} & \{1,2,4\} \\ \{2\} & \{1,3,4\} \\ \{1\} & \{2,3,4\}\end{matrix}\right\} \rightarrow {}^3C_2 B_1 = 3$

$k=3$ $\quad \left.\{1,2,3,4\}\right\} \rightarrow {}^3C_3 B_0 = 1$

$B_4 = 15$

$\boxed{B_0 = 1} \Longrightarrow \boxed{B_1 = 1} \Rightarrow$

$B_2 = B_1 + B_0 = 2 \Rightarrow \boxed{B_2 = 2}$

$B_3 = B_2 + 2B_1 + B_0$

$= 2 + 2 + 1 = 5$

$\boxed{\therefore B_3 = 5}$

$B_5 = B_4 + 4B_3 + 6B_2 + 4B_1 + B_0$

$= 15 + 20 + 12 + 4 + 1$

$\Rightarrow B_5 = 52$

26/3/24

→ **Revisiting Partial Order :** (POSET)

(i) Reflexive

(ii) Anti-symmetric

(iii) Transitive

→ **Trichotomy Property :** (TP)

A relation $R \subseteq A \times A$ Satisfies TP if

$\forall a, \forall b \in A$ either $(a,b) \in R$

or $(b,a) \in R$

or $a = b \in R$

→ **Total order :**

A relation R is total order if

(i) Partial order

(ii) TP

$A = \{1,2,3\}$

$R = \{(1,1), (2,2), (3,3), (1,2), (1,3)\}$

TP → Not Satisfied

• **Counting No. of TP relations :**

Given $|A| = n$

$|A \times A| = n^2$

Binary Relation $= 2^{n^2}$

Sol.

$\underbrace{\sqcup_{(1,n)} \quad \sqcup_{(2,n)} \quad \sqcup_{(3,n)} \quad \ldots \quad \sqcup_{(n,n)}}$

n diagonal elements - n boxes = 1

$A \times A = \{(1,1), (2,2), \ldots, (n,n)\}$

$$\underbrace{\boxed{\begin{matrix}(1,2)\\(2,1)\end{matrix}} \; \boxed{\begin{matrix}(1,3)\\(3,1)\end{matrix}} \; \cdots\cdots \; \boxed{\begin{matrix}(a,b)\\(b,a)\end{matrix}}}_{}$$

$a \neq b$

$\frac{n^2-n}{2}$ boxes – { 1^{st} element, 2^{nd} element

$$= 1 \times 2^{\frac{n^2-n}{2}}$$

$\therefore$ $2^{\frac{n^2-n}{2}}$ no. of TP relations

$\therefore$ No. of elements in a TP :

Reflexive

min $= n$ max $= n^2$

$$= \underbrace{n}_{\text{diagonals}} + \underbrace{\frac{n^2-n}{2}}_{\text{box.}}$$

$$= \frac{n^2+n}{2}$$

$\therefore$ No. of elements in Total Order :

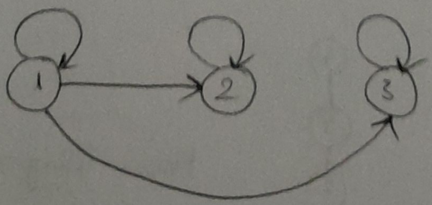
$$\frac{n^2+n}{2}$$

Ex. $R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (3,1)\}$

↳ Trichotomy Property ✓

Total Order ✗ ($\because$ Partial Order ✗ [Transitivity])

→ **Hasse Diagram** :

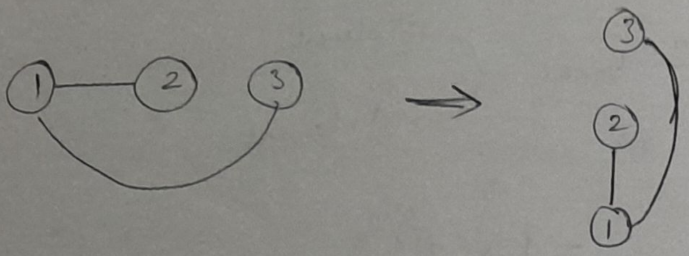


(1) No Reflexive arcs

(2) Undirected graph

~~(3) Not b~~

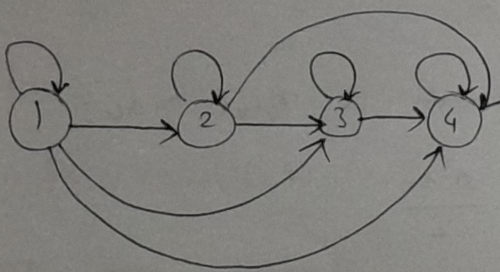
(3) No transitive arcs



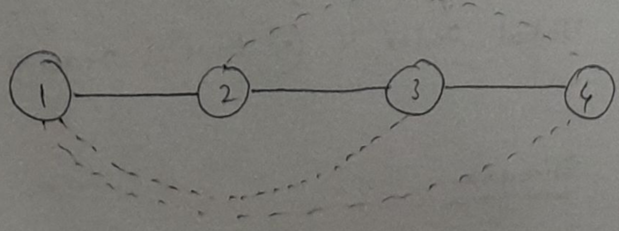
Hasse diagram represents Order among elements.

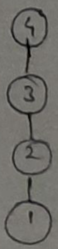
Ex $A = \{1,2,3,4\}$

$R = \{(1,1), (2,2), (3,3), (4,4), (3,4), (1,2), (1,3), (2,3), (2,4), (1,4)\}$



Hasse diagram:

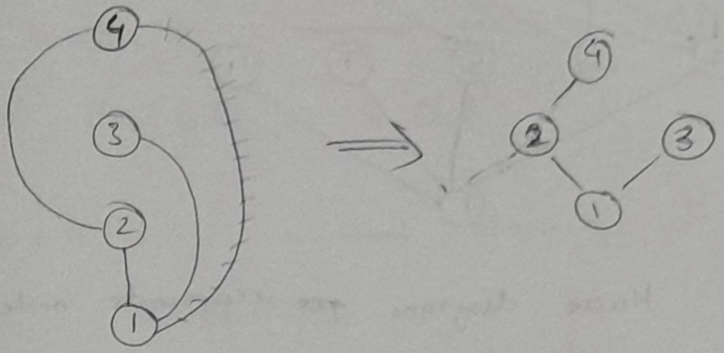


$\Rightarrow$  Hasse Diagram

→ Hasse Diagram for Partial Order:

$A = \{1, 2, 3, 4\}$

$R = \{(a, b) / a \text{ divides } b\}$



⇒ Hasse Diagram for Total order

$A = \{1, 2, 3, 4\}$

$R = \{(a, b) / a \leq b\}$

$1 \leq 2$ & $2 \leq 4$, $1 \leq 4$

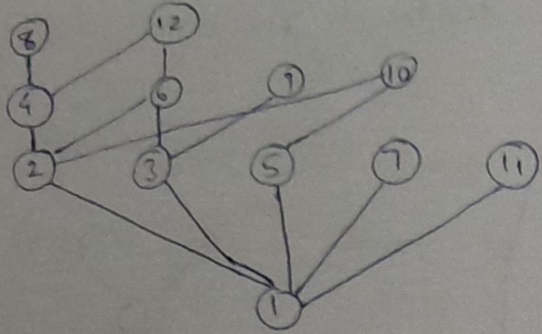
$\forall a, \forall b \in A$ – TP

[Linear Chain]

Hasse Diagram for Total order will always be Linear chain

Ex. $A = \{1, 2, 3, 4, \ldots 12\}$

27/3/21 $R = \{(a,b) \mid a \text{ divides } b\}$



Hasse diagram ~~[illegible]~~ represents order among elements.

→ <u>Greatest</u> & <u>least</u> <u>elements</u> <u>in</u> <u>POSET</u> :

→ let $(A, \preceq)$ a POSET and $B \subseteq A$ (~~[illegible]~~
($\preceq$ means some relation)

→ $b \in B$ is a greatest element if all $b' \in B$,
$b' \preceq b$ [b dominates all of B]

→ $b \in B$ is least element if all $b' \in B$,
$b \preceq b'$ [b is dominated by all of B]

B	Greatest	Least
$\{1,2,3,6\}$	$\{6\}$	$\{1\}$
$\{2,3,5\}$	NIL	NIL
$\{2,3,6\}$	$\{6\}$	NIL
$\{3,6,12\}$	$\{12\}$	$\{3\}$
$\{7\}$	$\{7\}$	$\{7\}$
$\{2,4,6,12\}$	$\{12\}$	$\{2\}$

→ Maximal & Minimal elements in POSET

→ Let $(A, \preceq)$ a POSET and $B \subseteq A$

($\preceq$ meaning some relation)

→ Let $b \in B$ is a ~~minimal~~ maximal element

if $\nexists\, b' \in B$, $b \preceq b'$

→ Let $b \in B$ is a minimal element, if $\nexists\, b' \in B$,

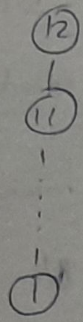
$b' \preceq b$

B	Maximal	Minimal
{1, 2, 3, 6}	{6}	{1}
{2, 3, 5}	{2, 3, 5}	{2, 3, 5}
{2, 3, 6}	{6}	{2, 3}
{3, 6, 12}	{12}	{3}
{7}	{7}	{7}
{2, 4, 6, 12}	{12}	{2}
A	{8, 12, 9, 10, 7, 11}	{1}

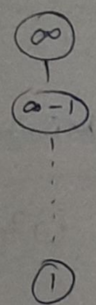
→ Differences :

Greatest & Least	Maximal & Minimal
(i) If exists, Unique element	(i) Multiple elements
(ii) Can be ϕ [NIL]	(ii) $\neq \phi$

Ex. $R = \{(a,b) / a \leq b\}$

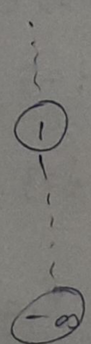


Ex. $A = (N, \leq)$



No Maximal

Ex. $A = (I, \geq)$



No Minimal

Upper and lower Bounds in POSET:

- Let $(A, \preceq)$ a POSET and $B \subseteq A$
  ($\preceq$ meaning some relation)

- $b \in A$ is an Upper bound for B if
  $\forall b' \in B$, $b' \preceq b$

- $b \in A$ is an LB for B if $\forall b' \in B$, $b \preceq b'$

B	UB	LB	LUB	GLB
{1, 2, 3, 6}	{6, 12}	{1}	{6}	{1}
{2, 4}	{4, 8, 12}	{1, 2}	{4}	{2}
{3, 5}	NIL	{1}	NIL	{1}
{2, 3}	{6, 12}	{1}	{6}	{1}
A	NIL	{1}	NIL	{1}
{5}	{5, 10}	{1, 5}	{5}	{5}
{3}	{3, 6, 9, 12}	{1, 3}	{3}	{3}

GLB & LUB: $\in B$ / $\in A \setminus B = A - B$

1/4/2

→ **Partial order :**

$(N, a \text{ divides } b)$

→ **Total order :**

$(N, a \leq b)$

→ **Well order :**

- Total order
- $\forall$ subset $A' \subseteq A$, $A' \neq \phi$
  if there exists a least element

Ex. $(N, a \leq b)$

Ex $(I, a \leq b)$ PO ✓
TO ✓
WO ✗ $(-\infty)$

FUNCTIONS

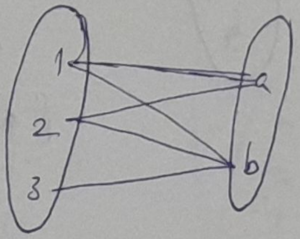
→ Given A, $A \times A$, $R \subseteq A \times A$

Given B, $A \times B$, $R \subseteq A \times B$

$A = \{1, 2, 3\}$

$B = \{a, b\}$

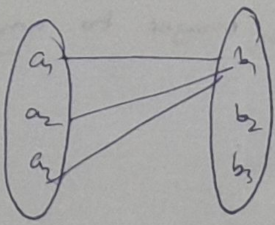
Then $R = \{(1,a), (2,b), (1,a), (2,b), (3,b)\}$



- A function is a selection $R \subseteq A \times B$

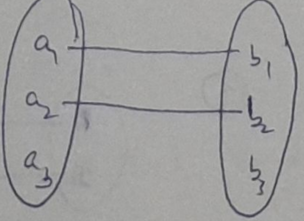
s.t. $\forall a \in A \;\; \exists!\, b \in B \,,\; (a,b) \in R$

Ex. 1



✓ Function

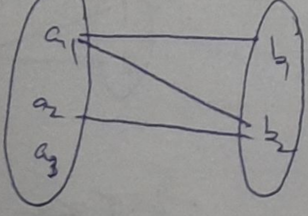
2



✗ Not a function

a_3 has no image

3

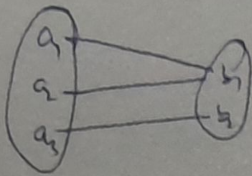


✗ Not a function

a_3 has no image &

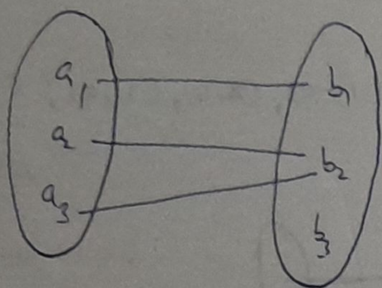
a_1 has 2 images

(4)



✓ function

→



$b_1 = \text{image}(a_1)$

$b_2 = \text{image}(a_2)$

$= \text{image}(a_3)$

$a_1 = \text{pre-image}(b_1)$

$\{a_2, a_3\} = \text{pre-image}(b_2)$

→ **Rules of not function:**

(i) If there is a lonely element in the domain, it is not a function.

