Ordered Basis

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Ordered Basis

If V is a finite-dimensional vector space, an ordered basis for V is a finite sequence of vectors which is L.I. and spans V.

Remark : Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V.
Let $\alpha \in V = \text{span } \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

$$\Longrightarrow \alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n - - - - (1)$$

The coordinate matrix of the vector α relative to the ordered basis B is

$$[\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$$

Claim : $[\alpha]_B$ is unique.
If not, there exist $y_j \in F$ such that

$$\alpha = y_1\alpha_1 + y_2\alpha_2 + \ldots + y_n\alpha_n - - - -(a)$$

From (a) and (1),

$$(x_1 - y_1)\alpha_1 + (x_2 - y_2)\alpha_2 + \ldots + (x_n - y_n)\alpha_n = 0$$

Since B is L.I., $\implies \quad x_1 - y_1 = x_2 - y_2 = \ldots = x_n - y_n = 0.$
$\implies x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$ and thus $[\alpha]_B$ is unique.

Example

Find the coordinate matrix of the vector $\alpha = (1, 2, 3)$ w.r.t. the ordered basis $B = \{\epsilon_1 = (1, 0, 0), \epsilon_2 = (0, 1, 0), \epsilon_3 = (0, 0, 1)\}$ and $B_1 = \{\alpha_1 = (1, 1, 1), \alpha_2 = (0, 1, 1), \alpha_3 = (0, 0, 1)\}$

Solution : Note that

$$\alpha = (1, 2, 3) = \epsilon_1 + 2\epsilon_2 + 3\epsilon_3, \quad \alpha = (1, 2, 3) = \alpha_1 + \alpha_2 + \alpha_3$$

$$[\alpha]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad [\alpha]_{B_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

What is the relation between $[\alpha]_B$ and $[\alpha]_{B_1}$?

Relation between $[\alpha]_B$ and $[\alpha]_{B_1}$?

$$\alpha_1 = \epsilon_1 + \epsilon_2 + \epsilon_3, \quad \alpha_2 = 0\epsilon_1 + \epsilon_2 + \epsilon_3, \qquad \alpha_3 = 0\epsilon_1 + 0\epsilon_2 + \epsilon_3$$

$$P_1 = [\alpha_1]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ P_2 = [\alpha_2]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ P_3 = [\alpha_3]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [P_1, P_2, P_3] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P[\alpha]_{B_1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [\alpha]_B$$

Verify that the matrix P is invertible and $[\alpha]_{B_1} = P^{-1}[\alpha]_B$?

Relation between $[\alpha]_B$ and $[\alpha]_{B_1}$?

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $B_1 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be two ordered bases of a finite-dimensional vector space V.Let $\alpha \in V$.

$$[\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}, \quad [\alpha]_{B_1} = \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix}$$

where

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \quad , \quad \alpha = \sum_{j=1}^{n} y_j \beta_j$$

contd..

Since $\beta_j \in V = \text{span } \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, there exists unique scalars P_{ij}, $1 \leq i \leq n$ such that

$$\beta_j = \sum_{i=1}^{n} P_{ij}\alpha_i \quad , 1 \leq j \leq n$$

$$\text{where } [\beta_j]_{B} = P_j = \begin{bmatrix} P_{1j} \\ P_{2j} \\ \ldots \\ P_{nj} \end{bmatrix}$$

contd.

$$\begin{aligned}\alpha &= \sum_{j=1}^{n} y_j \beta_j \\ &= \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{n} P_{ij} \alpha_i \right) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} y_j \right) \alpha_i\end{aligned}$$

We have, $\alpha = \sum_{i=1}^{n} x_i \alpha_i \Longrightarrow x_i = \sum_{j=1}^{n} P_{ij} y_j$, $1 \leq i \leq n$ (Thanks to unique coordinate matrix of α w.r.t. a basis B.)

contd.

$$x_i = \sum_{j=1}^{n} P_{ij} y_j \ , 1 \leq i \leq n$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

$$\implies X = PX' - - - - - (1)$$

where $X = [\alpha]_B$, $X' = [\alpha]_{B_1}$ and $P = [P_1, P_2, \dots, P_n]$.

contd.

