DIFFERENTIAL EQUATIONS (MA1001)

50 marks → End sem

25 marks → Mid Sem (SCORE WELL HERE)

25 marks → Continuous Evaluation (Assignment, test or some activity)

BOOKS →
- Simmons . G.F., Differential Equations, Tata McGraw Hill, 2003.
- Kreyzig . E., Advanced Engg. Mathematics, Wiley, 2007.

Concepts

Problems

CHECK SYLLABUS LATER.

Every system → defined by differential equations

differential equations → span across all fields.

↳ Toolkit to study behaviour of systems.

$y = f(x) \implies$ y : dependent variable
x : independent variable

$\Rightarrow \frac{dy}{dx} = f'(x) \longrightarrow$ rate of change of y wrt x.

$\vec{F} = m\vec{a}$

$\Rightarrow \vec{F} = m \cdot \frac{d^2y}{dt^2}$

for free fall, $\frac{d^2y}{dt^2} = g$

$\Rightarrow \vec{F_g} = m\frac{d^2y}{dt^2} = mg$

if there's air resistance,

$$\vec{F}_{drag} = -K\vec{v} = -K\frac{dy}{dt}$$

$\therefore$, $\vec{F} = \vec{F_g} - \vec{F}_{drag}$

$$\Rightarrow m\frac{d^2y}{dt^2} = mg - K\frac{dy}{dt}$$

diff eqn $\rightarrow$ involves one dependent variable & its derivatives wrt one/more independent variables.

ordinary diff. eqn $\rightarrow$ involves only one independent variable.
(all derivatives $\rightarrow$ ordinary)

partial diff. eqn $\rightarrow$ involves more than one independent variable.
(all derivatives $\rightarrow$ partial)

LEGENDRE

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0 \longrightarrow$$ ORDINARY DIFFERENTIAL EQUATIONS

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

BESSEL

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = 0$$

$$a^2\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2}\right) = \frac{\partial \omega}{\partial t}$$

$$a^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\frac{\partial^2 w}{\partial t^2}$$

ABOVE are LAPLACE equations

nth order :
ORDINARY DIFF
EQUATION

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right)=0$$

$$\downarrow$$

$$F(x, y, y', y'', \dots, y^{(n)})=0$$

Let solution by $y = y(x)$.
how to know if $y(x) \rightarrow$ solution?

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$: has a solution $y = e^{2x}$.

substitute & check:

$$\frac{d^2}{dx^2}(e^{2x}) - 5\frac{d}{dx}(e^{2x}) + 6e^{2x} = 0$$

$$\Rightarrow 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$$

$$\Rightarrow \boxed{0=0}$$

$y = e^{3x}$ also works!

$$\frac{d^2}{dx^2}(e^{3x}) - 5\frac{d}{dx}(e^{3x}) + 6e^{3x} = 0$$

$$\Rightarrow 9e^{3x} - 15e^{3x} + 6e^{3x} = 0$$

$$\Rightarrow \boxed{0=0}$$

generally, ∴ we can write the solution as:

$$y(x) = c_1e^{2x} + c_2e^{3x}$$

check:

$$\frac{d^2}{dx^2}\left(c_1e^{2x} + c_2e^{3x}\right) - 5\frac{d}{dx}\left(c_1e^{2x} + c_2e^{3x}\right) + 6\left(c_1e^{2x} + c_2e^{3x}\right) = 0$$

$$\Rightarrow \cancel{4c_1e^{2x}} + \cancel{9c_2e^{3x}} - \cancel{10c_1e^{2x}} - \cancel{15c_2e^{3x}} + \cancel{6c_1e^{2x}} + \cancel{6c_2e^{3x}} = 0$$

$$\Rightarrow \boxed{0=0}$$

$$\therefore \boxed{y(x) = c_1e^{2x} + c_2e^{3x}}$$

for n'th order diff. eqn → there'll be n arbitrary constants → & n linearly independent solutions.

↓

we can't write a given solution as a linear combo of other solutions ($c_1 f_1(x) + c_2 f_2(x) + \cdots = f_n(x)$)

→ no dependency of $f_n(x)$ wrt others.

$$\boxed{\text{H.W.} \quad \frac{dy}{dx} = \frac{y^2}{1-xy}}$$ → MUST SOLVE.

$\sin\left(\frac{d^2y}{dx^2}\right) + y = 0$ → order = 2, degree = N.A.

$\frac{d^2y}{dx^2} = -\sin^{-1} y$ → degree = 1, order = 2

$y'' + e^{y'} + 1 = 0$ → degree = N.A., order = 2.

$y'' + y' + \int y\,dx = 0$

$\Rightarrow y''' + y'' + y = 0$ → degree = 1, order = 3.

$$\left(\frac{d^2y}{dx^2}\right)^{3/2} + \left(\frac{d^3y}{dx^3}\right)^{2/3} = 0$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^{9} = \left(\frac{d^3y}{dx^3}\right)^{4}$$ order = 3, degree = 4

order → the max derivative in an eqn.

degree → power of highest order

$y'' + y' + \int P(x)\,dx = 0$

$\rightarrow y''' + y'' + P(x) = 0$ → order = 3, degree = 1

LINEAR & NON-LINEAR

if y → associated with transcendental functions

↳ NON-LINEAR. ↳ trig, inv. trig, powers, exp., log.

else, linear.

$y'' + P(x)\cdot y = e^y \rightarrow$ non-linear

$y'' + e^x y = x^2 \rightarrow$ linear

$y'' + y' + \sin^{-1} y = 0 \rightarrow$ non-linear

$$\frac{dy}{dx} = f(x)$$

$\Rightarrow dy = f(x)dx$

$\Rightarrow \int dy = y = \int f(x)dx$

Some cases

$\hookrightarrow$ not easy.

- $\int e^{-x^2} dx$
- $\int \frac{\sin x}{x} dx$

$$\frac{dy}{dx} = e^{3x} - x$$

$$dy = (e^{3x} - x)dx$$

$$\Rightarrow y = \int (e^{3x} - x)\,dx$$

$$\boxed{y = \frac{e^{3x}}{3} - \frac{x^2}{2} + c}$$

$$\boxed{(1+x^3)\frac{dy}{dx} = x}$$

$\rightarrow$ SOLVE as H.W.

$$\frac{x\,dx}{1+x^3} = \frac{x\,dx}{x^3\left(\frac{1}{x^3}+1\right)} = \frac{dx}{x^2\left(\frac{1}{x^3}+1\right)} = \frac{-d\left(\frac{1}{x}\right)}{\left(\frac{1}{x^3}+1\right)} = \frac{-dt}{1+t^3} = \frac{-dt}{(1+t)(1+t^2-t)}$$

try $t = \tan^2\theta$

SEPARABLE FUNCTIONS :

$$\frac{dy}{dx} = f(x)\cdot g(y)$$

$$\Rightarrow \frac{dy}{g(y)} = f(x)\,dx$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)dx + c$$

$$xy\cdot\frac{dy}{dx} = y - 1$$

$$\frac{y\,dy}{y-1} = \frac{dx}{x}$$

$$\frac{(y-1+1)\,dy}{y-1} = \frac{dx}{x}$$

$$\Rightarrow dy + \frac{dy}{y-1} = \frac{dx}{x} \Rightarrow \int dy + \int \frac{dy}{y-1} = \int \frac{dx}{x}$$

$\Rightarrow \quad y + \ln(y-1) = \ln x + c$

$$x^5 \frac{dy}{dx} + y^5 = 0$$

$$\Rightarrow \quad x^5 \frac{dy}{dx} = -y^5$$

$$\Rightarrow \quad \frac{dy}{y^5} = -\frac{dx}{x^5}$$

$$\Rightarrow \quad \int \frac{dy}{y^5} = -\int \frac{dx}{x^5}$$

$$\Rightarrow \quad \frac{y^{-5+1}}{-5+1} = -\left(\frac{x^{-5+1}}{-5+1}\right) + c$$

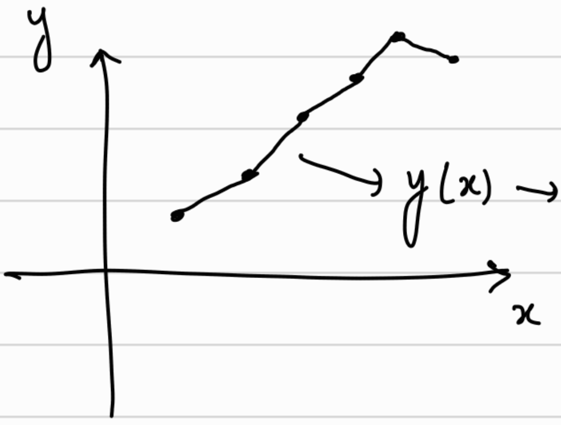
$$\Rightarrow \quad \frac{-1}{4y^4} = \frac{1}{4x^4} + c$$

$$\Rightarrow \quad \frac{-1}{y^4} = \frac{1}{x^4} + 4c$$

$$\Rightarrow \quad \frac{1}{y^4} + \frac{1}{x^4} + K_1 = 0 \quad ; \quad \boxed{K_1 = 4c}$$

GENERAL FORM of an ORDINARY DIFF. EQN

$$a_0(x)\left(y^{(n)}(x)\right)^m + a_1(x)\, y^{(n-1)}(x) + \cdots + a_k(x)\, y^{(n-k)}(x) + \cdots + a_n(x)\, y^{(0)}x = 0$$



the dl's along $\frac{dy}{dx}$'s trace the solution of the D.E.

$y(x) \rightarrow$ solution curve / integral curve

$(dy/dx = y(x))$

PICARD'S THEOREM

$f(x,y)$ & $\frac{\partial f}{\partial y} \rightarrow$ continuous on a rectangle R.

then $\forall\ (x_0, y_0) \in R$, a unique solution of the D.E. passes thru it.

Families of Curves / Orthogonal Trajectories

family of curves → y-coordinates displaced by some constant c.

(e.g.) $y = x^2 + c$
$x^2 + y^2 = R^2$

$x^2 + y^2 = R^2$
$2x + 2y \cdot \frac{dy}{dx} = 0$
$\Rightarrow 2y\frac{dy}{dx} = -2x$
$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{x}{y}}$

Orthogonal trajectories → slopes lar
$m_1 m_2 = -1$
$\Rightarrow m_2 \cdot \frac{dy}{dx} = -1$
$\Rightarrow \boxed{m_2 = -\frac{dx}{dy}}$

for concentric circles, the radius is the orthogonal trajectory. ANY line from origin traces a radius.

form: $y = cx.$
$\Rightarrow \frac{dy}{dx} = c$ $\therefore$, $\boxed{y = x\frac{dy}{dx}}$

one parameter family of curves

↳ diff. eqn corresponding to the orthogonal trajectory of any circle centered at origin.

$x^2 + y^2 - 2cx + c^2 - c^2 = 0$
$\Rightarrow (x-c)^2 + y^2 = c^2$ → circle centered at $(c, 0)$

$x^2 + y^2 = 2cx$
$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 2c$
$\Rightarrow \boxed{x + y \cdot \frac{dy}{dx} = c.}$
$\Rightarrow x^2 + y^2 = 2\left(x + y\frac{dy}{dx}\right)x$
$\Rightarrow x^2 + y^2 = 2x^2 + 2xy\frac{dy}{dx}$
$\Rightarrow x^2 - y^2 + 2xy\frac{dy}{dx} = 0 \Rightarrow \boxed{\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}}$

of the form $\boxed{\frac{dy}{dx} = f(x,y)}$

$$\frac{dy}{dx} = \frac{y}{2x} - \frac{x}{2y} = \frac{1}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{(y/x)}\right)$$

$$\boxed{z = \frac{y}{x} \Rightarrow y = zx \Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}}$$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{z}{2} - \frac{1}{2z}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{z}{2} - \frac{1}{2z} - z = -\frac{1}{2z} - \frac{z}{2}$$

$$= \frac{-1 - z^2}{2z}$$

$$\Rightarrow x\frac{dz}{dx} = -\frac{(z^2+1)}{2z}$$

$$\Rightarrow \int \frac{2z\,dz}{z^2+1} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln(z^2+1) = -\ln x + \ln A$$

$$\Rightarrow \ln(z^2+1) = \ln(A/x)$$

$$\Rightarrow z^2+1 = A/x$$

$$\Rightarrow \frac{y^2}{x^2} + 1 = A/x$$

$$\Rightarrow \boxed{y^2 + x^2 = Ax}$$

for a circle → find orthogonal trajectory:

$$x^2 + y^2 = c^2$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$ → tangent to circle at (x, y)

$$\Rightarrow -\frac{dx}{dy} = \frac{-x}{y}$$ → normal to circle at (x, y) [orthogonal trajectory]

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\Rightarrow \ln x = \ln y + \ln A$$

$$\Rightarrow \ln x = \ln(Ay)$$

$$\Rightarrow x = Ay$$

$$\Rightarrow y = \frac{1}{A}x = kx$$

$\boxed{\therefore\ y = kx}$ → family of orthogonal trajectories to a circle.

Similarly, orthogonal trajectories for :
$x^2+y^2=2cx$, we do -

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy} \rightarrow \text{tgt at } (x,y)$$

$$\Rightarrow -\frac{dx}{dy} = \frac{y^2-x^2}{2xy} \rightarrow \text{normal at } (x,y)$$

$$\Rightarrow \frac{dx}{dy} = \frac{x^2-y^2}{2xy} = \boxed{\frac{x}{2y} - \frac{y}{2x}} \leftarrow \text{Before solving this:}$$

$f(x,y) \rightarrow f(tx,ty) \rightarrow$ enables us to solve :

$x \rightarrow tx$ & $y \rightarrow ty$

$$f(tx,ty) = t^n f(x,y)$$
$$f(x,y) = x^2+y^2$$
$$f(tx,ty) = t^2x^2+t^2y^2 = t^2(x^2+y^2)$$
$$= t^2 f(x,y).$$
$$\therefore \boxed{f(tx,ty) = t^2 f(x,y)}$$

$M(x,y)dy + N(x,y)dx = 0$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-N(x,y)}{M(x,y)}} \longrightarrow$$ homogeneous if degree of $N(x,y) = M(x,y)$

$$= \frac{-N(tx,ty)}{M(tx,ty)}$$
$$= \frac{-t^n N(x,y)}{t^n M(x,y)}$$
$$\frac{dy}{dx} = -t^0 \cdot \frac{N(x,y)}{M(x,y)}$$

if not homogeneous, try :

$$z = y/x \Rightarrow y = zx$$
$$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\frac{dy}{dx} = f(x,y) = g(1, y/x) = g(1,z)$$

$$\Rightarrow z + x\frac{dz}{dx} = g(1,z)$$

$$\Rightarrow x\frac{dz}{dx} = g(1,z) - z = h(z)$$

$$\Rightarrow \frac{dz}{h(z)} = \frac{dx}{x} \Rightarrow \int \frac{dz}{h(z)} = \int \frac{dx}{x} = \ln x + c.$$

$$\therefore \boxed{\int \frac{dz}{h(z)} = \ln x + c}$$ → once solved, convert z in terms of x and y.

$M(x,y)\,dy + N(x,y)\,dx = 0$ → homogeneous if N and M are homogeneous fns. of the same degree.

→ homogeneous fns:
$f(tx,ty) = t^n f(x,y)$

lets solve:

$$\frac{dx}{dy} = \frac{x^2-y^2}{2yx}$$

check -
- $x^2-y^2 \equiv t^2(x^2-y^2)$; $2yx \equiv t^2 \cdot 2yx$
- $\dfrac{t^2(x^2-y^2)}{t^2 \cdot 2yx} = t^0 \cdot \dfrac{(x^2-y^2)}{2yx}$.
- hence, D.E. is homogeneous.

$$\Rightarrow \frac{dx}{dy} = \frac{y^2\left(\left(\frac{x}{y}\right)^2-1\right)}{2yx}$$

$$\frac{dx}{dy} = \frac{1}{2}\cdot\frac{\left(\left(x/y\right)^2-1\right)}{\left(\frac{x}{y}\right)}$$

$$\frac{x}{y} = z \Rightarrow x = yz$$
$$\Rightarrow \frac{dx}{dy} = z + y\frac{dz}{dy}$$

$$\Rightarrow z + y\frac{dz}{dy} = \frac{1}{2}\left[\frac{z^2-1}{z}\right]$$

$$\Rightarrow y\frac{dz}{dy} = \frac{z^2-1}{2z} - z = \frac{-z^2-1}{2z}$$

$$\Rightarrow y\frac{dz}{dy} = \frac{-(z^2+1)}{2z}$$

$$\Rightarrow \frac{2z\,dz}{(z^2+1)} = \frac{-dy}{y}$$

$$\Rightarrow \ln(z^2+1) = -\ln y + \ln A$$

$$\Rightarrow \ln(z^2+1) = \ln\left(A/y\right)$$

$$\Rightarrow z^2+1 = \frac{A}{y}$$

$$\Rightarrow \frac{x^2}{y^2}+1 = \frac{A}{y}$$

$$\Rightarrow \boxed{x^2+y^2 = Ay}$$

1. $y = e^{x^2}\int_0^x e^{-t^2}\,dt$

$$\frac{dy}{dx} = e^{x^2}\cdot e^{-x^2} + \left(\int_0^x e^{-t^2}dt\right)2xe^{x^2}$$

$$= 1 + 2x\cdot e^{x^2}\int_0^x e^{-t^2}dt = 1+2xy$$

$$\Rightarrow \frac{dy}{dx} = 2xy+1 \Rightarrow \boxed{y' = 2xy+1}$$

2. $y''-5y'+6y=0$; let $P(x) = y''-5y+6y$

(a) $y = e^{2x}$: $\frac{d^2}{dx^2}\left(e^{2x}\right) - 5\frac{d}{dx}\left(e^{2x}\right) + 6e^{2x} = P(x)$

to prove : $P(x)=0$.

$$\Rightarrow \frac{d}{dx}\left(e^{2x}\cdot 2\right) - 5\cdot 2e^{2x} + 6e^{2x} = P(x)$$

$$\Rightarrow 2\cdot 2e^{2x} - 10e^{2x} + 6e^{2x} = P(x)$$

$$\Rightarrow 4e^{2x} - 10e^{2x} + 6e^{2x} = P(x)$$

$$\Rightarrow 0 = P(x)$$

$$\Rightarrow P(x) = 0$$

(b) $y = c_1e^{2x} + c_2e^{3x}$: $a(x)$ T.P.T: $a(x)=0$

$$\frac{d^2}{dx^2}\left(c_1e^{2x}+c_2e^{3x}\right) - 5\cdot\frac{d}{dx}\left(c_1e^{2x}+c_2e^{3x}\right) + 6\left(c_1e^{2x}+c_2e^{3x}\right)$$

$$\Rightarrow \underline{4c_1e^{2x}} + \underline{9c_2e^{3x}} - \underline{10c_1e^{2x}} - \underline{15c_2e^{3x}} + \underline{6c_1e^{2x}} + \underline{6c_2e^{3x}} = a(x)$$

$$\Rightarrow (4-10+6)c_1e^{2x} + (9-15+6)c_2e^{3x}$$

$$\Rightarrow 0+0 = a(x)$$

$\therefore$, $\boxed{a(x)=0}$

3. $y = e^{mx}$: soln. of $2y'''+y''-5y'+2y=0$. find m.

$$\Rightarrow 2\cdot\frac{d^3}{dx^3}\left(e^{mx}\right) + \frac{d^2}{dx^2}\left(e^{mx}\right) - 5\frac{d}{dx}\left(e^{mx}\right) + 2e^{mx} = 0$$

$$\Rightarrow 2\cdot m^3e^{mx} + m^2e^{mx} - 5me^{mx} + 2e^{mx} = 0$$

$$\Rightarrow 2m^3 + m^2 - 5m + 2 = 0$$

$$\Rightarrow \quad 2m^3 + 4m^2 - 3m^2 - 6m + m + 2 = 0$$

$$\Rightarrow \quad 2m^2(m+2) - 3m(m+2) + (m+2) = 0$$

$$\Rightarrow \quad (2m^2 - 3m + 1)(m+2) = 0$$

$$\Rightarrow \quad m = -2 \quad \text{(or)} \quad 2m^2 - 3m + 1 = 0$$

$$\Rightarrow \quad 2m^2 - 2m - m + 1 = 0$$

$$\Rightarrow \quad 2m(m-1) - 1(m-1) = 0$$

$$\Rightarrow \quad (2m-1)(m-1) = 0$$

$$\Rightarrow \quad \boxed{m = \tfrac{1}{2} \quad \text{(or)} \quad m = 1}$$

$\therefore$ Solutions: e^{-2x}, $e^{\frac{1}{2}x}$, e^{x}

general solution: $y = c_1 e^{-2x} + c_2 e^{\frac{1}{2}x} + c_3 e^{x}$

check:

$$2\frac{d^3}{dx^3}\left(c_1 e^{-2x} + c_2 e^{\frac{1}{2}x} + c_3 e^{x}\right) + \frac{d^2}{dx^2}\left(c_1 e^{-2x} + c_2 e^{\frac{1}{2}x} + c_3 e^{x}\right)$$

$$-5\frac{d}{dx}\left(c_1 e^{-2x} + c_2 e^{\frac{1}{2}x} + c_3 e^{x}\right)$$

$$+ 2\frac{d}{dx}\left(c_1 e^{-2x} + c_2 e^{\frac{1}{2}x} + c_3 e^{x}\right)$$

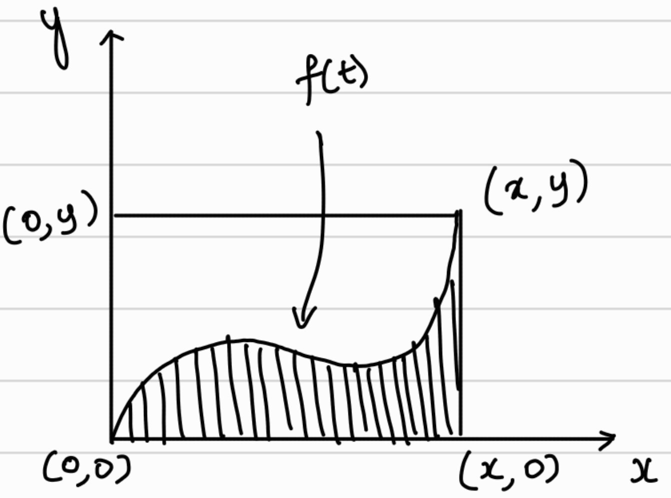
$$= 2\left[\cancel{-8c_1 e^{-2x}} + \cancel{\frac{c_2}{8} e^{\frac{1}{2}x}} + \cancel{c_3 e^{x}}\right] + \left[\cancel{4c_1 e^{-2x}} + \cancel{\frac{1}{4} c_2 e^{\frac{1}{2}x}} + \cancel{c_3 e^{x}}\right]$$

$$-5\left[\cancel{-2c_1 e^{-2x}} + \cancel{\frac{c_1}{2} e^{\frac{1}{2}x}} + \cancel{c_3 e^{x}}\right] + 2\left[\cancel{c_1 e^{-2x}} + \cancel{c_2 e^{\frac{1}{2}x}} + \cancel{c_3 e^{x}}\right]$$

$$= 0.$$

$\therefore$ general soln: $\boxed{y = c_1 e^{-2x} + c_2 e^{\frac{1}{2}x} + c_3 e^{x}}$

4.



