TUTORIAL - 1 ON LAPLACE TRANSFORMS

1 Show explicitly that Laplace transform of e^{x^2} does not exit.

2 Find the Laplace transform of the following functions.

(a) $f(x) = [x]$, where [x] denotes the greatest integer $\leq x$.

(b) $f(x) = e^{-x}\left(3\sinh(2x) - 5\cosh(2x)\right)$

(c) $f(x) = e^{x}x^{2}\sin(4t)$

(d) $f(x) = \displaystyle\int_0^x \int_0^v \frac{1-e^{-u}}{u} du dv$

3 Find inverse Laplace transform of the following functions.

(a) $F(p) = \dfrac{1}{p}\sin\left(\dfrac{1}{p}\right)$, (b) $F(p) = \cot^{-1}(p+1)$, (c) $F(p) = \dfrac{2p^2-4}{(p-3)(p^2-p-2)}$

4 Solve the following differential equation with the help of Laplace transform.

(a) $y'' + 2y' + 5y = 3e^{-x}\sin(x), \quad y(0) = 0, \quad y'(0) = 3$

(b) $xy'' + 2y' + xy = 0, \quad y(0) = 1, \quad y(\pi) = 0$

5 Solve $y' + 4y + 5\int_0^x y dx = e^{-x}, \quad y(0) = 0.$

6 Solve the integral equation $y(x) = x^3 + \displaystyle\int_0^x \sin(x-t)y(t)dt$ using Laplace transform.

TUTORIAL - 2 ON LAPLACE TRANSFORMS

1 Find the Laplace transform of the following functions.

$(a)\ f(x) = 4e^{-2x} + \sin(3x) - 4\cos(5x) + 12x^3 - 5$ $\quad (b)\ f(x) = \sin(5x)\cos(3x)$

$(c)\ f(x) = \cosh^2(4x)$ $\quad (d)\ f(x) = (x+2)^2 e^x$

$(e)\ f(x) = \cos(ax)\sinh(bx)$ $\quad (f)\ f(x) = e^{-2x}\cos^2(x)$

$$(g)\ f(x) = \begin{cases} \sin(x - \pi/3) & \text{if} \quad x > \pi/3 \\ 0 & \text{if} \quad x < \pi/3 \end{cases} \qquad (h)\ f(x) = \sin(ax) - ax\cos(ax) + \frac{\sin(x)}{x}$$

$(i)\ f(x) = xe^{ax} sin(bx)$ $\quad (j)\ f(x) = \int_0^x \frac{1 - e^{-u}}{u} du$

$(k)\ f(x) = \int_0^x \frac{\sin t}{t} dt$ $\quad (l)\ f(x) = erf(\sqrt{x})$

2 Dose the Laplace transform of foillowing function exist?

$(i)\ \frac{1}{x+2}$ $\qquad (ii)\ e^{x^2 - x}$ $\qquad (iii)\ \sin(x^2)$

3 Find $L\{sin\sqrt{t}; p\}$. Also obtain $L\left\{\frac{cos\sqrt{t}}{\sqrt{t}}; p\right\}$

4 Prove that $L\left\{\frac{cos(at) - cos(bt)}{t}; p\right\} = \frac{1}{2}\log\left(\frac{p^2 + b^2}{p^2 + a^2}\right)$ and deduce that

$$\int_0^\infty \frac{cos(6t) - cos(4t)}{t} dt = \log(2/3)$$

5 Prove that $L\left\{\frac{sin^2(x)}{x};p\right\} = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right)$ and deduce that

(i) $\int_0^\infty e^{-x}\frac{sin^2(x)}{x}dx = 0.25\log(5)$, (ii) $\int_0^\infty \frac{sin^2(x)}{x^2}dx = \pi/2$

6 Evaluate the following integral with the help of Laplace Transform.

$$(i) \quad \int_0^\infty x^3 e^{-x} sin(x)dx, \quad (ii) \quad \int_0^\infty \frac{e^{-x}\ sin(x)}{x}dx, \quad (iii) \quad \int_0^\infty xe^{-2x}cos(x)dx.$$

7 Prove that

$$L\{H(x-a);p\} = \frac{e^{-ap}}{p},$$

where $H(x-a)$ is Heaviside's unit step function.

8 Show that $L\{\delta(x-a);p\} = e^{-ap}$, where $\delta(x)$ is Dirac-delta function.

9 Prove that $L\{J_0(x);p\} = \frac{1}{\sqrt{p^2+1}}$, deduce that

(i) $L\{J_0(ax);p\} = \frac{1}{\sqrt{p^2+a^2}}$, (ii)$L\{xJ_0(ax);p\} = \frac{p}{(p^2+a^2)^{3/2}}$,

(iii) $\int_0^\infty J_0(x)dx = 1$,

(iv)$L\{J_1(x);p\} = 1 - \frac{p}{\sqrt{p^2+1}}$,

(v) $L\{xJ_1(x);p\} = \frac{1}{(p^2+1)^{3/2}}$

11 Find inverse Laplace transform of the following functions.

$(a)\ F(p) = \frac{1}{p^4} + \frac{3p}{p^2+16} + \frac{5}{p^2+4}$ $\quad (b)\ F(p) = \frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9}$

$(c)\ F(p) = \frac{p^2-1}{(p^2+1)^2}$ $\quad (d)\ F(p) = \frac{p}{(p+3)^{7/2}}$

$(e)\ F(p) = \frac{p}{(p+1)^5}$ $\quad (f)\ F(p) = \frac{1}{\sqrt{(2p+3)}}$

$(g)\ F(p) = \frac{1}{\sqrt{(p^2-4p+20)}}$ $\quad (h)\ F(p) = \log\left(\frac{p^2+a^2}{p^2+b^2}\right)$

$(i)\ F(p) = \frac{1}{p}\log\left(\frac{p+2}{p+1}\right)$ $\quad (j)\ F(p) = \frac{1}{(p+2)(p^2+4)}$

$(k)\ F(p) = \frac{p}{(p^2+a^2)^3}$ $\quad (l)\ F(p) = \cot^{-1}(p+1)$

$(m)\ F(p) = \frac{1}{p^3(p^2+1)}$ $\quad (l)\ F(p) = \frac{1}{(p+1)(p^2+1)}$

12 Solve the following differential equation with the help of Laplace transform.

(a) $y'' + y = x\cos(2x), \quad y(0) = y'(0) = 0$

(b) $y'' + 2y' + y = x, \quad y(0) = -3, \quad y(1) = -1$

(c) $xy'' + 2y' + xy = 0, \quad y(0) = 1, \quad y(\pi) = 0$

(d) $xy'' + (x-1)y' - y = 0, \quad y(0) = 5, \quad y(\infty) = 0$

(e) $y'' + 2y' + 10y = \delta(x), \quad y(0) = 0, \quad y'(0) = 0$ where $\delta(x)$ is Dirac-delta function.