Basis and Dimension

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Linearly Dependent (L.D.)

Let V be a vector space over the field F. A subset S of V is said to be linearly dependent if there exist distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$ and scalars $c_1, c_2, c_n \in F$, $c_i \neq 0$ for at least one i, such that

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0.$$

A set of which is not linearly dependent is called linearly independent (L.I.)

Remark

If S is a linearly independent set, then for any (finite) distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$,

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0 \implies c_1 = c_2 = \ldots = c_n = 0.$$

$$\implies \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_n \end{bmatrix} = 0$$

$$\implies \quad AX = 0 \text{ has only trivial solution } X = 0.$$

Remark contd.

Note: (i) If $AX = 0$ has only trivial solution, then columns of A forms a linearly independent set.
(ii) If A is an invertible matrix, then columns of A forms a linearly independent set (By note (i) and Theorem 13, chapter 1).

Note

1. **Any set which contains a linearly dependent set is linearly dependent.**
2. **Any subset of a linearly independent set is linearly independent.**
3. **Any set which contains the** 0 **vector is linearly dependent. Reason** $1.0 = 0$
4. **A set** S **of vectors is linearly independent if and only if each finite subset of** S **is linearly independent if and only if for any distinct vectors** $\alpha_1, \alpha_2, \ldots, \alpha_n$ **of** S**,** $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0 \Longrightarrow c_1 = c_2 = \ldots = c_n = 0$

Problem 1

Show that $\alpha_1 = (3, 0, -3), \alpha_2 = (-1, 1, 2), \alpha_3 = (4, 2, -2)$ and $\alpha_4 = (2, 1, 1)$ are linearly dependent (L.D.) on R^3.

Solution : Find sclars c_1, c_2, c_3, c_4 (at leaset one $c_i \neq 0$) such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$.

$$2\alpha_1 + 2\alpha_2 - \alpha_3 + 0\alpha_4 = 0$$

Problem 2

Show that $\epsilon_1 = (1,0,0), \epsilon_2 = (0,1,0)$ and $\epsilon_3 = (0,0,1)$ is a linearly independent (L.I.) subset of F^3.

Consider $c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 = 0$

$$\implies \quad (c_1, c_2, c_3) = (0,0,0)$$

$$\implies \quad c_1 = c_2 = c_3 = 0$$

Hence $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ is a $L.I.$ subset of F^3.

Note :

$\{\epsilon_1 = (1,0,0,\ldots,0,0), \epsilon_2 = (0,1,0,\ldots,0,0) \ldots, \epsilon_n = (0,0,0,\ldots,0,1)\}$ is a linearly independent subset of F^n.

Basis

Let V be a vector space over the field F. A set $\mathbb{B} \subseteq V$ is basis for V if

1. $\mathbb{B}$ **is a linearly independent subset of** V **and**
2. $V =$ **span** $\mathbb{B}$ $(= L(\mathbb{B})$ **).**

Note : A vector space V is finite dimensional if it has a finite basis.

Problem 3

Show that $\mathbb{B} =$
$\{\epsilon_1 = (1, 0, 0, \dots, 0, 0), \epsilon_2 = (0, 1, 0, \dots, 0, 0) \dots, \epsilon_n = (0, 0, 0, \dots, 0, 1)\}$
is a basis of F^n.

Solution :

Claim 1 : $\mathbb{B}$ is a linearly independent set in F^n.

Consider $c_1\epsilon_1 + c_2\epsilon_2 + \dots + c_n\epsilon_n = 0$

$$\implies (c_1, c_2, \dots, c_n) = (0, 0, \dots, 0)$$

$$\implies c_1 = c_2 = \dots = c_n = 0$$

$\implies \mathbb{B}$ is a $L.I.$ set.

Claim 2 : $F^n = \text{span } \mathbb{B}$.

