SEQUENCES

CONVERGENCE

if $\{a_n\} \to l$, $\forall \varepsilon > 0$, there exists a $N > 0$ s.t.

$$|a_n - l| < \varepsilon \quad \forall n \geq N$$

$\Rightarrow -\varepsilon < a_n - l < \varepsilon$

$\Rightarrow l - \varepsilon < a_n < l + \varepsilon$

$\Rightarrow \quad a_n \in (l - \varepsilon, l + \varepsilon)$ when $n \geq N$

$\hookrightarrow \boxed{\lim_{n \to \infty} a_n = l}$

TPT: $\lim_{n \to \infty} \frac{1}{n} = 0$. $(a_n = 1/n, \; l = 0)$

$\forall \varepsilon > 0$, there exists a N s.t

$\forall n \geq N$, $|a_n - l| < \varepsilon$

$\Rightarrow |1/n - 0| < \varepsilon$

$\Rightarrow \quad 1/n < \varepsilon \Rightarrow 1/\varepsilon < n$

$\therefore$, $\forall n > 1/\varepsilon$, $a_n \to l$.

$\Rightarrow \boxed{N = 1/\varepsilon}$.

as N exists, $1/n$ converges to 0

$\Rightarrow \lim_{n \to \infty} 1/n = 0$

T.P.T $1 < n^{1/n} < 2 \quad \forall n \geq 2$

let $h_n = n^{1/n} - 1$

$\Rightarrow n^{1/n} = 1 + h_n$

$\Rightarrow n = (1+h_n)^n$

$\Rightarrow n = \underline{1 + {}^nC_1 h_n + {}^nC_2 h_n^2 + \cdots}$

$\Rightarrow n > n h_n$

$\Rightarrow 1 > h_n$

$\Rightarrow 1 > n^{1/n} - 1$

$\Rightarrow 2 > n^{1/n}$

$n = 1 + h_n + n h_n^2 + {}^nC_2 h_n^3 + \cdots$

$\Rightarrow n > {}^nC_2 h_n^2$

$\Rightarrow 1 > \frac{n-1}{2} h_n^2$

$\sqrt{\frac{2}{n-1}}$

$\Rightarrow \sqrt{\frac{2}{n-1}} > h_n$

$\Rightarrow \sqrt{\frac{2}{n-1}} > n^{1/n} - 1$

$\Rightarrow \sqrt{\frac{2}{n-1}} < \varepsilon \Rightarrow \frac{2}{n-1} < \varepsilon^2$

$\Rightarrow \frac{n-1}{2} > 1/\varepsilon^2$

$$\Rightarrow n > \frac{2}{\varepsilon^2} + 1$$

$\therefore$, $N = 2/\varepsilon^2 + 1$, exists!

CONVERGENCE TO SAME LIMIT

let $a_n \rightarrow$ converge to l_1 & l_2.

$\Rightarrow \forall \frac{\varepsilon}{2} > 0$, there exists an $N_1 > 0$ s.t. & $N_2 > 0$ s.t

$\forall n \geq N_1$, $|a_n - l_1| < \varepsilon/2$

$\forall n \geq N_2$, $|a_n - l_2| < \varepsilon/2$

$\Rightarrow \forall n \geq N = \max(N_1, N_2) \Rightarrow |a_n - l_1| + |a_n - l_2| < \varepsilon.$

$$\Rightarrow |a_n - l_1 - a_n + l_2| < \varepsilon$$

$$\Rightarrow |l_2 - l_1| < \varepsilon$$

$$\Rightarrow \boxed{l_2 = l_1}$$

TPT: $(-1)^{n+1}$ diverges

$$(-1)^{n+1} = \begin{cases} 1 & , n \text{ is odd} \\ -1 & , n \text{ is even} \end{cases}$$

$\Rightarrow$ Let $(-1)^{n+1}$ converge to 1.

$\Rightarrow$ $\forall \varepsilon > 0$, there exists an $N > 0$ s.t.

