Implicit and Explicit Solutions

1. Verify that the following functions (explicit or implicit) are solutions of the corresponding differential equations:

(a) $y = x^2 + c$	$y' = 2x$
(b) $y = cx^2$	$xy' = 2y$
(c) $y^2 = e^{2x} + c$	$yy' = e^{2x}$
(d) $y = ce^{kx}$	$y' = ky$
(e) $y = c_1 \sin 2x + c_2 \cos 2x$	$y'' + 4y = 0$
(f) $y = c_1 e^{2x} + c_2 e^{-2x}$	$y'' - 4y = 0$
(g) $y = c_1 \sinh 2x + c_2 \cosh 2x$	$y'' - 4y = 0$
(h) $y = \arcsin xy$	$xy' + y = y'\sqrt{1 - x^2 y}$
(i) $y = x \tan x$	$xy' = y + x^2 + y^2$
(j) $x^2 = 2y^2 \ln y$	$y' = \dfrac{xy}{x^2 + y^2}$
(k) $y^2 = x^2 - cx$	$2xyy' = x^2 + y^2$
(l) $y = c^2 + c/x$	$y + xy' = x^4 (y')^2$
(m) $y = ce^{y/x}$	$y' = y^2 / \left(xy - x^2\right)$
(n) $y + \sin y = x$	$(y \cos y - \sin y + x) y' = y$
(o) $x + y = \arctan y$	$1 + y^2 + y^2 y' = 0$

2. Find the general solution of each of the following differential equations:

(a) $y' = e^{3x} - x$
(b) $y' = xe^{x^2}$
(c) $(1 + x)y' = x$
(d) $\left(1 + x^2\right) y' = x$
(e) $\left(1 + x^2\right) y' = \arctan x$
(f) $xy' = 1$
(g) $y' = \arcsin x$
(h) $y' \sin x = 1$
(i) $\left(1 + x^3\right) y' = x$
(j) $\left(x^2 - 3x + 2\right) y' = x$

3. For each of the following differential equations, find the particular solution that satisfies the given initial condition:

(a) $y' = xe^x$	$y = 3$ when $x = 1$
(b) $y' = 2 \sin x \cos x$	$y = 1$ when $x = 0$
(c) $y' = \ln x$	$y = 0$ when $x = e$
(d) $\left(x^2 - 1\right) y' = 1$	$y = 0$ when $x = 2$
(e) $x\left(x^2 - 4\right) y' = 1$	$y = 0$ when $x = 1$
(f) $(x + 1)\left(x^2 + 1\right) y' = 2x^2 + x$	$y = 1$ when $x = 0$

4. Show that the function

$$y = e^{x^2} \int_0^x e^{-t^2}\, dt$$

is a solution of the differential equation $y' = 2xy + 1$.

5. For the differential equation

$$y'' - 5y' + 4y = 0\,,$$

carry out the detailed calculations required to verify these assertions:
(a) The functions $y = e^x$ and $y = e^{4x}$ are both solutions.
(b) The function $y = c_1e^x + c_2e^{4x}$ is a solution for any choice of constants c_1, c_2.

6. Verify that $x^2y = \ln y + c$ is a solution of the differential equation $dy/dx = 2xy^2/(1 - x^2y)$ for any choice of the constant c.

7. For which values of m will the function $y = y_m = e^{mx}$ be a solution of the differential equation

$$2y''' + y'' - 5y' + 2y = 0\,?$$

Find three such values m. Use the ideas in Exercise 5 to find a solution containing three arbitrary constants c_1, c_2, c_3.

Answers to Odd problems

1. Work not shown.

3. (a) $y = xe^x - e^x + 3$;
(b) $y = \sin^2 x + 1$;
(c) $y = x\ln x - x$;
(d) $y = \frac{1}{2}\ln\left[\frac{3x - 3}{x + 1}\right]$;
(e) $y = \frac{1}{8}\ln\left[\frac{4 - x^2}{3x^2}\right]$;
(f) $y = \frac{1}{4}\ln\left[(x + 1)^2(x^2 + 1)^3\right] - \frac{1}{2}\arctan x + 1$.