$$A_R = xy$$

$$\int_0^x f(t)\,dt = A_R/3 = xy/3$$

$$\Rightarrow \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = \frac{d}{dx}\left(\frac{xy}{3}\right)$$

$$\Rightarrow f(x) = \frac{1}{3}\left(y + x\frac{dy}{dx}\right)$$

but, $y = f(x)$.

$$\therefore \quad f(x) = \frac{1}{3}\left(f(x) + xf'(x)\right)$$

5. $$\frac{8y^2-x^2}{4y^3-x^2y}\,dy = \frac{2x^2-4y^2}{4xy^2-x^3}\,dx$$

$$\Rightarrow \frac{x^2\left(8(y/x)^2-1\right)}{x^3\left(4(y/x)^3-(y/x)\right)}\,dy = \frac{x^2\left(2-4(y/x)^2\right)dx}{x^3\left(4(y/x)^2-1\right)}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{2-4(y/x)^2}{4(y/x)^2-1}\right)\times\left(\frac{4(y/x)^3-(y/x)}{8(y/x)^2-1}\right)$$

$$= \left(\frac{2-4(y/x)^2}{\cancel{4(y/x)^2-1}}\right)\times\left(\frac{\cancel{4(y/x)^2-1}}{8(y/x)^2-1}\right)\times(y/x)$$

let $y = xt \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$$\Rightarrow t + x\frac{dt}{dx} = \frac{2-4t^2}{8t^2-1}\times t$$

$$\Rightarrow x\frac{dt}{dx} = \frac{2t-4t^3}{8t^2-1} - t = \frac{2t-4t^3-8t^3+t}{8t^2-1}$$

$$= \frac{3t-12t^3}{8t^2-1}$$

$$\Rightarrow x\frac{dt}{dx} = \frac{3t(1-4t^2)}{8t^2-1}$$

$$\Rightarrow \frac{(8t^2-1)}{3t(1-4t^2)}\,dt = 3\cdot\frac{dx}{x}$$

$$\Rightarrow \frac{8t\,dt}{3(1-4t^2)} - \frac{dt}{3t(1-4t^2)} = 3\cdot\frac{dx}{x}$$

$$\Rightarrow -\int\frac{d(1-4t^2)}{3(1-4t^2)} - \int\frac{dt}{3t(1-2t)(1+2t)} = 3\int\frac{dx}{x}$$

$$\Rightarrow -\ln(1-4t^2) - \left[\int\frac{dt}{3t} - \frac{1}{6}\int\frac{d(1-2t)}{1-2t} - \frac{1}{3}\int\frac{dt}{1+2t}\right] = 3\ln x$$

$$\Rightarrow -\ln(1-4t^2) - \frac{1}{3}\left[\ln t - \frac{1}{2}\ln(1-2t) - \frac{1}{2}\ln(1+2t)\right] = 3\ln x$$

$$\frac{1}{3t(1-2t)(1+2t)} = \frac{A}{3t} + \frac{B}{1-2t} + \frac{C}{1+2t}$$

$A = 1$; $B = \frac{1}{3}$; $C = -\frac{1}{3}$

$\boxed{t = y/x}$

× ———————— × ———————— ×

① $(x^2-2y^2)dx + xy\,dy = 0$

$$\Rightarrow xy\,dy = (2y^2-x^2)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2-x^2}{xy} = \frac{2(y/x)^2-1}{(y/x)}$$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{2z^2-1}{z} \Rightarrow x\frac{dz}{dx} = \frac{2z^2-1-z^2}{z} = \frac{z^2-1}{z}$$

$$\Rightarrow \int \frac{z\,dz}{z^2-1} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\ln(z^2-1) = \ln x + c$$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{y^2}{x^2}-1\right) = \ln x + c$$

$$\Rightarrow \frac{1}{2}\left[\ln(y^2-x^2) - \ln x^2\right] = \ln x + c$$

$$\Rightarrow \frac{1}{2}\ln(y^2-x^2) = 2\ln x + c$$

$$\Rightarrow \ln(y^2-x^2) = 4\ln x + K \mathbin{/\!/}$$

(2) $x\sin(y/x)\frac{dy}{dx} = y\sin(y/x) + x$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin(y/x)+x}{x\sin(y/x)} = \frac{(y/x)\sin(y/x)+1}{\sin(y/x)}$$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{z\sin z + 1}{\sin z}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{z\sin z+1}{\sin z} - z = \frac{1}{\sin z}$$

$$\Rightarrow \int \sin z\,dz = \int \frac{dx}{x}$$

$$\Rightarrow -\cos z = \ln x + K$$

$$\Rightarrow -\cos(y/x) = \ln x + K$$

(3) $x^2\frac{dy}{dx} = 3(x^2+y^2)\tan^{-1}(y/x) + xy$

$$\Rightarrow \frac{dy}{dx} = 3(1+(y/x)^2)\tan^{-1}(y/x) + (y/x)$$

$$\Rightarrow z + x\frac{dz}{dx} = 3(1+z^2)\tan^{-1}z + z$$

$$\Rightarrow x\frac{dz}{dx} = 3(1+z^2)\tan^{-1}z$$

$$\Rightarrow \int \frac{dz}{(1+z^2)\tan^{-1}z} = 3\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{d(\tan^{-1}z)}{\tan^{-1}z} = 3\ln x + c$$

$$\Rightarrow \ln(\tan^{-1}z) = 3\ln x + c$$

$$\ln(\tan^{-1}(y/x)) = 3\ln x + c$$

(4) $x\frac{dy}{dx} = y + 2xe^{-y/x} \Rightarrow \frac{dy}{dx} = y/x + 2e^{-y/x}$

$$\Rightarrow z + x\frac{dz}{dx} = z + 2e^{-z}$$

$$\Rightarrow x\frac{dz}{dx} = 2e^{-z}$$

$$\Rightarrow \int \frac{dz}{e^{-z}} = 2\int\frac{dx}{x}$$

$$\Rightarrow e^{z} = 2\ln x + c$$

$$\Rightarrow e^{y/x} = 2\ln x + c \quad //$$

HOW TO SOLVE:

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$$

$x = x_0 + h \; ; \; y = y_0 + K$

$$\Rightarrow \frac{dy_0}{dx_0} = \frac{a(x_0+h)+b(y_0+K)+c}{a'(x_0+h)+b'(y_0+K)+c'}$$

$$= \frac{ax_0+by_0+ah+bK+c}{a'x_0+b'y_0+a'h+b'K+c'}$$

$$\boxed{\begin{aligned} ah+bK+c=0 \\ a'h+b'K+c'=0 \end{aligned}}$$

$ah + bK = -c \quad \Rightarrow \quad b'ah + b'bK = -b'c$

$a'h + b'K = -c' \quad \Rightarrow \quad ba'h + bb'K = -bc'$

$\Rightarrow \quad b'ah - ba'h = -b'c + bc'$

$\Rightarrow \quad h(b'a - ba') = bc' - b'c$

$$\Rightarrow \quad \boxed{h = \frac{bc'-b'c}{b'a-ba'}}$$

$$\boxed{K = \frac{-c-ah}{b}}$$

hence, we know h & K. just solve the eqn. normally...

$$\frac{dy_0}{dx_0} = \frac{ax_0+by_0}{a'x_0+b'y_0}$$

$\rightarrow$ after getting soln., substitute $x = x_0 + h$, $y = y_0 + K$

IFF $\frac{a}{a'} \,!\!= \frac{b}{b'} \,!\!= K$

$$\frac{dy}{dx} = \frac{2x+3y+4}{6x+9y+7}$$

$$2x+3y = z$$
$$\Rightarrow 2+3\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{1}{3}\left(\frac{dz}{dx}-2\right) = \frac{z+4}{3z+7}$$

$$\Rightarrow \frac{dz}{dx}-2 = \frac{3z+12}{3z+7}$$

$$\Rightarrow \frac{dz}{dx} = \frac{3z+12+6z+14}{3z+7} = \frac{9z+26}{3z+7}$$

$$\Rightarrow \left(\frac{3z+7}{9z+26}\right)dz = dx$$

$$\Rightarrow \frac{1}{3}\left(\frac{9z+26-5}{9z+26}\right)dz = dx$$

$$\Rightarrow \frac{1}{3}\int\left(1-\frac{5}{9z+26}\right)dz = \int dx$$

$$\Rightarrow \frac{1}{3}\left[\int dz - \frac{5}{9}\int\frac{d(9z+26)}{9z+26}\right] = x+C$$

$$\Rightarrow \frac{z}{3} - \frac{5}{9}\ln(9z+26) = x+C$$

$$\Rightarrow \frac{2x+3y}{3} - \frac{5}{9}\ln(18x+27y+26) = x+C \quad //$$

$$\frac{dy}{dx} = \frac{x+y+3}{x-3y+7}$$

$$x = w+h$$
$$y = z+k$$

$$\Rightarrow \frac{dz}{dw} = \frac{w+z+3+h+k}{w-3z+7+h-3k}$$

$$\begin{cases} 3+h+k = 0 \\ 7+h-3k = 0 \end{cases}$$

POSSIBLE!

$$\Rightarrow \frac{dz}{dw} = \frac{w+z}{w-3z}$$

$$\Rightarrow \frac{dz}{dw} = \frac{1+z/w}{1-3z/w}$$

$$z/w = t$$
$$z = wt$$
$$\frac{dz}{dw} = t + w\cdot\frac{dt}{dw}$$

$$\Rightarrow t + w\cdot\frac{dt}{dw} = \frac{1+t}{1-3t}$$

$$\Rightarrow w\cdot\frac{dt}{dw} = \frac{1+\cancel{t}-\cancel{t}+3t^2}{1-3t} = \frac{1+3t^2}{1-3t}$$

$$\Rightarrow \frac{(1-3t)}{(1+3t^2)} \cdot dt = \frac{dw}{w}$$

$$\Rightarrow \frac{dt}{1+3t^2} - \frac{1}{2} \cdot \frac{6t\,dt}{1+3t^2} = \frac{dw}{w}$$

$$\Rightarrow \frac{1}{\sqrt{3}} \int \frac{d\sqrt{3}t}{1+(\sqrt{3}t)^2} - \frac{1}{2}\int \frac{d(1+3t^2)}{1+3t^2} = \int \frac{dw}{w}$$

$$\Rightarrow \frac{1}{\sqrt{3}} \tan^{-1}\sqrt{3}t - \frac{1}{2} \cdot \ln(1+3t^2) = \ln w + c.$$

$$\Rightarrow \frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{3} \cdot z/w\right) - \frac{1}{2}\ln\left(1+3z^2/w^2\right) = \ln w + c$$

substitute z & w in terms of x & y

(a) $$\frac{dy}{dx} = \frac{x+y+4}{x-y-6} \longrightarrow \frac{x_0+y_0+h+k+4}{x_0-y_0+h-k-6}$$

$$\Rightarrow \frac{dy_0}{dx_0} = \frac{x_0+y_0}{x_0-y_0} = \frac{1+y_0/x_0}{1-y_0/x_0}$$

$$\Rightarrow z + x_0\frac{dz}{dx_0} = \frac{1+z}{1-z}$$

$$\Rightarrow x_0\frac{dz}{dx_0} = \frac{1+z}{1-z} - z = \frac{1+z-z+z^2}{1-z} = \frac{1+z^2}{1-z}$$

$$\Rightarrow \int \frac{(1-z)}{1+z^2}\,dz = \int \frac{dx_0}{x_0} \Rightarrow \tan^{-1}z - \frac{1}{2}\ln(1+z^2) = \ln x_0 + c$$

$$\Rightarrow \tan^{-1}(y_0/x_0) - \tfrac{1}{2}\ln\left(1+y_0^2/x_0^2\right) = \ln x_0 + c$$

$$\Rightarrow \tan^{-1}(y_0/x_0) - \frac{1}{2}\left[\ln(y_0^2+x_0^2) - \ln x_0^2\right] = \ln x_0 + c$$

$$\Rightarrow \tan^{-1}(y_0/x_0) - \frac{1}{2}\ln(x_0^2+y_0^2) = c$$

h & k can be solved for!

