OBJECTIVE:

* Solve $Ax = b$, where:

$$A = (a_{ij}) \in \mathbb{R}^{m \times n}, \quad b = (b_i) \in \mathbb{R}^m$$

are given.

We seek for : $x = (x_j) \in \mathbb{R}^n$

$$m\downarrow \underset{\xrightarrow{\quad n \quad}}{\left[\quad A \quad\right]} \cdot \left[\quad x \quad\right] \downarrow n \;=\; \left[\quad b \quad\right] \downarrow m$$

$$A = (a_{ij}) \in \mathbb{R}^{m \times n}, \quad x = x_j \in \mathbb{R}^n, \quad b = b_i \in \mathbb{R}^m$$

Check for "existence" and "uniqueness" of the solutions.

ELEMENTARY ROW COLUMN OPERATIONS

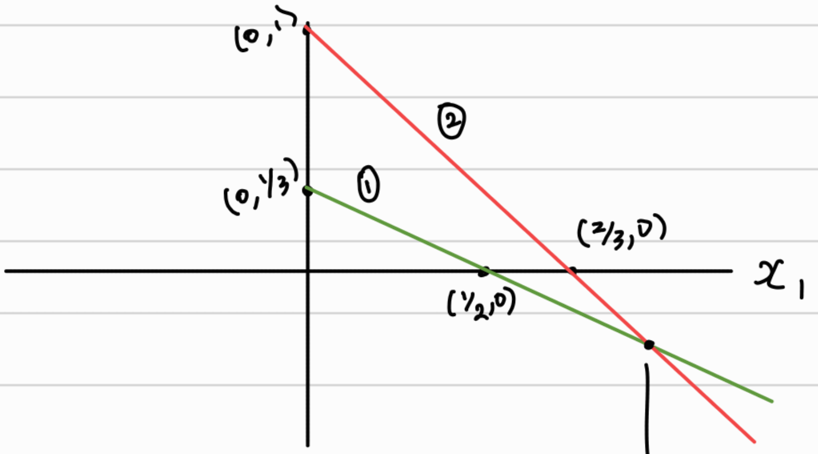
Motivation - use 2D matrices - explanation using geometry.

① $2x_1 + 3x_2 = 1$
② $3x_1 + 2x_2 = 2$

$\downarrow$ REPRESENT in $Ax = b$ form.

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

represent ① and ② in $x_1 x_2$ plane:



find coordinates. This is the solution.

x here is: $x = \begin{pmatrix} 4/5 \\ -1/5 \end{pmatrix}$

$$\left.\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 2 \end{aligned}\right\} \rightarrow$$ VERY EVIDENTLY, PARALLEL LINES... AND, is scaled by factor 2 fully. ∴, intersect at all pts. ∴, ∞ly many solutions (The lines are the same).

Solution set: $S = \left\{ (\alpha, \beta) : \beta = \frac{1}{3}(1 - 2\alpha),\ \alpha \in \mathbb{R} \right\}$

$$\left.\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 3 \end{aligned}\right\} \rightarrow$$ THESE ARE PARALLEL LINES, but coeffs. don't match! hence, shifted ||lel lines, i.e., they NEVER meet (or) meet at ∞.

recall: $2x_1 + 3x_2 = 1$ — ①

$3x_1 + 2x_2 = 2 \quad -(2)$

$3 \times (1) \equiv \quad 6x_1 + 9x_2 = 3$
$2 \times (2) \equiv \quad 6x_1 + 4x_2 = 4$

$3 \times (1) - 2 \times (2) \equiv \quad 5x_2 = -1 \quad \Rightarrow \boxed{x_2 = -1/5}$

$\therefore$, $(1) \Rightarrow \boxed{x_1 = 4/5}$

objective is "to reduce" $Ax = b$ — (I) to $Cx = d$ — (II)
where C has a simpler structure than A.

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \quad ; \quad C = \begin{pmatrix} 6 & 9 \\ 0 & 5 \end{pmatrix}$$ → upper triangular matrix. (entries below the principle diagonal are 0)

In general,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ 0 & 0 & c_{32} \cdots & c_{3n} \\ \vdots & & & \\ 0 & 0 & \cdots 0 & c_{nn} \end{pmatrix}$$

$Ax = b$ reduces to $Cx = d$, i.e.,

$c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n = d_1$ until you find all x_js.
$c_{22}x_2 + c_{23}x_3 + \cdots + c_{2n}x_n = d_2$

$$c_{33}x_3 + \cdots + c_{3n}x_n = d_3$$

$$\vdots \qquad \text{and so on}\ldots$$

$$c_{(n-1)(n-1)}x_{n-1} + c_{nn}x_n = d_{n-1}$$

$$c_{nn}x_n = d_n$$

sub back x_n, get x_{n-1}

$\downarrow$

get x_n

this is back-substitution (FRICK, RECURSION OCCURS EVERYWHERE- APPLY STACKS for this)

from A to C, we must do some operations - - - - -
say hi to elementary row operations . . .

Elementary Row Operations :

given $A \in \mathbb{R}^{m\times n}$ we perform the following operations.

① multiply a row by a non-zero constant.
② replace any row (s'th row) by (s'th row) + α * (t'th row)
↳ const . .

