Vectors :

Q) $\vec{A} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\vec{B} = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{C} = 2\vec{A} - 3\vec{B}$

$\Rightarrow \boxed{\vec{C} = 3\hat{i} + 13\hat{j} - 8\hat{k}}$

$\hat{c} = \dfrac{\vec{C}}{|\vec{C}|}$

$= \dfrac{3\hat{i} + 13\hat{j} - 8\hat{k}}{\sqrt{9+169+64}}$

$= \dfrac{3\hat{i} + 13\hat{j} - 8\hat{k}}{\sqrt{242}}$

$\boxed{\hat{c} = \dfrac{3\hat{i} + 13\hat{j} - 8\hat{k}}{11\sqrt{2}}}$

$\cos y = \dfrac{-8}{\sqrt{242}}$

$= \dfrac{-8}{11\sqrt{2}}$

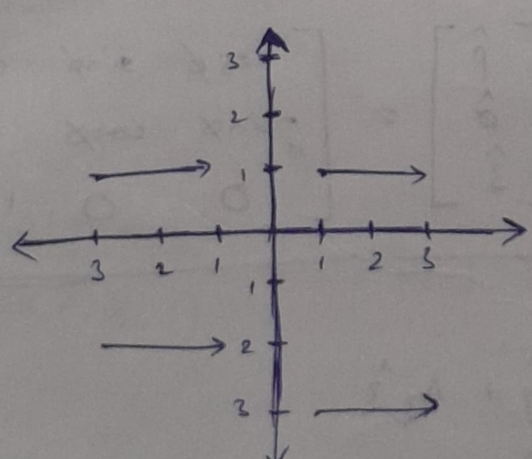
$\boxed{y = \cos^{-1}\left(\dfrac{-8}{11\sqrt{2}}\right)}$

Q)

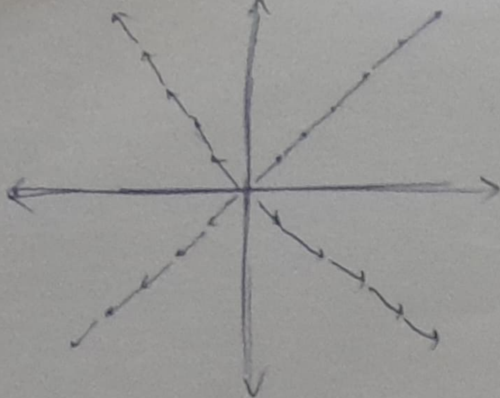
→ Vector field :

$\vec{F}(x, y)$

Ex. $\vec{F}(x, y) = 2\hat{i}$



Q) $\vec{F}(x, y) = x\hat{i} + y\hat{j}$

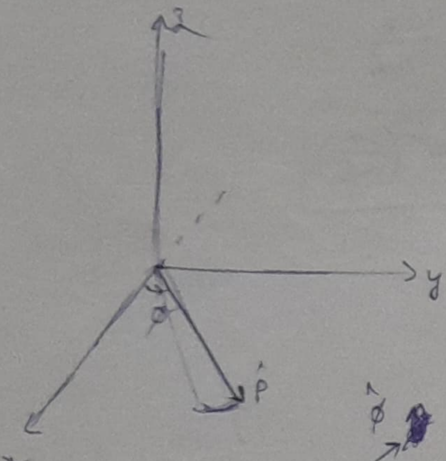


$$\frac{\hat{i}+\hat{j}}{\sqrt{2}}$$

$\sqrt{2}$

$$\frac{\hat{i}}{\sqrt{82}}, \frac{9\hat{j}}{\sqrt{82}}$$

$$\sqrt{\frac{1}{82}+\frac{81}{82}}=1$$



$$\hat{\phi}\cdot\hat{\rho}=0$$

$$\hat{\phi}\times\hat{\rho}=-\hat{z}$$

$$\hat{\phi}=-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}$$

$$\hat{\rho}=\cos\phi\,\hat{x}+\sin\phi\,\hat{y}$$

$$\hat{\rho}=\frac{\rho_x\hat{x}+\rho_y\hat{y}}{\rho}$$

$$\Rightarrow \hat{\rho}=\cos\phi\,\hat{x}+\sin\phi\,\hat{y}$$

$x=\rho\cos\phi$

$y=\rho\sin\phi$

$\rho=\sqrt{x^2+y^2}$

$\tan\phi=\frac{y}{x}$

$\Rightarrow \phi=\tan^{-1}\left(\frac{y}{x}\right)$

★
$$\boxed{\begin{bmatrix}\hat{\rho}\\ \hat{\phi}\\ \hat{z}\end{bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}}$$

$$\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$$
$$=A_\rho\hat{\rho}+A_\phi\hat{\phi}+A_z\hat{z}$$

$$\begin{bmatrix}A_\rho\\ A_\phi\\ A_z\end{bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}A_x\\ A_y\\ A_z\end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

☆

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}}$$

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28/08/23 :

ϕ = constant $\Rightarrow$ plane
ρ = constant $\Rightarrow$ cylinder

$$\left.\begin{matrix} A(\rho, \phi_1, z) \\ B(\rho, \phi_2, z) \end{matrix}\right\} \longrightarrow$$ Need to be converted to Cartesian Coordinate system for addition, subtraction, etc

Q) $A(x, y, z) \equiv A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\left.\begin{aligned} A_\rho &= x\cos\phi + y\sin\phi \\ A_\phi &= -x\sin\phi + y\cos\phi \\ A_z &= z \end{aligned}\right\} \Rightarrow \begin{aligned} A_\rho &= \rho\cos^2\phi + \rho\sin^2\phi = \rho \\ A_\phi &= -\rho\sin\phi\cos\phi + \rho\sin\phi\cos\phi = 0 \\ A_z &= z \end{aligned}$$

$A(x, y, z) \longrightarrow A(\rho, 0, z)$

P) $\vec{A} = 3\hat{\rho} + 2\hat{\phi} + 5\hat{z}$

defined at $P(3, \pi/6, 5)$

$\vec{B} = -2\hat{\rho} + 3\hat{\phi} - \hat{z}$

at $(4, \pi/3, 5)$

Find $\vec{C} = \vec{A} - \vec{B}$

at $(2, \pi/4, 3)$

12/4/23

Q) $\vec{A} = \frac{k}{\rho^2}\vec{a}_\rho + 5\sin 2\phi\, \vec{a}_z$ into rectangular coordinate system

Sol: $A_\rho = \frac{k}{\rho^2}$

$A_\phi = 0$

$A_z = 5\sin(2\phi)$

$A_x = \frac{k\cos\phi}{\rho^2}$

$A_y = \frac{k\sin\phi}{\rho^2}$

$A_z = 10\cos\phi\sin\phi$

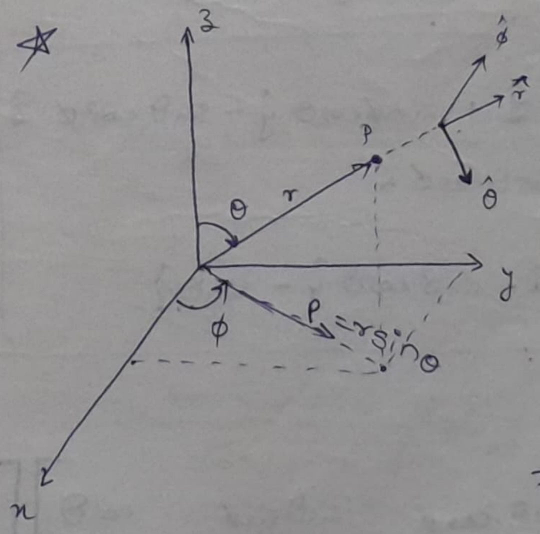
$\cos\phi = \frac{x}{\rho}$

$\sin\phi = \frac{y}{\rho}$

$\rho = \sqrt{x^2+y^2}$

$$\vec{A} = \frac{kx}{(x^2+y^2)^{3/2}}\hat{x} + \frac{ky}{(x^2+y^2)^{3/2}}\hat{y} + \frac{10xy}{x^2+y^2}\hat{z}$$

→ Spherical Coordinate System



(r, θ, ϕ)

$\rho = r\sin\theta$

$\vec{r} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$

$= r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} + 0\hat{z}$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$$\boxed{\hat{\phi} \times \hat{r} = \hat{\theta}}$$

$$\cancel{\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}}$$

$$\cancel{\hat{\theta} = \cos\phi\,\hat{x} - \sin\phi\,\hat{y}}$$

$$\cancel{\cos\hat{\theta} = -\cos\phi\,\hat{x} - \sin\phi\,\hat{y}}$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\cancel{\hat{\phi}\times\hat{r} = -\sin\theta\cos^2\phi\,\hat{z} + \sin\theta\sin^2\phi\,\hat{z} - \sin\phi\cos\theta\,\hat{y} - \sin\theta\cos\phi\sin\phi\,\hat{z} + \cos\theta\cos\phi\,\hat{x}}$$

$$\cancel{= \cos\theta\cos\phi\,\hat{x} - \sin\phi\cos\theta\,\hat{y}}$$

$x \to y \to z \to x$

$$\hat{\phi}\times\hat{r} = -\sin\theta\sin^2\phi\,\hat{z} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\cos^2\phi\,\hat{z} + \cos\theta\cos\phi\,\hat{x}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z}$$

☆
$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix} = \begin{bmatrix}\sin\theta\cdot\cos\phi & \sin\theta\cdot\sin\phi & \cos\theta\\ \cos\theta\cdot\cos\phi & \sin\phi\cdot\cos\theta & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\left.\begin{aligned} \vec{A} &= 10\hat{r} + 30\hat{\theta} - 10\hat{\phi} \\ \vec{B} &= -3\hat{r} - 10\hat{\theta} \pm 20\hat{\phi} \end{aligned}\right\}$$ Defined at same Point

(a) $2\vec{A} - 5\vec{B}$

$= 35\hat{r} + 110\hat{\theta} - 120\hat{\phi}$

(b) $\vec{A}\cdot\vec{B}$

$= -30 - 300 - 200$

$= -530$

(c) $\vec{A}\times\vec{B}$

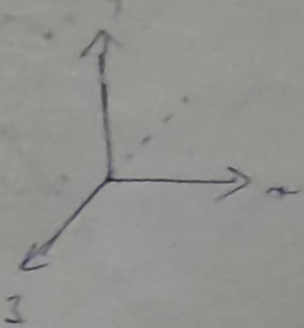
$= -100\hat{\phi} - 200\hat{\theta} + 90\hat{\phi} + 600\hat{r} + 30\hat{\theta} - 100\hat{r}$

$= 500\hat{r} - 170\hat{\theta} - 10\hat{\phi}$

1/9/23

(d) $|\vec{a}_B| = \dfrac{\vec{A}\cdot\vec{B}}{B} = \vec{A}\cdot\hat{B}$

(e) $\vec{a}_n = \left(\dfrac{\vec{A}\cdot\vec{B}}{B^2}\right)\vec{B}$



(f) $\vec{C} = \pm\dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$

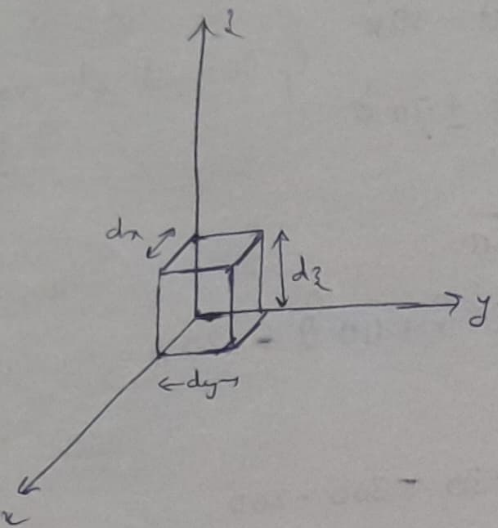
$= \pm\dfrac{500\hat{r} - 170\hat{\theta} - 10\hat{\phi}}{\sqrt{500^2 + 170^2 + 10^2}}$

Q) $\vec{P_1} = x\hat{i} + y\hat{j} + z\hat{k}$

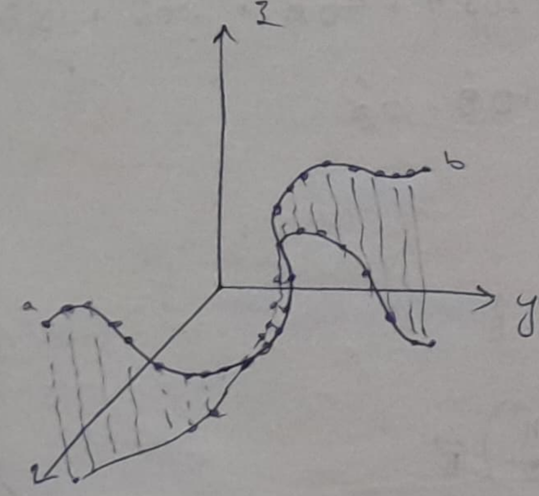
$\vec{P_2} = (x+dx)\hat{i} + (y+dy)\hat{j} + (z+dz)\hat{k}$

$\vec{P_2} - \vec{P_1} = \vec{dl} = (dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k}$

$d\vec{v} = d\bar{x} \cdot d\bar{y} \cdot d\bar{z}$



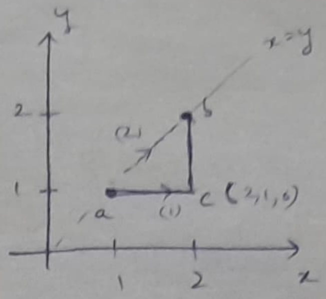
→ Line Integral :



$$\int_a^b v \cdot dl$$

$$\oint v \cdot dl$$

9)



$v = y^2\hat{x} + 2x(y+1)\hat{y}$

$a = (1,1,0)$

$b \equiv (2,2,0)$

$\oint \int \bar{v} \cdot d\bar{l}$

$= \int (y^2\hat{x} + 2x(y+1)\hat{y})(dx\hat{x} + dy\hat{y})$

$= \int y^2\, dx + 2x(y+1)\, dy$

$= \int y^2 dx + \int 2x(y+1)\, dy$

(i) $A \to C$:

$= \int_1^2 y^2 dx = 1(1) = 1$

$C \to B$:

$= \int_1^2 2x(y+1)\, dy$

$= 4\int_1^2 (y+1)\, dy = 4\left(\frac{y^2}{2} + y\right]_1^2$

$\cancel{4\left(\frac{y^2}{2}+y\right)}$ $= 4\left(4 - \frac{3}{2}\right)$

$= 10$

$(A \to C) + (C \to B) = \underline{\underline{11}}$

(ii) $A \to B$

$= \int_1^2 y^2 dx + \int_1^2 2x(y+1)\, dy$

$= \int_1^2 x^2 dx + \int_1^2 (2y^2 + 2y)\, dy$

$= \left(\frac{x^3}{3}\right]_1^2 + \left(\frac{2y^3}{3} + y^2\right]_1^2$

$= \frac{7}{3} + \left(\frac{16}{3} + 4\right) - \left(\frac{2}{3} + 1\right) = 10$ //

$B \to A : -10$

$(A \to C) + (C \to B) + (B \to A)$

$= 10 + 1 - 10$

$= \underline{1}$

$\oint \bar{v} \cdot d\bar{l} = 1$

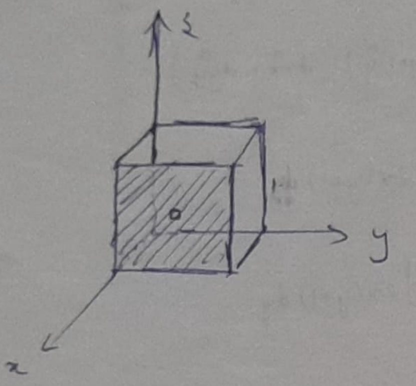
4/9/23:

→ Surface Integral:

~~$\oint v \cdot dA$ $\oint_s v \cdot da$~~

$$\int_s v \cdot dA \quad \text{(or)} \quad \oint v \cdot dA$$

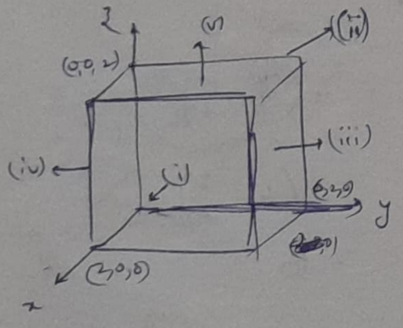
Ex.



$$d\vec{A} = dy \cdot dz\ \hat{x}$$

Q) $\vec{V} = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$

Calculate $\oint v \cdot dA$ for 5 surfaces (No bottom)



+ve ↑   +ve ⊙

(1) $dA = dy \cdot dz\ \hat{x}$

(i) $\oint v \cdot dA$

$\Rightarrow \oint (2xz)(dy\,dz)$

$\Rightarrow \oint 4z\,dy\,dz$

$\Rightarrow \int_{y=0}^{2}\int_0^2 (4z\,dz)\,dy \Rightarrow \int_0^2 [2z^2]_0^2\,dy \Rightarrow \int_0^2 8\,dy = \underline{16}$

(iii) $\oint v \cdot dA$ (ii) $\oint v \cdot dA = 0$

$\oint (x+2) \cdot dx\, dz$

$= \int_0^2 \int_0^2 (x+2)\, dx\, dz$ $\left.\frac{x^2}{2} + 2x\right]_0^2 = 2 + 4 = 6$

$= \int_0^2 6\, dz$

$= \underline{\underline{12}}$

(iv) $\oint v \cdot dA$

$= \oint (x+2)(-dx)\, dz$

$= - \oint (x+2)\, dx\, dz$

$= \underline{\underline{-12}}$

(v) $\oint v \cdot dA$

$= \oint y(z^2-3)\, dx\, dy$

$= \int_0^2 \oint_0^2 (y\, dy)\, dx$

$= \int_0^2 2\, dx$

$= \underline{\underline{4}}$

(i) $\rightarrow 16$
(ii) $\rightarrow 0$
(iii) $\rightarrow 12$
(iv) $\rightarrow -12$
(v) $\rightarrow 4$

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→ <u>Cylindrical Coordinates</u>:

$\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$dl_x = dx$

$dl_y = dy$

$dl_z = dz$

~~$\vec{dl} = dl_\rho \hat{\rho} + \ldots$~~

$\vec{dl} = dl_\rho\, \hat{\rho} + dl_\phi\, \hat{\phi} + dl_z\, \hat{z}$

$\vec{dl} = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}$

$\vec{ds_\rho} = \rho\, d\phi\, dz\,\hat{\rho}$

$\vec{ds_\phi} = d\rho\, dz\,\hat{\phi}$

$\vec{ds_z} = \rho\, d\rho\, d\phi\,\hat{z}$

$dl_\rho = d\rho$

$dl_\phi = \rho\, d\phi$

$dl_z = dz$

→ Spherical coordinates :

$\vec{dl} = dl_r\,\hat{r} + dl_\theta\,\hat{\theta} + dl_\phi\,\hat{\phi}$

$dl_\phi = r\sin\theta\, d\phi$

$dl_r = dr$

$dl_\theta = r\, d\theta$

$\vec{ds_r} = r^2\sin\theta\, d\theta\, d\phi\,\hat{r}$

$\vec{ds_\phi} = r\, d\theta\, dr\,\hat{\phi}$

$\vec{ds_\theta} = r\sin\theta\, dr\, d\phi\,\hat{\theta}$

$d\tau = \vec{dl_r}\cdot\vec{dl_\phi}\cdot\vec{dl_\theta}$

$= r^2\sin\theta\, dr\, d\theta\, d\phi$

Q) Sphere , $R = R$

$d\tau = R^2\sin\theta\, dR\cdot d\theta\cdot d\phi$

$\oint d\tau = \iiint R^2\sin\theta\, dR\, d\theta\, d\phi$

$= \int\left(\int\left(\int_0^R R^2\, dR\right)\sin\theta\, d\theta\right) d\phi$

$= \int_0^{2\pi}\left(\int_0^{\pi}\frac{R^3}{3}\sin\theta\, d\theta\right) d\phi$

$$\vec{dl}^{\,cart.} = dl_x\hat{i} + dl_y\hat{j} + dl_z\hat{k}$$
$$= dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\vec{dl}^{\,cyl.} = dl_\rho\hat{\rho} + dl_\phi\hat{\phi} + dl_z\hat{z}$$
$$= d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$$

$$\vec{dl}^{\,sph.} = dl_r\hat{r} + dl_\theta\hat{\theta} + dl_\phi\hat{\phi}$$
$$= dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}$$

$$\vec{dS}_z^{\,cart.} = dx\, dy\, \hat{z}$$
$$(dS_z = dl_x \cdot dl_y\, \hat{z})$$

$$\vec{dS}_\rho^{\,cyl.} = (dl_\phi \cdot dl_z)\hat{\rho}$$
$$\Rightarrow \vec{dS}_\rho = (\rho\, d\phi\, dz)\hat{\rho}$$

$$d\tau_{cart.} = dx \cdot dy \cdot dz$$
$$(d\tau^{cart.} = dl_x \cdot dl_y \cdot dl_z)$$

$$d\tau_{sph} = dl_r\, dl_\theta \cdot dl_\phi$$
$$= dr \cdot rd\theta \cdot r\sin\theta\, d\phi$$
$$= r^2\sin\theta\, d\theta\, d\phi\, dr$$

$$dS_r = dl_\theta \cdot dl_\phi \cdot \hat{r}$$
$$\oint dS_r = 0$$
$$\oint_S dS_r = \int r^2\sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$\tau = \int d\tau^{sph}$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{r=R} r^2\, dr\, d\phi\, d\theta\, \sin\theta$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{R^3}{3}\, d\phi\, d\theta\, \sin\theta$$

$$= \int_0^{\pi} \frac{2\pi}{3} R^3 \sin\theta\, d\theta$$

$$-\cos\theta\Big]_0^{\pi}$$
$$= \cos\theta\Big]_{\pi}^{0}$$
$$= 1-(-1)$$
$$= 2$$

$$= \frac{4\pi}{3}R^3$$

$$\boxed{\tau = \frac{4}{3}\pi R^3}$$

$\rho = 4 \times 10^{3} r \quad C/m^3$

$q = \int \rho \, d\tau$

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$\left[df = \left(\frac{df}{dx}\right) dx \quad \text{for one variable} \right]$$