$$\Rightarrow \tan^{-1}\left(\frac{y-k}{x-k}\right) - \frac{1}{2}\ln\left((x-h)^2+(y-k)^2\right) = c$$

(b) Similiar to (a)

(c) $$\frac{dy}{dx} = \frac{x+y+4}{x+y-6} \Rightarrow \frac{dz}{dx} - 1 = \frac{z+4}{z-6}$$

$z = x+y$

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$$\Rightarrow \frac{dz}{dx} = \frac{z+4}{z-6} + 1$$

$$= \frac{z+4+z-6}{z-6} = \frac{2z-2}{z-6}$$

$$= \frac{2(z-1)}{z-6}$$

$$\Rightarrow \int \frac{z-6}{z-1} \cdot dz = \int 2\,dx$$

$$\Rightarrow \int dz - 5\int \frac{d(z-1)}{z-1} = 2\int dx$$

$$\Rightarrow z - 5\ln(z-1) = 2x + c$$

$$\Rightarrow x+y - 5\ln(x+y-1) = 2x + c$$

$$\Rightarrow y - x - 5\ln(x+y-1) = c$$

EXACT EQUATIONS:

let $f(x,y)$ be a fn. of x and y.

then, Total derivative of $f(x,y)$ is:

$$\frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}dy = 0$$

The family $x^2y^3 = c$ has $2xy^3dx + 3x^2y^2dy = 0$ as its differential equation.

let $M(x,y)dx + N(x,y)dy = 0$

$$\Rightarrow \frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y)$$

if this is the case, then:
$Mdx + Ndy$ is called an exact differential
and $Mdx + Ndy = 0$ is an exact differential equation

check: $ydx + xdy = 0 \longrightarrow$ exact or not?

$$\frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}.dy = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = y \Rightarrow \partial f = y\partial x$$
$$\Rightarrow f = xy$$

$$\frac{\partial f}{\partial y} = x \Rightarrow \partial f = x\partial y$$
$$\Rightarrow f = xy.$$

Both generate the same fn.

$$\boxed{f(x,y) = xy} \; \boxed{\frac{\partial f}{\partial x} = y \; ; \; \frac{\partial f}{\partial y} = x}$$

$$xdy + ydx = 0$$

$$\Rightarrow xdy = -ydx \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$\ln y = -\ln x + K$

$\Rightarrow \ln(xy) = K$

$= \ln c$

$\Rightarrow \boxed{xy = c}$

$f(x,y)$

$$\frac{1}{y}\cdot dx - \frac{x}{y^2}\,dy = 0$$

$$\Rightarrow \frac{1}{y}dx + x\cdot d\left(\frac{1}{y}\right) = 0$$

$$\Rightarrow \frac{1}{y}\cdot dx = -x\cdot d\left(\frac{1}{y}\right) \Rightarrow \int \frac{d(1/y)}{(1/y)} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln\left(\frac{1}{y}\right) = -\ln x + c$$

$$\Rightarrow \ln\left(\frac{1}{y}\right) + \ln x = +c$$

$$\Rightarrow \ln\left(\frac{x}{y}\right) = c = \ln K$$

$$\Rightarrow \boxed{x/y = K}$$

$$\frac{\partial f}{\partial x} = M \Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

$$\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

if $\left.\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}\right\} \Rightarrow$ The diff. eqn. is EXACT.

$$y\,dx + (x^2y - x)\,dy = 0$$

$$\frac{\partial f}{\partial x} = y \quad ; \quad \frac{\partial f}{\partial y} = x^2y - x$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = 1 \quad ; \quad \frac{\partial^2 f}{\partial x \partial y} = 2xy - 1$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}} \longrightarrow \text{So, not exact}\ldots$$

$\boxed{\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}} \rightarrow$ sufficient & necessary condition for exactness...

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N$$

$$\Rightarrow f = \int M\,dx + g(y) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int M\,dx + g(y)\right) = N$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\int M dx\right) + g'(y) = N$$

$$\Rightarrow g'(y) = N - \frac{\partial}{\partial y}\left(\int M dx\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(g'(y)\right) = \frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\left(\int M dx\right)\right)$$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial}{\partial x \partial y}\left(\int M dx\right) = 0$$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial}{\partial y \partial x}\left(\int M dx\right) = 0$$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\Rightarrow \boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

$f = \int M dx + g(y)$; where $g(y) = \int \left(N - \frac{\partial}{\partial y}\left(\int M dx\right)\right)$

$\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

$$e^y dx + (xe^y + 2y) dy = 0$$
$$M dx + N dy = 0$$

$M = e^y$; $N = xe^y + 2y$

$\Rightarrow \frac{\partial M}{\partial y} = e^y$; $\frac{\partial N}{\partial x} = e^y$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\frac{\partial f}{\partial x} = e^y \Rightarrow f = \int e^y dx + g(y)$

$\frac{\partial f}{\partial y} = xe^y + 2y$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int e^y dx\right) + g'(y)$$

$$\Rightarrow N - \frac{\partial}{\partial y}\left(\int e^y dx\right) = g'(y)$$

$$\Rightarrow N - \frac{\partial}{\partial y}(xe^y) = g'(y)$$

$$\Rightarrow N - xe^y = xe^y + 2y - xe^y = g'(y)$$

$$\Rightarrow g'(y) = 2y$$

$$\Rightarrow g(y) = y^2$$

$\therefore\ f = \int e^y dx + g(y) = xe^y + y^2 + c$

$\therefore\ xe^y + y^2 + c = 0$

$\Rightarrow xe^y + y^2 = -c = c'$

$\therefore\ \boxed{xe^y + y^2 = c'}$

$(y - x^3)dx + (x + y^3)dy = 0$

$Mdx + Ndy = 0$

$M = y - x^3$

$\frac{\partial f}{\partial x} = y - x^3 \Rightarrow f(x,y) = xy - \frac{x^4}{4} + g(y)$

$\frac{\partial f}{\partial y} = x + g'(y) = N$

$\Rightarrow g'(y) = N - x = (x + y^3) - x = y^3$

$\Rightarrow g(y) = \frac{y^4}{4} + c$

$\therefore\ f(x,y) = xy - \frac{x^4}{4} + \frac{y^4}{4} + c$

$xy - \frac{x^4}{4} + \frac{y^4}{4} + c = 0$

$\Rightarrow xy - \frac{x^4}{4} + \frac{y^4}{4} = c'$

$(c = -c')$

$(y + y\cos xy)dx + (x + x\cos xy)dy = 0$

$Mdx + Ndy = 0$

$M = y + y\cos xy \quad ; \quad N = x + x\cos xy$

$\frac{\partial f}{\partial x} = y + y\cos xy \Rightarrow \int \partial f = \int y\,\partial x + \int \cos xy\, d(xy)$

$\Rightarrow f = xy + \sin xy + g(y)$

$\Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial}{\partial y}(\sin xy) + g'(y)$

$N = x + x\cos xy + g'(y)$

$\Rightarrow x + x\cos xy = x + x\cos xy + g'(y)$

$\Rightarrow g'(y) = 0$

$\Rightarrow \boxed{g(y) = K}$

$\therefore\ \boxed{f(x,y) = xy + \sin xy + K}$

$\Rightarrow xy + \sin xy + K = c'$

$\Rightarrow xy + \sin xy = -K = c$

$\Rightarrow xy + \sin xy = c$

what if not exact?

$$y\,dx + (x^2y - x)\,dy = 0$$

$$\Rightarrow \frac{y}{x^2}\,dx + \left(y - \frac{1}{x}\right)dy = 0$$

$$M\,dx + N\,dy = 0 \qquad \frac{\partial M}{\partial y} = \frac{1}{x^2} \;;\; \frac{\partial N}{\partial x} = +\frac{1}{x^2}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \rightarrow \text{So, exact!}$$

a fn. that tranforms a non-exact eqn. into an exact one, is called an integrating factor (WOW!!!)
another problem now → we should know if integrating factor exists or not → but thats not our headache → it'll be given if solution exists or not → so, the only problem is to find the integrating factor.

$$\underbrace{\mu(x,y)\cdot M(x,y)\,dx + \mu(x,y)\cdot N(x,y)\,dy}_{\hookrightarrow\ \text{EXACT!!}} = 0$$

$$\frac{\partial f}{\partial x} = \mu\cdot M \qquad \frac{\partial f}{\partial y} = \mu\cdot N$$

$$\Rightarrow \boxed{\left(\frac{\partial f}{\partial x}\right)\cdot\frac{1}{M} = \left(\frac{\partial f}{\partial y}\right)\cdot\frac{1}{N} = \mu}$$

$$\frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy = 0$$

how to find μ?

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \qquad M' = \mu\cdot M \;;\; N' = \mu\cdot N$$

$$\Rightarrow \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} \quad\Rightarrow\quad \frac{\partial(\mu\cdot M)}{\partial y} = \frac{\partial(\mu\cdot N)}{\partial x}$$

$$\Rightarrow M\frac{\partial \mu}{\partial y} + \mu\cdot\frac{\partial M}{\partial y} = N\cdot\frac{\partial \mu}{\partial x} + \mu\cdot\frac{\partial N}{\partial x}$$

$$\Rightarrow \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\cdot\frac{\partial \mu}{\partial x} - M\cdot\frac{\partial \mu}{\partial y}$$

$$\Rightarrow \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\mu}\left(N\cdot\frac{\partial \mu}{\partial x} - M\cdot\frac{\partial \mu}{\partial y}\right)$$

2 cases: (a) $\mu = \mu(x)$ & (b) $\mu = \mu(y)$

(a) $\underbrace{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}_{g(x)} = \frac{1}{\mu}\cdot\frac{\partial \mu}{\partial x} \rightarrow$ fn. of x

$\hookrightarrow$ do variable-separation

$$\Rightarrow \int g(x)\,dx = \int \frac{d\mu}{\mu}$$

$$\Rightarrow \int g(x)\,dx = \ln(\mu(x))$$

$$\Rightarrow \mu(x) = e^{\int g(x)\,dx}$$

(b) $\underbrace{-\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}_{h(y)} = \frac{1}{\mu}\cdot\frac{\partial \mu}{\partial y}$

$$\Rightarrow \boxed{\mu(y) = e^{\int h(y)\,dy}}$$

$$y\,dx + (x^2y - x)\,dy = 0$$

$M = y$; $N = x^2y - x$

$\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = 2xy - 1$

$$\boxed{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2 - 2xy}$$

$\frac{2-2xy}{x(xy-1)} = \frac{-2\cancel{(xy-1)}}{x\cancel{(xy-1)}} = -\frac{2}{x} \rightarrow$ purely fn. of x

$\frac{2-2xy}{-y} = -\frac{2}{y} + 2x \rightarrow$ not purely fn. of y.

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln(1/x^2)} = \frac{1}{x^2}$$

So, multiply by $1/x^2$.

$$y\,dx + (x^2y - x)\,dy = 0$$

$\Rightarrow \frac{y}{x^2}dx + (y - 1/x)\,dy = 0 \rightarrow$ now Apply exact!

$e^x dx + (e^x \cot y + 2y \operatorname{cosec} y)dy = 0$

- not homogeneous, exact.
- have to multiply by integrating factor.

$M dx + N dy = 0 \rightarrow$ not exact.

$\mu \cdot M dx + \mu \cdot N dy = 0 \rightarrow$ exact!

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \mu M \quad \text{and} \quad \frac{\partial f}{\partial y} = \mu N$$

AND.

