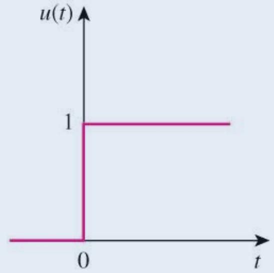
Electrical Circuits for Engineers (EC1000)

Lecture 06 (b)
First-Order Circuits
R-C & R-L with Unit Step Function
(Chapter 7)

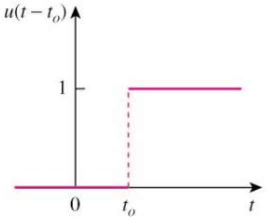
7.3 Unit-Step Function (1)

The **unit step function** $u(t)$ is 0 for negative values of t and 1 for positive values of t.

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

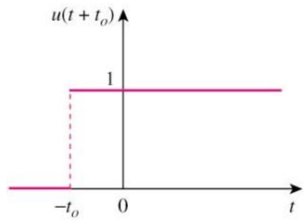


$$u(t - t_o) = \begin{cases} 0, & t < t_o \\ 1, & t > t_o \end{cases}$$



(a)

$$u(t + t_o) = \begin{cases} 0, & t < -t_o \\ 1, & t > -t_o \end{cases}$$

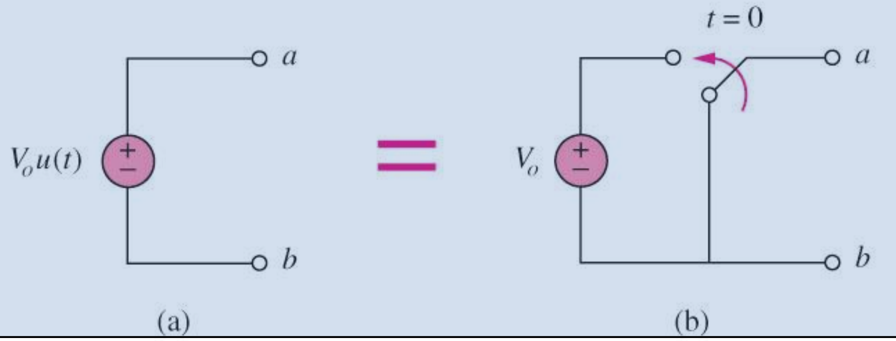


(b)

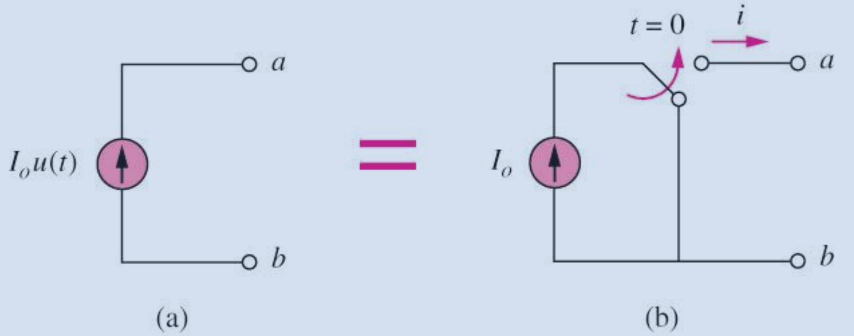
7.3 Unit-Step Function (2)

Represent an abrupt change for:

1. voltage source.