5. Work not shown.

7. $m = 1, 1/2, -2$; $y = c_1e^x + c_2e^{x/2} + c_3e^{-2x}$.

Method of Separation of Variables

1. Use the method of separation of variables to solve each of these ordinary differential equations.
   (a) $x^5 y' + y^5 = 0$
   (b) $y' = 4xy$
   (c) $y' + y \tan x = 0$
   (d) $(1 + x^2)\, dy + (1 + y^2)\, dx = 0$
   (e) $y \ln y\, dx - x\, dy = 0$
   (f) $xy' = (1 - 4x^2) \tan y$
   (g) $y' \sin y = x^2$
   (h) $y' - y \tan x = 0$
   (i) $xyy' = y - 1$
   (j) $xy^2 - y'x^2 = 0$

2. For each of the following differential equations, find the particular solution that satisfies the additional given property (called an *initial condition*):
   (a) $y'y = x + 1$ $\quad y = 3$ when $x = 1$
   (b) $(dy/dx)x^2 = y$ $\quad y = 1$ when $x = 0$
   (c) $\dfrac{y'}{1 + x^2} = \dfrac{x}{y}$ $\quad y = 3$ when $x = 1$
   (d) $y^2 y' = x + 2$ $\quad y = 4$ when $x = 0$
   (e) $y' = x^2 y^2$ $\quad y = 2$ when $x = -1$
   (f) $y'(1 + y) = 1 - x^2$ $\quad y = -2$ when $x = -1$

3. For the differential equation

$$\frac{y''}{y'} = x^2,$$

   make the substitution $y' = p$ to reduce the order. Then solve the new equation by separation of variables. Now resubstitute and find the solution y of the original equation.

4. Use the method of Exercise 3 to solve the equation

$$y'' \cdot y' = x(1 + x)$$

   subject to the initial conditions $y(0) = 1$, $y'(0) = 2$.

Answers to Odd problems

1. (a) $\dfrac{1}{y^4} = c - \dfrac{1}{x^4}$, $y = 0$;

   (b) $y = ce^{2x^2}$;

   (c) $y = c \cdot \cos x$;

   (d) $y = \dfrac{c - x}{1 + cx}$;

   (e) $y = e^{cx}$;

   (f) $y = \arcsin\left(cxe^{-2x^2}\right)$;

   (g) $y = \arccos\left(c - \dfrac{x^3}{3}\right)$;

   (h) $y = c \cdot \sec x$;

   (i) $y + \ln|y - 1| = \ln|x| + c$, $y = 1$;

   (j) $\dfrac{1}{y} = c - \ln|x|$, $y = 0$.

3. $y = \int ce^{x^3/3} dx$, where this integral is not an elementary function.

General and Particular Solutions

1. Find the general solution of each of the following first-order, linear ordinary differential equations:
   (a) $y' - xy = 0$
   (b) $y' + xy = x$
   (c) $y' + y = \dfrac{1}{1+e^{2x}}$
   (d) $y' + y = 2xe^{-x} + x^2$
   (e) $(2y - x^3)\,dx = x\,dy$
   (f) $y' + 2xy = 0$
   (g) $xy' - 3y = x^4$
   (h) $(1 + x^2)\,dy + 2xy\,dx = \cot x\,dx$
   (i) $y' + y\cot x = 2x\csc x$
   (j) $y - x + xy\cot x + xy' = 0$

2. For each of the following differential equations, find the particular solution that satisfies the given initial data:
   (a) $y' - xy = 0$ $\quad y = 3$ when $x = 1$
   (b) $y' - 2xy = 6xe^{x^2}$ $\quad y = 1$ when $x = 1$
   (c) $(x\ln x)y' + y = 3x^3$ $\quad y = 0$ when $x = 1$
   (d) $y' - (1/x)y = x^2$ $\quad y = 3$ when $x = 1$
   (e) $y' + 4y = e^{-x}$ $\quad y = 0$ when $x = 0$
   (f) $x^2y' + xy = 2x$ $\quad y = 1$ when $x = 1$

