The Ratio Test

Theorem (The Ratio Test)

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

(a) *the series* converges *if* $\rho < 1$,

(b) *the series* diverges *if* $\rho > 1$ *or is infinite,*

(c) *the test is inconclusive if* $\rho = 1$.

Proof

(a) $\rho < 1$.

Let r be a number between ρ and 1: $\rho < r < 1$.

Proof

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Let r be a number between ρ and 1: $\rho < r < 1$.
Then the number $\epsilon = r - \rho$ is positive.
Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho,$$

there is an integer N such that

$$n \geq N \quad \Rightarrow \quad \left| \frac{a_{n+1}}{a_n} - \rho \right| < \epsilon$$

This implies that, for $n \geq N$,

$$-\epsilon < \frac{a_{n+1}}{a_n} - \rho < \epsilon \quad \text{or} \quad \rho - \epsilon < \frac{a_{n+1}}{a_n} < \rho + \epsilon = r.$$

That is,

$$\begin{aligned}
a_{N+1} &< ra_N, \\
a_{N+2} &< ra_{N+1} < r^2 a_N, \\
a_{N+3} &< ra_{N+2} < r^3 a_N, \\
&\vdots \\
a_{N+m} &< ra_{N+m-1} < r^m a_N.
\end{aligned}$$

Consider the series $\sum c_n$, where $c_n = a_n$ for $n = 1, 2, \ldots, N$ and $c_{N+1} = ra_N$, $c_{N+2} = r^2 a_N, \ldots, c_{N+m} = r^m a_N, \ldots$.
Now $a_n \leq c_n$ for all n and

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n &= a_1 + a_2 + \ldots + a_{N-1} + a_N + ra_N + r^2 a_N + \ldots \\
&= a_1 + a_2 + \ldots + a_{N-1} + a_N(1 + r + r^2 + \ldots)
\end{aligned}$$

The geometric series $1 + r + r^2 + \ldots$ converges as $|r| < 1$.

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Now $a_n \leq c_n$ for all n and

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The geometric series $1 + r + r^2 + \ldots$ converges as $|r| < 1$. So $\sum c_n$ converges.
Since $a_n \leq c_n$, $\sum a_n$ also converges.

(b) $1 < \rho \leq \infty$.

From some index M on,

$$\frac{a_{n+1}}{a_n} > 1 \quad \text{and} \quad a_M < a_{M+1} < a_{M+2} < \dots .$$

So, the terms of the series do not approach zero as n becomes infinite. Hence the series diverges by the nth Term Test.

(c) $\rho = 1$.

Consider the following two series:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$: $\displaystyle\frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1.$

For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\displaystyle\frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1.$

In both cases, $\rho = 1$. But the first series diverges and the second converges.

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Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}.$$

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Solution: Here

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The series converges because here $\rho = 2/3 < 1$.

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Solution: For the series $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n!}$,

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}/(n+1)!}{a^n/n!} = \frac{a}{n+1} \to 0.$$

The series converges because here $\rho = 0 < 1$.

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Because the limit $\rho = 1$, we cannot decide whether the series converges from the ratio test.

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Hence the series diverges.

The Root Test

Theorem (The Root Test)

Let $\sum a_n$ *be a series with* $a_n \geq 0$ *for* $n \geq N$ *(*N *an integer) and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

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Which of the following series converges and which diverges?

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(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \to \frac{1}{2} < 1.$

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(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ diverges because $\displaystyle\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{(\sqrt[n]{n})^2} \to \frac{2}{1} > 1.$

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Thus here the limit is $\rho < 1$. Hence the series converges by the Root Test.

Alternating Series

Definition

A series in which the terms are alternatively positive and negative is called an **alternating series**.

Examples

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

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The first series, called the **alternating harmonic series**, converges.

The second series, a geometric series with common ratio $r = -1/2$, converges.

The third series diverges because the nth term does not approach zero.

Homework

Let $\{a_n\}$ be a sequence such that the subsequences $\{a_{2m}\}$ and $\{a_{2m+1}\}$ both converge to the same limit l.

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Theorem (The Alternationg Series Test (Leibniz's Theorem))

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \dots$$

converges if all three of the following conditions are satisfied:

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3. $u_n \to 0$.

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If n is an even integer, say $n = 2m$, then the sum of the first n terms is

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The first equality shows that the s_{2m} is the sum of m non-negative terms. Hence $s_{2m+2} \geq s_{2m}$.

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The first equality shows that the s_{2m} is the sum of m non-negative terms. Hence $s_{2m+2} \geq s_{2m}$.

The second equality shows that $s_{2m} \leq u_1$.

Proof:

Let us assume that $N = 1$.
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$$s_{2m+1} = s_{2m} + u_{2m+1} \to l + 0 = l.$$

Thus we have that $s_{2m} \to l$ and $s_{2m+1} \to l$.

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Hence $s_n \to l$. This means that the alternating series converges.

Example

The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

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