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= \underbrace{\nabla \vec{T}}_{\text{Gradient of } T} \cdot \vec{dl}$$

$$\boxed{dT = \vec{\nabla} T \cdot \vec{dl}}$$

$$dT = |\vec{\nabla} T| \cdot |\vec{dl}| \cos\theta$$

$$\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}$$

$$= \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right] \cdot T$$

$$= \nabla \vec{T}$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

9/9/21

$\vec{\nabla} \cdot \vec{v}$ → Divergence of Vector v

$\vec{\nabla} \times \vec{v}$ → Curl of Vector v

$$dT = \vec{\nabla} T \cdot d\vec{l}$$

Ex. $\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$

$$\vec{v_x} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

$$\vec{\nabla} \cdot \vec{v_x} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

→ $\vec{\nabla}\, k\, g(x,y,z)$

$= k\, \vec{\nabla} g(x,y,z)$

→ $\vec{\nabla}\left[f(x,y,z) \cdot g(x,y,z)\right]$

$= f(x,y,z) \cdot \vec{\nabla} g(x,y,z) + g(x,y,z)\, \vec{\nabla} f(x,y,z)$

$$\vec{\nabla} \vec{T} \cdot \vec{v} \neq \vec{\nabla}(\bar{T} \cdot \bar{v})$$

Q) $f(x,y,z) = x^2 + y^3 + z^4$ find $\vec{\nabla} f$ @ (2,1,0)

~~$\vec{\nabla} f(x,y,z) = 2x\hat{i} + 3y^2\hat{j} + 4z^3\hat{k}$~~

~~(2,1,0)~~ ~~@ (2,1,0)~~

~~$4\hat{i} + 3\hat{j}$~~ ~~$= 4+3+0$~~

~~$= 7$~~

Sol: $\vec{\nabla} f(x,y,z) = 2x\hat{i} + 3y^2\hat{j} + 4z^3\hat{k}$

@ (2,1,0)

$\Rightarrow 4\hat{i} + 3\hat{j}$

Ans

Q) $f(x) = x^2 + y^2 + z^2$

@ $(1,1,1)$

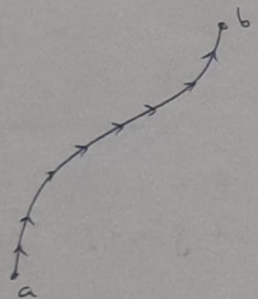
$\bar{\nabla} \bar{f} = ?$

Sol : $2\hat{x} + 2\hat{y} + 2\hat{z}$

$\Rightarrow 2\hat{i} + 2\hat{j} + 2\hat{k}$

— Fundamental Theorem of Gradient :

$$dT = \vec{\nabla} T \cdot d\vec{l}$$

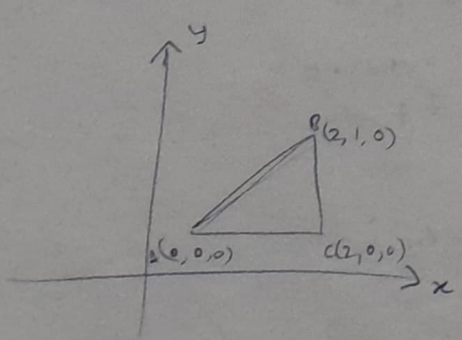


Path Independent

Conservative

$$\int_a^b dT = \int_a^b \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$$

Q) $T = xy^2$



$[2y = x]$

$T_b = 2 \times 1 = 2$

$T_a = 0$ $\boxed{T_b - T_a = 2}$

$$\int_{(0,0,0)}^{(2,1,0)} \left(y^2 \hat{i} + 2xy\, \hat{j}\right)\left(dx\, \hat{i} + dy\, \hat{j}\right)$$

$$= \int_{(0,0,0)}^{(2,1,0)} y^2\, dx + 2xy\, dy$$

$$= \int_{(0,0,0)}^{(2,1,0)} \frac{x^2}{4} dx + 4y^2\, dy$$

$$= \left.\frac{x^3}{12}\right]_0^2 + \left.\frac{4y^3}{3}\right]_0^1 = \frac{8}{12} + \frac{4}{3} = \frac{24}{12} = 2$$

Q) $\vec{V} = x\hat{x} + y\hat{y} + z\hat{z}$

@ (1, 2, 0)

$\vec{\nabla} \cdot \vec{V} = 1 + 2 = 3$

If $\vec{\nabla} \cdot \vec{V} = 0 \Rightarrow \vec{V}$ is a Solenoidal function/vector.

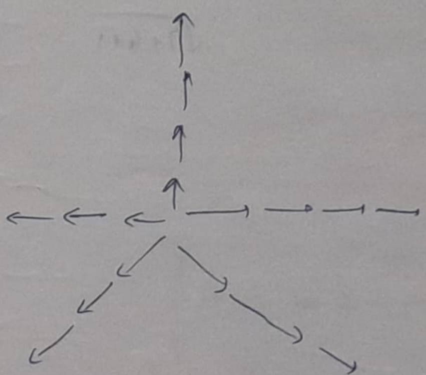
If $\vec{\nabla} \times \vec{V} = 0 \Rightarrow \vec{V}$ is a Irrotational vector

If $\vec{\nabla} \times \vec{V} \neq 0 \Rightarrow \vec{V}$ is a Rotational vector

No Divergence $\Rightarrow$ Solenoidal

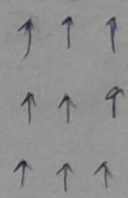
Divergence = + Divergence

Convergence = − Divergence



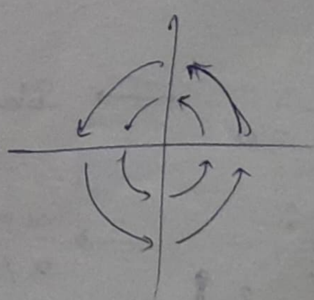
+Ve Divergence

Irrotational



Irrotational

Solenoidal



Rotational

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How do you find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$?

$$S(x, y, z) = x^3 + y^3 + 3xyz - 3$$

$$\vec{\nabla} S = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

$$= (3 - 6)\hat{i} + (12 - 3)\hat{j} + 6\hat{k}$$

$$= -3\hat{i} + 9\hat{j} + 6\hat{k}$$

$$\Rightarrow \vec{n} = 3(-\hat{i} + 3\hat{j} + 2\hat{k})$$

$$\hat{n} = \frac{3}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k})$$

$\sqrt{9+4+1}$

→ **Divergence:**

Ex. $f = x\hat{x} + y\hat{y} - \hat{z}$

$\vec{\nabla} \cdot \vec{f} = ?$ → Non-Solenoidal / Divergent

$1 + 1 = 2$

Ex. $f = 2\hat{y}$

$\vec{\nabla} \cdot \vec{f} = 0$ → Solenoidal

Ex. $f = x^2\hat{x}$

$\vec{\nabla} \cdot \vec{f} = 2x$ → $x = 0 \Rightarrow$ Solenoid
$x \neq 0 \Rightarrow$ Non-solenoidal / Divergent

→ Fundamental Theorem of divergences -

$$\int_V (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_{S'} v\cdot dA$$

[Gauss's Divergence Theorem]

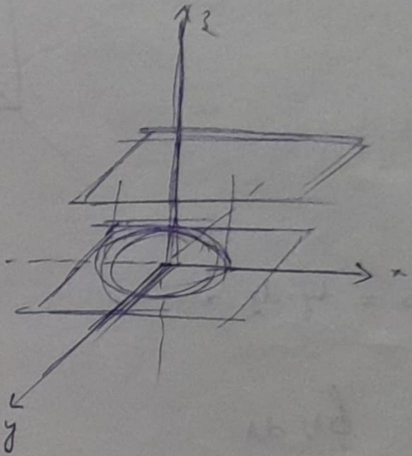
$\cancel{Ex.\ \vec{F} = x^4 y\,\hat{x} - 2x^3y^2\,\hat{y} + \hat{z}\,\hat{z}}$

Ex. $\vec{F} = x^4 y\,\hat{x} - 2x^3y^2\,\hat{y} + z^2\,\hat{z}$

$z = 0$

$z = h$

$x^2 + y^2 = R^2$



$$\oint_V (4x^3y - 4x^3y + 2z)\, d\tau$$

$$= \oint_V 2z\, d\tau$$

$d\tau = dl_\rho \cdot dl_\phi \cdot dl_z$

dl_ρ: $0 - R$, dl_ϕ: $0 - 2\pi$, dl_z: $0 - h$

$$= \oint_V 2z\cdot\rho\cdot d\rho\cdot d\phi\, dz$$

$\Rightarrow d\tau = d\rho\cdot\rho\, d\phi\cdot dz$

$= \rho\, d\rho\, d\phi\, dz$

$$= \int\left(\int\left(\int_0^h (2z\cdot dz)\, d\phi\right)\rho\cdot d\rho\right)$$

$$= \int\left(\int h^2\, d\phi\right)\rho\, d\rho$$

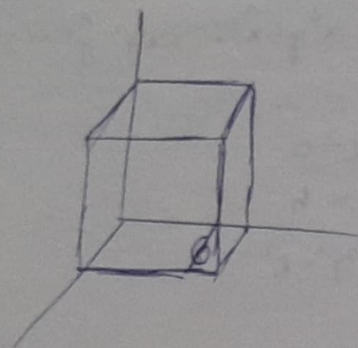
$$= \int_0^R h^2\left(\int_0^{2\pi} d\phi\right)\rho\, d\rho$$

$$= \int h^2\cdot 2\pi\rho\cdot d\rho = 2\pi h^2\cdot\frac{R^2}{2} = \pi R^2 h^2$$

Q) Check the divergence ~~function~~ theorem using the function

$$\vec{U} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$$

and a unit cube at the origin.



M-I

(i) $ds = dy\,dz\,\hat{x}$

$$\oint V \cdot dA$$

$$= \int y^2 \cdot dy \cdot dz$$

$$= \int_0^1 \left(\int_0^1 y^2 dy\right) dz$$

$$= \int_0^1 \frac{1}{3}\,dz$$

Flux $= \frac{1}{3}$

(ii) $ds = -dy \cdot dz\,\hat{x}$

Flux $= -1/3$

(iii) $ds = dx\,dz \cdot \hat{y}$

$$\Rightarrow \int (2xy + z^2)\,dx\,dz$$

$$\Rightarrow \int (2x + z^2)\,dx\,dz$$

(iv) $-\oint (z^2)\, dx\, dz$

(v) $ds = dx\, dy \cdot \hat{z}$

$\oint 2yz\; dx \cdot dy$

$= \oint 2y \cdot dx\, dy$

$= \int_0^1 \left(\int_0^1 2y\, dy\right) dx$

$= \int_0^1 1\, dx$

Flux = 1

(iii) $\Rightarrow \int 2x\, dx\, dz + \int z^2\, dx\, dz$

$\Rightarrow \int_0^1 \left(\int_0^1 2x\, dx\right) dz + \int_0^1 \left(\int_0^1 z^2\, dz\right) dx$

$\Rightarrow \int_0^1 1\, dz + \int_0^1 \frac{1}{3}\, dx$

$\Rightarrow 1 + \frac{1}{3} = \frac{4}{3}$

~~(iv) $\Rightarrow -\oint 2x\, dx\, dz - \int z^2\, dx\, dz$~~

(iv) $\Rightarrow -\oint z^2\, dx\, dz$

$\Rightarrow -\frac{1}{3}$

(vi) $\Rightarrow 0 =$ Flux $[\because z = 0]$

M-II $d\tau = dx \cdot dy \cdot dz$

~~$\int (\vec{\nabla} \cdot \vec{v})\, d\tau$~~

~~$\Rightarrow \int (2y + 2x + 2z)(dx \cdot dy \cdot dz)$~~

~~$\Rightarrow \int 2y \cdot (dx\, dy\, dz) + \int 2x (dx\, dy\, dz) + \int 2z (dx\, dy\, dz)$~~

~~$\Rightarrow 2 \int_0^1 \left(\int_0^1 \left(\int_0^1 2y\, dy\right) dz\right) dz$~~

~~$\Rightarrow 2 \int_0^1 \left(\int_0^1 1\, dx\right) dz$~~

~~$\Rightarrow 2$~~

$\int (\vec{\nabla} \cdot \vec{v})\, d\tau$

$= \int 2(x+y)\, d\tau$

$= \int 2(x+y) \cdot dx \cdot dy \cdot dz$

$$= 2 \int_0^1 \int_0^1 \int_0^1 (x+y)\, dx\, dy\, dz$$

$$= 2 \int_0^1 \int_0^1 \left(\frac{1}{2} + y\right) dy\, dz$$

$$= 2 \int_0^1 1\, dz$$

$$= 2$$

<u>M-I</u> : (i) + (ii) + (iii) + (iv) + (v) + (vi) = 2

<u>M-II</u> : 2

$$\oint_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{A}$$

Hence Proved

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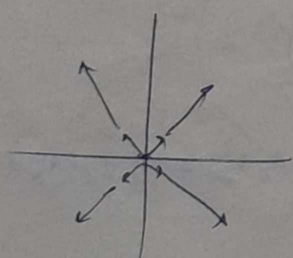
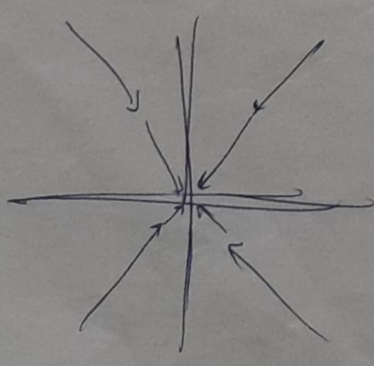


$\vec{\nabla} \cdot \vec{v} > 0$



$\vec{\nabla} \cdot \vec{v} < 0$

Q) $\vec{v} = x^4y^2\hat{x} + 2y^2x^3\hat{y} + z^2\hat{z}$

$\vec{\nabla}\cdot\vec{v} = 4x^3y^3 - 4yx^3 + 2z$

$\vec{\nabla}\cdot\vec{v} > 0$ @ $(0,0,1)$

$\therefore$ There's a source @ $(0,0,1)$ for $\vec{v}$

$\rightarrow$ $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ [cart.]

$= \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\cdot\frac{1}{\rho}\cdot\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$ [cyl]

$= \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$ [sph.]

Divergence:

$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$ [cart.]

$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\cdot\frac{\partial}{\partial\phi}(A_\phi) + \frac{\partial}{\partial z}(A_z)$

$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

[Bhag Guru, Ex. 2.20]

Q) $\vec{F} = \langle x^4y, -2x^3y^2, z^2\rangle$

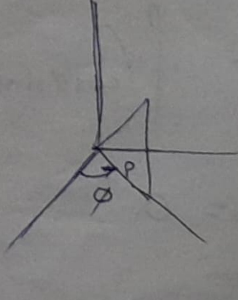
$z = 0$
$z = h$
$x^2+y^2 = R^2$

Sol:

$\rho = R$

for $x^2+y^2 = R^2$;

$ds = \rho\, d\phi\, dz\, \hat{\rho}$



for $z=0$;

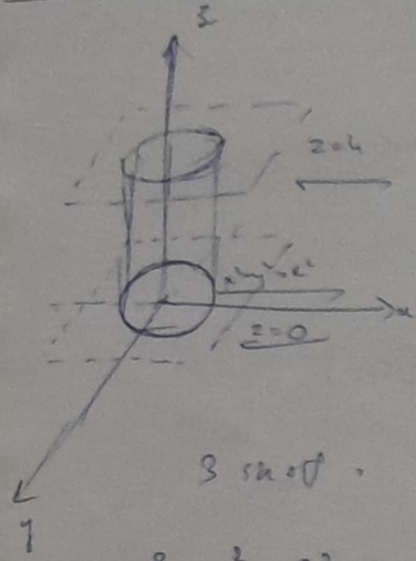
$ds = \rho\, d\phi\, d\rho\, \hat{z}$

for $z=h$;

$ds = -\rho\, d\rho\, d\phi\, \hat{z}$

~~$\Rightarrow \vec{F}\cdot d\vec{s} = x^4 y$~~

for divergence, $\iiint \vec{\nabla}\cdot\vec{F}\, d\tau = \iint_S \vec{F}\cdot d\vec{s}$

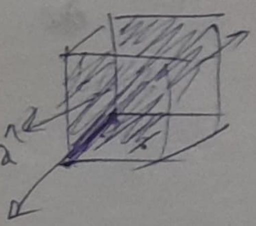


3 surf.

$\underline{\underline{x^2+y^2 = R^2}}$

$\rho^2 = R^2$

$\underline{\underline{\rho = R}}$ $\quad \hat{\rho} \downarrow ds_1$



$\hat{r}\, \underline{\underline{dl_\phi\, dl_z}}$

$\hat{\rho}\, \rho\, d\phi\, dz$

$ds_1 = \hat{\rho}\, \rho\, d\phi\, dz \qquad \rho = R \ -①$

$ds_2 = \hat{z}\, \rho\, d\phi\, d\rho \qquad z = h \ -②$

$\qquad\qquad\qquad\qquad\qquad z = 0 \ -③$

$ds_3 = -\hat{z}\, \rho\, d\phi\, d\rho$

$F = \langle 0, -2\rho^2 \cos\phi \sin\phi\, z, 2 + z^2 \rangle$

$F\cdot ds_3 = \iint -2\rho^2 \cos\phi \sin\phi\, z \cdot d\rho\, dz$

$F\cdot ds_2 = \iint \rho(2+z^2)\, d\phi\, d\rho$

$= \iint (2\rho + \rho z^2)\, d\phi\, d\rho$

$= \int_0^R \int_0^{2\pi} (2\rho + \rho h^2)\, d\phi\, d\rho$

$= \int_0^R 2\pi(2\rho + \rho h^2)\, d\rho$

$= 2\pi(2+h^2)\cdot \frac{R^2}{2}$

$F\cdot ds_2 = \pi R^2 (2+h^2)$

$F\cdot ds_3 = \int \left(\cos\phi \sin\phi \cdot (-2\rho^4)\cdot \int z\, dz\right)\cdot d\phi$

$= 0$

$\boxed{F\cdot ds_1 = 0}$

$\boxed{F\cdot ds_3 = 0}$

$\boxed{F\cdot ds_2 = \pi R^2 (2+h^2)}$

Curl :

$\vec{\nabla} \times \vec{A} \neq 0$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad [\text{Cart}]$$

$$\Rightarrow \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \quad [\text{Cyl}]$$

$$\Rightarrow \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\, \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

$[\text{Sph}]$

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(i) $r^2 = 1$

$\Rightarrow x^2 + y^2 + z^2 = 1$ (Sphere)

(ii) $r^2 = \csc^2\theta$

$\Rightarrow r^2 = \frac{1}{\sin^2\theta}$

$\Rightarrow r^2 \sin^2\theta = 1$

$\Rightarrow r^2 \sin^2\theta (\sin^2\phi + \cos^2\phi) = 1$

$\Rightarrow y^2 + x^2 = 1$

(OR) $r^2(1 - \cos^2\theta) = 1$

$\Rightarrow r^2 - r^2\cos^2\theta = 1$

$\Rightarrow x^2 + y^2 + z^2 - z^2 = 1$

$\Rightarrow x^2 + y^2 = 1$

Q) $\vec{V} = \rho \cdot z \cdot \hat{z}$

Prove divergence Theorem for cylinder bounded by radius $\rho = 2$ and $z = 0, z = 2$

$$\iiint (\vec{\nabla} \cdot \vec{V})\, d\tau = \iint \vec{V} \cdot d\vec{s}$$

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial z}(\rho z)$$

$$= \rho$$

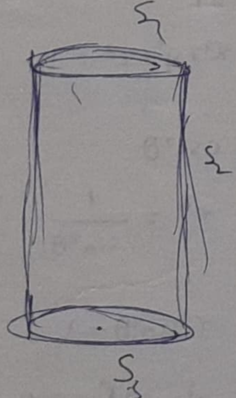
$$\iiint \rho \, d\tau$$

$$d\tau = d\rho \cdot \rho \, d\phi \, dz$$

$$\Rightarrow \int_{\rho=0}^{\rho=2}\int_{\phi=0}^{\phi=2\pi}\int_{z=0}^{z=2} \rho^2 \, d\rho \, d\phi \, dz$$

$$\Rightarrow \int_{z=0}^{z=2}\int_{\phi=0}^{\phi=2\pi} \frac{8}{3} \, d\phi \, dz$$

$$\Rightarrow \int_{z=0}^{z=2} \frac{16\pi}{3} \, dz = \frac{32\pi}{3}$$



$\cancel{dS_1} =$

$$dS_3 = \rho \, d\phi \, d\rho \, \hat{z}$$
$$dS_1 = -\rho \, d\phi \, d\rho \, \hat{z}$$
$$dS_2 = \rho \, d\phi \, dz \, \hat{\rho}$$

$$\iint \vec{V}\cdot d\vec{S_2} = 0$$

$$\iint \vec{V}\cdot d\vec{S_1} \Rightarrow \int_{z=0}^{z=2}\int_{\rho=0}^{\rho=2} -\rho^2 z \, d\phi \, d\rho \quad (z=2)$$

$$\Rightarrow \frac{32\pi}{3}$$

$$\iint \vec{V}\cdot d\vec{S_3} \Rightarrow \int_{z=0}^{z=0}\int_{\rho=0}^{\rho=2} \rho^2 z \, d\phi \, d\rho \quad (z=0)$$