3. The equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is known as *Bernoulli's equation*. It is linear when $n = 0$ or 1, otherwise not. In fact the equation can be reduced to a linear equation when $n > 1$ by the change of variables $z = y^{1-n}$. Use this method to solve each of the following equations:
   (a) $xy' + y = x^4y^3$
   (b) $xy^2y' + y^3 = x\cos x$
   (c) $x\,dy + y\,dx = xy^2\,dx$
   (d) $y' + xy = xy^4$

4. The usual Leibniz notation dy/dx implies that x is the independent variable and y is the dependent variable. In solving a differential equation, it is sometimes useful to reverse the roles of the two variables. Treat each of the following equations by reversing the roles of y and x:

(a) $(e^y - 2xy)y' = y^2$

(b) $y - xy' = y'y^2e^y$

(c) $xy' + 2 = x^3(y-1)y'$

(d) $f(y)^2\dfrac{dx}{dy} + 3f(y)f'(y)x = f'(y)$

5. We know from our solution technique that the general solution of a first-order linear equation is a family of curves of the form

$$y = c \cdot f(x) + g(x).$$

Show, conversely, that the differential equation of any such family is linear and first-order.

6. Show that the differential equation $y' + Py = Qy\ln y$ can be solved by the change of variables $z = \ln y$. Apply this method to solve the equation

$$xy' = 2x^2y + y\ln y.$$

7. One solution of the differential equation $y' \sin 2x = 2y + 2\cos x$ remains bounded as $x \to \pi/2$. Find this solution.

8. A tank contains 10 gal of brine in which 2 lb of salt are dissolved. New brine containing 1 lb of salt per gal is pumped into the tank at the rate of 3 gal/min. The mixture is stirred and drained off at the rate of 4 gal/min. Find the amount $x = x(t)$ of salt in the tank at any time t.

9. A tank contains 40 gal of pure water. Brine with 3 lb of salt per gal flows in at the rate of 2 gal/min. The thoroughly stirred mixture then flows out at the rate of 3 gal/min.

(a) Find the amount of salt in the tank when the brine in it has been reduced to 20 gallons.

(b) When is the amount of salt in the tank greatest?

Answers to Odd problems

1. (a) $y = ce^{x^2/2}$;
   (b) $y = 1 + ce^{-x^2/2}$;
   (c) $y = e^{-x} \arctan e^{x} + ce^{-x}$;
   (d) $y = x^2 e^{-x} + x^2 - 2x + 2 + ce^{-x}$;
   (e) $y = -x^3 + cx^2$;
   (f) $y = ce^{-x^2}$;
   (g) $y = x^4 + cx^3$;
   (h) $y = \dfrac{c + \ln(\sin x)}{1 + x^2}$;
   (i) $y = \left(x^2 + c\right) \csc x$;
   (j) $xy \sin x = \sin x - x \cos x + c$.

3. (a) $\dfrac{1}{y^2} = -x^4 + cx^2$;
   (b) $y^3 = 3 \sin x + 9x^{-1} \cos x - 18x^{-2} \sin x - 18x^{-3} \cos x + cx^{-3}$;
   (c) $1 + xy \ln x = cxy$;
   (d) $y^{-3} = 1 + ce^{3x^2/2}$.

5. Proof not shown.

7. $y = \tan x - \sec x$.

9. (a) 45 pounds;
   (b) after $\dfrac{40}{3}(3 - \sqrt{3}) \approx 16.9$ minutes.

Exact Differential Equations

Determine which of the following equations, in Exercises 1–19, is exact. Solve those that *are* exact.

