Phase and phase velocity:

A sinusoidal wave propagates through a medium is represented by,

$$\psi(x,t) = A\sin(kx - \omega t + \delta)$$

Phase $\Rightarrow \delta$, Phase angle $\Rightarrow \phi = (kx - \omega t + \delta)$

$$\boxed{\frac{\partial\phi}{\partial t} = -\omega} \qquad \boxed{\frac{\partial\phi}{\partial x} = k} \qquad \left.\frac{\partial x}{\partial t}\right|_{\phi} = \left.\frac{\partial\phi}{\partial t}\right|_{x} \Big/ \left.\frac{\partial\phi}{\partial x}\right|_{t}$$

Phase angle $\Rightarrow \phi = (kx + \omega t + \delta)$

$e^{i(x - v_p t + \delta)}$

The LHS is the speed of propagation of the condition of constant phase (velocity of the wave):

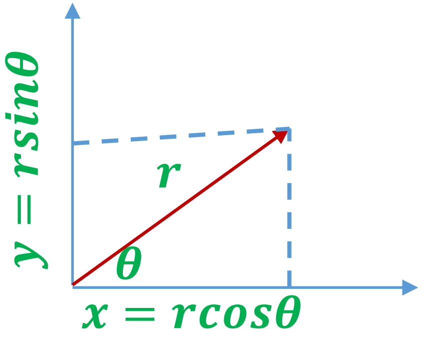
Phase velocity: $v_p = \mp\frac{\omega}{k}$

$$\boxed{\tilde{z} = x + iy}$$

$$\boxed{\tilde{z} = r(\cos\theta + i\sin\theta)}$$

$$\boxed{\tilde{z} = re^{i\theta}}$$

$$\boxed{r = \sqrt{x^2 + y^2}} \qquad \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$$



$$\boxed{Re(\tilde{z}) = r\cos\theta} \qquad \boxed{Im(\tilde{z}) = r\sin\theta}$$

$$\psi(x,t) = A\sin(kx - \omega t + \delta) = Im\{Ae^{i(kx - \omega t + \delta)}\}$$

$$\psi(x,t) = A\cos(kx - \omega t + \delta) = Re\{Ae^{i(kx - \omega t + \delta)}\}$$

$$\psi(x,t) = Ae^{i(kx - \omega t + \delta)} \text{ or } Ae^{i(kx + \omega t + \delta)}$$

Plane wave:

At any instant of time, the surfaces joining all points of equal phase are known as **wavefronts**

If at a given time, all the surfaces on which **a disturbance has constant phase** form a set of planes, each generally perpendicular to the propagation direction then the wave is known as a plane wave.

To derive the expression for **a plane** that is perpendicular to a given vector $\vec{k}$ and that passes through some point (x_0, y_0, z_0), we write,

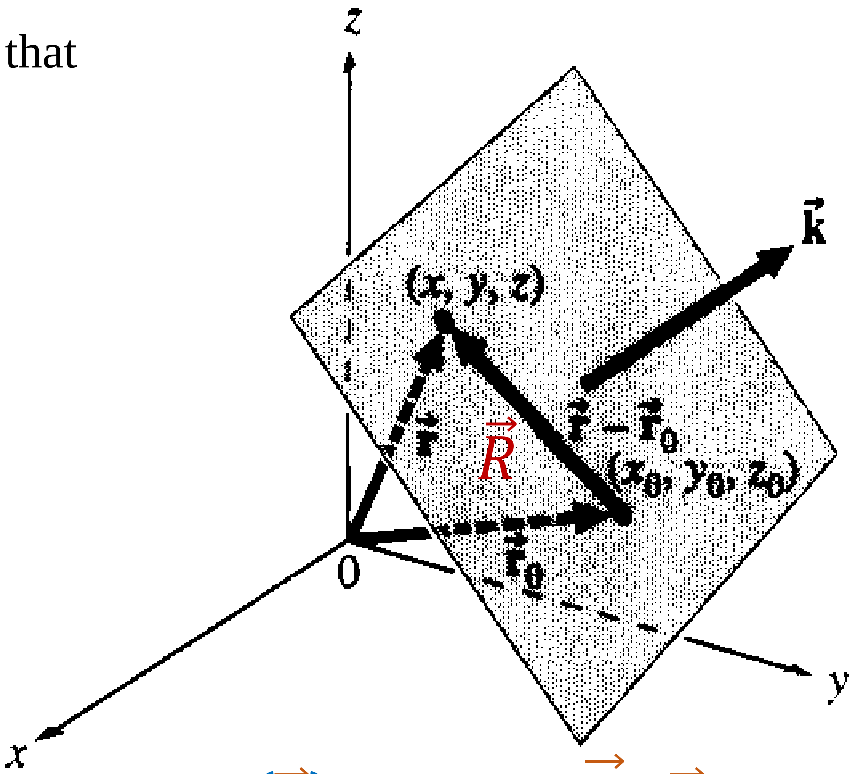
$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \vec{k} = k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k}$$

$$\vec{R} = \vec{r} - \vec{r_0} = (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} + (z - z_0)\hat{k}$$

$\vec{k} \cdot \vec{R} = 0 \Rightarrow$ for $\vec{R}$ perpendicular to $\vec{k}$

$$(x - x_0)k_x + (y - y_0)k_y + (z - z_0)k_z = 0$$

$$x_0k_x + y_0k_y + z_0k_z = xk_x + yk_y + zk_z$$



If we assume, for the fixed point $\vec{r_0}(x_0, y_0, z_0)$, $\vec{k} \cdot \vec{r_0} = constant$ (say)

$$\boxed{\vec{k} \cdot \vec{r} = a, \quad \text{constant}}$$

The plane is the locus of all points each of whose position vectors have a constant projection on $\vec{k}$.

$$\psi(\vec{r}) = A\sin(\vec{k} \cdot \vec{r})$$

$$\psi(\vec{r}) = A\cos(\vec{k} \cdot \vec{r})$$

$$\psi(\vec{r}) = Ae^{i\vec{k}\cdot\vec{r}}$$

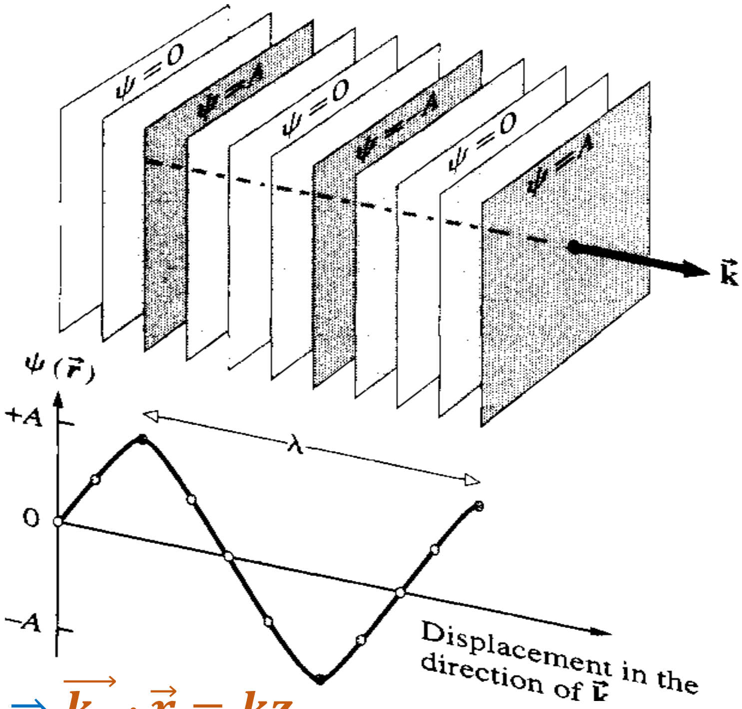
Plane wave:

$$\psi(\vec{r}) = A\sin(\vec{k}\cdot\vec{r}) \qquad \psi(\vec{r}) = Ae^{i\vec{k}\cdot\vec{r}}$$

The spatially repetitive nature of these harmonic functions can be expressed by

$$\psi(\vec{r}) = \psi\left(\vec{r} + \lambda \vec{k}/k\right) \qquad Ae^{i\vec{k}\cdot\vec{r}} = Ae^{i\vec{k}\cdot\vec{r}}e^{i\lambda k}$$

$$e^{i\lambda k} = 1 = e^{i2\pi} \qquad \boxed{\vec{k} = \frac{2\pi}{\lambda} \Rightarrow wave\ number}$$



The disturbance on a wavefront is constant, so that after a time ***dt***, if the front moves along k a distance $\boldsymbol{dr_k}$(r_k is the projection of $\vec{r}$ along $\vec{k}$ **),** we must have

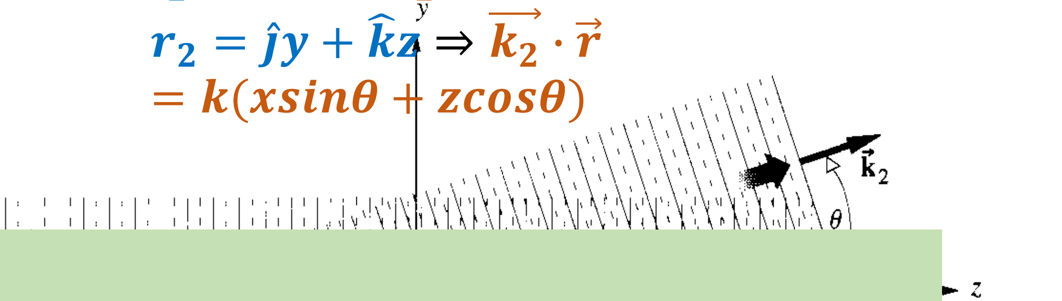
$$\psi(\vec{r},t) = \psi(r_k + dr_k, t + dt) = \psi(r_k, t)$$

$$\psi(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r}\mp\omega t)} = Ae^{i(kr_k + kdr_k \mp\omega t \mp \omega dt)} = Ae^{i(l}$$

$$kdr_k \mp \omega dt = 0 \qquad \boxed{\frac{dr_k}{dt} = \pm\frac{\omega}{k} = \pm v_p}$$

$$r_1 = \hat{k}z \Rightarrow \vec{k_1}\cdot\vec{r} = kz$$

$$r_2 = \hat{j}y + \hat{k}z \Rightarrow \vec{k_2}\cdot\vec{r} = k(x\sin\theta + z\cos\theta)$$



ψ_1

ψ_1

$\frac{2\pi}{k}$

The special significance of these waves are:

1. physically, sinusoidal waves can be generated relatively simply by using some form of harmonic oscillator;

2. ***any three-dimensional wave can be expressed as a combination of plane waves,*** each having a distinct amplitude and propagation direction.