Hence Proved

Q) $\vec{A} = (1+r)\hat{z}$

Calculate flux when radius = 1

$$\iint \bar{A}\cdot d\bar{S}$$

$dS = dl_\phi \, dl_\theta \, \hat{z}$

$= r\sin\theta \, d\phi \, . \, r\, d\theta \, \hat{z}$

$$= \iint (1+r)(r^2\sin\theta)\, d\phi\, d\theta$$

$$= \int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\pi/2} (r^2\sin\theta) + (r^2\sin\theta)\, d\theta\right) d\phi$$

$$= \int_{\phi=0}^{\phi=2\pi} (1+r)(r^2)\left(\int_{\theta=0}^{\theta=\pi/2}\sin\theta\, d\theta\right) d\phi$$

$$= \int_{\phi=0}^{\phi=2\pi}(1+r)\, r^2\, d\phi$$

$$= (1+r)(r^2).\ \phi\,(2\pi)$$

$$= 4\pi$$

$-\cos\theta\Big]_0^{\pi/2}$

$\cos\theta\Big]_{\pi/2}^{0}$

$1-0=1$

$$\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi/2}(1+r)\, r^2\sin\theta\, d\phi\, d\theta\, \cos\theta$$

$$= \int_{\phi=0}^{\phi=2\pi}(1+r)\, r^2\left(\int_{\theta=0}^{\theta=\pi/2}\sin\theta\cos\theta\, d\theta\right) d\phi$$

$$= \int_{\phi=0}^{2\pi}(1+r)\, r^2 \cdot \frac{1}{2}\, d\phi$$

$$= 2\pi$$

$$\int_0^{\pi/2}\frac{\sin 2\theta}{2}\, d\theta$$

$$= -\frac{\cos 2\theta}{4}\Big]_{\pi/2}^{0}$$

$$= \frac{1}{4}(1-(-1))$$

$$= \frac{1}{2}$$

Theorem :

STOKE'S THEOREM

$$\iint (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{l}$$

Revision

Q) $\vec{A} = \rho\hat{\rho} + \rho\cos\phi\,\hat{\phi} + z\hat{z}$

Rotational or not @ $\left(1, \frac{\pi}{4}, 1\right)$

$\longrightarrow$ Condition $\rightarrow$ $\vec{\nabla} \times \vec{v} \neq 0$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho & \rho^2\cos\phi & z^2 \end{vmatrix} = \cancel{\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 1 & \frac{1}{\sqrt{2}} & 1 \end{vmatrix}}$$

$$= \hat{z}(2\cos\phi)$$

$$= \frac{1}{\sqrt{2}}\hat{z} \times 2 = \sqrt{2}\,\hat{z}$$

$\rightarrow$ Rotational

REVISION

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cdot\cos\phi & \sin\theta\cdot\sin\phi & \cos\theta \\ \cos\theta\cdot\cos\phi & \cos\theta\cdot\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$\cancel{dl_\phi = r\sin\theta\, d\phi}$

$\cancel{dl}$

$dl_\rho = d\rho$	$dl_r = dr$
$dl_\phi = \rho\, d\phi$	$dl_\phi = r\sin\theta\, d\phi$
$dl_z = dz$	dl_θ

~~REVISION~~

$$\vec{\nabla} \times \bar{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

In cylindrical,

$\rho = \sqrt{x^2+y^2}$

$\tan\theta = y/x$

$\hat{\phi} \times \hat{\rho} = -\hat{z}$

In spherical,

$r = \sqrt{x^2+y^2+z^2}$

$\hat{\phi} \times \hat{r} = \hat{\theta}$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\cdot\sin\phi & \cos\theta \\ \cos\theta\cdot\cos\phi & \cos\theta\cdot\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$dl_\rho = d\rho$
$dl_\phi = \rho\, d\phi$
$dl_z = dz$

$dl_r = dr$
$dl_\theta = r\, d\theta$
$dl_\phi = r\sin\theta\, d\phi$

$$dT = \vec{\nabla} T \cdot \vec{dl}$$

$$\int_a^b dT = \int_a^b \vec{\nabla} T \cdot \vec{dl} = T_{(b)} - T_{(a)}$$ (Fundamental Th of Gradient)

$\vec{\nabla}\cdot\vec{v} = 0 \Rightarrow \vec{v}$: Solenoidal
$\vec{\nabla}\cdot\vec{v} \neq 0 \Rightarrow \vec{v}$: Divergent
[$\vec{\nabla}\cdot\vec{v} > 0$ @ P $\Rightarrow$ P is a source for $\vec{v}$.]

$\vec{\nabla}\times\vec{v} = 0 \Rightarrow \vec{v}$: Irrotational
$\vec{\nabla}\times\vec{v} \neq 0 \Rightarrow \vec{v}$: Rotational

$$\int_V (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_S v \cdot ds$$ (Fund. Theorem for Divergence)

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\cdot\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z} \quad \text{(cyl.)}$$

$$= \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta} + \phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \quad \text{(sph)}$$

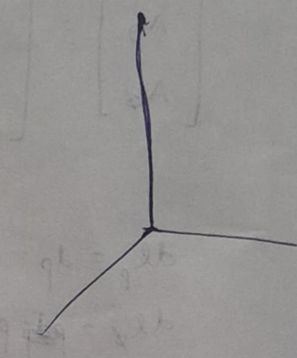
$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\cdot\frac{\partial}{\partial\phi}(A_\phi) + \frac{\partial}{\partial z}(A_z)$$

$$\iint (\vec{\nabla}\times\vec{F})\cdot d\vec{s} = \oint \vec{F}\cdot\vec{dl}$$ [Stoke's Theorem]

Q) $\vec{F} = x^2\hat{x} + 2x\hat{y} + z^2\hat{z}$

for surface, $4x^2+y^2=4$, $z=0$

$\vec{r} = \cos\theta\,\hat{x} + \sin\theta.\hat{y}$



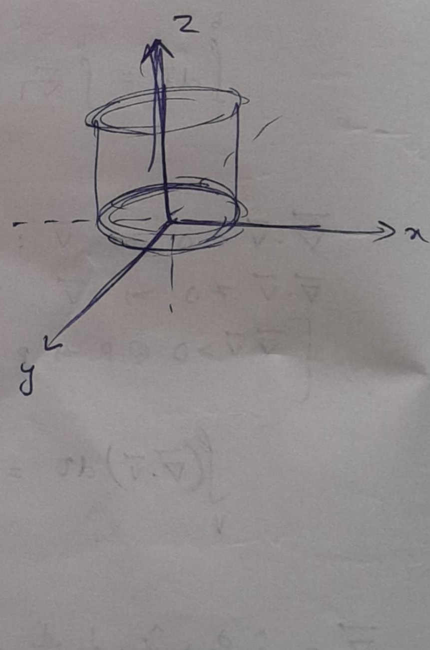
$\int \vec{F}\cdot d\vec{r}$

$\int (x^2\hat{x} + 2x\hat{y} + z^2\hat{z})\cdot(\cos\theta\,\hat{x} + \sin\theta)\,\hat{y}$

$\int x^2\cos\theta + 2x\sin\theta$

Q) $\vec{F} = \langle x^4y, -2x^2y^2, z^2 \rangle$

$z=0$
$z=h$
$x^2+y^2=R^2$



for $x^2+y^2=R^2$,

$\rho \Rightarrow$ const.

$\therefore ds = \rho\, d\phi\, dz\,\hat{\rho}$ — ①

for $z=0$,

$ds = \rho\, d\phi\, d\rho\,\hat{z}$ — ②

for $z=h$,

$ds = -\rho\, d\phi\, d\rho\,\hat{z}$ — ③

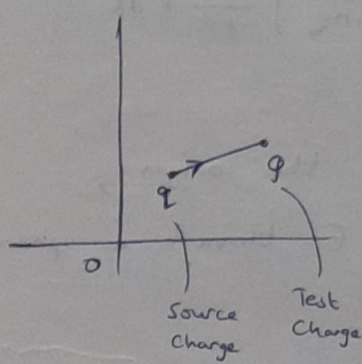
For Divergence, $\iiint \vec{\nabla}\cdot\vec{F}\,d\tau = \iint \vec{F}\cdot d\vec{s}$

Electrostatics :

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q\varphi}{r^2} \hat{r}$$

φ : Test Charge

ε_0 : 8.85×10^{-12} C^2/Nm^2 (Permittivity of free space)



q : rest

If we have several sources,

$$F(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i \varphi}{r_i^2} \hat{r}_i$$

$$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \dots \vec{F_n}$$

$$\therefore F(r) = \frac{\varphi}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r^2} \hat{r}_i$$

$$= \varphi \sum_{i=1}^{n} \frac{kq_i}{r_i^2} \hat{r}_i$$

$$F = -\vec{\nabla} V$$

$$F(r) = \varphi \times E$$

$$\therefore \vec{E} = \frac{kq}{r^2} \hat{r}$$

$$E(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{r^2} \hat{r} \, dq$$

$$\varphi = \int \lambda \, dl \quad (\text{for } 1D)$$
$$= \int \sigma \, dA \quad (\text{for } 2D)$$
$$= \int \rho \, dv \quad (\text{for } 3D)$$

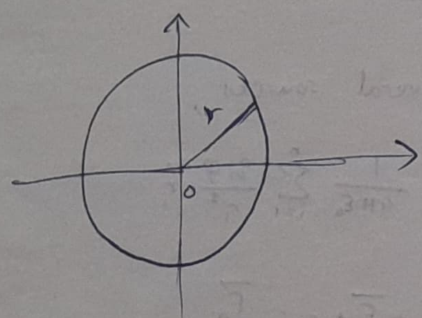
$$\frac{q}{L} = \frac{dq}{dl} \quad \text{(for 1D)}$$

$$\frac{q}{A} = \frac{dq}{dA} \quad \text{(for 2D)}$$

$$\frac{q}{V} = \frac{dq}{dV} \quad \text{(for 3D)}$$

$$E(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(r')}{r^2} \hat{r} \, dl'$$

Q) For a Point charge q at the origin, Calculate the flux of E through a spherical surface of radius r.



$$\text{Flux } \Phi = \oint E \cdot ds$$

$$\cancel{\oint\oint E \cdot ds = \iiint (\vec{\nabla} \cdot \vec{E}) \, d\tau}$$

$$\vec{E} = \frac{kq}{r^2} \hat{r}$$

$$\oint \bar{E} \cdot d\bar{A} = \int \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{r^2} \cdot \hat{r} \right) \cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int \frac{1}{4\pi\varepsilon_0} q \cdot \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\pi} \left(\int_0^{2\pi} (kq \, d\phi) \right) \sin\theta \, d\theta \quad \cancel{= \iiint \frac{\partial}{\partial r} \left(\frac{kq}{r^2} \right) d\tau}$$

$$= \int_0^{\pi} 2\pi kq \sin\theta \, d\theta \quad \cancel{= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} \frac{-2kq}{r^3} \times r^2 \sin\theta \, dr \, d\theta \, d\phi}$$

$$= \cancel{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{1}\left(\frac{-2kq}{r}\right)dr\right)\sin\theta \, d\theta \, d\phi}$$

$$\cancel{\int_0^1 \frac{-2kq}{r}\,dr} \qquad \cancel{2kq\,\ln(r)\Big]_1^0}$$

$$= \cancel{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}}$$

$$= 2\pi kq \int_0^{\pi} \sin\theta \, d\theta$$

$$-\cos\theta\Big]_0^{\pi} = \cos\theta\Big]_{\pi}^{0} = 1-(-1) = 2$$

$$= 2\pi kq \,(1+(-1))$$

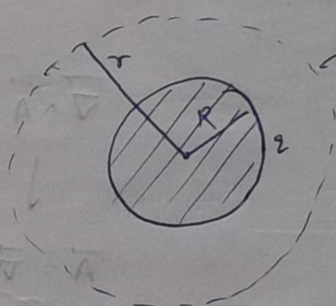
$$= q/\varepsilon_0$$

$$\phi = \iint E\cdot dS = \frac{q_{enc}}{\varepsilon_0}$$

$\phi \to 0$ to 2π

$\theta \to 0$ to π

Q) Find the field outside a uniformly charged solid sphere of Radius R and total charge q.



$$\cancel{\frac{kq}{r^2}}$$

$$\iint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \quad \text{(for a point charge)}$$

$$E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\Rightarrow E = \frac{q}{4\pi r^2 \cdot \varepsilon_0} = \frac{kq}{r^2}$$

$$\boxed{\therefore E = \frac{kq}{r^2}}$$

$$\iiint_V (\vec{\nabla}\cdot\vec{E})\, d\tau = \frac{1}{\varepsilon_0}\left(\iiint_V \rho\, d\tau\right)$$

$$\Rightarrow \int_V (\vec{\nabla}\cdot\vec{E})\, d\tau = \int_V \frac{\rho}{\varepsilon_0}\, d\tau$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\cancel{\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\frac{kq}{r^2}\hat{r} = \frac{-2kq}{r^3} = \frac{-2q}{4\pi\varepsilon_0 r^3}}$$

$$= \int_a^b \vec{E}\cdot d\vec{l}$$

$$= \int_a^b \frac{kq}{r^2}\, dr$$

$$= \left[-\frac{kq}{r}\right]_a^b$$

$$= \left.\frac{kq}{r}\right]_b^a$$

$$= \frac{kq}{a} - \frac{kq}{b}$$

$$= kq\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\frac{r^{-2+1}}{-2+1} = \frac{-1}{r}$$

$$\vec{\nabla}\times\vec{A} = 0$$

$$\downarrow$$

$$\vec{A} = \vec{\nabla}T$$

$a=b$ (Closed Path)

$$\iint_S (\vec{\nabla}\times\vec{E})\, d\vec{s} = 0 = \oint (\vec{E})\, dl$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = 0 \rightarrow \vec{\nabla}\times(\vec{\nabla}(\,,\,)) = 0$$

✓

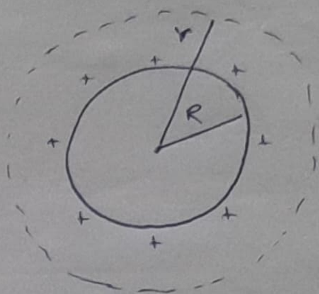
$$\boxed{\therefore \vec{E} = -\vec{\nabla}V}$$

Potential

$$\int_a^b (\nabla V \cdot \vec{dl}) = V_b - V_a$$

$$= -\int_a^b \vec{E} \cdot \vec{dl}$$

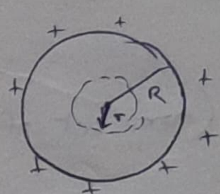
Q) E inside and outside a Spherical Shell



$$\vec{E} \cdot \vec{ds} = \frac{q_{enc}}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{kq}{r^2}$$



$$\vec{E} \cdot \vec{ds} = \frac{q_{enc}}{\varepsilon}$$

$$\Rightarrow \vec{E} \cdot \vec{ds} = 0$$

$$\Rightarrow E = 0$$

$V_{r>R}$, $r > R$, $E \neq 0$

$V_{r<R}$, $r < R$, $E_{in} = 0$

[Equipotential Surface]

Find $V_{r>R}$ and $V_{r<R}$ (H.W)

$$V_{(b)} - V_{(a)} = -\int_a^b E \cdot dl$$

for $r > R$, $a = \infty$, $b = r$

$$V_r - V_\infty = V_r - 0 = -\int_\infty^r \frac{kq}{r^2}\,dr = \frac{kq}{r}$$

for $r < R$, $a = R$, $b = r$

for $r < R$:

$$V_r - V_R = -\int_R^r 0\,dl$$

$$\Rightarrow V_r - V_R = 0$$

$$\Rightarrow \boxed{V_R = V_r = \frac{kq}{r}}$$

6/10/23

$$W = \int_a^b F \cdot dl$$

$$= -q \int_a^b E \cdot dl$$

$$= q(V_b - V_a)$$

$$W = q(V_r - V_\infty)$$

$$\boxed{W = q\,V(r)}$$

$$V_b - V_a = \frac{W}{q}$$

→ Calculating :

$$V(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r}$$ (for single Point charge)

$$V(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i}$$ (for set of point charges)

$$V(r) = \cancel{\frac{1}{4\pi\varepsilon_0}} \frac{1}{4\pi\varepsilon_0} \int_a^b \frac{1}{r}\,dq$$ (for continuous charge dist)

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{r}\,dE'$$ (for Volume charge)

→ Poisson's and Laplace's Equation :

$$\vec{E} = -\vec{\nabla} V$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \vec{\nabla} \times \vec{E} = 0$$

$$\boxed{\nabla^2 V = \frac{-\rho}{\varepsilon_0}}$$

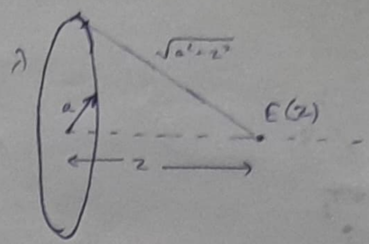
Poisson's Equation

If $\rho = 0$

$$\Rightarrow \boxed{\nabla^2 V = 0}$$

Laplace Equation

Q) Charged Ring → Radius 'a'.



$$E = \frac{kq}{\left(\sqrt{a^2+z^2}\right)^2}$$

$$q = \lambda \cdot 2\pi a$$

H.W: Find 'V'

$$\rightarrow E = \frac{k\lambda \times 2\pi a}{(a^2+z^2)}$$

$$\vec{E} = \frac{2k\pi\lambda a}{a^2+z^2}$$

Q) Infinitely long ~~cylindrical~~ cylinder along -ve z direction has $\rho = 3$ cm and contains a surface charge density

$$\rho_s = 2e^z \; nC/m^2$$

(a) Find Total Charge

HW (b) How much flux leaving the surface $\rho = 1$ cm,

$-2\text{ cm} < z < -1\text{ cm}$, $90° < \varphi < 180°$.

(i) $q = \iint \rho_s \, ds$

$ds = \rho \, d\phi \, dz$

$\phi \rightarrow 0$ to 2π

$z \rightarrow -\infty$ to 0

$$q = \iint 2e^z \rho \, d\phi \, dz$$

$$= \int 2\left(\int e^z dz\right) \rho \, d\phi$$

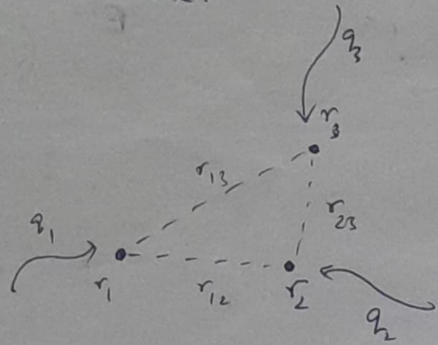
$$= \int_0^{2\pi} 6\left(\int_{-\infty}^{0} e^z dz\right) d\phi$$

$$= \int_0^{2\pi} 6 \, d\phi = 12\pi$$

$$e^z\Big]_{-\infty}^{0} = 1 - 0 = 1$$

$$\Rightarrow q = 12\pi$$

Q) Find Work Done.



$$W = W_1 + W_2 + W_3$$

$W_1 = 0 \because$ No other force

$$W_2 = q_2 V$$

$$= q_2 \cdot \frac{q_1}{4\pi\varepsilon_0 r_{12}}$$

$$W_3 = q_3 V + q_3 V$$

$$= q_3 \left[\frac{q_1}{4\pi\varepsilon_0 r_{13}} + \frac{q_2}{4\pi\varepsilon_0 r_{23}} \right]$$

$$W = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}} + \frac{q_2 q_3}{4\pi\varepsilon_0 r_{23}} + \frac{q_3 q_1}{4\pi\varepsilon_0 r_{31}}$$

$$= \frac{k q_1 q_2}{r_{12}} + \frac{k q_2 q_3}{r_{23}} + \frac{k q_3 q_1}{r_{31}}$$

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{r_{ij}}$$

$$\Rightarrow W = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

$$W = \frac{1}{2}\int \rho V \, d\tau$$

(or)

$$\boxed{W = \frac{\varepsilon_0}{2}\int E^2 \, d\tau}$$

$$W = \frac{1}{2}\int \sigma V \, dA$$

$[\because \rho = \varepsilon_0 \nabla \cdot E]$

H.W

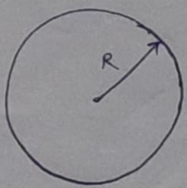
Q) Metallic Sphere

$r = 10\,cm$

$\sigma = 10 \cdot nC/m^2$

Calculate Electrical Energy Stored in the system.