③ interchange any 2 rows.

let e be a fn. $e : \mathbb{R}^{m\times n} \rightarrow \mathbb{R}^{m\times n}$ denotes a particular elementary row operation.

let $e_1 \equiv$ ①, $e_2 \equiv$ ② and $e_3 \equiv$ ③

$$e_1(A)_{ij} = \begin{cases} \alpha \cdot a_{sj} & \text{if } i = s \\ a_{ij} & \text{if } i \neq s \end{cases}$$

$$e_2(A)_{ij} = \begin{cases} a_{sj} + \alpha \cdot a_{tj} & \text{if } i = s \end{cases}$$

$$\begin{cases} \\ a_{ij} & \text{if } i \neq s \end{cases}$$

$$e_3(A)_{ij} = \begin{cases} a_{tj} & i = s \\ a_{sj} & i = t \\ a_{ij} & i \neq s,t \end{cases}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdot \\ \vdots & \vdots & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

let $a_{1K} \neq 0 \longrightarrow K = \min(\text{all } j\text{'s s.t. } a_{ij} \neq 0)$

$R_1 \longrightarrow \frac{1}{a_{1K}} R_1$

$$A = \begin{pmatrix} 0 \cdots 0 & 1 & \cdots & \cdot \\ \cdot & - & - & - \\ \cdot & - & - & - \\ \cdot & - & - & - \end{pmatrix}$$

make all other entries of K'th col of $A = 0$

$R_m \longrightarrow R_m - a_{mK} R_1 \quad \forall m\text{s}$ other than K.

$$A \sim \begin{pmatrix} 0 \cdots 0 & 1 & \cdot * - & - \\ * - * \cdots & 0 & - & * \\ \cdot - - \cdot & 0 & - & - \\ \cdot - - & \vdots & - & * \\ & 0 & & \end{pmatrix}$$

if $\exists$ a non-zero element in r'th row, it can't be in K^{th} col.

in 2nd row → ⓐ leading in left of K $(p < K)$
→ ⓑ leading in right of K $(q > K)$

ⓐ Leading Non-zero for $q > K$

$$A \sim \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & * & \cdots & & * \\ * & * & \cdots & * & 0 & * & * & \cdots & * \\ \vdots & & & & & & & & \\ * & \cdots & & * & 0 & * & * & \cdots & * \end{pmatrix}$$

→ Leading Non-zero in row 2.

$(a_{2q} \neq 0,\ q > K)$ → least among all j s.t. $a_{2j} \neq 0$.

$R_2 \to \frac{1}{a_{2q}} R_2$

$$\Rightarrow A \sim \begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & & * \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & * \\ * & \cdots & * & \vdots & * & \cdots & & * \\ * & \cdots & * & 0 & * & \cdots & & * \end{pmatrix}$$

→ $a_{2q} = 1$ //

make all ele. in col q other than a_{2q} 0.

$R_1 \to R_1 - \frac{a_{1q}}{a_{1K}} R_2$ { R_1 is the INITIAL MATRIX'S ROW }

$R_3 \to R_3 - a_{3q} R_2$

Similarly, for all i's $!= 2$

$$\Rightarrow \quad A \sim \begin{pmatrix} 0 & \cdots & 0 & 1 & * & 0 & \cdots & \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & * \\ \vdots & & & \vdots & & 0 & \cdots & * \\ * & \cdots & & 0 & & 0 & \cdots & * \end{pmatrix}$$

ⓑ Leading NON-ZERO in $p < K$.

$$A \sim \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & * & \cdots & * \\ * & * & \cdot & * & * & 0 & * & * & \cdots & * \\ \vdots \\ * & \cdots & & & * & 0 & * & * & \cdots & * \end{pmatrix}$$

Leading Non-zero in row 2. $(a_{2p} \neq 0, \; p < K)$
↓
least among all j s.t. $a_{2j} \neq 0$.

$$R_2 \to R_2 / a_{2p}$$

$$A \sim \begin{pmatrix} 0 & 0 & \cdot & 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & * & 0 & \cdots & & * \\ \cdots & & & & 0 & \cdots \\ \cdots & & & & 0 & \cdots \end{pmatrix}$$

since $p < K$, we can leave $i = 1$ untouched.
($a_{1p} = 0$ ALREADY, as a_{1K} is the 1st leading NON-ZERO).

$$\left.\begin{aligned} R_3 &\to R_3 - a_{3p} R_2 \\ R_m &\to R_m - a_{mp} R_2 \end{aligned}\right\} \quad R_2 \to \underline{\underline{\text{PIVOT ROW?}}}$$

$$A \sim \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 & \cdots & * \\ 0 & \cdots & 0 & 1 & * & 0 & \cdots & * \\ \cdots & & 0 & * & 0 & \cdots \end{pmatrix}$$

OBSERVED :

① if the leading non-zero entry occurs to the left / right of column K, the entries of the first row before & including 1 are unchanged - if $p < K$, you aren't going to operate on R_1, and if $q > K$, until q every entry in R_2 is 0, $\therefore$, no change even after operation.

By this, we are ticking 2 conditions for RRE. Now, we can interchange rows s.t. $c_1 < c_2 < \dots < c_i < \dots < c_m$, where c_i represents the col. in i'th row where the leading term is found.

$\therefore$, by induction theorem is true.