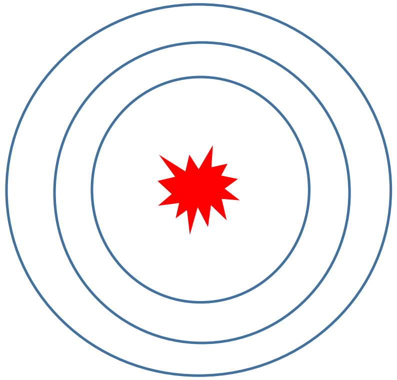
Spherical wave:

$$\frac{\partial^2\psi(x,y,z,t)}{\partial t^2} = v^2\nabla^2\psi(x,y,z,t)$$

$$\frac{\partial^2\psi(r,\theta,\phi,t)}{\partial t^2} = v^2\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\psi}{\partial r}\right] + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left[sin\theta\frac{\partial\psi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right\}$$

Since the system is spherically symmetric, $\psi(r,\theta,\phi,t) = \psi(r)$, only

$$\frac{\partial^2\psi}{\partial t^2} = v^2\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\psi}{\partial r}\right]\right\} = v^2\left\{\frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}\right\}$$



$$\frac{\partial^2(r\psi)}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2(r\psi)}{\partial t^2}$$

$$r\psi(r,t) = f(r-vt)$$

Wave travelling in +ve

$$\psi(r,t) = \frac{1}{r}f(r-vt)$$

Wave travelling in -ve

$$\psi(r,t) = \frac{1}{r}g(r+vt)$$

The general solution $\psi(r,t) = \frac{C}{r}f(r-vt) + \frac{D}{r}g(r+vt)$

Harmonic spherical wave (which is a special case of the general solution) is

$$\psi(r,t) = \frac{A}{r}sink(r\mp vt) = \frac{A}{r}e^{ik(r\mp vt)}$$

Superposition of wave:

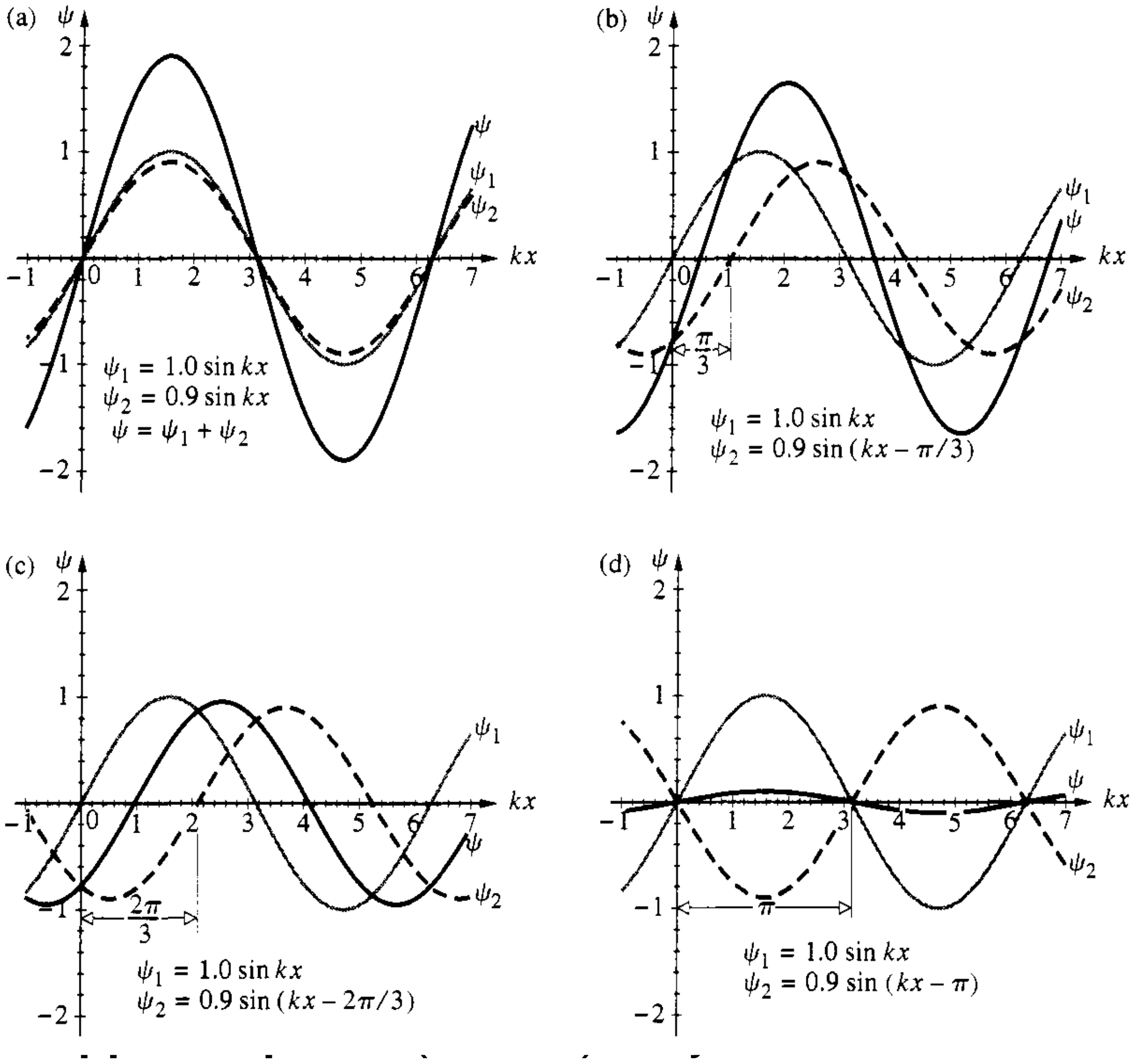
If $\boldsymbol{\psi_1(x,t)}$ and $\boldsymbol{\psi_2(x,t)}$ represents two harmonic, what is meant by $\boldsymbol{\psi_1(x,t)} + \boldsymbol{\psi_2(x,t)} = \boldsymbol{\psi(x,t)}$?

$$\boxed{\frac{\partial^2\psi_1(x,t)}{\partial t^2} = v^2\frac{\partial^2\psi_1(x,t)}{\partial x^2}} \qquad \boxed{\frac{\partial^2\psi_2(x,t)}{\partial t^2} = v^2\frac{\partial^2\psi_2(x,t)}{\partial x^2}}$$

$$\frac{\partial^2\psi_1(x,t)}{\partial t^2} + \frac{\partial^2\psi_2(x,t)}{\partial t^2} = v^2\frac{\partial^2\psi_1(x,t)}{\partial x^2} + v^2\frac{\partial^2\psi_2(x,t)}{\partial x^2}$$

$$\frac{\partial^2\{\boldsymbol{\psi_1(x,t)} + \boldsymbol{\psi_2(x,t)}\}}{\partial t^2} = v^2\frac{\partial^2\{\boldsymbol{\psi_1(x,t)} + \boldsymbol{\psi_2(x,t)}\}}{\partial x^2}$$

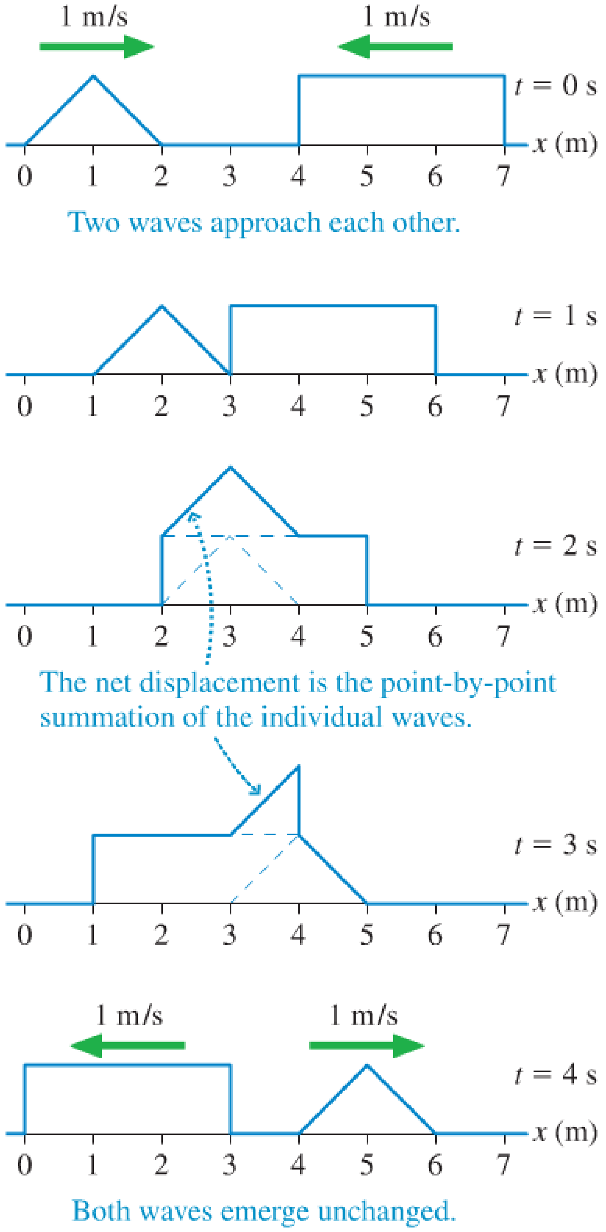
$$\frac{\partial^2\psi(x,t)}{\partial t^2} = v^2\frac{\partial^2\psi(x,t)}{\partial x^2}$$

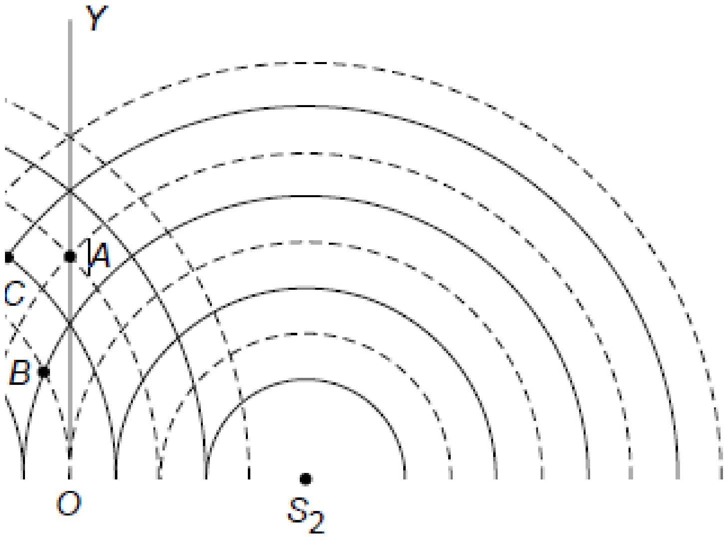


When the two waves are **in-phase** (δ =0), they interfere **const**
the amplitude of the individual waves. When the two waves ha
interfere **destructively** and cancel each other out.

Waves and Vibrations (PH2001)

Superposition of waves: Constructive and Destructive Interference





Two waves with the same amplitude, frequency, and wavelength are travelling in the same direction are,

$$\psi(x,t) = A\sin\{\omega t - (kx+\epsilon)\} \qquad \alpha = -(kx+\epsilon)$$

$$\psi_1(x,t) = A_1\sin(\omega t + \alpha_1)$$

$$\psi_2(x,t) = A_2\sin(\omega t + \alpha_2)$$

Due to the principle of superposition, the resulting wave displacement may be written as:

$$\boldsymbol{\psi(x,t) = \psi_1(x,t) + \psi_2(x,t)} \qquad \boldsymbol{\psi(x,t)} = A_1\sin(\omega t + \alpha_1) + A_2\sin(\omega t + \alpha_2)$$

$$\psi(x,t) = A_1\{\sin\omega t\cos\alpha_1 + \cos\omega t\sin\alpha_1\} + A_2\{\sin\omega t\cos\alpha_2 + \cos\omega t\sin\alpha_2\}$$

$$\psi(x,t) = \sin\omega t(A_1\cos\alpha_1 + A_2\cos\alpha_2) + \cos\omega t(A_1\sin\alpha_1 + A_2\sin\alpha_2)$$

The coefficients of $\sin\omega t$ and $\cos\omega t$ are independent of time. Let us consider,

$$A_1\cos\alpha_1 + A_2\cos\alpha_2 = A\cos\alpha \qquad A_1\sin\alpha_1 + A_2\sin\alpha_2 = A\sin\alpha$$

$\psi(x,t) = A\sin\{\omega t + \alpha\}$ $\alpha = -(kx + \epsilon)$

Waves and Vibrations (PH2001)

Superposition (same frequency)

$\psi_1(x,t) = A_1\sin(\omega t + \alpha_1)$

$\psi_2(x,t) = A_2\sin(\omega t + \alpha_2)$

Squaring and adding we get,

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1 sin\alpha_2)$$

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1)$$

Dividing, we get, $$tan\alpha = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2}$$