Q) Find Energy of Uniformly charged spherical shell



$$W = \frac{1}{2}\int \sigma V \, dA$$

$$V = \frac{kq}{R}, \quad \sigma = \frac{Q}{V}$$

$$W = V \cdot \frac{1}{2}\int \sigma \cdot dA$$

$$= \frac{kq}{2R}\int \sigma \cdot dA$$

$\sigma \cdot dA = dq$

$\int \sigma \, dA = q$

$$\Rightarrow \boxed{W = \frac{kq^2}{2R}}$$

$$\boxed{W = \frac{q^2}{8\pi\varepsilon R}}$$

(or)

$$W = \frac{\varepsilon_0}{2}\int E^2 \, d\tau$$

$\cancel{= \frac{\varepsilon_0}{2} k^2 q^2 \int \frac{d\tau}{r^4}}$

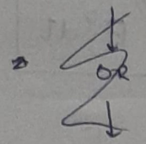
$\cancel{d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi}$

$\cancel{E = \frac{kq}{r^2}}$

$$\cancel{\Rightarrow W = \frac{\varepsilon_0}{2}(k^2q^2)\int_V \frac{r^2\sin\theta\, dr\, d\theta\, d\phi}{r^4}}$$

$\theta \to 0$ to π

$\phi \to 0$ to 2π → correct



$$\oint E \cdot ds = \frac{q}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E^2 = \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2$$

$$W = \frac{\varepsilon_0}{2}\int_R^\infty E^2\, 4\pi r^2\, dr$$

$$\Rightarrow W = \frac{\varepsilon_0}{2}\left(\int_R^\infty \frac{q^2 dr}{4\pi r^2}\right)\cdot\frac{1}{\varepsilon_0^2} = \frac{kq^2}{2}\int_R^\infty \frac{1}{r^2}\,dr = \frac{kq^2}{2R}$$

H.W

A) $W = \frac{1}{2}\int V\sigma\, dA$

$= \frac{1}{2} q_{net} \cdot V$ $\quad \because \sigma \to$ const.

$V \to$ const.

$= \frac{1}{2} \times 10 \times 4\pi\left(\frac{1}{100}\right)\cdot\frac{q}{4\pi\varepsilon_0 R} \times 10^{-9}$

$= \frac{\sigma A}{20R\varepsilon} \times 10^{-9}$

$q_{net} = \sigma . A$

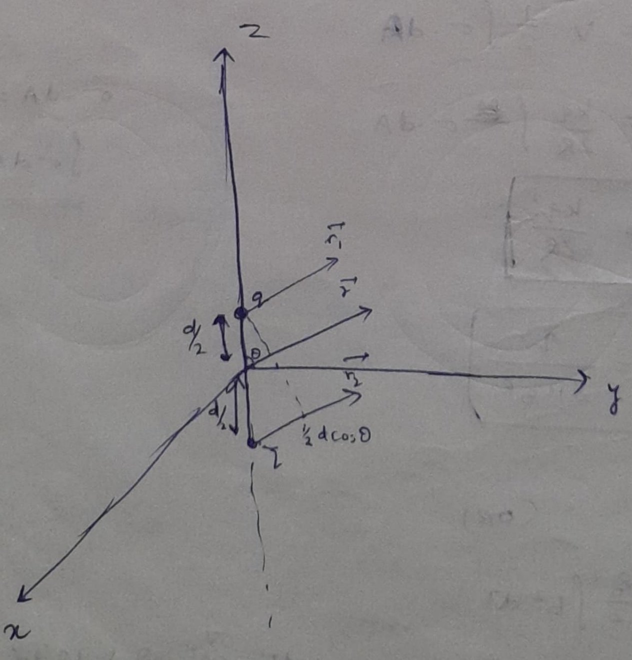
$\Rightarrow q_{net} = 1.257\ nC$

$V = \frac{kq}{R}$

$\Rightarrow V = 114.75\ V$

$$\boxed{W = 72.12\ J}$$

→ Principle of Superposition is not applicable for E.



$$V = \frac{q}{4\pi\varepsilon_0 r_1} - \frac{q}{4\pi\varepsilon_0 r_2}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$ [$r >> d$]

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$

$r_1 = r - \frac{d}{2}\cos\theta$

$r_2 = r + \frac{d}{2}\cos\theta$

$$\Rightarrow V = \frac{q}{4\pi\varepsilon_0}\left(\frac{\cancel{r} + \frac{d}{2}\cos\theta - \cancel{r} + \frac{d}{2}\cos\theta}{(r - \frac{d}{2}\cos\theta)(r + \frac{d}{2}\cos\theta)}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{d\cos\theta}{r^2 - \frac{d^2}{4}\cos^2\theta}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{d\cos\theta}{1 - \frac{d^2}{4r^2}\cos^2\theta}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}\left(d\cos\theta\right)$$ $\vec{p}$: -ve to +ve

$$= \frac{q\,d\cos\theta}{4\pi\varepsilon_0 r^2}$$ [$\vec{p} = q\vec{d}$]

$$= \frac{k d\cos\theta\, q}{r^2}$$

$$\Rightarrow V = \frac{kp\cos\theta}{r^2}$$

$$\Rightarrow \boxed{V = \frac{k\cdot(\vec{p}\cdot\hat{r})}{r^2}}$$

$E = -\nabla\cdot V$ [H.W]

Q) $d = 10^{-11}$ m, z axis symmetrically arranged

Centre of dipole : $z = 0$

$\vec{E}$ @ $P(3, 4, 12)$

A) $\vec{r} = 3\hat{x} + 4\hat{y} + 12\hat{z}$

$r = |\vec{r}| = \sqrt{9+16+144} = 13$

$$V = \frac{k.(\vec{p}\cdot\hat{r})}{r^2}$$

$\vec{p} = q.\vec{d}$

$= 1.6 \times 10^{-19} \times 10^{-11}$

$= 1.6 \times 10^{-30}\,\hat{z}$

$\hat{r} = \dfrac{3\hat{x} + 4\hat{y} + 12\hat{z}}{13}$

$$= \frac{9\times10^{9} \times \frac{12}{13} \times 1.6\times10^{-30}}{169}$$

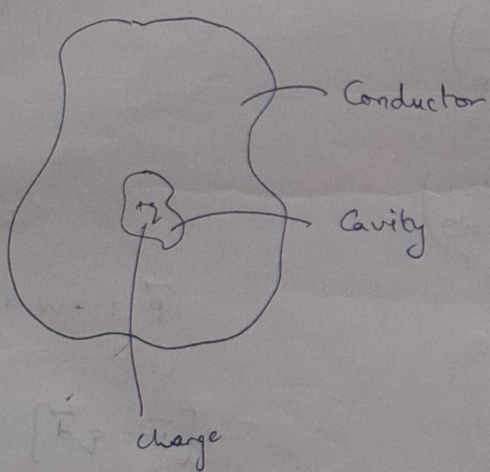
$= 0.07865 \times 10^{-21}$

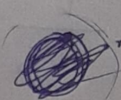
$= 7.865 \times 10^{-23}$ V

$\vec{F} =$ ___

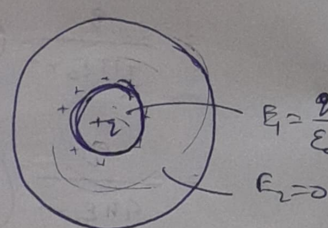
* [Conductors and Insulators PYQ] *

Q)

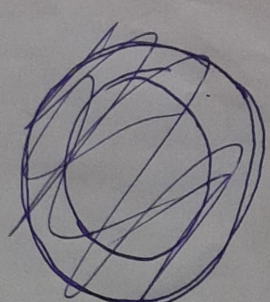


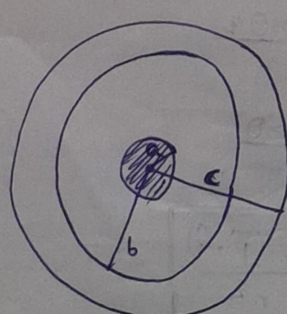


$\equiv$



Q)

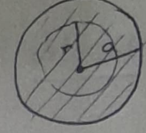




~~E.d~~ $E \cdot A = \dfrac{q_{enc}}{\varepsilon_0}$

$$\int E \cdot dA = \frac{q_{enc}}{\varepsilon_0}$$

for $r<a$:



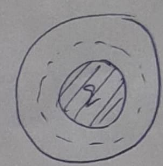
$$E \cdot 4\pi r^2 = \frac{\rho_v \cdot \frac{4}{3}\pi r^3}{\varepsilon_0}$$

$$\Rightarrow \boxed{E = \frac{\rho_v \cdot r}{3\varepsilon_0}}$$

$$\frac{q}{V} = \frac{dq}{dV} = \rho$$

for $a<r<b$:

$$\oint E \cdot dA = \frac{q_{enc}}{\varepsilon_0}$$



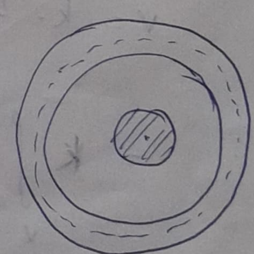
$$E \cdot 4\pi r^2 = \frac{\rho_v V}{\varepsilon_0}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{\rho_v \cdot \frac{4}{3}\pi a^3}{\varepsilon_0}$$

$$\Rightarrow \boxed{E = \frac{\rho_v \cdot a^3}{3 r^2 \varepsilon_0}}$$

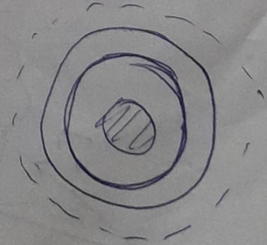
for $b<r<c$:

$$\boxed{E = 0}$$



for $r>c$:

$$\boxed{E = \frac{kq}{r^2}}$$

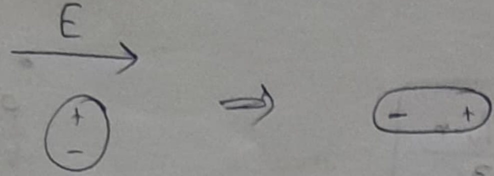


$$E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{kq}{r^2}$$

$$[\text{where } q = \rho_v V]$$

→ Induced Dipoles :



$p \propto E$

$\Rightarrow \boxed{p = \alpha E}$ → Polarisability

$\left[\frac{\alpha}{4\pi r^3}\right] \to [m^3]$

$\boxed{\tau = \vec{p} \times \vec{E}}$

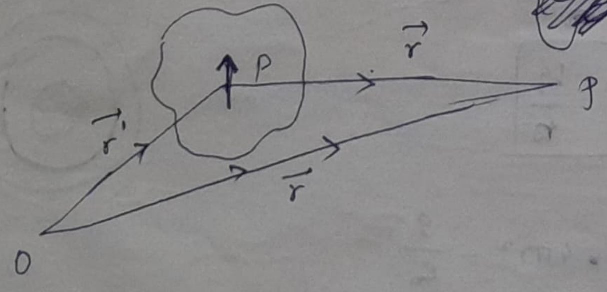
$\vec{p} = \int \vec{P}\, d\tau$

$\vec{P} = \frac{\vec{p}}{V}$

$V = k \cdot \frac{\vec{p} \cdot \hat{r}}{r^2}$

$= k \int_V \frac{\vec{P} \cdot \hat{r}}{r^2}\, d\tau'$

$\nabla'\left(\frac{1}{r}\right) = \frac{-1}{r^2}\hat{r}$



$\Rightarrow V = k \int_V \left(\vec{P} \cdot \nabla' \frac{1}{r}\right) d\tau$

$$\left[\text{As } \nabla' \cdot \frac{\vec{P}}{r} = \nabla'\left(\frac{1}{r}\right) \cdot \vec{P} + \frac{1}{r}(\nabla' \cdot P) \right]$$

$$\Rightarrow V = k \left[\int_V \nabla' \cdot \frac{P}{r} \, d\tau' - \int_V \frac{1}{r} (\nabla' \cdot P) \, d\tau' \right]$$

$$\Rightarrow V = k \left[\oint_S \frac{1}{r} \cdot P \cdot dA' - \int_V \frac{1}{r} \cdot (\nabla' \cdot P) \, d\tau' \right]$$

$$= k \left[\oint_S \frac{\sigma_b}{r} \, dA' - \oint_V \frac{\rho_b}{r} \, d\tau' \right]$$

	D	I
+	$\vec{P}$	$\nabla'(\frac{1}{r})$
−	$\nabla'(P)$	$\frac{1}{r}$

$$\left[\begin{aligned} \sigma_b &= \vec{P} \cdot \hat{n} \\ \rho_b &= -\vec{\nabla} \cdot \vec{P} \end{aligned} \right]$$

$$\rho = \rho_b + \rho_f$$

Electric Displacement, $D = \varepsilon_0 E + P$

Gauss Law :

$$\nabla \cdot D = \rho_f$$

$$\oint D \cdot dA = q_{f\,enc.}$$

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

$$\vec{D} = \varepsilon_0 (1 + \chi) \vec{E}$$

$$\Rightarrow \vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$$

Q) Point charge q enclosed in Linear Dielectric medium.

Calculate $\vec{E}$, $\vec{D}$ and $\vec{P}$,

Bound Surface Charge Density ρ_{sb},

Bound Volume Charge Density ρ_{vb}

$$\oint_S D \cdot d\vec{S} = q$$

$$\Rightarrow 4\pi r^2 D_r = q$$

$$\Rightarrow D_r = \frac{q}{4\pi r^2} \vec{a_r}$$

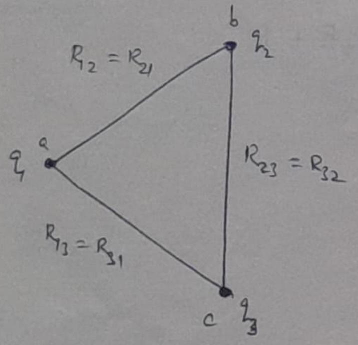
$$\& \quad E = \frac{q}{4\pi \varepsilon_0 r^2} \vec{a_r} \cdot \frac{1}{\varepsilon_r}$$

$$\overline{P} = \overline{D} - \varepsilon_0 \overline{E}$$

$$= \frac{q}{4\pi \varepsilon_r r^2} (\varepsilon_r - 1) \vec{a_r}$$

As $\nabla \cdot \overline{P} = 0$, $\rho_{vb} = 0$

→ Potential Energy :



$$W = W_3 + W_2 + W_1$$

$$= 0 + q_2 V_{b,c} + q_1 (V_{a,c} + V_{a,b})$$

$$= \frac{1}{4\pi \varepsilon_r} \left[\frac{q_2 q_3}{R_{23}} + \frac{q_1 q_3}{R_{13}} + \frac{q_1 q_2}{R_{12}} \right]$$

$$W = \frac{1}{2} \left[q_1 (V_{a,c} + V_{a,b}) + q_2 (V_{b,a} + V_{b,c}) + q_3 (V_{c,a} + V_{c,b}) \right]$$

Total Energy

$$W = \frac{1}{2} \left[q_1 V_1 + q_2 V_2 + q_3 V_3 \right] = \frac{1}{2} \sum_{i=1}^{3} q_i V_i$$

$$W = \frac{1}{2}\int_v \rho_v V \, dv$$

$dv \rightarrow$ Volume

$$\boxed{W = \frac{1}{2}\int_S \rho_s V \, ds}$$

From Gauss Law,

$$W = \frac{1}{2}\int_v V (\nabla \cdot \vec{D}) \, dv$$

As $V(\nabla \cdot \vec{D}) = \nabla \cdot (V\vec{D}) - \vec{D} \cdot (\nabla V)$

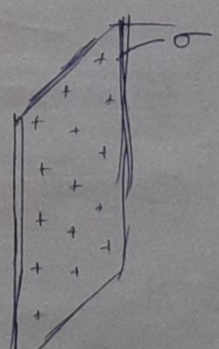
$$\Rightarrow W = \frac{1}{2}\left[\int_v \nabla (V \cdot \vec{D}) \, dv - \int_v \vec{D} \cdot (\nabla V) \, dv\right]$$

And $\int_v \nabla (V \cdot \vec{D}) \, dv = \oint_S V \cdot \vec{D} \cdot d\vec{S}$

$$W = -\frac{1}{2}\int_v D \cdot (\nabla V) \, dv = \frac{1}{2}\int \vec{D} \cdot \vec{E} \, dv$$

$\therefore W = \frac{1}{2}\vec{D} \cdot \vec{E} = \frac{1}{2}\varepsilon E^2 = \frac{1}{2\varepsilon} D^2$ (Energy /Unit Volume) [Energy Density]

→ Infinite Plane :

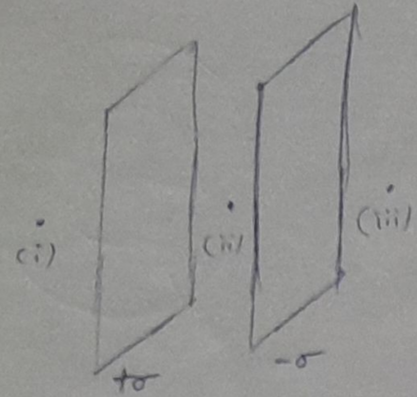


$$\oint E \cdot dA = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow \oint E \cdot dA = 2A|E|$$

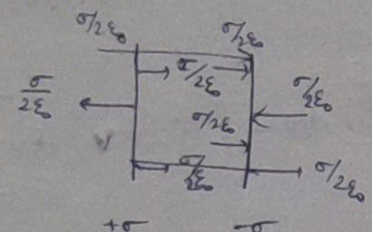
$$\Rightarrow 2A|E| = \frac{\sigma A}{\varepsilon_0}$$

Q)



Field in (i) and (iii) = 0

Field b/w plates [(ii)] = $\frac{\sigma}{\varepsilon}$



CAPACITORS :

$$C = \frac{Q}{V}$$

$$\Rightarrow \underline{Q = CV}$$

$$W = \frac{1}{2}CV^2$$

Q) 20nC @ (1,0,0)

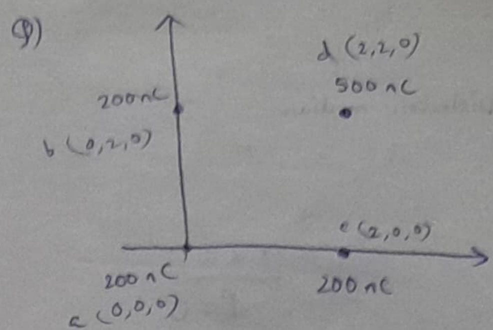
-20nC @ (0,1,0)

Determine $\vec{E}$ at (0,0,1)

$$E = \frac{kq}{r^2}$$

$$E_{net} = E_1 + E_2$$

9)



$$W = \frac{1}{2}\sum_{i=1}^{4} q_i V_i$$

$$W = \frac{1}{2}\left[q_1V_1 + q_2V_2 + q_3V_3 + q_4V_4\right]$$

$$= \frac{1}{2}\left[q_0(V_1+V_2+V_3) + \varphi V_4\right]$$

$q = 200\,nC$
$\varphi = 500\,nC$

$$V_1 = V_{(a,b)} + V_{(a,c)} + V_{(a,d)}$$

$$= \frac{kq_aq_b}{r_{ab}} + \frac{kq_aq_c}{r_{ac}} + \frac{kq_aq_d}{r_{ad}}$$

$$= \frac{kq^2}{2} + \frac{kq^2}{2} + \frac{kq\varphi}{2\sqrt{2}}$$

$$\Rightarrow V_1 = \frac{kq}{2}\left[2q + \frac{\varphi}{\sqrt{2}}\right]$$

$$V_2 = \frac{kq^2}{2} + \frac{kq^2}{2\sqrt{2}} + \frac{kq\varphi}{2}$$

$$\Rightarrow V_2 = \frac{kq}{2}\left[q + \frac{q}{\sqrt{2}} + \varphi\right]$$

$$V_3 = \frac{kq^2}{2} + \frac{kq^2}{2\sqrt{2}} + \frac{kq\varphi}{2}$$

$$\Rightarrow V_3 = \frac{kq}{2}\left[q + \frac{q}{\sqrt{2}} + \varphi\right]$$

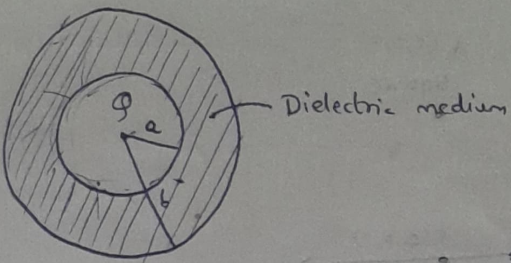
$$V_4 = \frac{kq\varphi}{2\sqrt{2}} + \frac{k\varphi q}{2} + \frac{k\varphi q}{2}$$

$$= \frac{kq\varphi}{2}\left[2 + \frac{1}{\sqrt{2}}\right]$$

$$W = \frac{1}{2}\left\{q\left[\frac{kq}{2}\left(4q + \sqrt{2}q + 2\varphi + \frac{\varphi}{\sqrt{2}}\right)\right] + \varphi\left[\frac{kq\varphi}{2}\left(2+\frac{1}{\sqrt{2}}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{kq^2}{2}\left(q(4+\sqrt{2}) + \varphi\left(2+\frac{1}{\sqrt{2}}\right)\right) + \frac{kq\varphi^2}{2}\left(2+\frac{1}{\sqrt{2}}\right)\right\}$$

q)