$|1 - l| < \varepsilon$

$\Rightarrow -\varepsilon < 1 - l < \varepsilon$

$\Rightarrow -\varepsilon < l - 1 < \varepsilon \Rightarrow 1 - \varepsilon < l < 1 + \varepsilon$

Let $(-1)^{n+1}$ converge to -1

$\forall \varepsilon > 0$, there exists an $N > 0$ st

$|-1 - l| < \varepsilon$

$\Rightarrow -\varepsilon < -1 - l < \varepsilon$

$\Rightarrow -\varepsilon < 1 + l < \varepsilon$

$\Rightarrow -1 - \varepsilon < l < \varepsilon - 1$

but, $\varepsilon < 1 \Rightarrow \varepsilon - 1$ is $-ve$.

$\Rightarrow$ acc. to 2 intervals, l lies b/w 2 positive no.s & 2 $-ve$ no.s, which is wrong.

$\therefore$, diverges

TPT: $\sqrt{n}$ diverges

let $\sqrt{n} \to l$.

$$\Rightarrow |\sqrt{n} - l| < \varepsilon$$

$$\Rightarrow -\varepsilon < \sqrt{n} - l < \varepsilon$$

$$\Rightarrow l - \varepsilon < \sqrt{n} < l + \varepsilon$$

$$\Rightarrow (l-\varepsilon)^2 < n < (l+\varepsilon)^2$$

but, n is not bounded.

as it goes to ∞.

$\therefore$, $\sqrt{n}$ is diverges.

DIVERGENCE

diverges (to $+\infty$) if $\forall M$, there exists a N s.t. $\forall n \geq N$, $a_n > M$.

$$\Rightarrow \lim_{n\to\infty} a_n = \infty \quad / \quad a_n \to \infty.$$

(or)

diverges (to $-\infty$) if $\forall m$, there exists a N s.t. $\forall n \geq N$, $a_n < m$.

$$\Rightarrow \lim_{n\to\infty} a_n = -\infty \quad / \quad a_n \to -\infty.$$

THEOREMS:

$a_n + b_n \to a+b$

$$a_n \to a \;;\; b_n \to b$$
$$\Rightarrow |a_n - a| < \varepsilon/2 \;\&\; |b_n - b| < \varepsilon/2$$
$$\Rightarrow |a_n - a| + |b_n - b| < \varepsilon$$
$$\Rightarrow |a_n + b_n - a - b| < \varepsilon$$
$$\Rightarrow |(a_n + b_n) - (a+b)| < \varepsilon$$
$$\Rightarrow \lim_{n\to\infty} a_n + b_n = a+b.$$

SIMILARLY for $a_n - b_n$

$Ka_n \to Ka$

$$|a_n - a| < \varepsilon/K$$
$$\Rightarrow K|a_n - a| < \varepsilon$$
$$\Rightarrow |Ka_n - Ka| < \varepsilon$$
$$\Rightarrow \lim_{n\to\infty} Ka_n = Ka.$$

$$b_n < M$$
$$\frac{b_n}{M} < 1$$

$a_n b_n \to ab$

$$P = |a_n b_n - ab_n + ab_n - ab|$$
$$= |b_n(a_n - a) + a(b_n - b)|$$
$$P = b_n(a_n - a) + a(b_n - b)$$
$$P < M(a_n - a) + a(b_n - b)$$

$$(a_n - a) < \frac{\varepsilon}{2M}$$

$$\boxed{b_n < M}$$

$$b_n - b < \frac{\varepsilon}{2a}$$

$$\Rightarrow P < \frac{M\varepsilon}{2M} + \frac{a\varepsilon}{2a}$$

$$\Rightarrow P < \varepsilon$$

$$\Rightarrow |a_n b_n - ab| < \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} a_n b_n = ab$$

$$\frac{a_n}{b_n} \longrightarrow \frac{a}{b}$$

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - a b_n}{b b_n}\right|$$

$$= \left|\frac{a_n b - ab + ab - a b_n}{b b_n}\right| = \left|\frac{b(a_n - a) - a(b_n - b)}{b b_n}\right|$$

$$P = \left|\frac{(a_n - a)}{b_n} - \frac{a}{b}\frac{(b_n - b)}{b_n}\right|$$

$$P = \left|\frac{(a_n - a)}{b_n} - \frac{a}{b}\frac{(b_n - b)}{b_n}\right| \quad b_n > m$$

$$P < \left|\frac{a_n - a}{m} - \frac{a}{b}\frac{(b_n - b)}{m}\right| \qquad a_n - a < 2m\varepsilon$$

$$b_n - b < \frac{bm\varepsilon}{a}$$

$$P < \left|\frac{2\cancel{m}\varepsilon}{\cancel{m}} - \frac{\cancel{a}}{\cancel{b}} \times \frac{\cancel{b}m\varepsilon}{m\cancel{a}}\right|$$