1. $\left(x + \frac{2}{y}\right) dy + y\,dx = 0$
2. $(\sin x \tan y + 1)\,dx + \cos x \sec^2 y\,dy = 0$
3. $(y - x^3)\,dx + (x + y^3)\,dy = 0$
4. $(2y^2 - 4x + 5)\,dx = (4 - 2y + 4xy)\,dy$
5. $(y + y\cos xy)\,dx + (x + x\cos xy)\,dy = 0$
6. $\cos x \cos^2 y\,dx + 2\sin x \sin y \cos y\,dy = 0$
7. $(\sin x \sin y - xe^y)\,dy = (e^y + \cos x \cos y)\,dx$
8. $-\frac{1}{y}\sin\frac{x}{y}\,dx + \frac{x}{y^2}\sin\frac{x}{y}\,dy = 0$
9. $(1 + y)\,dx + (1 - x)\,dy = 0$
10. $(2xy^3 + y\cos x)\,dx + (3x^2y^2 + \sin x)\,dy = 0$
11. $dx = \frac{y}{1 - x^2y^2}\,dx + \frac{x}{1 - x^2y^2}\,dy$
12. $(2xy^4 + \sin y)\,dx + (4x^2y^3 + x\cos y)\,dy = 0$
13. $\frac{y\,dx + x\,dy}{1 - x^2y^2} + x\,dx = 0$

Answers to Odd problems

1. $xy + \ln y^2 = c.$
3. $4xy - x^4 + y^4 = c.$
5. $xy + \sin xy = c.$
7. $xe^y + \sin x \cos y = c.$
9. Not exact.
11. $\ln\left[\frac{1 + xy}{1 - xy}\right] - 2x = c.$
13. $\ln\left[\frac{1 + xy}{1 - xy}\right] + x^2 = c.$
15. Not exact.
17. $x - y^2\cos^2 x = c.$
19. $x^3(1 + \ln y) - y^2 = c.$
21. $x^2y^2(4y^2 - x^2) = c.$

14. $2x\left(1+\sqrt{x^2-y}\right)dx=\sqrt{x^2-y}\,dy$

15. $(x\ln y+xy)\,dx+(y\ln x+xy)\,dy=0$

16. $\left(e^{y^2}-\csc y\csc^2 x\right)dx+\left(2xye^{y^2}-\csc y\cot y\cot x\right)dy=0$

17. $(1+y^2\sin 2x)\,dx-2y\cos^2 x\,dy=0$

18. $\dfrac{x\,dx}{\left(x^2+y^2\right)^{3/2}}+\dfrac{y\,dy}{\left(x^2+y^2\right)^{3/2}}=0$

19. $3x^2(1+\ln y)\,dx+\left(\dfrac{x^3}{y}-2y\right)dy=0$

20. Solve

$$\frac{y\,dx-x\,dy}{(x+y)^2}+dy=dx$$

as an exact equation by two different methods. Now reconcile the results.

21. Solve

$$\frac{4y^2-2x^2}{4xy^2-x^3}\,dx+\frac{8y^2-x^2}{4y^3-x^2y}\,dy=0$$

as an exact equation. Later on (Section 1.7) we shall learn that we may also solve this equation as a homogeneous equation.

22. For each of the following equations, find the value of n for which the equation is exact. Then solve the equation for that value of n.
(a) $\left(xy^2+nx^2y\right)dx+\left(x^3+x^2y\right)dy=0$
(b) $\left(x+ye^{2xy}\right)dx+nxe^{2xy}\,dy=0$

Orthogonal Trajectories

1. Sketch each of the following families of curves. In each case, find the family of orthogonal trajectories, and add those to your sketch.
   (a) $xy = c$
   (b) $y = cx^2$
   (c) $x + y = c$
   (d) $y = c(1 + \cos x)$
   (e) $y = ce^x$
   (f) $x - y^2 = c$

2. What are the orthogonal trajectories of the family of curves $y = cx^4$? What are the orthogonal trajectories of the family of curves $y = cx^n$ for n a positive integer? Sketch both families of curves. How does the family of orthogonal trajectories change when n is increased?

3. Sketch the family $y^2 = 4c(x + c)$ of all parabolas with axis the x-axis and focus at the origin. Find the differential equation of this family. Show that this differential equation is unaltered if dy/dx is replaced by $-dx/dy$. What conclusion can be drawn from this fact?