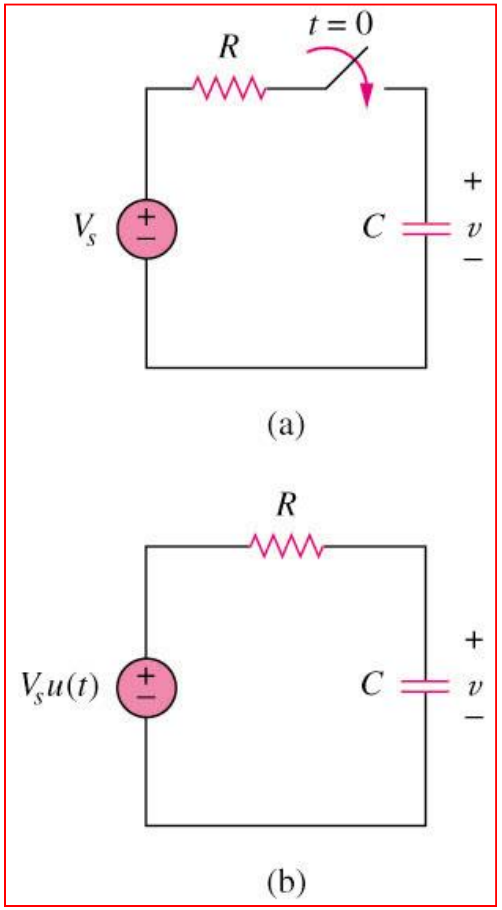


2. for current source:



7.4 The Step-Response of a RC Circuit (1)

The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- **Initial condition**:

  $v(0-) = v(0+) = V_0$

- Applying KCL,

$$c\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or

$$\frac{dv}{dt} = -\frac{v - V_s}{RC} u(t)$$

- Where u(t) is the unit-step function

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

Integrating both sides and introducing the initial conditions

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

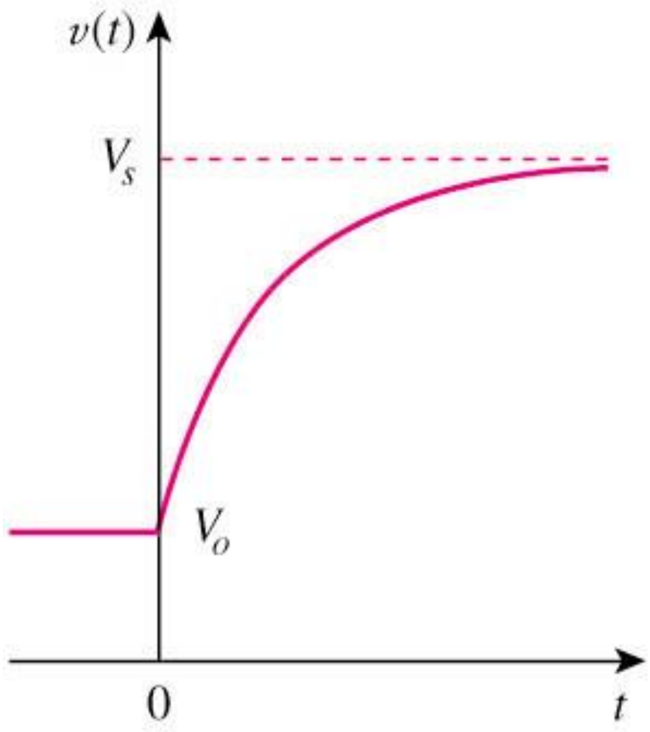
$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

7.4 The Step-Response of a RC Circuit (2)

Integrating both sides and considering the initial conditions, the solution of the equation is:



$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t > 0 \end{cases}$$

Final value at t -> ∞

Initial value at t = 0

Source-free Response

Complete Response = Natural response (stored energy) + Forced Response (independent source)

$= V_0 e^{-t/\tau} + V_s(1 - e^{-t/\tau})$

$$\text{Complete response} = \underbrace{\text{natural response}}_{\text{stored energy}} + \underbrace{\text{forced response}}_{\text{independent source}}$$

$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}$$

Three steps to find out the step response of an RC circuit:

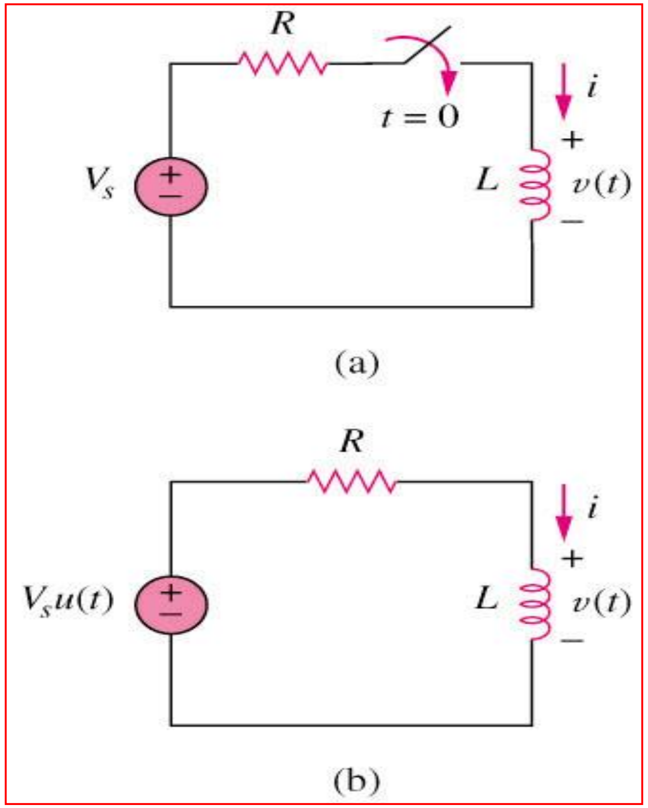
1. The initial capacitor voltage v(0).
2. The final capacitor voltage v(∞) — DC voltage across C.
3. The time constant τ.

$$v(t) = v(\infty) + [v(0+) - v(\infty)]e^{-t/\tau}$$

Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

7.5 The Step-response of a RL Circuit (1)

The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- **Initial current**
  $i(0-) = i(0+) = I_o$
- **Final inductor current**
  $i(\infty) = Vs/R$
- Time constant $\tau = L/R$

$$i(t) = \frac{V_s}{R} + (I_o - \frac{V_s}{R})e^{-\frac{t}{\tau}}u(t)$$

7.5 The Step-Response of a RL Circuit (2)

Three steps to find out the step response of an RL circuit:

1. The initial inductor current i(0) at t = 0+.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

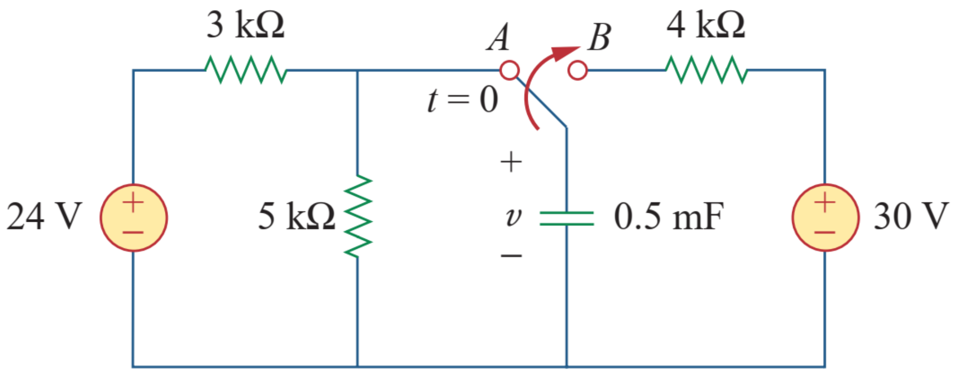
$$i(t) = i(\infty) + [i(0+) - i(\infty)]e^{-t/\tau}$$

Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

Example Problem

Problem 1

The switch in Fig. has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.