$\Rightarrow P < |\varepsilon| = \varepsilon$

$\therefore \left| \frac{a_n}{b_n} - \frac{a}{b} \right| < \varepsilon$

$\Rightarrow \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b}$.

CONVERGENCE $\Longrightarrow$ BOUNDED

let $\lim_{n \to \infty} a_n = a$

$\Rightarrow \forall \varepsilon > 0$, N exists. s.t.

$n \geq N \Longrightarrow |a_n - a| < \varepsilon$

$|a_n| = |a + (a_n - a)| \leq |a| + |a_n - a| \leq a + \varepsilon$

$M = \max\{|a_1|, |a_2|, \ldots |a_N|, |a| + \varepsilon\}$

$\Rightarrow |a_n| \leq M \;\forall n \geq N$.

$m = \min\{a_1, a_2, \ldots$
$|a| - \varepsilon, |a|, |a| + \varepsilon\}$

$\Rightarrow |a_n| \leq m$

$|a - b| \leq |a| - |b| \leq |a + b| \leq |a| + |b|$

$\Rightarrow |a_n| \geq |a| - \varepsilon$

$|a_n| = |-a_n| = |-a + (a - a_n)| \geq |-a| - |a - a_n|$

$|a| - |a_n - a| = |a| - \varepsilon$

$$\left.\begin{aligned} \forall n \geq N, \quad & a_n > \frac{M}{K} \\ \Rightarrow \quad & K a_n > M \end{aligned}\right\} \quad a_n \text{ div.}, \; K a_n \text{ div}$$

SANDWICH THEOREM

$$\boxed{a_n \leq x_n \leq b_n \qquad ; \; a_n \to s, \; b_n \to s}$$

$\forall \varepsilon > 0$, exists N_1 & N_2 s.t.

$$n \geq N_1 \Rightarrow \quad s - \varepsilon \leq a_n \leq s + \varepsilon$$
$$n \geq N_2 \Rightarrow \quad s - \varepsilon \leq b_n \leq s + \varepsilon$$

$N = \max(N_1, N_2)$

$$\Rightarrow \forall n \geq N, \quad s - \varepsilon \leq x_n \leq s + \varepsilon$$

$\boxed{|b_n| \leq c_n \quad \& \quad c_n \to 0}$ then

$$-c_n \leq b_n \leq c_n \quad \Rightarrow \boxed{b_n \to 0}$$

$\frac{(-1)^{n+1}}{n} \rightarrow$ converges to 0.

$$-\frac{1}{n} \leq \frac{(-1)^{n+1}}{n} \leq \frac{1}{n} \Rightarrow \boxed{\frac{(-1)^{n+1}}{n} \rightarrow 0}$$

$$\frac{1}{2^n} \rightarrow 0$$

$$0 \leq \frac{1}{2^n} \leq \frac{1}{n} \Rightarrow \boxed{\frac{1}{2^n} \rightarrow 0}$$

$$\frac{\cos n}{n} \rightarrow 0$$

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq +\frac{1}{n} \Rightarrow \boxed{\frac{\cos n}{n} \rightarrow 0}$$

CONTINUOUS FN. THEOREM

if $a_n \rightarrow l$, $f(a_n) \rightarrow f(l)$; $f \rightarrow$ cont.

L'HOSPITAL: $f(x) \rightarrow L \Rightarrow a_n \rightarrow L$
if $a_n = f(n)$;

SUBSEQUENCES

$\{a_n\} \to \ell \iff \{a_{n_k}\} \to \ell$ & $\{a_{2n-1}\}$ & $\{a_{2n}\} \to \ell$

$\not\Downarrow$

$\{a_n\} \to \ell$

$\{a_{2n}\} = a_2, a_4, a_6, \ldots, a_{2N_e}, \ldots$

$\{a_{2n-1}\} = a_1, a_3, a_5, \ldots, a_{2N_o-1}, \ldots$

TPT :

$a_n \to a$; $a_{n_k} \to a$

$$\underbrace{a_n \to a}$$

$\downarrow$

$\forall \varepsilon > 0, \exists P$ s.t.