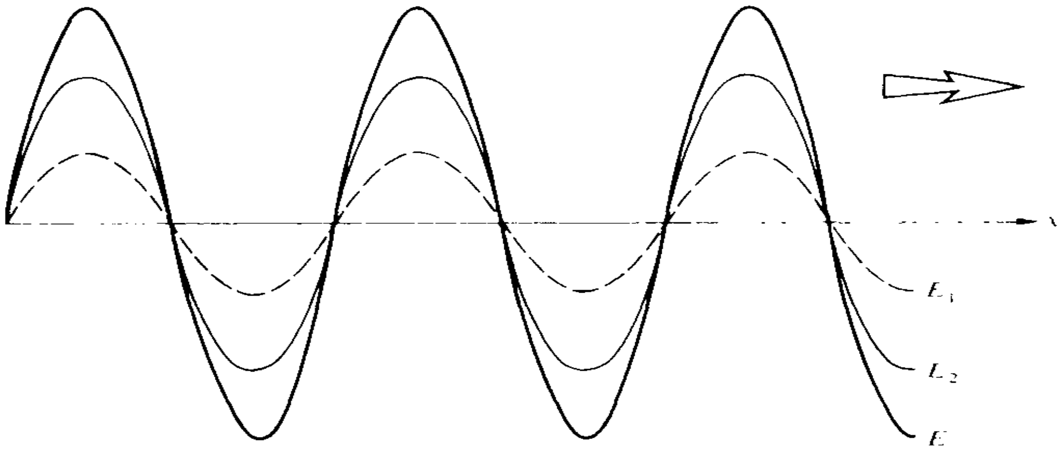
$$\boldsymbol{\psi(x,t) = \sin\omega t.\, Acos\alpha + \cos\omega t.\, Asin\alpha}$$

$$\boxed{\boldsymbol{\psi(x,t) = Asin(\omega t + \alpha)}}$$

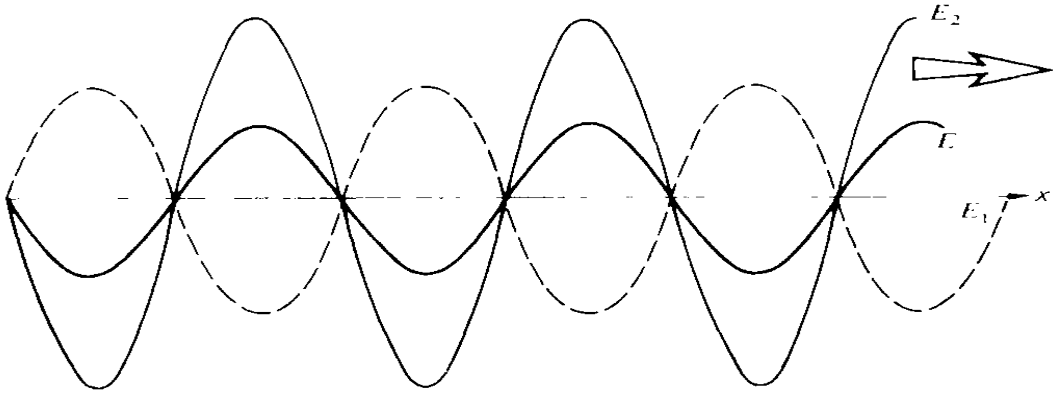
$$\delta = \alpha_2 - \alpha_1 = kx_2 + \epsilon_2 - kx_1 - \epsilon_1$$

$$\delta = \frac{2\pi}{\lambda}\Delta x + (\epsilon_2 - \epsilon_1)$$

$$\delta = (\epsilon_2 - \epsilon_1)$$
$= constant\ of\ time$
$\Rightarrow Coherent$



$E = E_1 + E_2$



$$\delta = \frac{2\pi}{\lambda}\Delta x$$

$$\boxed{n = \frac{c}{v} = \frac{\lambda_0}{\lambda}}$$

$$\delta = \frac{2\pi}{\lambda_0} n\Delta x$$

$\psi(x,t) = A\sin\{\omega t + \alpha\}$

$\alpha = -(kx + \epsilon)$

Waves and Vibrations (PH2001)

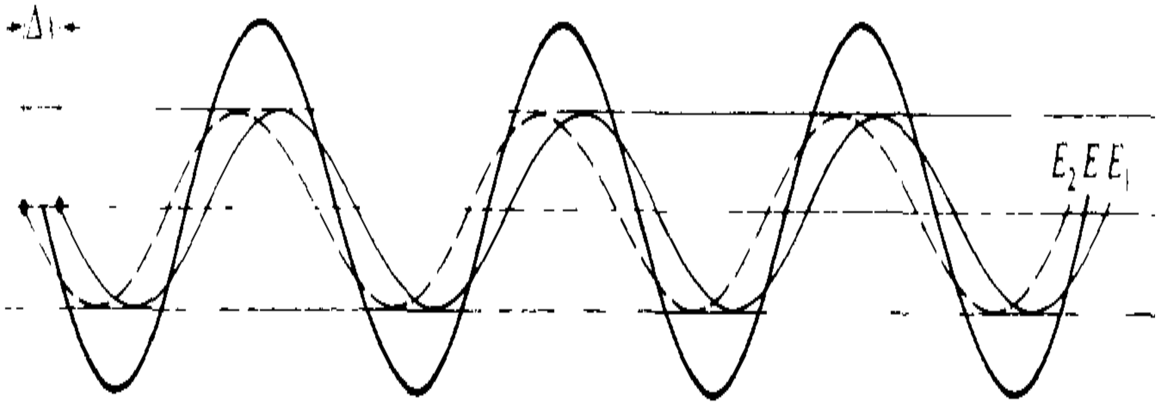
Superposition (same frequency)

$\psi_1(x,t) = A_1\sin(\omega t + \alpha_1)$

$\psi_2(x,t) = A_2\sin(\omega t + \alpha_2)$

$\boldsymbol{\psi(x,t) = Asin(\omega t + \alpha)}$

$\delta = \frac{2\pi}{\lambda}\Delta x + (\epsilon_2 - \epsilon_1)$



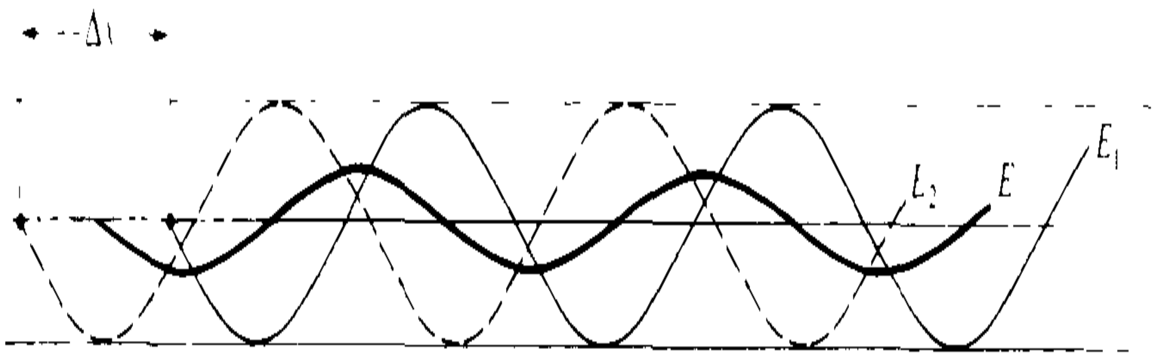
Let us consider two waves of same amplitude but having a path difference,$(x + \Delta x, x)$

$$\boldsymbol{\psi(x,t)} = A_0\sin(\omega t + \alpha_1) + A_0\sin(\omega t + \alpha_2)$$

$$= 2A_0 cos\left\{\frac{(\omega t + \alpha_2) - (\omega t + \alpha_1)}{2}\right\} \sin\left\{\frac{(\omega t + \alpha_2) + (\omega t + \alpha_1)}{2}\right\}$$

$$\boldsymbol{\psi(x,t)} = 2A_0 cos\left\{\frac{\alpha_2 - \alpha_1}{2}\right\} \sin\left\{\omega t + \frac{\alpha_2 + \alpha_1}{2}\right\}$$

$$\boxed{\boldsymbol{\psi(x,t)} = 2A_0 cos\left\{\frac{\delta}{2}\right\} \sin\left\{\omega t + \frac{\alpha_2 + \alpha_1}{2}\right\}}$$



If, $\epsilon_2 - \epsilon_1 = 0$, and path difference,$(x + \Delta x, x)$

$$\boxed{\boldsymbol{\psi(x,t)} = 2A_0 cos\left\{\frac{k\Delta x}{2}\right\} \sin\left\{\omega t - k(x + \frac{\Delta x}{2})\right\}}$$

$\psi(x,t) = A\sin\{\omega t + \alpha\}$ $\alpha = -(kx + \epsilon)$

Waves and Vibrations (PH2001)

Superposition (same frequency)

$$\psi_1(x,t) = A_1\sin(\omega t + \alpha_1)$$
$$\psi_2(x,t) = A_2\sin(\omega t + \alpha_2)$$

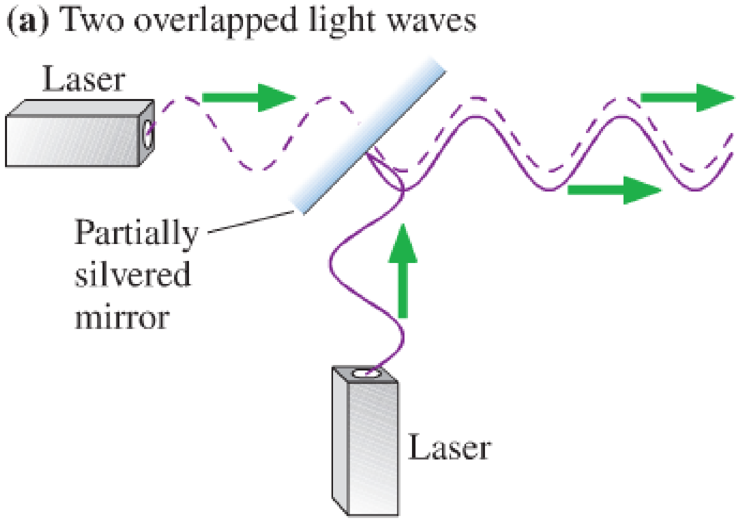
$$\boldsymbol{\psi(x,t) = A\sin(\omega t + \alpha)}$$

$$\delta = \frac{2\pi}{\lambda}\Delta x + (\epsilon_2 - \epsilon_1)$$

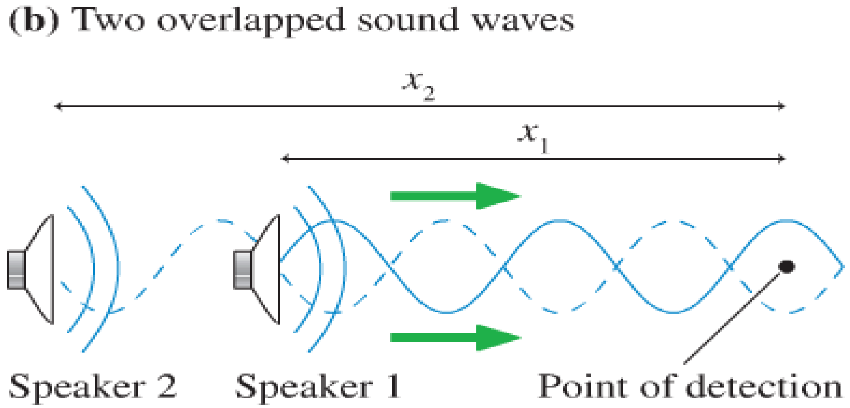
If, $\boldsymbol{\phi_0 = \epsilon_2 - \epsilon_1 = 0}$, and path difference,$(x + \Delta x, x)$

$$\delta = \frac{2\pi}{\lambda}\Delta x$$

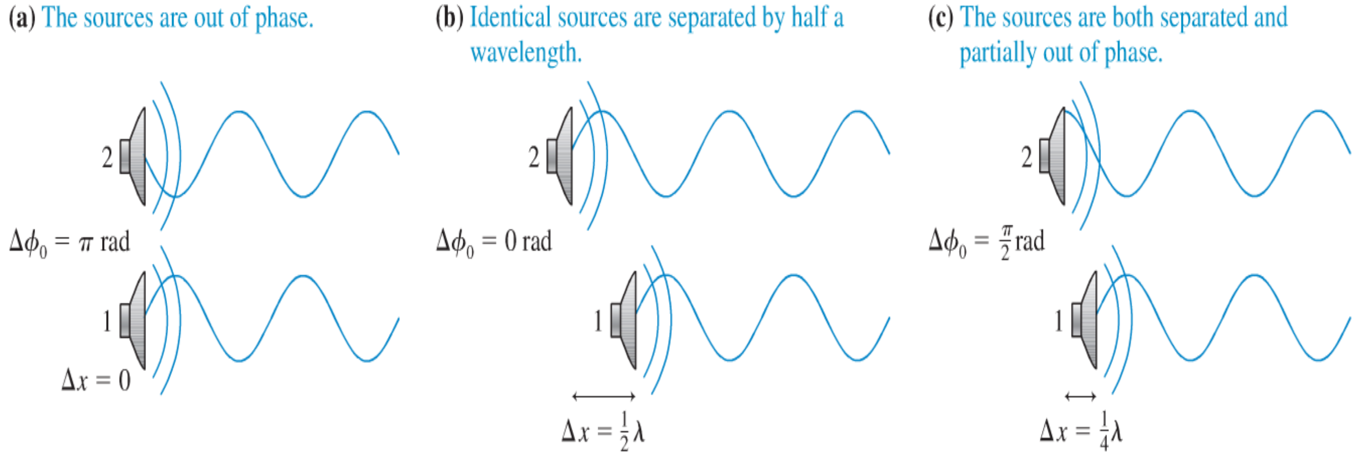
$$\boldsymbol{\psi(x,t) = 2A_0 cos\left\{\frac{k\Delta x}{2}\right\}\sin\left\{\omega t - k(x + \frac{\Delta x}{2})\right\}}$$