4. Find the curves that satisfy each of the following geometric conditions:
   (a) The part of the tangent cut off by the axes is bisected by the point of tangency.
   (b) The projection on the x-axis of the part of the normal between (x, y) and the x-axis has length 1.
   (c) The projection on the x-axis of the part of the tangent between (x, y) and the x-axis has length 1.
   (d) The part of the tangent between (x, y) and the x-axis is bisected by the y-axis.
   (e) The part of the normal between (x, y) and the y-axis is bisected by the x-axis.
   (f) The point (x, y) is equidistant from the origin and the point of intersection of the normal with the x-axis.

5. A curve rises from the origin in the x-y plane into the first quadrant. The area under the curve from $(0, 0)$ to (x, y) is one third of the area of the rectangle with these points as opposite vertices. Find the equation of the curve.

6. Find the differential equation of each of the following one-parameter families of curves:

(a) $y = x\sin(x+c)$

(b) All circles through $(1,0)$ and $(-1,0)$

(c) All circles with centers on the line $y = x$ and tangent to both axes

(d) All lines tangent to the parabolas $x^2 = 4y$ (*Hint*: The slope of the tangent line at $(2a, a^2)$ is a)

(e) All lines tangent to the unit circle $x^2 + y^2 = 1$

7. Use your symbol manipulation software, such as `Maple` or `Mathematica` or `MATLAB`, to find the orthogonal trajectories to each of these families of curves. Sketch the graphs.

(a) $y = \sin x + cx^2$

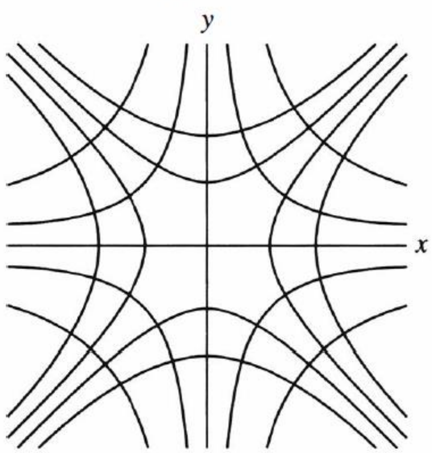
(b) $y = c\ln x + x, \qquad x > 0$

(c) $y = \dfrac{\cos x}{cx + \ln x}, \qquad x > 0$

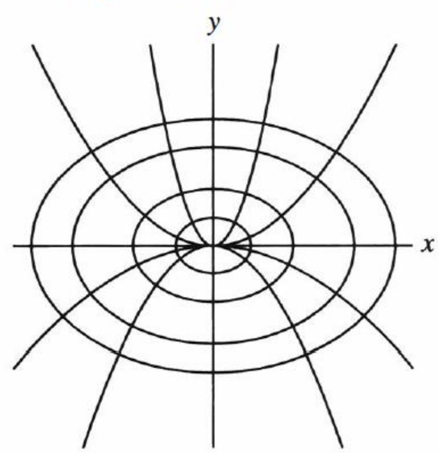
(d) $y = \sin x + c\cos x$

Answers to Odd problems

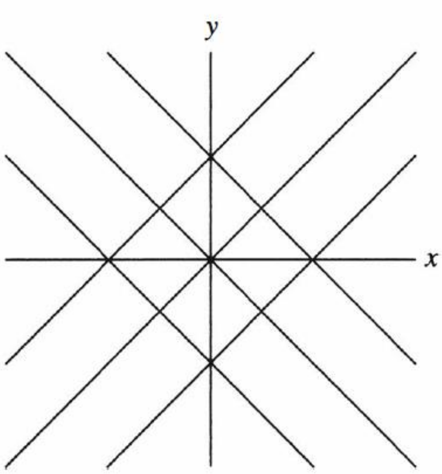
1. (a) $x^2 - y^2 = c$;



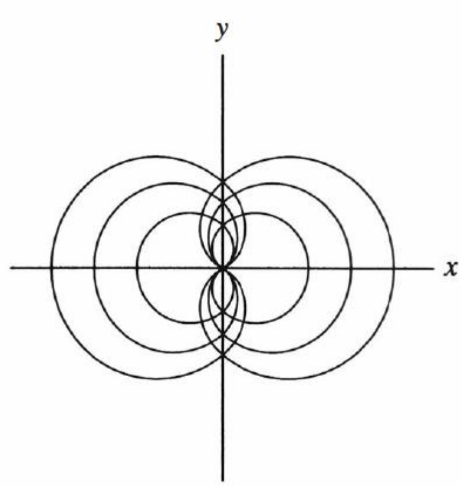
(b) $x^2 + 2y^2 = c^2$;



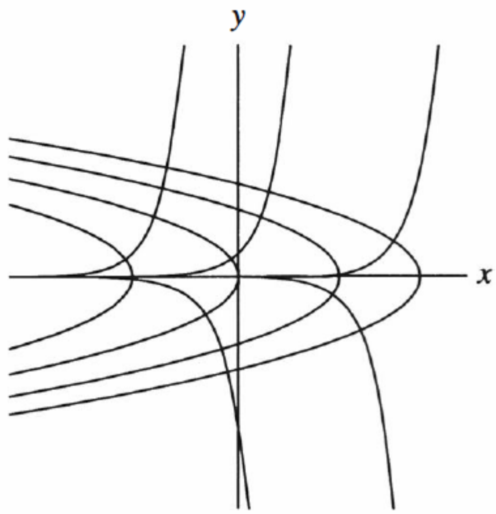
(c) $y = x + c$;



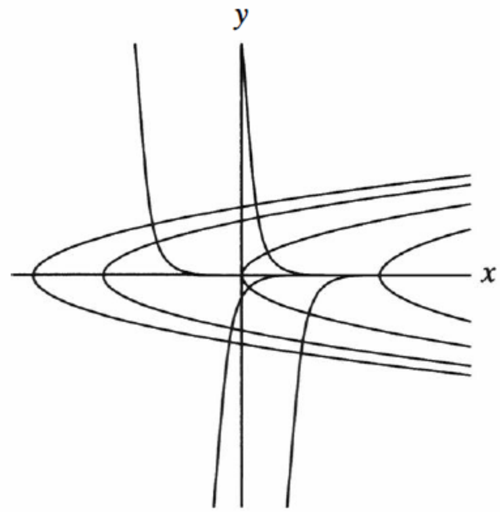
(d) $r = c(1 - \cos\theta)$;



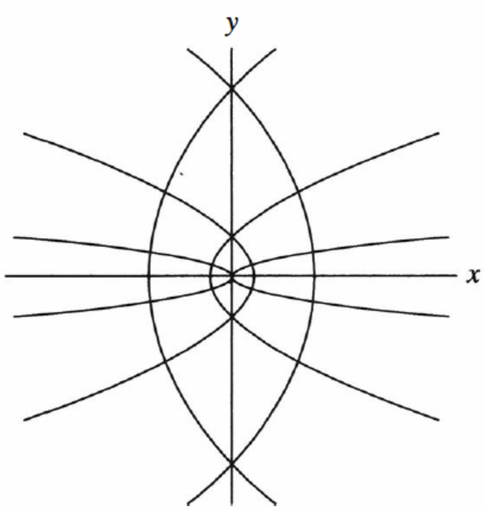
(e) $y^2 = -2x + c$;



(f) $y = ce^{-2x}$;



3.



$y^2 = 2xy\dfrac{dy}{dx} + y^2\left(\dfrac{dy}{dx}\right)^2$; the family is *self-orthogonal* in the sense that when a curve in the family intersects another curve in the family, it is orthogonal to it.

5. $y = cx^2$.