$\forall M \geq P, \ |a_n - a| < \varepsilon$

$\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, \ldots, a_{n_k}\}$

$\downarrow$

$1 \leq n_1 < n_2 < n_3 \ldots < n_k$

$1 \leq n_1 \Rightarrow 2 \leq n_2 \Rightarrow \ldots \Rightarrow i \leq n_i$

$M \geq P \Rightarrow n_k \geq P$ (remember sundar's approach)

$P \leq M \leq n_M \ \forall \ |x_{n_k} - l| < \varepsilon$

CAUCHY SEQUENCES (read once in manual)

SEQUENCE conv. iff cauchy.

let $\lim\limits_{n\to\infty} x_n = c$ & $\lim\limits_{m\to\infty} x_m = c$

$\Rightarrow |x_n - c| < \varepsilon/2$ & $|x_m - c| < \varepsilon/2$

$\forall n, m \geq N$

$\Rightarrow |x_m - x_n| = |x_m - c + c - x_n| = |(x_m - c) - (x_n - c)|$

$$\leq |x_m - c| + |x_n - c| = \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\therefore$, $|x_m - x_n| < \varepsilon$

if $(x_n)_{n\in(1,\infty)}$ cauchy,

$\Rightarrow \forall m, n \geq N$, $\quad |x_m - x_n| < 1 \; (\varepsilon)$

$\Rightarrow |x_n - x_{N+1}| < 1 \quad (\forall n \geq N)$

$\Rightarrow |x_n| - |x_{N+1}| < 1$

$\Rightarrow |x_n| < 1 + |x_{N+1}| \; (\forall n \geq N)$

Let $M = \max(|x_{N+1}|+1, |x_1|, |x_2|, \ldots, |x_N|)$

$\Rightarrow x_n \leq M$

$\forall \varepsilon > 0$, $\exists K_0$ s.t

$K > K_0 \Rightarrow |x_{n_K} - c| < \varepsilon/2$

& $\forall n \geq N$, $m, n > N$

$\Rightarrow |x_m - x_n| < \varepsilon/2$

Choose K: $\quad K > K_0$ && $n_K > N$

$\Rightarrow |x_n - c| \leq |x_n - x_{n_K}| + |x_{n_K} - c| < \varepsilon$

$\Rightarrow x_n \to c$ //

$\underbrace{\Sigma(a_n+b_n)}$ div if Σa_n is conv. & Σb_n is div.

$\downarrow$

$$S_n = \sum_1^n (a_k + b_k) = (a_1+b_1)+(a_2+b_2)+\cdots$$
$$+\cdots+(a_n+b_n)$$

$$S_n \geq \langle b_1+b_2+\cdots+b_n\rangle = B_n$$

but! $B_n \to$ div $\therefore$, S_n diverges.

how to prove $\Sigma(a_n - b_n)$ diverges?

$\Sigma \frac{1}{n!}$ bounded above by 3 (used geometric sum)

TAYLOR SERIES

$f(x) = e^x$; $f(0)=1$, $f'(0)=1$, $f''(0)=1$

$$T = \frac{f(0)\cdot x^0}{0!} + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)x^2}{2!} + \cdots$$

$$T = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots = e^x$$

$$S = \sum_{0}^{\infty} a_n c^n = a_0 + a_1 c + a_2 c^2 + \cdots$$

$|x| < |c|$

$$S = \underbrace{a_0 + |a_1 x| + \cdots + |a_{N-1} x^{N-1}|}_{\substack{\text{finite} \\ (S_0)}} + |a_N x^N| + \cdot$$

$|a_{N+1} x^{N+1}| + \cdots$

$$S < \underbrace{S_0}_{\text{conv.}} + \underbrace{\left|\frac{x}{c}\right|^N + \left|\frac{x}{c}\right|^{N+1} + \left|\frac{x}{c}\right|^{N+2} + \cdots}_{\text{conv.}}$$

