Electrical Circuits for Engineers (EC1000)

Lecture-10
AC Power Analysis
(Chapter 11)

AC Power Analysis

11.1 Instantaneous and Average Power (1)

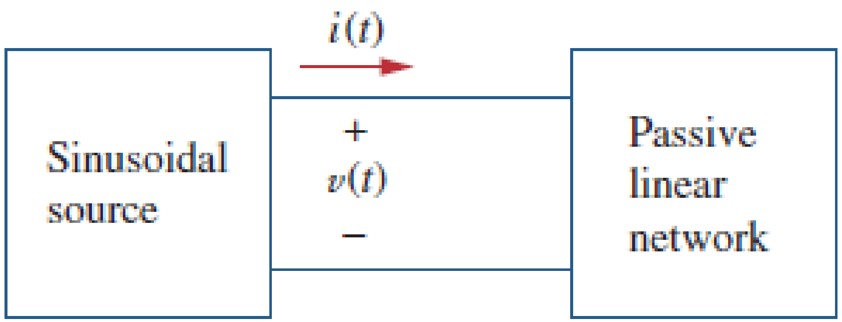
The instantaneously power, p(t)

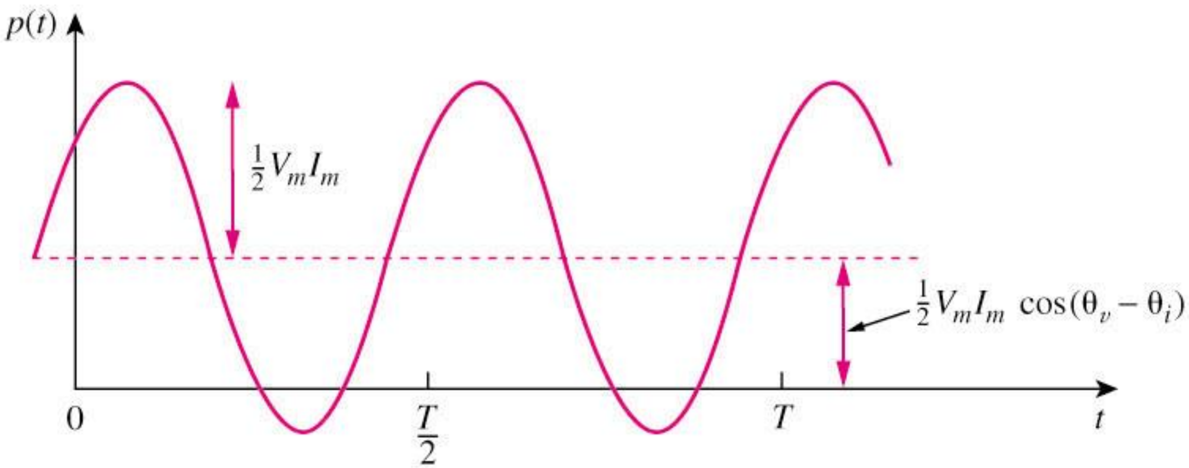
We can also think of the instantaneous power as the power absorbed by the element at a specific instant of time. Instantaneous quantities are denoted by lowercase letters.

$$p(t) = v(t)\,i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

$$= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

Constant power **Sinusoidal power at 2ωt**



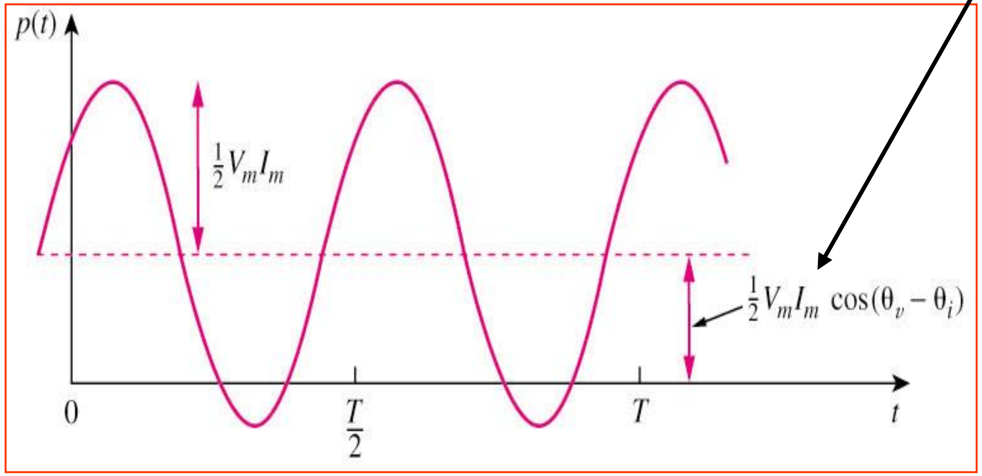


p(t) > 0: power is absorbed by the circuit; p(t) < 0: power is absorbed by the source.

11.1 Instantaneous and Average Power (2)

The average power, P, *is the average of the instantaneous power over one period.*

$$\boxed{P = \frac{1}{T}\int_0^T p(t)\,dt = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$



1. **P is not time dependent.**
2. **When $\theta_v = \theta_i$, it is a purely resistive load case.**
3. **When $\theta_v - \theta_i = \pm 90o$, it is a purely reactive load case.**
4. **P = 0 means that the circuit absorbs no average power.**

11.1 Instantaneous and Average Power (3)

Example 1 Calculate the instantaneous power and average power absorbed by a passive linear network if:

V(t)=120 cos (377t+45°)
i(t)=10 cos(377t-10°)

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

$$p(t) = 344.2 + 600\cos(754t + 35^\circ)\ \text{W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$

$$= 600\cos 55^\circ = 344.2\ \text{W}$$

which is the constant part of $p(t)$ above.

