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The geometric series

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converges absolutely because the corresponding series of absolute values

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converges.

In contrast, the alternating harmonic series does not converge absolutely: The corresponding series of absolute values is the divergent harmonic series.

Definition (Conditional Convergence)

A series that converges but does not converge absolutely is said to **converge conditionally**.

Example: The alternating harmonic series converges conditionally.

The Absolute Convergence Test

Theorem

If $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof:

For each n,

$$-|a_n| \leq a_n \leq |a_n| \qquad \text{so} \qquad 0 \leq a_n + |a_n| \leq 2|a_n|.$$

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But $a_n = (a_n + |a_n|) - |a_n|$. So, $\sum_{n=1}^{\infty} a_n$ can be expressed as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

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Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

Examples

(a) For $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \ldots$, the corresponding series of absolute values is the series

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Power Series

Definition (Power Series, Center, Coefficients)

A **power series about** $x = 0$ is a series of the form

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Here a is the **center** and $c_0, c_1, c_2, \ldots, c_n, \ldots$ are the **coefficients** of the power series. These are constants.

Example: A geometric series

Taking all the coefficients to be 1 gives the geometric power series

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This the geometric series with first term 1 and common ratio x.
It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots, \qquad -1 < x < 1.$$

Note

The power series

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gives the following polynomial approximations for the non-polynomial function $\frac{1}{1-x}$ for values of x near 0:

$$P_0(x) \quad = \quad 1$$

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The power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \ldots + \left(-\frac{1}{2}\right)^n (x-2)^n + \ldots$$

has center $a = 2$ and coefficients $c_0 = 1, c_1 = -1/2, c_2 = 1/4, \ldots, c_n = (-1/2)^n, \ldots$.

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The series converges for $|r| = \left|-\frac{x-2}{2}\right| < 1$ or $0 < x < 4$.

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So,

$$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \ldots + \left(-\frac{1}{2}\right)^n (x-2)^n + \ldots, \qquad 0 < x < 4.$$

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Testing Power Series for Convergence Using the Ratio Test

For what values of x does the following series converge?

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots .$$

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At $x = 1$, it becomes the alternating harmonic series. So, it converges (conditionally).

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Summary: the series converges for $-1 < x \leq 1$ and diverges for other values of x.

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$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots.$$

Solution: We apply the Ratio Test to the series $\sum_{n=1}^{\infty}|u_n|$, where u_n is the nth term of the given power series:

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$$

Thus the given power series converges absolutely for $x^2 < 1$.
It diverges for $x^2 > 1$ since the nth term does not converge to zero.

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Summary: The series converges for $|x| \leq 1$ and diverges otherwise.

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So, the series converges absolutely for all values of x.

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So, the nth term of the series does not converge to zero for $x \neq 0$. Hence the series diverges for all values of x except $x = 0$.

Theorem (The Convergence Theorem for Power Series)

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ *converges for* $x = c \neq 0$*, then it converges absolutely for all* x *with* $|x| < |c|$*.*

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Thus, it follows that if the power series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

Note

We prove a similar theorem for power series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

by considering the power series $\sum_{n=0}^{\infty} a_n y^n$:

From the theorem, the latter series converges for $y = c \neq 0$ implies that it converges for $|y| < |c|$.

This means that the given power series converges for x with $|x-a| < |c|$.

And so on.

The Radius of Convergence

Theorem

The convergence of the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ *is described by one of the following three possibilities:*

1. *There is a positive number* R *such that the series diverges for* x *with* $|x-a| > R$ *but converges absolutely for* x *with* $|x-a| < R$ *(i.e., for* $a-R < x < a+R$*). The series may or may not converge at either of the end points* $x = a-R$ *and* $x = a+R$*.*

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3. *The series converges at $x = a$ and diverges for all $x \neq a$ ($R = 0$).*

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converges if

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Definition

The **radius of convergence** of the power series $\sum_{n=0}^{\infty} a_n x^n$ is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

Example: Computing the Radius and Interval of Convergence

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots.$$

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Thus the interval of convergence is $|x-2| < 2$ or $0 < x < 4$.

Theorem (The Term-by-Term Differentiation Theorem)

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges for* $a-R<x<a+R$ *for some* $R>0$*, it defines a function* f*:*

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Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots, \qquad -1 < x < 1.$$

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for $a - R < x < a + R$.

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Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, \qquad -1 \leq x \leq 1.$$

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&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \qquad -1 < x < 1.
\end{aligned}$$

Theorem (The Multiplication Theorem for Power Series)

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$ $(R > 0)$

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Example

Multiply the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x} \quad (|x| < 1)$$

by itself to get a power series for $\frac{1}{(1-x)^2}$ for $|x| < 1$.

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$$\begin{aligned} c_n &= c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 + a_n b_0 \\ &= 1 + 1 + \ldots + 1 = n + 1 \end{aligned}$$

Then, by the Multiplication Theorem, the power series for $\dfrac{1}{(1-x)^2}$ is

$$\begin{aligned} A(x)\cdot B(x) &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1)x^n \\ &= 1 + 2x + 3x^2 + 4x^3 + \dots, \quad |x| < 1. \end{aligned}$$

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Suppose that

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and so on. The above equations all hold, in particular, at $x = a$.

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This implies that the sum function $f(x)$ has a unique power series expansion. (Why?)

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Note: The Maclaurin series generated by f is often called the Taylor series of f.

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Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $f(x)$?

Solution: We need to find $f(2), f'(2), f''(2), \ldots$.

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Thus the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$.

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$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

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Homework

Find the Taylor series generated by the following functions at $x = 0$: