Assignment 2

Assignment questions - Relations

1. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1), (1, 2), (1, 3), (2, 3),$ $(2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2), and (5, 4)$.
   Find a) R^2. b) R^3. c) R^4. d) R^5.
   Represent them as directed graph.

2. Let R be a nice reflexive symmetric binary relation defined on set A. The nice reflexive symmetric binary relation is a relation such that it is reflexive and contains exactly one symmetric pair. Count the number of nice reflexive symmetric binary relations.

3. Determine whether the relations represented by these zero-one matrices are partial orders.
   a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
   c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

4. (a.)Prove or disprove: If R and S are equivalence relations on A, then $R \circ S$ is an equivalence relation on A.
   (b.)Prove $R = \{(x, y)|x + y$ is an even integer$\}$ is an equivalence relation on $\mathcal{Z}$.

5. R is relation defined on A $=\{0, 1, 2, 3\}$. Let $R = \{(0, 1), (0, 2), (1, 1), (1, 3), (2, 2), (3, 0)\}$.
   Find the transitive closure of R.
   Find the symmetric closure of R.
   Find the reflexive closure of R.

6. Let S be a set with n elements and let a and b be distinct elements of S. How many relations R are there on S such that
   a) $(a, b) \in R$?
   b) $(a, b) \notin R$?

7. Let R be a relation from a set A to a set B. The complementary relation $\overline{R}$ is the set of ordered pairs $\{(a, b)|(a, b) \notin R\}$
   Show that the relation R on a set A is reflexive if and only if the inverse relation R^{-1} is reflexive.
   Show that the relation R on a set A is reflexive if and only if the complementary relation $\overline{R}$ is irreflexive.

8. Which of these are posets?
   a) $(R, =)$
   b) $(R, <)$
   c) $(R, \leq)$
   d) $(R, \neq)$
   Is there any total order relation among these?

9. Prove or disprove: If R is an equivalence relation on A, then $R \circ R$ is an equivalence relation on A.

10. Let R be the relation $\{(a, b)|a \neq b\}$ on the set of integers up to 15. What is the reflexive closure of R?
    Find the transitive closure of R.
    Find the symmetric closure of R.

Functions

11. Find an example of a function that is neither injective nor surjective.

12. Define functions f, g and h as follows:
    $f : \mathcal{R} \to \mathcal{R}, \forall x \in \mathcal{R}, f(x) = x^2$.
    $g : \mathcal{N} \to \mathcal{N}; \forall x \in \mathcal{N}, g(x) = x^2$.

$h : A \to B; \forall x \in A, h(x) = x^2$.
where $A = \{0, 1, 2, 3, 4\}$ and $B = \{0, 1, 4, 9, 16\}$
Which function is one-to-one?
Which function is onto?

13. Determine whether each of these functions from $\mathcal{Z}$ to $\mathcal{Z}$ is one-to-one.
a) $f(n) = n - 1$
b) $f(n) = n^2 + 1$
c) $f(n) = n^3$

14. If $f : R \to R$ and $g : R \to R$ are functions, then the function $(f + g) : R \to R$ is defined by the formula $(f + g)(x) = f(x) + g(x)$ for every real number x.

    (a) $f : R \to R$ and $g : R \to R$ are both one-to-one, is $f + g$ also one-to-one? Justify your answer.
    (b) $f : R \to R$ and $g : R \to R$ are both onto, is $f + g$ also onto? Justify your answer.

15. Suppose that f is a function from A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.

16. Let D be the set of all finite subsets of positive integers, and define $T : Z^+ \to D$ by the following rule: For every integer n, $T(n)$= the set of all of the positive divisors of n.

    (a) Is T one-to-one? Prove or give a counterexample.
    (b) Is T onto? Prove or give a counterexample.

17. Determine whether each of the following functions from Z to Z is one-to-one.

    (a) $f(n) = n + 7$
    (b) $f(n) = 2n - 3$
    (c) $f(n) = \lceil n/2 \rceil$

18. If X and Y are sets and $F : X \to Y$ is one-to-one and onto, then $F^{-1} : Y \to X$ is also one-to-one and onto.

19. Suppose $F : X \to Y$ is onto. Prove that for every subset $B \subseteq Y$, $F(F^{-1}(B)) = B$.

20. Give an example of finite sets A and B with $|A|, |B| \geq 4$ and a function $F : A \to B$ such that

    (a) F is one-to-one but not onto
    (b) F is onto but not one-to-one
    (c) F is onto and one-to-one.