11.1 Instantaneous and Average Power (4)

2. Calculate the instantaneous power and average power absorbed by a passive linear network if:

$$v(t) = 330\cos(10t + 20^\circ)\ \text{V} \qquad \text{and} \qquad i(t) = 33\sin(10t + 60^\circ)\ \text{A}$$

Answer: $3.5 + 5.445\cos(20t - 10^\circ)$ kW, 3.5 kW.

11.1 Instantaneous and Average Power (4)

Example 3. Find the average power delivered to the impedance Z=30-j70Ω when a voltage V=120 V is applied across it.

Solution:

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\underline{/0^\circ}}{30 - j70} = \frac{120\underline{/0^\circ}}{76.16\underline{/-66.8^\circ}} = 1.576\underline{/66.8^\circ}\text{ A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24\text{ W}$$

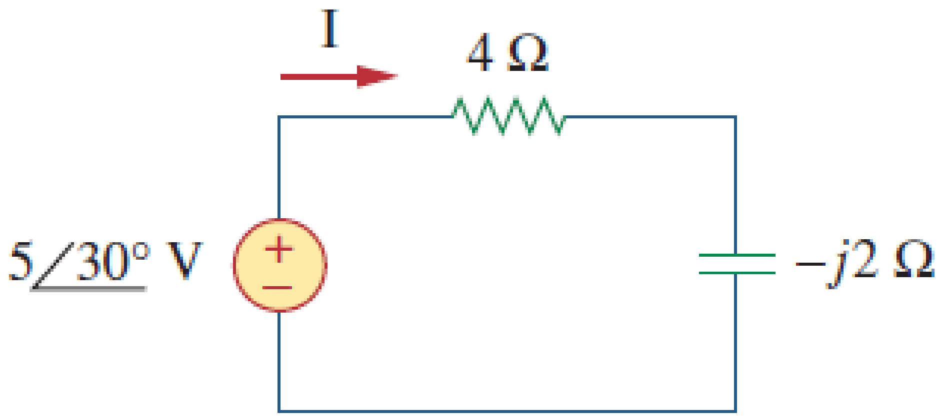
Example 4. A current $\mathbf{I} = 33\underline{/30^\circ}\text{ A}$ flows through an impedance $\mathbf{Z} = 40\underline{/-22^\circ}\ \Omega$. Find the average power delivered to the impedance.

Answer: 20.19 kW.

11.1 Instantaneous and Average Power (4)

Example 5

For the circuit shown in Figure, find the average power supplied by the source and the average power absorbed by the resistor.



Solution:

The current $\mathbf{I}$ is given by

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ \text{ A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I}_R = \mathbf{I} = 1.118\underline{/56.57^\circ} \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\underline{/56.57^\circ} \text{ V}$$

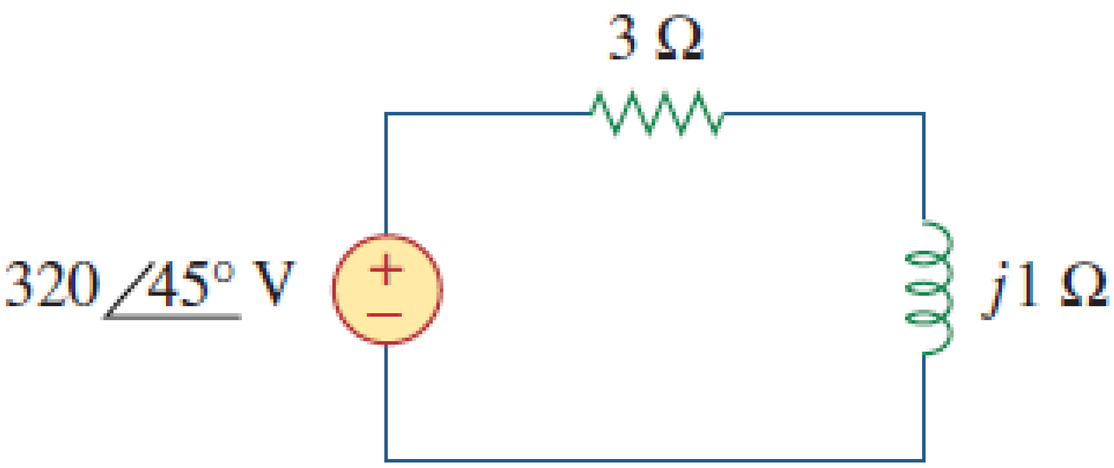
The average power absorbed by the resistor is

$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

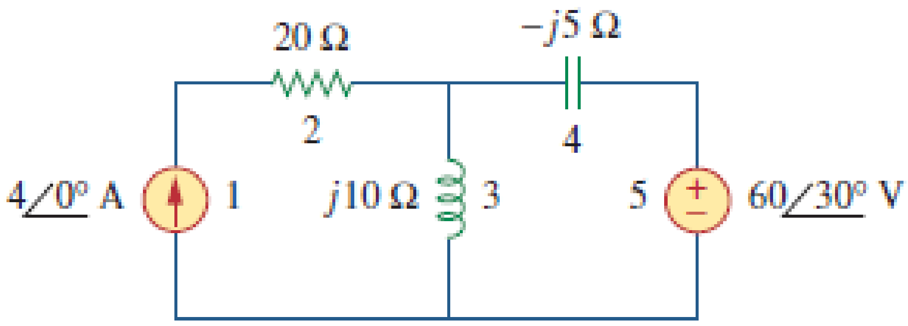
Example 6

For the circuit shown in Figure, find the average power observed by the resister and inductor. Find the average power supplied by the source.

