MA1000: Calculus

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Limit: Motivation

A rock breaks loose from the top of a cliff. What is its average speed

1. during the first two seconds of fall?
2. during the 1-second interval between second 1 and second 2?

Solution: Galileo's law: The distance fallen is proportional to the square of the time it has been falling. Indeed if y denotes the distance fallen in feet in t seconds, then

$$y = 16t^2.$$

1. The average speed during the first two seconds is

$$\frac{\Delta y}{\Delta t} = \frac{16(2^2) - 16(0^2)}{2 - 0} = 32 \text{ ft/sec}.$$

2. The average speed during the 1-second interval between second 1 and second 2 is

$$\frac{\Delta y}{\Delta t} = \frac{16(2^2) - 16(1^2)}{2 - 1} = 48 \text{ ft/sec}.$$

Limit: Motivation

Find the instantaneous speed of the rock at $t = 1$ and $t = 2$ seconds.

Solution: The average speed of the rock over a time interval $[t_0, t_0 + h]$ having length h is

$$\frac{\Delta y}{\Delta t} = \frac{16((t_0 + h)^2) - 16(t_0^2)}{h} \text{ ft/sec.}$$

To calculate the speed at t_0, we cannot simply substitute $h = 0$ in the above formula as we cannot divide by zero.
But we can use this formula to compute the the average speed over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. For $h \neq 0$, the above formula simplifies as follows:

$$\frac{\Delta y}{\Delta t} = \frac{16((t_0 + h)^2) - 16(t_0^2)}{h} = \frac{16(t_0^2 + 2t_0 h + h^2) - 16t_0^2}{h} = \frac{32t_0 h + 16h^2}{h} = 32t_0 + 16h.$$

Thus the instantaneous speed of the rock at $t_0 = 1$ second is 32 ft/sec
and the instantaneous speed of the rock at $t_0 = 2$ second is 64 ft/sec.

Limit: Motivation

Definition

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ of length $h \neq 0$ is

$$\frac{\Delta y}{\Delta t} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

Note: Geometrically, the rate of change of f over $[x_1, x_2]$ is the *slope* of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$.
What does happen when $x_2 = x_1$?
We rather see what happens when x_2 approaches x_1 as we cannot substitute x_1 for x_2.
When x_2 approaches x_1, $\frac{\Delta y}{\Delta t} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ could be approaching a finite value.

Limit: Motivation

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution: The given formula defines f for all real numbers x except $x = 1$. For $x \neq 1$, the formula simplifies as follows:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1, \qquad \text{for } x \neq 1.$$

For values of x close to 1, $f(x)$ is close to 2.
In this case, we write

$$\lim_{x \to 1} f(x) = 2.$$

Thus the graph of f is the line $y = x + 1$ with the point $(1, 2)$ removed.

Let

$$g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

What is $\lim_{x\to 1} g(x)$? Is it equal to $g(1)$?

Let

$$h(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

What is $\lim_{x\to 1} h(x)$? Is it equal to $h(1)$?

Examples

1. $\lim_{x \to 2} 24 = 24.$
2. $\lim_{x \to 3} 10 = 10.$
3. $\lim_{x \to 4} x^2 = 16.$
4. $\lim_{x \to 3} (3 - 5x) = -12.$
5. $\displaystyle\lim_{x \to -1} \frac{3 - 2x}{x - 1} = \frac{-5}{2}.$

More Examples

1. If f is the **identity function** $f(x) = x$, then for any value x_0

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

2. If f is the **constant function** $f(x) = k$, then for any value x_0

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

Nonexistence of Limit

Discuss the behavior of the following functions as $x \to 0$:

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

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Theorem (Limit Laws)

Let L, M, c be real numbers and let

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = M. \text{ Then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M.$
2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M.$
3. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L.$
4. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M.$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \qquad M \neq 0.$
6. *If r and s are integers with no common factors and $s \neq 0$, then*

   $$\lim_{x \to c} f(x)^{r/s} = L^{r/s}$$

   provided $L^{r/s}$ is a real number. (If s is even, then $L > 0$.)