For $t < 0$,

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$

For $t > 0$,

$$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V} \end{aligned}$$

At $t = 1$,

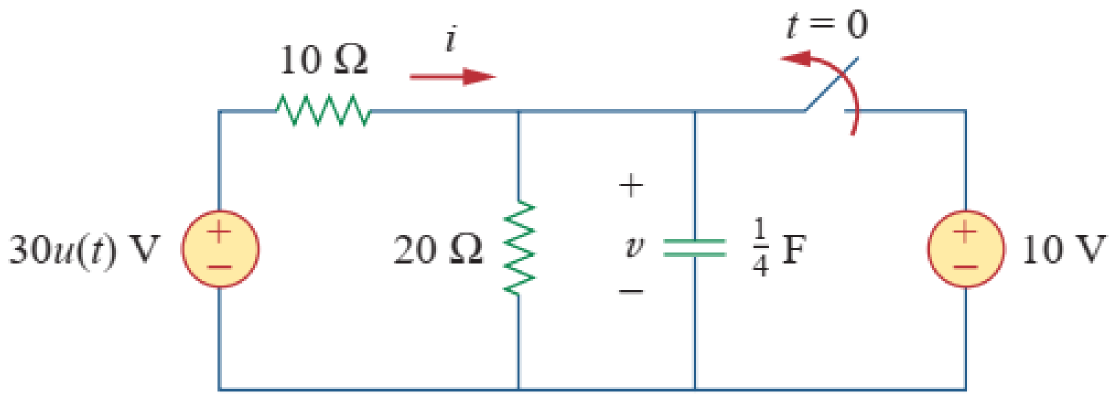
$$v(1) = 30 - 15e^{-0.5} = 20.9 \text{ V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97 \text{ V}$$

Problem 2

, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.



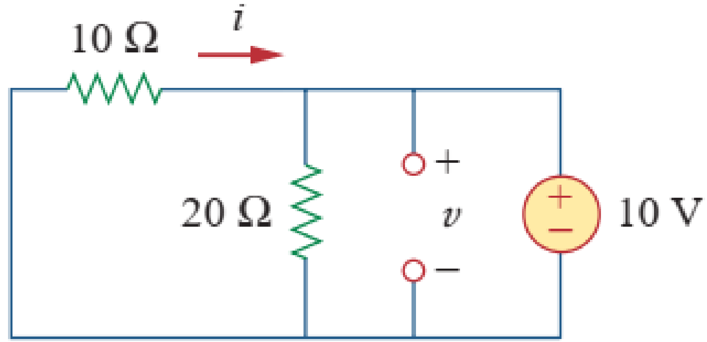
By definition of the unit step function,

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$, the switch is closed and $30u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit and should be regarded as

for $t < 0$.

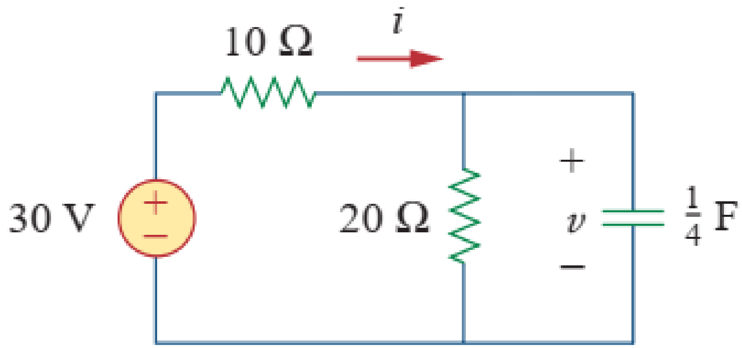
$$v = 10\ \text{V}, \qquad i = -\frac{v}{10} = -1\ \text{A}$$



Since the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = 10\ \text{V}$$

For $t > 0$,



We obtain $v(\infty)$ by using voltage division,

$$v(\infty) = \frac{20}{20 + 10}(30) = 20\ \text{V}$$

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \parallel 20 = \frac{10 \times 20}{30} = \frac{20}{3}\ \Omega \qquad \tau = R_{\text{Th}} C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3}\ \text{s}$$

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t})\text{ V}$$

$$i = \frac{v}{20} + C\frac{dv}{dt}$$
$$= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})\text{ A}$$

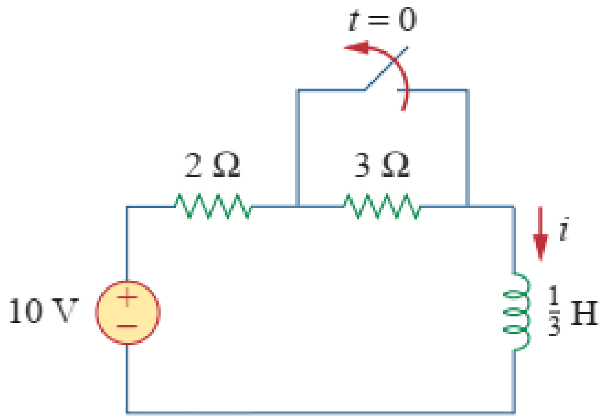
Answer

$$v = \begin{cases} 10\text{ V}, & t < 0 \\ (20 - 10e^{-0.6t})\text{ V}, & t \geq 0 \end{cases}$$

$$i = \begin{cases} -1\text{ A}, & t < 0 \\ (1 + e^{-0.6t})\text{ A}, & t > 0 \end{cases}$$