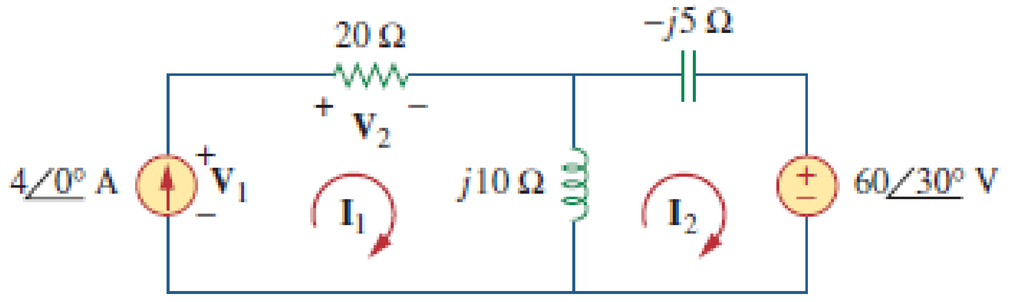


Answer: 15.361 kW, 0 W, 15.361 kW.

Example 7: Determine the average power generated by each source and the average power absorbed by each passive element in the circuit of Figure.



(a)



(b)

Solution:

We apply mesh analysis as shown in Fig. 11.5(b). For mesh 1,

$$\mathbf{I}_1 = 4 \text{ A}$$

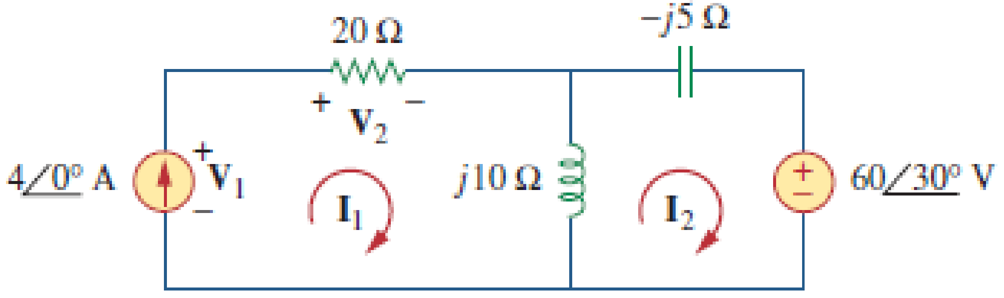
For mesh 2,

$$(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\underline{/30^\circ} = 0, \qquad \mathbf{I}_1 = 4 \text{ A}$$

or

$$j5\mathbf{I}_2 = -60\underline{/30^\circ} + j40 \quad \Rightarrow \quad \mathbf{I}_2 = -12\underline{/-60^\circ} + 8$$
$$= 10.58\underline{/79.1^\circ} \text{ A}$$

$$P_5 = \frac{1}{2}(60)(10.58)\cos(30° - 79.1°) = 207.8 \text{ W}$$



(b)

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 183.9 + j20 = 184.984\underline{/6.21°} \text{ V}$$

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21° - 0) = -367.8 \text{ W}$$

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

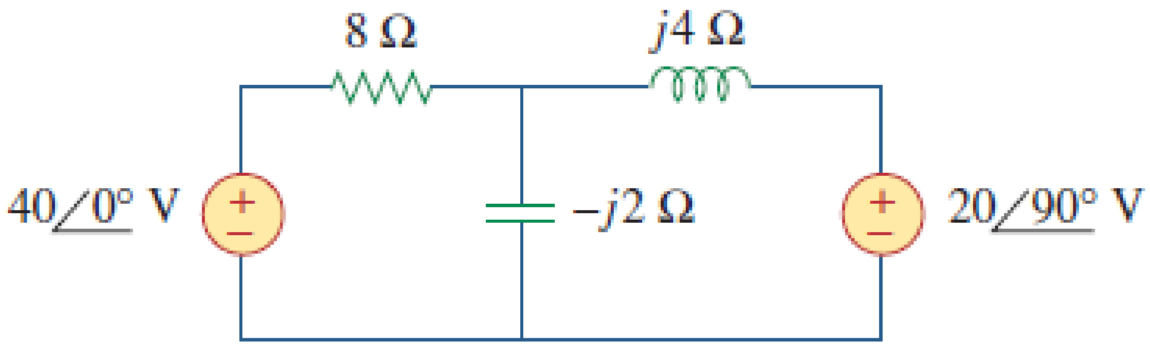
$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90°) = 0$$

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90° = 0$$

$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

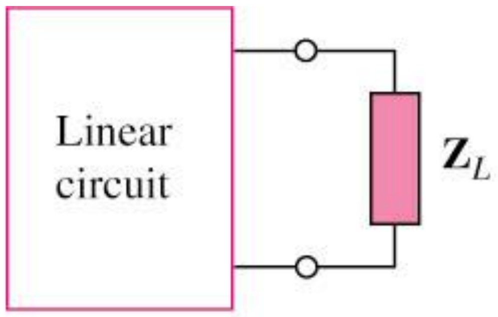
Practice Problem

Determine the average power generated by each source and the average power absorbed by each passive element in the circuit of Figure.