Since $\mathbb{B} \subseteq F^n$, $\quad \text{span } \mathbb{B} = L(\mathbb{B}) \subseteq F^n - - - (a)$

Let $x \in F^n$

$\implies x = (x_1, x_2, \ldots, x_n) = x_1\epsilon_1 + x_2\epsilon_2 + \ldots + x_n\epsilon_n \in \text{span } \mathbb{B}$

$$x \in F^n \implies x \in \text{span } \mathbb{B}$$

$$\implies \quad F^n \subseteq \text{span } \mathbb{B} - - - - (b)$$

From (a) and (b), $\quad F^n = \text{span } \mathbb{B}$.

By Claims 1 and 2, $\mathbb{B}$ is a basis of F^n

Note: $\mathbb{B} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ is called the standard basis of F^n.

Problem 3

Show that $\mathbb{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $\mathbb{B}_1 = \{(0,1,1),(1,0,1),(1,1,0)\}$ are basis for R^3.

Problem 4

Let $P \in F^{n\times n}$ be an invertible matrix. Let $P_1, P_2, \ldots, P_n$ be the columns of P. Show that $\mathbb{B} = \{P_1, P_2, \ldots, P_n\}$ is a basis of $F^{n\times 1}$.

Claim 1: $\mathbb{B}$ is a L.I. set.

Consider $x_1P_1 + x_2P_2 + \ldots + x_nP_n = 0$.

$$\implies \begin{bmatrix} P_1 & P_2 & \ldots & P_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = 0$$

$$\implies \quad PX = 0$$

$$\implies \quad X = 0, \quad (\text{ P is an invertible matrix})$$

Problem 4 contd.

$$\implies \quad x_1 = x_2 = \ldots = x_n = 0 \implies \mathbb{B} \text{ is a L.I. set}$$

Claim 2: $F^{n\times 1} = \text{ Span } \mathbb{B}$

We have $\mathbb{B} \subseteq F^{n\times 1}$. $\implies \quad \text{Span } \mathbb{B} \subseteq F^{n\times 1} - - - -(i)$

Let $Y \in F^{n\times 1}$. By Theorem 13 (Note that P is invertible), $PX = Y$ has a solution X for a each $Y \in F^{n\times 1}$.

$$Y = PX = x_1P_1 + x_2P_2 + \ldots + x_nP_n \in L(\{P_1, P_2, \ldots, P_n\})$$

$$\implies \quad Y \in \text{ Span } \mathbb{B}. \implies \quad F^{n\times 1} \subseteq \text{ Span } \mathbb{B} - - - (ii)$$

From (i) and (ii), $F^{n\times 1} = \text{ Span } \mathbb{B}$. By Claims 1 and 2, $\mathbb{B}$ (the set of all columns of P) is a basis of $F^{n\times 1}$.

Problem 5

Let $A \in F^{n \times n}$ and let $\{P_1, P_2, \ldots, P_n\}$ be columns of A. Prove that A is invertible if and only if $\{P_1, P_2, \ldots, P_n\}$ is a L.I. set.

Solution : A is invertible if and only if $AX = 0$ has only trivial solution $X = 0$ (Theorem 13, chapter 1) if and only if $x_1 P_1 + x_2 P_2 + \ldots + x_n P_n = 0$ has only trivial solution $x_1 = x_2 = \ldots = x_n = 0$ if and only if $\{P_1, P_2, \ldots, P_n\}$ is a L.I. set.

Note 1: (Visit previous lecture notes)

Find the solution space of the system $RX = 0$

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{aligned}\right\}$$

No. of non-zero rows of R, $r = 2$, **No. of variables,** $n = 5$

$k_1 = 2, k_2 = 4 \implies$ Pivot variables $= \{x_{k_1}, x_{k_2}\} = \{x_2, x_4\}$

No. of free variables $= n - r = 5 - 2 = 3$,

Free variables $= \{u_1, u_2, u_3\} = \{x_1, x_3, x_5\}$

$$\left.\begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{aligned}\right\} \implies \left.\begin{aligned} x_{k_1} + \sum_{j=1}^{n-r} C_{1j}u_j = 0 \\ x_{k_2} + \sum_{j=1}^{n-r} C_{2j}u_j = 0 \end{aligned}\right\} (general \;\; expression)$$