Find $i(t)$ in the circuit of Fig. or $t > 0$. Assume that the switch has been closed for a long time.

Problem 3



When $t < 0$, $\quad i(0^-) = \dfrac{10}{2} = 5 \text{ A}$

Since the inductor current cannot change instantaneously,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, the switch is open. The 2-Ω and 3-Ω resistors are in series, so that

$$i(\infty) = \frac{10}{2 + 3} = 2 \text{ A}$$

The Thevenin resistance across the inductor terminals is

$$R_{\text{Th}} = 2 + 3 = 5 \ \Omega$$

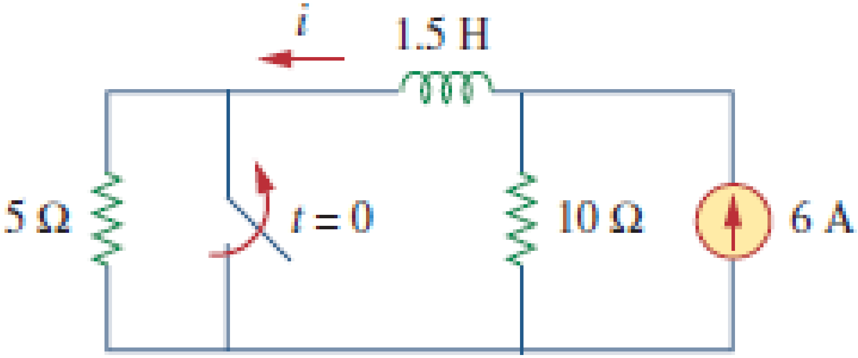
For the time constant,

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 2 + (5 - 2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0 \end{aligned}$$

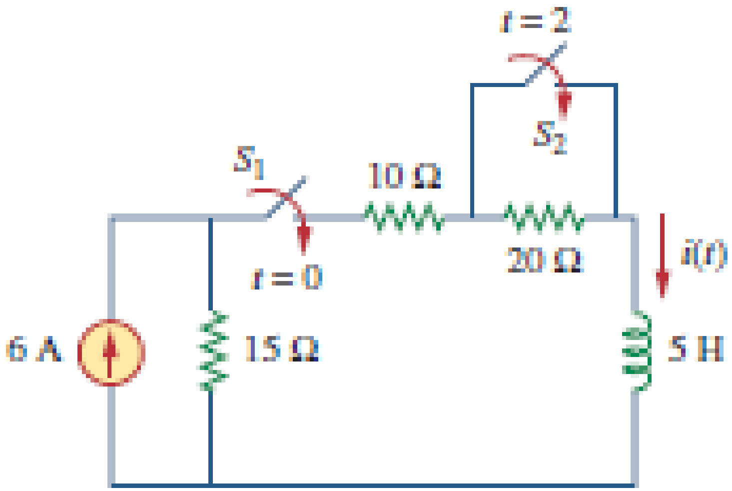
Practice Problem

1. The switch in figure has been closed for a long time. It opens at t=0. Find i(t) for t >0



Answer: $(4 + 2e^{-10t})$ A for all $t > 0$

2. Switch S_1 in figure is closed at t=0, and switch S_2 is closed at t=2s. Find i(t) for all t. Find i(1) and i(3).



Answer:

$$i(t) = \begin{cases} 0, & t < 0 \\ 2(1 - e^{-9t}), & 0 < t < 2 \\ 3.6 - 1.6e^{-5(t-2)}, & t > 2 \end{cases}$$

$i(1) = 1.9997$ A, $i(3) = 3.589$ A.