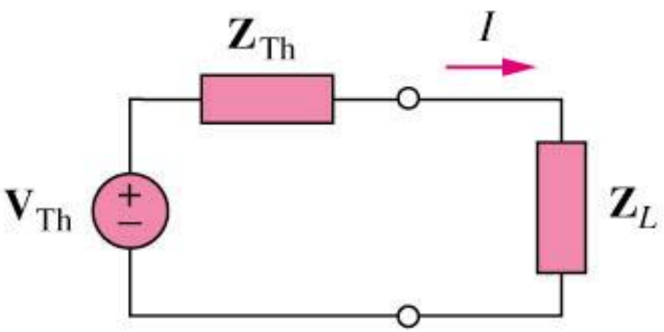


Answer: 40-V Voltage source: −60 W; j20-V Voltage source: −40 W; resistor: 100 W; others: 0 W.

11.2 Maximum Average Power Transfer (1)



(a)



(b)

$$Z_{TH} = R_{TH} + jX_{TH}$$

$$Z_L = R_L + jX_L$$

The maximum average power can be transferred to the load if

$$X_L = -X_{TH} \quad \text{and} \quad R_L = R_{TH}$$

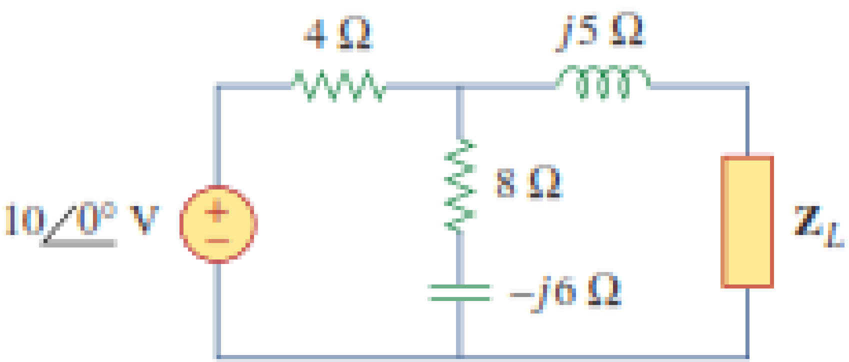
$$P_{max} = \frac{|V_{TH}|^2}{8R_{TH}}$$

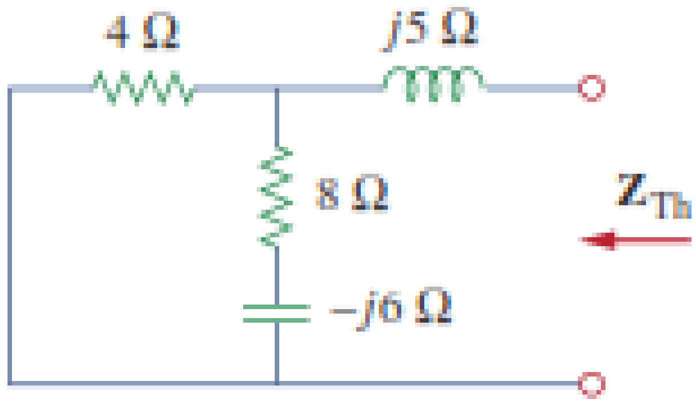
If the load is purely real, then $R_L = \sqrt{R_{TH}^2 + X_{TH}^2} = |Z_{TH}|$

11.2 Maximum Average Power Transfer (2)

Example 11.5

For the circuit shown below, find the load impedance Z_L that absorbs the maximum average power. Calculate that maximum average power.

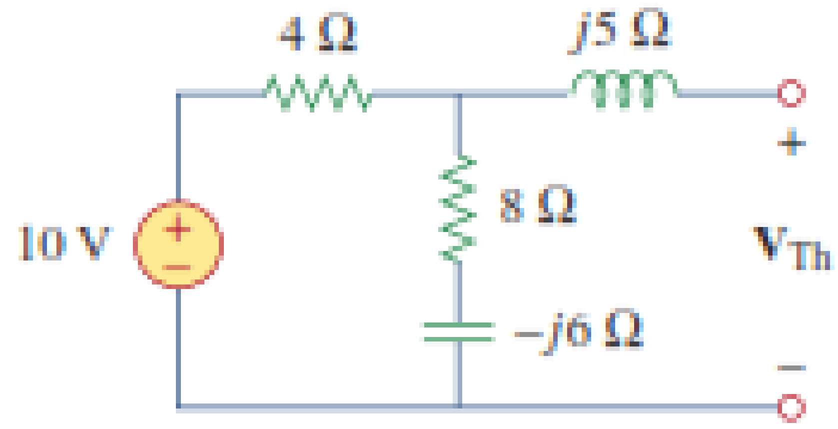




(a)

$$\mathbf{Z}_{Th} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

11.2 Maximum Average Power Transfer (2)



(b)

To find $\mathbf{V}_{\mathrm{Th}}$, consider the circuit in Fig. 11.8(b). By voltage division,

$$\mathbf{V}_{\mathrm{Th}} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\underline{/-10.3^\circ}\ \mathrm{V}$$

