Calculus:-

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1) Given $a_n = (1+\frac{3}{4n})^{\frac{8}{3}n}$

then $\lim_{n\to\infty} (1+3/4n)^{\frac{8}{3}n}$ [this of form $(1+0)^\infty$ as $(1+f(x))^{g(n)}$ where this is $e^{\lim_{n\to\infty} f(x)g(x)}$]

so, $\Rightarrow e^{\lim_{n\to\infty} \frac{3}{4n}\left(\frac{8^2}{3}n\right)}$

$\Rightarrow e^2$

$\therefore$ The limit of the sequences $a_n = (1+\frac{3}{4n})^{\frac{8}{3}n}$ is e^2//

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2) Given α is a rational number,

then $\lim_{n\to\infty} (\sin n!\alpha\pi) \Rightarrow \lim_{n\to\infty} (\sin(n!\alpha\pi)) \Rightarrow \sin(\infty)$

so, As we know $\sin n$ is a bounded function between [-1 1] so,

$\sin(\infty)$ we can't say as it value in [-1,1] so,

limit does not exist.

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3) Given $a_0=1$, $a_1=1$, for $n\geq 2$ let $a_n = a_{n-1}+a_{n-2}$, $\{a_n\}$ is a Fibonacci sequence.

then lets consider limit of $\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = L$

as $n\to\infty$ so,

we can cosider

$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = L$ — (1) we know $a_{n+1} = a_n + a_{n-1}$

$\Rightarrow \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{a_n+a_{n-1}}{a_n} \Rightarrow \lim_{n\to\infty}\left(1+\frac{a_{n-1}}{a_n}\right)$

$\Rightarrow 1+\frac{1}{L}$ — (2)

from (1) & (2) $L = 1+1/L$

$\Rightarrow L^2-L-1=0$, $\therefore$ By solving $L = \frac{1+\sqrt{5}}{2} = 1.6180349$//

$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1.6180349$

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4) Given sequence $\{a_n\}$, $a_1 = \sqrt{2}$

$a_n = \sqrt{2+\sqrt{a_{n-1}}}$ for $n \geq 2$

as $n \to \infty$, let $\lim a_n = L$, and we can write $\lim\limits_{n\to\infty} a_{n-1} = L$ so,

$L = \sqrt{2+\sqrt{L}} \Rightarrow (L^2 - 2)^2 = L$

$\Rightarrow L^4 - 4L^2 - L + 4 = 0$

Here roots are 1, 1.83 and two imaginary roots as $a_1 = \sqrt{2}$ so $L > \sqrt{2}$ so,

$\lim\limits_{n\to\infty} a_n = 1.83$

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5) Given $a_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \frac{1}{n^2+3} \cdots + \frac{1}{n^2+n}$

Then limit a_n (for $n \to \infty$) is

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{1}{n^2+1} \cdots + \frac{1}{n^2+n}\right) \Rightarrow$ $\left[\text{as } \lim\limits_{n\to\infty} \frac{1}{n^2} = 0\right]$

$\Rightarrow \lim\limits_{n\to\infty} \frac{1}{n^2+1} + \cdots + \lim\limits_{n\to\infty} \frac{1}{n^2+n}$

$\Rightarrow 0$

$\therefore$ The limit of the sequence $\{a_n\} = 0$

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6) Given $a_n \to a$, $b_n \to b$ and $b > a$, $S_n = \max\{a_n, b_n\}$

$\lim\limits_{n\to\infty} S_n = \max\{\lim\limits_{n\to\infty} a_n, \lim\limits_{n\to\infty} b_n\}$

$= \max\{a, b\}$

$= b$, (as $b > a$)

$\therefore$ As b is a finite number

$\therefore$ The $\{S_n\}$ converges to b

7) Given $\sum_{n=1}^{\infty} \frac{1}{n^p} \cos\left(\frac{1}{n}\right)$

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as we know $n \to \infty$, $\frac{1}{n} \to 0$, so $\cos(0)=1$

so we can only consider $\frac{1}{n^p}$ so, from integral Test

we show that $\int_1^{\infty} \frac{1}{n^p} = \left[\frac{n^{-p+1}}{-p+1}\right]_1^{\infty}$ i) let $0<p\leq 1$ then $\{A+\frac{1}{0}$ form$\}$