$\underline{r < a:}$

$\rho = 0$

$D = 0$

$E = 0$

$\underline{r > a:}$

$D \neq 0$

$$\Rightarrow D = \frac{q}{4\pi r^2}\hat{r}$$

$$\vec{D} = \varepsilon\vec{E}$$

$$= \varepsilon \times \frac{kq}{r^2}\hat{r}$$

$$= \varepsilon \times \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$= \frac{q}{4\pi r^2}\hat{r}$$

$\underline{a < r < b:}$

$$\vec{E} = \frac{q}{4\pi\varepsilon r^2}\hat{r}$$

$\underline{r > b:}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$\underline{r > b:}$

$$V = -\int_\infty^0 \vec{E}\cdot d\vec{l}$$

$$= -\left[\int_\infty^b \vec{E}\cdot d\vec{l} + \int_b^a \vec{E}\cdot d\vec{l} + \int_a^0 \vec{E}\cdot d\vec{l}\right]$$

$$\Rightarrow V = -\int_\infty^b \vec{E}\cdot d\vec{l} - \int_b^a \vec{E}\cdot d\vec{l}$$

$$= -\int_\infty^b \frac{q}{4\pi\varepsilon_0 r^2}dr - \int_b^a \frac{q}{4\pi\varepsilon r^2}dr$$

$$= \left(\left[\frac{1}{r}\right]_\infty^b + \left[\frac{1}{r}\right]_b^a\right)kq$$

$$= \frac{q}{4\pi\varepsilon_0 b} + \frac{q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Rightarrow V = \frac{q}{4\pi}\left(\frac{1}{b\varepsilon_0} + \frac{1}{a\varepsilon} - \frac{1}{b\varepsilon}\right)$$

$\underline{\underline{\text{Total:}}}$

$$W = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}\,d\tau$$

$$= \frac{1}{2}\int_{V_1}\vec{D}\cdot\vec{E}\,d\tau + \frac{1}{2}\int_{V_2}\vec{D}\cdot\vec{E}\,d\tau + \frac{1}{2}\int_{V_3}\vec{D}\cdot\vec{E}\,d\tau$$

$$= \frac{1}{2}\left[\cancel{\int_0^a \vec{D}\cdot\vec{E}\,d\tau}^{\,0} + \int_a^b \vec{D}\cdot\vec{E}\,d\tau + \int_b^\infty \vec{D}\cdot\vec{E}\,d\tau\right]$$

$$= \frac{1}{2}\left[\int_a^b \vec{D}\cdot\vec{E}\,d\tau + \int_b^\infty \vec{D}\vec{E}\,d\tau\right]$$

$$= \frac{1}{2}\left[\int_a^b DE\,4\pi r^2\,dr + \int_b^\infty DE\,4\pi r^2\,dr\right]$$

$$= \frac{1}{2}\left[\int_a^b \frac{q^2}{4\pi\varepsilon r^2}dr + \int_b^\infty \frac{q^2}{4\pi\varepsilon_0 r^2}dr\right]$$

→ Magnetostatics :

Lorentz Force Law :

$$F = q(\vec{v} \times \vec{B})$$

Net Force (if $E \neq 0$) :

$$F = q(\vec{E} + \vec{v} \times \vec{B})$$

1 T = 10^4 Gauss
(SI)   (CGS)

Q) proton, $B = 1T$
$v = 10^6$ m/s

$$\frac{mv^2}{r} = q(\vec{v} \times \vec{B})$$

$$\frac{mv^2}{r} = qvB \quad (v \perp B)$$

$$\Rightarrow \frac{mv}{r} = qB$$

$$\Rightarrow \boxed{r = \frac{mv}{qB}} = 1\,cm \quad (10^{-2}\,m)$$

$$R = \frac{mv}{qB} = \frac{1.6 \times 10^{-27} \times 10^6}{1.6 \times 10^{-19}} = 10^{-2}\,m = 1\,cm$$

Circular Path. How? Proof (HW)

Q) q → velocity 'v'

moves 'dl' in 'dt'.

Work done by Magnetic force, F_{mag}

$$\cancel{dF = q(d\vec{v} \times \vec{B})} \qquad (\vec{v} \perp \vec{B})$$

$$\cancel{= q(d\vec{v})B}$$

$$\cancel{= \frac{q\,dv\,B}{dt}} \qquad \int dF = \int \frac{qB}{d'}$$

$$W = \int \vec{F} \cdot \vec{dl} \qquad \vec{dl} = \vec{v} \cdot dt$$

$$= \int q(\vec{v} \times \vec{B}) \cdot \vec{v}\,dt = 0$$

$\rightarrow \quad I = \lambda v$

$$\cancel{\text{mag.}}\ F_{mag.} = \int (\vec{v} \times \vec{B})\, dq$$

$$= \int (\vec{v} \times \vec{B})\, \lambda\, d\vec{l}$$

$$= \int (\vec{I} \times \vec{B})\, d\vec{l}$$

$$F_{mag.} = \int I\, (d\vec{l} \times \vec{B})$$

$$J = \rho v$$

$$F_{mag} = \int (\vec{v} \times \vec{B})\, \rho\, d\tau$$

$$= \int (\vec{J} \times \vec{B})\, d\tau$$

<u>H.W</u> : σ, $F_{mag.} = ?$ $\qquad k = \sigma v$ [$k \rightarrow$ current per unit length]

$$\Rightarrow F_{mag} = \int (\vec{k} \times \vec{B})\, dS$$

$\rightarrow$ Continuity Eq.

$$I = \int_S J \cdot dA_{\perp} = \int_S J \cdot dA$$

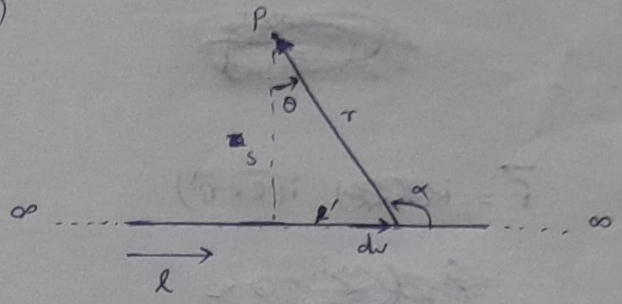
$$\int_S J \cdot dA = \int_V (\nabla \cdot J)\, d\tau$$

$$\int_V (\nabla \cdot J)\, d\tau = -$$

→ <u>Biot-Savart's law</u>:

$$B(r) = \frac{\mu_0}{4\pi}\int \frac{i \times \hat{r}}{r^2}\, dl'$$

Q)



$$B = \frac{\mu_0 i}{4\pi}\int \frac{\vec{dl'} \times \hat{r}}{r^2}$$

$$= \frac{\mu_0 i}{4\pi}\int \frac{(dl')\sin\alpha}{r^2}$$

$$\alpha = 90^\circ + \theta$$

$$= \frac{\mu_0 i}{4\pi}\int \frac{\cos\theta}{r^2}\, dl'$$

$$\cancel{= \frac{\mu_0 i}{4\pi}\int_{-\infty}^{\infty} \frac{s}{r^2}\, dl'}$$

$$\cancel{= \frac{\mu_0 i s}{4\pi}\int_{-\infty}^{\infty} \frac{1}{r^3}\, dl'}$$

$$\tan\theta = \frac{l'}{s}$$

$$l' = s\tan\theta$$

$$dl' = s(\sec^2\theta)d\theta$$

$$= \frac{\mu_0 i s}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{r^2} \times s\sec^2\theta\, d\theta$$

$$= \frac{\mu_0 i s}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{r^2\cos\theta}\, d\theta$$

$$r = s\sec\theta$$

$$= \frac{\mu_0 i}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{s}\, d\theta$$

$$= \frac{\mu_0 i}{4\pi s}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{\mu_0 i}{4\pi s}(\sin\theta]_{-\pi/2}^{\pi/2}$$

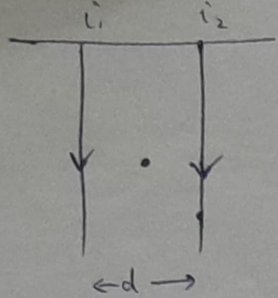
$$= \frac{\mu_0 i}{4\pi s}(1-(-1))$$

$$\boxed{|\vec{B}| = \frac{\mu_0 i}{2\pi s}}$$ [for only long wire]

$$B = \frac{\mu_0 i}{4\pi s}(\sin\theta]_{\theta_1}^{\theta_2}$$

$$B = \frac{\mu_0 i}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$

Q)



$$B_1 = \frac{\mu_0 i_1}{2\pi d} \odot$$

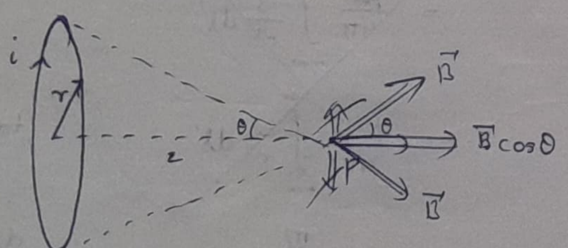
$$\vec{F} = i(\vec{l} \times \vec{B})$$

$$\Rightarrow \vec{F} = i_2\left(\vec{l} \times \frac{\mu_0 i_1}{2\pi d}\right)$$

$$f = \frac{\vec{F}}{l} = I_2\left(\frac{\mu_0 i_1}{2\pi d}\right)$$

$$= \frac{\mu_0 i_1 i_2}{2\pi d}$$

Q)



$$\vec{B} = \int_{-\theta}^{\theta} \vec{B}\cos\theta$$

if $z \to \infty$,

$$B = \frac{\mu_0 i}{2z^3}$$

$$= \int \frac{\mu_0 i}{4\pi z}\left(\frac{d\vec{l} \times \hat{r}}{r^2}\right)\cos\theta$$

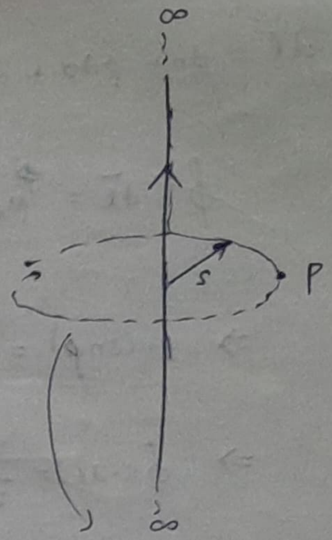
$\rightarrow$ $\vec{M} = i\vec{A}$

$$\Rightarrow \vec{M} = i(2\pi r^2)$$

$$\vec{B} = \frac{\mu_0 \vec{M}}{2\pi z^3}$$

$$\tau = \vec{M} \times \vec{B}$$

Magnetic Monopoles cannot exist.



Amperecal Loop

$$\oint \vec{B} \cdot d\vec{l}$$

$$= \oint \frac{\mu_0 i}{2\pi s} \cdot dl$$

$$= \frac{\mu_0 i}{2\pi s} \oint dl$$

$$\oint dl = 2\pi r$$

$$= \frac{\mu_0 i}{2\pi s} \cdot 2\pi s$$

$$= \mu_0 i$$

$$\boxed{\oint \vec{B} \cdot dl = \mu_0 i} \Rightarrow \boxed{\oint \vec{B} \, dl = \mu_0 i_{enc.}}$$

i enclosed by the loop

$\cancel{\vec{B}}$

$$\oint_l \vec{B} \cdot d\vec{l} = \int_s (\vec{\nabla} \times \vec{B}) \, d\vec{s}$$

$$i = \int \vec{J} \cdot d\vec{s}$$

$$\int_s (\vec{\nabla} \times \vec{B}) \, d\vec{s} = \int_s \mu_0 J \cdot d\vec{s}$$

$$\boxed{\therefore \vec{\nabla} \times \vec{B} = \mu_0 j_{enc}}$$

$\vec{\nabla} \times \vec{B} \neq 0 \Rightarrow$ Rotational

$$\vec{B} = \frac{\mu_0 i}{2\pi s} \hat{z}$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial}{\partial z} \frac{\mu_0 i}{2\pi s} \hat{z} = 0$$

7/11/23

$dl = dr + \rho d\phi + dz$

$$\oint \bar{B}\cdot d\bar{l} = \mu_0 i_{enc}$$

$\vdots$

$\Rightarrow B\cdot(2\pi\rho) = \mu_0 i_{enc}$

$\Rightarrow \oint B\cdot dl = \frac{\mu_0 i}{2\pi}\int \frac{1}{\rho}\cdot \rho d\phi$

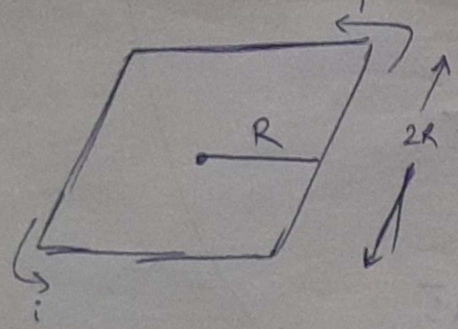
$= \frac{\mu_0 i}{2\pi}$

$\frac{\mu_0 \times \frac{I}{\pi a^2} \times \pi r^2}{2\pi r}$

$\rightarrow i_{enc} = \int J\cdot dA$

$$\int(\nabla\times\vec{B})\, ds = \int \mu_0 \bar{J}\cdot dS \Rightarrow \bar{\nabla}\times\bar{B} = \mu_0 J$$

Q)

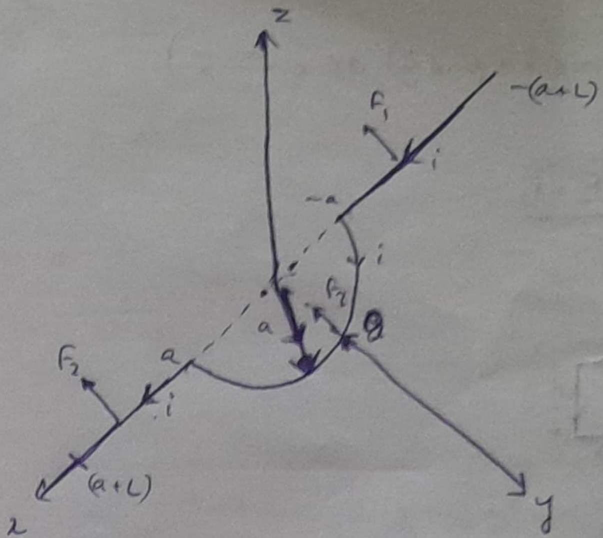


$B_1 = \frac{\mu_0 i}{4\pi R}\left(\frac{1}{\sqrt{2}} - \left(\frac{-1}{\sqrt{2}}\right)\right)$

$B_{net} = 4B_1 = \frac{\mu_0 i}{\pi R}(\sqrt{2})$

$$\boxed{B_{net} = \frac{\mu_0 i \times \sqrt{2}}{\pi R}\hat{k}}$$

Q)



$\vec{B} = B\vec{a_z}$

~~F~~ $dF_1 = i(dl \times \vec{B})$

$dF_1 = iB \cdot dl$

$\Rightarrow F_1 = \int_{-(a+L)}^{-a} iB\, dl$

$= iBl\Big]_{-(a+l)}^{-a}$

$= iB(-a+a+l)$

$\boxed{F_1 = iBl \;\Big|\; -\hat{y}}$ $[\hat{x} \times \hat{z} = -\hat{y}]$

$F_2 = \int_{a}^{a+l} iB\, dl$

$= iB(l\Big]_{a}^{a+l}$

$\boxed{F_2 = iBl \;\Big|\; -\hat{y}}$

$\cancel{dF_3 = \int_{-a}^{a} i(B\vec{a_z})}$

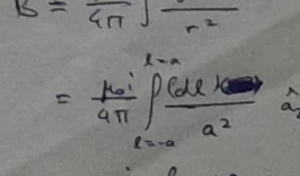
$dF_3 = i(\vec{dl} \times \vec{B})$

$F_3 = i\left(\vec{dl} \times \frac{\mu_0 i}{4a}\hat{z}\right)$

& $\vec{dl} = dl\cos\theta\,\hat{x} + dl\sin\theta\,\hat{y}$

$\rho = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y}$

$dl = \rho\, d\phi$



$\vec{B} = \frac{\mu_0 i}{4\pi}\int \frac{\vec{dl} \times \hat{r}}{r^2}$

$= \frac{\mu_0 i}{4\pi}\int_{l=-a}^{l=a} \frac{dl}{a^2}\,\hat{a_z}$

$= \frac{\mu_0 i}{4\pi}\int \frac{a\,d\theta}{a^2}\,\hat{a_z}$

$= \frac{\mu_0 i}{4\pi a}\int_{-\pi/2}^{\pi/2} d\theta\,\hat{a_z}$

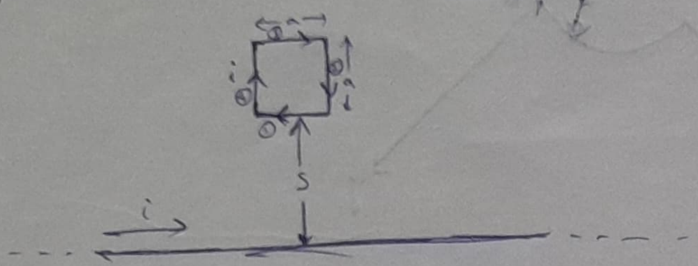
$= \frac{\mu_0 i}{4\pi a}(\theta\Big]_{-\pi/2}^{\pi/2}\,\hat{a_z}$

$\vec{B} = \frac{\mu_0 i}{4a}\hat{z}$

$$dF_3 = i\int_0^{\pi}(\rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y})\,d\phi \times \frac{\mu_0 i}{4a}\hat{z}$$

$$\underline{f = f_1 + f_2 + f_3}$$

Q)



$$f_1 = -\frac{\mu_0 i^2}{2\pi s} \qquad (\text{Attractive} \to +\text{ve})$$

$$f_3 = \frac{\mu_0 i^2}{2\pi(s+a)}$$

$$f_1 + f_3 = \frac{\mu_0 i^2}{2\pi s}\left(\frac{1}{1+\frac{a}{s}} - 1\right)$$

$$= \frac{\mu_0 i^2}{2\pi s}\left(\frac{\cancel{1} - \cancel{1} - a/s}{1 + a/s}\right)$$

$$= \frac{\mu_0 i^2}{2\pi s}\left(\frac{-a/s}{1+a/s}\right)$$

$$= \frac{\mu_0 i^2}{2\pi s}\left(\frac{-a}{a+s}\right)$$

$$\boxed{f_{net} = \frac{\mu_0 i^2 a}{2\pi s(s+a)} \quad (\text{Repulsive})}$$

f_{net} on Infinite wire $= \frac{\mu_0 i^2 a}{2\pi s(s+a)}\,-\hat{j}$

f_{net} on Square wire $= \frac{\mu_0 i^2 a}{2\pi s(s+a)}\,\hat{j}$ (Per unit length)

$$\boxed{F_{net} = \frac{\mu_0 i^2 a^2}{2\pi s(s+a)}\,\hat{j}}$$

→ <u>Magnetic Flux</u> :

$$\phi = \oint_s \vec{B} \cdot \vec{ds}$$

For a closed Surface,

$$\oint \vec{B} \cdot \vec{ds} = 0 \longrightarrow \oint (\vec{\nabla} \cdot \vec{B}) \, d\tau = 0$$

→ Magnetic Monopoles do not exist.

→ North and south poles cannot exist seperately.

→ No. of field lines leaving North Pole = No. of field lines entering South Pole

→ B is Solenoidal
$\because (\vec{\nabla} \cdot \vec{B} = 0)$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$$

$$\begin{bmatrix} \nabla \times E = 0 \\ \Downarrow \\ E = -\nabla V \end{bmatrix}$$

Q) Which A produces B along $-x$ direction?

~~(a)~~ $\frac{x}{2} \hat{y}$

(b) $z\hat{x} + x\hat{z}$

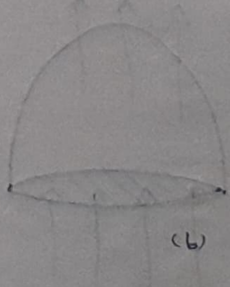
(c) $x\hat{x} + y\hat{y}$

(d) $-x\hat{x} - y\hat{y} - z\hat{z}$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

(a) $\Rightarrow \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{x}{2} & 0 \end{vmatrix} = \hat{x}\left(-\frac{\partial}{\partial z}\left(\frac{x}{2}\right)\right) + \hat{z}\left(\frac{\partial}{\partial x}\left(\frac{x}{2}\right)\right)$

$= -\hat{x}$ ✓



(b)

$\Rightarrow \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & x \end{vmatrix} = \hat{y}(0) = 0$

(c)

$$\overline{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0$$

$= 0$

(d)

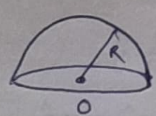
$$\overline{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & -y & -z \end{vmatrix} = 0$$

Q) $\overline{B} = B\vec{a_z}$

Calculate Flux through Hemisphere Radius $= R$,

Centre 'O'.