The load impedance draws the maximum power from the circuit when

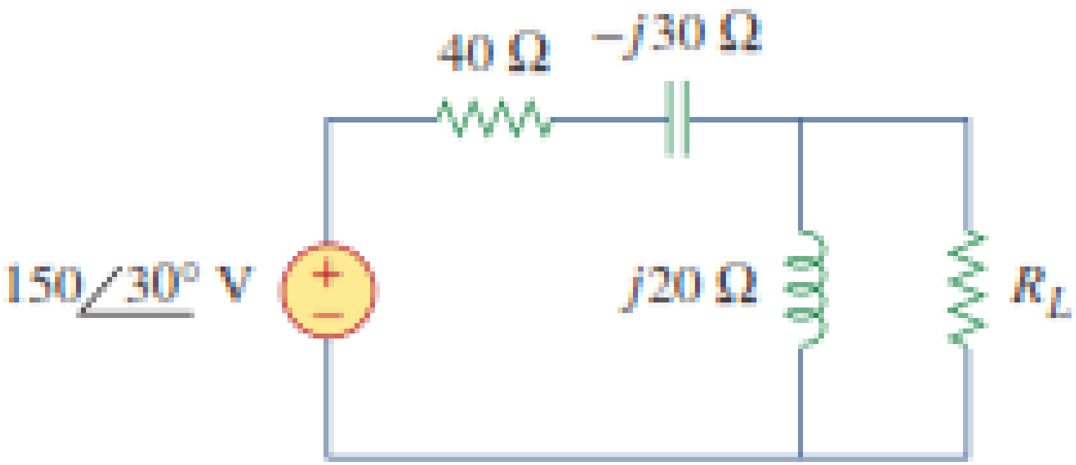
$$\mathbf{Z}_L = \mathbf{Z}_{\mathrm{Th}}^* = 2.933 - j4.467\ \Omega$$

$$P_{\max} = \frac{|\mathbf{V}_{\mathrm{Th}}|^2}{8R_{\mathrm{Th}}} = \frac{(7.454)^2}{8(2.933)} = 2.368\ \mathrm{W}$$

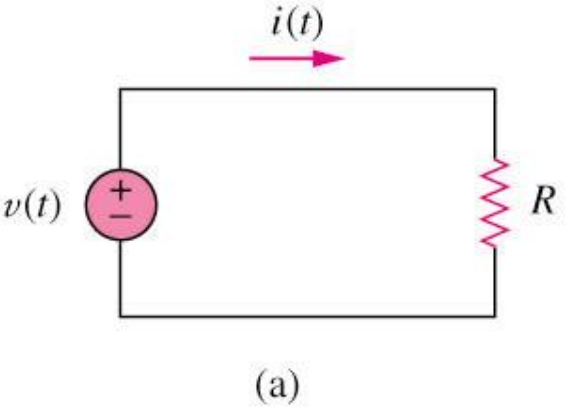
11.2 Maximum Average Power Transfer (2)

Example: 11.6: In the circuit in Figure, find the value of R_L that will absorb the maximum average power. Calculate that power.

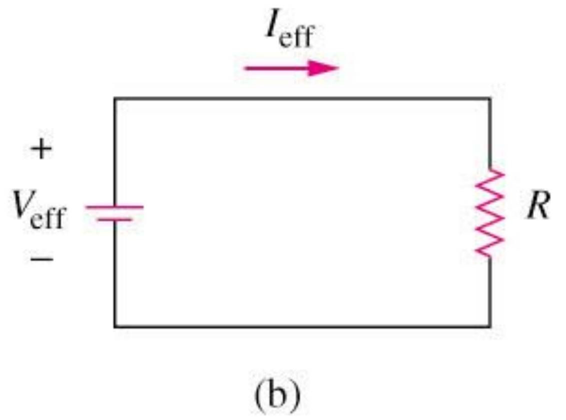
Ans: 39.29 watts



11.3 Effective or RMS Value (1)



(a)



(b)

The total power dissipated by R is given by:

$$P = \frac{1}{T}\int_0^T i^2 R dt = \frac{R}{T}\int_0^T i^2 dt = I_{rms}^2 R$$

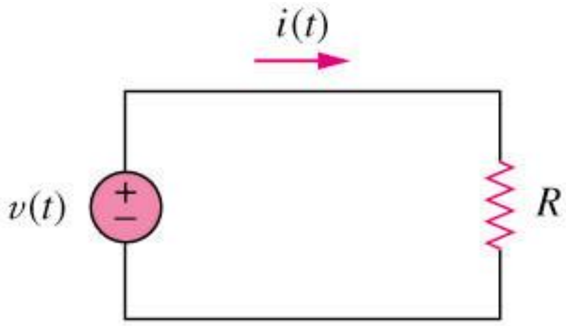
Hence, I_{eff} is equal to:

$$\boxed{I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2\, dt} = I_{rms}}$$

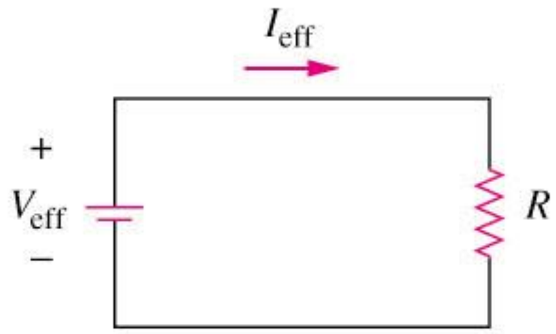
The rms value is a constant itself which depending on the shape of the function i(t).

