# Engineering Electromagnetics

## Lecture 20

16/10/2023

*by*

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# Capacitors


By applying external energy→ charge transfer→ potential diff

Since **E** is proportional to $Q$, so also is $V$. The constant of proportionality is called the **capacitance** of the arrangement:

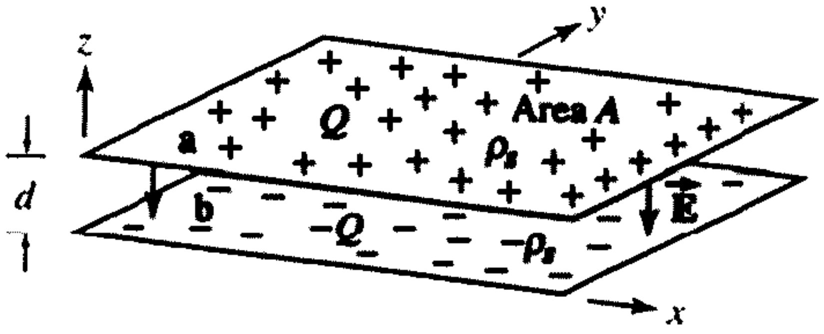
$$\boxed{C \equiv \frac{Q}{V}}.$$

Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation of the two conductors. In SI units, $C$ is measured in **farads** (F); a farad is a coulomb-per-volt. Actually, this turns out to be inconveniently large; more practical units are the microfarad ($10^{-6}$ F) and the picofarad ($10^{-12}$ F).

Griffith book

# Problem-1

Two parallel conducting plates, each of area $A$, and separated by a distance $d$, as shown in Figure 3.36, form a parallel-plate capacitor. The charge on the top is $+Q$ and that on the other plate is $-Q$. What is its capacitance? Also express the energy stored in the medium in terms of the capacitance of the system.



# Solution-1

Let us assume that the separation between the plates is very small compared to their other dimensions. Therefore, we can neglect the edge effects (fringing) and assume that the charge is uniformly distributed over the inner surface of each plate. The electric field intensity between the conductors is

$$\vec{\mathbf{E}} = -\frac{\rho_s}{\epsilon}\vec{\mathbf{a}}_z \quad \text{and} \quad \rho_s = \frac{Q}{A}$$

where $Q$ is the charge on top plate $a$ at $z = d$, $A$ is the surface area of each plate, and $\epsilon$ is the permittivity of the medium. Note that the charge on plate $b$ at $z = 0$ is $-Q$.

The potential of plate $a$ with respect to plate $b$ is

$$V_{ab} = -\int_b^a \vec{\mathbf{E}} \cdot \vec{d\ell} = \frac{\rho_s}{\epsilon}\int_0^d dz = \frac{\rho_s d}{\epsilon} = \frac{Qd}{\epsilon A}$$

Thus, the capacitance of the parallel-plate capacitor is

$$C = \frac{Q}{V_{ab}} = \frac{\epsilon A}{d} \tag{3.75a}$$

The energy stored in the system is

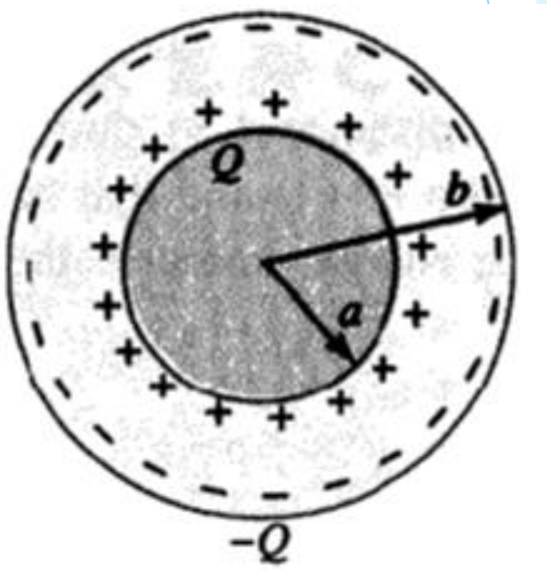
$$W = \frac{1}{2}\int_v \epsilon E^2\, dv = \frac{1}{2}\frac{Ad}{\epsilon}\rho_s^2 = \frac{1}{2}\frac{d}{\epsilon A}Q^2$$

$$= \frac{1}{2C}Q^2 = \frac{1}{2}CV_{ab}^2$$

These are the basic circuit equations for the energy stored in a capacitor. • • •

