Inner product spaces

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Inner product spaces

Objective. Introducing (1) length and (2) angle on vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

Definition. Let F be the field of real numbers or the field of complex numbers, and V a vector space over F. An **inner product** on V is a function $\langle , \rangle : V \times V \longrightarrow F$ such a way that for all $\alpha, \beta, \gamma \in V$ and all $c \in F$,

(1) $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$;

(2) $\langle c\alpha, \beta \rangle = c\langle \alpha, \beta \rangle$;

(3) $\langle \beta, \alpha \rangle = \overline{\langle \alpha, \beta \rangle}$; (Complex conjugate)

(4) $\langle \alpha, \alpha \rangle > 0$ if $\alpha \neq \mathbf{0}$.

Example 1. Standard inner product

Let $V = F^n = \{(x_1, x_2, \dots, x_n) : x_i \in F\}$. Let $\alpha = (x_1, x_2, \dots, x_n)$, $\beta = (y_1, y_2, \dots, y_n)$ and $\gamma = (z_1, z_2, \dots, z_n)$ be vectors in V.

$$\langle \alpha, \beta \rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n} = \sum_{j=1}^{n} x_j\overline{y_j}$$

Show that $\langle , \rangle$ is an inner product.

(1) $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$;

$\alpha + \beta = (x_1 + y_1, \dots, x_n + y_n)$.

$$\langle \alpha + \beta, \gamma \rangle = (x_1 + y_1)\overline{z_1} + (x_2 + y_2)\overline{z_2} + \cdots + (x_n + y_n)\overline{z_n} = \sum_{j=1}^{n}(x_j + y_j)\overline{z_j}$$

$$\langle \alpha + \beta, \gamma \rangle = \sum_{j=1}^{n} x_j\overline{z_j} + \sum_{j=1}^{n} y_j\overline{z_j} = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle.$$

(2) $\langle c\alpha, \beta\rangle = c\langle\alpha, \beta\rangle$;

$c\alpha = (cx_1, cx_2, \ldots, cx_n)$

$$\langle c\alpha, \beta\rangle = \sum_{j=1}^{n} cx_j\overline{y_j} = c\sum_{j=1}^{n} x_j\overline{y_j} = c\langle\alpha, \beta\rangle$$

(3) $\langle\beta, \alpha\rangle = \overline{\langle\alpha, \beta\rangle}$;

$$\overline{\langle\alpha, \beta\rangle} = \overline{\sum_{j=1}^{n} x_j\overline{y_j}} = \sum_{j=1}^{n} y_j\overline{x_j} = \langle\beta, \alpha\rangle$$

(4) $\langle\alpha, \alpha\rangle > 0$ if $\alpha \neq \mathbf{0}$

$$\langle\alpha, \alpha\rangle = \sum_{j=1}^{n} x_j\overline{x_j} = \sum_{j=1}^{n} |x_j|^2 > 0, \quad \text{provided } \alpha \neq \mathbf{0}$$

Example 2. Let $V = \mathbb{R}^2$. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ be two vectors in $\mathbb{R}^2$. Define

$$\langle \alpha, \beta \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2.$$

Prove that $\langle , \rangle$ is an inner product.

Definition. Let A be an $m \times n$ matrix over the field F. We define the **conjugate transpose** or **adjoint** of A to be the $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}}$ for all i, j.

Example 3. Let $V = F^{n \times n} = M_{n \times n}(F)$. Let A, B be two matrices in $F^{n \times n}$. Define

$$\langle A, B \rangle = \mathrm{tr}\,(B^* A).$$

Prove that $\langle , \rangle$ is an inner product on $F^{n \times n}$.

Example 4. Let $V = C[0,1]$ be the vector space of all continuous functions from the interval $[0,1]$ to $\mathbb{R}$. For $f, g \in C[0,1]$, define

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Prove that $\langle , \rangle$ is an inner product on $C[0,1]$.

Example 5. Let $V = C[0,1]$ be the vector space of all continuous functions from the interval $[0,1]$ to $\mathbb{C}$. For $f, g \in C[0,1]$, define

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt.$$

Prove that $\langle , \rangle$ is an inner product on $C[0,1]$.

Property 1. For all $\alpha, \beta \in V$ and $c \in F$, the followings are true.

