IIITDM KANCHEEPURAM

MA1000 Calculus
Problem Set 5

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1. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in \mathbb{R}$ *if and only if* for every $\epsilon > 0$ there corresponds a $\delta > 0$ such that
$$|x - x_0| < \delta \quad \implies \quad |f(x) - f(x_0)| < \epsilon.$$

2. Provide a function $f(x)$ that is continuous only at the origin.

3. Prove: (a) $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$; (b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

4. Show that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ for $0 < x < \dfrac{\pi}{2}$.

5. Show that the point $(2, 4)$ lies on the curve $x^3 + y^3 - 9xy = 0$. Find the tangent and normal to the curve at $(2, 4)$.

6. Examine the function $f(x) = \begin{cases} x^m \sin \dfrac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$ for derivability at the origin. Also find the values of m for which f' is continuous at the origin.

7. Find the linearization of $f(x) = \sqrt{9 + x^2}$ at $a = -4$.

8. (a) Let $f(x) = x^{1/3}(x-4)$. Find the intervals on which the function is either increasing or decreasing. Also find the local extreme values.
   (b) Suppose the derivative of the function $y = f(x)$ is $y' = (x-1)^2(x-2)(x-4)$. At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

9. Establish the following inequalities.

   (a) $1 + x < e^x < 1 + xe^x, \quad \forall\ x$
   (b) $\dfrac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \dfrac{v-u}{1+u^2}$, if $0 < u < v$. Also deduce that
$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

10. (a) Let $f(x)$ be a differentiable function. Show that between two zeros of $f(x)$ there exists at least one zero of $f'(x)$.

    (b) Prove that c is the geometric mean of a and b in the Rolle's theorem for the function $f(x) = \ln\left(\dfrac{x^2+ab}{x(a+b)}\right)$ in $[a, b]$ where $a > 0$.

    (c) Show that for any real number k the polynomial $f(x) = x^3 + x + k$ has exactly one real root.

    (d) If c is a point at which Rolle's theorem holds for the function $f(x) = \ln\left(\dfrac{x^2+\alpha}{7x}\right)$ on the interval $[3, 4]$, where $\alpha \in \mathbb{R}$, then find the value of c.

11. Show that the value of c in the conclusion of the mean value theorem for $f(x) = x^2$ on any interval $[a, b]$ is the arithmetic mean of a and b.

12. A twice differentiable function $f(x)$ is such that $f(a) = f(b) = 0$ and $f(x) > 0$ for $a < x < b$. Prove that there is at least one value x_0 between a and b for which $f''(x_0) < 0$.

13. Prove that if $f'(x) = 0$ on an interval (a, b), then $f(x)$ is a constant function.