$1-p$ is >0

so, $\left[\frac{n^{-p+1}}{-p+1}\right]_1^{\infty} = \infty$

not converges

ii) if $p>1$

$1-p<0$

so, $\left[\frac{n^{-p+1}}{-p+1}\right]_1^{\infty} = 0$

converges

so, $\sum_{n=1}^{\infty} \frac{1}{n^p}\cos\left(\frac{1}{n}\right)$ converges for $p>1$, Diverges for $0<p\leq 1$

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8) Given Σa_n coverges and $a_n \geq 0$

we know from nth root test Σa_n coverges if $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$

Σa_n converges for $\rho<1$ so, Here

$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho \quad (\rho<1)$

squaring on both sides

$\lim_{n\to\infty} \sqrt[n]{a_n^2} = \rho^2 \quad \left(\text{as } \begin{matrix}\rho<1 \\ \rho^2<1\end{matrix}\right)$

So, $\rho^2<1$ so, Σa_n^2 also converges from nth root test.

Hence proved.

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9) Given a_n converges, $a_n \geq 0$

from this we can know

$a_n \leq a_n + a_{n+1}$
$a_{n+1} \leq a_n + a_{n+1}$ $\overset{\text{Multiply}}{\Rightarrow}$ $a_n a_{n+1} \leq (a_n + a_{n+1})^2$

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$\because \sqrt{a_n a_{n+1}} \leq a_n + a_{n+1}$

Apply sigma

$$\Sigma \sqrt{a_n a_{n+1}} \leq \Sigma a_n + \Sigma a_{n+1}$$

Given Σa converges of Σa_{n+1} also converges, so $\Sigma (a_n + a_{n+1})$ also converges.

From above

$$\Sigma \sqrt{a_n a_{n+1}} \text{ also converges}$$

Hence proved.

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10) Given $1 + e^b + e^{2b} + \dots = 9$

This of form infinite G.P where $r = e^b$, $a = 1$

$\because 1 + e^b + e^{2b} + \dots = \frac{1}{1-e^b} = 9$

$1 = 9 - 9e^b$

$e^b = 8/9 \Rightarrow b = \ln(8/9)$

$b = -0.1178$

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11) Given infinite series $1 + 2r + r^2 + \dots$

This can be written as

$(1 + r + r^2 + \dots) + (r + r^3 + \dots)$

$\Rightarrow$ This are two G.P, If this two infinite G.P are converges then their sum also converges

So, this two G.P are converges for $r \in (-1, 1)$

$\therefore$ The Given series converges for $r \in (-1,1)$ and sum of the series is $\left(\frac{1}{1-r}\right) + \left(\frac{r}{1-r^2}\right) \Rightarrow \frac{1+2r}{1-r^2}$ (where $r \in (-1,1)$)

12) As x is an finite value

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$\sum \frac{1}{nx} \Rightarrow \frac{1}{x} \sum \frac{1}{n}$

we know $\sum \frac{1}{n}$ is a Diverging series so,
for any value (finite) value of x $\sum \frac{1}{nx}$ is Diverging series.

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13) Here for let $\sum a_n$ and $\sum b_n$ both be $(-1)^n \frac{1}{\sqrt{n}}$ (convergence)

$\therefore$ $\sum a_n b_n = \sum \frac{1}{n}$ which is Diverging

So, $\sum a$ & $\sum b$ both are converging and $\sum a_n b_n$ be Diverging.

