Vector Spaces

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Mathematical structures in linear algebra

(1) **Field** (See Chapter 1)

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(2) **Vector Space**

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(3) **...........**

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$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma, \ \text{ for all} \alpha, \beta, \gamma \in V$$

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Show that $\langle F^n, F, +, . \rangle$ is a vector space.

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Reason : F is a field and $x_i + y_i = y_i + x_i$

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Note : (i) R^n is called the Euclidean vector space

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Note that $F^{n\times n}$ is not a field

Example 3 : The set of all real valued continuous functions defined on $[0, 1]$

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Example 4 : The space of polynomial functions over a field

Assignment

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Let $c_1 = c_2 = 1, \alpha = (1,1)$

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$$0 = c0 + (c0 + -(c0)), \quad (\textit{Associative})$$

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Qn. Show that $0\alpha = 0$ for all $\alpha \in V$, where 0 is the additive identity in the field F and 0 is the zero vector in the vector space V

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$\Longrightarrow$ additive inverse of α, $\quad -\alpha = (-1)\alpha$

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Show that $(x, y, z) \in \mathbb{R}^3$ is a linear combination of vectors $\alpha = (1, 1, 1)$, $\beta = (0, 1, 1)$ and $\gamma = (0, 0, 1)$.

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$\Longrightarrow a + b = 2$ **and** $a + b = 3$**, lead us to a contradiction.**

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By Theorem 12, A is invertible ($A \sim I$).

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By Theorem 12, A is invertible ($A \sim I$). By Theorem 13, the system $AX = Y$ has a solution X for all Y. Hence for every $Y^t = (x, y, z) \in \mathbb{R}^3$, there exists $X^t = (a, b, c)$ such that

$$a(1, 0, -1) + b(0, 1, 1) + c(1, 1, 1) = (x, y, z)$$

Matrix multiplication and linear combination

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(3) Every row of AB is a linear combination of rows of B