These two waves are in phase ($\delta = 0 = \Delta x$) and of equal amplitude $\boldsymbol{A_0}$, will give constructive interference. This will lead to a combined amplitude $\boldsymbol{A = 2A_0}$



These two waves are out of phase ($\delta = n\pi = \frac{2\pi}{\lambda}\Delta x$) and of equal amplitude $\boldsymbol{A_0}$, will give destructive interference. This will lead to a combined



 $\psi(x,t) = A\sin\{\omega t-(kx+\epsilon)\}$ $\alpha = -(kx+\epsilon)$ Waves and Vibrations (PH2001)

Superposition of waves: Many waves

$$\boldsymbol{\psi(x,t) = A\sin(\omega t+\alpha)}$$

$\psi_1(x,t) = A_1\sin(\omega t+\alpha_1)$ $\psi_2(x,t) = A_2\sin(\omega t+\alpha_2)$ $\psi_3(x,t) = A_3\sin(\omega t+\alpha_3)$

Due to the principle of superposition, the resulting wave displacement may be written as:

$$\psi(x,t) = A_1\{\sin\omega t\cos\alpha_1 + \cos\omega t\sin\alpha_1\} + A_2\{\sin\omega t\cos\alpha_2 + \cos\omega t\sin\alpha_2\} + A_3\{\sin\omega t\cos\alpha_3 + \cos\omega t\sin\alpha_3\}$$

$$\psi(x,t) = \sin\omega t\,(A_1\cos\alpha_1 + A_2\cos\alpha_2 + A_3\cos\alpha_3) + \cos\omega t(A_1\sin\alpha_1 + A_2 sin\alpha_2 + A_3\sin\alpha_3)$$

$$Acos\alpha = A_1\cos\alpha_1 + A_2\cos\alpha_2 + A_3\cos\alpha_3$$

$$Asin\alpha = A_1\sin\alpha_1 + A_2\sin\alpha_2 + A_3\sin\alpha_3$$

$$tan\alpha = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2 + A_3\sin\alpha_3}{A_1\cos\alpha_1 + A_2\cos\alpha_2 + A_3\cos\alpha_3}$$

$$\boldsymbol{\psi(x,t) = A\sin(\omega t+\alpha)}$$

$$A^2 = A_1^2 + A_2^2 + A_3^2 + 2A_1A_2\cos(\alpha_1-\alpha_2) + 2A_1A_2\cos(\alpha_1-\alpha_3) + 2A_3A_2\cos(\alpha_2-\alpha_3)$$

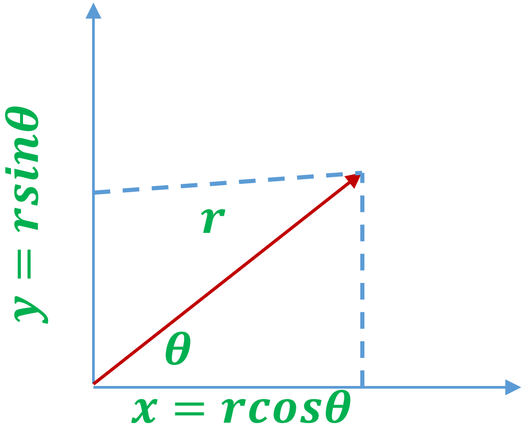
For multiple wave,

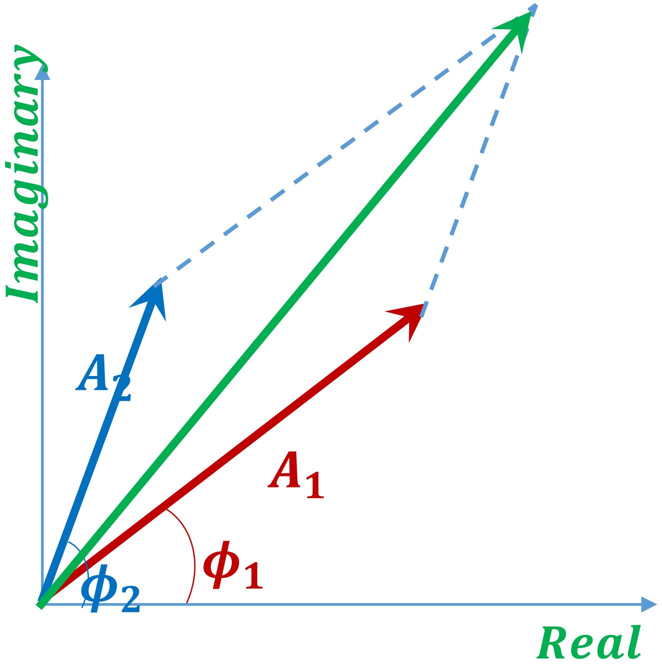
$$A^2 = \sum_i^M A_i^2 + 2\sum_{j>i}^M\sum_i^M A_iA_j\cos(\alpha_i-\alpha_j)$$

$$tan\alpha = \frac{\sum_i^M A_i\sin\alpha_i}{\sum_i^M A_i\cos\alpha_i}$$

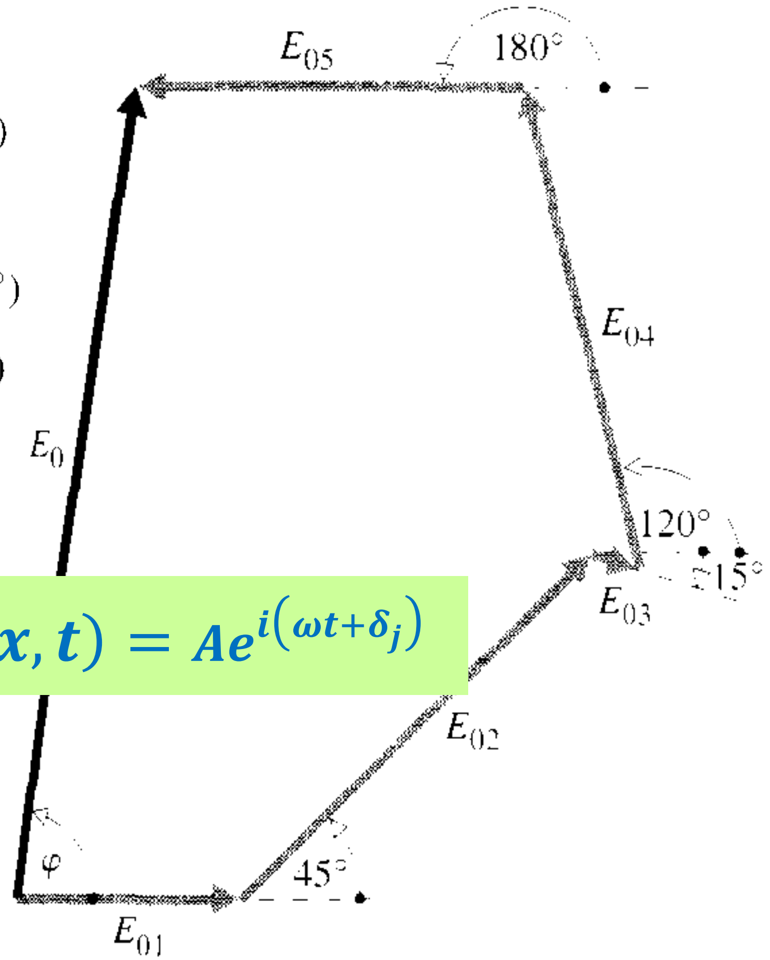
Superposition: Phasor

$r = \sqrt{x^2 + y^2} \quad \theta = tan^{-1}\left(\frac{y}{x}\right)$





$$E_1 = 5 \sin \omega t$$
$$E_2 = 10 \sin (\omega t + 45°)$$
$$E_3 = \sin (\omega t - 15°)$$
$$E_4 = 10 \sin (\omega t + 120°)$$
$$E_5 = 8 \sin (\omega t + 180°)$$



$$\psi_j(x,t) = Ae^{i(\omega t + \delta_j)}$$

$$\tilde{z} = x + iy \quad = r(cos\theta + isin\theta)$$

$$\boxed{\tilde{z} = re^{i\theta}}$$

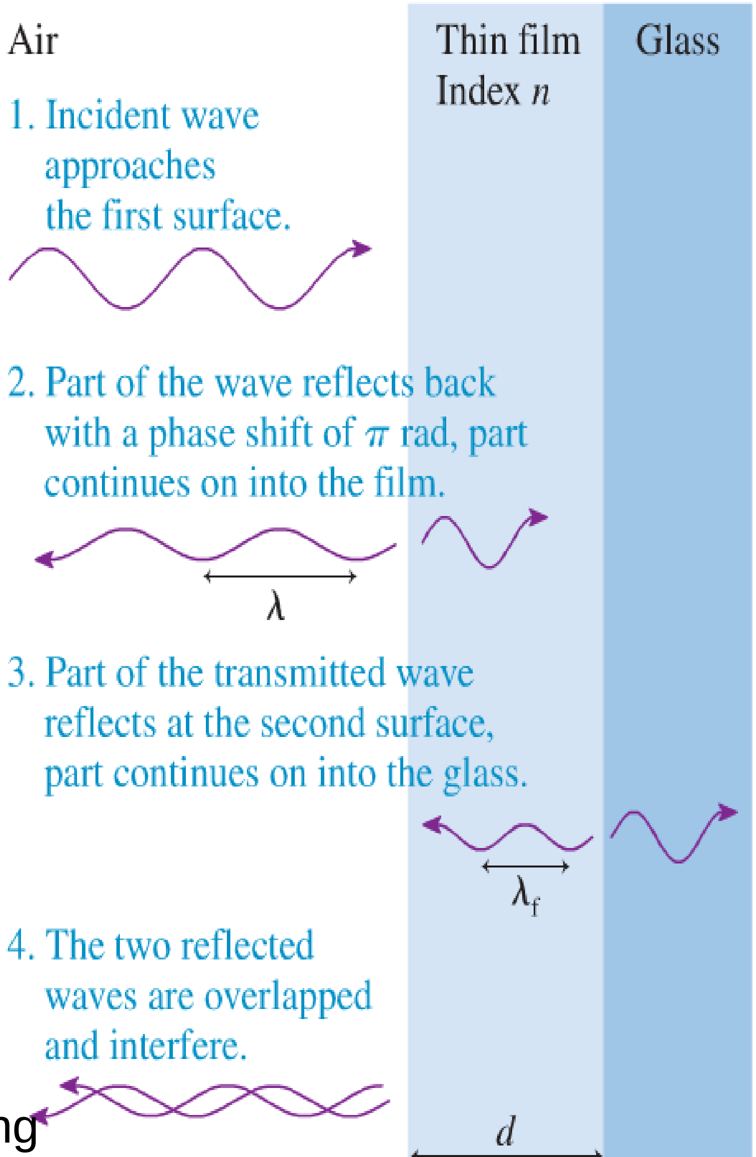
$$\boxed{Re(\tilde{z}) = rcos\theta} \quad \boxed{Im(\tilde{z}) = rsin\theta}$$

