Problem: 1

① Let $\mathbb{F} = \mathbb{C}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}\right\} \text{ — } (S_1)$$

$$\left.\begin{aligned} 3x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}\right\} \text{ — } (S_2)$$

② Test the following systems of equations as in exercise ①.

$$\left.\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= 0 \end{aligned}\right\} \text{ — } (S_1)$$

$$\left.\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}\right\} \text{ — } (S_2)$$

③ Test the following systems of equations as in exercise ②.

$$\left.\begin{aligned} 2x_1 + (-1+i)x_2 + x_4 &= 0 \\ 3x_2 - 2i x_3 + 5x_4 &= 0 \end{aligned}\right\} \text{ --- } (S_1)$$

$$\left.\begin{aligned} (1+i/2)x_1 + 8x_2 - i x_3 - x_4 &= 0 \\ \tfrac{2}{3}x_1 - \tfrac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}\right\} \text{ --- } (S_2)$$

(4) Let $\mathbb{F} = \{0, 1\}$ be a set. Prove that the set $\mathbb{F}$ is a field with the following addition and multiplication.

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

(5) Find all solutions to the system of equations over $\mathbb{C}$.

$$(1-i)x_1 - i x_2 = 0$$
$$2x_1 + (1-i)x_2 = 0.$$

(6) If $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}$, find all solutions of $AX = 0$ by row-reducing A.

(7) If $A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$, find all solutions of $AX = 2X$ and all solutions of $AX = 3X$.

(8) Find the row-reduced echelon matrix of

$$A = \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix}$$

(9) Prove that the following two matrices are not row-equivalent:

$$\begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix}.$$

(10) Find all solutions of the following system of equations by row-reducing the coefficient matrix:

$$\frac{1}{3}x_1 + 2x_2 - 6x_3 = 0$$

$$-4x_1 + 5x_3 = 0$$

$$-3x_1 + 6x_2 - 13x_3 = 0$$

$$-\frac{7}{3}x_1 + 2x_2 - \frac{8}{3}x_3 = 0.$$

(11) Find a row-reduced echelon matrix which is row-equivalent to

$$A = \begin{pmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{pmatrix}.$$

(12) Consider the system of equations

$$x_1 - x_2 + 2x_3 = 1$$
$$2x_1 + 2x_3 = 1$$
$$x_1 - 3x_2 + 4x_3 = 2.$$

Does this system have a solution? If so, describe explicitly all solutions.

(13) Show that the system

$$x_1 - 2x_2 + x_3 + 2x_4 = 1$$
$$x_1 + x_2 - x_3 + x_4 = 2$$
$$x_1 + 7x_2 - 5x_3 - x_4 = 3$$

has no solution.

(14) Find all solutions of

$$2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 = -2$$
$$x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 = -2$$
$$2x_1 - 4x_3 + 2x_4 + x_5 = 3$$
$$x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 = -7$$

(15) Let

$$A = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}.$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?

(16) Solve

$$x_1 + x_2 + 3x_3 - x_4 = 0$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1 - 2x_2 + x_3 - x_4 = 1$$
$$4x_1 + x_2 + 8x_3 - x_4 = 0$$

(17) Solve

$$x_1 + 2x_2 + 2x_4 = 6$$
$$3x_1 + 5x_2 - x_3 + 6x_4 = 17$$
$$2x_1 + 4x_2 + x_3 + 2x_4 = 12$$
$$2x_1 - 7x_3 + 11x_4 = 7.$$

(18) Solve

$$2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1$$
$$x_1 - x_2 + x_3 + 2x_4 - x_5 = 2$$
$$4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6.$$

(19) Solve

$$3x_1 - x_2 + x_3 - x_4 + 2x_5 = 5$$
$$x_1 - x_2 - x_3 - 2x_4 - x_5 = 2$$
$$5x_1 - 2x_2 + x_3 - 3x_4 + 3x_5 = 10$$
$$2x_1 - x_2 - 2x_4 + x_5 = 5$$

(20) Solve

$$3x_1 - x_2 + 2x_3 + 4x_4 + x_5 = 2$$
$$x_1 - x_2 + 2x_3 + 3x_4 + x_5 = -1$$
$$2x_1 - 3x_2 + 6x_3 + 9x_4 + 4x_5 = -5$$
$$7x_1 - 2x_2 + 4x_3 + 8x_4 + x_5 = 6$$

(21) Solve

$$2x_1 + 3x_3 - 4x_5 = 5$$
$$3x_1 - 4x_2 + 8x_3 + 3x_4 = 8$$
$$x_1 - x_2 + 2x_3 + x_4 - x_5 = 2$$
$$-2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 = -8$$

(22) Solve

$$x_1 - x_2 + 2x_3 + 3x_4 = -7$$
$$2x_1 - x_2 + 6x_3 + 6x_4 = -2$$
$$-2x_1 + x_2 - 4x_3 - 3x_4 = 0$$
$$3x_1 - 2x_2 + 9x_3 + 10x_4 = -5.$$