The effective of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

11.3 Effective or RMS Value (2)



(a)



(b)

The rms value of a sinusoid $i(t) = I_m \cos(\omega t)$ is given by:

$$\boxed{I_{rms}^{2} = \frac{I_m}{\sqrt{2}}}$$

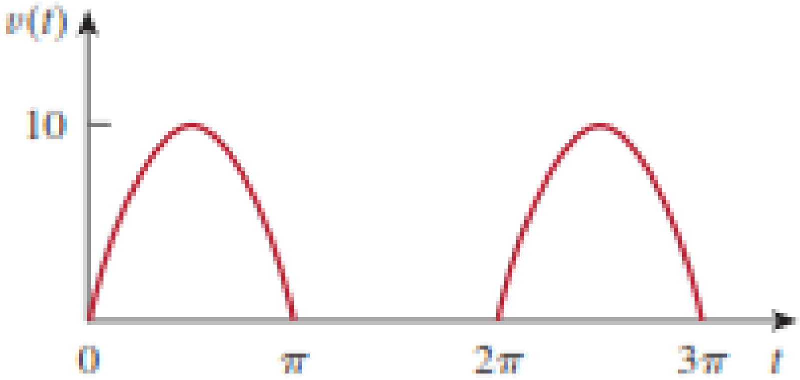
The average power can be written in terms of the rms values:

$$\boxed{P = \frac{1}{2} V_m\, I_m \cos(\theta_v - \theta_i) = V_{rms}\, I_{rms} \cos(\theta_v - \theta_i)}$$

<u>Note</u>: If you express amplitude of a phasor source(s) in rms, then all the answer as a result of this phasor source(s) must also be in rms value.

11.3 Effective or RMS Value (2)

The waveform shown in Figure is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a10-Ohm resistor.



Solution:

The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10 \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi} (10 \sin t)^2\,dt + \int_{\pi}^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence,

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^{\pi} \frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi}$$

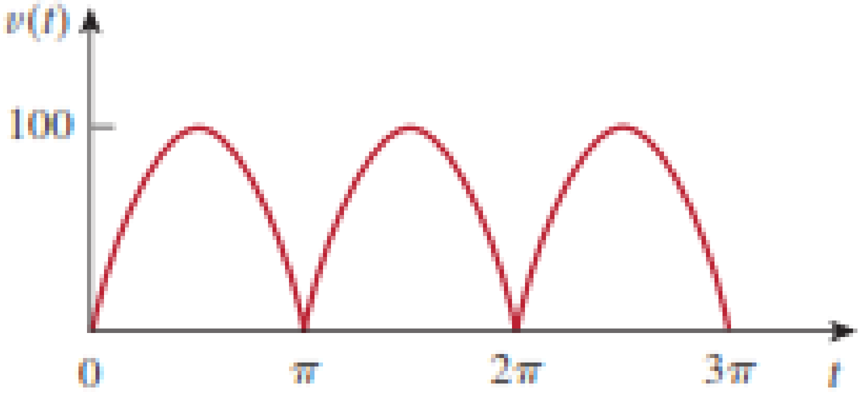
$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\text{rms}} = 5\text{ V}$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5\text{ W}$$

11.3 Effective or RMS Value (2)

The waveform shown in Figure is a full-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 6-Ohm resistor.



Answer: **70.71 V, 833.3 W.**

11.4 Apparent Power and Power Factor (1)

Apparent Power, S, is the product of the r.m.s. values of voltage and current.
It is measured in volt-amperes or VA to distinguish it from the average or real power which is measured in watts.

$$P = \underbrace{V_{rms} I_{rms}}_{\text{Apparent Power, } S} \cos(\theta_v - \theta_i) = S \underbrace{\cos(\theta_v - \theta_i)}_{\text{Power Factor, pf}}$$

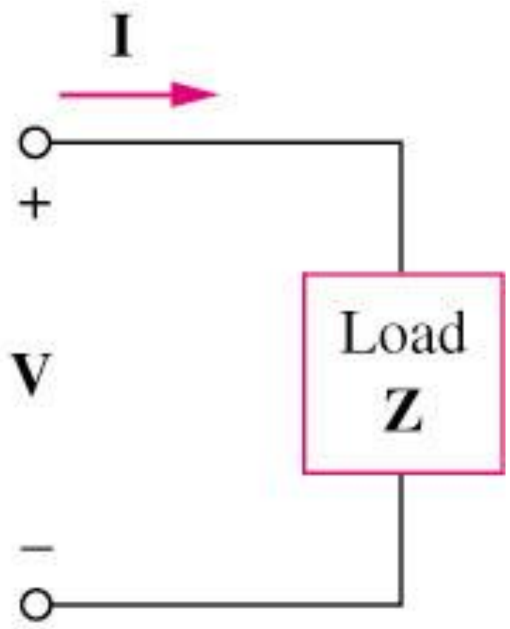
Power factor is the cosine of the phase difference between the voltage and current. It is also the cosine of the angle of the load impedance.

