Basis and Dimension

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If S is a linearly independent set, then for any (finite) distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$,

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Note: (i) If $AX = 0$ has only trivial solution, then columns of A forms a linearly independent set.
(ii) If A is an invertible matrix, then columns of A forms a linearly independent set (By note (i) and Theorem 13, chapter 1).

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4. **A set** S **of vectors is linearly independent if and only if each finite subset of** S **is linearly independent if and only if for any distinct vectors** $\alpha_1, \alpha_2, \ldots, \alpha_n$ **of** S**,** $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0 \Longrightarrow c_1 = c_2 = \ldots = c_n = 0$

Problem 1

Show that $\alpha_1 = (3, 0, -3), \alpha_2 = (-1, 1, 2), \alpha_3 = (4, 2, -2)$ and $\alpha_4 = (2, 1, 1)$ are linearly dependent (L.D.) on R^3.

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Solution : Find sclars c_1, c_2, c_3, c_4 (at leaset one $c_i \neq 0$) such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$.

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$$2\alpha_1 + 2\alpha_2 - \alpha_3 + 0\alpha_4 = 0$$

Problem 2

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Note :

$\{\epsilon_1 = (1,0,0,\ldots,0,0), \epsilon_2 = (0,1,0,\ldots,0,0) \ldots, \epsilon_n = (0,0,0,\ldots,0,1)\}$ is a linearly independent subset of F^n.

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Note : A vector space V is finite dimensional if it has a finite basis.

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Note: $\mathbb{B} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ is called the standard basis of F^n.

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Show that $\mathbb{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $\mathbb{B}_1 = \{(0,1,1),(1,0,1),(1,1,0)\}$ are basis for R^3.

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$$\implies \quad X = 0, \quad (\text{ P is an invertible matrix})$$

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We have $\mathbb{B} \subseteq F^{n\times 1}$. $\implies \quad \text{Span } \mathbb{B} \subseteq F^{n\times 1} - - - -(i)$

Let $Y \in F^{n\times 1}$. By Theorem 13 (Note that P is invertible), $PX = Y$ has a solution X for a each $Y \in F^{n\times 1}$.

$$Y = PX = x_1P_1 + x_2P_2 + \ldots + x_nP_n \in L(\{P_1, P_2, \ldots, P_n\})$$

$$\implies \quad Y \in \text{ Span } \mathbb{B}. \implies \quad F^{n\times 1} \subseteq \text{ Span } \mathbb{B} - - - (ii)$$

From (i) and (ii), $F^{n\times 1} = \text{ Span } \mathbb{B}$.

Problem 4 contd.

$$\implies \quad x_1 = x_2 = \ldots = x_n = 0 \implies \mathbb{B} \text{ is a L.I. set}$$

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From (i) and (ii), $F^{n\times 1} = \text{ Span } \mathbb{B}$. By Claims 1 and 2, $\mathbb{B}$ (the set of all columns of P) is a basis of $F^{n\times 1}$.

Problem 5

Let $A \in F^{n\times n}$ and let $\{P_1, P_2, \ldots, P_n\}$ be columns of A. Prove that A is invertible if and only if $\{P_1, P_2, \ldots, P_n\}$ is a L.I. set.

Problem 5

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Solution : A is invertible if and only if $AX = 0$ has only trivial solution $X = 0$ (Theorem 13, chapter 1) if and only if $x_1P_1 + x_2P_2 + \ldots + x_nP_n = 0$ has only trivial solution $x_1 = x_2 = \ldots = x_n = 0$

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Note 1: (Visit previous lecture notes)

Find the solution space of the system $RX = 0$

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}\right\}$$

No. of non-zero rows of R, $r = 2$, **No. of variables,** $n = 5$

$k_1 = 2, k_2 = 4 \Longrightarrow$ Pivot variables $= \{x_{k_1}, x_{k_2}\} = \{x_2, x_4\}$

No. of free variables $= n - r = 5 - 2 = 3$,

Free variables $= \{u_1, u_2, u_3\} = \{x_1, x_3, x_5\}$

$$\left.\begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j &= 0 \\ x_{k_2} + \sum_{j=1}^{n-r} C_{2j} u_j &= 0 \end{aligned}\right\} \text{(general expression)}$$

Note 1 contd.

$$\left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\} \Longrightarrow \left.\begin{array}{c} x_{k_1} + \displaystyle\sum_{j=1}^{n-r} C_{1j} u_j = 0 \\ x_{k_2} + \displaystyle\sum_{j=1}^{n-r} C_{2j} u_j = 0 \end{array}\right\} \textit{general expression})$$

Set the free variables as :

$u_1 = x_1 = a,\ u_2 = x_3 = b,\quad u_3 = x_5 = c$

$\Longrightarrow\quad x_2 = 3b - \frac{1}{2}c,\quad x_4 = -2c$

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c)\ :\ a, b, c \in \mathrm{R}\right\}$

Note 1 contd. (back to chapter one !)