Bounded by $z = 0$



$$\text{Flux} = \int B \cdot ds$$

$$ds = dl_\rho \cdot dl_\phi \, \hat{z}$$
$$= d\rho \cdot \rho \, d\phi \, \hat{z}$$

$$= \int (B)\hat{z} \cdot (d\rho \cdot \rho \, d\phi)\hat{z}$$

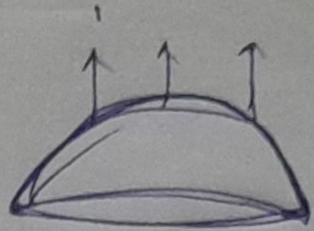
$$= \iint B\rho \, d\rho \cdot d\phi$$

$$= \int B\left(\int_0^R \rho \, d\rho\right) d\phi$$

$$= \int_0^{2\pi} B \cdot \frac{R^2}{2} \, d\phi$$

$$= B\pi R^2$$

H.W



Calculate flux

$[\int A \cdot dl]$

$$\text{Flux} = \int B \cdot dS$$

$$= \int (B\,\hat{z}) \cdot (d\rho \cdot \rho\, d\phi)\,\hat{z}$$

$$= \int_{2\pi}^{0} \int_{0}^{R} B\rho \; d\rho \cdot d\phi$$

$$= \int_{2\pi}^{0} B \frac{R^2}{2} \, d\phi$$

$$= -B\pi R^2$$

$$\text{Total flux} = B\pi R^2 - B\pi R^2 = 0$$

$$\oint \vec{B} \cdot \overline{dS} = 0$$

$B = \nabla \times A$

$\Rightarrow \nabla \times B = \nabla \times (\nabla \times A)$

$= \nabla(\nabla \cdot A) - \nabla^2 A$

$\nabla^2 A = \frac{-\rho}{\varepsilon_0}$

[Poisson's Equation : $\nabla^2 V = \frac{-\rho}{\varepsilon_0}$
(In Electrostatics)]

$\Rightarrow \nabla^2 A = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla} \times \vec{B}$

$= \vec{\nabla}(0) - \vec{\nabla} \times \vec{B}$

$= -\vec{\nabla} \times \vec{B}$

B → Solenoidal
A → Solenoidal
↳ (By Definition)

$\boxed{\nabla^2 A = -\mu_0 J_{enc}}$

$\boxed{\nabla^2 A = -\mu_0 J}$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

$\Rightarrow \iint_S (\vec{\nabla} \times \vec{B}) ds = \mu_0 \iint_S J \cdot ds$

$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 J}$

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$\vec{A} = \frac{\mu_0}{4\pi} \oint \frac{i \, d\vec{l}'}{R}$

$\left[J = \frac{i}{A} \right]$

$= \frac{\mu_0}{4\pi} \oint \frac{J A \, d\vec{l}}{R}$

$= \frac{\mu_0}{4\pi} \oint_V \frac{J \cdot d\tau}{R}$

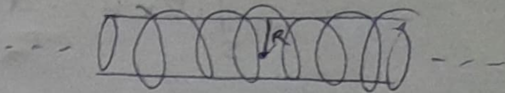
$\cancel{\phi} = \oint B$

$\phi = \int_S \vec{B} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{A}) \, d\vec{s}$

$\Rightarrow \phi = \oint \vec{A} \cdot d\vec{l}$

→ Infinite Solenoid with 'n' turns per unit length with current carrying.

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 i_{enc}$$



⇒ [On one turn]

$$\Rightarrow \oint \bar{B}\cdot\bar{dl} = \mu_0 N i_{enc}$$

(on N turns)

$$\Rightarrow \oint B\cdot L = \mu_0 N i$$

$$\Rightarrow B = \frac{\mu_0 N i}{L} = \mu_0 n i \qquad \left(\frac{N}{L} = n\right)$$

$$\boxed{B = \mu_0 n i}$$

For finding A:

$$\phi = \oint_l \bar{A}\cdot\bar{dl} = \int \bar{B}\cdot\bar{dS}$$

$$\Rightarrow A\cdot(2\pi R) = \mu_0 n i (\pi R^2)$$

$$\Rightarrow A(2) = \mu_0 n i R$$

$$\Rightarrow \boxed{A = \frac{\mu_0 n i R}{2}}$$

[s = R]

$$A(2\pi s) = \mu_0 n i (\pi s^2)$$

$$\Rightarrow \boxed{A = \frac{\mu_0 n i s}{2}}$$

[s < R]

→ Magnetic Field Intensity:

(i) $D = \epsilon E_0$

(ii) $B = \mu_0 H$

$$\oint \bar{B}\cdot\bar{dl} = \mu_0 i_{enc}$$

$$\oint \bar{H}\cdot\bar{dl} = i_{enc}$$

$$\Rightarrow \oint_l H \cdot dl = I$$

$$\Rightarrow \int_s (\bar{\nabla} \times \bar{H}) \cdot d\bar{s} = I$$

$$I = \int_s J \cdot ds$$

$$\Rightarrow \int_s (\bar{\nabla} \times \vec{H}) \cdot \vec{ds} = \int_s J \cdot ds$$

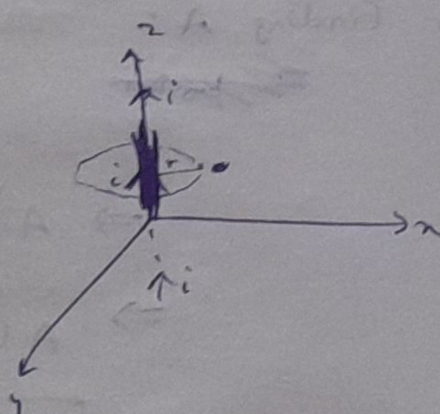
$$\Rightarrow \boxed{(\bar{\nabla} \times \bar{H}) = J}$$

Q) Very Long, very thin, straight wire along z axis Carries i in z direction.

Find Magnetic field Intensity @ any point in free space

A)

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 i$$



$$\Rightarrow B \cdot 2\pi r = \mu_0 i$$

$$\Rightarrow \mu_0 H \cdot 2\pi r = \mu_0 i$$

$$\Rightarrow \boxed{H = \frac{i}{2\pi r}} \; \vec{z} \times \hat{r}$$

$$i = 2\pi r H$$

$$\boxed{H = \frac{i}{2\pi \rho} \hat{\phi}}$$

→ Magnetisation:

$$\vec{P} = \frac{\vec{p}}{V}$$

$$M = \frac{m}{V}$$

$$\Rightarrow p = \int P\,d\tau$$

$$\Rightarrow m = \int_V M\,d\tau$$

$$V = k\,\frac{\vec{p}\cdot\hat{r}}{r^2}$$

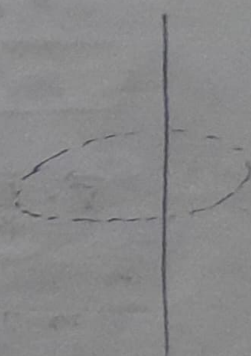
$\cancel{\nabla =}$

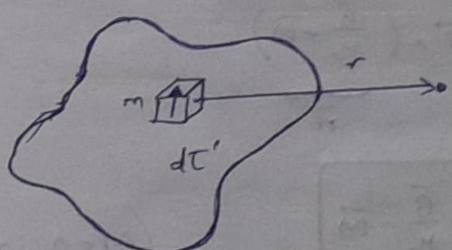
$$A = \frac{\vec{m}\times\hat{r}}{r^2}\cdot\frac{\mu_0}{4\pi}$$

→ Bound Charges:

$$A(r) = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\hat{r}}{r^2}$$

$$A(r) = \frac{\mu_0}{4\pi}\int\frac{M(r')\times\hat{r}}{r^2}\,d\tau'$$





$$\left[\because m = \int_V M\,d\tau\right]$$

$$\nabla'\cdot\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}$$

$$\Rightarrow A(r) = \frac{\mu_0}{4\pi}\int\left[M(r')\times\nabla'\left(\frac{1}{r}\right)\right]d\tau'$$

$$= \frac{\mu_0}{4\pi}\left[\int\frac{1}{r}\left(\nabla'\times M(r')\right)d\tau' - \int\nabla'\times\left(\frac{M(r')}{r}\right)d\tau'\right]$$

$$= \underbrace{\frac{\mu_0}{4\pi}\int\frac{1}{r}\left(\nabla'\times M(r')\right)d\tau'}_{\vec{J}_B = \vec{\nabla}\times\vec{M}} + \underbrace{\frac{\mu_0}{4\pi}\oint\frac{1}{r}\left(M(r')\times dA'\right)}_{K_b = \vec{M}\times\hat{n}}$$

$$A(r) = \frac{\mu_0}{4\pi}\int_V \frac{J_b(r')}{r}\, d\tau' + \frac{\mu_0}{4\pi}\oint_S \frac{K_b(r')}{r}\, dA'$$

Volume Bound Current : $J_b = \nabla \times M$

Surface Bound Current : $K_b = M \times \hat{n}$

$$\left[\begin{array}{l} \rho_b = -\nabla\cdot P \\ \sigma_b = \vec{P}\cdot\hat{n} \end{array}\right]$$

$$J = J_b + J_f$$

$H \to B$ & m

$$\cancel{\Rightarrow J = \nabla\times M + M\times\hat{n}}$$
$$\cancel{= \nabla\times M - \hat{n}\times M}$$
$$\cancel{= M\times\hat{n} - M\times\nabla}$$
$$\cancel{= M\times(\hat{n} - \vec{\nabla})}$$
$$\cancel{=}$$

$$\left[\begin{array}{l} \oint B\cdot dl = \mu_0 i \\ \vec{\nabla}\times\vec{B} = \mu_0 j \end{array}\right]$$

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$$\vec{\nabla}\times\frac{\vec{B}}{\mu_0} = \vec{\nabla}\times M + J_f$$

$$\Rightarrow J_f = \vec{\nabla}\times\underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)}_{\vec{H}}$$

$$\boxed{\therefore \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}}$$

($M = 0$ in free space)

$$\Rightarrow \vec{B} = \mu_0(\vec{H} + \vec{M})$$

$$\boxed{\therefore \vec{B} = \mu_0(\vec{H} + \vec{M})}$$

Linear Media : $M \propto H$

$$\Rightarrow \vec{M} = \chi_m \vec{H}$$

$$\vec{B} = \mu_0(\vec{H} + \chi_m\vec{H})$$

$$= \underbrace{\mu_0(1+\chi_m)}_{\mu}\vec{H}$$

$$\mu = \mu_0(1+\chi_m)$$

And $\mu = \mu_0\cdot\mu_r$

$$\boxed{\therefore \vec{B} = \mu\vec{H}}$$

$$\boxed{\therefore \mu_r = 1 + \chi_m}$$

Q) $l = 5\text{ cm} = 5\times10^{-2}\text{ m}$

$A = 2\text{ cm}^2 = 2\times10^{-4}\text{ m}^2$

Magnetic Moment (Net) = 1 Am^2

Magnetization (M) = ?

A)

$$M = \frac{m}{V}$$

$$\Rightarrow M = \frac{1}{10\times10^{-6}} = 10^5$$

$$\boxed{M = 10^5\ A/m}$$

Q) $l = 0.25\text{ m}$

$N = 400$ turns

$i = 15\text{ A}$

Magnetic field strength / Intensity (H) = ?

Compute flux Density, B, if the coil is in vacuum

A)

$$B = \mu H$$

$$\oint B.dl = \mu_0 Ni$$

$[n = \frac{N}{l}]$ $[B = \mu_0 ni]$

$$\Rightarrow B(0.25) = \mu_0 \times 400 \times 15$$

$$\Rightarrow B = 24\times10^3 \times \mu_0$$

$$\boxed{B = 24\mu_0\times10^3\ T}$$

$\mu_0 = 4\pi\times10^{-7}$

$$\Rightarrow \boxed{B = 0.03015\ T}$$

$$H = \frac{B}{\mu_0} = \frac{96\times10^3}{4}\ T$$

$$\boxed{H = 24\times10^3\ A/m}$$

Q) $B = 0.63\text{ T}$

$H = 5\times10^5\ A/m$

(a) $\mu = B/H = 0.126\times10^{-5}$

$= 1.26\times10^{-6}\ H/m$ $\Rightarrow$ $\boxed{\mu_0 = 1.26\times10^{-6}\ H/m}$

(b) $B = \mu H$

$\mu = \mu_0 \mu_r$

$\Rightarrow \mu_r = \frac{\mu}{\mu_0} \cancel{= \frac{1.26 \times 10^{-6}}{8.854 \times 10^{-12}}} = \frac{1.26 \times 10^{-6}}{4\pi \times 10^{-7}}$

$\cancel{= 0.1423 \times 10^{6}} = 1.002676$

$\cancel{\boxed{\Rightarrow \mu_r = 1.423 \times 10^{5}}}$ $\boxed{\mu_r = 1.002676 \ H/m}$

$\mu_r = 1 + \chi_m$

$\Rightarrow \chi_m = \mu_r - 1$

$\boxed{\therefore \chi_m = 0.002676}$ $\longrightarrow$ Diamagnetic

~~$\chi_m > 0$ & Large $\Downarrow$ Ferromagnetic~~

$\chi_m > 0$ & Large : Paramagnetic

$\chi_m > 0$ & small (very) : Diamagnetic

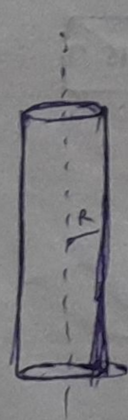
Q) Long Copper Rod

Radius R

Free current I (Uniform)

Find H inside & Outside.

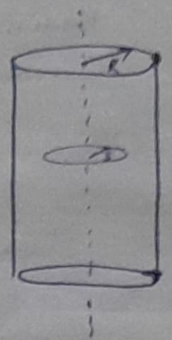
A)



Find H @ Radius < R

& H @ Radius > R

for H, s < R :



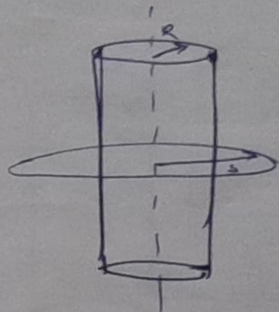
$$\oint H \cdot dl = I$$

$$\Rightarrow \oint H \cdot dl = \left(\frac{I}{\pi R^2}\right) \pi s^2$$

$$\Rightarrow H \cdot 2\pi s = \frac{I s^2}{R^2}$$

$$\Rightarrow \boxed{H = \frac{I s}{2\pi R^2} \hat{\phi}} \quad \forall s < R$$

for H, s > R :



$$\oint H \cdot dl = I$$

$$\Rightarrow H \cdot 2\pi s = I$$

$$\Rightarrow \boxed{H = \frac{I}{2\pi s} \hat{\phi}} \quad \forall s > R$$

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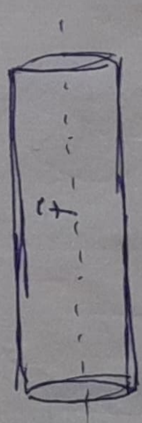
Q)



$r = 2$ cm

$i = 10$ A

Capacitor in Gap @ 1 cm

$$H \cdot 2\pi s = \frac{I s}{R^2}$$

$$\Rightarrow H \cdot 2\pi = \frac{I}{4}$$

$$\therefore \boxed{H = \frac{I}{8\pi}} \Rightarrow \boxed{H = \frac{500}{4\pi}}$$

$$\boxed{B = \frac{\mu I}{8\pi}}$$

$$\cancel{B = \frac{5\mu_0}{4\pi}}$$

$$\boxed{B = \frac{125}{\pi}}$$

Q) Vector Potential of Infinite solenoid → n turns per unit length

Radius R

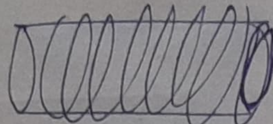
Current I

$$\int_s B \cdot ds = \oint A \cdot dl$$

$n \to 1$

$nl \to l$

$$(\mu_0 n I) \cdot \pi s^2 = A \cdot 2\pi s$$



$$\Rightarrow A = \frac{\mu_0 n I s}{2}$$

$$\boxed{\therefore A = \frac{\mu_0 n I R}{2}}$$

Q) Flux Density (B) = ?

$l = 0.25$

$N = 400$ turns } Solenoid

$i = 15\ A$

Magnetization (M) = ?

Magnetic Susceptibility $(\chi_m) = 3.13 \times 10^{-4}$

A) $\vec{B} = \mu \vec{H}$

$$\vec{B} = \mu n i$$

$$= \mu_0 \mu_r n i$$

$$= \mu_0 (1 + \chi_m) n i$$

$$= 4\pi \times 10^{-7} \times (1 + 3.13 \times 10^{-4}) \times \frac{400}{0.25} \times 15$$

$$= 0.03016 \times (1 + 3.13 \times 10^{-4})$$

$$= 0.03016873$$

$$\Rightarrow \boxed{\vec{B} = 3.01687 \times 10^{-2}\ T}$$

$\vec{H} = ni$

$= \frac{400}{0.25} \times 15$

$\boxed{\vec{H} = 24000 \ A/m}$

$B \to T$

$\mu \to H/m$

$H = B/\mu$

$\vec{M} = \chi_m \vec{H}$

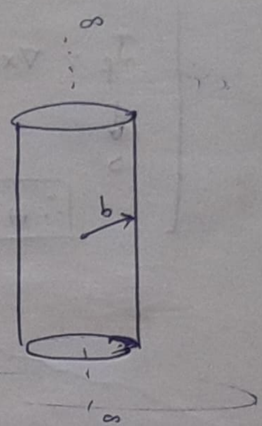
$= 3.13 \times 10^{-4} \times 24000$

$= 75120 \times 10^{-4}$

$\boxed{\vec{M} = 7.512 \ A/m}$

(4.)

Q)



$M = br\hat{z}$

[no free current]

Find (a) H, B_{in}, B_{out}

(b) J_b, K_b

~~$\vec{M} = \vec{\chi}_m \vec{H}$~~

~~$H = br\hat{z}$~~

$\vec{M} = \chi_m \vec{H}$

(b) $J_B = \vec{\nabla} \times \vec{M}$

$K_B = \vec{M} \times \hat{n}$

$$J_B = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ M_\rho & \rho M_\phi & M_z \end{vmatrix} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & b\rho \end{vmatrix}$$

$= \frac{1}{\rho}(-b)\rho\hat{\phi}$

$\boxed{\therefore J_b = -b\hat{\phi}}$

$$K_b = br\,\hat{z}\times\hat{r}$$

$$= b\rho\,\hat{z}\times\hat{\rho}$$

$$= b\rho\,\hat{\phi}$$

P

z ϕ

$$\therefore K_b = b\rho\,\hat{\phi}$$

$$\boxed{\therefore K_b = br\,\hat{\phi}}$$

(a)

$$\oint H\cdot dl = I_{free\ enc}$$

$$= 0$$

$$\boxed{\therefore H = 0} \quad \text{always}$$

$$\left[\, J_f = \nabla\times H \;\downarrow\; 0 \qquad \boxed{\therefore H = 0} \,\right]$$

Outside:

$$\vec{B} = \mu_0(H+M)$$

$$\Rightarrow \vec{B} = \mu_0(0+M)$$

$$\therefore B = \mu_0 M$$

But Nothing to Magnetise,

$$\therefore M = 0$$

$$\boxed{\therefore B = 0}$$

Inside:

$$\vec{B} = \mu_0(H+M)$$

$$\therefore \vec{B} = \mu_0(0+M)$$

$$\therefore B = \mu_0 M$$

1 hr → 20m (Written)

$$\boxed{\therefore \vec{B} = \mu_0 br\,\hat{z}}$$

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As $\oint B \cdot dS = 0$

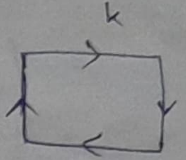
[Boundary Conditions]

$\oint (B_1^{\perp} - B_2^{\perp}) \cdot ds = 0$

$\Rightarrow B_1^{\perp} = B_2^{\perp}$

$\boxed{\therefore B_1^{\perp} = B_2^{\perp}}$ $\Rightarrow$ Normal ~~~~ Components are continuous

As $\oint B \cdot dl \neq 0$



$\oint B \cdot dl = \mu_0 i$

$\left[i = \oint k \cdot dl \right]$

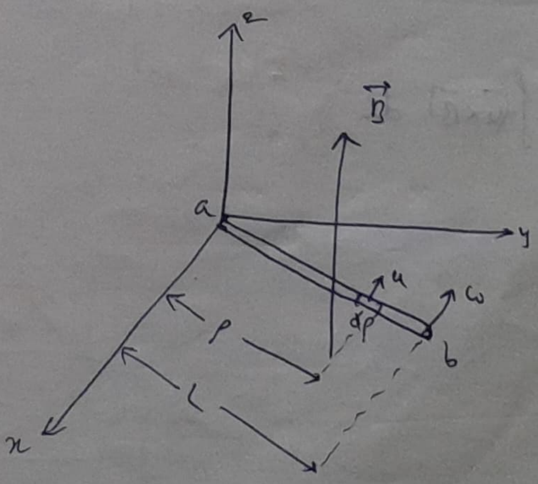
$\Rightarrow \oint B \cdot dl = \mu_0 \oint k \cdot dl$

$\int (B_1^{\parallel} - B_2^{\parallel}) \cdot dl = \oint (\mu_0 k)\, dl$

$\Rightarrow B_1^{\parallel} - B_2^{\parallel} = \mu_0 k$

$\boxed{\therefore B_1^{\parallel} - B_2^{\parallel} = \mu_0 k}$

Q) Copper strip $\rightarrow$ length L



Find Induced EMF.