$$\psi(x,t) = Ae^{i(kx - \omega t + \delta)} = Ae^{i\phi}$$

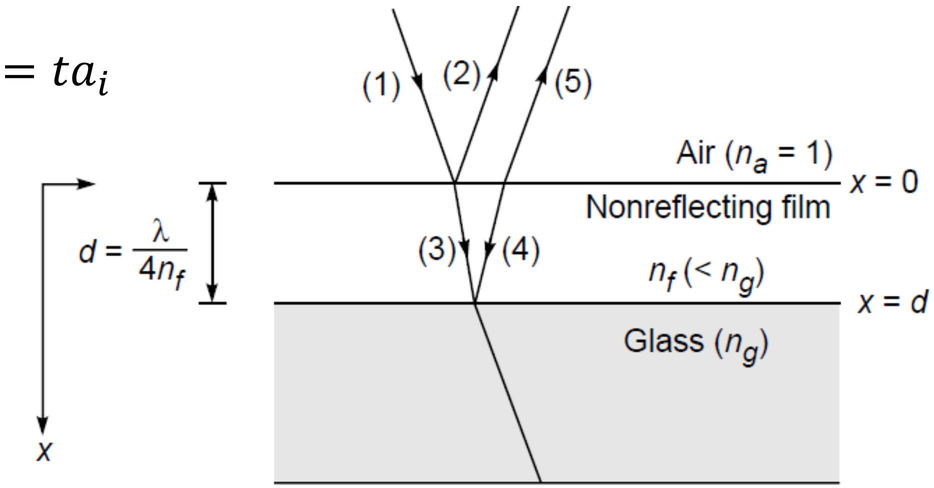
The complex amplitude is known as a **phasor,** and it's specified by its magnitude and phase ($A\angle\phi$)

Superposition: Anti-reflective coating

From Fresnel's law of EM wave: $a_r = \frac{n_1 - n_2}{n_1 + n_2} a_i = r a_i, \quad a_t = \frac{2n_1}{n_1 + n_2} a_i = t a_i$



Ref: Neil Alberding



For destructive interference between (2) and (5),

$$2n_f d = \frac{1}{2}\lambda \Rightarrow d = \frac{\lambda}{4n_f}$$

For $\mathrm{MgF_2}, n_f = 1.38, \lambda = 5000A^0 \Rightarrow d = 0.9 \times 10^{-5} cm$

$$a_2 = \frac{n_a - n_f}{n_a + n_f} a_1, \; a_3 = \frac{2n_a}{n_a + n_f} a_1, a_4 = \frac{n_f - n_g}{n_f + n_g} a_3, a_5 = \frac{2n_f}{n_f + n_a} a_4$$

For complete destructive interference between (2) and (5),

$$a_2 = a_5 \Rightarrow \frac{n_f - n_a}{n_f + n_a} = \frac{n_g - n_f}{n_g + n_f}, \qquad \Rightarrow n_f = \sqrt{n_a n_g}$$

Superposition: Division of wavefront

At A, $\psi_1 = a\cos(\omega t)$; $\boxed{\psi_2 = a\cos(\omega t)\,; \quad x = 0}$

$\boldsymbol{\psi_A = 2a\cos\omega t}$ **=> amplitude is $2a$**

At $t = \frac{T}{4} = \frac{\pi}{2\omega}$, $\quad \psi_A$**=0**

At C, $S_1C - S_2C = \lambda\ (assume)$

$\psi_C = \psi_1 + \psi_2 = a\cos(\omega t) + a\cos(\omega t - \pi)$;

$\psi_C = a\cos(\omega t) + a\cos(\omega t) = 2a\cos\omega t$
Constructive Interference

At a general point P,

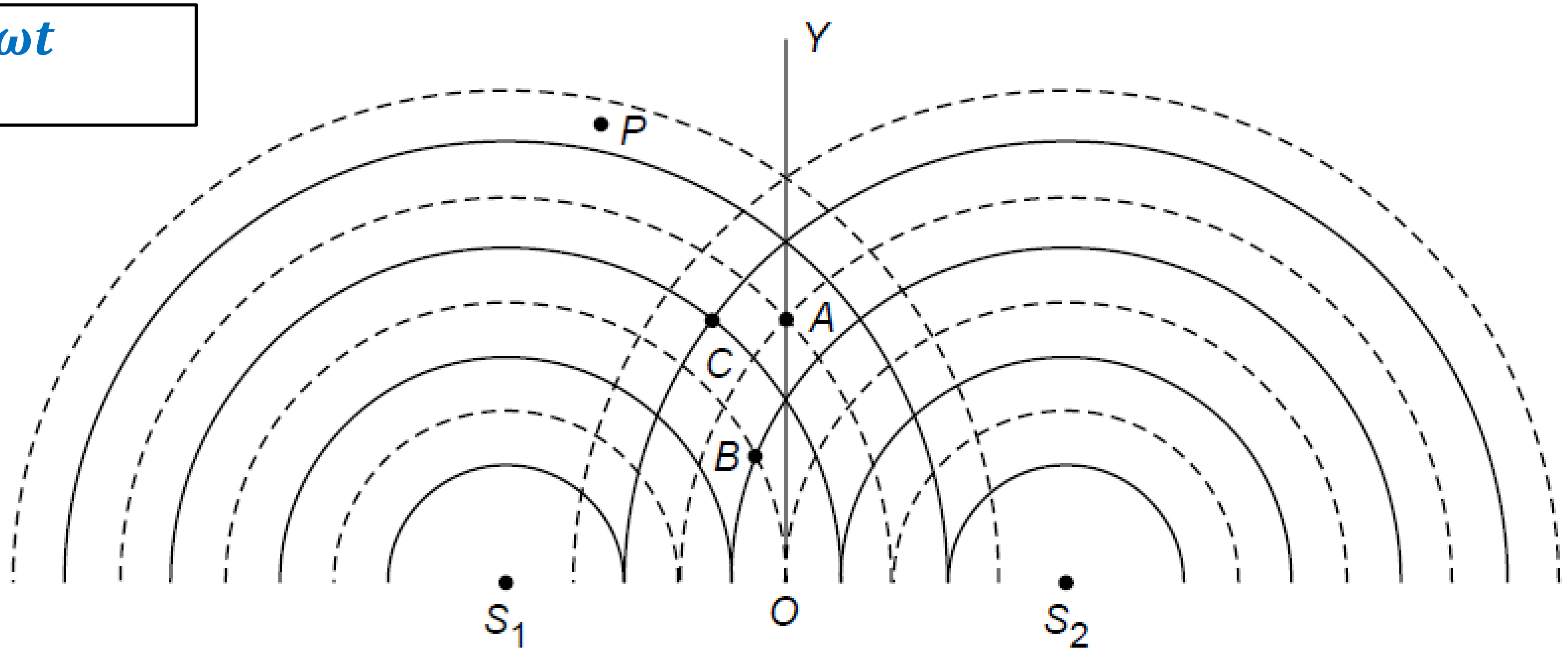
$S_1P - S_2P = n\lambda$
Constructive Interference

$S_1P - S_2P = \{n + \frac{1}{2}\}\lambda$
Destructive Interference

At B, $S_1B - S_2B = \frac{\lambda}{2}\ (assume)$

$\psi_B = \psi_1 + \psi_2 = a\cos(\omega t) + a\cos(\omega t - \pi)$;

$\psi_B = a\cos(\omega t) - a\cos(\omega t) = 0$ for all time
Destructive interference



Lloyd Mirror Expt: Division of wavefront

S and S' are two coherent sources of light.

$\theta_i \approx \pi/2$ (grazing angle)

At a general point P,

$$SOP - SP = n\lambda$$

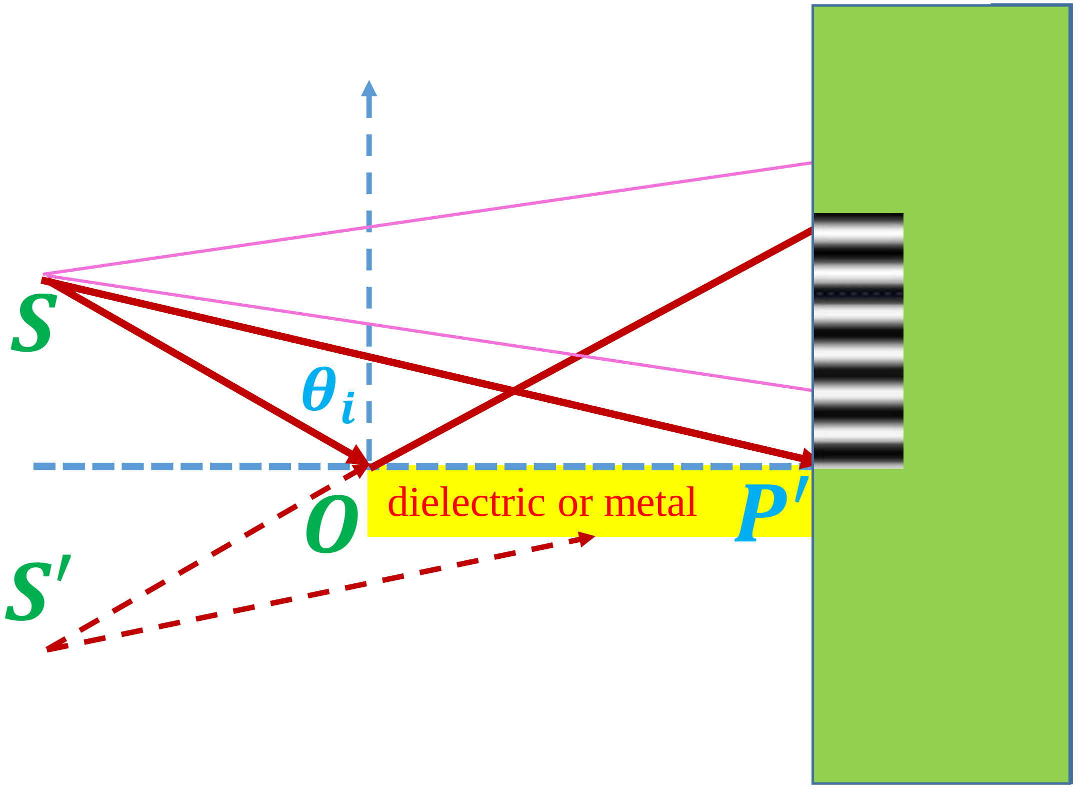
Constructive Interference

$$SOP - SP = \{n + \frac{1}{2}\}\lambda$$

Destructive Interference

Central dark fringe

The distinguishing feature of this device is that at glancing incidence ($\theta_i = \pi/2$) the reflected beam undergoes a 180^0 phase shift



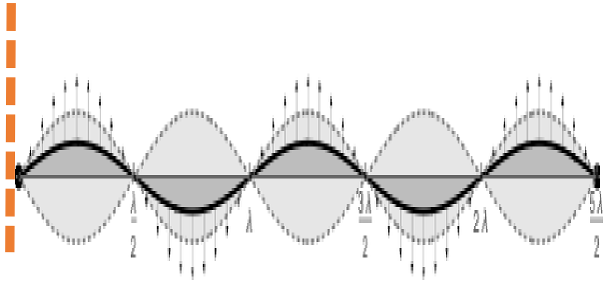
Waves and Vibrations (PH2001)

Superposition: Standing wave

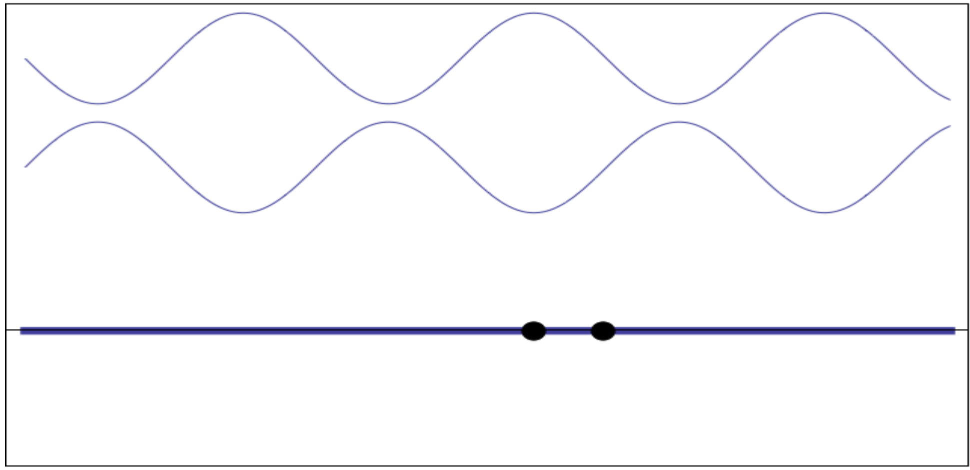
Two harmonic waves of the **same frequency** propagating in opposite directions.

$$\psi_I(x,t) = A_I \sin\{kx + \omega t + \alpha_I\}$$

$$\psi_R(x,t) = A_R \sin\{kx - \omega t + \alpha_R\}$$



A situation of practical concern arises when the incident wave is reflected backward off some sort of mirror; a rigid wall will do for sound waves or a conducting sheet for EM waves.