Example: Limit Laws

Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$ and the properties of limits to find the following limits.

1. $\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 = c^3 + 4c^2 - 3.$
2. $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5} = \dfrac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)} = \dfrac{c^4 + c^2 - 1}{c^2 + 5}.$
3. $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2} (4x^2 - 3)} = \sqrt{13}.$

Theorem (Limits of Polynomials)

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, *then*

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \ldots + a_0.$$

Theorem (Limits of Rational Functions)

If $P(x)$ *and* $Q(x)$ *are polynomials and* $Q(c) \neq 0$, *then*

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Theorem (The Sandwich Theorem for Limits of Functions)

Suppose that $g(x) \leq f(x) \leq h(x)$ in some open interval containing c except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Example:

Let

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{4} \quad \text{for all } x \neq 0.$$

Since $\lim_{x \to 0} (1 - \frac{x^2}{4}) = \lim_{x \to 0} (1 + \frac{x^2}{4}) = 1$, by the Sandwich Theorem, $\lim_{x \to 0} u(x) = 1$ no matter how complicated u is.

Examples

1. From the definition of $\sin\theta$, we have that $-|\theta| \leq \sin\theta \leq |\theta|$.
   Also $\lim\limits_{\theta\to 0}(-|\theta|) = \lim\limits_{\theta\to 0}|\theta| = 0$.
   Hence, by the Sandwich Theorem,
   $$\lim_{\theta\to 0}\sin\theta = 0.$$
2. From the definition of $\cos\theta$, $0 \leq 1-\cos\theta \leq |\theta|$. Hence $\lim\limits_{\theta\to 0}(1-\cos\theta) = 0$ or
   $$\lim_{\theta\to 0}\cos\theta = 1.$$
3. For any function $f(x)$, if $\lim\limits_{x\to c}|f(x)| = 0$, then $\lim\limits_{x\to c} f(x) = 0$ since $-|f(x)| \leq f(x) \leq |f(x)|$ for all x.

Theorem

If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$, and the limits f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

The Precise Definition of Limit

Guess the value of

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}.$$

For some values of x close to zero (like $\pm 1, \pm 0.5, \pm 0.1$), it is close to 0.05.

For some other values of x close to zero (like $\pm 0.0005, \pm 0.0001, \pm 0.00001, \pm 0.000001$), it is close to 0.

Does the limit exist? If so, what is it?

The limit exists and it is 0.05. But how can we be sure?

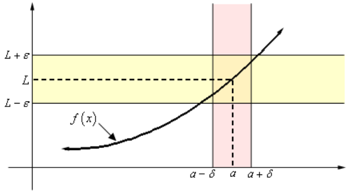
Definition (Limit of a Function)

Let $f(x)$ be defined on an open interval containing x_0 except possibly at x_0. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$



Note: The figure is from the internet.

Example

Show that

$$\lim_{x \to 1} \left(\frac{3}{2}x - 1 \right) = \frac{1}{2}.$$

Solution: Here $x_0 = 1$, $f(x) = \dfrac{3}{2}x - 1$ and $L = \dfrac{1}{2}$.
Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - \frac{1}{2}| < \epsilon.$$

We work backward:

$$\begin{aligned}
\left| \left(\frac{3}{2}x - 1 \right) - \frac{1}{2} \right| \quad &= \quad \left| \frac{3}{2}x - \frac{3}{2} \right| < \epsilon \\
&\Leftrightarrow \quad \frac{3}{2}|x - 1| < \epsilon \\
&\Leftrightarrow \quad |x - 1| < \frac{2}{3}\epsilon
\end{aligned}$$

Let us take $\delta = \frac{2}{3}\epsilon$. Let $0 < |x-1| < \delta = \frac{2}{3}\epsilon$. Then

$$\left| \left(\frac{3}{2}x - 1 \right) - \frac{1}{2} \right| = \left| \frac{3}{2}x - \frac{3}{2} \right| = \frac{3}{2}|x-1| < \frac{3}{2}\left(\frac{2}{3}\epsilon \right) = \epsilon.$$

Thus $\lim\limits_{x \to 1} \left(\frac{3}{2}x - 1 \right) = \frac{1}{2}$.