11.4 Apparent Power and Power Factor (2)

Purely resistive load (R)	$\theta_v - \theta_i = 0, \quad Pf = 1$	P/S = 1, all power are consumed
Purely reactive load (L or C)	$\theta_v - \theta_i = \pm 90^o$, $pf = 0$	P = 0, no real power consumption
Resistive and reactive load (R and L/C)	$\theta_v - \theta_i > 0$ $\theta_v - \theta_i < 0$	• Lagging - inductive load • Leading - capacitive load

11.5 Complex Power (1)

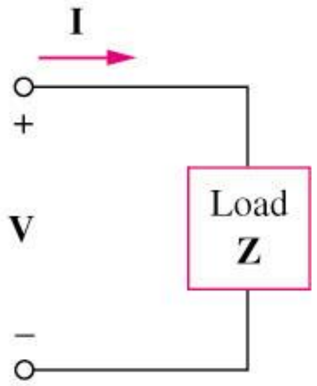
Complex power **S** is the product of the voltage and the complex conjugate of the current:



$$\mathbf{V} = V_m \angle \theta_v \qquad \mathbf{I} = I_m \angle \theta_i$$

$$\frac{1}{2} \mathbf{V}\,\mathbf{I}^* = V_{rms}\, I_{rms} \angle \theta_v - \theta_i$$

11.5 Complex Power (2)



$$S = \frac{1}{2} V I^* = V_{rms} I_{rms} \angle \theta_v - \theta_i$$

$$\Rightarrow S = \underbrace{V_{rms} I_{rms} \cos(\theta_v - \theta_i)}_{P} + j\, \underbrace{V_{rms} I_{rms} \sin(\theta_v - \theta_i)}_{Q}$$

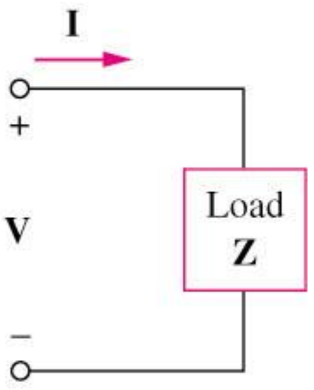
$$\boldsymbol{S} = \mathbf{P} + \mathbf{j} \; \mathbf{Q}$$

P: is the average power in watts delivered to a load and it is the only useful power.

Q: is the reactive power exchange between the source and the reactive part of the load. It is measured in VAR.

- Q = 0 for *resistive loads* (unity pf).
- Q < 0 for *capacitive loads* (leading pf).
- Q > 0 for *inductive loads* (lagging pf).

11.5 Complex Power (3)



$$\Rightarrow S = \underbrace{V_{rms}\, I_{rms} \cos(\theta_v - \theta_i)} + j\, \underbrace{V_{rms}\, I_{rms} \sin(\theta_v - \theta_i)}$$

$$\mathbf{S} = P + j\,Q$$

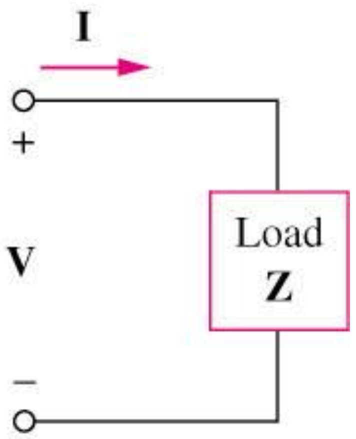
<u>Apparent Power,</u> $S = |\mathbf{S}| = \text{Vrms*Irms} = \sqrt{P^2 + Q^2}$

<u>Real power,</u> $P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$

<u>Reactive Power,</u> $Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$

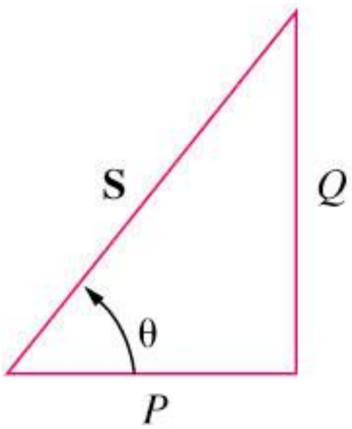
<u>Power factor,</u> $pf = P/S = \cos(\theta_v - \theta_i)$

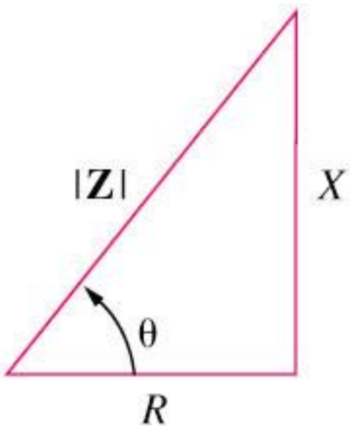
11.5 Complex Power (4)