We assume, $\boldsymbol{A_I = A_R = A,\ \alpha_I = 0}$

BC: at the mirror, $x = 0$, the **resulting disturbance is zero**,

$$\psi(0,t) = \mathbf{0} = \psi_I(x,0) + \psi_R(x,0)$$

$$\psi_I(0,t) = A\sin\{\omega t\} \Rightarrow \psi_R(0,t) = -A\sin\{\omega t\} = A\sin\{-\omega t + 0\},\ \alpha_R = 0$$

$$\psi(x,t) = \psi_I(x,t) + \psi_R(x,t) = A[\sin\{kx + \omega t\} + \sin\{kx - \omega t\}]$$

$$\psi(x,t) = 2A\sin\{kx\}\cos\{\omega t\}$$

$\boldsymbol{\psi(x,t) = 0,\ for\ kx = n\pi,\ ie,\ x = \frac{n\lambda}{2}}$ => NODES

Half-way, maxima are located. $\boldsymbol{x = \frac{n\lambda}{4}}$ => ANTINODES

$\omega t = n\pi$ MAX $\qquad \omega t = (2n+1)\frac{\pi}{2}$ MIN

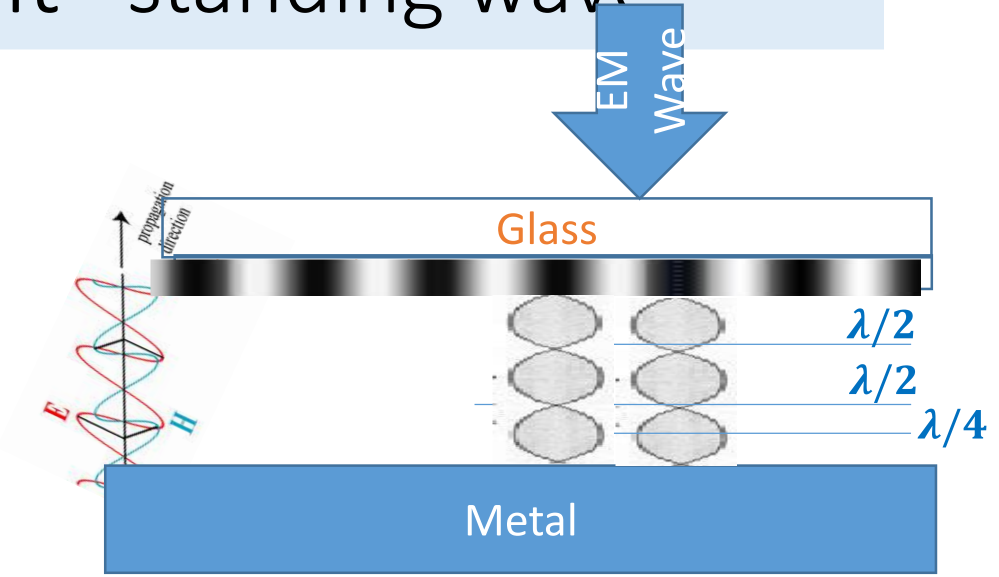
Summary

- A standing wave is a superposition of two waves travelling in opposite directions with same frequency.
- Constructive interference creates antinodes, destructive interference creates nodes.
- Nodes on a standing wave are spaced λ/2 apart and never move
- Antinodes are halfway between nodes.

TPS

Waves and Vibrations (PH2001)

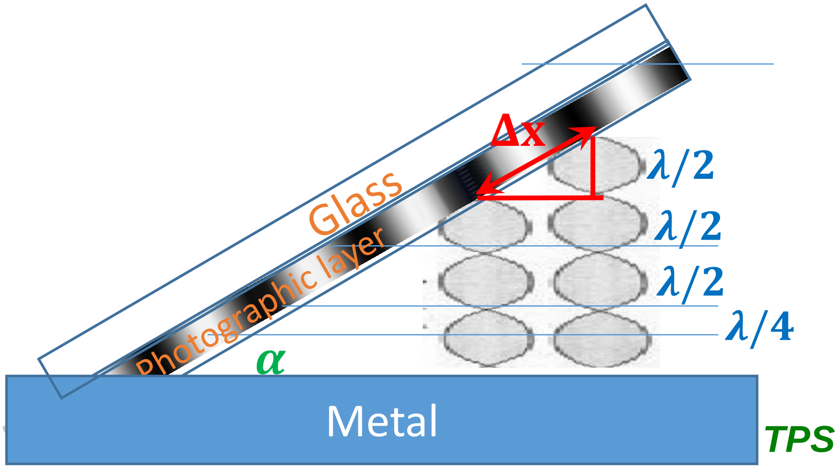
Superposition: Wiener's Experiment--standing wave



James Clerk Maxwell had demonstrated theoretically in 1865 that the electromagnetic field equations allowed wave solutions which propagated at the velocity of light.

Otto Wiener first demonstrated the existence of standing light waves (500nm). Photographic layer to be roughly 1/30th of the wavelength of the light

Wiener had demonstrated, in the context of the electromagnetic theory, that the electric field is the 'active ingredient' in light waves.



We assume, $(\lambda/2)/(\Delta x) = \sin\alpha$

For very small α, $\Delta x = \dfrac{\lambda}{2\alpha}$

Waves and Vibrations (PH2001)

Superposition of two sine waves with different frequencies: Beats

Practically, disturbances of any kind, are not strictly of a single frequency (monochromatic). It is realistic, to consider quasi-monochromatic wave, which is composed of a narrow range of frequencies.
The study of such light will lead us to the important concepts of bandwidth and coherence time.

Let us consider the composite disturbance arising from a combination of the coherent waves of slightly different frequencies and same amplitude

$$\psi_1(x,t) = A_0 \sin(k_1 x - \omega_1 t) \qquad k_1 > k_2$$

$$\psi_2(x,t) = A_0 \sin(k_2 x - \omega_2 t) \qquad \omega_1 > \omega_2$$

$$\omega = \frac{1}{2}(\omega_1 + \omega_2) \Rightarrow avg\ freq$$

$$\omega_m = \frac{1}{2}(\omega_1 - \omega_2) \Rightarrow modulation\ freq$$

$$k = \frac{1}{2}(k_1 + k_2) \Rightarrow avg\ \text{prop no.}$$

$$k_m = \frac{1}{2}(k_1 - k_2) \Rightarrow modulation\ \text{prop no.}$$

$$\psi(x,t) = A_0 \sin(k_1 x - \omega_1 t) + A_0 sin(k_2 x - \omega_2 t)$$

$$\psi(x,t) = 2A_0 cos\left\{\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right\} sin\left\{\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right\}$$

$$\psi(x,t) = 2A_0 cos\left\{\frac{(k_1 - k_2)x - (\omega_1 - \omega_2)t}{2}\right\} \sin\left\{\frac{(k_1 + k_2)x - (\omega_1 + \omega_2)t}{2}\right\}$$

$$\boldsymbol{\psi(x,t) = 2A_0 cos\{k_m x - \omega_m t\} sin\{kx - \omega t\}}$$

$$\boldsymbol{\psi(x,t) = A sin\{kx - \omega t\}} \qquad \boldsymbol{A = 2A_0 cos\{k_m x - \omega_m t\}}$$

The total disturbance may be regarded as a traveling wave of frequency $\boldsymbol{\omega}$ having a time-varying (modulated) amplitude $\boldsymbol{A}$

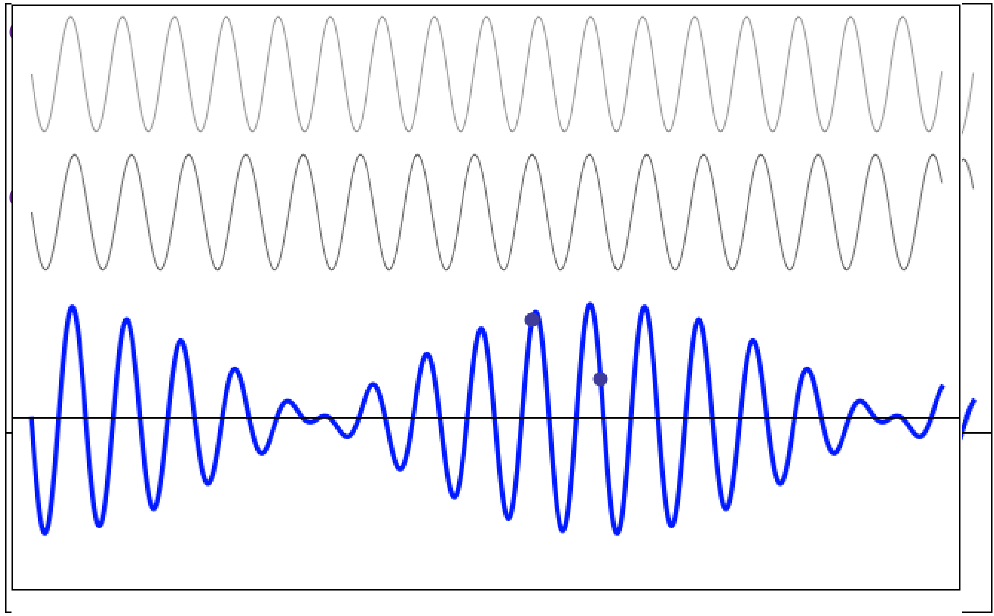
Waves and Vibrations (PH2001)

Beats:

$\omega = \frac{1}{2}(\omega_1 + \omega_2)$ $k = \frac{1}{2}(k_1 + k_2)$ $\omega_m = \frac{1}{2}(\omega_1 - \omega_2)$ $k_m = \frac{1}{2}(k_1 - k_2)$

$$\psi(x,t) = A sin\{kx - \omega t\}, \qquad A = 2A_0 cos\{k_m x - \omega_m t\}$$

In applications of interest here, ω_1 and ω_2, will always be rather large and are comparable to each other,

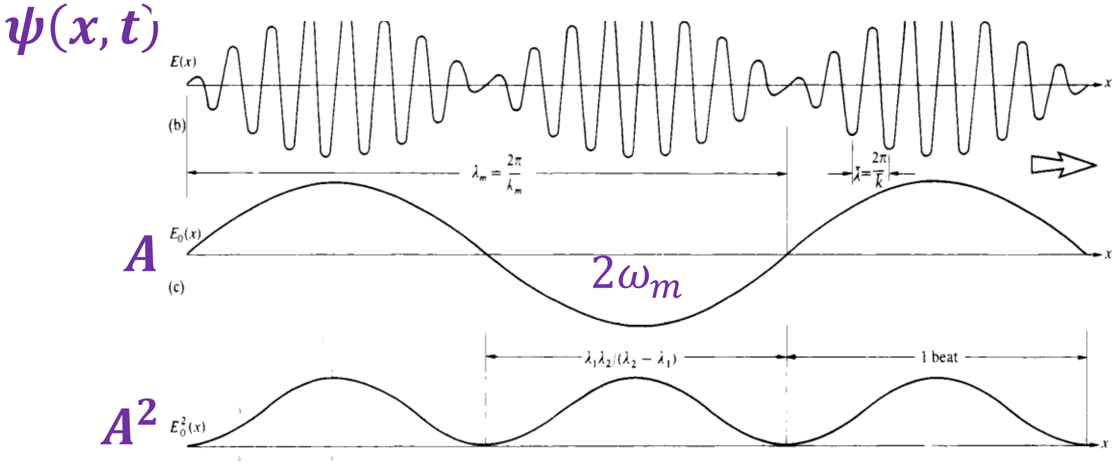


Amplitude will change **slowly,** whereas $\psi(x,t)$ will vary **quite rapidly**