$|x| < |c|$

$\frac{|x|}{|c|} < 1$

$\therefore$, $S \to$ conv ABSOLUTELY

$\forall\ |x| < |c|$

$\therefore$

$S = \sum a_n x^n \to$ diverges for $x = d$

$\forall\ |x| > |d|$,

S diverges

$$S = \underbrace{a_0 + a_1 x + a_2 x^2}_{S_0} + \underbrace{\cdots + a_N x^N + a_{N+1} x^{N+1}}_{+ \cdots}$$

$$S > a_0 + a_1 d + a_2 d^2 + \cdots + a_N d^N + a_{N+1} d^{N+1}$$

$$S > \overset{\text{CONV}}{\widetilde{S_0}} + a_{N+1} d^{N+1} + a_{N+2} d^{N+2} + \cdots$$

LIMITS & CONTINUITY

SANDWICH THEOREM

$g(c) = L$

$$g(x) \leq f(x) \leq h(x)$$

$\lim_{x \to c} g(x) = L$

$\lim_{x \to c} h(x) = L$

$h(c) = L$

$$\underbrace{\lim_{x \to c} g(x)} \leq \lim_{x \to c} f(x) \leq \underbrace{\lim_{x \to c} h(x)}$$

$$|g(x) - L| < \varepsilon \quad |x - c| < \delta_1$$

$$\Rightarrow L - \varepsilon < g(x) \leq L + \varepsilon$$

$$L - \varepsilon \leq h(x) \leq L + \varepsilon$$

$$|x - c| < \delta_2$$

$$L - \varepsilon \leq g(x) \leq h(x) \leq L + \varepsilon$$

$$\Rightarrow L - \varepsilon \leq f(x) \leq L + \varepsilon$$

$$\Rightarrow |f(x) - L| < \varepsilon \longrightarrow \delta = \min(\delta_1, \delta_2) > |x - c|$$

$f(x) \leq g(x)$ $\quad \lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} g(x) = M$$

TPT: $L \leq M$.

assume $L > M$.

$\Rightarrow \quad L - M > 0$

let $\varepsilon = L - M$.

$$|g(x) - f(x) - (M - L)| < \varepsilon = L - M$$

$\Rightarrow \quad g(x) - f(x) - M + L < L - M$

$\Rightarrow \quad g(x) - f(x) < 0$

$\Rightarrow \quad g(x) < f(x) \longrightarrow$ NOT TRUE!

$\therefore, \quad \underline{\underline{M \geq L}}$

$|g(x) - M| < \varepsilon/2$

$|f(x) - L| < \varepsilon/2$

$$\underline{\underline{|g(x) - f(x) - (M - L)| < \varepsilon}}$$

TPT : $\lim_{x \to x_0} x = x_0$

to prove: $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$\forall\; 0 < |x - x_0| < \delta$, $|x - x_0| < \varepsilon$

$\Rightarrow$ if $\varepsilon = \delta$, proof holds good..

$\therefore$, $\boxed{\lim_{x \to x_0} x = x_0}$

TPT : $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$$

TP : $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$. s.t.

$|f(x) - 4| < \varepsilon \quad \forall\; 0 < |x - 2| < \delta$

$\Rightarrow |x^2 - 4| < \varepsilon$

$\Rightarrow 4 - \varepsilon < x^2 < 4 + \varepsilon$

$\forall\; \varepsilon \leq 4$ $\quad \Rightarrow \sqrt{4 - \varepsilon} < |x| < \sqrt{4 + \varepsilon}$

$x > 0$

$\Rightarrow \sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$

open interval, excluding 2 .