Homework

Prove using the definition of limit that (a) $\lim_{x \to x_0} x = x_0$ and (b) $\lim_{x \to x_0} k = k$.

Solution: (a) Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |x - x_0| < \epsilon.$$

The above implication will hold if δ equals ϵ or any other positive number less than ϵ. This prove that $\lim_{x \to x_0} x = x_0$.

(b) Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |k - k| < \epsilon.$$

Since $|k - k| = 0$, we can use any positive number as δ and the implication will hold. Thus $\lim_{x \to x_0} k = k$.

Example

Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Solution: Given $\epsilon > 0$, we must show that there is a $\delta > 0$ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \epsilon.$$

We first solve the inequality $|f(x) - 4| < \epsilon$ to find an open interval containing $x_0 = 2$ on which the inequality holds for all $x \neq x_0$:

$$\begin{aligned} |f(x) - 4| < \epsilon \quad &\Leftrightarrow \quad |x^2 - 4| < \epsilon \\ &\Leftrightarrow \quad -\epsilon < x^2 - 4 < \epsilon \\ &\Leftrightarrow \quad 4 - \epsilon < x^2 < 4 + \epsilon \\ &\Leftrightarrow \quad \sqrt{4 - \epsilon} < |x| < \sqrt{4 + \epsilon} \quad \text{assuming } \epsilon \leq 4. \end{aligned}$$

Thus for $x \neq 2$ such that $\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}$, we have $|f(x) - 4| < \epsilon$.
We have thus found an open interval containing 2 (but excluding 2) for which $|f(x) - 4| < \epsilon$.

We now choose a $\delta > 0$ such that the centered interval $(2-\delta, 2+\delta)$ is contained in the above interval.
For this, we take $\delta = \min\{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$. This implies that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \epsilon.$$

If $\epsilon > 4$, then we take δ to be the distance from $x_0 = 2$ to the nearer endpoint of the interval $(0, \sqrt{4+\epsilon})$: That is $\delta = \min\{2, \sqrt{4+\epsilon} - 2\}$.

Using the Definition of Limit to Prove Theorems

Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, prove that

$$\lim_{x\to c}(f(x)+g(x)) = L+M.$$

Solution: Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that

$$0 < |x-c| < \delta \quad \Rightarrow \quad |f(x)+g(x)-(L+M)| < \epsilon.$$

Now

$$|f(x)+g(x)-(L+M)| = |(f(x)-L)+g(x)-M| \le |f(x)-L| + |g(x)-M|.$$

Since $\lim_{x\to c} f(x) = L$, corresponding to $\epsilon/2 > 0$, there exists a $\delta_1 > 0$ such that

$$0 < |x-c| < \delta_1 \quad \Rightarrow \quad |f(x)-L| < \epsilon/2.$$

Similarly, since $\lim_{x \to c} g(x) = M$, corresponding to $\epsilon/2 > 0$, there exists a $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \quad \Rightarrow \quad |g(x) - M| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$.
Let x be such that $0 < |x - x_0| < \delta$. Then $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$ and hence

$$|f(x) + g(x) - (L + M)| = |(f(x) - L) + g(x) - M| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Homework

Given that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ and that $f(x) \leq g(x)$ in an open interval containing c except possibly at c prove that $L \leq M$.

One-Sided Limits

If $\lim_{x \to c} f(x) = L$, then it means that as x approaches c from either left or from right $f(x)$ approaches L.
In other words, both the left-hand and right-hand limits exist and equal L.
That is,

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

The converse also is true: If the left-hand and right-hand limits of a function exist as x approaches c and are equal, then $\lim_{x \to c} f(x)$ exists and equals the common left-hand and right-hand limits.