A)

$$\vec{F} = q(\vec{v} \times \vec{B})$$

$$\Rightarrow \vec{E} = \vec{v} \times \vec{B}$$

$$\mathcal{E} = \int (\vec{v} \times \vec{B}) \cdot dl$$

$[v = \rho\omega]$

$$= \int ((\rho\omega)\hat{\phi} \times B\hat{z}) \cdot dl$$

z ρ ϕ

$$= \int (\rho\omega B)\hat{\rho} \cdot (d\rho)\hat{\rho}$$

$$= \int_0^L \rho\omega B \, d\rho$$

$$= \left.\frac{\rho^2\omega B}{2}\right]_0^L = \frac{L^2\omega B}{2}$$

$$\therefore \mathcal{E} = \frac{1}{2} B\omega L^2$$

$$\left[\mathcal{E} = -\frac{d\phi}{dt} = \frac{dW}{dq}\right]$$

22/11/23 :

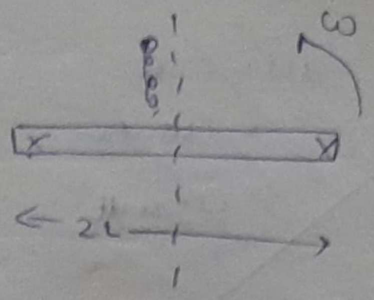
$$\boxed{\mathcal{E} = -\frac{d\phi}{dt} = \frac{dW}{dq}}$$

$$\frac{F \cdot dl}{q} = \int E \cdot dl$$

$$\mathcal{E} = \int (\vec{u} \times \vec{B}) \, dl$$

4)

i)



$$\mathcal{E} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

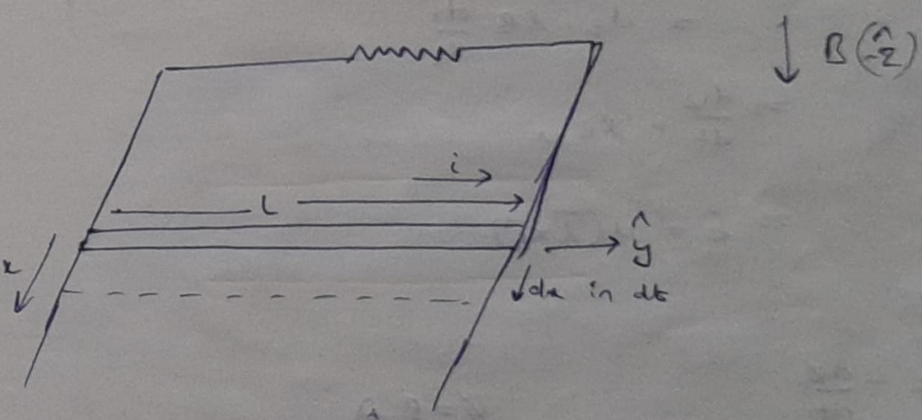
$$= \int ((\rho\omega)\hat{\phi} \times B\hat{z})\, d\vec{l} + \int (-\rho\omega)\hat{\phi} \times (B\hat{z})(-d\vec{l})$$

$$= \int_0^L \rho\omega B\, d\rho + \int_0^{-L} \rho\omega B\, d\rho$$

$$= \frac{\rho\omega L^2}{2} - \frac{\rho\omega L^2}{2}$$

$$= 0$$

ii)



$$F_m = i(\vec{l} \times \vec{B})$$

$$= i(l\hat{y} \times B(-\hat{z}))$$

$$= -ilB\hat{x}$$

$$\boxed{\therefore F_m = -ilB\hat{x}}$$

$$\boxed{F_{ext} = ilB\hat{x}}$$

Method-1

$$W = \int F \cdot dx$$

where $dW = F \cdot dx$

$$= i l B \cdot dx$$

$$= \frac{dq}{dt} l B \, dx \qquad (u : \hat{x})$$

$$= dq \cdot l B \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dW}{dt} = l B u$$

$$\Rightarrow \mathcal{E} = u(\vec{l} \times \vec{B})$$

Method-2

$$\mathcal{E} = -\frac{d\phi}{dt}$$

$$\phi = B \cdot A$$

$$= B \cdot (-\hat{z}) \cdot (l \, dx) \, \hat{z}$$

$$= -B l \, dx$$

$$= -\frac{-B l \, dx}{dt}$$

$$= B l (u)$$

$$\therefore \mathcal{E} = \vec{u}(\vec{l} \times \vec{B})$$

$$\cancel{P} = iV$$

$$i = \frac{\mathcal{E}}{R} = \frac{B u l}{R}$$

$$P = i^2 R$$

$$\boxed{P = \frac{B^2 u^2 l^2}{R}}$$

23/11/23

→ $\varepsilon = \oint \vec{E} \cdot \vec{dl} = -\frac{d\phi}{dt}$

~~$\oint \vec{E} \cdot dl = -\iint \frac{\partial B}{\partial t} \cdot dA$~~

$\phi = B \cdot A$

$\Rightarrow \oint \vec{E} \cdot \vec{dl} = \frac{-d}{dt}(B \cdot A)$

Assume B → function of time &
A → constant

$\Rightarrow \oint \vec{E} \cdot \vec{dl} = -A \frac{dB}{dt}$

$\Rightarrow \iint (\vec{\nabla} \times \vec{E}) \, ds = \iint -\frac{\partial B}{dt} \, \partial A$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial B}{dt}}$$

↳ $\vec{\nabla} \times \vec{E} \neq 0 \Rightarrow \vec{E}$ is non conservative
if time varying $\vec{B}$ exists

→ <u>Maxwell's Equation</u>:

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ $\qquad$ $\vec{\nabla} \cdot \vec{B} = 0$

As $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

$\vec{\nabla} \times \vec{H} = \vec{j}$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{j}$

$\Rightarrow 0 = \vec{\nabla} \cdot \vec{j}$

$\therefore \vec{\nabla} \cdot j = 0$

$\vec{\nabla} \cdot j = -\frac{\partial \rho_f}{dt}$

Always 0 $\qquad$ Not always 0.

$\therefore$ Not True

$\vec{\nabla} \times \vec{H} = \bar{j} + \bar{G}$

$\Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{j} + \vec{\nabla} \cdot \vec{G}$

$\Rightarrow 0 = \vec{\nabla} \cdot \vec{j} + \vec{\nabla} \cdot \vec{G}$

$\therefore \vec{\nabla} \cdot G = -\vec{\nabla} \cdot \vec{j} = +\frac{\partial \rho_f}{\partial t}$

$\left[\vec{\nabla} \cdot \vec{D} = \rho_f\right]$

$$\left[\begin{aligned} D &= \varepsilon_0 \vec{E} + \vec{P} \\ \vec{\nabla} \cdot \vec{D} &= \vec{\nabla} \cdot (\varepsilon_0 \vec{E} + \vec{P}) \\ &= \varepsilon_0 \frac{\rho}{\varepsilon_0} + (-\rho_b) \\ &= \rho - \rho_b \\ \therefore \vec{\nabla} \cdot \vec{D} &= \rho_f \end{aligned}\right]$$

$\vec{\nabla} \cdot \vec{G} = \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{D})$

$= \nabla \cdot \frac{\partial}{\partial t} \vec{D}$

$$\boxed{\therefore G = \frac{\partial \vec{D}}{\partial t}}$$

$$\boxed{\vec{\nabla} \times \vec{H} = j + \frac{\partial \vec{D}}{\partial t}}$$

As ~~D~~ D → changing with time

$\Rightarrow \therefore$ E exists and changes

$[\because D = \varepsilon E]$

$$\left.\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \frac{\rho}{\varepsilon_0} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= \frac{-\partial B}{\partial t} \\ \vec{\nabla} \times \vec{H} &= j + \frac{\partial \vec{D}}{\partial t} \end{aligned}\right] \text{Maxwell's Equations}$$

$\Rightarrow \vec{\nabla} \times \vec{H} = j_c + j_D$

$j_c = \sigma E$

σ → Conductivity

j_c → Conduction ~~Cu~~ Current density

$J_D = \frac{\partial D}{\partial t} = \varepsilon_0 \varepsilon_r \frac{\partial E}{\partial t}$

J_D → Displacement current density

Q) $\sigma = 0.11$ S/m

$\varepsilon_r = 1.2$

$t = 5s \Rightarrow$ Calculate J_C, J_D

$E = \cos(0.1t)$ V/m

$$J_C = \sigma E$$

$$= 0.11 \cos(0.1t)$$

$$= 0.11 \cos(0.5)$$

$$= \cancel{0.1x}$$

radians

$$\frac{D}{180} = \frac{R}{\pi}$$

$$J_D = \varepsilon_0 \varepsilon_r \cdot \frac{\partial E}{\partial t}$$

$$= 8.85 \times 10^{-12} \times 1.2 \times (-\sin(0.1)t) \cdot 0.1$$

$$= -10.62 \times 10^{-13} \times \sin(0.1t)$$

$$= (-1.062 \times 10^{-12}) \sin(0.1t)$$

radians

$$\cancel{= -(0.00834 \times 10^{-12})}$$

$$\cancel{= -8.34 \times 10^{-15}}$$

29/11/23

→ Self Induction :

$\lambda = N\phi$ → Flux linkage through each turn

No. of turns

Number of Flux Linkage

$$L = \frac{\lambda}{i}$$

Self Inductance

$\lambda = N\phi \propto B$

$\phi \propto i$

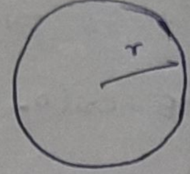
$\Rightarrow \phi = Li$

$$\mathcal{E} = N \frac{d\phi}{dt}$$

$$\frac{d\lambda}{di} = L \Rightarrow d\lambda = L\, di$$

$\phi = N\phi\ \alpha$;

$\phi = BA$



$N\phi = Li \Rightarrow L = \frac{N\phi}{i}$

$$\left[\begin{array}{l}\mathcal{E} = N\frac{d\phi}{dt} \\ \mathcal{E} = L\frac{di}{dt}\end{array}\right]$$

$\because \lambda = N\phi$

$\phi = B\cdot A = \mu_0 n i \cdot \pi r^2$

$= \mu_0 \frac{N}{l} i \pi r^2$

~~n: no. of~~

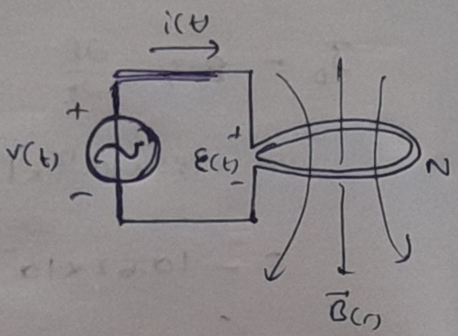
n: n turns per unit length

$\left[n = \frac{N}{l}\right]$

$\lambda = N\phi$

$= \frac{\mu_0 N^2 i \pi r^2}{l}$

$$\boxed{\therefore L = \frac{\mu_0 N^2 \pi r^2}{l}}$$



9) $\frac{N_1}{N_2} = \frac{1}{1}$

$\frac{l_1}{l_2} = \frac{2}{1}$

$\frac{r_1}{r_2} = \frac{3}{2}$

$\frac{L_1}{L_2} = ?$

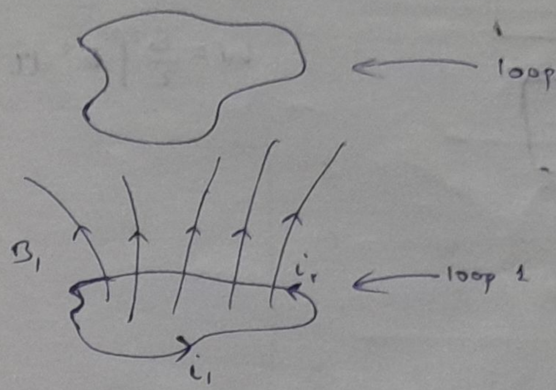
$\frac{N_1^2 \times r_1^2}{l_1} : \frac{N_2^2 \times r_2^2}{l_2}$

$\frac{1 \times 9}{2} : \frac{1 \times 4}{1}$

$\frac{9}{2} : 4$

$9 : 8$ (d)

→ Mutual Induction :



$\phi_2 \propto B_1$

& $B_1 \propto i_1$

$\Rightarrow \phi_2 \propto i_1$

$\boxed{\therefore \phi_2 = M i_1}$

M: Mutual Inductance

& $\phi_1 = M i_2$

$\boxed{\mathcal{E}_1 = -M \frac{d(i_2)}{dt}}$

$\mathcal{E}_2 = -\frac{d\phi_2}{dt}$

$= -\frac{d(Mi_1)}{dt}$

$= M \cdot \frac{-d(i_1)}{dt}$

$\boxed{\therefore \mathcal{E}_2 = -M \cdot \frac{d(i_1)}{dt}}$

→ Energy in Magnetic field :

$W_{ext} = (-\mathcal{E})(q)$

$\Rightarrow \frac{dW}{dt} = (-\mathcal{E}) \frac{dq}{dt}$

$= -(\mathcal{E})(i)$

$\therefore \frac{dW}{dt} = Li \frac{di}{dt}$

$\Rightarrow W = \int_0^i Li \frac{di}{dt} dt = \int_0^i Li \, di = \frac{Li^2}{2}$

$\boxed{\therefore W = \frac{1}{2} Li^2}$

$$\boxed{\therefore W = \frac{1}{2\mu_0}\int_v B^2 d\tau}$$

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau$$

$$\boxed{\therefore W = \frac{1}{2}\int_v H^2 d\tau}$$

Q) Long cylinder

$R = 20\,cm = 1/5$

$n = 200$

$i \rightarrow$ constant

Calculate Inductance

A) $\frac{\mu_0 N^2 \pi R^2}{l} \Rightarrow L$

$n = \frac{N}{l}$

$$\Rightarrow L = \frac{\mu_0 N^2 \times \pi (1/25)}{l}$$

$$= \frac{\mu_0 n^2 l^2 \times \pi/25}{l}$$

$$L = \frac{\mu_0 n^2 l \pi}{25}$$

$$= \frac{200 \times 200 \times 4\pi \times 10^{-7} \times \pi}{25} l$$

$$\Rightarrow \frac{L}{l} = 6.31 \times 10^{-3}\ H/m$$

$$= 6.31\ mH/m$$

$L = \frac{N\phi}{i}$

$\Rightarrow L = \frac{200 \times l \times \mu_0 \times 200\, i A}{i}$

$= (200 \times 200) l \mu_0 \times \pi r^2$

$= \frac{\pi}{25} \times 200 \times 200\ l \mu_0$

$= 5026.55\ l\mu_0$

Q) $\vec{H} = H_0 \sin\theta\, \vec{a_y}$ A/m

$\theta = \omega t - \beta z$

[β: constant]

find (i) D (ii) E

$\vec{H} = H_0 \sin(\omega t - \beta z)\, \vec{a_y}$

$\vec{\nabla} \times \vec{H} = \vec{\nabla} \times \left[H_0 \sin(\omega t - \beta z)\, \vec{a_y} \right]$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (\;) & 0 \end{vmatrix}$$

$\vec{\nabla} \times \vec{H} = \hat{x}\left(-\frac{\partial}{\partial z} (H_0 \sin(\omega t - \beta z))\right)$

$= (\hat{x}) (H_0 \cos(\omega t - \beta z))\, \beta$

$\therefore\; j_D = H_0 \beta \cos(\omega t - \beta z) \cdot \hat{x} \quad — (1)$

$\because j_c = 0$ in free space

$$\boxed{\frac{\partial \vec{D}}{\partial t} = H_0 \beta \cos(\omega t - \beta z)\, \hat{x}}$$

& $\therefore\; \varepsilon \frac{d\vec{E}}{dt} = H_0 \beta \cos(\omega t - \beta z)\, \hat{x} \quad — (2)$

$$\boxed{\frac{d\vec{E}}{dt} = \frac{H_0 \beta}{\varepsilon_0 \varepsilon_r} \cos(\omega t - \beta z)\, \hat{x}}$$

Solving

$\int d\vec{D} = \int H_0 \beta \cos(\omega t - \beta z)\, dt$

$\Rightarrow \vec{D} = \frac{\beta H_0 \sin(\omega t - \beta z)}{\omega}\, \hat{x}$

$$\boxed{\therefore \vec{D} = \frac{\beta H_0}{\omega} \sin(\omega t - \beta z)\hat{x}} \Rightarrow \boxed{\vec{D} = \beta \frac{H_0}{\omega} \sin \theta\, \hat{x}}$$

Similarly

$\Rightarrow \vec{E} = \int \frac{H_0 \beta}{\varepsilon_0} \cos(\omega t - \beta z)\, \hat{x}\, dt \Rightarrow \frac{H_0 \beta}{\varepsilon_0\, \omega} \sin(\omega t - \beta z)\, \hat{x}$

$$\boxed{\therefore \overline{E} = \frac{\beta H_0}{\varepsilon_0 \omega} \sin(\omega t - \beta z)\ \hat{x}}$$

$$\boxed{\vec{E} = \frac{\beta H_0}{\varepsilon_0 \omega} \sin\theta\ \hat{x}}$$

→ Inst. power Density

$\Rightarrow (E/\text{time})/\text{Area}$ → Vector quantity

[Poynting vector]

↳ Power ~~[illegible]~~ flowing out per unit Area

$$\vec{S} = \vec{E} \times \vec{H}$$

$$\overline{S} = \left(\frac{\beta H_0}{\varepsilon_0 \omega} \sin\theta\right)\hat{x} \times \left(H_0 \sin(\theta)\right)\hat{y}$$

$$= \frac{\beta H_0^2}{\varepsilon \omega} \sin^2\theta\ \hat{z}$$

$$\boxed{\therefore \overline{S} = \frac{\beta H_0^2}{\varepsilon_0 \omega} \sin^2(\omega t - \beta z)\ \hat{z}}$$

→ **Power** :

$$P = \overline{F} \cdot \overline{v}$$

$$P = q(\overline{E} + \vec{u} \times \overline{B}) \cdot \vec{u}$$

$$= q\vec{u} \cdot \vec{E}$$

$$dP = dq\,(\vec{u} \cdot \vec{E})$$

$$dq = \rho_v\, dV$$

$$= \rho_v\, \vec{u} \cdot \vec{E}\, dv$$

$$\Rightarrow dP = \vec{E} \cdot \rho_v \vec{u}\, dv$$

$$\rho_v u = \frac{dq}{L \times A} \times \frac{L}{dt} = \vec{J}$$

$\Rightarrow dP = \vec{E} \cdot \vec{J}\, dV$

$\Rightarrow \frac{dP}{dV} = \vec{E} \cdot \vec{J}$ ⟶ Power per unit vol.

