Tut 1 Ch1 L1 L2 L3 L4 L5

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L1 Fields

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1. 0 is in F
2. 1 is in F
3. If x and y are in F then so is $x + y$
4. If x is in F then so is $-x$
5. If x and y are in F then so is xy
6. If $x \neq 0$ is in F then so is x^{-1}

Solution 1 cont.

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For **5**, suppose $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$. Then
$xy = (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \in F$.

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$$xy = (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \in F.$$

For **6** , suppose $x = a + b\sqrt{2}$ where at least one of a or b is not zero. Let $n = a^2 - 2b^2$. Let $y = a/n + (-b/n)\sqrt{2} \in F$. Then $xy = \frac{1}{n}(a + b\sqrt{2})(a - b\sqrt{2}) = \frac{1}{n}\left(a^2 - 2b^2\right) = 1$. Thus $y = x^{-1}$ and $y \in F$. $\square$

Question 2

Question 2: Let F be a set that contains exactly two elements, 0 and 1. Define addition and multiplication by the tables

Addition

$+$	0	1
0	0	1
1	1	0

Multiplication

.	0	1
0	0	0
1	0	1

Show that F is a field.

Question 3

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Question 4

Question 4: Prove that each field of characteristic zero contains a copy of the rational number field.

Question 5

Question 5: Let F be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \qquad & 3x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 0 \qquad & x_1 + x_2 &= 0 \end{aligned}$$

Solution: Yes, the two systems are equivalent. We show this by writing each equation of the first system in terms of the second, and conversely.

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Solution 5 cont.

$$
\begin{aligned}
3x_1 + x_2 &= \frac{1}{3}(x_1 - x_2) + \frac{4}{3}(2x_1 + x_2) \\
x_1 + x_2 &= \frac{-1}{3}(x_1 - x_2) + \frac{2}{3}(2x_1 + x_2) \\
x_1 - x_2 &= (3x_1 + x_2) - 2(x_1 + x_2) \\
2x_1 + x_2 &= \frac{1}{2}(3x_1 + x_2) + \frac{1}{2}(x_1 + x_2)
\end{aligned}
$$

$\square$

Question 6

Question 6: Test the following systems of equations as in previous question 5.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ 1x_1 + x_2 + 1x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{aligned}$$

Question 7

Question 7: Test the following systems as in Question 5.

$$2x_1 + (-1+i)x_2 \quad + x_4 = 0 \quad (1+i/2)x_1 + 8x_2 - ix_3 - x_4 = 0$$
$$3x_2 - 2ix_3 + 5x_4 = 0 \quad (2/3)x_1 - (1/2)x_2 + x_3 + 7x_4 = 0$$

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$$3x_2 - 2ix_3 + 5x_4 = 0 \quad (2/3)x_1 - (1/2)x_2 + x_3 + 7x_4 = 0$$

Solution: These systems are not equivalent.Call the two equations in the first system E_1 and E_2 and the equations in the second system E_1' and E_2'. Then if $E_2' = aE_1 + bE_2$ since E_2 does not have x_1 we must have $a = 1/3$. But then to get the coefficient of x_4 we'd need $7x_4 = \frac{1}{3}x_4 + 5bx_4$. That forces $b = \frac{4}{3}$. But if $a = \frac{1}{3}$ and $b = \frac{4}{3}$ then the coefficient of x_3 would have to be $-2i\frac{1}{3}$ which does not equal 1 . Therefore the systems cannot be equivalent.

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Solution: Write the two systems as follows:

$$\begin{array}{cc}
a_{11}x + a_{12}y = 0 & b_{11}x + b_{12}y = 0 \\
a_{21}x + a_{22}y = 0 & b_{21}x + b_{22}y = 0 \\
\vdots & \vdots \\
a_{m1}x + a_{m2}y = 0 & b_{m1}x + b_{m2}y = 0
\end{array}$$

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Each system consists of a set of lines through the origin $(0,0)$ in the xy plane. Thus, the two systems have the same solutions if and only if they either have $(0,0)$ as their only solution or if both have a single line $ux + vy = 0$ as their common solution.

Solution 8 cont.

In the latter case all equations are simply multiples of the same line, so clearly the two systems are equivalent. So assume that both systems have $(0, 0)$ as their only solution. Assume without loss of generality that the first two equations in the first system give different lines. Then

$$\frac{a_{11}}{a_{12}} \neq \frac{a_{21}}{a_{22}} \tag{1}$$

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We need to show that there's a (u, v) which solves the following system:

$$a_{11}u + a_{12}v = b_{i1}$$

$$a_{21}u + a_{22}v = b_{i2}$$

Solution 8 cont.

Solving for u and v we get

$$u = \frac{a_{22}b_{i1} - a_{12}b_{i2}}{a_{11}a_{22} - a_{12}a_{21}}$$
$$v = \frac{a_{11}b_{i2} - a_{21}b_{i1}}{a_{11}a_{22} - a_{12}a_{12}}$$

By (1) $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

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By (1) $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Thus both u and v are well-defined. So, we can write any equation in the second system as a combination of equations in the first. Analogously, we can write any equation in the first system in terms of the second.

