Theorem: (The set of feasible solutions to an LPP is a convex set)

Proof:- Let the LPP be to determine X so that maximize $Z = CX$, $X^T \in \mathbb{R}^n$, $C \in \mathbb{R}^n$

subject to constraints

$$\boxed{Ax = b}, \quad x \geq 0$$

Let X_1 & X_2 be two feasible solutions of

$Ax_1 = b$; $Ax_2 = b$; $X_1 \geq 0$ & $X_2 \geq 0$

$$X = \lambda X_1 + (1-\lambda) X_2, \quad 0 \leq \lambda \leq 1.$$

Clearly, $Ax = A(\lambda x_1 + (1-\lambda) x_2)$

$= \lambda A x_1 + (1-\lambda) A x_2$

$= \lambda b + (1-\lambda) b = b$

Again, since $x_1 \geq 0$, $x_2 \geq 0$

$\lambda, 1-\lambda \geq 0$, $\boxed{\therefore X \geq 0}$

Hence X is also feasible solution to the problem. Thus the set

$S = \{X \mid x \text{ is a feasible sol}^n \text{ to LPP}\}$

is convex set.

(FS to BFS)

Theorem:- If an LPP has a FS then it also has an BFS

Proof :- Let the LPP be to determine X so as to maximize

$$Z = cX \quad ; \; c, X^T \in \mathbb{R}^n$$

Subject to the constraints

$$Ax = b, \quad x \geq 0$$

where A is an $m \times n$ real Matrix and b, c are $m \times 1$ and $1 \times n$ real matrices respectively.

Let $\rho(A) = m$ (Rank of matrix)

Since there exist a feasible solution.

$\Rightarrow$ System is consistent

$\Rightarrow$ $\boxed{\rho(A, b)} = \boxed{\rho(A)}$ & $m < n$

Let $x = (x_1, x_2, \ldots, x_n)$ be a feasible solution so that $x_j \geq 0$ for all j.

$\Rightarrow$ Suppose x has 'p' positive components. & $n-p$ zero.

Relabel the components such that the first 'p' components are positive & correspondingly columns of A have been relabelled accordingly then $\rightarrow$ $\boxed{a_1} x_1 + \boxed{a_2} x_2 + \cdots + \boxed{a_p} x_p = b$

$a_1, a_2, \ldots, a_p$ are 1st p columns of A.

Two cases now do arise:

$\boxed{\text{Case-I}}$ The vectors $a_1, a_2, \ldots, a_p$ are LI (rank $-m$)

$\boxed{\text{Then } p \leq m.}$

$\boxed{\text{If } p = m}$ solution is non-degenerate BFS, with $x_1, x_2, \ldots, x_p$ as basic var

$\boxed{\text{If } p < m}$ then set $\{a_1, a_2, \ldots, a_p\}$ can be extended to $\{a_1, a_2, \ldots, a_p, a_{p+1}, \ldots, a_m\}$ to form a basis for the columns of A.

Then, we have

$$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = b$$

where $x_j = 0$ for $j = p+1, p+2, \dots, m$

Thus we have, in this case, a degenerate basic feasible solution with (m-p of the basic) variables zero.

$\boxed{\text{Case-II}}$ The set $\{a_1, a_2, \dots, a_p\}$ is LD.

Obviously $p > m$.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ be a set of constants (not all zero) such that

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_p a_p = 0$$

Suppose $\alpha_r \neq 0$

then $$\boxed{a_r = -\sum_{j \neq r}^{p} \frac{\alpha_j a_j}{\alpha_r}}$$

$$\boxed{\sum_{j=1}^{p} a_j x_j = b}$$

$$\Rightarrow \sum_{\substack{j=1 \\ j \neq r}}^{p} a_j x_j + a_r x_r = b$$

$$\Rightarrow \sum_{\substack{j=1 \\ j \neq r}}^{p} a_j x_j + x_r \left(- \sum_{\substack{j=1 \\ j \neq r}}^{p} \frac{\alpha_j}{\alpha_r} a_j \right) = b$$

$$\Rightarrow \quad \| \quad - \sum_{\substack{j=1 \\ j \neq r}}^{p} \left(\frac{\alpha_j x_r}{\alpha_r} \right) a_j = b$$

$$\sum_{\substack{j=1 \\ j \neq r}}^{p} \left(x_j - \frac{x_r}{\alpha_r} \alpha_j \right) a_j = b \quad \underbrace{}_{p-1}$$

Start $\rightarrow$ $\displaystyle\sum_{j=1}^{p} x_j a_j = b$ $\rightarrow$ p-variable.