↳ do for all m rows.

↳ $A \sim$ Row - Reduced - Echelon Matrix (RRE)

$Rx = 0$:

$$\begin{pmatrix} 0 & 1 & 4 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \underline{\underline{\text{RRE}}}.$$

$$x_2 + 4x_3 + \tfrac{1}{2}x_5 = 0$$
$$x_4 + 5x_5 = 0$$

choose $x_1 = \alpha$, $x_2 = \beta$ and $x_5 = \gamma$

as arbitrary values

$$x_3 = \frac{1}{4}\left(-\frac{1}{2}x_5 - x_2\right)$$
$$= \frac{1}{4}\left(-\frac{1}{2}\gamma - \beta\right)$$

$$x_4 = -5\gamma$$

$$S = \left\{\left(\alpha, \beta, \frac{1}{4}\left(-\frac{1}{2}\gamma - \beta\right), -5\gamma, \gamma\right) : \alpha, \beta, \gamma \in \mathbb{R}\right\}$$

$R \in \mathbb{R}^{m \times n}$ is a RRE matrix.

$Rx = 0$.

* Original unknowns : $x_1, x_2, \cdots, x_n$.

* let $i = 1, 2, 3, \cdots$ are the non-zero rows of R.

* let $c_1, c_2, c_3, \cdots, c_r$ are the cols. corresponding to the leading non-zero entries of the rows $1, 2, 3, \cdots, r$. w.r.t. $c_1 < c_2 < \cdots < c_r$.

* relabel unknowns $\equiv$ NON-ZERO ROWS.
  $x_{c_1}, x_{c_2}, \cdots, x_{c_r}$.

  remaining variables / unknowns denoted by
  $v_1, v_2, \cdots, v_{n-r}$.

$Rx = 0$:

$$x_{c_1} + \sum_{j=1}^{n-r} \alpha_{ij} v_j = 0$$

$$x_{c_2} + \sum_{j=1}^{n-r} \alpha_{2j} v_j = 0$$

$$\vdots$$

$$x_{c_r} + \sum_{j=1}^{n-r} \alpha_{rj} v_j = 0$$

if $v_1, v_2, \ldots, v_{n-r}$ are assigned arbitrary values of $x_{c_1}, x_{c_2}, \ldots x_{c_r}$ can be obtained.

START- Spanning Sets

let $S \subset V$. The span of $S \rightarrow sp(S)$ is the set of all Linear combinations of elements of S.

i.e., $$sp(S) = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_K v_K \;\middle|\; \alpha_i \in \mathbb{F} \text{ and } v_i \in S \right\}$$

t.p.t. $sp(S) \longrightarrow$ SUBSPACE.

a) closed under addn.

b) closed under multipln.

let $u, \omega \in sp(S)$

$$u = \sum_{i=1}^{K} \alpha_i u_i \quad ; \quad v = \sum_{i=1}^{r} \beta_i v_i$$

$\alpha_i, \beta_i \in \mathbb{F}$ and $u_i, v_i \in S$

($|u|$ need not be equal to $|v|$)

($|u|$ need not be equal to $|w|$)

$$u + w = \sum_{i=1}^{K} \alpha_i u_i + \sum_{i=1}^{\gamma} \beta_i v_i$$

let $u_1, u_2, \ldots u_K, v_1, v_2, \ldots, v_\gamma$
$\equiv x_1, x_2, x_3, \ldots, x_{K+\gamma}$
&
$\alpha_1, \alpha_2, \ldots, \alpha_K, \beta_1, \beta_2, \ldots, \beta_\gamma$
$\equiv \gamma_1, \gamma_2, \ldots, \gamma_{K+\gamma}$.

$\therefore$, $u + w = \sum_{i=1}^{K+\gamma} \gamma_i x_i$

where $x_i \in S$ and $\gamma_i \in \mathbb{F}$.

$\therefore$, $u + w \in S$

let $u \in sp(S)$ and $c \in \mathbb{F}$.

$u = \sum_{i=1}^{\gamma} \alpha_i u_i$, $\alpha_i \in \mathbb{F}$ & $u_i \in S$

$$\Rightarrow cu = c \cdot \sum_{i=1}^{\gamma} \alpha_i u_i = \sum_{i=1}^{\gamma} c(\alpha_i u_i)$$

$$= \sum_{i=1}^{\gamma} (c\alpha_i) u_i$$

let $c\alpha_i = \beta_i$

$$= \sum_{i=1}^{\gamma} \beta_i u_i$$

$\beta_i \in \mathbb{F}$
$u_i \in S$ $\therefore$, $cu \in sp(S)$.