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14) a) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

$\Rightarrow$ By integral test

$$\int_1^{\infty} \frac{\ln n}{n^2} \Rightarrow \left[\ln n\left(\frac{-1}{n}\right) - \int \frac{1}{n}\left(\frac{-1}{n}\right)\right]_1^{\infty}$$

$$\Rightarrow \left[\frac{-\ln n}{n}\right]_1^{\infty} - \left[\frac{1}{n}\right]_1^{\infty}$$

$\left[\lim_{n\to\infty} \frac{\ln n}{n} = 0 \text{ By L-Hospital rule}\right]$

$$[0-0] - [0-1]$$

$$\Rightarrow 1$$

$\therefore \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converged

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14) b) $\sum_{n=2}^{\infty} \frac{1}{\ln(\ln n)}$

we know $\ln \ln n < n$

$$\frac{1}{\ln(\ln n)} > \frac{1}{n} \Rightarrow \sum \frac{1}{\ln(\ln(n))} > \sum \frac{1}{n}$$

we know $\sum \frac{1}{n}$ Diverges

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so, $\sum_{n=2}^{\infty} \frac{1}{\ln(\ln n)}$ also Diverges.

14) $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$

lets try ratio test

$$\lim_{n\to\infty} \left(\frac{2^{n+1}}{3+4^{n+1}}\right) \times \frac{3+4^n}{2^n} \Rightarrow \lim_{n\to\infty} 2\left(\frac{3+4^n}{3+4^{n+1}}\right)$$

This of form $\left(\frac{\infty}{\infty}\right)$ so, let $f(x) = \frac{3+4^x}{3+4^{x+1}}$ By using L-Hospital rule

so, $2 \lim_{x\to\infty}\left(\frac{4^x \ln 4}{4^{x+1}\ln 4}\right) \Rightarrow 2\left(\frac{1}{4}\right) = \frac{1}{2}$ $(\rho<1)$

$\therefore$ so $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$ converges

14) d) By n^{th} root test

$$\lim_{n\to\infty} \left[\left[\frac{(n+1)^{n+1}}{(n+1)} - \frac{n+1}{n}\right]^{-n}\right]^{1/n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{\left[(n+1)^n - 1 - \frac{1}{n}\right]} \Rightarrow \lim_{n\to\infty} \frac{1}{\left[(n+1)^n - 1\right]}$$

Here let $(n+1)^n = y$

$n \ln(n+1) = \ln y$

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(n+1)}{\frac{1}{n}}$ $\left(\frac{\infty}{0}\right)$

L.Hospital

$y = e^{-\infty}$

$y = 0$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{[0-1]} = -1, \ (\rho<1)$$

$\therefore$ $\sum_{n=1}^{\infty} \left[\frac{(n+1)^{n+1}}{n+1} - \frac{n+1}{n}\right]^{-n}$ is converges.

14) e) $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

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lets try ratio test

$\lim_{n\to\infty} \frac{(n+1)5^n(5)}{(2n+5)\ln(n+2)} \cdot \frac{(2n+3)(\ln(n+1))}{n5^n}$

$\Rightarrow \lim_{n\to\infty} \frac{(n+1)}{n} \frac{2n+3}{2n+5} \frac{\ln(n+1)}{\ln(n+2)} (5)$

[Here for $\frac{\ln(n+1)}{\ln(n+2)}$ we can try L-Hospital Rule
$\lim_{x\to\infty} = \frac{x+2}{x+1} \Rightarrow \lim_{x\to\infty} \frac{1+2/x}{1+1/x} = 1$]

$\Rightarrow \lim_{n\to\infty} (1+1/n) \frac{2+3/n}{2+5/n} (1)(5)$

$\Rightarrow \lim_{n\to\infty} (1)(1)(1)(5) \Rightarrow 5 \;(\rho>1)$

so, Divergence $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$ Diverges.