# Problem-2

A spherical capacitor is formed by two concentric metallic spheres of radii $a$ and $b$, as shown in Figure 3.37 (see below). The charge on the inner sphere is $+Q$ and that on the outer sphere is $-Q$. Determine the capacitance of the system. What is the capacitance of an isolated sphere? Assuming the earth to be an isolated sphere of radius $6.5 \times 10^6$ meters, calculate its capacitance. Deduce an approximate expression for the capacitance when the separation between the spheres is very small as compared to their radii.



**Solution** For a uniform charge distribution over the spheres, the electric field intensity, from Gauss's law, within the spheres is

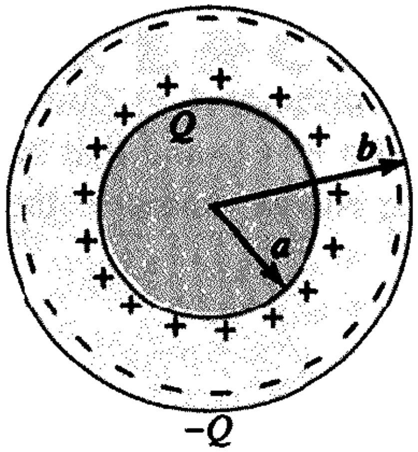



Figure 3.37 A spherical capacitor

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon r^2}\vec{\mathbf{a}}_r$$

The potential of the inner sphere with respect to the outer sphere is

$$V_{ab} = -\frac{Q}{4\pi\epsilon}\int_b^a \frac{1}{r^2}dr = \frac{Q}{4\pi\epsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Hence, the capacitance of the system is

$$C = \frac{Q}{V_{ab}} = \frac{4\pi\epsilon ab}{b-a} \tag{3.75b}$$

By setting $b \to \infty$, we obtain the capacitance of an isolated sphere as $C = 4\pi\epsilon a$. Substituting the values for earth with $\epsilon = \epsilon_0$, we have

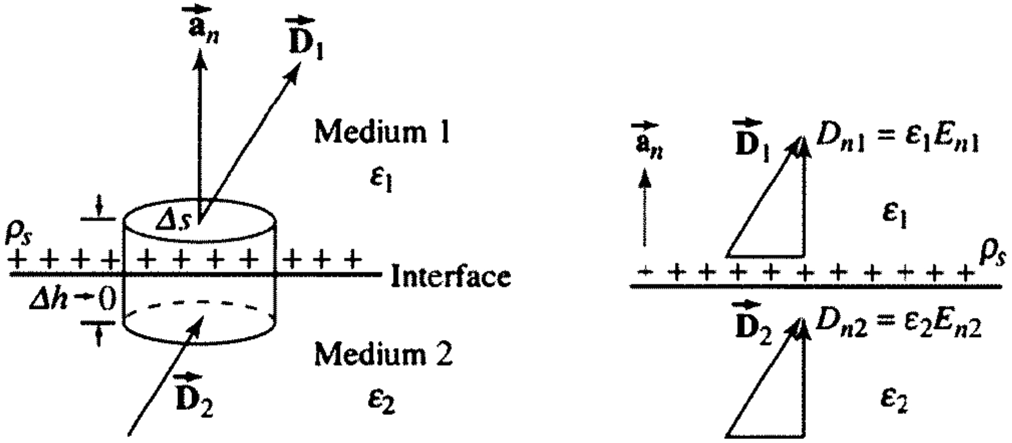
$$C = \frac{6.5 \times 10^6}{9 \times 10^9} = 0.722 \times 10^{-3} \quad \text{or} \quad 722\,\mu\text{F}$$