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\Rightarrow M \cdot \frac{\partial \mu}{\partial y} + \mu \cdot \frac{\partial M}{\partial y} = N \cdot \frac{\partial \mu}{\partial x} + \mu \cdot \frac{\partial N}{\partial x}$$

$$\Rightarrow \mu \cdot \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N \cdot \frac{\partial \mu}{\partial x} - M \cdot \frac{\partial \mu}{\partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{\mu}\left(N \cdot \frac{\partial \mu}{\partial x} - M \cdot \frac{\partial \mu}{\partial y}\right)$$

assume $\rightarrow$ μ: fn. of x $\Rightarrow \dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = \dfrac{N}{\mu} \cdot \dfrac{\partial \mu}{\partial x}$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$$

$$\Rightarrow g(x) = \frac{1}{\mu} \cdot \frac{d\mu}{\partial x}$$

$$\Rightarrow \int g(x)dx = \int \frac{d\mu}{\mu}$$

$$\Rightarrow \ln \mu = \int g(x)\,dx$$

$$\Rightarrow \mu(x) = e^{\int g(x)\,dx}$$

similarly for y:

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{1}{\mu} \cdot M \cdot \frac{\partial \mu}{\partial y}$$

$$\Rightarrow -\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\mu} \cdot \frac{\partial \mu}{\partial y} = h(y)$$

$$\Rightarrow \int h(y)\,dy = \int \frac{d\mu}{\mu} \quad \Rightarrow \ln \mu = \int h(y)\,dy$$

$$\Rightarrow \mu(y) = e^{\int h(y)dy}.$$

whichever fn. is purely $f(x)$ or $f(y)$ $\rightarrow$ I.F.

$\downarrow$

μ

lets use the same logic:

$$e^x dx + (e^x \cot y + 2y \operatorname{cosec} y)\, dy = 0$$

$\downarrow$ M $\quad\quad$ $\downarrow$ N.

let I.F. be μ.

$$\mu(x) = e^{\int g(x)dx} \quad ; \text{ where } g(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$g(x) = \frac{1}{e^x \cot y + 2y \operatorname{cosec} y}\left(\cancel{\frac{\partial}{\partial y}(e^x)}^{\,0} - \frac{\partial}{\partial x}\left(e^x \cot y + \cancel{2y \operatorname{cosec} y}^{\,0}\right)\right)$$

$$= \frac{-e^x \cot y}{e^x \cot y + 2y \operatorname{cosec} y} \rightarrow \text{not purely fn. of } x.$$

$$h(y) = -\frac{1}{M}\cdot\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{e^x}\left(\cancel{\frac{\partial}{\partial y}(e^x)}^{\,0} - \frac{\partial}{\partial x}\begin{pmatrix} e^x \cot y \\ + 2y \operatorname{cosec} y\end{pmatrix}\right)$$

$$= -\frac{1}{e^x}\cdot\left(-e^x \cdot \cot y\right)$$

$$= \cot y$$

$\therefore$, $h(y) = \cot y$

$$I.F. = e^{\int \cot y\, dy} = e^{\int \frac{\cos y}{\sin y} dy} = e^{\int \frac{d \sin y}{\sin y}} = e^{\ln \sin y} = \sin y.$$

$\therefore$, exact equation:

$$e^x \sin y\, dx + \sin y \cdot (e^x \cot y + 2y \operatorname{cosec} y)\, dy = 0$$

$$\Rightarrow e^x \sin y\, dx + (e^x \cos y + 2y)\, dy = 0$$

$$\frac{\partial f}{\partial x} = e^x \sin y$$

$$\Rightarrow f(x,y) = e^x \sin y + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = e^x \cos y + g'(y) = N$$

$$\Rightarrow g'(y) = N - e^x \cos y$$

$$= e^x \cos y + 2y - e^x \cos y$$

$$\Rightarrow g'(y) = 2y$$

$$\Rightarrow g(y) = y^2.$$

$\therefore$, $f(x,y) = e^x \sin y + y^2 + c' = 0$

$$\Rightarrow e^x \sin y + y^2 = -c' = c \quad\quad \therefore, \boxed{e^x \sin y + y^2 = c}$$

① Show : if $\dfrac{\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)}{(Ny - Mx)} = g(z)$, where $z = xy$

then $\mu = e^{\int g(z)dz}$ is an I.F. of $Mdx + Ndy = 0$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\mu}\left(N \cdot \frac{\partial \mu}{\partial x} - M \cdot \frac{\partial \mu}{\partial y}\right)$$

$$= \frac{1}{\mu} \cdot \left(N \cdot \frac{\partial \mu}{\partial z} \cdot \frac{\partial z}{\partial x} - M \cdot \frac{\partial \mu}{\partial z} \cdot \frac{\partial z}{\partial y}\right)$$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\mu} \cdot \frac{\partial \mu}{\partial z} \cdot (Ny - Mx)$$

$$\Rightarrow \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{(Ny - Mx)} = \frac{1}{\mu} \cdot \frac{\partial \mu}{\partial z}$$

$$\Rightarrow g(z) = \frac{1}{\mu} \cdot \frac{\partial \mu}{\partial z}$$

$$\Rightarrow \int g(z)dz = \int \frac{d\mu}{\mu}$$

$$\Rightarrow \ln \mu = \int g(z)dz \Rightarrow \boxed{\mu(z) = e^{\int g(z)dz}}$$

LINEAR DIFFERENTIAL EQUATIONS

$$\frac{dy}{dx} = p(x)y + q(x)$$

$$\frac{d^2y}{dx^2} = p(x) \cdot \frac{dy}{dx} + q(x)y + r(x)$$

$$\boxed{\frac{d^ny}{dx^n} = p(x) \cdot \frac{d^{n-1}y}{dx^{n-1}} + q(x) \cdot \frac{d^{n-2}y}{dx^{n-2}} + r(x) \cdot \frac{d^{n-3}y}{dx^{n-3}} + \cdots + z(x) \cdot y + a(x)}$$

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$\Rightarrow \frac{dy}{dx} + p(x) \cdot y - q(x) = 0$$

$$\Rightarrow dy + (p(x)y - q(x))dx = 0$$

$$\Rightarrow \underbrace{\left(p(x)\cdot y - q(x)\right)}_{M} dx + \underbrace{1}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = p(x)$$

$$\frac{1}{N} = 1. \qquad \therefore \boxed{\mu(x) = e^{\int p(x)\,dx}}$$

$$\boxed{\text{derive : } \quad y e^{\int P dx} = \int Q e^{\int P dx} dx + c}$$

$$\frac{dy}{dx} + \underbrace{\frac{1}{x}}_{P} y = \underbrace{3x}_{Q}$$

$$y \cdot e^{\int \frac{1}{x} dx} = \int 3x \cdot e^{\int \frac{1}{x} dx} dx + c$$

$$\Rightarrow y \cdot e^{\ln x} = \int 3x \cdot e^{\ln x} dx + c$$

$$\Rightarrow y \cdot x = \int 3x^2 dx + c$$

$$\Rightarrow y \cdot x = x^3 + c$$

$$\Rightarrow \boxed{y = x^2 + cx^{-1}}$$

BERNOULLI'S EQUATIONS

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n$$

$$\Rightarrow \frac{1}{y^n}\cdot\frac{dy}{dx} + p(x)\cdot y^{1-n} = Q(x)$$

$$\Rightarrow \frac{1}{1-n}\cdot\frac{d}{dx}\left(y^{1-n}\right) + p(x)\, y^{1-n} = Q(x)$$

$$\Rightarrow \frac{1}{1-n}\cdot\frac{dz}{dx} + p(x)\cdot z = Q(x)$$

$$\Rightarrow \frac{dz}{dx} + P(x)\cdot(1-n)\cdot z = Q(x)\cdot(1-n)$$

$$\Rightarrow \frac{dz}{dx} + P_0(x)\cdot z = Q_0(x)$$

$$\Rightarrow z \cdot e^{\int P_0(x)dx} = \int Q_0 e^{\int P_0(x)dx} dx + c$$

$$\Rightarrow z = e^{-\int P_0(x)dx} \cdot \left[\int Q_0 e^{\int P_0(x)dx} \cdot dx + c\right]$$

$$\Rightarrow y^{1-n} = e^{-(1-n)\int P(x)\,dx} \cdot \left[(1-n) \int Q(x) \cdot e^{(1-n)\int P(x)\,dx} \cdot dx + c \right]$$

$$\Rightarrow y = \sqrt[1-n]{e^{-(1-n)\int P(x)\,dx} \cdot \left[(1-n) \int Q(x) \cdot e^{(1-n)\int P(x)\,dx}\, dx + c \right]}$$

① $(e^y - 2xy) \cdot \frac{dy}{dx} = y^2$ $\longrightarrow$ $\boxed{\text{SOLVE FIRST USING EXACT and then APPLY BERNOULLI'S EQUATION}}$

① $\underbrace{(x^3+xy^3)}_{M}dx + \underbrace{3y^2}_{N}dy = 0$

$$\frac{\partial M}{\partial y} = 3xy^2 \qquad \frac{\partial N}{\partial x} = 0$$

$$e^{\int g(x)dx} = \mu(x) \; ; \qquad g(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$= \frac{1}{3y^2}\left(3xy^2\right) = x.$$

$$\therefore \; \mu(x) = e^{\int x\,dx} = e^{x^2/2}$$

$$M' = e^{x^2/2}\cdot(x^3+xy^3) \quad ; \quad N' = 3y^2\cdot e^{x^2/2}$$

$$\frac{\partial f}{\partial x} = x^3\cdot e^{x^2/2} + xe^{x^2/2}y^3$$

$$\Rightarrow f = \int x^3 e^{x^2/2}dx + y^3\int xe^{x^2/2}dx$$

$$= \int x^2\cdot e^{x^2/2}\cdot d(x^2/2) + y^3\int e^{x^2/2}d(x^2/2)$$

$$= \int x^2\cdot d(e^{x^2/2}) + y^3\cdot e^{x^2/2} + g(y)$$

$$= x^2e^{x^2/2} - 2\int e^{x^2/2}d(x^2/2) + y^3e^{x^2/2} + g(y)$$

$$f = x^2e^{x^2/2} - 2e^{x^2/2} + y^3e^{x^2/2} + g(y)$$

$$\frac{\partial f}{\partial y} = 3y^2\cdot e^{x^2/2} + g'(y) = N$$

$$\Rightarrow g'(y) = N - 3y^2e^{x^2/2} = 3y^2e^{x^2/2} - 3y^2e^{x^2/2} = 0$$

$$\Rightarrow \boxed{g(y) = K}$$

$$\Rightarrow f = x^2e^{x^2/2} - 2e^{x^2/2} + y^3e^{x^2/2} + K = 0$$

$$\Rightarrow \boxed{x^2e^{x^2/2} - 2e^{x^2/2} + y^3e^{x^2/2} = C}$$

② $M\,dx + N\,dy = 0$

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \Rightarrow \mu\cdot\frac{\partial M}{\partial y} + M\cdot\frac{\partial \mu}{\partial y} = \mu\cdot\frac{\partial N}{\partial x} + N\cdot\frac{\partial \mu}{\partial x}$$

$$\Rightarrow \mu\cdot\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \left(N\cdot\frac{\partial \mu}{\partial x} - M\cdot\frac{\partial \mu}{\partial y}\right)$$

$$\Rightarrow \mu = \frac{\left(N\cdot\frac{\partial\mu}{\partial x} - M\cdot\frac{\partial\mu}{\partial y}\right)}{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$$

if $\mu = f(z)$, $z = x+y$

$\Rightarrow dz = dx + dy$

$$= \frac{\left(N\cdot\frac{\partial\mu}{\partial z}\cdot\frac{\partial z}{\partial x} - M\cdot\frac{\partial\mu}{\partial z}\cdot\frac{\partial z}{\partial y}\right)}{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$$

$$\Rightarrow \mu = \frac{\frac{\partial\mu}{\partial z}(N-M)}{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}} \Rightarrow \int \underbrace{\boxed{\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N-M}}}_{h(z)} \cdot dz = \int \frac{\partial\mu}{\mu} = \ln\mu$$

$$\Rightarrow \boxed{\mu = e^{\int h(z)\,dz}}$$

③ $y' = 2xy + 1$

$\Rightarrow \frac{dy}{dx} = 2xy + 1$

$\Rightarrow dy = (2xy+1)\,dx$

$\Rightarrow (2xy+1)\,dx + (-1)\,dy = 0$

$$h(x) = \frac{1}{N}\cdot\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{\partial M}{\partial y} = -2x$$