$\therefore$, $sp(S) \longrightarrow$ CLOSED under addn. & multipln.
$\therefore$, $sp(S) \rightarrow$ SUBSPACE of V

1. $S = \{x_1, x_2\}$; $x_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$ + $x_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$Sp(S) \longrightarrow$?

if $x \in Sp(S)$,

$$x = \alpha_1 x_1 + \alpha_2 x_2 \longrightarrow \alpha_1, \alpha_2 \in \mathbb{R}$$

$$\Rightarrow x = \alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 1\\0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & x_1\\0 & 1 & x_2\\1 & 0 & x_3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & x_1\\0 & 1 & x_2\\0 & -1 & x_3 - x_1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & x_1 - x_2\\0 & 1 & x_2\\0 & 0 & x_3 - x_1 + x_2\end{array}\right]$$

$$\Rightarrow 0 = x_3 - x_1 + x_2 \Rightarrow x_1 = x_2 + x_3.$$

$$Sp(S) = \{x \in \mathbb{R}^3 \mid x_1 = x_2 + x_3\}$$

2. $S = \{x_1, x_2, x_3\}$; $x_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $x_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$x_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$

We seek b s.t. $A\alpha = b$, where:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 0 & 1 & b_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & -1 & 1 & b_3 - b_1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 - b_2 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 1 & \frac{b_3 - b_1 + b_2}{2} \end{array}\right) \leftarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 - b_2 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 2 & b_3 - b_1 + b_2 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 - b_2 + (\quad) \\ 0 & 1 & 0 & b_2 - (\quad) \\ 0 & 0 & 1 & \frac{b_3 - b_1 + b_2}{2} \end{array}\right)$$

$$A\alpha = b \longrightarrow R\alpha = d$$

$$\downarrow$$

* n(nonzero rows) $= 3 =$ rows of r
* System has soln. for all $b \in \mathbb{R}^3$

$\therefore$, $Sp(S) = \mathbb{R}^3$

Let $A \in \mathbb{R}^{m \times n}$. The row space of $A \rightarrow$ subspace of all linear combinations of rows of A.

row space of $A \longrightarrow$ subspace of $\mathbb{R}^n$.

Similarly column space of A is defined.

col. space of $A \longrightarrow$ subspace of $\mathbb{R}^m$.

Row. space of $A = Sp(A_i)$; $i \in \{1, 2, \cdots, m\}$

↳ its subspace of $\mathbb{R}^n$. why?

* let $\alpha, \beta \in R \cdot S$.

for α : choose ANY K rows of A, $K \in \{1, 2, \cdots, m\}$

β : choose ANY l rows of A, $l \in \{1, 2, \cdots, m\}$

let the chosen K rows of A in α be $\alpha_1, \alpha_2, \cdots, \alpha_K$ & l rows of A in β be $\beta_1, \beta_2, \cdots, \beta_l$.

$\therefore$, $\alpha = \sum_{i=1}^{K} u_i \alpha_i$ & $\beta = \sum_{i=1}^{l} v_i \beta_i$

so, $\alpha, \beta \rightarrow$ some ARBITRARY L.C.s of rows of A, that belongs to R.S.

$$\alpha + \beta = \sum_{i=1}^{K} u_i \alpha_i + \sum_{i=1}^{l} v_i \beta_i \quad ; \; u_i, v_i \in \mathbb{R}$$

let $\alpha + \beta = \gamma$, $\{u_1, u_2, \cdots, u_K, v_1, v_2 \cdots, v_l\}$ be $\{w_1, w_2, \cdots, w_K, w_{K+1}, w_{K+2}, \cdots w_{K+l}\}$

AND $\{\alpha_1, \alpha_2, \cdots \alpha_K, \beta_1, \beta_2, \cdots, \beta_l\}$ be $\{\gamma_1, \gamma_2, \cdots \gamma_{K+l}\}$

$\therefore, \quad \alpha+\beta=\gamma=\sum_{i=1}^{k+l} w_i \gamma_i$

looking at the form $\rightarrow$ evident that $\gamma \rightarrow$ some L.C. of rows of A

$\therefore, \gamma \in$ R.S. of A.

$\therefore, \quad \alpha, \beta \in$ R.S. of A

$\Rightarrow \quad \alpha+\beta \in$ R.S. of A.

* let $\alpha \in$ R.S. of A.

from prev. step, we can write α as:

$$\alpha = \sum_{i=1}^{k} u_i \alpha_i \quad , \quad \text{where } \alpha_i \in \text{Rows of } A$$

$u_i \in \mathbb{R}$.

$$c\alpha = c \cdot \sum_{i=1}^{k} u_i \alpha_i$$

$$= c \cdot (u_1\alpha_1 + u_2\alpha_2 + \cdots + u_k\alpha_k)$$

$$= c\cdot(u_1\alpha_1) + c\cdot(u_2\alpha_2) + \cdots + c\cdot(u_k\alpha_k)$$

$$= (cu_1)\alpha_1 + (cu_2)\cdot\alpha_2 + \cdots + (cu_k)\cdot\alpha_k$$

$$= v_1\alpha_1 + v_2\alpha_2 + \cdots + v_k\alpha_k$$

$$c\alpha = \sum_{i=1}^{k} v_i \alpha_i$$

$u_i \in \mathbb{R}$

$cu_i \in \mathbb{R}$

$c\alpha \rightarrow$ combo of Rows of A

$\therefore, \quad c\alpha \in$ R.S.

as $\alpha+\beta \in$ R.S. & $c\alpha \in$ R.S., R.S. of A $\rightarrow$ subspace of $\mathbb{R}^n$.

Similar proof can be presented for col. space

$\therefore$, Col. space of A $\rightarrow$ subspace of $\mathbb{R}^m$

LINEAR INDEPENDENCE / DEPENDENCE

Consider a set of vectors V:

$$v_1, v_2, \dots, v_K \in V$$

* $v_1, v_2, \dots, v_K \rightarrow$ linearly dependent if $\exists\, \alpha_i \neq 0$ s.t.