$\Rightarrow |f(x)-4| < \varepsilon \rightarrow$ true in range.

choose $\delta > 0$ s.t. $(2-\delta, 2+\delta)$ is in range.

$$\delta = \min\{2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2\}$$

$$\Rightarrow 0 < |x-2| < \delta$$

$$\Rightarrow |f(x)-4| < \varepsilon$$

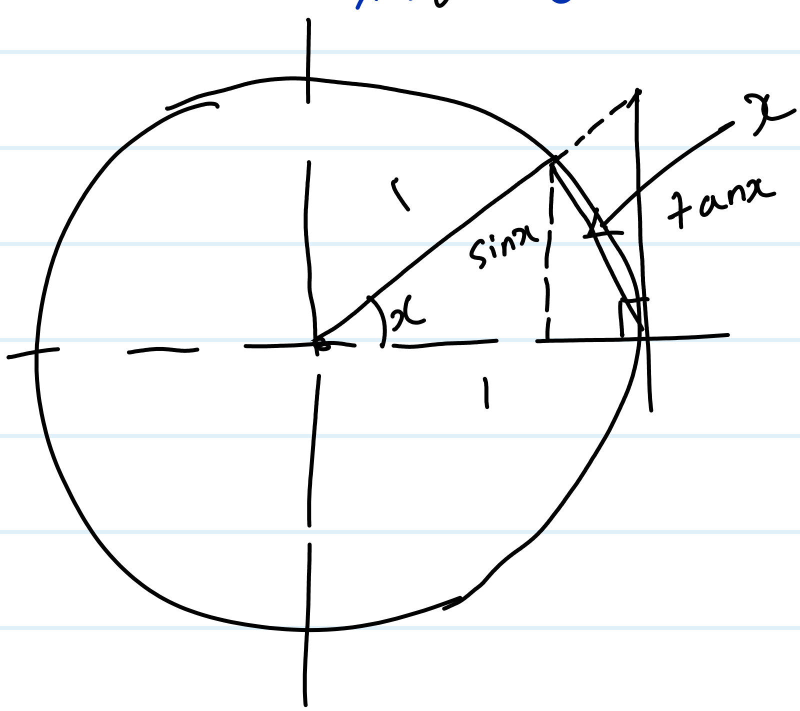
if $\varepsilon > 4$, $\delta = \min(2, \sqrt{4+\varepsilon}-2)$

$$\therefore 0 < |x-2| < \delta$$

$$\Rightarrow |f(x)-4| < \varepsilon$$

Prove : $\sin 1/x$ has no limit as $x \to 0$.

Prove : $\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1$



$$\frac{1}{2}\sin x \leq \frac{1}{2}x \leq \frac{1}{2}\tan x$$

$$\frac{\cancel{\sin x}}{\sin x} \leq \frac{x}{\sin x} \leq \frac{\cancel{\sin x}}{\cos x}$$

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

$$\sin x/x \to 1$$

TPT: $f \cdot g \to LM$, $f \to L$ & $g \to M$

$P = |f \cdot g - LM|$

$P = |f(x) \cdot g(x) - Lg(x) + Lg(x) - LM|$

$P = |g(x)(f(x) - L) + L(g(x) - M)|$

$P < |g(x)(f(x) - L)| + |L(g(x) - M)|$

$\Rightarrow P < |M(f(x) - L)| + |L(g(x) - M)|$ $\quad g(x) \leq M$

$\Rightarrow P < \left|\cancel{M} \times \frac{\varepsilon}{2\cancel{M}}\right| + \left|\cancel{L}\frac{\varepsilon}{2\cancel{L}}\right|$ $\quad |g(x) - M| \leq \frac{\varepsilon}{2L}$

$|f(x) - L| \leq \frac{\varepsilon}{2M}$

$\Rightarrow P < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\Rightarrow P < \varepsilon$

$\therefore |f(x) g(x) - LM| < \varepsilon$

TPT: $\lim_{x \to c} f(x) = f(c)$

$LHL = RHL = f(c)$

$\Rightarrow \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

① $\lim_{x \to c^-} f(x) = f(c)$

$\Rightarrow \forall \varepsilon > 0, \exists \delta' > 0$ s.t.

$|f(x) - f(c)| < \varepsilon \quad \forall\, 0 < c - x < \delta'$

② $\lim_{x \to c^+} f(x) = f(c)$

$\Rightarrow \forall \varepsilon > 0, \exists \delta'' > 0$ s.t.

$|f(x) - f(c)| < \varepsilon \quad \forall\, 0 < x - c < \delta''$

$\Rightarrow$ Let $\delta = \min(\delta', \delta'')$

$\therefore, \; |f(x) - f(c)| < \varepsilon \quad \forall\, 0 < |x - c| < \delta$

So, $\lim_{x \to c} f(x) = f(c)$.