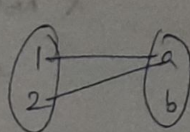
(ii) If there are multiple images to any single element.

→ **Counting no. of functions:**

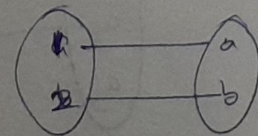
$|A| = 2$ & $|B| = 2$

$A = \{1, 2\}$ $B = \{a, b\}$

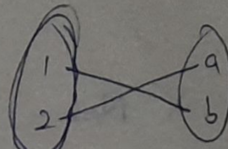
(1)



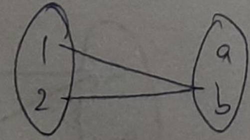
(2)



(3)



(4)



$2^2 = 4$

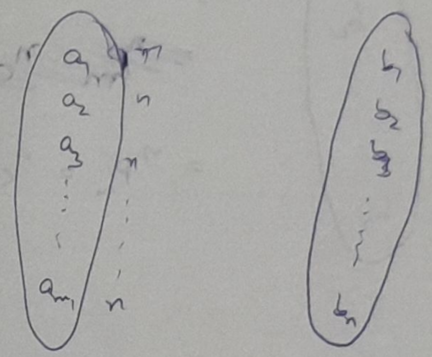
Total no. of Binary Relations $= 2^{2 \cdot 2} = \underline{16}$

If $|A| = 3$, $|B| = 2$

No. of functions $= 8$

$\therefore$ No. of functions, given $|A| = m$ & $|B| = n$, is n^m.

i.e. $|A| = m$ & $|B| = n$ & $A \to B$



No. of Binary relations $= 2^{nm}$

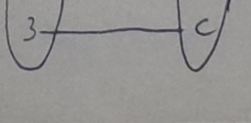
No. of functions $= n^m$

$= |B|^{|A|}$

→ <u>One-to-one</u> [Injective]

Given $|A| = m$
$|B| = n$

A function is 1-1 $\longleftrightarrow$ $\nexists a\ \nexists b$ S.T. $f(a) = f(b)$
$a \neq b$

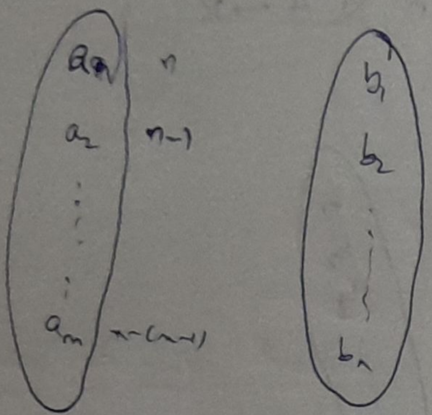


function ✓
one-one ✓

→ Observation on 1-1 function :

$f : A \to B$ $|B| \geq |A|$, Then only 1-1 fns are possible otherwise no 1-1 fns are possible

$|A| = m$, $|B| = n$



No. of 1-1 fns :

$= n(n-1)(n-2) \cdots n-(m-1)$

$= {}^nP_m = {}^nC_m \times m!$

3/4/24

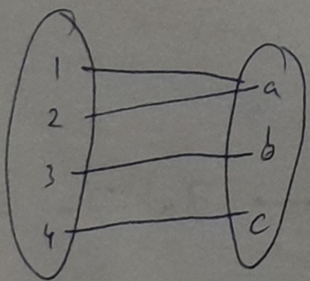
→ Onto function :

① function f

② $\forall b \in B$, $\exists a \in A$ and $(a, b) \in f$

Ex.

①



Function ✓

1-1 ✗

Onto ✓

Condition :

$|A| > |B|$

→ Observation on Onto:

- Necessary : $|A| \geq |B|$, Then only Onto functions are possible otherwise not possible.

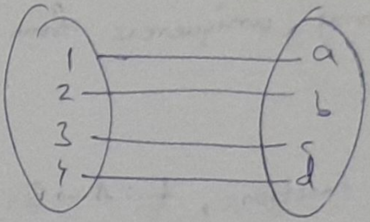
→ Bijective :

① Function f

② 1-1

③ Onto

Ex



function ✓
1-1 ✓
Onto ✓
Bijective ✓

Necessary :

$|A| = |B|$, Then only Bijective functions are possible, otherwise not.

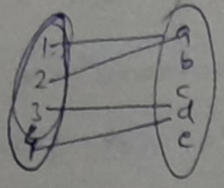
→ Observations & Remarks on Functions :

$f : A \to B$

→ 1-1 function : $|B| \geq |A|$
→ Onto function : $|B| \leq |A|$
→ Bijective function : $|B| = |A|$

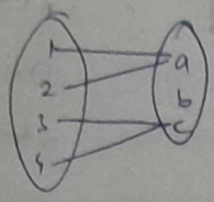
} Necessary but not sufficient

Ex. ① $|B| \geq |A|$ ② $|B| \leq |A|$ ③ $|B| = |A|$



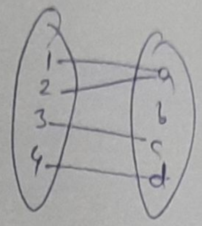
Function ✓

1-1 ✗



Function ✓

Onto ✗



Function ✓

Bij ✗

→ 1-1 function :

$|B| \geq |A|$

1-1 functions bring uniqueness from domain to codomain

Observation :

For any 1-1 function, $f : A \to B$

The function $g : A \to f(A)$ is always Bijective fn

Bijective function → Inverse is Possible

→ Onto functions :

$f : A \to B$, $|B| \leq |A|$

onto functions bring Uniqueness from co-domain to Domain (Pre-images)

Observation :

A function is onto if Range = Co-domain

Ex. ① $f : N \to N$

$f(x) = x+1$ → One-One ✓

Onto ✗ 1 is not having Pre Image

② $f: N \to N$

$f(x) = x \longrightarrow$ One-one ✓

$1 \Rightarrow$: Onto ✓

Bij. ✓

③ $f: Z \to Z$

$f(x) = x^2 \longrightarrow$ One-one ✗

onto ✗

④ $f: Z \to Z$

$f(x) = x^3 \longrightarrow$ One-one ✓

onto ✗

⑤ $f: N \to N$

$f(x) = x+2$ if $x \geq 5$

$f(x) = x-2$ if $x \leq 5$

function ✗ 1, 2 are not having any images & 5 has two images

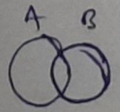
→ **Principle of Inclusion & Exclusion:**

Claim: How many integers from 1 to 100 divisible by 3 or 5.

C_3 = No. of Ints divisible by 3 $= \lfloor \frac{100}{3} \rfloor = 33$ [3, 6, 9, 12, ...]

No. of Ints divisible by 5 $= \lfloor \frac{100}{5} \rfloor = 20$ [5, 10, 15, ...]

No. of Ints divisible by 3 & 5 i.e 15 $= \lfloor \frac{100}{15} \rfloor = 6$ [15, 30, 45, ...]



$33 + 20 - 6 = \underline{47}$

9) from 1 to 400, 3 (or) 5 (or) 7

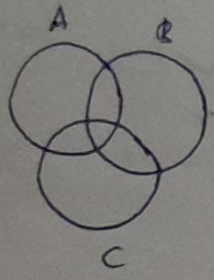
$\left\lfloor \frac{400}{3} \right\rfloor = \underline{133}$ $\left\lfloor \frac{400}{15} \right\rfloor = \underline{26}$

$\left\lfloor \frac{400}{5} \right\rfloor = \underline{80}$ $\left\lfloor \frac{400}{35} \right\rfloor = \underline{11}$

$\left\lfloor \frac{400}{7} \right\rfloor = \underline{57}$ $\left\lfloor \frac{400}{21} \right\rfloor = \underline{19}$

$\left\lfloor \frac{400}{105} \right\rfloor = 3$

$133 + 80 + 57 - 26 - 11 - 19 + 3 = \underline{\underline{217}}$



3/4/24

→ Counting Derangements:

Given a set $\{1, 2, 3, \ldots n\}$

How many permutations are there with no element appears in its natural position in the permutati

Sol: For any i, element i should not appear in position i.

$\{1, 2\}$ σ 1 2 ✗

2 1 ✓

$\{1, 2, 3\}$ σ 1 2 3 ✗

1 3 2 ✗

2 1 3 ✗

2 3 1 ✓

3 1 2 ✓

3 2 1 ✓

A_i : No. of σ ∋ i appear in position i

A_1 : $\underbrace{1 - 2 - -}_{\sigma}$ (n-1)!

A_2 : $\underbrace{- 2 - - -}_{\sigma}$ (n-1)!

A_i : $\underbrace{- - i - -}_{\sigma}$ (n-1)!

A_{12} : 1 2 $\underbrace{- - -}_{\sigma}$ (n-2)!

A_{23} : $\underbrace{- 2 3 - - -}_{\sigma}$ (n-2)!

A_{34} : $\underbrace{- - 3 4 - - -}_{\sigma}$ (n-2)!

Derangements : Total no. of permutations − Permutations not satisfying the property.

$$= n! - \Big[\underbrace{{}^nC_1 (n-1)!}_{\substack{\text{All } \sigma \\ \text{elements} \\ \text{in pos "i"}}} - \underbrace{{}^nC_2 (n-2)!}_{\substack{1 \\ 1243}} + {}^nC_3 (n-3)! \ldots + (-1)^{n+1}\, {}^nC_n (n-n)!\Big]$$

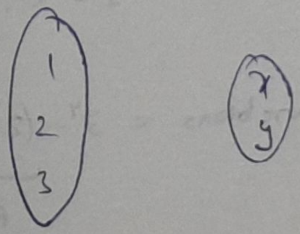
1243 ↓ Included twice. So exclude once.

$$\therefore D_n = n! - \Big[{}^nC_1 (n-1)! - {}^nC_2 (n-2)! + {}^nC_3 (n-3)! - \ldots + (-1)^{n+1}\, {}^nC_n (n-n)!\Big]$$

→ **Counting ~~Bo~~ Onto Functions** :

Derangements = Total no. of σ − σ not satisfying property

Onto functions = Total no. of σ − Non. onto. fns

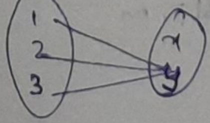


$|A| \geq |B|$

$\# \text{fns} = |B|^{|A|} = 2^3 = 8$

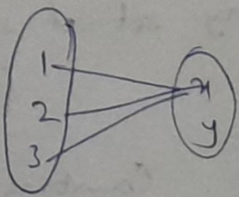
Non - Onto functions :

①

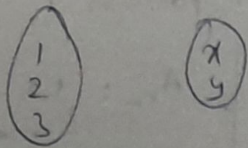


fn with no preimage for 'x'

②

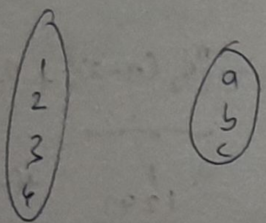


fn with no preimage for 'y'



