Engineering Electromagnetics

Lecture 9

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by

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Gradient

- Suppose, now, that we have a function of three variables→ T (x, y, z)
- "How fast does T vary?"

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

This tells us how *T* changes when we alter all three variables by the infinitesimal amounts *dx, dy, dz*. Notice that we do *not* require an infinite number of derivatives—*three* will suffice: the *partial* derivatives along each of the three coordinate directions.

$$dT = \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}})$$

$$= (\nabla T) \cdot (d\mathbf{l}), \quad \longleftarrow \quad d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Gradient of **T**

Scalar/Vector?

Geometrical interpretation

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T||d\mathbf{l}|\cos\theta$$ where θ is the angle between ∇T and $d\mathbf{l}$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Q: $f(x, y, z) = x^2 + y^3 + z^4$ at a point (2,1,0)

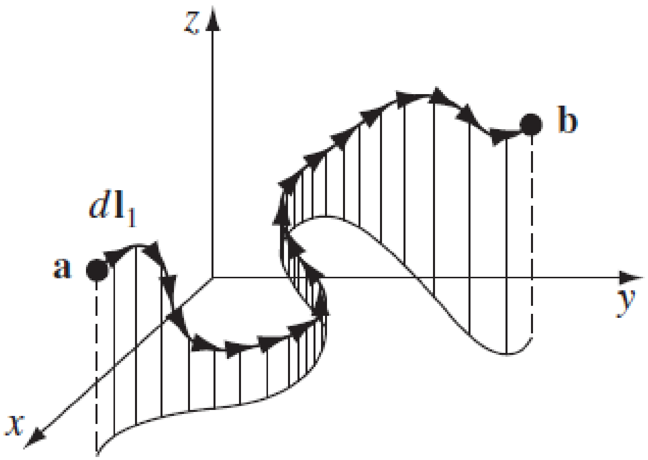
Q: $f(x, y, z) = x^2 + y^2 + z^2$ at (1,1,1)

*https://qsstudy.com/physics/physical-significance-gradient#:~:text=Gradient%20tells%20you%20how%20much,change%20of%20a%20particular%20function.

Solutions

- f (x, y, z) = $x^2 + y^3 + z^4$ at a point (2,1,0)
- $\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} = 2x\ \hat{x} + 3y^2\hat{y} + 4z^3\hat{z} = 4\hat{x} + 3\hat{y}$
- f(x, y, z) = $x^2 + y^2 + z^2$ at (1,1,1)
- $\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} = 2x\hat{x} + 2y\hat{y} + 2z\hat{z} = ?$

Fundamental theorem for gradient



Conservative Fields

Suppose we have a scalar function of three variables $T(x, y, z)$. Starting at point **a**, we move a small distance $d\mathbf{l}_1$ (Fig. 1.26). According to Eq. 1.37, the function T will change by an amount

$$dT = (\nabla T) \cdot d\mathbf{l}_1.$$

Now we move a little further, by an additional small displacement $d\mathbf{l}_2$; the incremental change in T will be $(\nabla T) \cdot d\mathbf{l}_2$. In this manner, proceeding by infinitesimal steps, we make the journey to point **b**. At each step we compute the gradient of T (at that point) and dot it into the displacement $d\mathbf{l}$... this gives us the change in T. Evidently the *total* change in T in going from **a** to **b** (along the path selected) is

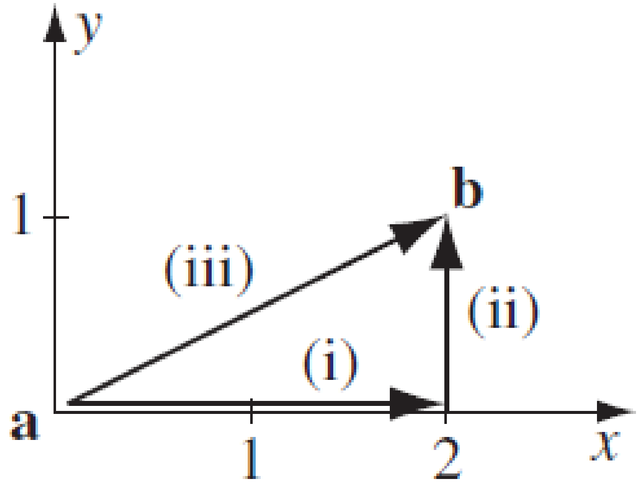
$$\boxed{\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).} \tag{1.55}$$

This is the **fundamental theorem for gradients**;

Geometrical Interpretation: Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up (that's the left side of Eq. 1.55), or you could place altimeters at the top and the bottom, and subtract the two readings (that's the right side); you should get the same answer either way (that's the fundamental theorem).