TPT: $\sin x$ is continuous $\forall x$

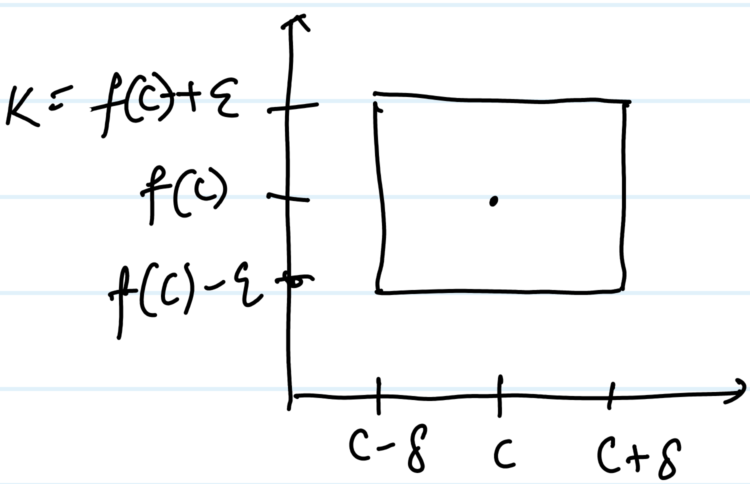
INTERMEDIATE VALUE THEOREM

if $a < x_0 < b$ and $f(a) < y_0 < f(b)$,
then $f(x_0) = y_0$ for $x_0 \in [a, b]$

Let $f(a) < K < f(b)$.
Let $H = \{x : f(x) < K\}$
$c = \sup(H)$
claim: $f(c) = K$.

Why?

① Let $f(c) < K$:



Let $K = f(c) + \varepsilon$
$\Rightarrow K - f(c) = \varepsilon$

for continuity, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon = K - f(c) \quad \forall\, 0 < |x - c| < \delta$$

$$\Rightarrow f(x) - f(c) < K - f(c)$$

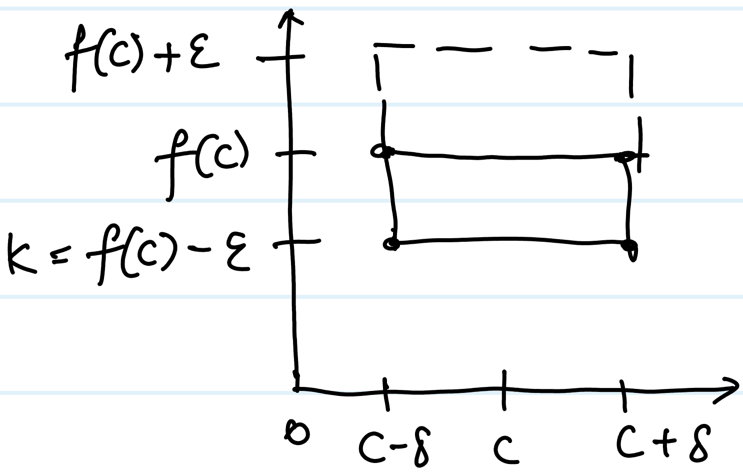
$$\Rightarrow f(x) < K$$

but! $\forall x \in (c, c+\delta)$,

$f(x) > K$ (defn. of sup)

$\therefore$ $f(c) \not< K$.

② $f(c) > K$:



for continuity, $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(x) - f(c)| < \varepsilon \quad \forall\ 0 < |x-c| < \delta$

$\Rightarrow |f(x) - f(c)| < f(c) - K \quad \forall\ 0 < |x-c| < \delta$

if we consider $x \in (c-\delta, c)$

for some x, $f(c) > f(x)$

$\Rightarrow \cancel{f(c)} - f(x) < \cancel{f(c)} - K$

$\Rightarrow f(x) > K$.

but!! $\forall x \in (c-\delta, c)$,

$f(x) < K$!!

$\therefore$, $f(x) = K$.