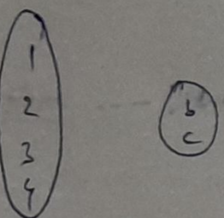
functions with no preimages for both x and y } — No function exists

No of onto functions $= 8 - 2 = 6$



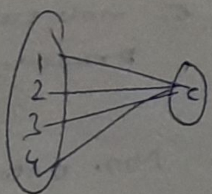
$= 3^4$ — Non-onto functions

{ fns with no preimage (a)
+ fns with no preimage (b)
+ fns with no preimage (c)
+ (a, b) + (b, c) + (c, a)



— 2^4

fns with no preimage (b) $= 2^4$
fns with no preimage (c) $= 2^4$
fns with no preimage (a) $= 2^4$



fns with no preimages (a, b) $= 1$
fns with no preimages (b, c) $= 1$
fns with no preimages (c, a) $= 1$

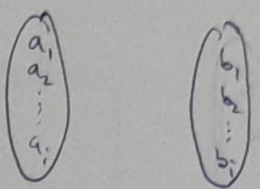
∴ Total no. of Onto Functions $= 3^4 - \underbrace{(2^4 + 2^4 + 2^4)}_{\text{Exclusion}} + \underbrace{1 + 1 + 1}_{\text{Inclusion}}$

$= 3^4 - {}^3C_1 \cdot 2^4 + {}^3C_2 \cdot 1^4$

Any set of size $|A| = n$, $|B| = m$
$[n \geq m]$

Counting no. of onto functions :-

$$m^n - \Big[\underbrace{{}^mC_1(m-1)^n}_{\text{Functions with exactly one } b_i \text{ not having preimage}} - {}^mC_2(m-2)^n + {}^mC_3(m-3)^n - \cdots + \cdots + (-1)^m\, {}^mC_{m-1}(1)^n\Big]$$



$|B|^{|A|} = (m-1)^n$

$a_1 \;\; b_1$

b_2

$a_n - b_n$

No preimage (b_1, b_2)

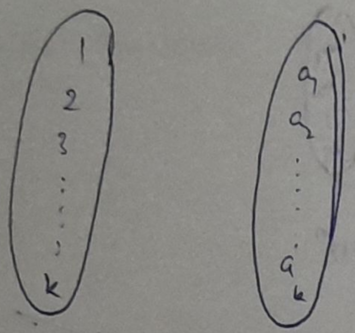
$\therefore$ Included twice

8/4/24

→ Finite and Infinite Sets :

- **Definition** : A set 'A' is finite if $\exists$ bijection from $\{1, 2, 3, \dots k\}$ to A, $k = |A|$

Example :



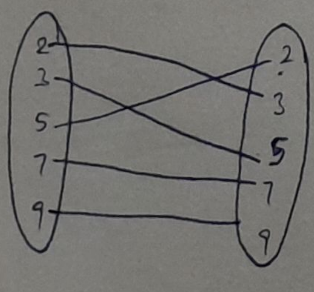
Observation : If there is no bijection, Then A is infinite set.

Ex. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ are Infinite sets

- **Definition** : A set 'A' is infinite if $\exists f : A \rightarrow A$ s.t, $f : 1-1$ & $f(A) \subset A$

~~Observation~~

Example :



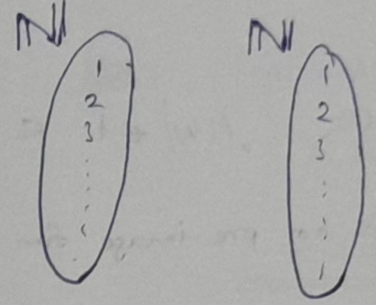
$f(A) = A$, Not $f(A) \subset A$

$\therefore$ Not an Infinite set

Observation:

① It is not possible to satisfy above definition for finite sets

② Any one-one function, $f: A \to A$ is also a bijective function



$f(x) = x$

$f: 1 \to 1$ ✓

$f(N) \subset N$ ✗

$[\because f(N) = N]$

$\therefore$ Not an Infinite set

If $f(x) = x+1$

$f: 1 \to 1$ ✓

$f(N) \subset N$ ✓

$\because f(N) = N - \{1\}$

$\therefore$ Infinite set

If $f(x) = x^2$

$f: 1 \to 1$ ✓

$f(N) \subset N$ ✓

$\therefore$ Infinite set

Claim:

N is an Infinite set

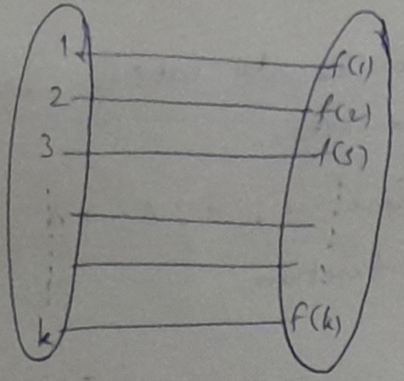
Proof by Contradiction:

Assume N is a finite set.

$\therefore$ Should agree the definition

Assume k is the cardinality of N.

i.e. $k = |N|$



$\max\left(f(1), f(2), f(3), \ldots, f(k)\right) + 1 \in N$

$\Rightarrow$ But there is no pre-image for above element.

$\therefore$ Not onto function

$\therefore$ Not Bijective function

$\therefore$ Not a finite set

i.e. Contradicting our Assumption

$\therefore$ N is not a finite set.

$\therefore$ N is an Infinite set.

Claim:

If A' is an Infinte set & $A' \subset A$, then A is an Infinite set

Proof:

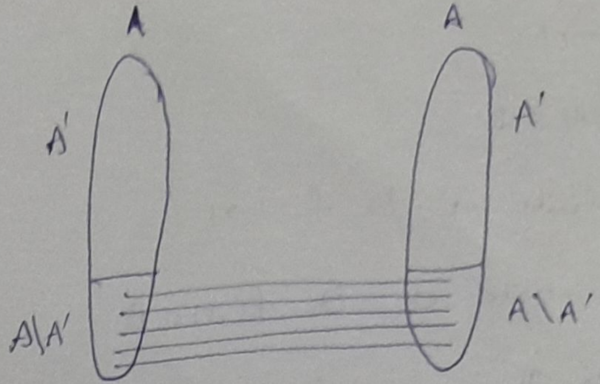
Given:

$g: A' \longrightarrow A'$	$f: A \to A$
$g: 1 \to 1$	$f: 1 \to 1$
$g(A') \subset A'$	$f(A) \subset A$

$\left.\right\}$ Need to be proved

$f(x) = x$ if $x \in A \setminus A'$

$f(x) = g(x)$ if $x \in A'$



$$f(A) = A \setminus A' \cup g(A)$$
$$= A \setminus A' \cup (\subset A')$$
$$\therefore f(A) \subset A \quad — (3)$$

Hence, A is an Infinite set.

__Claim:__

If A is an Infinite set, $f : A \to B$ is a 1-1 function, then B is an infinite set.

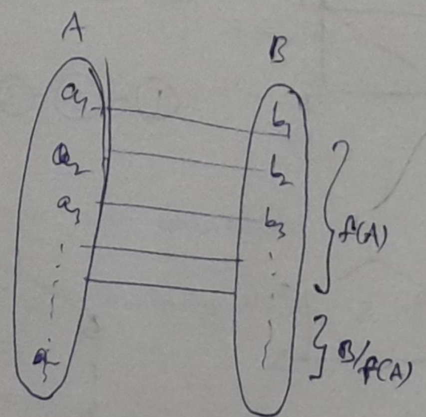
__Solution:__

Given A is an Infinite set

$g : A \to A$

$g : 1 \to 1$

$g(A) \subset A$



$f : A \to B$

$f : A \to f(A)$

Bijective function

$\therefore$ Inverse is possible w.r.t $A \to f(A)$

i.e. $f(A) \to A$

Since $f: 1 \to 1$

$\therefore f(A) \subset B$

A is infinite set & $f: 1 \to 1$

$\therefore f(A)$ is infinite set & $f(A) \subset B$

$\therefore$ Then B is an infinite set.

9/4/24

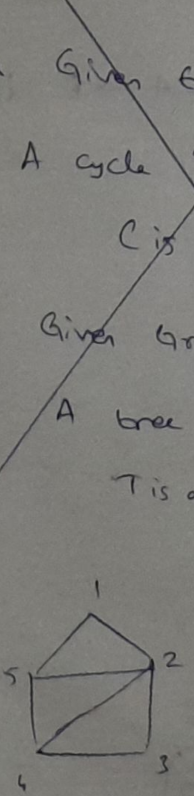
Spanning Tree / Cycle / Path :

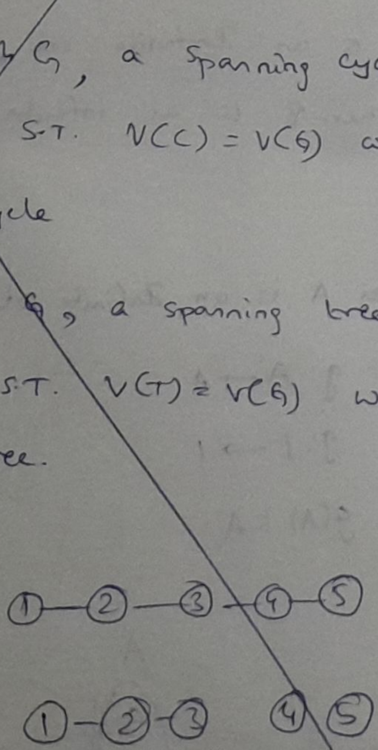
In a Given Graph G, a Spanning path is a subgraph H S.T. $V(H) = V(G)$ & H is a Path.

In a Given Graph G, a Spanning cycle :
A cycle 'C' S.T. $V(C) = V(G)$ where C is a cycle

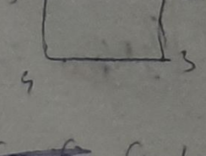
In a Given Graph G, a spanning tree :
A tree T S.T. $V(T) = V(G)$ where T is a tree.

G

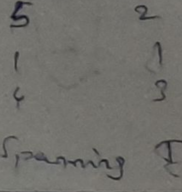




Spanning path

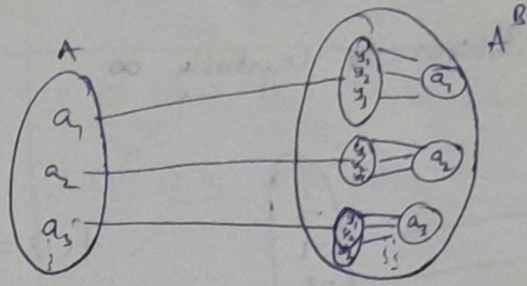


~~Graph~~ Spanning Cycle



Spanning Tree

③ A^B, All functions from $B \to A$



$f(a_i) = (g(a_i) = a_i)$ → Connecting element of A with element of A^B where the element of B is connected to the same element of A.

④ Power set of A

$f(a_i) = \{a_i\}$

$A \to P(A)$: 1-1 $\qquad f(a) \subset P(A)$

$A \not\subset P(A)$

P(A) is an ∞ set.

Theorem 2:

$P(A) = \{ \emptyset ; \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\} \}$

↳ '1-1' mapping is provided

$A \not\subset P(A)$ because in A $\{1,2,3\}$ are 3 different elements.

Not seperated by $\{1\}, \{2\}, \{3\}$

But in P(A), $\{1,2,3\}$ is one single element ∴ Theorem can't be used.

Infinite sets

→ Countable
→ Uncountable

Countable ↳ A set is countable if

- ∃ enumeration of A
- Listing of A
- Ordering of A

A set A is countable if ∃ bijection, $A \leftrightarrow \mathbb{N}$

Observations:

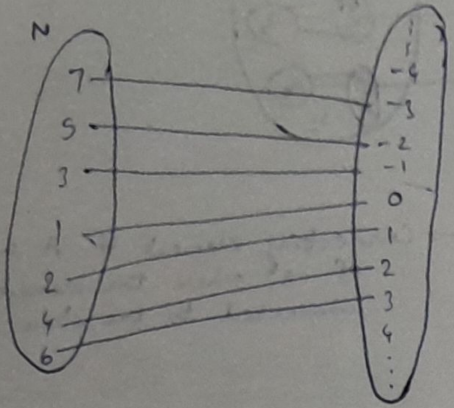
① All finite sets are countable

② Countable
- → Countably finite ∴ ∃ Enumeration, & ∃ bijection $\{1,2,3,\dots k\}$ to A
- → Countably Infinite ∴ ∃ enumeration & ∃ Bijection to $\mathbb{N}$

10/4/24 :

<u>Claim</u> : Set of Integers is Countably ∞

<u>Proof</u> :



$f(1) = 0$

$f(x) = \frac{x}{2}$ if x is even

$f(x) = \frac{1-x}{2}$ if x is odd

$N \rightarrow I$ is one-one function $|A| \leq |B|$

$I \rightarrow N$ is one-one function $|B| \leq |A|$

$N \leftrightarrow I$, There exists a bijection

$|I| = |N|$ $\quad [|A| = |B|]$

Both are countably ∞.

