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MA1000 Calculus - End Semester Examination (B Batch)

March 1, 2021

Time: 9:30 - 12:00 | Answer All Questions | Marks: 40

1. Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ converges. (4)

2. For the following series find the interval of convergence and, within the interval of convergence, the sum of the series as a function of x:
(a) $\displaystyle\sum_{n=0}^{\infty}\frac{(x-1)^{2n}}{4n}$; (b) $\displaystyle\sum_{n=0}^{\infty}(\ln x)^n$. (6)

3. Prove using the $\epsilon-\delta$ definition that $\lim_{x\to 3}(x^2+2x)=15$. (4)

4. Prove that a function f is continuous at a point a if and only if for every $\epsilon>0$ there exists a $\delta>0$ such that (3)
$$|x-a|<\delta \quad\Rightarrow\quad |f(x)-f(a)|<\epsilon.$$

5. Let f and g be continuous on a closed interval $[a,b]$ with $g(a)\neq g(b)$. If both f and g are differentiable on the open interval (a,b) and the derivatives $f'(x)$ and $g'(x)$ are not zero for any x in (a,b), then prove that there exists a point c in (a,b) such that (4)
$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}.$$

6. Find the absolute maximum and minimum values of $f(x)=3x^{2/3}$ defined on the interval $-27\leq x\leq 8$. (3)

7. Consider a bounded function $f:[a,b]\to\mathbb{R}$. Prove that its lower Riemann integral is less than or equal to its upper Riemann integral. (3)

8. Consider $f(x)=x^2$ on the interval $[1,3]$. Find a sequence of partitions P_n of $[1,3]$ such that $\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}R(f,P_n)$. Hence prove that the function is Riemann integrable. Prove also that this common limit equals the Riemann integral $\displaystyle\int_1^3 x^2dx$. (7)

9. Consider the xy-plane. Prove or disprove the following statements:
(a) $\{(x,y)\mid x^2+y^2=1 \text{ and } y\geq 0\}$ is a closed set;
(b) $\{(x,y)\mid x^2+y^2=1 \text{ and } y>0\}$ is an open set. (2)

10. Prove: The function $f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ is continuous everywhere. (4)