Note 1 contd.

$$\left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\} \Longrightarrow \left.\begin{array}{c} x_{k_1} + \displaystyle\sum_{j=1}^{n-r} C_{1j}u_j = 0 \\ x_{k_2} + \displaystyle\sum_{j=1}^{n-r} C_{2j}u_j = 0 \end{array}\right\} \textit{general expression})$$

Set the free variables as :

$u_1 = x_1 = a,\ u_2 = x_3 = b,\quad u_3 = x_5 = c$

$\Longrightarrow \quad x_2 = 3b - \frac{1}{2}c, \quad x_4 = -2c$

Solution set $S = \left\{ \left(a, 3b - \frac{1}{2}c, b, -2c, c\right) \ : \ a, b, c \in \mathrm{R} \right\}$

Note 1 contd. (back to chapter one !)

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c) \ : \ a, b, c \in \mathrm{R}\right\}$

$$S = \left\{ a(1,0,0,0,0) + b(0,3,1,0,0) + c(0,-\frac{1}{2},0,-2,1) \ : \ a, b, c \in \mathrm{R} \right\}$$

$$= \text{Span of } \left\{ (1,0,0,0,0), (0,3,1,0,0), (0,-\frac{1}{2},0,-2,1) \right\}$$

Dimension of $S = dim\ S = 3 = n - r$ **(Information for future)**

Alternate way to find a basis of S

$$\left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\} - -(i)$$

Note that $\{x_2, x_4\}$ are pivot variables and $\{x_1, x_3, x_5\}$ are free varaiables.

Set $x_1 = 1, x_3 = 0, x_5 = 0. \implies x_2 = 0, x_4 = 0$

Let $E_1 = (1, 0, 0, 0, 0)$

Set $x_1 = 0, x_3 = 1, x_5 = 0. \implies x_2 = 3, x_4 = 0$

Let $E_3 = (0, 3, 1, 0, 0)$

Set $x_1 = 0, x_3 = 0, x_5 = 1. \implies x_2 = -\frac{1}{2}, x_4 = -2$

Let $E_5 = (0, -\frac{1}{2}, 0, -2, 1)$

Clearly, $S = \text{Span}\,\{E_1, E_3, E_5\}$ (See the previous slide)

Prove that $\{E_1, E_3, E_5\}$ is a linearly independet set.

Prove that $\{E_1, E_3, E_5\}$ is a linearly independet set.

Hence $\{E_1, E_3, E_5\}$ is a basis of S.

Please read chater 2, example 15 for details.

Problem 5 (assignment)

Let W be set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfies

$$\begin{aligned}
2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \qquad\quad &= 0 \\
x_1 \qquad\quad + \tfrac{2}{3}x_3 \qquad\quad - x_5 &= 0 \\
9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0
\end{aligned}$$

Find a basis of W.

(2) Find a basis of the vector space of all polynomials over the field F (see chapter 2, example 16)

Theorem 4

Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$. Then any linearly independent set of vectors in V is finite and contains no more than m elements.

Proof: We have

$$V = \text{span } \{\beta_1, \beta_2, \ldots, \beta_m\} - - - -(i)$$

It is enough to prove that

if $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq V$ is an arbitrary L.I. set, then $n \leq m$.

We prove by method of contradiction. Assume that $m < n$.

By (i), $\alpha_1 = A_{11}\beta_1 + A_{21}\beta_2 + \ldots + A_{m1}\beta_m$

$\alpha_2 = A_{12}\beta_1 + A_{22}\beta_2 + \ldots + A_{m2}\beta_m$

$$\alpha_j = A_{1j}\beta_1 + A_{2j}\beta_2 + \ldots + A_{mj}\beta_m = \sum_{i=1}^{m} A_{ij}\beta_i \ , \quad j = 1, 2, \ldots, n$$

Consider the homogeneous system

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = 0 - - - - - - -(ii)$$

$$\implies \sum_{j=1}^{n} x_j\alpha_j = 0$$

$$\implies \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} A_{ij}\beta_i \right) = 0$$

$$\implies \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}x_j \right) \beta_i = 0$$

Theorem 4 contd.