<u>Claim</u> : Set of Positive Rational numbers is countably ∞

<u>Proof</u> : $\frac{p}{q}$, $q \neq 0$

$p \in N$

$q \in N$

$\frac{1}{2} \equiv (1, 2)$

$\frac{1}{3} \equiv (1, 3)$

$$
\begin{array}{c|ccccc}
 & & & q \in N & & \\
\hline
 & (1,1) & (1,2) & (1,3) & (1,4) & \cdots \infty \\
 & (2,1) & (2,2) & (2,3) & (2,4) & \\
 & (3,1) & & & & \\
p \in N & (4,1) & & & & \\
 & (5,1) & & & & \\
 & \vdots & & & & \\
 & \infty & & & &
\end{array}
$$

$$|Q^+| = |N \times N|$$

$$N \longleftrightarrow N \times N$$

(Bijection)

$$|N| = |N \times N|$$

Similarly, Set of Negative Rational Numbers is countably ∞

__Claim:__

Set of all rational numbers is countably ∞

__Proof:__

$$Q = Q^+ \cup Q^-$$

$$
\begin{array}{c|l}
Q & \\
\hline
Q^+ & Q_1^+, Q_2^+, Q_3^+, \cdots \infty \\
Q^- & Q_1^-, Q_2^-, Q_3^-, \cdots \infty
\end{array}
$$

#STUDent

Claim :

$\Sigma = \{a, b\}$

Σ^* is countably ∞

Proof :

Σ^* = Set of all strings over $\{a, b\}$

Lexicographic Ordering	Standard Ordering
ϕ	ϕ
a	a
aa	b
aaa	aa
aaaa	ab
⋮	ba
b	bb
bb	⋮
bbb	
bbbb	
⋮	

Enumeration is not possible

Some string of size 'n'

$2^0 + 2^1 + 2^2 + \ldots\ldots 2^{n-1} + \text{Something} < 2^n$

Claim:

Set of all C-programs is contably ∞.

Proof:

Set of C-programs contains ASCII charecters

Σ = ASCII charecters

Set of C-programs = Σ^*

C-programs:	Standard ordering:
$\emptyset$ } size 0	size 0
256 } size 1	size 1
	size 2

Claim: Set of Real Numbers is Uncountabley ∞.

Proof: in $[0,1]$, Assume that it is contably ∞.

a_1	$0.a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15} \cdots \infty$
a_2	$0.a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25} \cdots \infty$
a_3	$0.a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35} \cdots \infty$
a_4	$\vdots$
a_5	$\vdots$
a_6	$\vdots$

$a_{ij} \in [0,9]$

$b = 0.y_1\ y_2\ y_3\ y_4 \cdots$

$y_i \in [0,9]$

$y_1 = 0$ if $a_{11} \neq 0$

$y_1 \neq 0$ if $a_{11} = 0$

$y_i = 0$ if $a_{ii} \neq 0$

$y_i \neq 0$ if $a_{ii} = 0$

$b \neq a_1$ Even though

$b \neq a_2$ $b \in [0,1]$ Enumeration is not possible.

$b \neq a_3$

$\therefore$ Our Assumption is wrong.

$\therefore$ $[0,1]$ is Uncountable

$[0,1] \subset R \longrightarrow R$ is also uncountable.

Claim:

Countable Union of Countable sets is Countable.

$$\left.\begin{array}{l} A = A_1 \cup A_2 \cup A_3 \cup \dots A_k \\ A = A_1 \cup A_2 \cup A_3 \cup \dots A_{\mathbb{N}} \end{array}\right\} \text{Countable}$$

$$\text{Here each } A_i \text{ is Countable} \longrightarrow \left\{\begin{array}{l} \text{Countably finite} \\ \text{Countably infinite} \end{array}\right.$$

Ex. $A = N \cup I \cup N \times N \cup \{1,2,3\} \cup \{a,b,c,d\}$

A	A is countable				
A_1	a_{11}	a_{12}	a_{13}	a_{14}	---
A_2	a_{21}	a_{22}	a_{23}	a_{24}	---
A_3					
A_4					
⋮					

We can establish Bijection

$A \leftrightarrow N$

$\therefore$ A is countably ∞

Claim:

Set of Real Numbers (Uncountable)

Set of Rational (C∞)

Set of Irrational (Uncountable)

Proof by Contradiction:

Let's Assume that set of Irrational numbers is C∞.

Set of Real Numbers = Set of ~~Reta~~ Rational $\cup$ Set of Irrational

C∞ = C∞ $\cup$ C∞

$\therefore$ Our Assumption is wrong

$\therefore$ Set of Irrational numbers ~~is~~ is uncountable

Claim:

Powerset ~~of~~ (N) is uncountable.

Proof:

Ex. $A = \{1,2,3\}$

$p(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{3,3\}, \{1,3\}, \{1,2,3\}\}$

P(A)	1	2	3
ϕ	0	0	0
$\{1\}$	1	0	0
$\{2\}$	0	1	0
⋮	⋮	⋮	⋮
$\{1,2,3\}$	1	1	1

Let's assume that P(N) is countable

There exists some enumeration

P(N)	1	2	3	- - - - ∞
A_1	0	0	0	- - - - ∞
A_2	1	0	0	
A_3	[illegible]	[illegible]	0	
A_4	1	0	1	

$S = \{ i / (A_i, i) = 0 \}$

$i \in S$ if $i \notin A_i$

$i \notin S$ if $i \in A_i$

Even though $S \in P(N)$

$\therefore$ Enumeration is not possible

$\therefore$ Our Assumption is wrong. $\Rightarrow$ P(N) is uncountable

Claim:

Powerset (Σ^*) is Uncountable

Solution:

Let's Assume that $P(\Sigma^*)$ is countable

$P(\Sigma^*)$	w_1	w_2	w_3	w_4	—
A_1	0	0	0	0	— —
A_2	0	1	0	0	— —
A_3	1	0	0	0	—
A_4	1	0	0	1	—
⋮					

$\Sigma = \{a, b\}$

$\Sigma^* = \{\emptyset, a, b, aa, ab, \ldots\}$

$\Sigma^* = \{w_1, w_2, w_3, \ldots\}$

$S = \{ w_i \,/\, (A_i, w_i) = 0 \}$

$w_i \in S$ if $w_i \notin A_i$

$w_i \notin S$ if $w_i \in A_i$

$S \notin A_1$

$S \notin A_2$

$S \notin A_3$

$S \in P(\Sigma^*)$ but there is no enumeration

$\therefore P(\Sigma^*)$ is uncountable.

Observations:

① $N \leftrightarrow A$ A is Countably ∞

Any $C\infty \longleftrightarrow A \longrightarrow$ If $\exists$ bijection

(2) $\mathbb{R} \leftrightarrow A \longrightarrow A$ is uncountable

Any Uncountable $\leftrightarrow A$

Ex. $[0,1] \longleftrightarrow [10,15]$

$$f(x) = 5x + 10$$

Ex. $[3,5] \leftrightarrow [10,15]$

$$\frac{x - \min}{\max - \min} \Rightarrow \frac{x-3}{5-3} = \frac{x-3}{2}$$

$$\left.\begin{array}{l} x=3 \Longrightarrow \frac{3-3}{2} = 0 \\ x=5 \Rightarrow \frac{5-3}{2} = 1 \end{array}\right\} [0,1]$$

$$[0,1] \times 5 + 10$$

$$\therefore \left(\frac{x-3}{2}\right) \times 5 + 10$$

$$\Rightarrow \frac{5}{2}(x-3) + 10$$

$$\Rightarrow \frac{5}{2}x + \frac{5}{2}$$

$$\Rightarrow \frac{5}{2}(x+1)$$

Ex. $[-3,3]$ to $[-4,4]$

$$\frac{x+3}{6} \longrightarrow \begin{array}{l} x=-3 \Rightarrow 0 \\ x=3 \Rightarrow 1 \end{array} \checkmark$$

$$\left(\frac{x+3}{6}\right) \times 8 + -4$$

$\Rightarrow (x+3) \times \frac{4}{3} - 4$

$\Rightarrow \underline{\frac{4}{3}x}$

~~A) Graph Theory~~

15/4/24

GRAPH THEORY

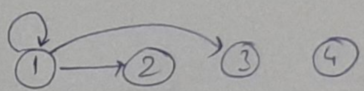
Notations in Graphs : Set of Vertices : $V(G)$

Set of Edges : $(E(G) \subseteq V(G) \times V(G))$

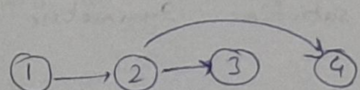
[Binary Relation]

Ex $V(G) = \{1, 2, 3, 4\}$

(1) $E(G) = \{(1,1), (1,2), (1,3)\}$

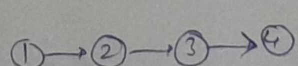


(2) $E(G) = \{(1,2), (2,3), (2,4)\}$



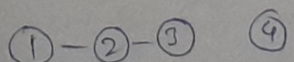
$$\text{Graphs} \begin{cases} \text{Directed} \\ \text{Undirected} \end{cases}$$

Directed :



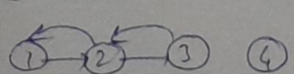
$\{(1,2), (2,3), (3,4)\}$

Undirected :



$\{(1,2), (2,1), (2,3), (3,2)\}$

=



Undirected is a Special case of Directed Graph.

Observation : $V(G)$ - Set

$$E(G) \subseteq V(G) \times V(G)$$

Any Binary Relation $\equiv$ A direct Graph

G1 $V(G) = \{1, 2, 3\}$

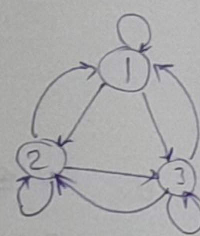
$E(G) = \emptyset$

σ 1 2 3

G2 $E(G) = \{(1,2), (2,3)\}$

$1 \rightarrow 2 \rightarrow 3$

G3 $E(G) = V(G) \times V(G)$



Observation : Undirected

$1 - 2$ $\{(1,2), (2,1)\}$

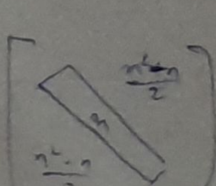
Undirected Graph satisfies Symmetric property in Binary relation.

$a - b$ $\{(a,b), (b,a)\}$

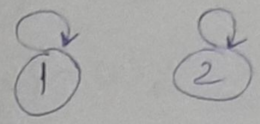
No. of Undirected graphs = No. of Symmetric binary relations

$$= 2^n \times 2^{\frac{n^2-n}{2}}$$

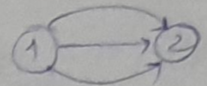
$$= 2^{\frac{n^2+n}{2}}$$



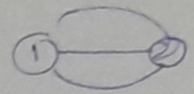
Simple Graph : A Graph with no self loops & parallel edges



(Self loop)



Parallel Edges (directed)



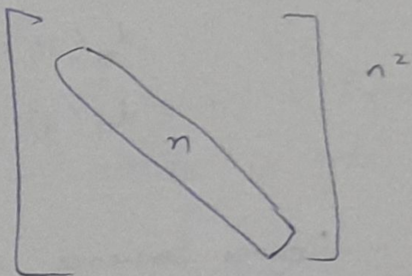
(Undirected)

No. of diff. Simple directed Graphs

= No. of Irreflexive Binary Relations

$= 1 \times 2^{n^2-n}$

$= 2^{n^2-n}$

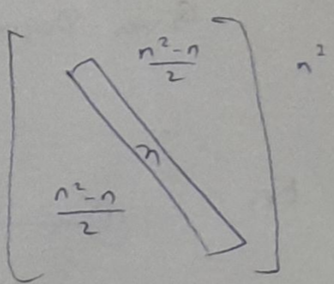


No. of Different Simple Undirected Graphs

= No. of Irreflexive Symmetric Binary Relations

$= 1 \times 2^{\frac{n^2-n}{2}}$

$= 2^{\frac{n^2-n}{2}}$



Focus : Simple Undirected Graphs

Definition : Neighbourhood of a vertex $v \in V(G)$

$$N_G(v) = \{ u \,/\, (v,u) \in E(G) \}$$

Undirected is a Special case of Directed Graph.