Problem-1

Example 1.9. Let $T = xy^2$, and take point **a** to be the origin $(0, 0, 0)$ and **b** the point $(2, 1, 0)$. Check the fundamental theorem for gradients.

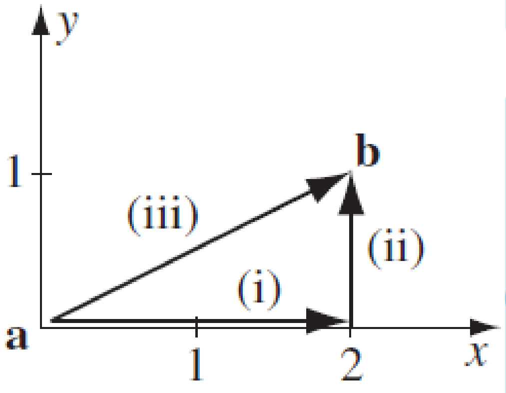


Solution

Although the integral is independent of path, we must *pick* a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii) (Fig. 1.27). As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\nabla T = y^2\,\hat{\mathbf{x}} + 2xy\,\hat{\mathbf{y}}$.

(i) $y = 0$; $d\mathbf{l} = dx\,\hat{\mathbf{x}}$, $\nabla T \cdot d\mathbf{l} = y^2\,dx = 0$, so

$$\int_{\mathrm{i}} \nabla T \cdot d\mathbf{l} = 0.$$



(ii) $x = 2$; $d\mathbf{l} = dy\,\hat{\mathbf{y}}$, $\nabla T \cdot d\mathbf{l} = 2xy\,dy = 4y\,dy$, so

$$\int_{\mathrm{ii}} \nabla T \cdot d\mathbf{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2.$$

The total line integral is 2. Is this consistent with the fundamental theorem? Yes: $T(\mathbf{b}) - T(\mathbf{a}) = 2 - 0 = 2$.

The Del operator

The gradient has the formal appearance of a vector, ∇, "multiplying" a scalar T:

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) T$$

(For once, I write the unit vectors to the *left*, just so no one will think this means $\partial\hat{\mathbf{x}}/\partial x$, and so on—which would be zero, since $\hat{\mathbf{x}}$ is constant.) The term in parentheses is called **del**:

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.}$$

it does not "multiply"

it is an instruction to *differentiate* what follows.

The Del operator

Now, an ordinary vector $\mathbf{A}$ can multiply in three ways:

1. By a scalar a : $\mathbf{A}a$;
2. By a vector $\mathbf{B}$, via the dot product: $\mathbf{A} \cdot \mathbf{B}$;
3. By a vector $\mathbf{B}$ via the cross product: $\mathbf{A} \times \mathbf{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the **divergence**)
3. On a vector function $\mathbf{v}$, via the cross product: $\boxed{\nabla \times \mathbf{v} \text{ (the \textbf{curl})}}$.

The Divergence

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.}$$

From the definition of ∇ we construct the divergence:

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.\end{aligned}$$

Vector or scalar?

Divergence: examples

- If $\boldsymbol{f} = x\hat{x} + y\hat{y} - \hat{z}$, $\nabla . f =?$
- If $\boldsymbol{f} = 2\hat{y}$, $\nabla . f =?$
- If $\boldsymbol{f} = x^2\hat{x}, \nabla . f =?$

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.}$$

Solution

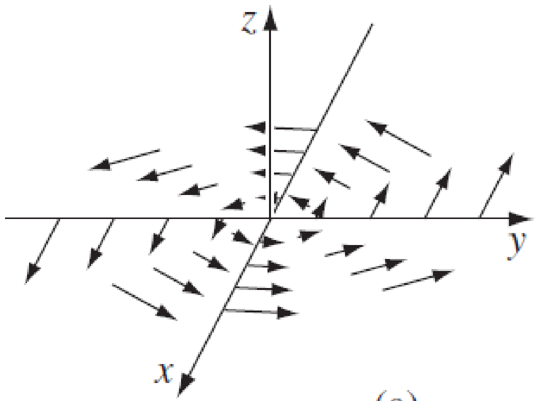
- $f = x\hat{x} + y\hat{y} - \hat{z}$, $\nabla . f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} =$
- $f = 2\hat{y}$, $\nabla . f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} =$
- $f = x^2\hat{x}$, $\nabla . f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} =$