Consider $\displaystyle\sum_{j=1}^{n} A_{ij}x_j = 0, i = 1, 2, \ldots, m - - - -(iii)$

The system (iii) is a homogeneous linear system with m equations and n variables. Since $m < n$, the system (iii) has a non-trivial solution say $x_1^*, x_2^*, \ldots, x_n^*$ (at least one $x_j^* \neq 0$) such that

$\displaystyle\sum_{j=1}^{n} A_{ij}x_j^* = 0, i = 1, 2, \ldots, m - - - (iv)$

$$x_1^*\alpha_1 + x_2^*\alpha_2 + \ldots + x_n^*\alpha_n = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j^*\right)\beta_i \quad (see \ \ (ii))$$

$$x_1^*\alpha_1 + x_2^*\alpha_2 + \ldots + x_n^*\alpha_n = \sum_{i=1}^{m}(0)\,\beta_i = 0 \quad (see \ \ (iv))$$

Theorem 4 contd.

Hence

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = 0 - - - - - - -(ii)$$

has a non-trivial solution $x_1^*, x_2^*, \ldots, x_n^*$ (at least one $x_j^* \neq 0$).
A contradiction to the assumption that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a L.I. set.
Therefore, $n \leq m$. This completes the proof.

Corollary 1 If V is a finite dimensional vector space, then any two bases of V have the same (finite) number of elements.

Proof: Since V is a finite dimensional vector space, it has a finite basis say

$$B_1 = \{\beta_1, \beta_2, \ldots, \beta_m\}.$$

Hence B_1 is a L.I. set and span $B_1 = V$. Let $B_2 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be another basis of V. That is B_2 is a L.I. set and span $B_2 = V$.

Since span $B_1 = V$ and B_2 is a L.I. set, $n \leq m - - - (a)$ (by Theorem 4).

Since span $B_2 = V$ and B_1 is a L.I. set, $m \leq n - - - (b)$ (by Theorem 4).

By (a) and (b), $m = n$. $\Longrightarrow |B_1| = |B_2|$. It completes the proof.

Dimension

The dimension of a finite dimensional vector space V is the number of elements in a basis for V.

1. Conisder the vector space F^n. Let $\mathbb{B} = \{\epsilon_1 = (1,0,0,\dots,0,0), \epsilon_2 = (0,1,0,\dots,0,0),\dots, \epsilon_n = (0,0,0,\dots,0,1)\}$ is a basis of F^n.

$$\text{Dimension of } F^n = dim(F^n) = |\mathbb{B}| = n$$

2. Let r be the number of non-zero rows of a row-reduced echelon matrix $R \in F^{m\times n}$. Show that dimension of the solution space of the homogeneous system of linear equations $RX = 0$ is of dimension $n - r$. (Assignment)
3. Show that dimension of $F^{m\times n} = mn$ (Assignment).
4. The dimension of zero space is zero.

Corollary 2: Let V be a finite dimensional vector space and let $n = \dim\ V$. Then

1. any subset of V which contains more than n vectors is a L.D.;
2. no subset of V which contains fewer than n vectors can span V.

Proof. Let $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be a basis of V. Then (i) B is L.I. and (ii) $V = \text{span } B$.
Proof of (1). Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_p\} \subseteq V$.
If S is a linearly independent set, (by Theorem 4) $p \leq n$.
Therefore, if $p > n$, then S is L.D. This proves (1).
Proof of (2). Suppose that $V = \text{span } \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$. Since B is a linearly independent set, (by Theorem 4) $n \leq p$. $\Longrightarrow$ Any set of vectors which spans V contains at least n vectors. This proves (2).

Lemma

Let S be a linearly independent subset of a vector space V. Suppose that there is a vector $\beta \in V - L(S)$. Then $S \cup \{\beta\}$ is a L.I. subset of V.