1. $$\langle \alpha, c\beta \rangle = \overline{c}\langle \alpha, \beta \rangle.$$
2. $$c\langle \alpha, \beta \rangle = \langle \alpha, \overline{c}\beta \rangle.$$

Proof of (1).

$$\langle \alpha, c\beta \rangle = \overline{\langle c\beta, \alpha \rangle} = \overline{c\langle \beta, \alpha \rangle} = \overline{c}\ \overline{\langle \beta, \alpha \rangle} = \overline{c}\ \langle \alpha, \beta \rangle.$$

Property 2.

$$\langle \alpha, \beta + \gamma \rangle = \langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle.$$

Proof.

$$\begin{aligned} \langle \alpha, \beta + \gamma \rangle &= \overline{\langle \beta + \gamma, \alpha \rangle} \\ &= \overline{\langle \beta, \alpha \rangle + \langle \gamma, \alpha \rangle} \\ &= \overline{\langle \beta, \alpha \rangle} + \overline{\langle \gamma, \alpha \rangle} \\ &= \langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle. \end{aligned}$$

Property 3.

$$\langle \alpha, 0 \rangle = \langle 0, \alpha \rangle = 0. \qquad \text{(Homework)}$$

Property 4.

$$\langle \alpha, \alpha \rangle = 0 \text{ if and only if } \alpha = 0. \qquad \text{(Homework)}$$

Inner Product Spaces

An inner product space is a real or complex vector space, together with a specified inner product on that space.

Examples

1. $\mathbb{R}^n$ is an inner product space with the standard inner product, which is the dot product.
2. $\mathbb{C}^n$ is an inner product space with the standard inner product defined in Example 1.
3. $F^{n\times n}$ is an inner product space with the inner product defined in Example 3.
4. $C[0,1]$ is an inner product space with the inner product defined in Example 4.

Definition. A finite-dimensional real inner product space is called a Euclidean space.

Definition. A complex inner product space is called a unitary space.

Norm (length)

Definition.
The norm (length) of a vector α in an inner product space is defined by

$$\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}\ , \quad \text{(Positive square root).}$$

Example Consider the standard inner product on $\mathbb{R}^n$

$$\langle \alpha, \alpha \rangle = \sum_{j=1}^{n} x_j \overline{x_j} = \sum_{j=1}^{n} x_j x_j = \sum_{j=1}^{n} x_j^2$$

$$\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle} = \sqrt{\sum_{j=1}^{n} x_j^2} \quad \text{(length of the vector } \alpha\text{).}$$

Property 5. In an inner product space the parallelogram law holds. That is

$$\|\alpha+\beta\|^2+\|\alpha-\beta\|^2 = 2\left(\|\alpha\|^2+\|\beta\|^2\right).$$

Proof.

$$\|\alpha+\beta\|^2 = \langle \alpha+\beta, \alpha+\beta\rangle \; = \langle \alpha, \alpha+\beta\rangle + \langle \beta, \alpha+\beta\rangle$$

$$= (\langle \alpha,\alpha\rangle + \langle \alpha,\beta\rangle) + (\langle \beta,\alpha\rangle + \langle \beta,\beta\rangle)$$

$$= \|\alpha\|^2 + \|\beta\|^2 + \langle \alpha,\beta\rangle + \langle \beta,\alpha\rangle$$

$$= \|\alpha\|^2 + \|\beta\|^2 + \langle \alpha,\beta\rangle + \overline{\langle \alpha,\beta\rangle}$$

$$\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 + 2\ \mathrm{Re}\ \langle \alpha,\beta\rangle. \tag{1}$$

Property 5 contd.

Using similar arguments, we can obtain

$$\|\alpha - \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 - 2 \,\mathrm{Re}\, \langle \alpha, \beta \rangle. \tag{2}$$

From equations (1) and (2) we get

$$\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2\left(\|\alpha\|^2 + \|\beta\|^2\right).$$

The sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.

Problem 1

Show that if $F = \mathbb{R}$, then

$$\langle \alpha, \beta \rangle = \frac{1}{4}\|\alpha + \beta\|^2 - \frac{1}{4}\|\alpha - \beta\|^2.$$

Theorem 1

If V is an inner product space, then for any vectors $\alpha, \beta \in V$ and any scalar c,

(i) $\|c\alpha\| = |c| \; \|\alpha\|$;

(ii) $\|\alpha\| > 0$ for $\alpha \neq 0$;

(iii) $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \; \|\beta\|$; (Cauchy-Schwarz inequality)

(iv) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$. (Triangle inequality)

Orthogonal vectors

Let V be an inner product space and $\alpha, \beta \in V$. We say α is orthogonal to β if $\langle \alpha, \beta \rangle = 0$.

Example Consider the Euclidean space R^3 and the standard basis $B = \{\epsilon_1 = (1, 0, 0), \epsilon_2 = (0, 1, 0), \epsilon_3 = (0, 0, 1)\}$.

$$\langle \alpha, \beta \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3, \text{ where } \alpha, \beta \in R^3.$$

$$\langle \epsilon_1, \epsilon_2 \rangle = 1 \times 0 + 0 \times 1 + 0 \times 0 = 0.$$

This means ϵ_1 and ϵ_2 are orthogonal to each other.

Similarly, we can verify that $\langle \epsilon_1, \epsilon_3 \rangle = \langle \epsilon_2, \epsilon_3 \rangle = 0$. Thus, $\epsilon_1, \epsilon_2, \epsilon_3$ are orthogonal to each other.

Any set which has this property is called an orthogonal set.