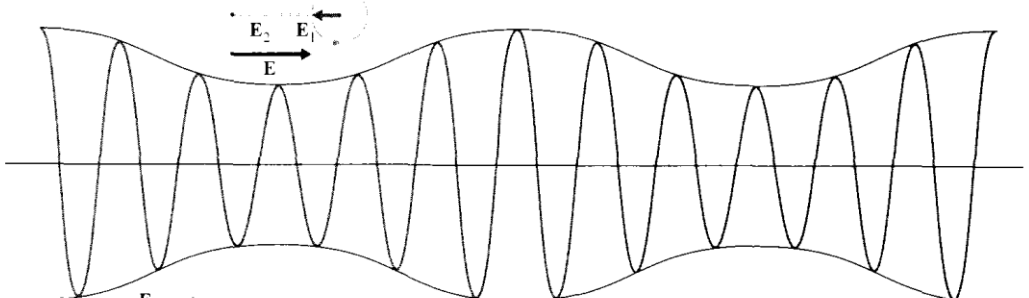
$$A^2 = 4A_0^2 cos^2\{k_m x - \omega_m t\} = 2A_0^2[1 + cos\{2k_m x - 2\omega_m t\}]$$

$$A^2 = 2A_0^2 + 2A_0^2 cos\{2k_m x - 2\omega_m t\}]$$

$$2\omega_m = \omega_1 - \omega_2 \Rightarrow Beat\ frequency$$



If, $A_1 \neq A_2$

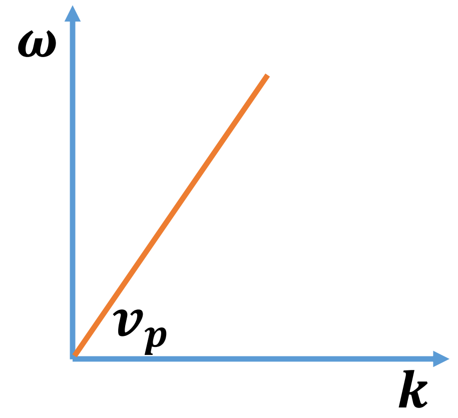


Waves and Vibrations (PH2001)

Dispersion:

$$\omega = \sqrt{\frac{T_0}{\rho_0}}k \qquad \omega = \sqrt{\frac{Y}{\rho}}k \qquad \omega = \sqrt{\frac{\gamma P}{\rho}}k$$

Relation giving ω as a function of wave number k is known as dispersion relation.



$$v_p = \frac{\omega}{k} \Rightarrow \omega = kv_p$$

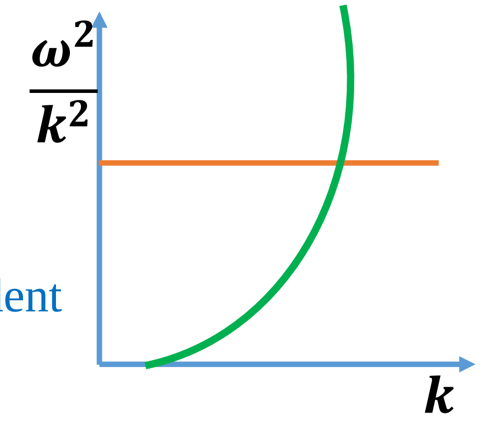
For a Piano the dispersion relation is , $\dfrac{\omega^2}{k^2} = v_p^2 + \beta k^2$

Smaller value of β indicates more flexibility of the string

Waves obey simple dispersion relation, $\dfrac{\omega}{k} = constant$, is known as nondispersive.

When $\dfrac{\omega}{k}$ depends on k the wave is dispersive. Vacuum is the only truly nondispersive environment. $\dfrac{\omega}{k} = \dfrac{c}{n}$ $\qquad n = \dfrac{c}{v_p}$

$$n = \frac{ck}{\omega} = n(\omega)$$



Dispersion corresponds to the phenomena where the refractive index of a medium is frequency dependent

The specific relationship between ω and k determines the phase velocity of a wave (v_p). In a nondispersive medium $\dfrac{\omega}{k} = con$ and a plot of ω versus k is a straight line. The frequency and wavelength change so as to keep v_p constant. All waves travel with the same phase speed in a nondispersive medium. By contrast, in a dispersive medium (anything other than vacuum) every wave propagates at a speed that depends on its frequency.

Waves and Vibrations (PH2001)

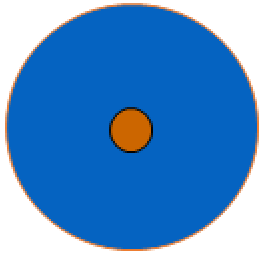
Dispersion:

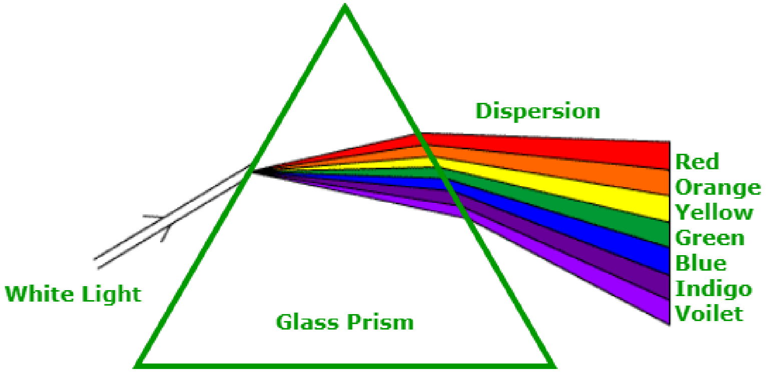
$$v_p = \frac{\omega}{k} \quad n = \frac{c}{v_p}$$

Polarization: Development of induced dipole moment (per unit volume) in a material in the presence of an external electric field is known as polarization.

P* α *E* ➔ *P=αE;

***α—polarizability,* measure of ability to get polarized**





$$F = -\beta x$$

$$m_e \frac{d^2x}{dt^2} = -\beta x$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

$$\boxed{x = x_0 cos\omega_0 t}$$

$$\omega_0^2 = \frac{\beta}{m_e}$$

$$m_e \frac{d^2x}{dt^2} = -\beta x + q_e E_0 cos\omega t$$

$$\frac{d^2x}{dt^2} = -\omega_0^2 x + (q_e/m_e) E_0 cos\omega t$$

$$\boxed{x(t) = x_0 cos\omega t}$$

$$\boxed{x(t) = \frac{(q_e/m_e)}{\omega_0^2 - \omega^2} E(t)}$$

Dispersion:

$$x(t) = \frac{(q_e/m_e)}{\omega_0^2 - \omega^2} E(t)$$

$$\frac{N\alpha}{3\varepsilon_0} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

Waves and Vibrations (PH2001

Clausius-Mossotti relation.

$P = (q_e x)N, \qquad N \to no.\,density$

$$P = \frac{(q_e^2/m_e)N}{\omega_0^2 - \omega^2} \boldsymbol{E}$$

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}$$

$$\epsilon = \epsilon_0 + P/E$$

$$\epsilon = \epsilon_0 + \frac{(q_e^2/m_e)N}{\omega_0^2 - \omega^2}$$

$$v = \sqrt{\frac{1}{\epsilon\mu}}, \quad n = \frac{c}{v} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\epsilon_r}$$

$\mu_r \approx 1$ *in most of the cases*

$$n^2 - 1 = \frac{q_e^2 N}{\epsilon_0 m_e}\left(\frac{1}{(\omega_0^2 - \omega^2)}\right)$$

$$\frac{n^2 - 1}{n^2 + 2} = \frac{q_e^2 N}{3\epsilon_0 m_e} \cdot \frac{1}{\omega_0^2 - \omega^2}$$

Considering local field, $E_L = E + \frac{P}{3\epsilon_0}$

$$P = \frac{(q_e^2/m_e)N}{\omega_0^2 - \omega^2}\left(E + \frac{P}{3\epsilon_0}\right)$$

Considering N molecules per unit volume, each with f_j oscillators having natural frequencies $\omega_{0j}, j = 1,2,3 \ldots$

$$P = \frac{(q_e^2/m_e)N}{\omega_0^2 - \omega^2}\left(\frac{P}{\{\epsilon_0(\epsilon_r - 1)\}} + \frac{P}{3\epsilon_0}\right)$$

$$\frac{n^2 - 1}{n^2 + 2} = \frac{q_e^2 N}{3\epsilon_0 m_e} \cdot \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2}$$

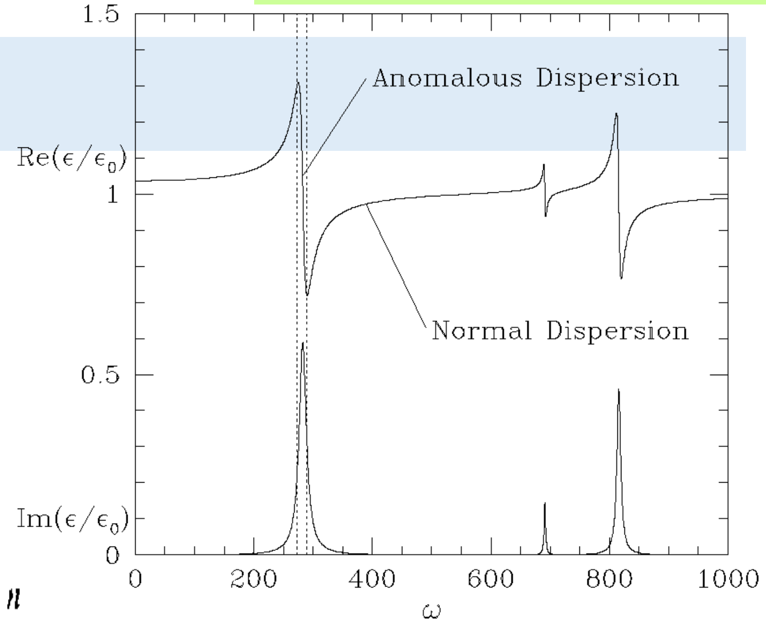
$$1 = \frac{(q_e^2/m_e)N}{\omega_0^2 - \omega^2} \cdot \frac{3 + \epsilon_r - 1}{3\epsilon_0(\epsilon_r - 1)}$$

$$\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{q_e^2 N}{3\epsilon_0 m_e} \cdot \frac{1}{\omega_0^2 - \omega^2}$$

Dispersion:

$$\frac{n^2-1}{n^2+2} = \frac{q_e^2 N}{3\epsilon_0 m_e} \cdot \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2}$$

Waves and Vibrations (PH2001



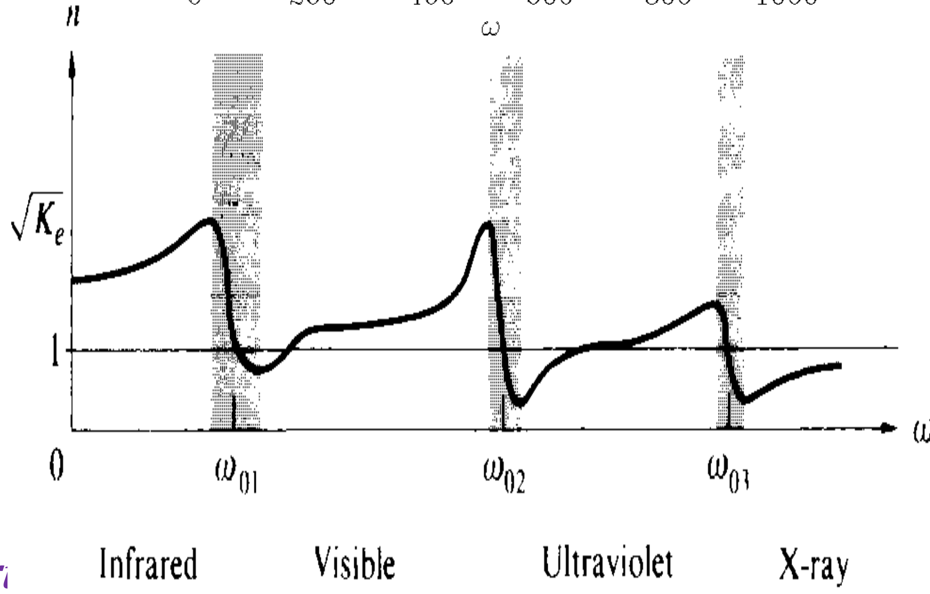
Considering damping effect ($\approx m_e \nu \frac{dx}{dt}$),
the dispersion relation is written as

$$\frac{n^2-1}{n^2+2} = \frac{q_e^2 N}{3\epsilon_0 m_e} \cdot \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 + i\nu\omega}$$

If damping is small $\nu \to 0$

If, $\omega_{0j}^2 \gg \omega^2$, n *is a constant of frequency*

For colourless/transparent materials $\omega_0 = 5\omega_{visible}$



As, ω *approaches* ω_{0j}, (ω *not very close to* ω_{0j}),
$(\omega_{0j}^2 - \omega^2)$ *decreases and*
n *increases* $\Rightarrow$ ***Normal Dispersion***---- $\frac{dn}{d\omega} > \mathbf{0}$.