Theorem

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c-} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c-} f(x) = L \quad \textit{and} \quad \lim_{x \to c^+} f(x) = L.$$

One-Sided Limits: Examples

1. The function $f(x) = \dfrac{x}{|x|}$ has limit 1 as x approaches 0 from the right and has limit -1 as x approaches 0 from the left. Thus the left-hand and right-hand limits exists as x approaches 0 but they are not equal. So, $\lim_{x \to 0} f(x)$ does not exist.
2. The function $f(x) = \sqrt{4 - x^2}$ is defined at all points on the closed interval $[-2, 2]$. We have
$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$
This function does not have a left-hand limit at $x = -2$ and a right-hand limit at $x = 2$. Thus it does not have ordinary two-sided limits at either -2 or 2.

The Precise Definition

Definition (Right-Hand, Left-Hand Limits)

We say that $f(x)$ has **right-hand limit** L **at** x_0, and write

$$\lim_{x \to x_0^+} f(x) = L,$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that $f(x)$ has **left-hand limit** L **at** x_0, and write

$$\lim_{x \to x_0^-} f(x) = L,$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Example

Prove using the definition of limit that

$$\lim_{x\to 0^+} \sqrt{x} = 0.$$

Solution: Let $\epsilon > 0$ be given. Here $x_0 = 0$ and $L = 0$. So, we want to find a $\delta > 0$ such that

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon \quad \text{or} \quad 0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \epsilon.$$

Now $\sqrt{x} < \epsilon$ if $0 < x < \epsilon^2$. So, we take $\delta = \epsilon^2$. And have that

$$0 < x < \delta = \epsilon^2 \quad \Rightarrow \quad \sqrt{x} < \epsilon.$$

This shows that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Homework

1. Show that $f(x) = \sin(\frac{1}{x})$ has no limit as x approaches zero from either side.
2. Prove that $\lim_{x \to 0} \frac{\sin \theta}{\theta} = 1$.

Continuity

Definition (Continuous at a Point)

A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

A function $y = f(x)$ is **continuous at a left end point point** a or is **continuous at a right end point point** b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b), \qquad \text{respectively.}$$

Examples

1. The identity function $f(x) = x$ is continuous at every point in the interval $(-\infty, \infty)$.
2. The constant function $f(x) = k$ is continuous at every point in the interval $(-\infty, \infty)$.
3. The function $f(x) = \dfrac{1}{x}$ is continuous at every point in the interval $(0, \infty)$.
4. The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point in the interval $[-2, 2]$. In particular, it is **right-continuous** at $x = -2$ and is **left-continuous** at $x = 2$.
5. The unit step function $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ is continuous at all points except 0. It is right-continuous at $x = 0$ but is not left-continuous there. So, it is not continuous at $x = 0$. It has a jump discontinuity at $x = 0$.

Homework: Discuss the continuity of the greatest integer function $f(x) = \lfloor x \rfloor$; that is, $f(x)$ is the greatest integer less than or equal to x for every x. (It is right-continuous at every integer n but is not left-continuous.)

Types of Discontinuity

Definition

- A function $f(x)$ is said to have a **removable discontinuity** at $x = c$ if $\lim_{x \to c} f(x)$ exists but f is not defined at c or $f(c)$ is defined at c but is not equal to the limit at c.
- A function $f(x)$ is said to have a **jump discontinuity** at $x = c$ if the left-hand and right-hand limits exist at $x = c$ but they are not equal.
- A function $f(x)$ is said to have an **infinite discontinuity** at $x = c$ if the left-hand and/or the right-hand limits at c are $\pm\infty$.
- A function $f(x)$ is said to have an **oscillating discontinuity** at $x = c$ if the function oscillates too much as x approaches c and so does not have a limit at $x = c$.

Examples

- Removable discontinuity: The function $f(x) = \begin{cases} x+3, & x \neq 0 \\ 1, & x = 0 \end{cases}$ has a removal discontinuity at $x = 0$.
- Jump discontinuity: The unit step function $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ has a jump discontinuity at $x = 0$.
- Infinite discontinuity: The function $f(x) = \dfrac{1}{x^2}$ has an infinite discontinuity at $x = 0$.
- Oscillating discontinuity: The function $f(x) = \sin(\frac{1}{x})$ has an oscillating discontinuity at $x = 0$.