MAXIMA MINIMA THEOREM

f: attains local max / min, for $x_0 \in (a,b)$
then $f'(x_0) = 0$

proof:

assume $f(x) \rightarrow$ local max at x_0

$$\Rightarrow \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) < 0$$

$$(f(x_0+h) < f(x_0))$$

&

$$\lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) > 0$$

$$(f(x_0+h) < f(x_0) \text{ \& } h < 0)$$

$\Rightarrow$ $f'(x_0) > 0$ and $f'(x_0) < 0$

$\Rightarrow$ $f'(x_0) = 0$ {neither +ve nor -ve} similary for minimum....

LAGRANGE'S MVT

$f : [a,b] \to R$ diff·able.
then for $x_0 \in (a,b)$,

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

use: $f(x) \in [m, M]$ if f : cont.

let $h : [a,b] \to R$ exist s.t.
$h(t) = [f(b)-f(a)]t - (b-a)f(t)$
$\Rightarrow h'(t) = [f(b)-f(a)] - (b-a)f'(t)$
but! $h'(x_0) = 0$ for some
$x_0 \in (a,b)$ as $h(a) = h(b)$
$\Rightarrow 0 = [f(b)-f(a)] - (b-a)f'(x_0)$

$$\Rightarrow \boxed{f'(x_0) = \frac{f(b)-f(a)}{b-a}}$$

if $f(x) \to$ diffable & $f'(x) > 0$
$\forall x \in (a,b)$, then $f(x) \to \uparrow\uparrow$.

let $(x,y) \in (a,b)$
$\Rightarrow$ $f : [x,y]$ is cont. & diffable

$\Rightarrow$ $\exists$ $z \in (x,y)$ s.t. (use LMVT)

$$f'(z) = \frac{f(y) - f(x)}{y - x} > 0$$

$$\Rightarrow \boxed{f(y) > f(x)}$$

INTERMEDIATE VALUE PROP (DERIV)

$f : [a,b] \to \mathbb{R}$ diff.able &
$f'(a) < \lambda < f'(b)$, then
for $x_0 \in (a,b)$, $f'(x_0) = \lambda$.

consider $h(t) = f(t) - \lambda t$

$h(t) \to$ cont. & diff. $\forall x \in (a,b)$.

$\Rightarrow h(x) \in (m, M) \quad \forall x \in (a,b)$

$\Rightarrow h'(t) = f'(t) - \lambda \Rightarrow h'(x_0) = f'(x_0) - \lambda = 0$

$\Rightarrow \boxed{f'(x_0) = \lambda}$ $\quad (h'(x_0) = 0$

for $x_0 \in (a,b))$

* We have to show that the maxima and minima cannot simultaneously occur at $x = a$ (and) $x = b$.

$$h'(a) = \lim_{h \to 0^+} \frac{h(a+h) - h(a)}{h}$$

h ? ? ? }

$$= \lim_{h \to 0^+} \frac{f(a+h) - \lambda(a+h) - f(a) + \lambda a}{h}$$

$$= \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0^+} \frac{-\lambda(a+h) + \lambda a}{h}$$

$$= f'(a) - \lambda < 0 \,//$$

$$h'(b) = \lim_{h \to b^-} \frac{h(b+h) - h(b)}{h}$$

$$= \lim_{h \to b^-} \frac{f(b+h) - \lambda(b+h) - f(b) + \lambda b}{h}$$

$$= \lim_{h \to b^-} \frac{f(b+h) - f(b)}{h} + \lim_{h \to b^-} \frac{\cancel{\lambda b} - \cancel{\lambda b} - \lambda \cancel{h}}{\cancel{h}}$$

$$= f'(b) - \lambda > 0 \quad \left\{ \begin{array}{l} \text{But } f'(b) - \lambda > 0 \\ \text{by definition} \end{array} \right\}$$

Now we know that both the maxima and minima cannot simultaneously be present at a & b.

$\Rightarrow$ The minima (or) the maxima must occur between (a, b)

$$\Rightarrow \boxed{f'(x) = \lambda}$$

// hence proved

$M_i(f^2) - m_i(f^2)$

$$\leq X^2 - Y^2$$
$$= (X+Y)(X-Y)$$
$$\leq 2B(X-Y)$$
$$= 2B(f(x) - f(y))$$
$$= 2B(M_i(f) - m_i(f))$$

$\therefore$, $M_i(f^2) - m_i(f^2) \leq 2B(M_i(f) - m_i(f))$