If the separation between the two spheres is very small; i.e., $d = b - a$ and $d \ll a$, we can approximate $ab \approx a^2$, and the capacitance of the system becomes

$$C = \frac{4\pi\epsilon a^2}{b-a} = \frac{\epsilon A}{d} \tag{3.75c}$$

where $A = 4\pi a^2$ is the surface area of the inner sphere. • • •

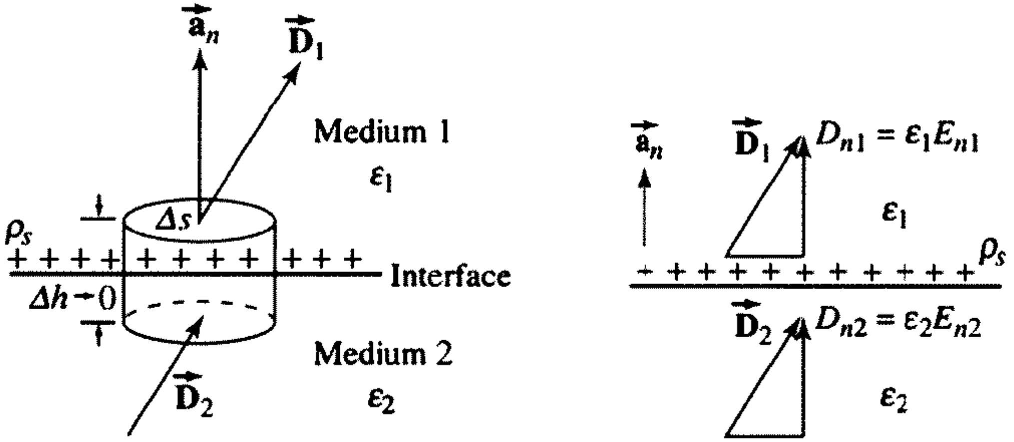
# Boundary condition




a) Boundary conditions　　b) Normal components

Let us apply Gauss's law to find the boundary condition pertaining to the normal component of the electric flux density at an interface, as exhibited in Figure 3.33a. We have constructed a Gaussian surface in the form of a pillbox, with half in medium 1, and the other half in medium 2. Each flat surface is so small that the electric flux density in each medium is essentially constant over the surface in that medium. We also assume that the area of the curved surface is negligibly small as the height of the pillbox $\Delta h$ shrinks to zero. We shall also assume that there exists a free surface charge density $\rho_s$ at the interface.

# Boundary condition




a) Boundary conditions

b) Normal components

If the surface area is $\Delta s$, the total charge enclosed by the pillbox is $\rho_s\ \Delta s$. Applying Gauss's law, we get

$$\vec{\mathbf{D}}_1 \cdot \vec{\mathbf{a}}_n\, \Delta s - \vec{\mathbf{D}}_2 \cdot \vec{\mathbf{a}}_n\, \Delta s = \rho_s\, \Delta s$$

or

$$\vec{\mathbf{a}}_n \cdot (\vec{\mathbf{D}}_{.} - \vec{\mathbf{D}}_2) = \rho_s$$

or

$$D_{n1} - D_{n2} = \rho_s$$

where $\vec{\mathbf{a}}_n$ is the unit vector normal to the interface pointing from medium 2 to medium 1. $D_{n1}$ and $D_{n2}$ are the components of the $\vec{\mathbf{D}}$ field normal to the interface in medium 1 and medium 2, respectively, as shown in Figure 3.33b. Equation (3.70) states that *the normal components of the electric flux density are discontinuous if a free surface charge density exists at the interface.*

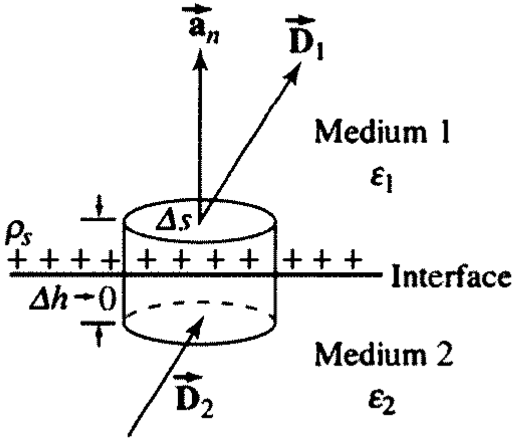
Since $\mathbf{D} = \epsilon \mathbf{E}$, we can also write (3.70) in terms of the normal components of the $\vec{\mathbf{E}}$ field. That is,

$$\vec{\mathbf{a}}_n \cdot (\epsilon_1 \vec{\mathbf{E}}_1 - \epsilon_2 \vec{\mathbf{E}}_2) = \rho_s \tag{3.70c}$$

