Newton's Method

$\boxed{1}$ In the problem of minimizing a func f of single variable.

$\boxed{2}$ Assume that at each measurement point $x^{(k)}$ we can calculate $f(x^{(k)})$, $f'(x^{(k)})$, and $f''(x^{(k)})$.

$\boxed{3}$ We can fit a quadratic function through $x^{(k)}$ that matches its 1st and 2nd derivatives with that function f.

$$q(x) = f(x^{k}) + f'(x^{(k)})(x - x^{(k)}) + f''(x^{(k)})(x - x^{(k)})^2$$

$$q(x^{(k)}) = f(x^{(k)})$$

$$q'(x^{(k)}) = f'(x^{(k)}) \quad \text{and} \quad \underline{q''(x^{(k)}) = f''(x^{(k)})}$$

Minimizing 'f' $\longrightarrow$ minimizing 'q'

i.e; $q'(x) = f'(x^{(k)}) + f''(x^{(k)})(x - x^{(k)})$

Setting $x = x^{(k+1)}$ we Obtain.

$$\boxed{x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}} \quad \text{———} \ (*)$$

At point $x^{(k)}$ the first & second derivatives:

$$f'(x^{(k)}) = \frac{f(x^{(k)} + \Delta x^{(k)}) - f(x^{(k)} - \Delta x^{(k)})}{2\Delta x^{(k)}} \quad \text{— (I)}$$

$$f''(x^{(k)}) = \frac{f(x^{(k)} + \Delta x^{(k)}) - 2f(x^{(k)}) + f(x^{(k)} - \Delta x^{(k)})}{(\Delta x^{(k)})^2} \quad \text{— (II)}$$

The parametre $\Delta x^{(k)}$ is usually taken to be small value. In all our calculations we assign $\Delta x^{(k)}$ to be about 1 % of $x^{(k)}$

$$\Delta x^{(k)} = \begin{cases} 0.01\,|x^{(k)}|, & \text{if } x^{(k)} > 0.01 \\ 0.0001, & \text{otherwise} \end{cases} \quad \text{— (**)}$$

Algorithm (Newton's Method)

$\boxed{\text{Step-1}} \rightarrow$ Choose initial guess $x^{(1)}$, & a small number 'ϵ', Set $k=1$.
Compute $f'(x^{(k)})$

$\boxed{\text{Step-2}} \rightarrow$ compute $f''(x^{(k)})$

$\boxed{\text{Step-3}} \rightarrow$ $\boxed{\text{Calculate } (*)}$ $x^{(k+1)} = x^{(k)} - \dfrac{f'(x^{(k)})}{f''(x^{(k)})}$

then compute $f'(x^{(k+1)})$

$\boxed{\text{Step-4}} \rightarrow$ If $|f'(x^{(k+1)})| < \epsilon$ $\boxed{\text{Terminate}}$

$\boxed{\text{Else}}$ — $k = k+1$ and go to $\boxed{\text{Step-2}}$

Example - 2.5.1

Consider the minimization problem:

$$f(x) = x^2 + \frac{54}{x}, \quad x \in (0,5)$$

$\boxed{\text{Step-1}}$ → Initial guess $x^{(1)} = 1$ (personal choice)

Termination factor $\epsilon = 10^{-3}$

Iteration count $k = 1$

$\boxed{\text{using } (**) \quad \Delta x^{(1)} = 0.01}$ Numerical Derivative -52.005 (by Eqn I)

Exact Derivative -52

verify → $f'(1) = 2x - \frac{54}{x^2} = 2 \cdot 1 - \frac{54}{1} = -52$

$\boxed{\text{Step-2}}$ → Exact 2nd Derivative at $x^{(1)}$ is 110

By Eqn (II) → Numerical derivative $f''(x^{(1)}) = 110.011$

which is close to exact value.

Step-3 →

$$x^{(2)} = x^{(1)} - f'(x^{(1)}) / f''(x^{(1)})$$

$$= 1 - (-52.005) / 110.011$$

$$= 1.473 \Rightarrow \boxed{f'(x^{(2)}) = -21.944}$$

Step-4 → Since $f'(x^{(2)}) \not< \epsilon$

We apply increament K to 2 and go-to $\boxed{\text{step-2}}$

$\boxed{\text{This completes One iteration of Newton-Rapson method}}$

Step-2 → compute : $f''(x^{(2)}) = 35.796$

Numerically

Step-3 → Using ⊛ $x^{(3)} = 2.086$ & $f'(x^{(3)}) = -8.239$ $\begin{pmatrix} \text{By} \\ \text{I} \end{pmatrix}$

$\boxed{\text{Step-4}} \rightarrow$

Since $f'(x^{(3)}) \not< \epsilon$, we set $k=3$

More to $\boxed{\text{step-2}}$

This is end of 2nd iteratin

$\boxed{\text{Step-2}} \rightarrow$ $f''(x^{3}) = 13.899$ (by - II)

$\boxed{\text{Step-3}} \rightarrow$ New point $x^{(4)} = 2.679$

$f'(x^{(4)}) = -2.167$

$\boxed{\text{Step-4}} \rightarrow$ $\boxed{f'(x^{(4)}) \not< \epsilon}$, agin porceed to $\boxed{\text{step 2}}$

3-function evaluation at each iteraton

The iteration stop at $x^{(7)} = 3.0001$, $f'(x^{(7)}) = -4(10)^{-8} < \epsilon$

$$z = x_2 - \frac{f'(x_2)}{[f'(x_2) - f'(x_1)] / (x_2 - x_1)}$$