Note that $X = PX' - -(1)$, $\alpha = \sum_{i=1}^{n} x_i\alpha_i$, and $\alpha = \sum_{j=1}^{n} y_j\beta_j$

Claim (1): $X = 0 \Longleftrightarrow X' = 0$

Proof : $X = 0 \Longleftrightarrow x_1 = x_2 = \ldots = x_n = 0$

$\Longleftrightarrow \alpha = 0, (B$ is a L.I. set)

$\Longleftrightarrow y_1 = y_2 = \ldots = y_n = 0, (B_1$ is a L.I. set).

$\Longleftrightarrow X' = 0$

Claim (2) : P is an invertible matrix.

Proof : $PX' = 0 \Longrightarrow X = 0 \Longrightarrow X' = 0(\textit{By Claim}\ (1))$

Hence the homogeneous system $PX' = 0$ has only trivial solution $X' = 0$ and thus P is invertible.

Theorem 7

Let V be a n-dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $B_1 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be two ordered bases of V. Then there is a unique, necessarily invertible, $n \times n$ matrix P with entries in F such that
(i) $[\alpha]_B = P[\alpha]_{B_1}$ **and (ii)** $[\alpha]_{B_1} = P^{-1}[\alpha]_B$ **for every $\alpha \in V$.**
The columns of P are given by $P_j = [\beta_j]_B$, $j = 1, 2, \ldots, n$.
Proof : (See the previous slides.)

Theorem 8 (Assignment)

Note : For a given ordered basis B and an invertible matrix P, it is possible to contruct another ordered basis B_1 of a finite-dimensional vector space V.

An example (Theorem 8)

Find an ordered basis for R^4. Let $B = \{\alpha_1 = (0, 1, 1, 1), \alpha_2 = (1, 0, 1, 1), \alpha_3 = (1, 1, 0, 1), \alpha_4 = (1, 1, 1, 0)\}$ be an ordered basis for R^4 and let P be an invertible matrix, where

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}$$

Solution

$$[\beta_1]_B = P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [\beta_2]_B = P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, [\beta_3]_B = P_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix},$$

$$[\beta_4]_B = P_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\beta_1 = 1\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\alpha_4 = (1, 1, 2, 2)$$

$$\beta_2 = 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 1\alpha_4 = (2, 2, 1, 1)$$

solution contd.

$$\beta_3 = 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 4\alpha_4 = (4, 5, 5, 1)$$

$$\beta_4 = 0\alpha_1 + 0\alpha_2 + 0\alpha_3 + 2\alpha_4 = (2, 2, 2, 0)$$

Row rank / Column rank

Let $A \in F^{m \times n}$, let $\{R_1, R_2, \ldots, R_m\}$ be the rows of A and let $\{C_1, C_2, \ldots, C_n\}$ be columns of A.
Row space of $A = \text{span } \{R_1, R_2, \ldots, R_m\}$
Column space of $A = \text{span } \{C_1, C_2, \ldots, C_n\}$
Row rank of $A =$ dim (row space of A)
Column rank of $A =$ dim (column space of A)

Row-equivalent matrices admit same row space

Let $A, B \in F^{m\times n}$ be two row-equivalent matrices. Then there exists an invertible $n \times n$ matrix P such that $B = PA$. So every row of B is a linear combination rows of A.

$\implies$ row space of $B \subseteq$ row space of $A - - - (1)$

$$A = P^{-1}B$$

So every row of A is a linear combination of rows of B.

$\implies$ row space of $A \subseteq$ row space of $B - - - (2)$

From (1) and (2), row space of $A =$ row space of B

Basis of a row-reduced echelon matrix

Let $R \in F^{m \times n}$ be row-reduced echelon matrix. Then non-zero rows of R forms a basis of row space of R.(Assignment)

$$\text{Row rank of } \; R = \text{No. of non-zero rows of } R$$