So, B is an orthogonal set.

Orthogonal set and Orthonormal set

Definition. Let V be an inner product space. A set $S \subseteq V$ is called an orthogonal set if $\langle \alpha, \beta \rangle = 0$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$.

Definition. An orthonormal set is an orthogonal set S with the additional property that $\|\alpha\| = 1$ for all $\alpha \in S$.

Example. Observe that

$$\|\epsilon_1\| = \sqrt{\langle \epsilon_1, \epsilon_1 \rangle} = \sqrt{1 \times 1 + 0 \times 0 + 0 \times 0} = 1,$$

Similarly we can verify that $\|\epsilon_2\| = 1$ and $\|\epsilon_3\| = 1$. Thus, B is an orthonormal set.

Gram-Schmidt orthogonalization process

Input: A basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ of an inner product space V.

Output: An orthogonal basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of V.

$$\alpha_1 = \beta_1$$

$$\alpha_2 = \beta_2 - \frac{\langle \beta_2, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1$$

$$\alpha_3 = \beta_3 - \frac{\langle \beta_3, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle \beta_3, \alpha_2 \rangle}{\|\alpha_2\|^2}\alpha_2$$

$$\alpha_4 = \beta_4 - \frac{\langle \beta_4, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle \beta_4, \alpha_2 \rangle}{\|\alpha_2\|^2}\alpha_2 - \frac{\langle \beta_4, \alpha_3 \rangle}{\|\alpha_3\|^2}\alpha_3$$

and so on...

Problem 3

Find an orthogonal basis of $\mathbb{R}^3$ with standard inner product from the basis $B = \{\beta_1 = (3,0,4), \beta_2 = (-1,0,7), \beta_3 = (2,9,11)\}$ using Gram-Schimdt process.

Solution.

$$\alpha_1 = \beta_1 = (3,0,4); \quad \|\alpha_1\|^2 = 3^2 + 0^2 + 4^2 = 25$$

$$\begin{aligned}
\alpha_2 &= \beta_2 - \frac{\langle \beta_2, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 \\
&= (-1,0,7) - \frac{\langle (-1,0,7),(3,0,4) \rangle}{25}(3,0,4) \\
&= (-1,0,7) - \frac{(-1 \times 3 + 0 \times 0 + 7 \times 4)}{25}(3,0,4) \\
&= (-4,0,3).
\end{aligned}$$

$$\|\alpha_2\|^2 = (-4)^2 + 0^2 + 3^2 = 25$$

Problem 3 contd.

$$
\begin{aligned}
\alpha_2 = \beta_3 - \frac{\langle \beta_3, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle \beta_3, \alpha_2 \rangle}{\|\alpha_2\|^2}\alpha_2 & \\
= \quad & (2, 9, 11) - \frac{\langle (2, 9, 11), (3, 0, 4) \rangle}{25}(3, 0, 4) - \frac{\langle (2, 9, 11), (-4, 0, 3) \rangle}{25}(-4, 0, 3) \\
= \quad & (2, 9, 11) - 2(3, 0, 4) - (-4, 0, 3) \\
= \quad & (0, 9, 0).
\end{aligned}
$$

$$\|\alpha_3\|^2 = 81$$

Problem 3 contd.

$$B' = \{\alpha_1 = (3,0,4), \alpha_2 = (-4,0,3), \alpha_3 = (0,9,0)\}$$

is an orthogonal basis of $\mathbb{R}^3$.

Verification

$$\langle \alpha_1, \alpha_2 \rangle = 3 \times (-4) + 0 \times 0 + 4 \times 3 = 0$$

$$\langle \alpha_1, \alpha_3 \rangle = \langle \alpha_2, \alpha_3 \rangle = 0$$

Note that B' is a L.I. subset of $\mathbb{R}^3$ and its an orthogonal basis of $\mathbb{R}^3$.

Problem 3 contd.

$$B'' = \left\{ \frac{\alpha_1}{\|\alpha_1\|}, \frac{\alpha_2}{\|\alpha_2\|}, \frac{\alpha_3}{\|\alpha_3\|} \right\}$$

is an orthonormal basis of $\mathbb{R}^3$.

$$B'' = \left\{ \frac{1}{5}(3,0,4), \frac{1}{5}(-4,0,3), (0,1,0) \right\}.$$

Problem 4

Using Gram-Schimdt process, find an orthonormal basis for the Euclidean space $\mathbb{R}^3$ from the following ordered basis

$$B = \{\beta_1 = (1,1,1), \beta_2 = (0,1,1), \beta_3 = (0,0,1)\}.$$

Solution.

$$\alpha_1 = \beta_1 = (1,1,1,), \qquad \|\alpha_1\|^2 = 3$$

$$\alpha_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \|\alpha_2\|^2 = \frac{2}{3}$$

$$\alpha_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right), \quad \|\alpha_3\|^2 = \frac{1}{2}$$

The orthonormal basis:

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$