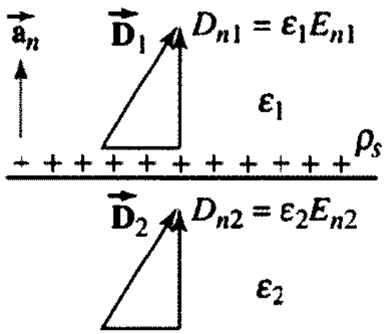
or

$$\epsilon_1 E_{n1} - \epsilon_2 E_{n2} = \rho_s \tag{3.70d}$$

# Boundary condition



a) Boundary conditions



b) Normal components

Since $\mathbf{D} = \epsilon \mathbf{E}$, we can also write (3.70) in terms of the normal components of the $\vec{\mathbf{E}}$ field. That is,

$$\vec{\mathbf{a}}_n \cdot (\epsilon_1 \vec{\mathbf{E}}_1 - \epsilon_2 \vec{\mathbf{E}}_2) = \rho_s$$

or

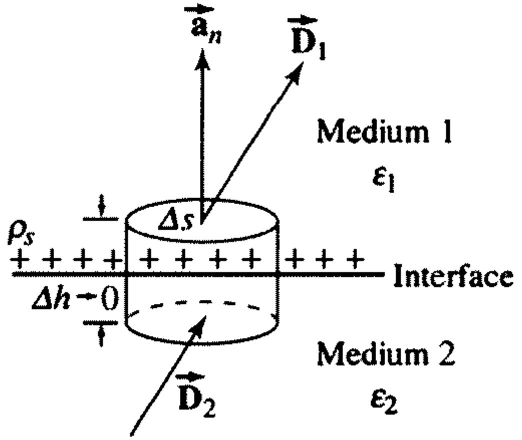
$$\epsilon_1 E_{n1} - \epsilon_2 E_{n2} = \rho_s$$

When the interface is between two different dielectrics, we do not expect any free surface charge density at the boundary unless the charge is deliberately placed there. Ruling out the possibility of such an intentional placement of a charge, we find that *the normal components of the electric flux density are continuous across a dielectric boundary*; i.e.,

$$D_{n1} = D_{n2} \tag{3.71a}$$

or

$$\epsilon_1 E_{n1} = \epsilon_2 E_{n2} \tag{3.71b}$$




a) Boundary conditions

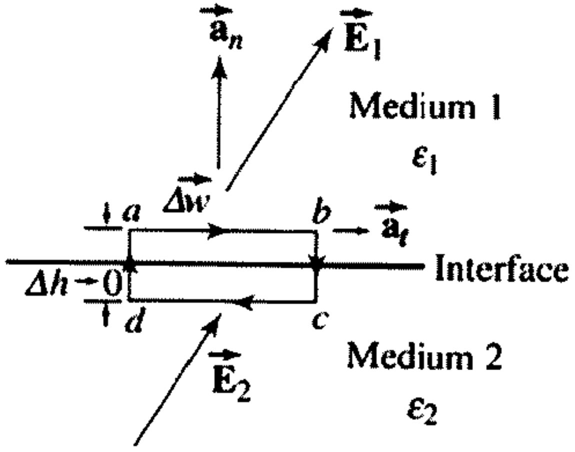
If medium 2 is a conductor, the electric flux density $\vec{\mathbf{D}}_2$ must be zero under static conditions. For the normal component of the electric flux density $\vec{\mathbf{D}}_1$ to exist in medium 1, there must be a free surface charge density on the conductor's surface in harmony with (3.70). That is,

$$\vec{\mathbf{a}}_n \cdot \vec{\mathbf{D}}_1 = D_{n1} = \rho_s \tag{3.72a}$$

$$\epsilon_1 E_{n1} = \rho_s \tag{3.72b}$$

*The normal component of the electric flux density in a dielectric medium just above the surface of a conductor is equal to the surface charge density on the conductor.*