Solution set $S = \left\{(a, 3b - \frac{1}{2}c, b, -2c, c) \ : \ a, b, c \in \mathrm{R}\right\}$

$$S = \left\{ a(1,0,0,0,0) + b(0,3,1,0,0) + c(0,-\frac{1}{2},0,-2,1) \ : \ a,b,c \in \mathrm{R} \right\}$$

$$= \text{Span of } \left\{ (1,0,0,0,0), (0,3,1,0,0), (0,-\frac{1}{2},0,-2,1) \right\}$$

Dimension of $S = dim\ S = 3 = n - r$ **(Information for future)**

Alternate way to find a basis of S

$$\left.\begin{array}{l} x_2 - 3x_3 + \frac{1}{2}x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right\} - -(i)$$

Note that $\{x_2, x_4\}$ are pivot variables and $\{x_1, x_3, x_5\}$ are free varaiables.

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Set $x_1 = 1, x_3 = 0, x_5 = 0.$

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Let $E_1 = (1, 0, 0, 0, 0)$

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Set $x_1 = 0, x_3 = 0, x_5 = 1. \implies x_2 = -\frac{1}{2}, x_4 = -2$

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Clearly, $S = \text{Span}\,\{E_1, E_3, E_5\}$ (See the previous slide)

Prove that $\{E_1, E_3, E_5\}$ is a linearly independet set.

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Problem 5 (assignment)

Let W be set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfies

$$\begin{aligned}
2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \qquad\quad &= 0\\
x_1 \qquad\quad + \tfrac{2}{3}x_3 \qquad\quad - x_5 &= 0\\
9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0
\end{aligned}$$

Find a basis of W.

(2) Find a basis of the vector space of all polynomials over the field F (see chapter 2, example 16)

Theorem 4

Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$.

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By (i), $\alpha_1 = A_{11}\beta_1 + A_{21}\beta_2 + \ldots + A_{m1}\beta_m$

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By (i), $\alpha_1 = A_{11}\beta_1 + A_{21}\beta_2 + \ldots + A_{m1}\beta_m$

$\alpha_2 = A_{12}\beta_1 + A_{22}\beta_2 + \ldots + A_{m2}\beta_m$

$$\alpha_j = A_{1j}\beta_1 + A_{2j}\beta_2 + \ldots + A_{mj}\beta_m$$

$$\alpha_j = A_{1j}\beta_1 + A_{2j}\beta_2 + \ldots + A_{mj}\beta_m = \sum_{i=1}^{m} A_{ij}\beta_i \ , \quad j = 1, 2, \ldots, n$$

$$\alpha_j = A_{1j}\beta_1 + A_{2j}\beta_2 + \ldots + A_{mj}\beta_m = \sum_{i=1}^{m} A_{ij}\beta_i \ , \quad j = 1, 2, \ldots, n$$

Consider the homogeneous system

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = 0 - - - - - - -(ii)$$

$$\implies \sum_{j=1}^{n} x_j\alpha_j = 0$$

$$\alpha_j = A_{1j}\beta_1 + A_{2j}\beta_2 + \ldots + A_{mj}\beta_m = \sum_{i=1}^{m} A_{ij}\beta_i \ , \quad j = 1, 2, \ldots, n$$

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$$\implies \sum_{j=1}^{n} x_j\alpha_j = 0$$

$$\implies \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} A_{ij}\beta_i \right) = 0$$

$$\alpha_j = A_{1j}\beta_1 + A_{2j}\beta_2 + \ldots + A_{mj}\beta_m = \sum_{i=1}^{m} A_{ij}\beta_i \ , \quad j = 1, 2, \ldots, n$$

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Theorem 4 contd.

Consider $\displaystyle\sum_{j=1}^{n} A_{ij}x_j = 0, i = 1, 2, \ldots, m - - - -(iii)$

Theorem 4 contd.

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Theorem 4 contd.