The Curl

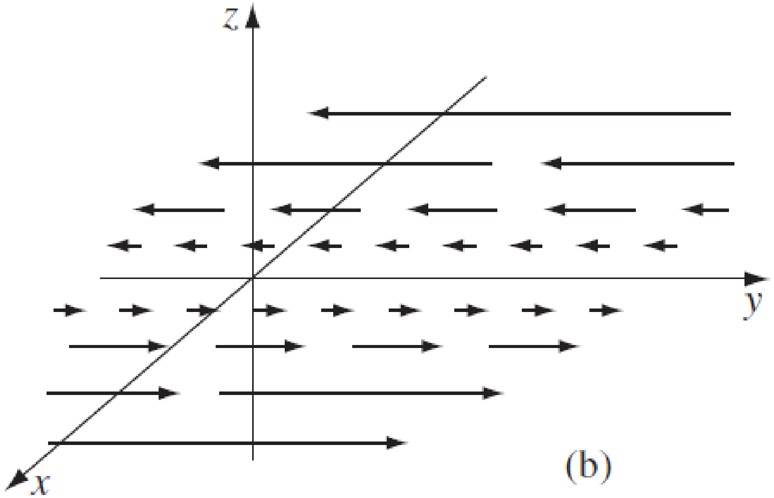
From the definition of ∇ we construct the curl:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$



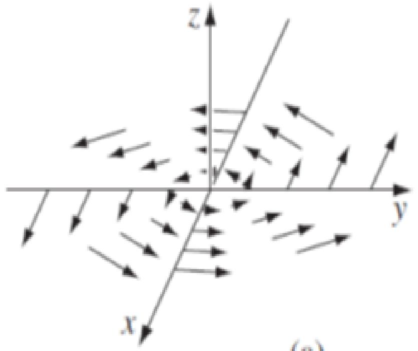
(a)



(b)

Geometrical interpretation

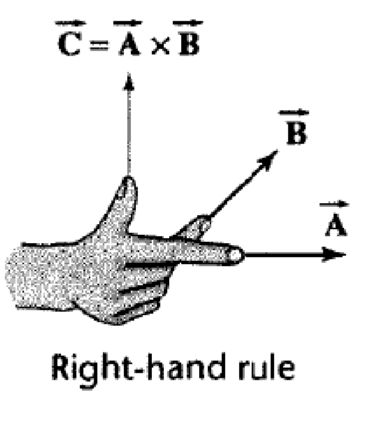
- The name curl is also well chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ swirls around the point in question.



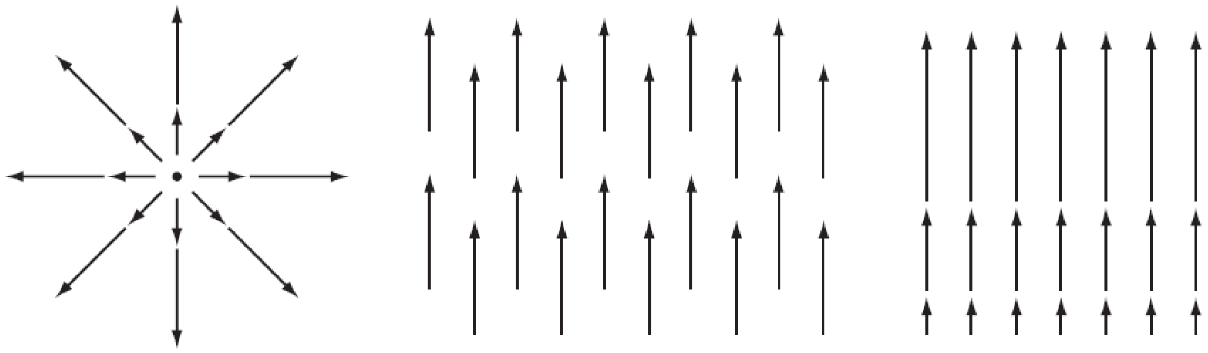
(a)

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.$$

- a substantial curl, pointing in the z direction, as the natural right-hand rule would suggest.



- Curl? →



(a) (b) (c)

Summary

there are three ways the operator ∇ can act:

1. On a scalar function T : $\boxed{\nabla T \text{ (the gradient)}}$;
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the **divergence**)
3. On a vector function $\mathbf{v}$, via the cross product: $\boxed{\nabla \times \mathbf{v} \text{ (the } \textbf{curl}\text{)}}$.

Thank You