[Power Density]

$J = \vec{\nabla} \times H - \frac{\partial \vec{D}}{\partial t}$

$\Rightarrow \frac{dP}{dV} = \vec{E} \cdot \left(\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \right)$

(Derivation won't be asked in exams)

$$\left[\begin{aligned} W_m &= \frac{1}{2} \vec{B} \cdot \vec{H} = \frac{1}{2} \mu H^2 \\ W_e &= \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{1}{2} \varepsilon E^2 \end{aligned} \right]$$

$$\oint S \cdot ds + \int_v J \cdot E\, dV = - \int (W_e + W_m)\, dV$$

Q) $\vec{E} = E \cos(\omega t - kz)\, \vec{a}_x$ V/m

E : peak value

k : constant

$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E\cos(\omega t - kz) & 0 & 0 \end{vmatrix}$$

$$= \hat{y} \left(\frac{\partial}{\partial z} \left(E \cos(\omega t - kz) \right) \right)$$

$$\therefore \boxed{ -\frac{dB}{dt} = E k \sin(\omega t - kz)\, \hat{y} }$$

$$\frac{\partial B}{\partial t} = -Ek\sin(\omega t - kz)$$

$$B = \int -Ek\sin(\omega t - kz)\, dt$$

$$= \left(\frac{Ek}{\omega}\right)\cos(\omega t - kz)$$

$$\boxed{\therefore B = \frac{Ek}{\omega}\cos(\omega t - kz)\,\hat{y}}$$

$$\vec{H} = \frac{\vec{B}}{\mu}$$

$$\boxed{\therefore \vec{H} = \frac{Ek}{\omega\mu}\cos(\omega t - kz)\,\hat{y}\ A/m} \quad -(a)$$

$$\vec{S} = \vec{E}\times\vec{H}$$

$$= \left(E\cos(\omega t - kz)\,\hat{x}\right)\times\left(\frac{Ek}{\omega\mu}\cos(\omega t - kz)\,\hat{y}\right)$$

$$\boxed{\vec{S} = \frac{E^2k}{\omega\mu}\cos^2(\omega t - kz)\,\hat{z}}\ W/m^2 \quad -(b)$$

Along z-axis $\rightarrow$ Power flow

$$\langle S_{avg}\rangle = \frac{1}{T}\int_0^T S\, dt$$

$$= \frac{1}{T}\int_0^T \frac{E^2k}{\omega\mu}\cos^2(\omega t - kz)\, dt$$

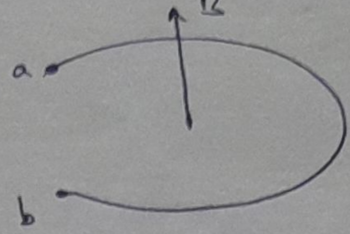
$$= \frac{E^2k}{\omega\mu T}\int_0^T \cos^2(\omega t - kz)\, dt = \frac{E^2k}{\omega\mu T}\times\frac{T}{2}$$

$$= \frac{E^2k}{2\mu\omega}\ W/m^2$$

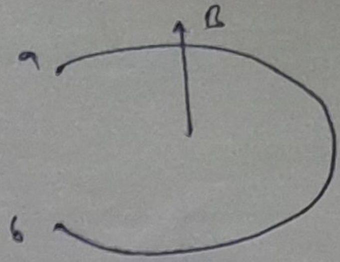
$$\left[\underbrace{\int_0^T \cos^2(\omega t - kz)dt}_{x} + \underbrace{\int_0^T \sin^2(\omega t - kz)\, dt}_{x} = T \right]$$

$$x = T/2$$

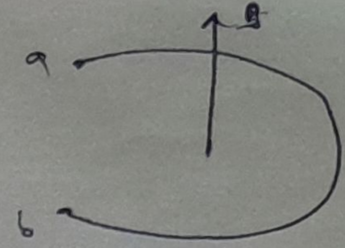
$$\boxed{\therefore \langle S_{avg}\rangle = \frac{E^2k}{2\mu\omega}}\ W/m^2$$



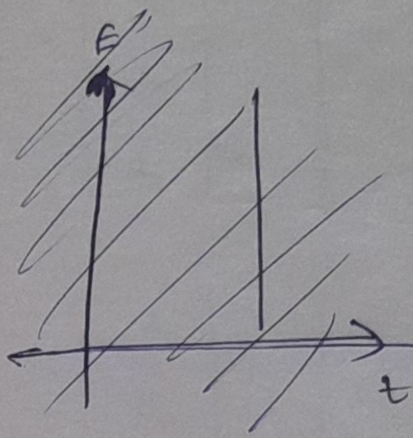
B is constant

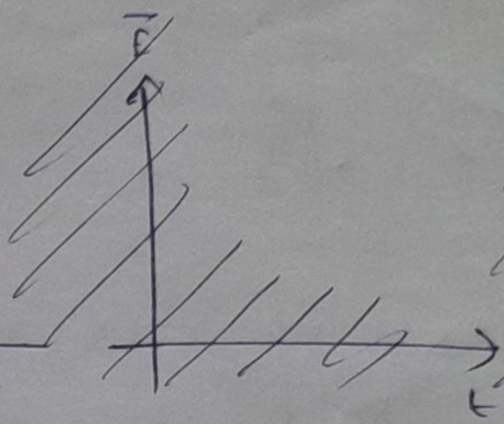


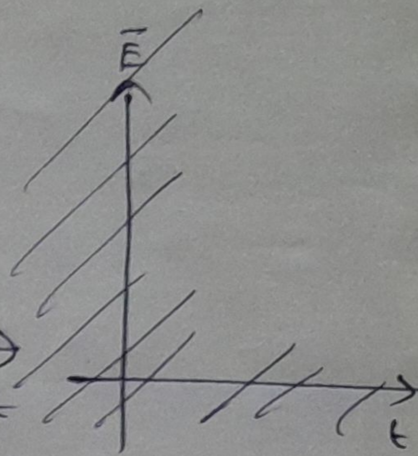
B increases with time

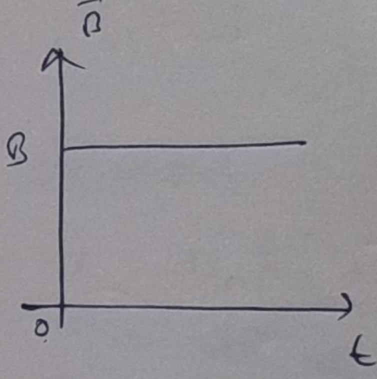


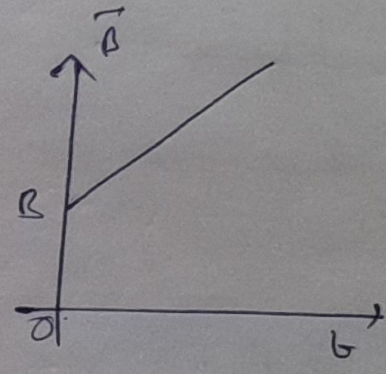
B decreases with time

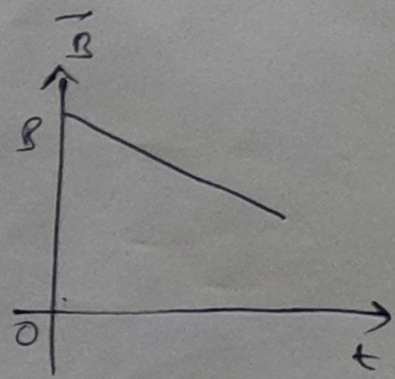


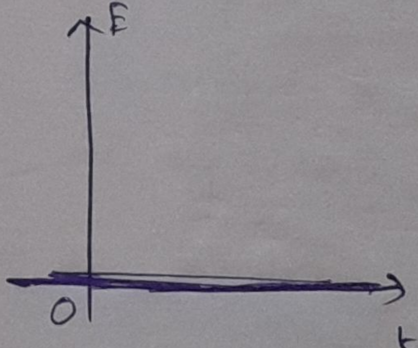


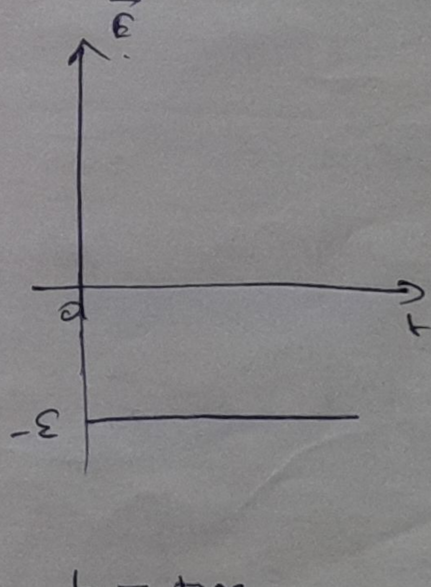


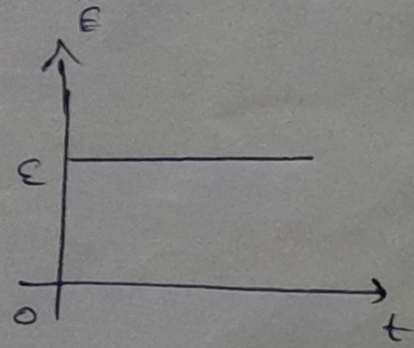












$$\left(\mathcal{E} = -\frac{\partial B}{\partial t} A\right)$$

b, a are equipotential

b – +ve
a – -ve

b – -ve
a – +ve

① (i) $\vec{\nabla}\cdot\vec{E} = \frac{-\rho}{\varepsilon_0}$

(ii) $\vec{\nabla}\cdot\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$

(iv) $\vec{\nabla}\times\vec{B} = \mu_0 J + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$

(i) $\oint_S D\cdot dA = Q_{f\,enc}$

(ii) $\oint_S B\cdot dA = 0$

Closed Surface S

(iii) $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S B\cdot dA$

$\oint \vec{H}\cdot d\vec{l} = I_{f\,enc} + \frac{d}{dt}\int_S D\cdot dA$

②

(i) $\vec{\nabla}\cdot\vec{D} = \rho_f$

(ii) $\vec{\nabla}\cdot\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$

(iv) $\vec{\nabla}\times\vec{H} = J_f + \frac{\partial D}{\partial t}$

Boundary Condition: [If no free charge ~~of~~ current $\Rightarrow$ All are equal]

Normal Component: $D_1^{\perp} - D_2^{\perp} = \sigma_f \Rightarrow \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$

Tangential Components: $E_1^{\parallel} - E_2^{\parallel} = 0$

Normal Components: $B_1^{\perp} - B_2^{\perp} = 0$

Tangential Components: $H_1^{\parallel} - H_2^{\parallel} = K_f\times\hat{n} \Rightarrow \frac{B_1^{\parallel}}{\mu_1} - \frac{B_2^{\parallel}}{\mu_2} = K_f\times\hat{n}$

$W_e = \frac{\varepsilon_0}{2}\int E^2\, d\tau$

$W_M = \frac{1}{2\mu_0}\int B^2\, d\tau$

$W = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$

$(\hat{n})$

Polarisation vector defines plane of vibration

$$\hat{n}\cdot\hat{z} = 0$$

Plane Wave :

$$B_0 = \frac{E_0}{c}$$

$$\left[\vec{k} = \frac{2\pi}{\lambda}\,\overrightarrow{\text{wave}}\right]$$

$$E = E_0\cos(k\cdot r - \omega t + \delta)\,\hat{n}$$

$$B = \frac{1}{c}E_0\cos(k\cdot r - \omega t + \delta)\,(\hat{k}\times\hat{n})$$

$$[\hat{n}\cdot\hat{k} = 0]$$

Q) Our planet is recieving 10 cal/m² per sec Energy . Calculate E_0 and B_0

$$\vec{S} = \vec{E}\times\vec{H} = \vec{E}\times\frac{\vec{B}}{\mu_0}$$

$$\Rightarrow \vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

$$\Rightarrow S = \frac{1}{\mu_0}\times B_0 c B_0$$

$$\Rightarrow B_0 = \sqrt{\frac{\mu_0 S}{c}}$$

$$E = E_0\cos(\quad)\,\hat{n}$$

$$B = B_0\cos(\quad)\,(\hat{k}\times\hat{n})$$

$$V = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$$

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

For most material,

$$\mu \approx \mu_0$$

$$\Rightarrow n \approx \sqrt{\varepsilon_r}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) d\tau - \frac{1}{\mu_0}\oint_S (\vec{E}\times\vec{B})\cdot dA$$

$$\Rightarrow S = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

Waves:

Displacement of Particle ⊥ Wave direction : Transverse

Wave : Disturbances

Result is particle going up & down

Sinusoidal Waves:

$$y(x,t) = a\cos k(x - vt)$$

$-ve$ indicates propagation in $+x$ direction

dist. b/w mean position to max height

@ $x = x_1$, Amplitude $= a$

@ $x = x_2$, Amplitude $= a$

$$\lambda = x_2 - x_1$$

x_1 & x_2 - consecutive

$$\omega = kv \quad \& \quad T = \frac{2\pi}{\omega}$$

Derivation:

$$\varphi(x,t) = f(x')$$

$$x' = x \mp vt$$

$$\frac{\partial\varphi}{dx} = \frac{df}{dx}$$

$$\Rightarrow \quad \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{v^2} \frac{\partial \varphi^2}{\partial t^2}$$

$$E = A \cos(kx - \omega t)\, \hat{z}$$

$$f_V(z,t) = A e^{i(kz - \omega t)} \hat{x} \quad : \ x \text{ polarised}$$

$$f_H(z,t) = A e^{i(kz - \omega t)} \hat{y} \quad : \ y \text{ polarised}$$

$$\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

$$\frac{1}{v^2} = \mu_0 \varepsilon_0$$

$$\Rightarrow \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \ m/s$$

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

Formulae List

→ **Cylindrical:**

① $\rho = \sqrt{x^2+y^2}$ $\qquad$ $\begin{bmatrix} x = \rho\cos\phi \\ y = \rho\sin\phi \end{bmatrix}$

$\tan\phi = \dfrac{y}{x}$

②
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

③
(i) $\vec{dl} = dl_\rho\,\hat{\rho} + dl_\phi\,\hat{\phi} + dl_z\,\hat{z}$

$\Rightarrow \vec{dl} = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}$

(ii) $d\vec{s}_\rho = \rho\, d\phi\, dz\,\hat{\rho}$

$d\vec{s}_\phi = d\rho \cdot dz\,\hat{\phi}$

$d\vec{s}_z = \rho\, d\rho\, d\phi\,\hat{z}$

(iii) $d\tau = \rho\, d\rho\, d\phi\, dz$

④ $\vec{\nabla}\cdot\vec{A} = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho A_\rho) + \dfrac{1}{\rho}\dfrac{\partial}{\partial\phi}(A_\phi) + \dfrac{\partial}{\partial z}(A_z)$ $\quad$ [Divergence]

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$ [Curl]

→ Spherical :

① $r = \sqrt{x^2+y^2+z^2}$ $\qquad [\rho = r\sin\theta]$

②

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

③ (i) $\vec{dl} = dr\,\hat{r} + r\sin\theta\, d\phi\,\hat{\phi} + r\,d\theta\,\hat{\theta}$

(ii) $d\vec{S}_r = r^2\sin\theta\, d\theta\, d\phi\,\hat{r}$

$d\vec{S}_\phi = r\,d\theta\, dr\,\hat{\phi}$

$d\vec{S}_\theta = r\sin\theta\, dr\, d\phi\,\hat{\theta}$

(iii) $d\tau = r^2\sin\theta\, dr\, d\phi\, d\theta$

④ $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(A_\phi)$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{A}_r & r\hat{A}_\theta & r\sin\theta\,\hat{A}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

Note : For sphere / Hemisphere, Limits

$\phi \to 0$ to 2π

$\theta \to 0$ to π [Sphere]

0 to $\pi/2$ [Hemisphere]

→ Theorems :

① Fundamental Theorem of Gradient :

$$\int_a^b dT = \int_a^b \vec{\nabla} T \cdot \vec{dl} = T(b) - T(a)$$

② Fundamental Theorem of Divergence :

$$\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{A}$$

③ Stoke's Theorem :

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s} = \oint \vec{F} \cdot \vec{dl}$$

Note :

① $\vec{\nabla} \cdot \vec{v} = 0 \Rightarrow \vec{v}$ is Solenoidal function

② $\vec{\nabla} \times \vec{v} = 0 \Rightarrow \vec{v}$ is Irrotational

$\vec{\nabla} \times \vec{v} \neq 0 \Rightarrow \vec{v}$ is Rotational

Electrostatics

① $\vec{F} = \dfrac{kQq}{r^2}\,\hat{r}$

② $\vec{E}(r) = k\displaystyle\int \frac{dq}{r^2}\,\hat{r}$ $\qquad \left[\, Q = \int \lambda\, dl = \int \sigma\, dA = \int \rho\, dV \,\right]$

$\qquad = k \displaystyle\int \frac{\lambda(r')}{r^2}\,\hat{r}\, dl'$

③ $\text{Flux} = \displaystyle\oint \vec{E}\cdot d\vec{S} = \frac{q_{enc}}{\varepsilon_0}$ → q_{free} enclosed

④ $E = -\vec{\nabla} V$ $\qquad \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ — ①

$\qquad \vec{\nabla}\times\vec{E} = 0$

$V_b - V_a = -\displaystyle\int_a^b \vec{E}\cdot d\vec{l} = \frac{W}{q}$

⑤ $\nabla^2 V = \dfrac{-\rho}{\varepsilon_0} \Longrightarrow \nabla^2 V = 0$

(Poisson's Equation) $\qquad$ (Laplace Equation) → $[\rho = 0]$

⑥ $W = \dfrac{1}{2}\displaystyle\int \rho V\, d\tau$

$\quad = \dfrac{1}{2}\displaystyle\int \sigma V\, dA$

$\quad = \dfrac{1}{2}\varepsilon_0 \displaystyle\int E^2\, d\tau$

$W = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} q_i V_i$

$W = \dfrac{1}{2}(\vec{D}\cdot\vec{E})$

$\quad = \dfrac{1}{2}\displaystyle\int_V \vec{D}\cdot\vec{E}\, d\tau$

⑦ Dipole :

$V = \dfrac{k(\vec{p}\cdot\hat{r})}{r^2}$

$\vec{p} = \alpha\vec{E}$

$\tau = \vec{p}\times\vec{E}$

$\vec{p} = \displaystyle\int \vec{P}\, d\tau$

⑨ Boundary conditions

$D_1^{\perp} - D_2^{\perp} = \sigma_f$

$E_1^{\parallel} = E_2^{\parallel}$

⑧ Bound Charges

$\sigma_b = \vec{P}\cdot\hat{n}$

$\rho = \rho_b + \rho_f \begin{cases} \rho_b = -\vec{\nabla}\cdot\vec{P} \\ \rho_f = \vec{\nabla}\cdot\vec{D} \end{cases}$

$\quad \hookrightarrow \oint D\cdot dA = Q_{free\,enc}$

$\left.\begin{array}{l} \vec{D} = \varepsilon\vec{E} \\ \vec{D} = \varepsilon_0\vec{E} + \vec{P} \end{array}\right\} \Rightarrow \vec{P} = \varepsilon_0 \chi \vec{E}$

$\varepsilon_r = 1 + \chi$

10. $c = \frac{q}{V}$

$W = \frac{1}{2} c V^2$

MAGNETOSTATICS

1. $\vec{F} = q(\vec{v} \times \vec{B})$

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ $[1\,T = 10^4\ \text{Gauss}]$

$\vec{F} = I(\vec{l} \times \vec{B})$

2. $\vec{F} = \int (\vec{J} \times \vec{B})\, d\tau$

$= \int (\vec{K} \times \vec{B})\, ds$

$\vec{\nabla} \times \vec{B} = \mu_0 J_{enc}$

$\vec{\nabla} \cdot \vec{B} = 0$ — ②

$\vec{B} = \vec{\nabla} \times \vec{A}$

3. $B(r) = \frac{\mu_0}{4\pi} \int \frac{i \times \hat{r}}{r^2}\, dl'$

4. $\vec{M} = i\vec{A}_{rea}$

$\vec{\tau} = \vec{M} \times \vec{B}$

5. $\oint \vec{B} \cdot d\vec{l} = \mu_0 i_{enc} \Rightarrow \oint \vec{H} \cdot d\vec{l} = i_{enc}$ (free current) $\boxed{\because B = \mu H}$

$\oint \vec{B} \cdot d\vec{S} = 0$

6. $\nabla^2 A = -\mu_0 J_{enc}$

$A = \frac{\mu_0}{4\pi} \oint \frac{i\, d\vec{l}'}{R}$

$= \frac{\mu_0}{4\pi} \oint_V \frac{J\, d\tau}{R}$

7. $\int \vec{B} \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l} = \text{Flux}$

8. $\vec{\nabla} \times \vec{H} = J_{free} = J_c + J_D = \sigma E + \frac{\partial \vec{D}}{\partial t}$ — ③

(Bound Charges)

$\left.\begin{array}{l} J_b = \vec{\nabla} \times \vec{M} \ \&\ J_f = \vec{\nabla} \times \vec{H} \\ K_b = \vec{M} \times \hat{n} \end{array}\right\} J = J_b + J_f$

$\vec{M} = \chi_m \vec{H}$

$\mu_r = 1 + \chi_m$

$\vec{B} = \mu_0 (1 + \chi_m) \vec{H}$

$M = \frac{m}{V}$, i.e. $m = \int_V M\, d\tau$

$A = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

9 Boundary Conditions :

$$B_1^{\perp} = B_2^{\perp}$$

$$H_1'' - H_2'' = k_f \times \hat{n}$$

10 $$\mathcal{E} = -\frac{d\phi}{dt} = \frac{dW}{dq}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} \quad - (3)$$

$$\mathcal{E} = \int (\vec{v} \times \vec{B}) \cdot \vec{dl}$$

11 $L = \frac{\lambda}{i}$ | $\mathcal{E} = N\frac{d\phi}{dt}$

$N\phi = Li$ | $\mathcal{E} = L\frac{di}{dt}$

$[d\lambda = L\, di]$

$$\therefore \underbrace{N\phi = Li}_{=\lambda} \quad \& \quad \phi = B \cdot A$$

For Circular Coil, $L = \frac{\mu_0 N^2 \pi r^2}{l}$

12 $\phi_2 = Mi_1$

$\phi_1 = Mi_2$

$$\mathcal{E}_1 = -M\frac{d(i_2)}{dt}$$

$$\mathcal{E}_2 = -M\frac{d(i_1)}{dt}$$

13 $$W = \frac{1}{2}Li^2$$

$$= \frac{1}{2\mu_0}\int_V B^2 d\tau \implies W = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\tau}\right)$$

14 $$\vec{S} = \vec{E} \times \vec{H}$$

$$\frac{dP}{dV} = \vec{E} \cdot \vec{J}$$

$$\hookrightarrow \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t}$$

15 $$\langle S_{avg} \rangle = \frac{1}{T}\int_0^T S \cdot dt$$

Maxwells Equations :

1, 2, 3, 4

(1) $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

(2) $\vec{\nabla} \cdot \vec{B} = 0$

(3) $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$

(4) $\vec{\nabla} \times \vec{H} = j + \frac{\partial \vec{D}}{\partial t}$

EM Wave

① $\hat{n} \cdot \hat{z} = 0$

$\hookrightarrow$ Polarisation Vector

② Plane Wave :

$$B_0 = \frac{E_0}{c}$$

$$E = E_0 \cos(\hat{k}\cdot\hat{r} - \omega t + \delta)\ \hat{n}$$

$$B = B_0 \cos(\hat{k}\cdot\hat{r} - \omega t + \delta)\ (\hat{k}\times\hat{n})$$

$$[\hat{k}\cdot\hat{n} = 0]$$

$$\vec{k} = \frac{2\pi}{\lambda}\ \overrightarrow{\text{wave}}$$

③ $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

For most materials,

$$\mu \approx \mu_0$$

$$\therefore n \cong \sqrt{\varepsilon_r}$$