$\alpha_j = 0 \rightarrow x_j - 0 \geq 0$

$\alpha_j < 0 \rightarrow x_j - \frac{x_r}{\alpha_r} \alpha_j \geq 0$

$\alpha_j > 0 \rightarrow x_j - \frac{x_r}{\alpha_r} \alpha_j \geq 0$

$\Rightarrow x_j \geq \frac{x_r}{\alpha_r} \alpha_j \Rightarrow \frac{x_j}{\alpha_j} \geq \frac{x_r}{\alpha_r}$ $\quad j = 1 \ldots p,\ j \neq r$

$$\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j} , \underline{\alpha_j > 0} \right\}$$

Which j is our 'r'

p +ve Variable. $x_1, x_2, \ldots, x_p$

$p-1$ +ve Var $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{p-1}$

$$\hat{x}_1 = x_1 - \frac{x_r}{\alpha_r}\alpha_1$$

$$\hat{x}_2 = x_2 - \frac{x_r}{\alpha_r}\alpha_2 \quad \ldots \quad \hat{x}_p = x_p - \frac{x_r}{\alpha_r}\alpha_p$$

$a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_p$

$(p-1)$ columns

If this $p-1$ columns LI then stop

If $p-1$ column not LI $\longrightarrow$ LD

$$\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_{p-1} a_{p-1} = 0$$

some $\beta_j \neq 0$

$p - i = M$

Ex:- Let $x_1 = 2$, $x_2 = 4$ & $x_3 = 1$ be a feasible solution to the system

$$\begin{cases} 2x_1 - x_2 + 2x_3 = 2 \\ x_1 + 4x_2 = 18 \end{cases}$$

Reduce this FS to a BFS.

2×3

Solution:-

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 4 & 0 \end{bmatrix}, \quad b = \begin{pmatrix} 2 \\ 18 \end{pmatrix}$$

$$\text{Rank } A = 2$$

$$a_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$p = 3 > \boxed{\text{rank} = m = 2}$

$$\underline{\alpha_1 a_1 + \alpha_2 a_2} \; \underline{+ \alpha_3 a_3} = 0$$

for some $\alpha_i \neq 0$

$\alpha_3 = -1$

$$a_3 = \alpha_1 a_1 + \alpha_2 a_2$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2\alpha_1 - \alpha_2 = 2 \\ \alpha_1 + 4\alpha_2 = 0 \end{cases}$$

$\alpha_1 = 8/9$

$\alpha_2 = -2/9$

$$\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j} , \underline{\underline{\alpha_j > 0}} \right\}$$

$$= \min \left\{ \frac{2}{8/9} \right\} = 9/4$$

$j=1 \rightarrow r=1$

$$\hat{x}_1 = x_1 - \frac{x_1}{\alpha_1} \alpha_1 = \underline{\underline{0}}$$

$$\hat{x}_2 = x_2 - \frac{x_1}{\alpha_1} \alpha_2 = 4 - \frac{9}{4}\left(-\frac{2}{9}\right) = 9/2$$

$$\hat{x}_3 = x_3 - \left(\frac{x_1}{\alpha_1}\right) \alpha_3 = 1 - \frac{9}{4}(-1) = \frac{13}{4}$$

BFS.

$\boxed{Ex}$ $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ is FS

of system

$$\begin{cases} x_1 + x_2 + 2x_3 = 4 \\ 2x_1 - x_2 + x_3 = 2 \end{cases}$$

Reduce FS to BFS

How to find an Improve BFS

M, W, Thu, Fri, SAT