Question 1: Find all solutions to the systems of equations

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Solution: The matrix of coefficients is

$$\begin{bmatrix} 1-i & -i \\ 2 & 1-i \end{bmatrix}$$

Row reducing

$$\rightarrow \begin{bmatrix} 2 & 1-i \\ 1-i & -i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1-i \\ 0 & 0 \end{bmatrix}$$

Row Reducing

Thus, $2x_1 + (1-i)x_2 = 0$. Thus, for any $x_2 \in \mathrm{C}$, $\left(\frac{1}{2}(i-1)x_2, x_2\right)$ is a solution and these are all solutions.

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Question 2: If

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}$$

find all solutions of $AX = 0$ by row-reducing A.

Row Reducing

Question 3: If

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

find all solutions of $AX = 2X$ and all solutions of $AX = 3X$. (The symbol cX denotes the matrix each entry of which is c times the corresponding entry of X.)

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Solution: The system $AX = 2X$ is

$$\begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which is the same as

Row Reducing

$$
\begin{aligned}
6x - 4y &= 2x \\
4x - 2y &= 2y \\
-x + 3z &= 2z
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
4x - 4y &= 0 \\
4x - 4y &= 0 \\
-x + z &= 0
\end{aligned}
$$

The matrix of coefficients is

$$
\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}
$$

Row Reducing

Thus, the solutions are all elements of F^3 of the form (x, x, x) where $x \in F$. The system $AX = 3X$ is

$$\begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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$$\begin{aligned} 6x - 4y &= 3x \\ 4x - 2y &= 3y \\ -x + 3z &= 3z \end{aligned}$$

which is equivalent to

$$\begin{aligned} 3x - 4y &= 0 \\ x - 2y &= 0 \\ -x &= 0 \end{aligned}$$

Row Reducing

The matrix of coefficients is

$$\begin{bmatrix} 3 & -4 & 0 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Thus, the solutions are all elements of F^3 of the form $(0, 0, z)$ where $z \in F$.

Try Yourself

Question 4: Find a row-reduced matrix that is row-equivalent to

$$A = \begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}$$

Row Equivalent

Question 5: Prove that the following two matrices are not row-equivalent:

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}$$

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Solution: Call the first matrix A and the second matrix B. The matrix A is row-equivalent to

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

and the matrix B is row-equivalent to

$$B' = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$AX = 0$ and $A'X = 0$ have the same solutions.

Solution

and the matrix B is row-equivalent to

$$B' = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$AX = 0$ and $A'X = 0$ have the same solutions. Similarly $BX = 0$ and $B'X = 0$ have the same solutions. Now if A and B are row-equivalent then A' and B' are row equivalent.

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Try Yourself

Question 6: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 matrix with complex entries. Suppose that A is row-reduced and also that $a + b + c + d = 0$. Prove that there are exactly three such matrices.

Row Operation are Inter Related

Question 7: Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

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Solution: Write the matrix as

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

WOLOG we'll show how to exchange rows R_1 and R_2. First add R_2 to R_1 :

Solution

$$\begin{bmatrix} R_1 + R_2 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

Solution

$$\begin{bmatrix} R_1 + R_2 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

Next, subtract row one from row two:

$$\begin{bmatrix} R_1 + R_2 \\ -R_1 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}$$

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Next, add row two to row one again

Solution

$$\begin{bmatrix} R_2 \\ -R_1 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

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$$\begin{bmatrix} R_2 \\ -R_1 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

Finally, multiply row two by -1 :

$$\begin{bmatrix} R_2 \\ R_1 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}$$

Try Yourself

Question 8: Consider the system of equations $AX = 0$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix over the field F. Prove the following:

1. If every entry of A is 0 , then every pair (x_1, x_2) is a solution of $AX = 0$.
2. If $ad - bc \neq 0$, the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.
3. If $ad - bc = 0$ and some entry of A is different from 0 , then there is a solution $\left(x_1^0, x_2^0\right)$ such that (x_1, x_2) is a solution if and only if there is a scalar y such that $x_1 = yx_1^0, x_2 = yx_2^0$.

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Tut Ch1 L4 Row Reduced Echelon Form

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What are the solutions of $AX = 0$?

Solution: A row reduces as follows:

$$\rightarrow \begin{bmatrix} 1 & -i \\ 1 & 1 \\ i & 1+i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -i \\ 0 & 1+i \\ 0 & i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -i \\ 0 & 1 \\ 0 & i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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Thus, the only solution to $AX = 0$ is $(0, 0)$.

Try Yourself

Question 2: Describe explicitly all 2×2 row-reduced echelon matrices.

Solution of System

Question 3: Consider the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_3 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2 \end{aligned}$$

Does this system have a solution? If so, describe explicitly all solutions.