Definition

A function $f(x)$ is said to be **continuous on an interval** if it is continuous at every point of the interval.

Examples:

- The identity function $f(x) = x$ is continuous on the interval $(-\infty, \infty)$.
- The constant function $f(x) = k$ is continuous on the interval $(-\infty, \infty)$.
- The function $f(x) = \sqrt{4 - x^2}$ is continuous on the interval $[-2, 2]$.
- The function $f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$.
- The function $f(x) = 1/x$ is continuous on the interval $(0, \infty)$ but is not continuous on the interval $(-\infty, \infty)$.

Theorem

If f and g are continuous at $x = c$, then the following combinations are also continuous at $x = c$:

1. *Sum:* $f + g$.
2. *Difference:* $f - g$.
3. *Constant Multiple:* $k \cdot f$.
4. *Product:* $f \cdot g$.
5. *Quotient:* $\dfrac{f}{g}$.
6. *Powers:* $f^{r/s}$, *provided it is defined on an open interval containing c, where r and s are integers.*

Proof:

The theorem follows from the theorem on limits of sum of functions, difference of functions, etc. For instance, we prove Part (1) as follows:

$$\begin{aligned}
\lim_{x\to c}(f+g)(x) &= \lim_{x\to c}(f(x)+g(x)) \\
&= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \\
&= f(c)+g(c) \\
&= (f+g)(c)
\end{aligned}$$

Examples

1. Every polynomial $P(x)$ is continuous because $\lim_{x \to c} P(x) = P(c)$.
2. Every rational function $\dfrac{P(x)}{Q(x)}$ is continuous wherever it is defined (i.e., $Q(c) \neq 0$).
3. The absolute value function $f(x) = |x|$ is continuous at every value of x. If $x > 0$, $f(x) = x$, a polynomial. If $x < 0$, $f(x) = -x$, another polynomial. At the origin, $\lim_{x \to 0} |x| = 0 = |0|$.

Note

A function $f(x)$ is continuous at $x = c$ if and only if

$$\lim_{x \to c} f(x) = f(c) \quad \text{or} \quad \lim_{h \to 0} f(c+h) = f(c).$$

Homework: Prove that $\sin x$ and $\cos x$ are continuous at every x. (Use the facts that $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} \cos x = 1$.)

The Continuity of Composite Functions

Theorem

If f is continuous at c and g continuous at $f(c)$, then the composite function $g \circ f$ is continuous at c.

Homework: Show that the following functions are continuous wherever they are defined.

1. $f(x) = \sqrt{x^2 - 2x - 5}$.
2. $f(x) = \left| \dfrac{x-2}{x^2-2} \right|$.

Continuous Extension to a Point

The function $f(x) = \dfrac{\sin x}{x}$ is defined and continuous at every point except $x = 0$. But it has a finite limit, namely 1, as $x \to 0$. So, we define a new function

$$F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$F(x)$ is defined and continuous at every x. In particular, it is continuous at $x = 0$ because

$$\lim_{x \to 0} F(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 = F(0).$$

$F(x)$ is called the **continuous extension of** $f(x)$ **to** $x = c$.

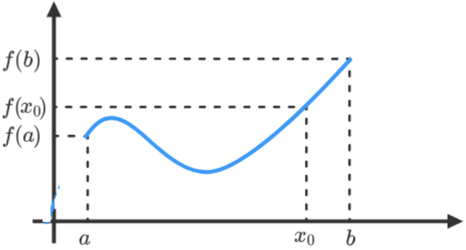
Homework: Show that

$$f(x) = \frac{x^2 + x - 2}{x^2 - 1}$$

has a continuous extension to $x = 1$ and find the extension.

Theorem (Intermediate Value Theorem for Continuous Functions)

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(x_0)$ for some x_0 in $[a, b]$.



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