Observation : $V(G)$ - Set

$E(G) \subseteq V(G) \times V(G)$

Any Binary Relation = A direct Graph

G1 $V(G) = \{1, 2, 3\}$

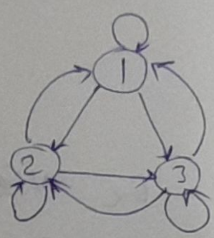
$E(G) = \emptyset$

1 2 3

G2 $E(G) = \{(1,2), (2,3)\}$

$1 \to 2 \to 3$

G3 $E(G) = V(G) \times V(G)$



Observation : Undirected

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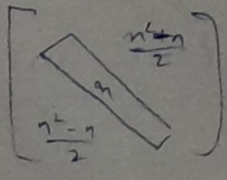
Undirected Graph satisfies Symmetric property in Binary relation.

a — b $\{(a,b), (b,a)\}$

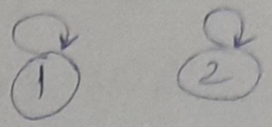
No. of Undirected graphs = No. of Symmetric binary relations

$$= 2^n \times 2^{\frac{n^2-n}{2}}$$

$$= 2^{\frac{n^2+n}{2}}$$

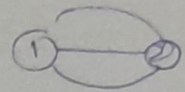


Simple Graph : A Graph with no self loops & parallel edges



(Self loop)

Parallel Edges (directed)



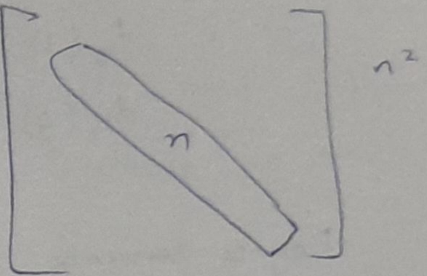
(Undirected)

No. of diff. simple directed Graphs

= No. of Irreflexive Binary Relations

$= 1 \times 2^{n^2-n}$

$= 2^{n^2-n}$

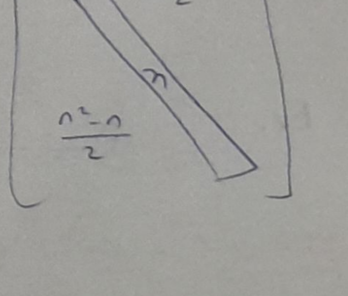


No. of Different Simple Undirected Graphs

= No. of Irreflexive Symmetric Binary Relations

$= 1 \times 2^{\frac{n^2-n}{2}}$

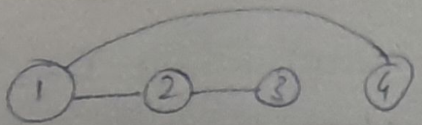
$= 2^{\frac{n^2-n}{2}}$



Focus : Simple Undirected Graphs

Definition : Neighbourhood of a vertex $v \in V(G)$

$$N_G(v) = \{u \mid (v,u) \in E(G)\}$$



$N_G(1) = \{2,4\}$

$N_G(2) = \{1,3\}$

$N_G(3) = \{2\}$

$N_G(4) = \{1\}$

$V(G) = \{1,2,3,4\}$

$E(G) = \{(1,2),(2,3),(1,4),$
$(2,1),(3,2),(4,1)\}$

Degree of Vertex, $d_G(v) = |N_G(v)|$

$d_G(1) = 2$

$d_G(2) = 2$

$d_G(3) = 1$

$d_G(4) = 1$

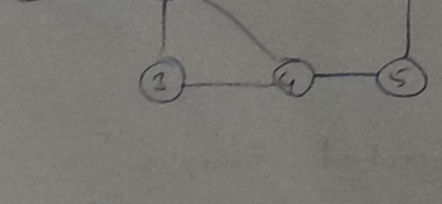
Degree sequence of a Graph $= \{d_G(1), d_G(2), \ldots\}$

$= \{2,2,1,1\}$

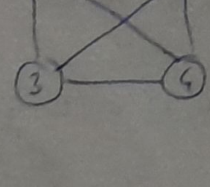
Given a Graph $\Rightarrow$ D_G degree sequence exists

Given Degree Sequence, does there exists a graph.

Ex.

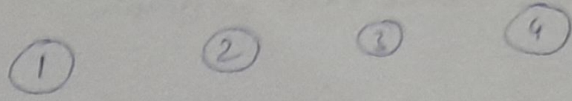


$D(G) = \{1,3,2,3,2,1\}$



$D(G) = \{3,3,3,3\}$

If G is given $\Rightarrow$ $D(G)$ exists



Ex. $(1, 1, 3, 4)$ — Graph is not possible

① Observation 1 : $\forall v \in V(G)$ $\quad 0 \le d_G(v) \le n-1$

$|V(G)| = n$

$V(G) = 4$
$d_G(V) = 4$ $\Big\} \Rightarrow$ Graph is not possible

② Observation 2 :

For any Graph G, $S = \sum d_G(v_i)$ is Even.

Sum of degrees of vertices is even.

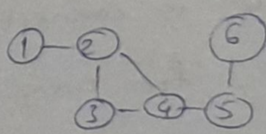
③ Observation 3 :

Sum of Degrees $= 2 \times |E(G)|$

$= 2 \times$ no. of Edges

$$\sum_{i=1}^{n} d_G(v_i) = 2 \times |E(G)|$$

 $(2, 2, 2) = \underline{\underline{6}}$



3 Edges

$1 + 3 + 2 + 3 + 2 + 1 = \underline{\underline{12}}$

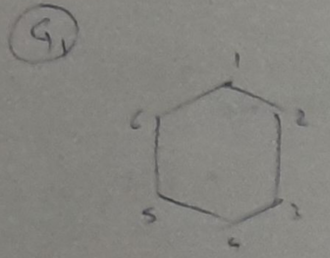
④ Observation 4 :

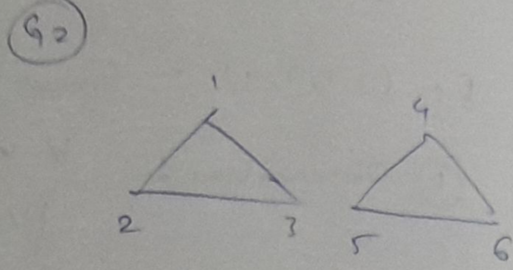
There will be even no. of vertices having odd degree

$$\underbrace{\sum_{i=1}^{n} d_G(v_i)}_{\text{even}} = \underbrace{\sum_{d_G(v):\text{odd}} d_G(v)}_{\text{even}} + \underbrace{\sum_{d_G(v):\text{Even}} d_G(v)}_{\text{even}}$$

16/4/24

(2,2,2,2,2,2) - Does exist a graph

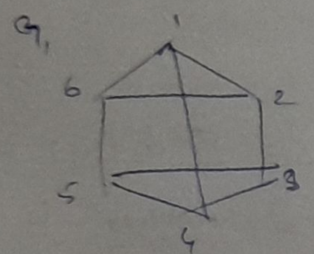


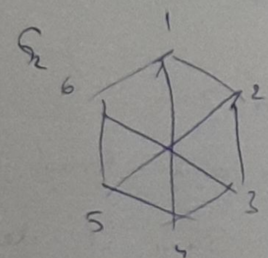


Connected

Disconnected

(3,3,3,3,3,3)





Both are connected

Triangles

~~Connected~~ 1

Connected:

A Graph is connected if $\forall u, v$, there exists a path (u, v) in G.

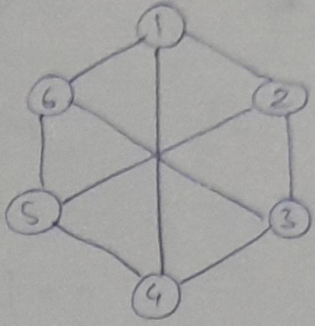
→ G_1 is connected because there exists path for every pair of vertices in G_1

→ G_2 is not connected (Disconnected)

Note: If G is disconnected, then G is a collection of connected components

↳ ~~Maxmal~~ Maximal connected Subgraphs

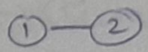
Subgraph: H is a subgraph of G ~~H ⊆ G~~
($H \subseteq G$)

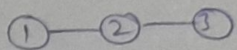


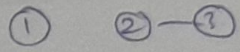
G

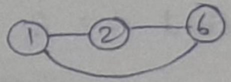
If $V(H) \subseteq V(G)$

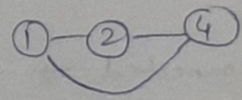
$E(H) \subseteq E(G) \longrightarrow H \subseteq G$

H_1  ✓

H_2  ✓

H_3  ✓

H_4  ✗

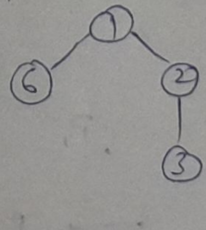
H_5  ✗

H is an Induced Subgraph of G

if $V(H) \subseteq V(G)$

$(u,v) \in E(H)$ iff $(u,v) \in E(G)$

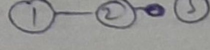
H_1



Sub ✓

Ind Sub ✗

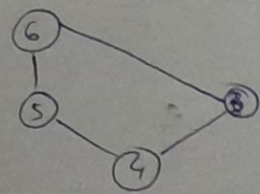
H_2



Sub ✓

Ind. Sub ✗

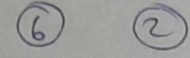
H_3



Sub ✓

2nd sub ✓

H_4

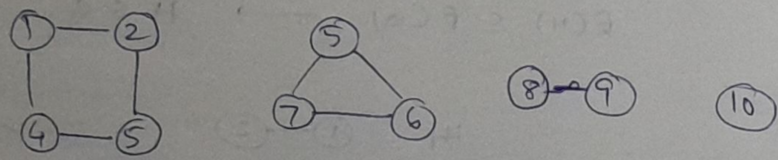


Sub ✓

2nd. sub ✓

Note:

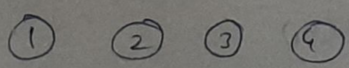
Graph G



G has 4 Connected Components

If $G \to$ connected $\Rightarrow$ G has exactly one connected component (G itself)

If $G \to$ disconnected $\Rightarrow$ G has ≥ 2 connected components



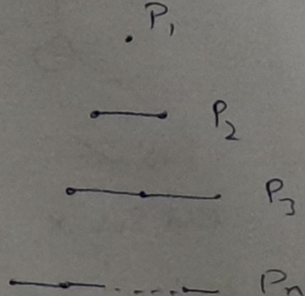
4 components

→ Some Special Graphs :

① Path Graphs :

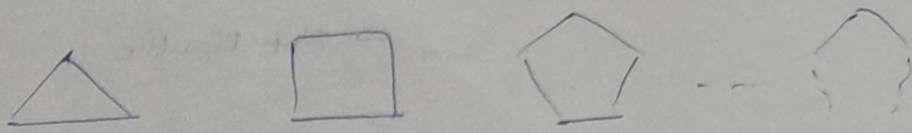
$|V(P_n)| = n$

$|E(P_n)| = n-1$



② Cycle Graphs :

If Closed ~~Circuit~~ Circuit



C_3 C_4 C_5 C_n

$|V(C_n)| = n$

$|E(C_n)| = n$

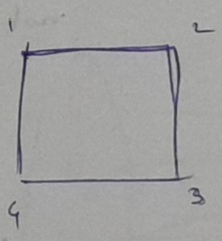
③ Bipartite Graphs :

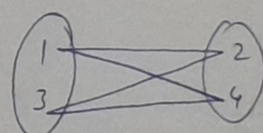
G is ~~Bi~~ Bipartite if $V(G) = V_1 \cup V_2$

S.T. $V_1 \cap V_2 = \phi$

$\forall (u,v) \in E(G)$ $u \in V_1$ and $v \in V_2$

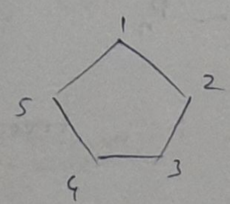
Ex.