Consider $\sum_{j=1}^{n} A_{ij}x_j = 0, i = 1, 2, \ldots, m - - - -(iii)$

The system (iii) is a homogeneous linear system with m equations and n variables.

Theorem 4 contd.

Consider $$\sum_{j=1}^{n} A_{ij}x_j = 0, i = 1, 2, \ldots, m - - - -(iii)$$

The system (iii) is a homogeneous linear system with m equations and n variables. Since $m < n$, the system (iii) has a non-trivial solution say $x_1^*, x_2^*, \ldots, x_n^*$ (at least one $x_j^* \neq 0$) such that

$$\sum_{j=1}^{n} A_{ij}x_j^* = 0, i = 1, 2, \ldots, m - - - (iv)$$

Theorem 4 contd.

Consider $\displaystyle\sum_{j=1}^{n} A_{ij}x_j = 0, i = 1, 2, \ldots, m - - - -(iii)$

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$$x_1^*\alpha_1 + x_2^*\alpha_2 + \ldots + x_n^*\alpha_n$$

Theorem 4 contd.

Consider $\sum_{j=1}^{n} A_{ij}x_j = 0, i = 1, 2, \ldots, m - - - -(iii)$

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$$x_1^*\alpha_1 + x_2^*\alpha_2 + \ldots + x_n^*\alpha_n = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}x_j^* \right) \beta_i \quad (see \quad (ii))$$

$$x_1^*\alpha_1 + x_2^*\alpha_2 + \ldots + x_n^*\alpha_n = \sum_{i=1}^{m} (0)\, \beta_i$$

Theorem 4 contd.

Consider $\displaystyle\sum_{j=1}^{n} A_{ij}x_j = 0, i = 1, 2, \ldots, m - - - -(iii)$

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$\displaystyle\sum_{j=1}^{n} A_{ij}x_j^* = 0, i = 1, 2, \ldots, m - - - (iv)$

$$x_1^*\alpha_1 + x_2^*\alpha_2 + \ldots + x_n^*\alpha_n = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j^*\right)\beta_i \quad (see \;\; (ii))$$

$$x_1^*\alpha_1 + x_2^*\alpha_2 + \ldots + x_n^*\alpha_n = \sum_{i=1}^{m}(0)\,\beta_i = 0 \quad (see \;\; (iv))$$

Theorem 4 contd.

Hence

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = 0 - - - - - - -(ii)$$

has a non-trivial solution $x_1^*, x_2^*, \ldots, x_n^*$ (at least one $x_j^* \neq 0$).

Hence

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = 0 - - - - - - - (ii)$$

has a non-trivial solution $x_1^*, x_2^*, \ldots, x_n^*$ (at least one $x_j^* \neq 0$). A contradiction to the assumption that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a L.I. set.

Theorem 4 contd.

Hence

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has a non-trivial solution $x_1^*, x_2^*, \ldots, x_n^*$ (at least one $x_j^* \neq 0$).
A contradiction to the assumption that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a L.I. set.
Therefore, $n \leq m$. This completes the proof.

Corollary to Theorem 4

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Proof: Since V is a finite dimensional vector space, it has a finite basis say

$$B_1 = \{\beta_1, \beta_2, \ldots, \beta_m\}.$$

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By (a) and (b), $m = n$. $\Longrightarrow |B_1| = |B_2|$. It completes the proof.

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$$S_m = S_0 \cup \{\beta_1, \beta_2, \ldots, \beta_m\}$$

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$S_1 = S_0 \cup \{\beta_1\} = \{(1,1,1),(1,1,0)\}$ is a L.I. subset of R^3.

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Let $S_0 = \{(1,1,1)\}$. Find a basis for R^3 which contains S_0.

Solution :

$$L(S_0) = \{a(1,1,1) \ : a \in R\} = \{(a,a,a) \ : \ a \in R\}$$

Clearly, $\beta_1 = (1,1,0) \notin L(S_0)$. By Theorem 5,
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$S_2 = S_1 \cup \{\beta_2\} = \{(1,1,1),(1,1,0),(1,0,0)\}$ is a L.I. set. Verify that $L(S_2) = R^3$. Hence, S_2 is a basis for R^3.

Corollary to Theorem 5

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Corollary 2 : In a finite-dimensional vector space V every non-empty linearly independent set of vectors is part of a basis.

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Let $A \in F^{n \times n}$, and suppose the row vectors of A form a linearly independent set of vectors in F^n.