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_K v_K = 0 \qquad (i \in \{1,2,\dots,K\})$$

let $\alpha_s != 0$

$$\Rightarrow \quad \alpha_s v_s = -(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{s-1} v_{s-1} + \alpha_{s+1} v_{s+1} + \dots + \alpha_K v_K)$$

$$\Rightarrow \quad v_s = \left(\frac{-\alpha_1}{\alpha_s}\right) v_1 + \left(\frac{-\alpha_2}{\alpha_s}\right) v_2 + \dots + \left(\frac{-\alpha_K}{\alpha_s}\right) v_K$$

$$= \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_K v_K$$

$v_1, v_2, \dots, v_K \rightarrow$ linearly independent (if they aren't dependent)

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_K v_K = 0 \longrightarrow \forall_{i \in \{1,2,\dots,K\}} \; \alpha_i = 0.$$

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$$

$$\Rightarrow \quad \begin{pmatrix} \alpha_1 + \alpha_2 \\ \dots \end{pmatrix} \quad \begin{pmatrix} 0 \\ \dots \end{pmatrix}$$

$$\begin{pmatrix} \alpha_2+\alpha_3 \\ \alpha_1+\alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad \begin{aligned} \alpha_2 &= -\alpha_1 \\ \alpha_3 &= -\alpha_2 = \alpha_1 \end{aligned}$$

$$\alpha_1+\alpha_3=0 \Rightarrow 2\alpha_1 = 0$$
$$\Rightarrow \alpha_1 = 0$$
$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$\therefore$, x_1, x_2 & $x_3 \longrightarrow$ Linearly independent vectors

PROPERTIES :

1. Empty sets are linearly independent.

2. let $X = \{x_1, x_2, \cdots, x_m\}$, where one of the x_is is the ZERO VECTOR. then, X is a linearly dependent set.

   $\hookrightarrow$ $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_i v_i + \cdots + \alpha_m x_m = 0$
   let $v_i \rightarrow$ ZERO VECTOR, $\alpha_i != 0$.
   let other α_is $= 0$.
   $\Rightarrow 0\cdot x_1 + 0\cdot x_2 + \cdots + \alpha_i \cdot 0 + \cdots + 0\cdot x_m = 0$
   $\hookrightarrow$ HOLDS GOOD.

3. let $X = \{x_1, x_2, \cdots, x_m\}$ be linearly independent & $A \subseteq X$. then, $A \rightarrow$ linearly independent.

   proof by contradiction: assume $A \rightarrow$ linearly dependent. $\Rightarrow \exists \beta_l != 0$ s.t.
   $\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_l a_l + \cdots + \beta_K a_K = 0$

choose $\beta_l != 0$; $\beta_1, \beta_2, \cdots, \beta_{l-1}, \beta_{l+1}, \cdots, \beta_K = 0$

in X : $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m = 0$

$$\Rightarrow \underbrace{\sum_{i=1}^{K} \beta_i a_i}_{} + \underbrace{\sum_{i=1}^{m-K} \gamma_i c_i}_{} = 0$$

→ choose all γ_is $= 0$

→ $\exists \beta_l != 0$, other $\beta_s = 0$

as $\exists \beta_l != 0$ in a_i summation, i choose ALL OTHER β_is & γ_is to be ZERO, as I want at least one coeff. not ZERO for proof to work.

$$\Rightarrow \quad 0 + 0 = 0$$

BUT! if this happens X → linearly dependent → contradicts given.

∴ A → linearly independent.

4. let $X = \{x_1, x_2, \cdots, x_m\}$ be given s.t. $\phi = A \subseteq X$ is linearly dependent. Then X is linearly dependent

(above proof CAN BE USED?)

two vectors x & y → L. dep. iff. one is multiple of the other.

if $x = \alpha y$, $\alpha \in \mathbb{F}$ then :

$$x = \alpha y$$
$$\Rightarrow x - \alpha y = \alpha y - (\alpha y)$$

$\Rightarrow \quad x - \alpha y = 0$
$\Rightarrow \quad 1 \cdot x + (-\alpha) y = 0$
$\Rightarrow \quad \beta \cdot x + (-\alpha) y = 0$, where $\beta != 0$

$\therefore$, $\{x, y\} \rightarrow$ L. dependent

if $\{x, y\} \rightarrow$ linearly dependent

$\Rightarrow \quad \alpha x + \beta y = 0$, where $\alpha != 0$ or $\beta != 0$ or both.

$\Rightarrow \quad x = \frac{-\beta}{\alpha} \cdot y$, if $\alpha != 0$

if $\alpha = 0$, $\beta y = 0$, $\beta \neq 0$
$\alpha \cdot x = 0$ $\quad \Rightarrow y = 0 = \alpha \cdot x$

THEOREM : Non-zero vectors $V_1, V_2, \ldots, V_n$ are linearly dependent iff. at least one vector in this set is a L. combo of the preceeding VECTORS.

$\exists K, \quad 1 < K \leq n$ s.t.

$$V_K = \alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_{K-1} V_{K-1} \quad (*)$$

Proof : assume (*) holds.