As, $\omega \to \omega_{0j}$, $(\omega_{0j}^2 - \omega^2) \to 0$, ν *becomes significant, the oscillator start*
amplitude vibrations resulting absorption of energy from the wave $\Rightarrow$ ***Anomalous Dispersion***

Regions immediately around each ω_{0j} , $\frac{dn}{d\omega} < \mathbf{0}$, absorption of energy is observed and is known as ***Absorption Band***

$$\omega_m = \frac{1}{2}(\omega_1 - \omega_2) \qquad k_m = \frac{1}{2}(k_1 - k_2)$$

Waves and Vibrations (PH2001)

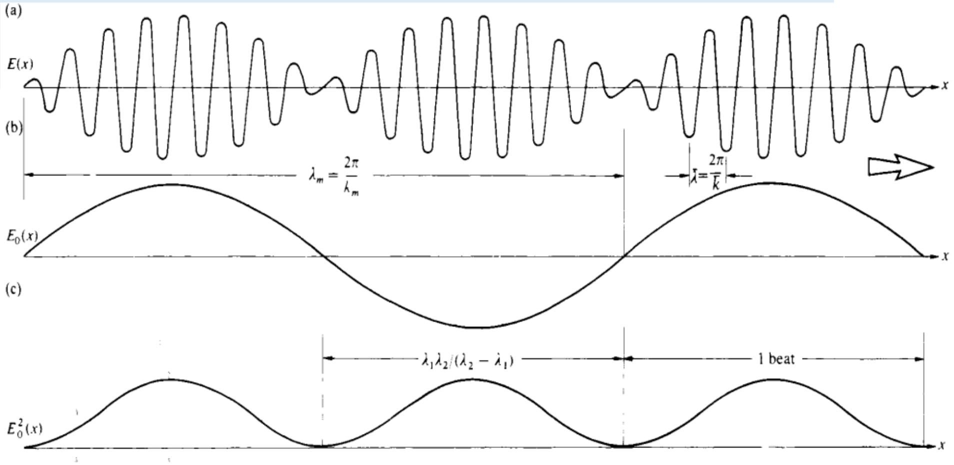
Group Velocity

When a number of different-frequency harmonic waves superimpose to form a composite disturbance, the resulting modulation envelope will travel at a speed different from that of the constituent waves.

$\psi(x,t)$

A

A^2

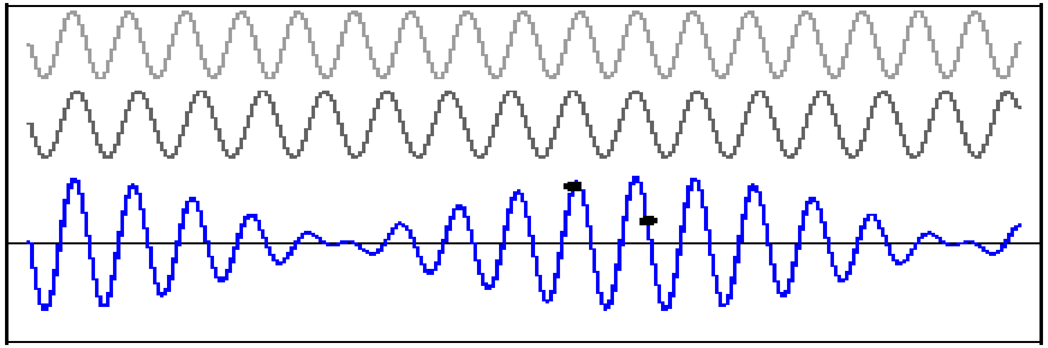


We need to recognize some constant feature in the shape of a pulse (ex. leading edge/crest). The rate at which that feature moves to be the velocity of the group of waves as a whole (v_g).

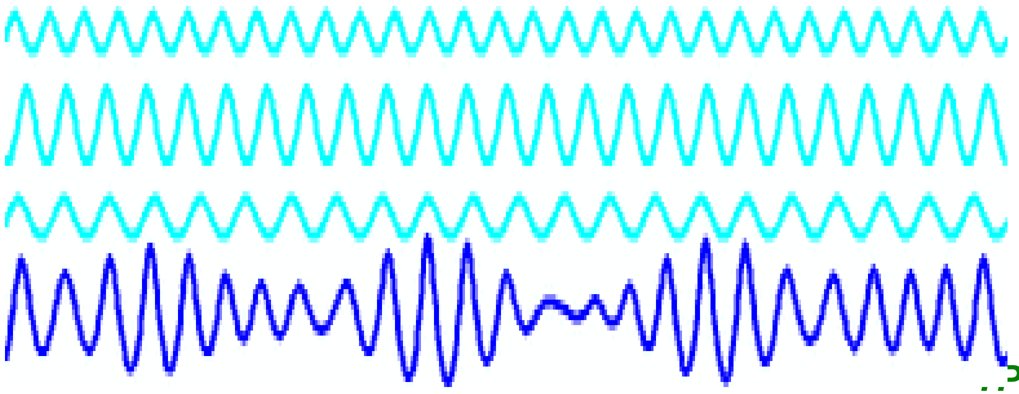
$$\psi(x,t) = A\sin\{kx - \omega t\}, \quad A = 2A_0\cos\{k_m x - \omega_m t\}$$

$$v_g = \frac{\omega_m}{k_m} = \frac{\frac{1}{2}(\omega_1 - \omega_2)}{\frac{1}{2}(k_1 - k_2)} = \frac{\Delta\omega}{\Delta k}$$

$$v_g = \left(\frac{d\omega}{dk}\right)_{\omega_{avg}}$$



when the frequency range $\Delta\omega$, centred at ω_{avg} then the ratio of the difference may be written as derivative.



The modulation advances at a rate dependent on the phase of the envelope $\{k_m x - \omega_m t\}$, i.e. at what distance and time the crest will repeat.

$$k_m x - \omega_m t = const$$

$$k_m dx - \omega_m dt = 0$$

$$\frac{x}{t} = \frac{\omega_m}{k_m} = \left(\frac{d\omega}{dk}\right)_{\omega_{avg}} = v_g$$

Waves and Vibrations (PH2001)

Group Velocity

$$v_g = \left(\frac{d\omega}{dk}\right)_{\omega_{avg}}$$

$$k(\omega) = \frac{\omega}{c} n(\omega)$$

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{1}{c}\left\{n(\omega) + \omega\frac{dn}{d\omega}\right\} \rightarrow$$

For free space, $\mathbf{n}(\omega) = 1$, $v_g = 1$

$$\frac{c}{v_g} = n_g = n(\omega) + \omega\frac{dn}{d\omega}$$

$\mathbf{n_g}$ is known as Group Index

$$\omega = \frac{2\pi c}{\lambda_0}, \qquad \lambda_0 \Rightarrow free\ space\ wavelength$$

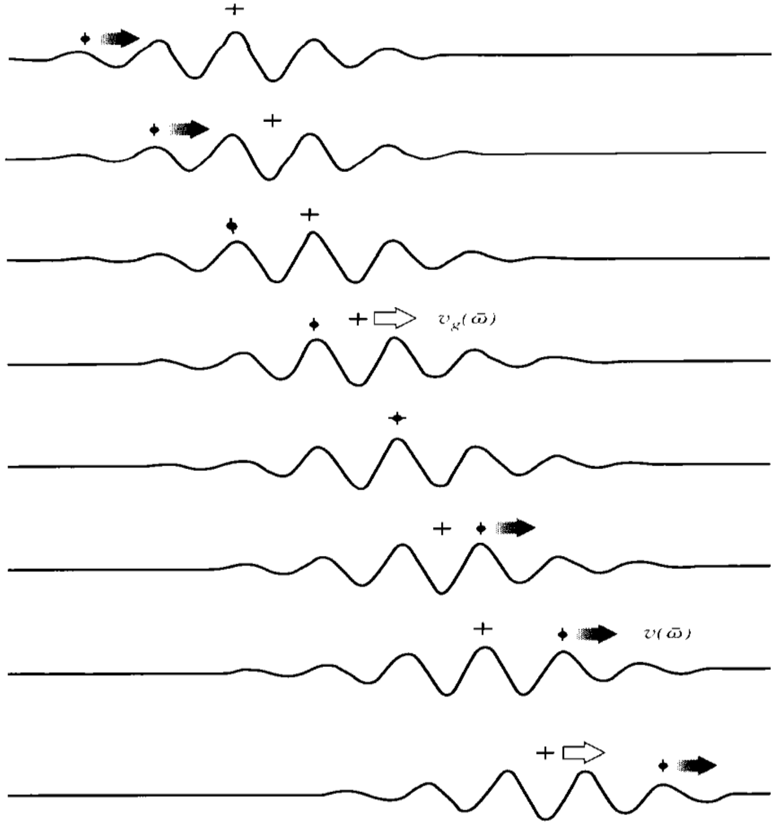
Now, $\frac{dn}{d\omega} = \frac{dn}{d\lambda_0}\frac{d\lambda_0}{d\omega} = -\frac{\lambda_0^2}{2\pi c}\frac{dn}{d\lambda_0}$

$$\frac{c}{v_g} = n(\lambda_0) + \frac{2\pi c}{\lambda_0}\left(-\frac{\lambda_0^2}{2\pi c}\frac{dn}{d\lambda_0}\right)$$

$$n_g = n(\lambda_0) - \lambda_0\frac{dn}{d\omega} \qquad n_g = n(\nu) + \nu\frac{dn}{d\nu}$$

Waves and Vibrations (PH2001)

Modulation, Wave packet



Modulation means to change something (amplitude/frequency/phase) about it in the way that can be decoded at a distant receiver. For example in amplitude modulation a series of dots and dashes (in Morse code) are sent where each pattern of dots and dashes represents a letter of alphabet.

Any real wave is finite in spatial extent. It turned on (or received) at some specific time and, presumably, shut off at some later time. A real wave is therefore actually a pulse, though it could be a rather long one. As we're about to learn any such pulse is identical to a superposition of numerous different-frequency sine waves (often called Fourier components), each with a specific amplitude and phase. Accordingly, envision not just two constituent wa\es as in Fig (7.16), but upwards of a thousand, all with different frequencies. If, as is certainly possible, the sinusoids cancel each other everywhere except over a region where they are in-phase, or nearly so, the resulting disturbance will resemble a localized pulse, often called a ***Wave packet*** to remind us that it's just that.