$\therefore h(x) = 2x$;)

$$I.F. = e^{-\int 2x\,dx} = e^{-x^2}$$

$$\left(2xye^{-x^2/2} + e^{-x^2}\right)dx + \left(-e^{-x^2/2}\right)dy = 0$$

D.K.

$$\frac{\partial f}{\partial x} = 2xye^{-x^2/2} + e^{-x^2/2} \Rightarrow f = \int 2xye^{-x^2}\,dx + \boxed{\int e^{-x^2/2}\,dx}$$

$+ g(y)$

④

$$\frac{dy}{dx} + y = |x|$$

$\forall\, x > 0$: $\frac{dy}{dx} + y = x \Rightarrow y\cdot e^{\int dx} = \int x e^{\int dx}\,dx + C$

$\Rightarrow ye^x = \int xe^x\,dx + C = \int x\,d(e^x) + C$

$= xe^x - e^x + C$

$\Rightarrow \boxed{y = x - 1 + C_1 e^{-x}}$

$\forall\, x < 0$: $\frac{dy}{dx} + y = -x$

$\Rightarrow ye^{\int dx} = \int -xe^{\int dx}\,dx + C_2$

$\Rightarrow ye^{x} = -\int xe^{x}dx + c_2$

$y(-1) = 2 + c_2 e$

$\Rightarrow \boxed{c_2 = -2/e}$

$\Rightarrow ye^{x} = -xe^{x} + x + c_2$

$\Rightarrow y = -x + 1 + c_2 e^{-x}$

$c_2 = c_1$

$\Rightarrow y(1) = \frac{-2}{e} \cdot e^{-1} = -2/e^2$

(5) $xdy + ydx = xy^2 dx$

$\Rightarrow x\frac{dy}{dx} + y = xy^2$

$\Rightarrow \frac{dy}{dx} + y/x = y^2$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1$

$\Rightarrow \frac{-d}{dx}\left(\frac{1}{y}\right) + \frac{1}{x}\cdot\left(\frac{1}{y}\right) = 1$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{y}\right) + \left(\frac{-1}{x}\right)\left(\frac{1}{y}\right) = -1$

$\frac{1}{y}\cdot e^{\int -1/x\, dx} = \int -1\cdot e^{\int -1/x\, dx} dx + c$

$\Rightarrow \frac{1}{y}\cdot\frac{1}{x} = -\int \frac{1}{x}dx + c$

$\Rightarrow \frac{1}{y}\cdot\frac{1}{x} = -\ln x + c$

$\Rightarrow \frac{1}{y} = -x\ln x + cx \Rightarrow \boxed{y = \frac{1}{cx - x\ln x}}$

(6) $\frac{1}{N}\cdot\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x) = 0$

$\Rightarrow \mu(x) = e^{\int g(x)dx} = e^{c_1} = k_1$

$\frac{-1}{M}\cdot\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y) = 0$

$\mu(y) = e^{\int g(y)dy} = e^{c_2} = k_2$

→ solve without fn. assumption...

$\therefore \boxed{\mu(x) \text{ (or) } \mu(y) = K}$

SECOND ORDER DIFFERENTIAL EQUATIONS

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

$$y'' + P(x)y' + Q(x)y = R(x)$$

$P(x)$, $Q(x)$, $R(x) \longrightarrow$ fns. of x (or) constants.

$$x^2y'' + 2xy - 2y = 0 \Rightarrow y'' + y'\underbrace{\left(\frac{2}{x}\right)}_{P(x)} + y\underbrace{\left(-\frac{2}{x^2}\right)}_{Q(x)} = \underbrace{0}_{R(x)}$$

HOW TO SOLVE? → make use of Existence and Uniqueness Theorem

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

first solve: $\frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)\cdot y = 0$

solution: $y_g = c_1y_1 + c_2y_2$

$y_p(x) \rightarrow$ solution for: $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$

$$y(x) = y_g(x) + y_p(x)$$

let $P(x)$, $Q(x)$, $R(x) \rightarrow$ continuous in $[a,b]$.

if $x_0 \in [a,b]$ and $y_0 = y(x_0)$ and $y_0' = y'(x_0)$ then

$y(x) \rightarrow$ UNIQUE solution of: $\frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)y = R(x)$

$y'' + y = 0$, $y(0) = 0$ and $y'(0) = 1$.

$\Rightarrow y = \sin x$ as $y(0) = \sin 0 = 0$ and $y' = \cos x$

$\quad$ & $y'(0) = \cos 0 = 1$.

& &

$$(\sin x)'' + \sin x = 0$$
$$\Rightarrow (\cos x)' + \sin x = 0$$
$$\Rightarrow -\sin x + \sin x = 0 \Rightarrow \boxed{0 = 0}$$

general solution of $y''+y=0$ is:

$$y = c_1 \sin x + c_2 \cos x$$

$y(0)=0 \Rightarrow$ $c_2 = 0$

$y'(0)=1 \Rightarrow y' = c_1 \cos x - \cancel{c_2 \sin x}^{\to 0} = c_1 \cos x$

$\Rightarrow \boxed{1 = c_1}$

$\therefore$, $\boxed{y = \sin x}$

$y(0)=1 \Rightarrow$ $y = c_1 \sin x + c_2 \cos x$

$\Rightarrow \boxed{c_2 = 1}$

$y'(0)=0 \Rightarrow$ $y' = c_1 \cos x - c_2 \sin x$

$\Rightarrow$ $0 = c_1 \cos 0 - \sin 0$

$\Rightarrow$ $c_1 = 0$

$\therefore$, $\boxed{y = \cos x}$

consider: $y'' + P(x)y' + Q(x)y = R(x)$.

if $y_g \to$ general ($y''+P(x)y'+Q(x)y=0$) solution of homogeneous eqn

$y_p \to$ particular solution of the given complete, non-homogeneous soln.

then $y_g + y_p \to$ general solution of the complete equation.

if y_1 and $y_2 \to$ 2 soln.s of homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0$$

then,

$c_1 y_1(x) + c_2 y_2(x) \to$ solution of the equation too!

$$\left(c_1 y_1(x) + c_2 y_2(x)\right)'' + P(x)\left(c_1 y_1(x) + c_2 y_2(x)\right)' + Q(x)\left(c_1 y_1(x) + c_2 y_2(x)\right)$$

$$= c_1\left(y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x)\right) + c_2\left(y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x)\right)$$

$$= c_1 \cdot 0 + c_2 \cdot 0$$

$$= 0$$

$c_1 y_1(x) + c_2 y_2(x) \longrightarrow$ linear combo of y_1 and $y_2 \to$ solution!

(y_1 & $y_2 \to$ solutions)

if $y_1 = K y_2$,

same soln!

then $c_1 K y_2(x) + c_2 y_2(x) = (c_1 K + c_2) y_2(x) = c\, y_2(x)$

∘∘ general soln → linear combo. of LINEARLY independent solutions . . . . .

$$y'' + y' = 0$$

$$\Rightarrow (y'+y)' = 0$$

$$\Rightarrow y' + y = K$$

$$\Rightarrow \frac{dy}{dx} = K - y$$

$$\Rightarrow \int \frac{dy}{K-y} = \int dx \quad \Rightarrow -\ln(K-y) = x$$

$$\Rightarrow \ln(K-y) = -x$$

$$\Rightarrow K - y = e^{-x}$$

$$\Rightarrow y = K - e^{-x}$$

$K \to C_1$

$-1 \to C_2$ $\quad \Rightarrow \boxed{y = C_1 + C_2 e^{-x}}$

$$x^2 y'' + 2xy' - 2y = 0$$

$$y'' + \frac{2}{x} y' - \frac{2}{x^2} y = 0$$

$$y'' = \frac{2}{x^2} y - \frac{2}{x} y' = 2\left(y \cdot \tfrac{1}{x^2} - \tfrac{1}{x} \cdot y'\right)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 2 \cdot \frac{d\left(-y/x\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2y}{x} + K$$

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = K$$

$$y \cdot e^{\int 2/x \, dx} = \int K \cdot e^{\int 2/x \, dx} dx$$

$$yx^2 = K \int x^2 dx = \frac{Kx^2}{2} + c$$

$$\Rightarrow yx^2 = \frac{Kx^3}{2} + c \quad \Rightarrow \quad y = \frac{Kx}{2} + cx^{-2}$$

$$\Rightarrow y = C_1 x + C_2 x^{-2}$$

if $f(x) \equiv 0$, then dependent . . . .

we know y_1 and $y_2 \to$ 2 linearly independent solutions.
$y(x_0) = y_0$ and $y_0' = y'(x_0)$ $\quad (y'' + Py' + Qy = R)$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 \quad -①$$
$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$$
$$\Rightarrow y_0' = c_1 y_1'(x_0) + c_2 y_2'(x_0) \quad -②$$

① and ② $\to$ can be solved to get c_1 and c_2 . . . . .

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 \quad -①$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0' \quad -②$$

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0 \quad (\text{WHY ????})$$

$$\Rightarrow y_1(x_0) \cdot y_2'(x_0) - y_2(x_0) \cdot y_1'(x_0) \neq 0$$

$W(x) = y_1(x) \cdot y_2'(x) - y_2(x) \cdot y_1'(x) \to$ if this $\neq 0$, then y_1 & y_2 are linearly independent . . . .

if n fns $\downarrow$

$$\begin{vmatrix} a & b & \cdots \\ a' & b' & \cdots \\ a'' & b'' & \cdots \\ \vdots & \vdots & \\ a^{(n-1)} & b^{(n-1)} & \cdots \end{vmatrix} \leftarrow n \times n \text{ matrix} \quad \neq 0$$

$W(x) \to$ Wronskian

LEMMA 1 :

if y_1 and y_2 are any 2 solutions of :
$$y'' + P(x)y' + Q(x) \cdot y = 0$$
then
$$W = 0 \, / \, W \neq 0 \quad \forall x \in [a, b].$$
$$W = Y_1 Y_2' - Y_2 Y_1'$$

$$W = Y_1 Y_2' - Y_2 Y_1'$$
$$W' = Y_1 Y_2'' + Y_1' Y_2 - (Y_2 Y_1'' + Y_2' Y_1')$$
$$= Y_1 Y_2'' + \cancel{Y_1' Y_2} - Y_2 Y_1'' - \cancel{Y_2' Y_1'}$$
$$\boxed{W' = Y_1 Y_2'' - Y_2 Y_1''}$$

$$Y_1'' + P(x)Y_1' + Q(x)Y_1 = 0 \quad \times Y_2$$
$$Y_2'' + P(x)Y_2' + Q(x)Y_2 = 0 \quad \times Y_1$$

$$\Rightarrow Y_2Y_1'' + P(x)\cdot Y_2Y_1' + Q(x)Y_2\cdot Y_1 = 0 \quad - (1)$$

and

$$Y_1Y_2'' + P(x)Y_1Y_2' + Q(x)Y_1Y_2 = 0 \quad - (2)$$

$$\Rightarrow \{Y_2Y_1'' - Y_1Y_2''\} + P(x)\cdot\{Y_2Y_1' - Y_1Y_2'\} + Q(x)\cdot\underbrace{\cancel{\{Y_2Y_1 - Y_1Y_2\}}}_{\to 0} = 0$$

$$\Rightarrow -W' + P(x)\cdot(-W) = 0$$

$$\Rightarrow W' + P(x)\cdot W = 0 \quad - (3)$$

$$\Rightarrow \frac{dW}{dx} = -P(x)\cdot W$$

$$\Rightarrow \frac{dW}{W} = -P(x)\,dx$$

$$\Rightarrow \ln W = -\int P(x)dx + K$$

$$\Rightarrow W = e^{-\int P(x)dx + K} = e^K \cdot e^{-\int P(x)dx} = Ae^{-\int P(x)dx}$$

A happens to be $\underbrace{W(x_0) = W_0}$

↳ initial condition..