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15) a) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$

Here $a_n = \frac{1}{n^3 3^n}$ $\therefore$ Radius of convergence $= \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{1}{n^3 3^n} \times (n+1)^3 3^{n+1}\right)$

$R = \lim_{n\to\infty}\left(\left(\frac{1+n}{n}\right)^3 \times 3\right) \Rightarrow R = \lim_{n\to\infty}\left(3 \times (1+1/n)^3\right) = 3$

Interval of convergence:-

At $|x-1| < |3|$
$\Rightarrow x \in (-2, 4)$

At $x-1=3$
$x=4$
$\sum_{n=1}^{\infty} \frac{1}{n^3} \Rightarrow$ converges

At $x-1=-3$
$x=-2$
Alternating seres converges

$\therefore$ The Interval of convergence $-2 \le x \le 4$

(i) Absolutely converges for $x \in [-2, 4]$
(ii) No value of x for which this converges conditionaly

15) b) $\sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{\sqrt{n}+3}$

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Here $a_n = \frac{(-1)^n}{\sqrt{n}+3}$

$\therefore$ Radius of convergence $= \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$R = \left|\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}+3}(\sqrt{n+1}+3)\right)\right|$ [from L-H rule]

let $f(x) = \frac{\sqrt{x+1}+3}{\sqrt{x}+3} \Rightarrow \lim_{x\to\infty} \frac{\cancel{2}\sqrt{x}}{\cancel{2}\sqrt{x+1}} \Rightarrow \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{1/2}$

$\Rightarrow 1$

$\therefore \underline{R=1}$

Interval of convergence

At $|x| < |1|$
$x \in (-1, 1)$

At $x = 1$
Alternating series converges

At $\underline{x=-1}$
Again Alternating series converges

$\therefore$ Interval of convergence $-1 \le x \le 1$

i) Absolutely converges at $x \in (-1, 1)$

ii) conditionaly converges at $x = -1, 1$

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15) c) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$

Radius of convergence $= \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^{n^2} \left(\frac{n+2}{n+1}\right)^{(n+1)^2}\right|$

Here $\left(\frac{n}{n+1}\right)^{n^2}$ is in 1^∞ form so, $e^{\lim_{n\to\infty} \ln(n/n+1) n^2} \Rightarrow e^{\lim_{n\to\infty} \frac{\ln(n/n+1)}{1/n^2}}$

$\Rightarrow e^{\lim_{n\to\infty} \frac{\cancel{n+1}}{\cancel{n}} \left(\frac{-1}{3n^{3/2}}\right) \left(\frac{(n+1)1-(n)}{(n+1)^2}\right)} \Rightarrow e^{\lim_{n\to\infty} \frac{-3n^2}{(n+1)}} \Rightarrow e^{-\infty} = 0$

$\therefore R = 0$ $\quad\therefore$ The Intervel of convergence is $x = 0$

i) Absolutly converges at $x = 0$

ii) conditionaly converges at no value of x

16) Given

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$f(x) = \cos(2x + \pi/2) = -\sin(2x)$, $x = \pi/4$

Taylor series at $x = \pi/4$ (a)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

Here $a = \pi/4$

$$\sum_{n=0}^{\infty} \frac{\sin^n(2a)}{n!}(x-a)^n = -1 - 2\cos(2x)(x-a) + \frac{4}{2!}\sin(2x)(x-a)^2 + \dots$$

$$= -1 + \frac{2^2}{2!}(x-a)^2 - \frac{2^4}{4!}(x-a)^4 \dots \frac{(-1)^n 2^{2n}(x-a)^{2n}}{(2n)!} + \dots$$

17) Given $f(x) = \dfrac{1}{(1-x)^3}$

Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$= \frac{1}{(1-x)^3} + (-1)\left[-\frac{3x}{(1-x)^4} + \frac{3,4x^2}{(1-x)^5} \dots \frac{(-1)^n x^n (3,4,5 \dots n+2)}{(1-x)^{n+3}}\right]$$

$(a=0)$

$$= \frac{1}{(1-x)^3} + (-1)\left[-3x + 3,4x^2 - 3.4.5x^3 + \dots + (-1)^n (3,4,5 \dots n+2)x^n\right]$$