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Solution: The augmented coefficient matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right]$$

We row reduce it as follows:

Solution

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution

Thus, the system is equivalent to

$$x_1 + x_3 = 1/2$$
$$x_2 - x_3 = -1/2$$

Thus the solutions are parameterized by x_3. Setting $x_3 = c$ gives $x_1 = 1/2 - c, x_2 = c - 1/2$. Thus the general solution is

$$\left(\frac{1}{2} - c, c - \frac{1}{2}, c\right)$$

for $c \in \mathbb{R}$.

Try Yourself

Question 3: Show that the system

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + 7x_2 - 5x_3 - x_4 &= 3 \end{aligned}$$

has no solution.

Condition for consistency

Question 4: Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

For which (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

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For which (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

Solution: The matrix A is row reduced as follows:

$$\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 0 & 8 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 7 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, for every (y_1, y_2, y_3) there is a (unique) solution.

Question 5: An $n \times n$ matrix A is called upper-triangular if $a_{ij} = 0$ for $i > j$, that is if every entry below the main diagonal is 0 . Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from zero.

Solution: Suppose that $a_{ii} \neq 0$ for all i. Then we can divide row i by a_{ii} to give a row-equivalent matrix that has all ones on the diagonal. Then by a sequence of elementary row operations, we can turn all off-diagonal elements into zeros. We can therefore row-reduce the matrix to be equivalent to the identity matrix. By Theorem 12 page 23, A is invertible.

Solution

Now, suppose that some $a_{ii} = 0$. If all a_{ii} 's are zero then the last row of the matrix is all zeros. A matrix with a row of zeros cannot be row-equivalent to the identity so cannot be invertible.

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Thus, we can assume there's at least one i such that $a_{ii} \neq 0$. Let i' be the largest such index, so that $a_{\dot{P}'i'} = 0$ and $a_{ii} \neq 0$ for all $i > i'$. We can divide all rows with $i > i'$ by a_{ii} to give ones on the diagonal for those rows. We can then add multiples of those rows to row i' to turn row i' into an entire row of zeros. Since again A is row-equivalent to a matrix with an entire row of zeros, it cannot be invertible.

Question 6: Prove if A is an $m \times n$ matrix and B is an $n \times m$ matrix and $n < m$, then AB is not invertible.
Solution: There are n columns in A so the vector space generated by those columns has a dimension no greater than n. All columns of AB are linear combinations of the columns of A. Thus the vector space generated by the columns of AB is contained in the vector space generated by the columns of A.

Thus, the column space of AB has a dimension no greater than n. Thus the column space of the $m \times m$ matrix AB has a dimension less or equal to n and $n < m$. Thus the columns of AB generate a space of dimension strictly less than m. Thus AB is not invertible.

The Inverse of an Upper Triangular Matrix is Upper Triangular

- Let A be an invertible upper triangular matrix.

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- Let $A = [a_{ij}]$ be an $n \times n$ upper triangular matrix, meaning:

$$a_{ij} = 0 \quad \text{for} \quad i > j.$$

- This implies that all elements below the main diagonal are zero.

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- We need to prove that the inverse of A, denoted as $A^{-1} = [b_{ij}]$, is also upper triangular, i.e., $b_{ij} = 0$ for $i > j$.
- We know that by the definition of an inverse matrix:

$$AA^{-1} = I_n,$$

where I_n is the identity matrix.

The Inverse of an Upper Triangular Matrix is Upper Triangular

- Writing out the matrix product $AA^{-1} = I_n$ explicitly:

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta (1 if $i = j,$ 0 otherwise).

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- This system of equations holds for each entry in the resulting identity matrix.
- We will show that the entries of A^{-1} below the diagonal are zero. Let $i > j$, and examine the (i, j)-th entry of the matrix product:

$$\sum_{k=1}^{n} a_{ik} b_{kj} = 0, \quad \text{since} \quad \delta_{ij} = 0 \text{ for } i \neq j.$$

The Inverse of an Upper Triangular Matrix is Upper Triangular

- Since $i > j$, notice that $a_{ik} = 0$ for all $k < i$, because A is upper triangular. Therefore, the sum simplifies to:

$$\sum_{k=i}^{n} a_{ik} b_{kj} = 0.$$

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- But for $k = i$, we know that $a_{ii} \neq 0$ (since A is invertible, all diagonal entries of A are non-zero), and thus the equation becomes:

$$a_{ii} b_{ij} + \sum_{k=i+1}^{n} a_{ik} b_{kj} = 0.$$

The Inverse of an Upper Triangular Matrix is Upper Triangular

- Now, because A is upper triangular, $a_{ik} = 0$ for $k > i$, so:

$$a_{ii} b_{ij} = 0.$$

  Since $a_{ii} \neq 0$, it follows that $b_{ij} = 0$ for $i > j$.

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  Since $a_{ii} \neq 0$, it follows that $b_{ij} = 0$ for $i > j$.
- Thus, the elements of A^{-1} below the diagonal are zero, meaning that A^{-1} is also upper triangular.