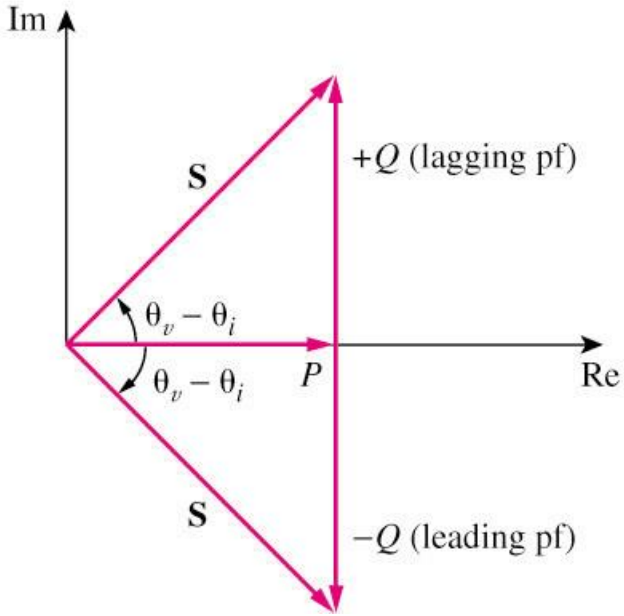


$$\Rightarrow S = \underbrace{V_{rms}\, I_{rms}\, \cos(\theta_v - \theta_i)}_{} + j\, \underbrace{V_{rms}\, I_{rms}\, \sin(\theta_v - \theta_i)}_{}$$

$$S = P + j\, Q$$







Power Triangle **Impedance Triangle** **Power Factor**

Example Problem.1

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:

The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20^\circ - 10^\circ) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\angle -20^\circ}{4\angle 10^\circ} = 30\angle -30^\circ = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30^\circ) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C} \Rightarrow C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

Practice Problem.1

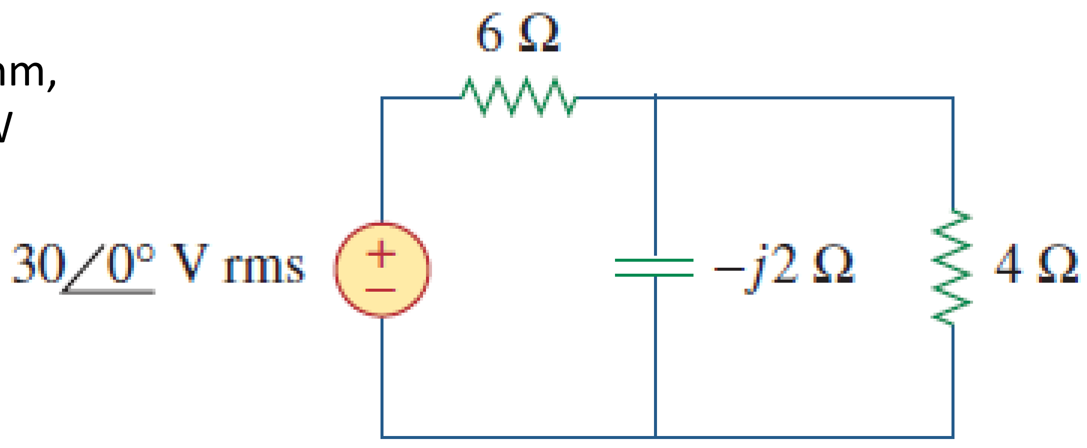
Obtain the power factor and the apparent power of a load whose impedance is $\mathbf{Z} = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 320 \cos(377t + 10^\circ)$ V.

Answer: 0.8321 lagging, $710\angle 33.69^\circ$ VA

Practice Problem.2

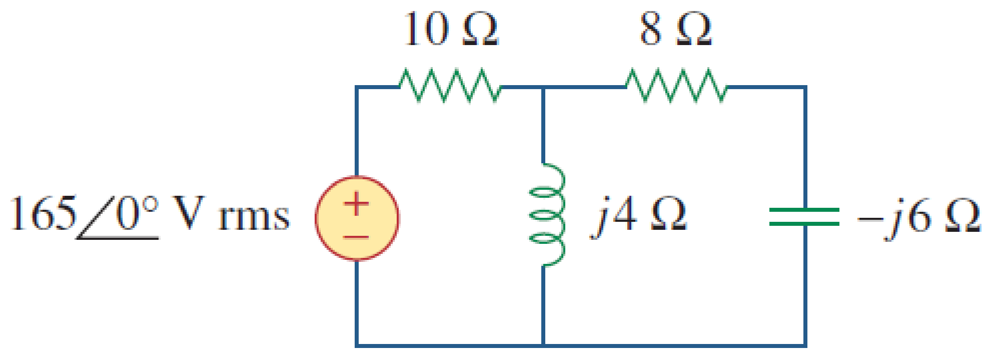
2. Determine the power factor of the entire circuit of Figure as seen by the source. Calculate the average power delivered by the source.

Ans: Z= 7∠-13.24 Ohm,
0.9734 leading, 125 W



Practice Problem.3

3. Determine the power factor of the entire circuit of Figure as seen by the source. Calculate the average power delivered by the source.



Answer: 0.936 lagging, 2.008 kW

Example Problem on Complex Power

The voltage across a load is $v(t) = 60\cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\angle -10^\circ, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\angle +50^\circ$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = \left(\frac{60}{\sqrt{2}}\angle -10^\circ\right)\left(\frac{1.5}{\sqrt{2}}\angle -50^\circ\right) = 45\angle -60^\circ \text{ VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45 \text{ VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\underline{/-60^\circ} = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\underline{/-10^\circ}}{1.5\underline{/+50^\circ}} = 40\underline{/-60^\circ}\ \Omega$$

which is a capacitive impedance

Practice Problem on Complex Power

For a load, $\mathbf{V}_{rms} = 110\angle 85^\circ$ V, $\mathbf{I}_{rms} = 0.4\angle 15^\circ$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Answer: (a) $44\angle 70^\circ$ VA, 44 VA, (b) 15.05 W, 41.35 VAR, (c) 0.342 lagging, $94.06 + j258.4\ \Omega$.