LEMMA 2 :

y_1 and y_2 are any 2 solutions of :

$y'' + P(x)y' + Q(x)y = 0$.

P.T. $W=0$ iff y_1 and y_2 are linearly dependent.

y_1 and y_2 are linearly dependent :

$$y_2 = ky_1 \; ; \; y_2' = ky_1'$$

$$W = Y_1Y_2' - Y_2Y_1'$$
$$= K(Y_1Y_1' - Y_1Y_1')$$
$$= K\cdot 0 = 0.$$

$\therefore$, $\boxed{W=0}$

$W=0 \Rightarrow$ linear dependency

$$y_2 = ky_1 \quad \Rightarrow \quad \frac{y_2}{y_1} = k$$

$$\Rightarrow \left(\frac{y_2}{y_1}\right)' = 0$$

$W = 0$

$\Rightarrow y_1 y_2' - y_2 y_1' = 0$

$\Rightarrow \dfrac{y_1 y_2' - y_2 y_1'}{y_1^2} = \dfrac{0}{y_1^2} = 0$ (assume $y_1 \neq 0$ in a subpart of the given interval)

$\Rightarrow \left(\dfrac{y_2}{y_1}\right)' = 0$

$\Rightarrow \dfrac{y_2}{y_1} = k \Rightarrow \boxed{y_2 = k y_1}$ → since its true for pts in the subinterval, by existence & uniqueness theorem, this must be true for the entire interval.

$\left.\begin{array}{l} y_2' = k \cdot y_1'(x_0) \\ y_2 = k y_1(x_0) \end{array}\right\} \to$ if TRUE, then the solution holds good!
(existence & uniqueness theorem)

SHOW THAT: $y = C_1 \sin x + C_2 \cos x$ is the GENERAL SOLUTION of $y'' + y = 0$ on any interval. Also find y_p if $y(0) = 2$ and $y'(0) = 3$.

1. Check: y_1 and y_2 are solutions: DONE!

2. $W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$
$= -\sin^2 x - \cos^2 x$
$= -1(\sin^2 x + \cos^2 x)$
$= -1 \cdot 1 = -1$.

Thus, $\sin x$ & $\cos x$ → L. Independent.
then solve for $y(0)$ and $y'(0)$.

Show: $c_1 e^x + c_2 e^{2x}$ → general soln. of $y'' - 3y' + 2y = 0$
& find y_p if $y(0) = -1$ and $y'(0) = 1$.

a) Check → e^x: $e^x - 3e^x + 2e^x = 0$ ✓
e^{2x}: $4e^{2x} - 6e^{2x} + 2e^{2x} = 0$ ✓

b) $W = e^x \cdot 2e^{2x} - e^{2x} \cdot e^x = e^{2x} \cdot e^x$ → NEVER ZERO!
$\therefore$, e^x and e^{2x} are L. Independent.

c) $y(0) = -1$ $y'(0) = 1$
$\Rightarrow -1 = c_1 + c_2$ — 1 $\Rightarrow c_1 + 2c_2 = 1$. — 2
solve 1 and 2.

The use of Known solution to find another

let y_1 be soln. of : $y'' + P(x)y' + Q(x)y = 0$

let $y_2 = vy_1$, where $v \rightarrow$ non. const. fn. $\{v(x)\}$.

if $y_2 \rightarrow$ solution, then:

$$(vy_1)'' + P(x)\cdot(vy_1)' + Q(x)(vy_1) = 0$$

$$\Rightarrow (vy_1' + y_1v')' + P(x)\cdot(vy_1' + y_1v') + Q(x)(vy_1) = 0$$

$$\Rightarrow (vy_1'' + v'y_1' + y_1'v' + y_1v'') + P(x)\cdot(vy_1' + y_1v') + Q(x)\cdot(vy_1) = 0$$

$$\Rightarrow (vy_1'' + 2v'y_1' + y_1v'') + P(x)\cdot(vy_1' + y_1v') + Q(x)\cdot(vy_1) = 0$$

$$\Rightarrow \cancel{v(y_1'' + P(x)y_1' + Q(x)\cdot y_1)}^{\,0} + 2v'y_1' + P(x)\cdot y_1v' + y_1v'' = 0$$

$$\Rightarrow y_1v'' + v'(P(x)y_1 + 2y_1') = 0$$

$$\Rightarrow v'' + v'\left(P(x) + \frac{2y_1'}{y_1}\right) = 0$$

$$\Rightarrow t' + t\left(P(x) + \frac{2y_1'}{y_1}\right) = 0$$

$$\Rightarrow t' = -t\left(P(x) + \frac{2y_1'}{y_1}\right)$$

$$\Rightarrow dt = -t\left(P(x) + \frac{2y_1'}{y_1}\right)dx$$

$$\Rightarrow \frac{dt}{t} = -\left(P(x) + \frac{2y_1'}{y_1}\right)dx$$

$$\Rightarrow \ln t = -\int\left(P(x) + \frac{2y_1'}{y_1}\right)dx$$

$$\Rightarrow \ln t = -\int P(x)\,dx - 2\int \frac{dy_1}{y_1}$$

$$= -\int P(x)\,dx - 2\ln y_1$$

$$\Rightarrow \ln t = -\int P(x)\,dx - \ln y_1^2$$

$$\Rightarrow t = e^{-\left(\int P(x)\,dx + \ln y_1^2\right)}$$

$$= e^{-\int P(x)\,dx} \cdot e^{-\ln y_1^2}$$

$$t = \frac{1}{y_1^2} \cdot e^{-\int P(x)\,dx}$$

$$\Rightarrow \boxed{V = \int \frac{1}{y_1^2} \cdot e^{-\int P(x)dx} \cdot dx}$$

SOLVE : $x^2 y'' + xy' - y = 0$

$$\Rightarrow y'' + \frac{1}{x} y' - \frac{y}{x^2} = 0$$

by inspection: $y = x$ is 1 solution.

$$V(x) = \int \frac{1}{y_1^2(x)} \cdot e^{-\int P(x)dx} dx$$

$$= \int \frac{1}{x^2} \cdot e^{-\int 1/x \, dx} dx = \int \frac{1}{x^2} \cdot \frac{1}{x} dx$$

$$= \int x^{-3} dx = \frac{x^{-2}}{-2}$$

$$\boxed{V(x) = \frac{x^{-2}}{2}}$$

2nd Linearly Indep $\rightarrow$ $V(x) y_1(x) = -\frac{x^{-2}}{2} \cdot x = \frac{1}{2x}$ //

can be part of C_1.

$\therefore$, 2 solns : x and $\frac{1}{x}$

$$\boxed{y(x) = c_1 x + c_2 x^{-1}}$$

$$y'' + P(x) y' + Q(x) y = 0 \qquad \{R(x) = 0\}$$

$$\boxed{\begin{array}{c} V(x) = \int \frac{1}{y_1^2(x)} \cdot e^{\int -P(x)dx} dx \\ y_2(x) = v(x) \cdot y_1(x) \end{array}}$$

if $P(x)$ and $Q(x)$ are constants,

$y'' + p \cdot y' + q y = 0 \longrightarrow$ possible solution : e^{mx}

$$\Rightarrow m^2 \cdot e^{mx} + p \cdot m e^{mx} + e^{mx} = 0$$

$$\Rightarrow (m^2 + pm + q) \cdot e^{mx} = 0$$

$\Rightarrow \quad m^2 + pm + q = 0 \quad \Rightarrow \quad m = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$

$y'' + py' + qy = 0 \xrightarrow[\text{solutions}]{} e^{m_1 x}$ (or) $e^{m_2 x}$

where $m_1, m_2 \equiv \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$

possibilities : ① $p^2 - 4q > 0 \rightarrow$ 2 distinct solns. (linearly indep)

② $p^2 - 4q < 0 \rightarrow$ 2 imaginary solns.

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-p \pm i\sqrt{4q - p^2}}{2}$$

$$= \frac{-p}{2} \pm \frac{i\sqrt{4q - p^2}}{2}$$

$$= a \pm ib$$

$\therefore$ $\boxed{\text{2 solns : } e^{(a+ib)x} \text{ (or) } e^{(a-ib)x}}$

$$\Rightarrow y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 e^{ax} \cdot e^{ibx} + c_2 \cdot e^{ax} \cdot e^{-ibx}$$

$$y(x) = e^{ax}\left[c_1 e^{ibx} + c_2 e^{-ibx}\right]$$

$$= e^{ax}\left[c_1 (\cos bx + i \sin bx) + c_2 (\cos bx - i \sin bx)\right]$$

$$= e^{ax}\left[\cos bx (c_1 + c_2) + \sin bx (ic_1 - ic_2)\right]$$

$$= e^{ax}\left[c_1' \cos bx + c_2' \sin bx\right]$$

③ $p^2 - 4q = 0$

$\Rightarrow \quad m_1 = m_2 \Rightarrow$ ONLY 1 L.Indep. of the form!

So, use:

$$y_2(x) = y_1(x) \cdot v(x).$$

where $v(x) = \displaystyle\int \frac{1}{y_1^2(x)} \cdot e^{\int -p(x)\,dx}\, dx$

$$= \int \frac{1}{y_1^2(x)} \cdot e^{-px}\, dx$$

$\boxed{m_1 = \dfrac{-p}{2}}$

$$= \int \frac{1}{e^{2m_1 x}} e^{-px}\, dx$$

$$= \int e^{-(p + 2m_1)x}\, dx$$

$$= \int e^{-(p-p)x}\, dx$$

$= x$

$\therefore\ y_2(x) = xe^{m_1 x}$ and $y_1(x) = e^{m_1 x}$

$\therefore\ y(x) = C_1 e^{m_1 x} + C_2 \cdot x e^{m_1 x}$
$= e^{m_1 x}(c_1 + c_2 x)$

auxiliary equation: $(m^2 + pm + q)\, e^{mx} = 0$ → eqn. obtained when soln. substituted

$y'' + 2y' + 1y = 0$, with $2 \downarrow p$, $1 \downarrow q$

$m^2 \cdot e^{mx} + 2me^{mx} + e^{mx} = 0$
$\Rightarrow m^2 + 2m + 1 = 0$
$\Rightarrow (m+1)^2 = 0$
$\Rightarrow m = -1$

e^{-x}.

$\int \frac{1}{e^{-2x}} \cdot e^{\int -2\,dx}\, dx$

$= \int \frac{1}{e^{-2x}} \cdot e^{-2x}\, dx = x$

$$\boxed{y(x) = C_1 e^{-x} + C_2 x e^{-x}}$$

$y_2(x) = xe^{-x}$

$y'' - 9y' + 20y = 0$
$\Rightarrow m^2 - 9m + 20 = 0$
$\Rightarrow m^2 - 5m - 4m + 20 = 0$
$\Rightarrow m(m-5) - 4(m-5) = 0$
$\Rightarrow (m-5)(m-4) = 0$
$\Rightarrow m = 5$ and $m = 4$.

$$\therefore\ \boxed{y(x) = c_1 e^{5x} + c_2 e^{4x}}$$

$y'' + y' = 0$

$\Rightarrow m^2 + m = 0$

$\Rightarrow m(m+1) = 0$

$\Rightarrow m = 0$ (or) $m = -1$

$$\boxed{y(x) = c_1 + c_2 e^{-x}}$$

$2y'' + 5xy' + 6y = 0 \longrightarrow$ CAN'T TALK ABOUT AUXILIARY EQN here as $P(x)$ and $Q(x)$ shd. be constants.