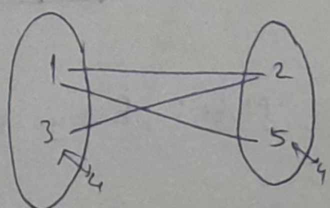




$C_4 \longrightarrow$ Bipartite

Ex.





C_5

Not Bipartite

4 will be problem

As $3 \rightarrow 4$ & $4 \rightarrow 5$

[Edge within same set]

Observations : All Even cycles are Bipartite

$C_{2n} \longrightarrow$ Bipartite

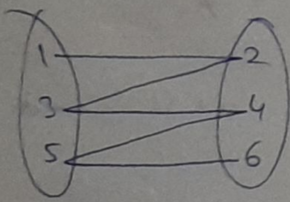
$C_{2n+1} \rightarrow$ Not Bipartite

Theorem : G is bipartite iff G has no odd cycles.

$\Longleftrightarrow$ G is bipartite iff G is odd cycle free.

$\Longleftrightarrow$ G is bipartite iff does not have C_{2n+1} by Subgraph.

Ex. Path Graphs



All path graphs are Bipartite graphs.
But the converse need not be true.

④ <u>Tree</u> :

Connected Acyclic Graph

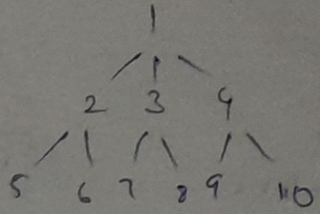
$|V(T_n)| = n$

$|E(T_n)| = n-1$

Ex.

<u>Note</u> : Every path Graph is a tree

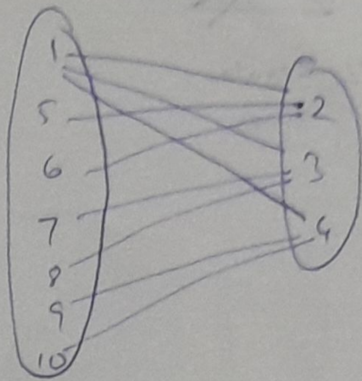
Ex.



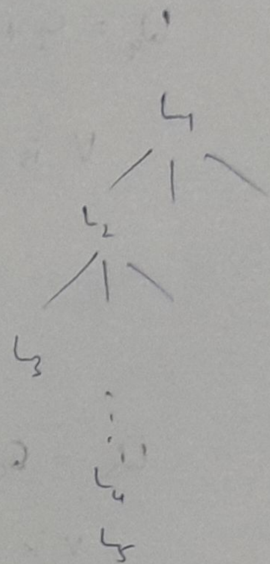
Every tree may not be path Graph

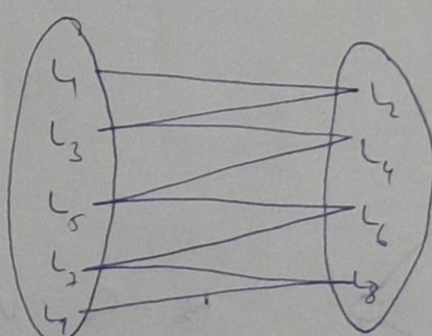
Note : Every tree is a Bipartite Graph
but the converse may not be true.

Ex.



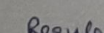
Proof :



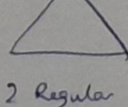


→ **K-Regular Graph :**

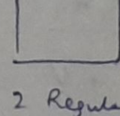
$$\forall v, \; d_G(v) = k$$



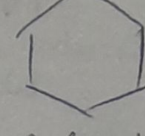
1 Regular Graph



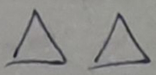
2 Regular Graph



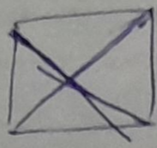
2 Regular Graph



2 Regular Graph

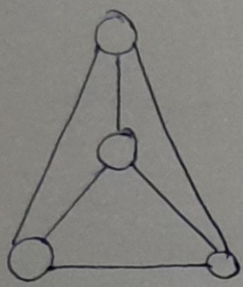


2 Regular Graph



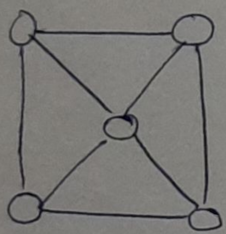
3 Regular Graph

→ **Wheel Graph:**



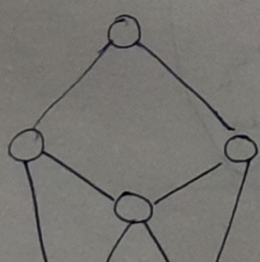
$W_4 = C_3 + V$

V is a Universal to all



$W_5 = C_4 + V$

V is Universal to all



$W_6 = C_5 + V$

V is a Universal to all

Note: Non-Bipartite

$|V(W_n)| = n$

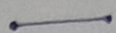
$W_n + C_{n-1} + V$

$|E(W_n)| = n-1 + n-1 = 2n-2$

→ Complete Graph:

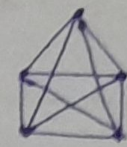
$E(G) = {}^nC_2 = \frac{n(n-1)}{2} = k_n$

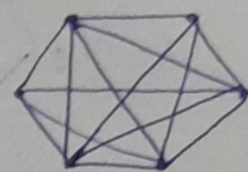
k_1 

k_2 

k_3 

k_4 

k_5 

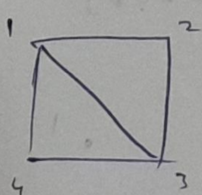
k_6 

$|V(k_n)| = n$

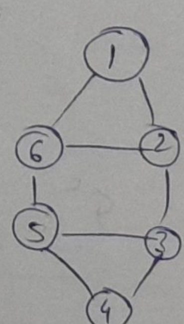
$|E(k_n)| = \frac{n(n-1)}{2}$

→ Induced ~~Graph~~ Cycle:

A chord in a cycle is an edge joining a pair of non-consecutive vertices in a cycle C.

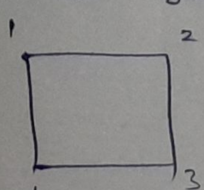


C_4 {1,3} is a chord

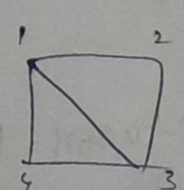


C_6 : {2,6} {3,5} chords

Induced Cycle: A cycle without any chord.

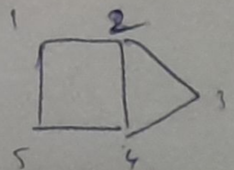


Induced C_4 ✓



Induced C_4 ✗

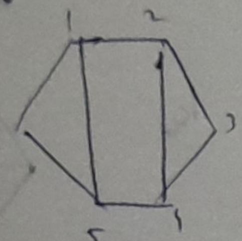
Induced C_3 ✓



Induced C_5 ✗

Induced C_4 ✓

Induced C_3 ✓



$C_6 \begin{Bmatrix} 2,4 \\ 3,5 \end{Bmatrix}$ - chords

Induced C_6 ✗

Induced C_5 ✗

Induced C_4 ✓

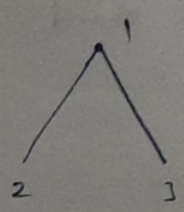
Induced C_3 ✓

→ **Complement of a Graph:**

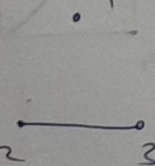
Given $G = (V(G), E(G)) = (V, E)$

$$V(G^c) = V(G)$$
$$E(G^c) = \{(u,v) \mid (u,v) \notin E(G)\}$$

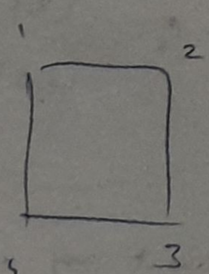
G :



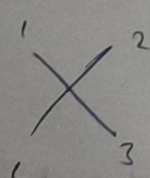
G^c :



G :



G^c :



$$|V(G)| = |V(G^c)| = n$$
$$|E(G)| = l \implies |E(G^c)| = {}^nC_2 - l$$

→ Self - Complementary Graph :

If G and G^c are same
↓
Structurally same

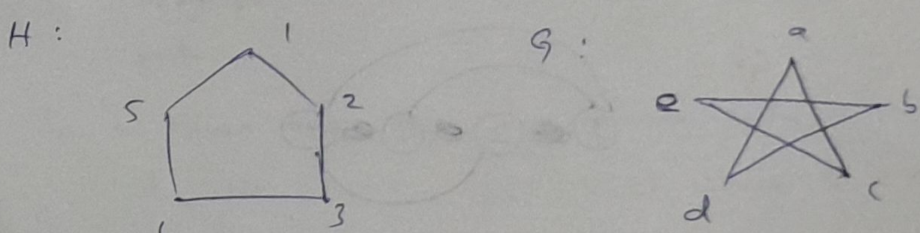
i.e. G and G^c are Isomorphic to each other.

Isomorphism :

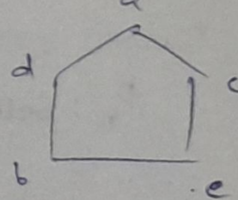
G and H are isomorphic if
$\exists$ bijection $V(H) \leftrightarrow V(G)$

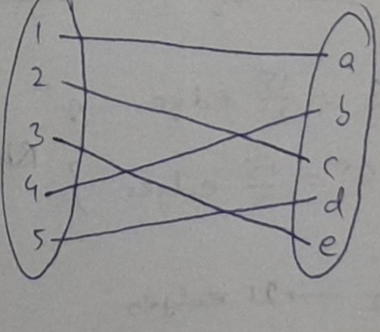
$f : G \to H$

$\forall (u, v)$, $(u,v) \in E(G)$ iff $(f(u), f(v)) \in E(H)$



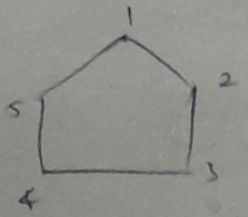
≅



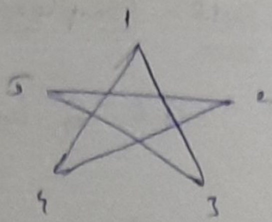


$f(1) = a$
$f(2) = c$
$f(3) = e$
$f(4) = b$
$f(5) = d$

G:



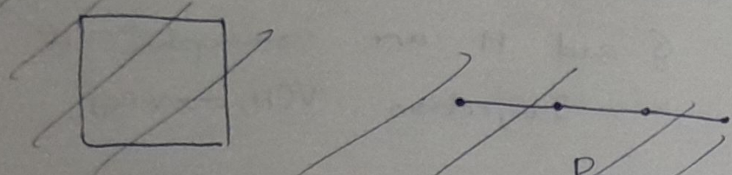
G^c:



Self Complementary Graph

~~Ex~~

~~G:~~



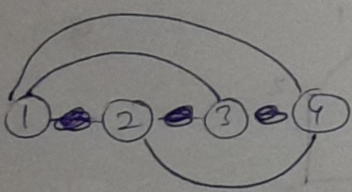
~~P_4~~

Ex

G

1 — 2 — 3 — 4 $\simeq P_4$

G^c:



$\cong$

3 — 1 — 4 — 2

P_4 : Self Complementary Graphs

Ex. Graph on 6 vertices — ${}^6C_2 = 15$ edges

$$\left.\begin{array}{l} G = \frac{15}{2} \text{ edges} \\ G^c = \frac{15}{2} \text{ edges} \end{array}\right\} \text{Not possible}$$

${}^7C_2 \longrightarrow 21$ edges

$\therefore$ 7 vertices Graph – Not Possible

P_3 : 1 - 2 - 3 G_1

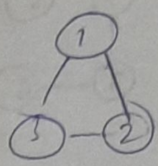
1 - 3 - 2 G_2

2 - 1 - 3 G_3

No. of $= {}^{n}P_2$

$= \frac{n!}{2!}$

G_3 :



$\equiv$ 1 - 2 - 3 - 1

No. of. $= {}^{n-1}P_2$

$= \frac{(n-1)!}{2!}$

17/4/24

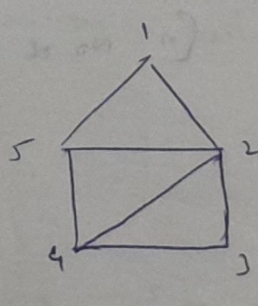
Spanning Tree / Cycle / Paths :

In a Given Graph 'G', A spanning path is a subgraph 'H' S.T $V(H) = V(G)$ & H is a path.