$\Rightarrow \alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_{K-1} V_{K-1} - 1 \cdot V_K + 0 \cdot V_{K+1} + \cdots + 0 \cdot V_n = 0$
$\Rightarrow \{\alpha_1, \alpha_2, \cdots \alpha_n\} \rightarrow$ L. dependent.

Converse $\rightarrow$ let $\{v_1, v_2, \ldots, v_n\} \rightarrow$ linearly dep.

let $K \rightarrow$ largest index in i s.t. $\alpha_K = 0$, $1 < K \leq n$

let $\alpha_i = 0 \;\; \forall i \in \{k+1, k+2, \cdots, n\}$

$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$

$\Rightarrow \quad v^k = -\frac{1}{\alpha_k}(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{k-1} v_{k-1})$

$\Rightarrow \quad v^k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{k-1} v_{k-1}$

↳ L. combo of vectors preceeding it //.

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \in \mathbb{R}^2$

seeking $\alpha, \beta \in \mathbb{R}$ s.t.

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Rightarrow \quad \alpha + \beta = 1 \qquad \Rightarrow \quad \beta = 2$ & $\alpha = -1$

$\alpha + 2\beta = 3 \qquad$ $\therefore$, $\exists$ at least 1 non-zero scalar s.t.

$\alpha v_1 + \beta v_2 = v_3$

$\therefore$, L. Dep

DEFN: A finite subset $S \subseteq V$ ($\{v_1, v_2, \cdots, v_k\}$) is called a spanning set of V if for every $v \in V$ there are scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$, $\alpha_i \in \mathbb{F}$ s.t.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$.

let $\quad S = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\} \subset \mathbb{R}^3$

take any $X \in \mathbb{R}^3$. $X = (x_1, x_2, x_3)$. then

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

DEFN: let $V \to$ vector space. A subset B of V is called basis for V if:
a) $B \to$ linearly indep.
b) $B \to$ spans V.

Theorem: let $X = \{x_1, x_2, \cdots, x_m\}$ &
$Y = \{y_1, y_2, \cdots, y_n\}$ such that X is linearly independent & Y is spanning subset in V. then, $m \leq n$.

Proof: consider $\{x_m, y_1, y_2, \cdots y_n\}$

$y_1, y_2, \cdots, y_n \to$ form spanning subset of V
$\therefore$, $x_m \to$ L.C. of $y_1, y_2, \cdots, y_n$
$\Rightarrow \{x_m, y_1, y_2, \cdots, y_n\} \to$ Linear dep. subset
$\Rightarrow \exists$ one y_j thats a Linear combo. of vectors preceeding it (including x_m)
let $Y_1 = \{x_m, y_1, y_2, \cdots, y_n\} - \{y_j\}$

$\to$ has n vectors
$\to$ spans V.

* take any $x \in V$
* $x \to$ linear combo of $y_1, y_2, \cdots, y_n$
* in Y_1, y_j is not present, BUT! y_j is a linear combo of

$x_m, y_1, y_2, \cdots y_{j-1}$, hence
can be replaced by them in
the expression of x
* true for arbitrary x, hence
true for all. $Y_1 \rightarrow$ spans V.

consider $\{x_{m-1}, x_m, y_1, y_2, \cdots, y_n\} - \{y_j\}$

$\hookrightarrow$ linearly dep. ($x_{m-1} \in V$, vectors in Y
can represent it in linear combo, but
there's a y_j that's absent, which can be
replaced by a linear combo of $x_m, y_1, y_2, \cdots y_{j-1}$

now, $Y_2 = \{x_{m-1}, x_m, y_1, y_2, \cdots, y_n\} - \{y_j, y_r\}$

$\hookrightarrow$ has 'n' vectors
$\hookrightarrow$ by same argument for Y_1, Y_2 also
spans V.

repeat procedure for m times. then,
$Y_m \rightarrow$ has n vectors & spans V.
Also, $X \subseteq Y_m$. $\therefore$, $m \leq n$.

THEOREM: let S and T be basis for V with
$S = \{u_1, u_2, \cdots, u_m\}$ and
$T = \{v_1, v_2, \cdots, v_n\}$. Then $m = n$.

PROOF: given $S \rightarrow$ basis, $S \rightarrow$ linearly independent.
$T \rightarrow$ basis, $T \rightarrow$ spanning set.
then, $m \leq n$

given $T \to$ basis, $T \to$ linearly independent.
$S \to$ basis, $S \to$ spanning set.
then, $n \leq m$.

$\Rightarrow \quad m = n$ //

<u>DEFN</u> : let $V \to$ vector space with a finite basis. The dimension of V is the no. of elements in any basis of V. If $V \to$ has no finite basis, then $V \to$ infinite dimension.

$V = \{0\} \to$ zero-dimensional $\longrightarrow$ DOESN'T HAVE a basis, as zero can only be formed by Linearly dependent sets!

$\mathbb{R}^n$

$B = \{e_1, e_2, \cdots, e_n\}$, $\dim(\mathbb{R}^n) = n$.
$\hookrightarrow$ STANDARD BASIS.

