Corollary:-

If $z_j - c_j = 0$ for at least one j for which $y_{ij} > 0$, $i = 1, 2, \ldots, m$; then another basic feasible solution is obtained which gives an unchanged value of objective function

Proof: $\hat{z} = z_0 - (z_j - c_j)\dfrac{x_{Br}}{y_{rj}} = z_0$

Unbounded Solution.

Theorem:- Let there exist a BFS of LPP. If at least for one j, for which $y_{ij} \leq 0$ $(i = 1, 2, \ldots, m)$ and $z_j - c_k < 0$ then there doesn't exist any optimal soln to LPP.

Proof :- consider LPP

$$\max \; z = CX$$

$$S.T.C \quad AX = b \;, \; X^T \in \mathbb{R}^n$$

A is $m \times n$ & b is $m \times 1$ real matrices respectively.

Let $\boxed{\rho(A) = m}$ & X_B is BFS so that

$$\boxed{B X_B = b} \text{ and } \boxed{X_B \geq 0}$$

with the value of objective function

$$z_0 = c_B X_B = \sum_{i=1}^{m} c_{Bi} X_{Bi}$$

Now $b = B X_B + \lambda a_j - \lambda a_j$, wher $a_j \in A$ & λ is a scalar.

$$= \sum_{i=1}^{m} x_{Bi} b_i + \lambda a_j - \lambda \sum_{i=1}^{m} y_{ij} b_j$$

Then $b = \sum_{i=1}^{m} (x_{Bi} - \xi y_{ij}) b_i + \xi a_i$

If $\xi > 0$, then $(x_{Bi} - \xi y_{ij}) \geq 0$

since $y_{ij} \leq 0$.

This shows that there exist a FS whose $(m+1)$ components may be strictly positive..

$\longrightarrow$ That can not be a BS

The value of objective function for these $(m+1)$ variables is given by

$$\hat{z} = \sum_{i=1}^{m} c_{Bi} (x_{Bi} - \xi y_{ij}) + \xi c_j$$

$$= \sum_{i=1}^{m} c_{Bi} x_{Bi} - \xi \left(\sum_{i=1}^{m} c_{Bi} y_{ij} - c_j \right)$$

$$= z_0 - \xi (z_j - c_j)$$

but $z_j - c_j < 0$
& $\xi > 0$

$\therefore \hat{z} \rightarrow +\infty$ as $\xi \rightarrow +\infty$

Hence there is no limit to the optimum value z $\Rightarrow$ LPP have unbounded solution.

Remark – (a) $\exists$ 'j' for which $y_{ij} \leq 0$. $(i = 1, 2, \ldots, m)$

And $z_j - c_j > 0$; Then $\hat{z} \rightarrow -\infty$

(b) $\exists$ 'j' for which $y_{ij} \leq 0$

& $z_j - c_j = 0$, then $\hat{z} = z_0$

Condition for optimality

Theorem:- A sufficient condition for a basic feasible solution to an LPP to be an optimum (maximum) is the $z_j - c_j \geq 0$

for all j for which the column vector $a_j \in A$ is not in the basis B

Proof $\Rightarrow$ Let the LPP be to determine X so as to

$$\text{Max } Z = CX \ , \ C, X^T \in \mathbb{R}^n$$

S.T.C $\quad Ax = b$ and $X \geq 0$

where A is $m \times n$ and b is $m \times 1$ real matrices respectively.

Let $\rho(A) = m$ then we have a submatrix B such that $\quad BX_B = b, \ X_B \geq 0$

$$\& \quad Z_0 = C_B X_B$$

Given :- for all j for which $a_j \notin B$ we have $z_j - c_j \geq 0$. Let $a_j = b_j$ for all j for which $a_j \in B$

$$y_j = B^{-1} b_j = e_j \text{, the unit vector}$$

$$z_j - c_j = c_B y_j - c_j = c_B e_j - c_j$$

$$= c_{B_i} - c_j = 0$$

Thus $z_j - c_j \geqslant 0$ for all j for which $a_j \in A$.

Now, let x be a feasible solution.

Then $\sum_{j=1}^{n} (z_j - c_j) x_j \geqslant 0$

$$\Rightarrow \sum_{j=1}^{n} z_j \cdot x_j \geqslant \sum_{j=1}^{n} c_j x_j$$

or $\sum_{j=1}^{n} (c_B y_j) x_j \geqslant \sum_{j=1}^{n} c_j x_j$ (since $\underline{z_j = c_B y_j}$)

We know $z_j = c_B y_j = \sum_{i=1}^{m} c_{B_i} y_{ij}$

$$\sum_{j=1}^{n} (c_B y_j) x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} c_{B_i} y_{ij} \right) x_j$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \boxed{c_{B_i}} y_{ij} x_j$$

$\rightarrow$ constant (j free)

$$= \sum_{i=1}^{m} c_{B_i} \underline{\sum_{j=1}^{n} y_{ij} x_j}$$

i.e. $\displaystyle\sum_{i=1}^{m} c_{B_i} \sum_{j=1}^{n} y_{ij} x_j \geq \sum_{j=1}^{n} c_j x_j$ —— $(*)$

for all j for which $a_j \notin B$

<u>Now since</u>, $X_B = \underline{B^{-1}}(AX) = \underline{(B^{-1}A)}\, X = Y_{m\times n} X_{n\times 1}$

$A_{m\times n}$, $B_{m\times m}$, $B^{-1}_{m\times m}$ $[Y_{m\times n} = B^{-1}A]$

or $\quad x_{B_i} = \sum_{j=1}^{n} y_{ij} x_j \quad (\text{for } i = 1, 2, \ldots, m)$

Now inequation $(*)$ can be written as.

$$\sum_{i=1}^{m} c_{B_i} x_{B_i} \geq \sum_{j=1}^{n} c_j x_j$$

or $\quad \boxed{c_B x_B \geq cx}$

or $\quad z_0 \geq z^*$

where z^* is the value of the objective function for the feasible solution x.

Hence z_0 is the optimum solution for which $z_j - c_j \geq 0$ for all j such that $a_j \notin B$. □