In a Given Graph 'G', A spanning cycle :
A cycle 'C' S.T. $V(C) = V(G)$ where C is a cycle

In a Given Graph 'G', A spanning tree :
A tree T S.T. $V(T) = V(G)$ where T is a tree.

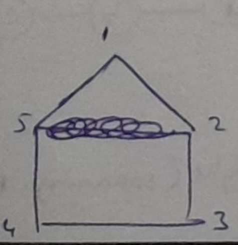
G :



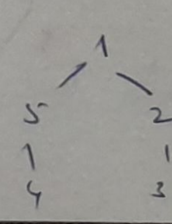
1 - 2 - 3 - 4 - 5

~~1 - 2 - 3 - 4 - 5~~

[Spanning Paths]

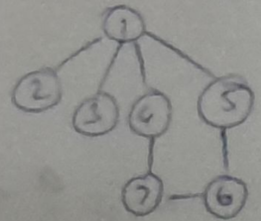


[Spanning Cycle]



[Spanning Tree]

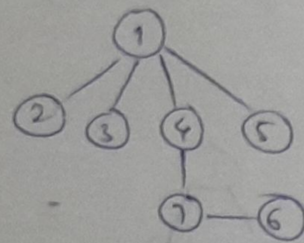
G:



Spanning path - Not possible

Spanning cycle - Not possible

Spanning Tree - Possible



Observations:

① If G is connected, then ∃ spanning tree

② If G is connected, Spanning path (or) Spanning cycle may or may not be possible

③ If G is connected, Then ∃ ≥ 1 Spanning Tree

No. of Undirected spanning paths $= \frac{n!}{2}$

[n: No. of vertices]

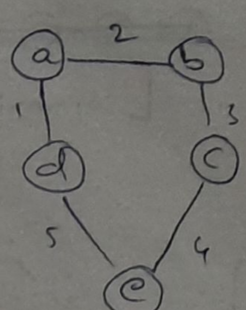
No. of Spanning cycles $= \frac{(n-1)!}{2}$ [Undirected]

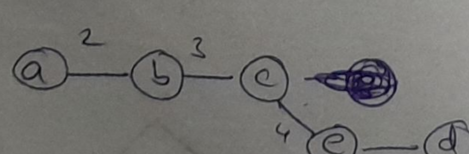
[n: No. of vertices]

No. of spanning Trees $= n^{n-2}$

[n: No. of vertices]

Ex.





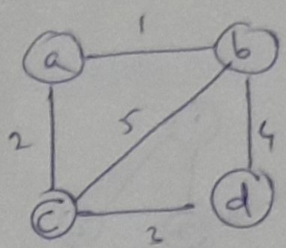
Weight (spanning path) = 14

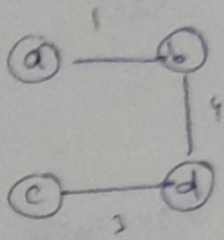
Weight (spanning tree) = 10

i.e. shortest path

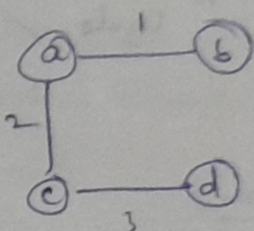
Minimal Spanning Tree

Ex.





Weight (sp. path) = 8



weight (sp. path) = 6

Note :

There can be multiple minimum spanning paths, spanning Trees depending on weights of the edges.

→ Hamitonian Graph : is called as

Hamiltonian path : A path containing all vertices (Spanning path)

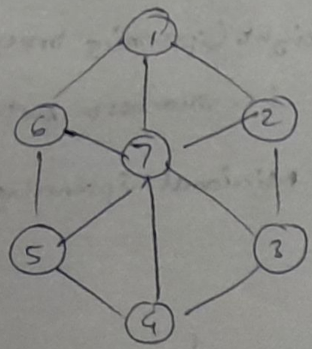
Hamiltonian cycle : A ~~path~~ cycle containing all vertices (spanning cycle)

A graph is Hamiltonian path if ∃ spanning path.

A graph is Hamiltonian cycle if ∃ spanning cycle.

Hamiltonian cycle : Each vertex should appear exactly once. Start & end points are same.

Ex. W_7



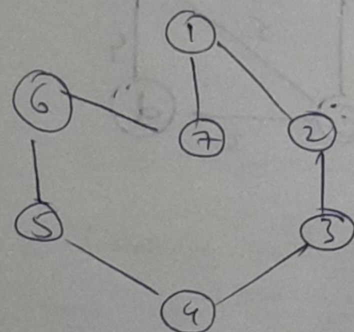
1 - 2 - 3 - 4 - 5 - 6 - 7

∃ spanning path

∴ H path ✓

∃ sp. cycle ✓

H cycle ✓

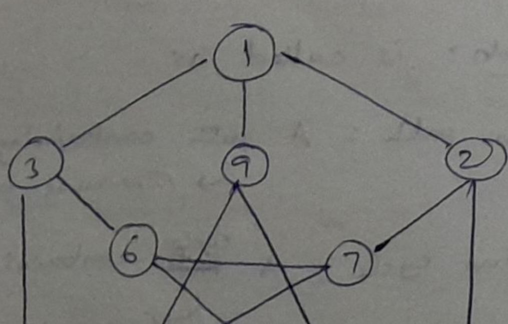


<u>Observation</u>:

If G is H cycle ⇒ G is H path

If G is H path ⇒ G is H cycle [Not necessary]

Counter-Example:



Peterson Graph

~~Peterson Graph~~

~~3-deg graph~~

Property : 3-Regular Graph

H path ✓

1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10

H cycle ×

sp. cycle is not possible

→ P_n — H path ✓
H cycle ✗

K_n — H path ✓
H cycle ✓

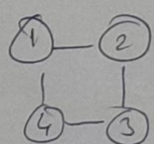
C_n — H path ✓
H cycle ✓

W_n — H path ✓
H cycle ✓

<u>Observation</u>: If G is H path / H cycle, then G is connected.

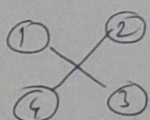
Ex 1

G



H path ✓
H cycle ✓

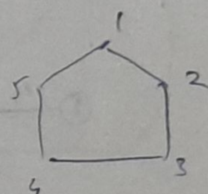
G^c



H path ✗
H cycle ✗

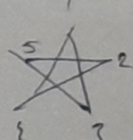
Ex 2.

G



H path ✓
H cycle ✓

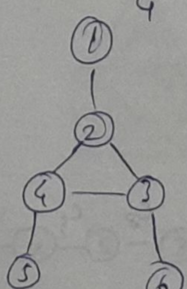
G^c



H path ✓
H cycle ✓

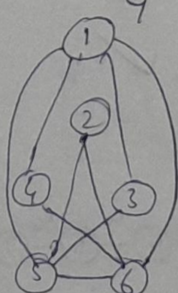
Ex. 3

G



H path ✗
H cycle ✗

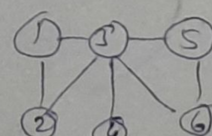
G^c



H path ✓
H cycle ✓

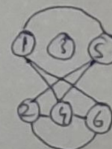
Ex 4

G



H path ✗
H cycle ✗

G^c



H path ✗
H cycle ✗

22/4/24

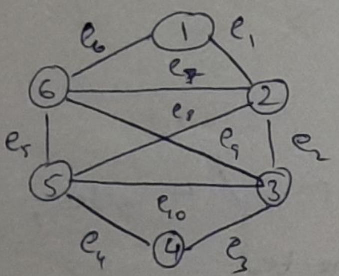
Eulerian Graph :

Eulerian Path :
A sp. path contains all Edges S.T each edge is listed exactly one

Eulerian Cycle :

A sp. cycle contains all Edges S.T each edge is listed exactly once. vertices may be repeated

Ex.

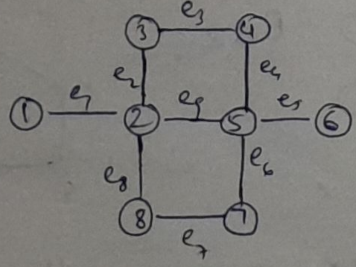


$1 \xrightarrow{e_1} 2 \xrightarrow{e_2} 3 \xrightarrow{e_3} 4 \xrightarrow{e_4} 5 \xrightarrow{e_8} 2 \xrightarrow{e_7} 6 \xrightarrow{e_9} 3 \xrightarrow{e_{10}} 5 \xrightarrow{e_5} 6 \xrightarrow{e_6} 1$

Eulerian Cycle ✓

Eulerian Path ✗

Ex.



Eulerian cycle ✗

Eulerian Path ✓

: $1 \xrightarrow{e_1} 2 \xrightarrow{e_2} 3 \xrightarrow{e_3} 4 \xrightarrow{e_4} 5 \xrightarrow{e_6} 7 \xrightarrow{e_7} 8 \xrightarrow{e_8} 2 \xrightarrow{e_9} 5 \xrightarrow{e_5} 6$

Note :

(1) For Eulerian Cycle, Degrees of all vertices must be even.

For Eulerian Path, No Restrictions.

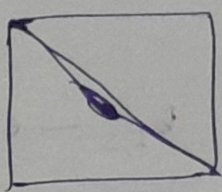
(2) If a Graph G is Eulerian Path / Cycle, then Graph G is connected

→ **Planar Graph :**

A Graph with plane drawing

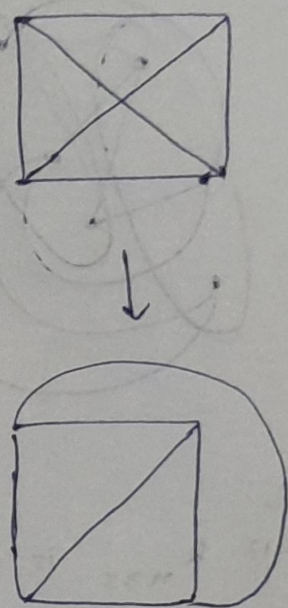
↳ Drawing of Edges S.T. no two edges cross each other

Ex.



Planar Graph

Ex.

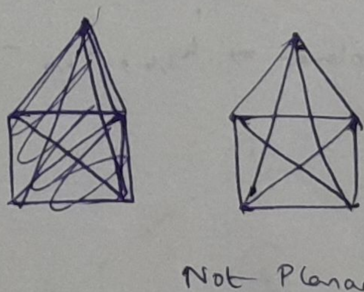


K_4

Planar Graph

$\therefore$ k_4 Also → Planar Graph

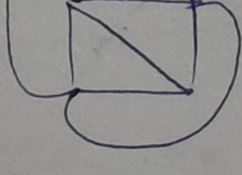
Ex



k_5 - 10 Edges

Not Planar Graph

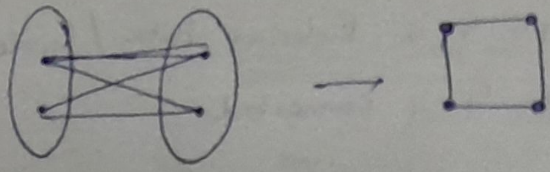
$\therefore$



Max Edges : 9

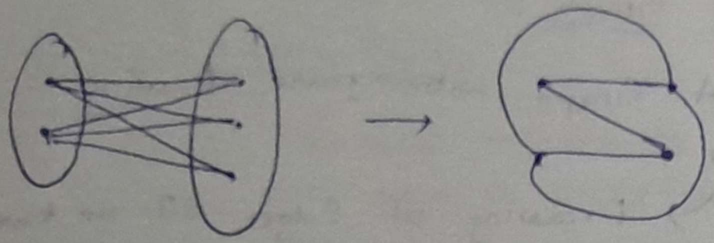
$\therefore$ $k_5 - e$ = Planar Graph

$k_{2,2}$



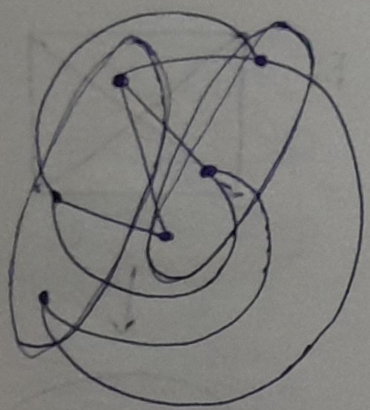
Planar

$k_{2,3}$

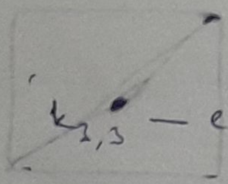


Planar

$k_{3,3}$



$k_{3,3} - e$ is planar



Note:

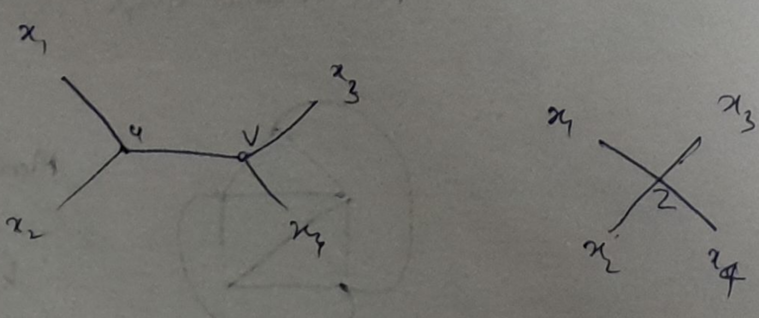
(1) $k_{n \geq 5}$ is non-planar

(2) If $H \leq G$ and H is non-planar, then G is non-planar

(3) k_5 – Non-planar $\Rightarrow$ k_6 – Non-planar $\Rightarrow$ k_7

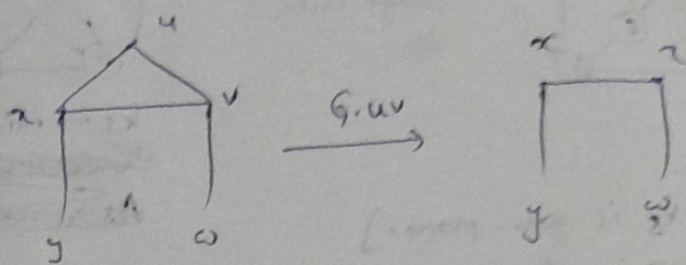
(4) $k_{3,3}$ is ~~not~~ non-planar $\Rightarrow$ $k_{3,4}$, $k_{6,6}$ – Non-planar

→ Edge Contradiction:



$G \cdot uv \left\{ \begin{array}{l} \text{delete } \{u, v\} \\ \text{Merge } N_G(u),\ N_G(v) \end{array} \right.$

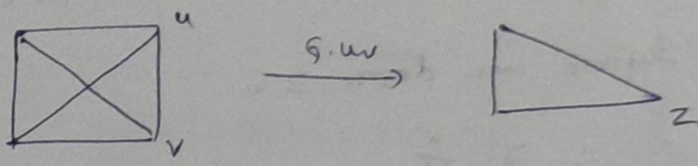
$\dot{z} \qquad N_G(z) = N_G(u) \cup N_G(v)$



$P_n \cdot uv \Longrightarrow P_{n-1}$

$C_n \cdot uv \Longrightarrow C_{n-1}$

$K_n \cdot uv \Longrightarrow K_{n-1}$



K_4

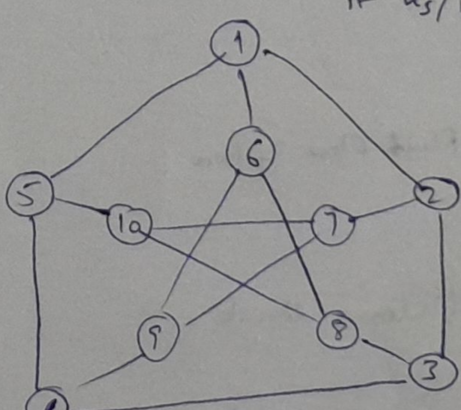
23/4/24

→ Planar Graph Observations :

① Karatowski's result :

G is planar iff G has no $\underbrace{K_5 \text{ (or) } K_{3,3} \text{ minor}}$

if $K_5/K_{3,3}$ is obtained through a sequence of edge contractions



Peterson's Graph

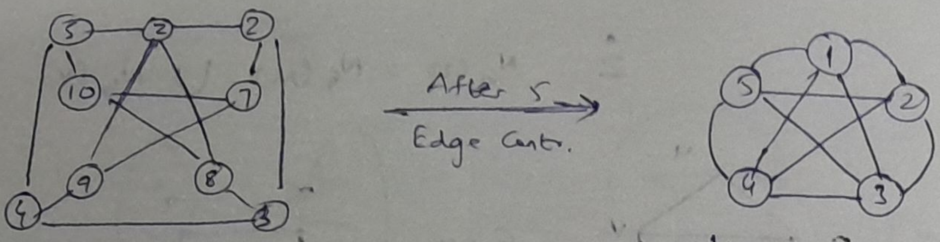
$\left.\begin{array}{l} K_{2,2} \\ K_{3,3}-e \end{array}\right\}$ Planar

$\left.\begin{array}{l} K_{3,3} \\ K_{4,4} \\ \vdots \end{array}\right\}$ Non Planar

$\left.\begin{array}{l} K_4 \\ K_5-e \end{array}\right\}$ Planar

$\left.\begin{array}{l} K_5 \\ K_6 \\ \vdots \end{array}\right\}$ Non Planar

After Edge Contraction b/w 1,6 :



K5 → minor obtained

~~[illegible]~~

~~As K5 → non-planar~~,

[As K5 is non-planar]

∴ Original Graph → Non-planar

② If G is planar, then $|E(G)| \le 3|V(G)| - 6$

$\Rightarrow \boxed{e \le 3n - 6}$

③ Minimum Degree of any Planar Graph ≤ 5.

Suppose min. degree ≥ 6

$\therefore \exists \sum_{i=1}^{n} d(i) \ge 6n$

$\sum d = 2|E(G)|$

$\therefore E(G) \ge 3n$

Contradicts Assumption

→ Real Life Applications of Planar Graph:

① CPU Connection

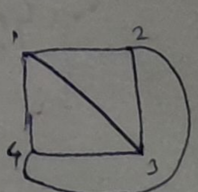
② Maze Design

③ Pipeline / Drain / Fluid Flow System

④ Railway Tracks

→ Planar Graph : face : Closed Circuit

Ex.



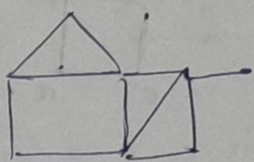
V	E	F
4	6	3 Interior + 1 Exterior = 4

Ex. 

V	E	F
5	4	1 Ext

Ex. 

V	E	F
5	5	2 (1 Int + 1 Ext)

Ex. 

V	E	F
8	11	(4 Int + 1 Ext) 5

Equation :

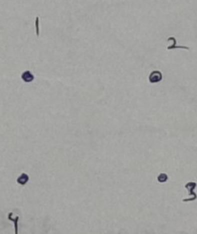
$$V - E + F = 2$$

↳ Euler's Planarity formula

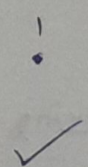
$$|V(G)| - |E(G)| + F(G) = 2$$

Only Works for :

Connected Planar Graphs.

Ex 

X



$1 - 0 + 1 = 2$

→ Vertex Coloring :

Colouring → Vertex Colouring

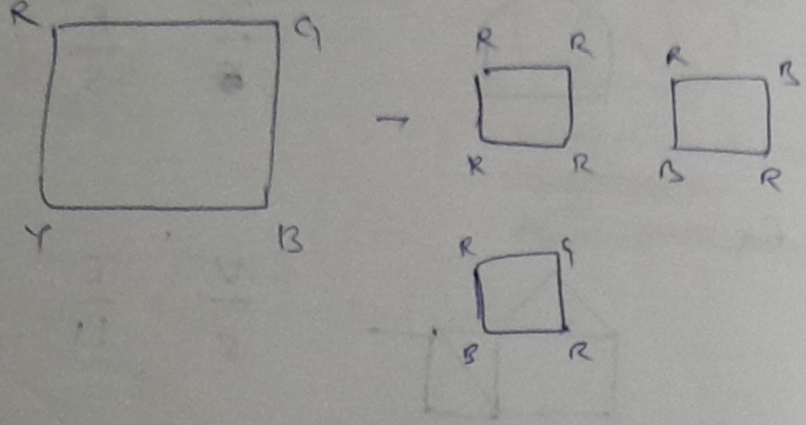
Colouring → Edge Colouring

My S
My P

~~[illegible]~~ day

Vertex Colouring : Assignment of Colours to Vertices

Ex

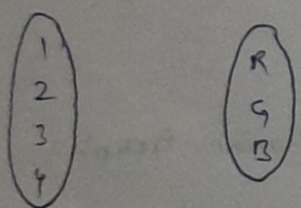


5 vertices — 4 colourings

$\therefore 4^5 = 4 \times 4 \times 4 \times 4 \times 4$

Note :

Colouring is a function from vertices to colours



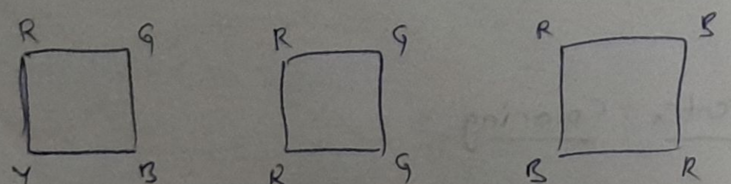
$\Longrightarrow 3^4$

~~Proper~~

Proper Colouring :

Adjacent vertices recieve different colours

Ex



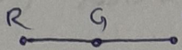
Properly Coloured | Not Properly Coloured | Properly Coloured

Chromatic ~~colouring~~ number :

The minimum no. of colours required to properly colour the graph.

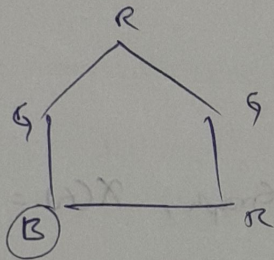
$\chi(C_4) = 2$

→ $\chi(P_n) = 2$, $n > 1$



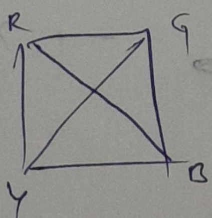
→ $\chi(C_n) = 3$, if $n = 2k$
$\quad\quad\quad\;\; 2$, if $n = 2k+1$ $\Big\}$ $k \in N$

Ex.:

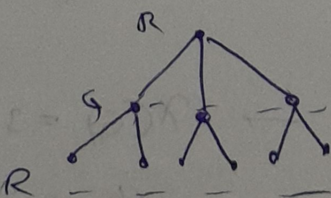


χ — C_5

→ $\chi(K_n) = n$



→ $\chi(T_n) = 2$



→ $\chi(\text{Bipartite}) = 2$

Note:

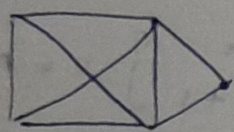
① G is Bipartitite $\equiv$ G is odd cycle tree $\equiv$ $\chi(G) = 2$

Observations:

① If $k_r \in G$, Then $\chi(G) \geq r$

Complete Graph of r vertices (Clique)

Ex.



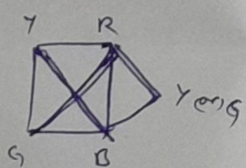
(G)

k_4 :

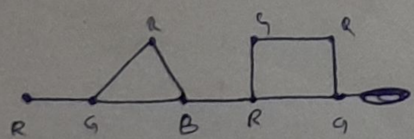
$k_4 \in G$

$\therefore \chi(G) \geq 4$

For this Graph, $\chi(G) = 4$

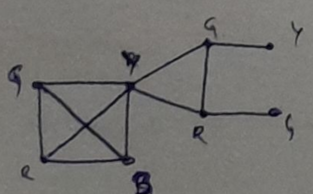


Ex. ①



$\rightarrow$ $\chi(G) = 3$

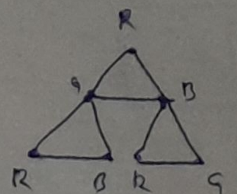
Ex. ②



$\rightarrow$ $\chi(G) = 4$

(k_4 exists)

Ex. ③



$\rightarrow$ $\chi(G) = 3$

(k_3 exists)

Note:

$\chi(G) \geq W(G)$ — Clique number

$W(G)$: Maximally connected ~~subgraph~~ complete subgraph