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$$F^n = W = \text{span } \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

Corollary 3 Theorem 5 contd.

Since $\epsilon_1 = (1, 0, \ldots, 0) \in F^n = \text{span } \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$,

$$\epsilon_1 = B_{11}\alpha_1 + B_{12}\alpha_2 + \ldots + B_{1n}\alpha_n$$

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Similarly, $\epsilon_2 = (0, 1, \dots, 0), \dots, \epsilon_n = (0, 0, \dots, 1) \in F^n$,

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$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \dots \\ \epsilon_n \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{bmatrix}$$

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$$\implies I = BA$$

Hence B is a left inverse of A and thus A is invertible.

Sum of Subsets

Definition: If $\mathrm{S}_1,\ \mathrm{S}_2, \ldots, \mathrm{S}_\mathrm{k}$ are subsets of a vector space V, the set of all sums

$$\alpha_1 + \alpha_2 + \cdots + \alpha_\mathrm{k}$$

of vectors α_i in S_i is called the sum of the subsets $\mathrm{S}_1,\ \mathrm{S}_2, \ldots, \mathrm{S}_k$ and is denoted by

$$\mathrm{S}_1 + \mathrm{S}_2 + \cdots + \mathrm{S}_\mathrm{k}$$

or by

$$\sum_{i=1}^{k} S_i$$

Sum of Subspaces

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \cdots + W_k$$

is easily seen to be a subspace of V containing each subspace W_i. From this it follows, as in the proof of Theorem 3, that W is the subspace spanned by the union of $W_1, W_2, \ldots, W_k$.

Example 9

Let F be a subfield of the field C of complex numbers, and let V be the vector space of all 2×2 matrices over F. Let W_1 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

where x, y, z are arbitrary scalars in F. Finally, let W_2 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

where x and y are arbitrary scalars in F. Then W_1 and W_2 are subspaces of V(**Verify!**).

Example 9

Also

$$V = W_1 + W_2$$

because

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}.$$

The subspace $W_1 \cap W_2$ consists of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

Dimension of $(W_1 + W_2)$

Theorem 6: If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $\mathrm{W}_1 + \mathrm{W}_2$ is finite-dimensional and

$$\dim \mathrm{W}_1 + \dim \mathrm{W}_2 = \dim \left(\mathrm{W}_1 \cap \mathrm{W}_2\right) + \dim \left(\mathrm{W}_1 + \mathrm{W}_2\right).$$

Dimension of $(W_1 + W_2)$

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$$\dim \mathrm{W}_1 + \dim \mathrm{W}_2 = \dim(\mathrm{W}_1 \cap \mathrm{W}_2) + \dim(\mathrm{W}_1 + \mathrm{W}_2).$$

Proof: By Theorem 5 and its corollaries, $W_1 \cap W_2$ has a finite basis $\{\alpha_1, \ldots, \alpha_k\}$ which is part of a basis

$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m\} \text{ for } W_1$$

and part of a basis

$$\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n\} \text{ for } W_2.$$

Proof

The subspace $W_1 + W_2$ is spanned by the vectors

$$\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_\mathrm{m}, \quad \gamma_1, \ldots, \gamma_\mathrm{n}$$

and these vectors form an independent set. For suppose

$$\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0.$$

Then

$$-\sum z_r \gamma_r = \sum x_i \alpha_i + \sum y_j \beta_j$$

which shows that $\sum z_r \gamma_r$ belongs to W_1. As $\sum z_r \gamma_r$ also belongs to W_2 it follows that

$$\sum z_r \gamma_r = \sum c_i \alpha_i$$

for certain scalars $c_1, \ldots, c_k$.

Proof

Because the set

$$\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_{\mathrm{n}}\}$$

is independent, each of the scalars $z_r = 0$. Thus

$$\sum x_i \alpha_i + \sum y_j \beta_j = 0$$

and since

$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_{\mathrm{m}}\}$$

is also an independent set, each $x_i = 0$ and each $y_j = 0$. Thus,

$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_{\mathrm{m}}, \gamma_1, \ldots, \gamma_{\mathrm{n}}\}$$

is a basis for $W_1 + W_2$.

Proof

Finally

$$\begin{aligned}\dim W_1 + \dim W_2 &= (k+m) + (k+n) \\ &= k + (m+k+n) \\ &= \dim(W_1 \cap W_2) + \dim(W_1 + W_2)\,.\end{aligned}$$

Verify Theorem 6 by Example 9.