# Tangential component of E



a) Boundary conditions

$$\vec{E}_1 \cdot \vec{\Delta w} - \vec{E}_2 \cdot \vec{\Delta w} = 0$$

or

$$(\vec{E}_1 - \vec{E}_2) \cdot \vec{\Delta w} = 0$$

# Boundary value problems

- From charge distribution → electric field
- In practice: 1st E needs to be calculated, no information about charge distribution, may be only at the boundary

$$\nabla^2 V = -\rho_v/\epsilon \quad \textit{Poisson's equation}$$

There are some problems in electrostatics that involve charge distributions on the surface of conductors. In these cases, the free volume charge density is zero in the region of interest. Thus, in the region where $\rho_v$ vanishes, (3.83) reduces to

$$\nabla^2 V = 0$$

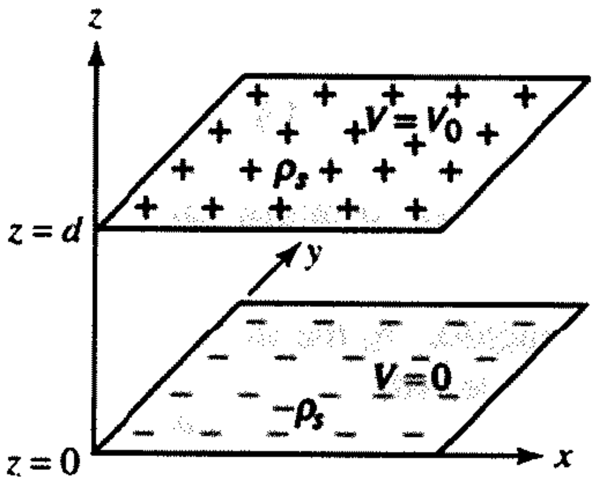
and is called *Laplace's equation*.

**To calculate E**

In a charge-free region, we will seek a potential function $V$ that satisfies Laplace's equation subjected to the boundary conditions. Once the potential function in the region is known, the electric field intensity $\vec{\mathbf{E}}$ can be determined as $\vec{\mathbf{E}} = -\nabla V$.

# Laplace's equation

The two metal plates of Figure 3.40 having an area $A$ and a separation $d$ form a parallel-plate capacitor. The upper plate is held at a potential of $V_0$, and the lower plate is grounded. Determine (a) the potential distribution, (b) the electric field intensity, (c) the charge distribution on each plate, and (d) the capacitance of the parallel-plate capacitor.



Since the two metal plates (conductors) form equipotential surfaces in the $xy$ plane at $z = 0$ and $z = d$, we expect that the potential $V$ must be a function of $z$ only. For the charge-free region between the plates, Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial z^2} = 0$$

with a solution

$$V = az + b$$

where $a$ and $b$ are constants to be evaluated from the knowledge of boundary conditions.

When $z = 0$, $V = 0 \Rightarrow b = 0$. The potential distribution within the plates now becomes

$$V = az$$

However, when $z = d$, $V = V_0$ suggests that $a = V_0/d$. Thus, the potential varies linearly in a parallel-plate capacitor as

$$V = \frac{z}{d} V_0$$

We can now compute the electric field intensity as

$$\vec{\mathbf{E}} = -\nabla V = -\vec{\mathbf{a}}_z \frac{\partial V}{\partial z} = -\frac{V_0}{d} \vec{\mathbf{a}}_z$$

and the electric flux density is

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}} = -\frac{\epsilon V_0}{d} \vec{\mathbf{a}}_z$$

Since the normal component of the $\vec{\mathbf{D}}$ field must be equal to the surface charge density on a conductor, the surface charge density on the lower plate is

$$\rho_s|_{z=0} = -\frac{\epsilon V_0}{d}$$

and that on the upper plate is

$$\rho_s|_{z=d} = \frac{\epsilon V_0}{d}$$

The total charge on the upper plate is

$$Q = \frac{\epsilon V_0 A}{d}$$

Thus, the capacitance of the parallel plate capacitor is

$$C = \frac{Q}{V_0} = \frac{\epsilon A}{d}$$

• • •

# Thank You