$y'' + 4y' + 4y = 0$

$\Rightarrow m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0$

$\Rightarrow \boxed{m = -2}$

e^{-2x}.

$$\int \frac{1}{(e^{-2x})^2} \cdot e^{\int -4dx} dx$$

$$= \int \frac{1}{e^{-4x}} \cdot e^{-4x} dx = x.$$

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

$y'' + py' + qy = R(x) \rightarrow$
- e^{ax} (1)
- $\sin bx / \cos bx$ (2)
- $a_0 + a_1 x + \cdots a_n x^n$ (3)

(1) $y'' + Py' + qy = e^{ax}$: The solution to this is $y_p(x) \rightarrow$ particular soln ...

$y_p(x) = Ae^{ax}$

$\Rightarrow Ae^{ax}(a^2 + pa + q) = e^{ax}$

$\Rightarrow \boxed{A = \frac{1}{a^2 + pa + q}} \rightarrow$ A is the unknown. found by Computing this expression. ONE PROBLEM – if denominator is ZERO.

denominator is ZERO if 'a' is the root of the auxiliary eqn of : $y'' + Py' + qy = 0$ $\{m^2 + pm + q = 0\}$

if $y_p(x)$ and $y_g(x)$ have a common solution, then:

we multiply x with $y_g(x)$.

REFER SLIDES → FOR $R(x)$

$y'' + 3y' - 10y = 6e^{4x}$

$m^2 + 3m - 10 = 0$

$m^2 + 5m - 2m - 10 = 0$

$m(m+5) - 2(m+5) = 0 \Rightarrow (m-2)(m+5) = 0$

$\Rightarrow m = 2$ (or) $m = -5$.

$y_p = Ae^{4x}$ → solution!

$$A = \frac{6}{a^2 + ap + q} = \frac{6}{16 + 4\times3 - 10} = \frac{6}{18} = \frac{1}{3}$$

$\therefore$, $y_p = \frac{1}{3} e^{4x}$

$y_g = y_c(x) + y_p(x)$

REVISION:

$y'' + y' = \sin x$

Step ①: compare with → $y'' + P(x)y' + Q(x)y = R(x)$

$P(x) = 1$, $Q(x) = 0$, $R(x) = \sin x$

Step ②: Check if we can use method of undetermined coefficients - YES, as $P(x)$ & $Q(x)$ → constants.

we know - $y_g(x) = y_c(x) + y_p(x)$

STEP ③: find $y_c(x)$

$y'' + y' = 0$

$\Rightarrow m^2 + m = 0$

$\Rightarrow m(m+1) = 0$

$\Rightarrow m = 0$ (or) $m = -1$

$\Rightarrow y_c(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{0\cdot x} + c_2 e^{-1\cdot x}$

$\Rightarrow \boxed{y_c(x) = c_1 + c_2 e^{-x}}$

Step (4) : find $y_p(x)$

$R(x) = \sin bx$ (or) $\cos bx$, $\quad b = 1$.

we know $\rightarrow$ $y_p(x) = A\sin bx + B\cos bx$
$= A\sin x + B\cos x$

substitute $y_p(x)$ in original eqn:

$$(A\sin x + B\cos x)'' + (A\sin x + B\cos x)' = \sin x$$
$$\Rightarrow -A\sin x - B\cos x + A\cos x - B\sin x = \sin x$$
$$\Rightarrow -(A+B)\sin x + (A-B)\cos x = 1 \times \sin x + 0 \times \cos x$$

$\Rightarrow -(A+B) = 1$ and $A - B = 0$
$\Rightarrow -(A+A) = 1$ $\quad\Rightarrow \boxed{A = B}$
$\Rightarrow \boxed{A = B = -\frac{1}{2}}$

$\therefore$ $y_p(x) = A\sin x + B\cos x$
$\boxed{y_p(x) = -\frac{1}{2}(\sin x + \cos x)}$ $\quad \boxed{y_c(x) = c_1 + c_2 e^{-x}}$

$y_g(x) = y_c(x) + y_p(x)$

$$\boxed{y_g(x) = c_1 + c_2 e^{-x} - \frac{1}{2}(\sin x + \cos x)}$$

VARIATION of PARAMETERS

in $y'' + P(x)y' + Q(x)y = R(x)$, till know we've solved only if $P(x)$ and $Q(x)$ are constants. What if they arent?

$$y_p(x) = v_1(x)\cdot y_1(x) + v_2(x)\cdot y_2(x)$$

Substitute y_p in original eqn. before that:

$$y_p' = v_1'y_1 + y_1'v_1 + v_2'y_2 + y_2'v_2$$

$\boxed{\text{let } v_1'y_1 + v_2'y_2 = 0}$ — (a) $\rightarrow$ ASSUMPTION. WHY?

$\Rightarrow y_p' = v_1y_1' + v_2y_2'$ — (1)
$\Rightarrow y_p'' = v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2'$ — (2)

substitute (1) and (2) in original equation.

$$y'' + P(x)y' + Q(x)y = R(x)$$

$$\Rightarrow v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' + P(x)[v_1 y_1' + v_2 y_2'] + Q(x)[v_1 y_1 + v_2 y_2] = R(x)$$

$$\Rightarrow \underbrace{(v_1 y_1'' + P(x)\cdot v_1 y_1' + Q(x)\cdot v_1 y_1)}_{0} + \underbrace{(v_2 y_2'' + P(x) v_2 y_2' + Q(x) v_2 y_2)}_{0} + v_1' y_1' + v_2' y_2' = R(x)$$

$$\Rightarrow 0 + 0 + v_1' y_1' + v_2' y_2' = R(x)$$

$$\Rightarrow \boxed{v_1' y_1' + v_2' y_2' = R(x)} \quad - (b)$$

solve (a) and (b)

$$v_1' y_1' + v_2' y_2' = R(x)$$
$$v_1' y_1 + v_2' y_2 = 0$$

$$ax + by = k_1$$
$$cx + dy = k_2$$

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-R(x)\cdot y_2}{W}$$

$$x = \frac{\begin{vmatrix} k_1 & b \\ k_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & k_1 \\ c & k_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$\Rightarrow \boxed{v_1 = -\int \frac{R(x)\cdot y_2}{W}\,dx}$$

similarly, $\boxed{v_2 = +\int \frac{R(x)\cdot y_1}{W}\,dx}$

$$y_p(x) = v_1(x)\cdot y_1(x) + v_2(x)\cdot y_2(x)$$

$$\boxed{y_p(x) = -y_1(x)\cdot \int \frac{R(x)\cdot y_2(x)}{W(x)}\,dx + y_2(x)\int \frac{R(x)\cdot y_1(x)}{W(x)}\,dx}$$

$$y'' + 9y = 2\sin 3x + 4\sin x - 26e^{-2x} + 27x^3$$

1 $\quad y'' + 9y = 0 \quad \Rightarrow \quad m^2 + 9 = 0$

$$\Rightarrow \quad m = \pm 3i$$

$$y_c(x) = c_1 e^{i(3x)} + c_2 e^{-i(3x)}$$
$$= c_1(\cos 3x + i\sin 3x) + c_2(\cos 3x - i\sin 3x)$$
$$= (c_1 + c_2)\cos 3x + (c_1 - c_2)i\sin 3x$$
$$= c_1' \cos 3x + c_2' \sin 3x$$

(2) $y'' + 9y = 2\sin 3x$

$y = A\sin 3x + B\cos 3x \rightarrow$ linearly dependent with $y_c(x)$.

$\therefore\ y_{p_1}(x) = x\left(c_1'\cos 3x + c_2'\sin 3x\right)$

$y' = x\left(-3c_1'\sin 3x + 3c_2'\cos 3x\right) + \left(c_1'\cos 3x + c_2'\sin 3x\right)$

$y'' = x\left(-9c_1'\cos 3x - 9c_2'\sin 3x\right) + \left(-3c_1'\sin 3x + 3c_2'\cos 3x\right) + \left(-3c_1'\sin 3x + 3c_2'\cos 3x\right)$

$y'' + 9y = 2\sin 3x$

$\Rightarrow \cancel{x(-9c_1'\cos 3x - 9c_2'\sin 3x)} + (-3c_1'\sin 3x + 3c_2'\cos 3x) + (-3c_1'\sin 3x + 3c_2'\cos 3x) + \cancel{9x(c_1'\cos 3x + c_2'\sin 3x)} = 2\sin 3x$

$\Rightarrow \quad -6c_1'\sin 3x + 6c_2'\cos 3x = 2\sin 3x$

$2 = -6c_1'$

$\Rightarrow \boxed{c_1' = -1/3}\ ,\ \boxed{c_2' = 0}$

$\therefore\ y_{p_1}(x) = -\dfrac{x}{3}\cos 3x$

$y'' + 9y = 4\sin x$

$y = A\sin x + B\cos x$

$\Rightarrow -A\sin x - B\cos x + 9(A\sin x + B\cos x) = 4\sin x$

$\Rightarrow \quad 8A\sin x + 8B\cos x = 4\sin x$

$\Rightarrow \quad A = 1/2$ and $B = 0$

$\Rightarrow \quad \boxed{y_{p_2}(x) = \tfrac{1}{2}\sin x}$

$y'' + 9y = -26e^{-2x}$

$4Ae^{-2x} + 9Ae^{-2x} = -26e^{-2x}$

$13A\cancel{e^{-2x}} = -26\cancel{e^{-2x}}$

$\Rightarrow \boxed{A = -2}$

$\therefore\ y_{p_3}(x) = -2e^{-2x}$

$y'' + 9y = 27x^3$

$a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$(2a_2 + 6a_3 x) + 9a_0 + 9a_1 x + 9a_2 x^2 + 9a_3 x^3 = 27x^3$

$27x^3 = 9a_3 x^3 \Rightarrow a_3 = 3$

$6a_3 x + 9a_1 x = 0$

$\Rightarrow 18x = -9a_1 x$

$\Rightarrow \boxed{a_1 = -2}$

$\therefore\ y_{p_4}(x) = 3x^3 - 2x$

$$y(x) = c_1' \cos 3x + c_2' \sin 3x + x(c_1' \cos 3x + c_2' \sin 3x) - \frac{x}{3}\cos 3x + \frac{1}{2}\sin x - 2e^{-2x} - 2x + 3x^3$$

Variation of parameters :

$$V_1' = \frac{\begin{vmatrix} 0 & Y_2 \\ R(x) & Y_2' \end{vmatrix}}{W} \quad ; \quad V_2' = \frac{\begin{vmatrix} Y_1 & 0 \\ Y_1' & R(x) \end{vmatrix}}{W}$$

$y'' + y = \operatorname{cosec} x$

$y_c(x) = c_1 \cos x + c_2 \sin x$

$$W = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = +\cos^2 x + \sin^2 x = 1$$

$y_p(x) = V_1(x)\cdot Y_1(x) + V_2(x)\cdot Y_2(x)$

$V_1(x) = -\int \frac{Y_2(x)\cdot R(x)}{W}\,dx$ and $V_2(x) = \int \frac{Y_1(x)R(x)}{W}\,dx$

$= -\int \frac{\sin x \cdot \operatorname{cosec} x}{1}\,dx$ and $V_2(x) = \int \frac{\cos x \cdot \operatorname{cosec} x}{1}\,dx$

$= -x$ and $\ln \sin x$

$\therefore\ \boxed{y_p(x) = -x\cos x + \sin x \ln \sin x}$

$$y_g(x) = -x\cos x + \sin x \ln(\sin x) + c_1 \cos x + c_2 \sin x$$