<u>THEOREM</u> : let $V \to$ finite dimensional vector space (i.e., V has a finite basis). Let S be a subset of V. Then

a) if S is linearly independent, then $|S| \leq \dim V$

b) if $\dim V < |S|$, then $S \to$ linearly dependent.

c) if $S = \{u_1, u_2, \cdots, u_n\}$, $n = \dim V$ is linearly independent, then $S \to$ spanning set of V.
$\hookrightarrow$ BASIS

Proof :

a) let $\dim V = n \rightarrow$ basis of V has n elements.

$S = \{v_1, v_2, \cdots, v_m\} \rightarrow$ linearly indep.

then, $m \leq n$ (elements in any basis of V)

$\Rightarrow m \leq n$

comes from $X \rightarrow$ linearly indep
$Y \rightarrow$ spanning set set proof.
if we keep replacing y_js with x_is, we get that cardinality of X (linearly indep) is lesser than / equal to spanning set.

$\underbrace{|S|}_{\text{linearly indep}} \leq \underbrace{\dim V}_{\text{spanning set}}$

(b) if $|S| > \dim V \rightarrow$ L. dep. (Contrapositive of (a))

(c) $S = \{u_1, u_2, \cdots, u_n\}$, where $n = \dim V$
& $S \rightarrow$ linearly indep.

if $sp(S) \mathrel{!}= V$, $\exists\, v \in V$ s.t. $v \notin sp(S)$.
then the set,

$S \cup \{x\} = \{u_1, u_2, \cdots, u_n, x\} \rightarrow$ is linearly independent, as u_is are L. indep, & x can't be rep. using u_is as per assumption.

$|S \cup \{x\}| = n+1 \rightarrow$ linearly indep.

but : as per (a), the cardinality of a linearly indep. set can be AT MOST $\dim V = n$.

$\therefore$, assumption is false. $\therefore$, $Sp(S) = V$.

S is BASIS.

Theorem : let $W \to$ subspace of a finite dimensional vector space V. Then, $\dim W \leq \dim V$.

Proof : let $B \to$ basis of W. Then B is a linearly independent set in W. This means that B is linearly independent in V.
W.K.t. cardinality of any linearly indep. set in V CAN'T exceed $\dim V$ (the same X, Y set proof as before). $\Rightarrow |B| \leq \dim V$

$\therefore$, $\dim V \leq \dim W$.

Theorem : let $A \in \mathbb{R}^{n \times n}$. if the columns of A are linearly indep., then A is invertible.

Proof : let $A_1, A_2, \ldots, A_n$ denote the cols. of matrix A, i.e, $A = \{A_1, A_2, \ldots, A_n\}$.
A matrix is invertible iff. the non-homogenous eqn. $AX = b$ has a soln. $\forall b$.

Consider the system :

$$AX_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

claim $\longrightarrow$ above system has a soln.

Linearly Independent subset of V having cardinality equal to $\dim V$ must be a basis of V!

$\therefore$, set of cols. of $A \to$ basis of $\mathbb{R}^n$.

$\hookrightarrow$ use it to write e_is

e_1: $\exists$ scalars $x_1, x_2, \ldots x_n \in \mathbb{R}$ s.t.

$$e_1 = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n.$$

$$= (A_1\ A_2\ A_3 \cdots A_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= AX_1$$

$$\Rightarrow AX_1 = e_1$$

Similarly, $AX_i = e_i$ have a soln. $\forall\, i \in 2 \leq i \leq n$

Let $X = (X_1, X_2, \ldots, X_n) \in \mathbb{R}^{n \times n}$

consider
$$\begin{aligned} AX &= A(X_1, X_2, \ldots, X_n) \\ &= (AX_1, AX_2, \ldots, AX_n) \\ &= (e_1, e_2, \ldots, e_n) \\ &= I. \end{aligned}$$

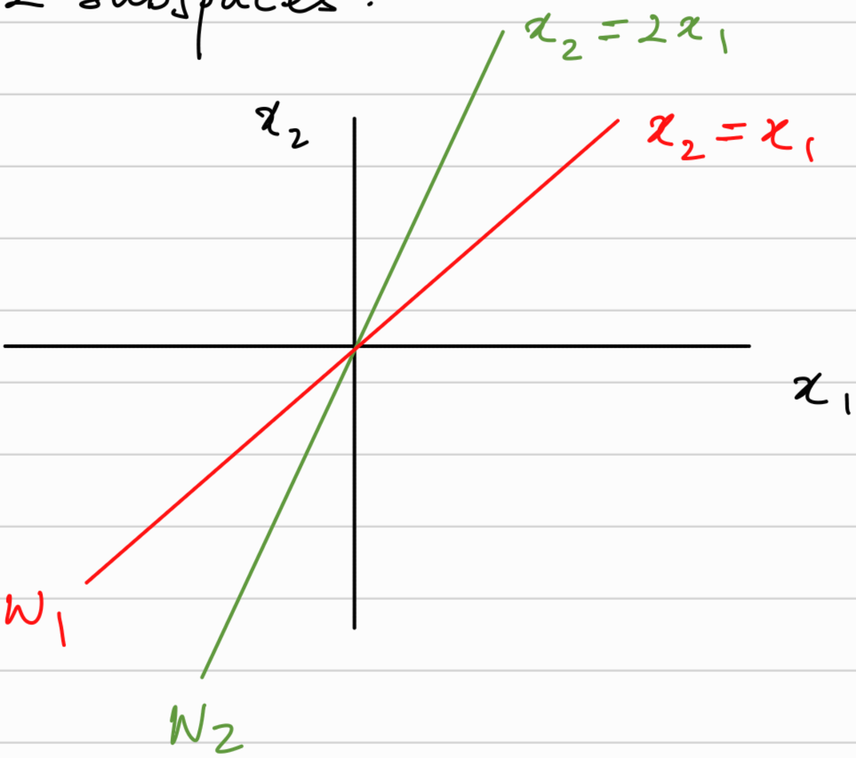
$\therefore$, $AX = I$

$\Rightarrow A \to$ has right-inverse

Since A is also square, A is invertible.