Proof: Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct vectors in S and that

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n + b\beta = 0 - - - - - - (i)$$

Then $b = 0$; otherwise $\beta = -\frac{c_1}{b}\alpha_1 - \frac{c_2}{b}\alpha_2 - \ldots - \frac{c_n}{b}\alpha_n \in L(S)$, a contradiction.

$$(i) \implies c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0$$

Since S is a L.I. set, $c_1 = c_2 = \ldots = c_n = 0 = b$. Thus $S \cup \{\beta\}$ is a L.I. set in V.

Theorem 5

If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is a part of a (finite) basis for W.

Proof: Since V is a finite-dimensional vector space,
$n = \dim\ V < \infty$. Suppose that S_0 is a L.I. subset of W.

Claim 1: $|S_0|$ is finite.

Since $S_0 \subseteq W \subseteq V$, S_0 is a L.I. subset of V and thus by Corollary 2 to Theorem 4, $|S_0| \leq \dim\ V = n$.

Claim 2: S_0 is a part of a (finite) basis for W.

We extend S_0 to a basis for W, as follows.

If $W = \text{span } S_0$ ($= L(S_0)$), S_0 is a basis for W and we are done. If not, there exists a non-zero vector $\beta_1 \in W - L(S_0)$. Let $S_1 = S_0 \cup \{\beta_1\}$. By previous lemma, S_1 is a L.I. subset of W.

Theorem 5 contd

If $W = \text{span } S_1$, we are done. If not, apply the previous lemma to obtain a $\beta_2 \in W - L(S_1)$ such that $S_2 = S_1 \cup \{\beta_2\}$ is a L.I. set. If we continue in this way, then (in not more than dim V steps) we reach a L.I. set

$$S_m = S_0 \cup \{\beta_1, \beta_2, \ldots, \beta_m\}$$

which is basis for W.

Example

Let $S_0 = \{(1,1,1)\}$. Find a basis for R^3 which contains S_0.
Solution :

$$L(S_0) = \{a(1,1,1)\ : a \in R\} = \{(a,a,a)\ :\ a \in R\}$$

Clearly, $\beta_1 = (1,1,0) \notin L(S_0)$. By Theorem 5,
$S_1 = S_0 \cup \{\beta_1\} = \{(1,1,1),(1,1,0)\}$ is a L.I. subset of R^3.

$$L(S_1) = \{a(1,1,1) + b(1,1,0) = (a+b, a+b, a)\ :\ a, b \in R\}$$

Clearly $\beta_2 = (1,0,0) \notin L(S_1)$. By Theorem 5,
$S_2 = S_1 \cup \{\beta_2\} = \{(1,1,1),(1,1,0),(1,0,0)\}$ is a L.I. set. Verify that $L(S_2) = R^3$. Hence, S_2 is a basis for R^3.

Corollary 1 : If W is a proper subspace of a finite-dimensional vector space V, then W is finite-dimensional and dim $W <$ dim V.
Proof: (assignment)
Corollary 2 : In a finite-dimensional vector space V every non-empty linearly independent set of vectors is part of a basis.

Corollary 3 to Theorem 5

Let $A \in F^{n\times n}$, and suppose the row vectors of A form a linearly independent set of vectors in F^n. Then A is invertible.

Proof : Let

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{bmatrix}$$

where $\alpha_i \in F^n$. Let $W = \text{span}\,\{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Since $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a L.I. set, dim $W = n$. Since dim $W = n =$ dim F^n and $W \subseteq F^n$, by Corollary 1 to Theorem 5, $W = F^n$.

$$F^n = W = \text{span}\ \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

Corollary 3 Theorem 5 contd.