7. Answers will vary.

Homogeneous Differential Equations

1. Verify that each of the following equations is homogeneous, and then solve it.

(a) $(x^2 - 2y^2)\,dx + xy\,dy = 0$

(b) $x^2y' - 3xy - 2y^2 = 0$

(c) $x^2y' = 3(x^2 + y^2) \cdot \arctan \dfrac{y}{x} + xy$

(d) $x\left(\sin \dfrac{y}{x}\right)\dfrac{dy}{dx} = y \sin \dfrac{y}{x} + x$

(e) $xy' = y + 2xe^{-y/x}$

(f) $(x - y)\,dx - (x + y)\,dy = 0$

(g) $xy' = 2x - 6y$

(h) $xy' = \sqrt{x^2 + y^2}$

(i) $x^2y' = y^2 + 2xy$

(j) $(x^3 + y^3)\,dx - xy^2\,dy = 0$

2. Use rectangular coordinates to find the orthogonal trajectories of the family of all circles tangent to the y-axis at the origin.

3. (a) If $ae \neq bd$ then show that h and k can be chosen so that the substitution $x = z - h$, $y = w - k$ reduces the equation

$$\frac{dy}{dx} = F\left(\frac{ax + by + c}{dx + ey + f}\right)$$

to a homogeneous equation.

(b) If $ae = bd$ then show that there is a substitution that reduces the equation in (a) to one in which the variables are separable.

4. Solve each of the following equations:

(a) $\dfrac{dy}{dx} = \dfrac{x + y + 4}{x - y - 6}$

(b) $\dfrac{dy}{dx} = \dfrac{x + y + 4}{x + y - 6}$

(c) $(2x - 2y)\,dx + (y - 1)\,dy = 0$

(d) $\dfrac{dy}{dx} = \dfrac{x + y - 1}{x + 4y + 2}$

(e) $(2x + 3y - 1)\,dx$
$-4(x + 1)\,dy = 0$

5. By making the substitution $z = y/x^n$ (equivalently $y = zx^n$) and choosing a convenient value of n, show that the following differential equations can be transformed into equations with separable variables, and then solve them:

(a) $\dfrac{dy}{dx} = \dfrac{1 - xy^2}{2x^2 y}$

(b) $\dfrac{dy}{dx} = \dfrac{2 + 3xy^2}{4x^2 y}$

(c) $\dfrac{dy}{dx} = \dfrac{y - xy^2}{x + x^2 y}$

6. Show that a straight line through the origin intersects all integral curves of a homogeneous equation at the same angle.

7. Use your symbol manipulation software, such as `Maple` or `Mathematica` or `MATLAB`, to find solutions to each of the following homogeneous equations (note that these would be difficult to do by hand):

(a) $y' = \sin[y/x] - \cos[y/x]$

(b) $e^{x/y}\, dx - \dfrac{y}{x}\, dy = 0$

(c) $\dfrac{dy}{dx} = \dfrac{x^2 - xy}{y^2 \cos(x/y)}$

(d) $y' = \dfrac{y}{x} \cdot \tan[y/x]$

Answers to Odd problems

1. (a) $y^2 = x^2 + cx^4$;

(b) $y = \dfrac{cx^3}{1 - cx^2}$;

(c) $y = x \tan cx^3$;

(d) $\cos(y/x) + \ln cx = 0$;

(e) $y = x \ln\left(\ln cx^2\right)$;

(f) $x^2 - 2xy - y^2 = c$;

(g) $y = \dfrac{2}{7}x + \dfrac{c}{x^6}$;

(h) $y\sqrt{x^2 + y^2} + x^2 \ln\left[y + \sqrt{x^2 + y^2}\right] - 3x^2 \ln x + y^2 = cx^2$;

(i) $y = cx^2/(1 - cx)$;

(j) $y^3 = x^3 \ln cx^3$.

3. (a) $h = \dfrac{ce - bf}{ae - bd}, k = \dfrac{af - cd}{ae - bd}$; (b) $z = dx + ey$.

5. (a) $n = -1/2,\ x = ce^{xy^2}$;

(b) $n = 3/4,\ 2 + 5xy^2 = cx^{5/2}$;

(c) $n = -1,\ x = cye^{xy}$.

7. Answers will vary.