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18) a) Given

$\tan^{-1}(3x^{4})$ let $3x^{4} = y$

we know derivative of $\tan^{-1}y = \frac{1}{1+y^2}$

$\therefore \tan^{-1}y = \int \frac{1}{1+y^2}\,dy = \int \frac{1}{1-(-y^2)}\,dx$

$= \int 1+(-y^2)+(-y^2)^2+(-y^2)^3+\dots\,dy$

$= \int \sum_{n=0}^{\infty} (-y^2)^n\,dy$

$= \int 1-y^2-y^4-y^6+\dots\,dy$

$\tan^{-1}(y) = C + y - \frac{1}{3}y^3 + \frac{1}{5}y^5 \dots$

let $y=0$

then $\tan^{-1}(y)=0$, $C=0$

$\therefore \tan^{-1}(y) = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 \dots + \frac{(-1)^n}{2n+1}y^{2n+1}\dots$ (Taylor series) $x=0$ (Maclaurin)

Here $y=3x^{4}$, n starts from 0

$\tan^{-1}(3x^{4}) = 3x^{4} - \frac{1}{3}(3x^{4})^3 + \frac{1}{5}(3x^{4})^5 \dots$

18) b) Given $\frac{1}{1+\frac{3}{4}x^3}$ let $\frac{3}{4}x^3 = y$

then $\frac{1}{1+y} = \frac{1}{1-(-y)} = 1 - y + y^2 - y^3 \dots + (-1)^n y^n \dots$

$\therefore$ Taylor series ($x=0$) (Maclaurin)

$\frac{1}{1+y} = 1 - y + y^2 - y^3 \dots + (-1)^n y^n$ (n starts from 0)

Here $y = \frac{3}{4}x^3$

$\frac{1}{1+\frac{3}{4}x^3} = 1 - \frac{3}{4}x^3 + \left(\frac{3}{4}x^3\right)^2 - \left(\frac{3}{4}x^3\right)^3 \dots$

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$x^2\cos(x^2)$ RollNo:- EC20B1050

$\Rightarrow$ lets first find Tylor series of $\cos(x^2)$ ($x=0$, Maclaurin)

$$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} \cdots$$

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} \cdots$$

$\therefore$ Tyler series ($x=0$) $= x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} + \cdots$

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19) b) $x\ln(1+2x)$

let first see tyler series of $\ln(1+2x)$ ($x=0$, Maclaurin)

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \cdots$$

$$= 2x - \frac{4x^2}{2} + \frac{8x^3}{3} + \cdots$$

Tyler series ($x=0$) $\Rightarrow 2x^2 - \frac{4x^3}{2} + \frac{8x^4}{3} + \cdots$

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19) c) $\cos^2 x$

we know $\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$

let first see tyler series of $\cos(2x)$ ($x=0$, Maclaurin)

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} \cdots$$

$$= 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} \cdots$$

Tyler series:- ($x=0$) $\frac{1}{2} + \frac{1}{2} - \frac{2x^2}{2!} + \frac{8x^4}{4!} + \cdots$

$$1 - \frac{2x^2}{2!} + \frac{8x^4}{4!} + \cdots //$$

20) Given

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$$\sin^2 x = \frac{1-\cos 2x}{2}$$

For

Maclaurin series of $\sin^2 x$ let first see Maclaurine of $\cos(2x)$

$$\cos(2x) = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} + \cdots$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 - \cdots\right)$$

$$\sin^2 x = \frac{2}{2!}x^2 - \frac{8}{4!}x^4 + \cdots$$

$$\sin^2 x = x^2 - \frac{8}{4!}x^4 + \cdots$$

Deffinentiate the series

$$2\sin x\cos x = 2x - \frac{8}{3!}x^3 + \cdots \quad // \quad \text{—(1)}$$

Maclaurin series of $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$

$$\sin(2x) = 2x - 8\frac{x^3}{3!} \cdots \quad \text{—(2)}$$

from (1) & (2)

$$\sin(2x) = 2\sin x\cos x = 2x - \frac{8x^3}{3!} + \cdots \quad //$$