Since $\epsilon_1 = (1, 0, \ldots, 0) \in F^n = \text{span } \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$,

$$\epsilon_1 = B_{11}\alpha_1 + B_{12}\alpha_2 + \ldots + B_{1n}\alpha_n$$

Similarly, $\epsilon_2 = (0, 1, \ldots, 0), \ldots, \epsilon_n = (0, 0, \ldots, 1) \in F^n$,

$$\epsilon_2 = B_{21}\alpha_1 + B_{22}\alpha_2 + \ldots + B_{2n}\alpha_n$$

$$\epsilon_n = B_{n1}\alpha_1 + B_{n2}\alpha_2 + \ldots + B_{nn}\alpha_n$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \dots \\ \epsilon_n \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{bmatrix}$$

$$\implies I = BA$$

Hence B is a left inverse of A and thus A is invertible.

Sum of Subsets

Definition: If $\mathrm{S}_1,\ \mathrm{S}_2,\ldots,\mathrm{S}_\mathrm{k}$ are subsets of a vector space V, the set of all sums

$$\alpha_1 + \alpha_2 + \cdots + \alpha_\mathrm{k}$$

of vectors α_i in S_i is called the sum of the subsets $\mathrm{S}_1,\ \mathrm{S}_2,\ldots,\mathrm{S}_k$ and is denoted by

$$\mathrm{S}_1 + \mathrm{S}_2 + \cdots + \mathrm{S}_\mathrm{k}$$

or by

$$\sum_{i=1}^{k} S_i$$

Sum of Subspaces

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \cdots + W_k$$

is easily seen to be a subspace of V containing each subspace W_i. From this it follows, as in the proof of Theorem 3, that W is the subspace spanned by the union of $W_1, W_2, \ldots, W_k$.

Example 9

Let F be a subfield of the field C of complex numbers, and let V be the vector space of all 2×2 matrices over F. Let W_1 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

where x, y, z are arbitrary scalars in F. Finally, let W_2 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

where x and y are arbitrary scalars in F. Then W_1 and W_2 are subspaces of V(**Verify!**).

Example 9

Also

$$V = W_1 + W_2$$

because

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}.$$

The subspace $W_1 \cap W_2$ consists of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem 6: If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $\mathrm{W}_1 + \mathrm{W}_2$ is finite-dimensional and

$$\dim \mathrm{W}_1 + \dim \mathrm{W}_2 = \dim (\mathrm{W}_1 \cap \mathrm{W}_2) + \dim (\mathrm{W}_1 + \mathrm{W}_2).$$

Proof: By Theorem 5 and its corollaries, $W_1 \cap W_2$ has a finite basis $\{\alpha_1, \ldots, \alpha_k\}$ which is part of a basis

$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m\} \text{ for } W_1$$

and part of a basis

$$\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n\} \text{ for } W_2.$$

Proof

The subspace $W_1 + W_2$ is spanned by the vectors

$$\alpha_1, \dots, \alpha_k, \quad \beta_1, \dots, \beta_{\mathrm{m}}, \quad \gamma_1, \dots, \gamma_{\mathrm{n}}$$

and these vectors form an independent set. For suppose

$$\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0.$$

Then

$$-\sum z_r \gamma_r = \sum x_i \alpha_i + \sum y_j \beta_j$$

which shows that $\sum z_r \gamma_r$ belongs to W_1. As $\sum z_r \gamma_r$ also belongs to W_2 it follows that

$$\sum z_r \gamma_r = \sum c_i \alpha_i$$

for certain scalars $c_1, \dots, c_k$.

Proof

Because the set

$$\{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_{\mathrm{n}}\}$$

is independent, each of the scalars $z_r = 0$. Thus

$$\sum x_i \alpha_i + \sum y_j \beta_j = 0$$

and since

$$\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{\mathrm{m}}\}$$

is also an independent set, each $x_i = 0$ and each $y_j = 0$. Thus,

$$\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{\mathrm{m}}, \gamma_1, \dots, \gamma_{\mathrm{n}}\}$$

is a basis for $W_1 + W_2$.

Proof

Finally

$$\begin{aligned}\dim W_1 + \dim W_2 &= (k+m) + (k+n) \\ &= k + (m+k+n) \\ &= \dim\left(W_1 \cap W_2\right) + \dim\left(W_1 + W_2\right).\end{aligned}$$

Verify Theorem 6 by Example 9.