SUM of 2 subspaces:

$V = \mathbb{R}^2$



$W_1, W_2 \rightarrow$ subspaces (any line passing thru origin is a subspace)

$\dim(W_1) = 1, \quad \dim(W_2) = 1$

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2$ here

$= 1 + 1$

$= 2.$

$\Rightarrow W_1 + W_2 = \mathbb{R}^2$?

* Show that $W_1 + W_2$ has a basis consisting of 2 elec.
* show that $W_1 + W_2$ is a linearly indep. set with 2 elements.

take $u \in W_1$ & $v \in W_2$.

* $(u, v) \rightarrow$ linearly indep. as they lie on diff. lines.
* (u, v) span $W_1 + W_2$

$\ast \quad (u, v)$ span $W_1 + W_2$

$\hookrightarrow \alpha \in W_1 = a \cdot u \quad , \quad \beta \in W_2 = b \cdot v$

$\therefore, \quad \alpha + \beta = a \cdot u + b \cdot v$

$\Rightarrow \quad W_1 + W_2 = \mathbb{R}^2 \quad (???)$

THEOREM: let W_1, W_2 be subspaces of a finite dimensional vector space V s.t. $W_1 \cap W_2 = \{0\}$. Then,

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2$$

Proof: let $B_1 = \{u_1, u_2, \ldots, u_\ell\}$ be a basis of W_1
$B_2 = \{v_1, v_2, \ldots, v_K\}$ be a basis of W_2

$\dim(W_1) = \ell, \quad \dim(W_2) = K$

claim: $B = B_1 \cup B_2 = \{u_1, u_2, \ldots, u_\ell, v_1, v_2, \ldots, v_K\}$ is the basis of $W_1 + W_2$.

$\ast$ let $z \in W_1 + W_2$. We must show that $z \in Sp(B)$

$\Rightarrow \exists x \in W_1$ & $\exists y \in W_2$ s.t.

$z = x + y$

$x = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_\ell u_\ell \ ; \ u_1, u_2, \ldots u_\ell \in W_1$

$y = \gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_K v_K \ ; \ v_1, v_2, \ldots, v_K \in W_2$.

$z = x + y$

$= \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_\ell u_\ell + \gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_K v_K$

$\Rightarrow z \in Sp(B)$

* Show that B → linearly indep.

consider :

$$\underbrace{(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_\ell u_\ell)}_{u \in W_1} + \underbrace{(\delta_1 v_1 + \delta v_2 + \cdots + \delta_K v_K)}_{v \in W_2} = 0$$

$\Rightarrow \quad u + v = 0$

$\Rightarrow \quad u = -v$

* this means u & v are linearly dep, & lie on same line

$\Rightarrow (u, v) \in W_1 \cap W_2 = \{0\}$

$\Rightarrow (u, v) = (0, 0)$

as $u = 0$,

$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_\ell u_\ell = 0$

$\Rightarrow \alpha_1, \alpha_2, \cdots, \alpha_\ell = 0.$

Sim.,

$\delta_1, \delta_2, \cdots \delta_K = 0$

$\therefore$, B is the basis of $W_1 + W_2$

$B = \{u_1, u_2, \cdots, u_\ell, v_1, v_2, \cdots, v_K\}$

$\dim(W_1 + W_2) = \ell + K$

$= \dim(W_1) + \dim(W_2)$

THEOREM : let V → finite dimensional vector space. and S → linearly indep. subset of V. Then S can be extended to a basis of V.

$\Rightarrow \exists B$ of V s.t. $S \subseteq B$.

Proof : * if $sp(S) = V$ → then S is a spanning

subset of $V \rightarrow$ S is also linearly indep,
hence $S \rightarrow$ also a basis.

* if $sp(S) \,!= V \rightarrow$ then $\exists\, x_1 \in V$ s.t. $x_1 \notin sp(S)$

let $S = \{u_1, u_2, \cdots, u_\ell\}$

$S \cup \{x\} = \{u_1, u_2, \cdots, u_\ell, x_1\} = S_1$

$\hookrightarrow$ linearly indep. set as $u_1, u_2, \cdots, u_\ell$ are linearly indep. & $x_1 \notin sp(S)$, meaning x_1 isn't a L.C. of ele of S.

$\hookrightarrow$ if $sp(S_1) = V$, then S_1 is a basis & $S \subset S_1$. else, repeat!

after $\dim V - \dim sp(S)$

The above process must terminate after atmost $\dim V$ steps, as $V \rightarrow$ finite dim &

$\dim sp(S) < \dim V$ $\{\dim W \leq \dim V$, if $W_1 \rightarrow$ subspace of $V\}$

at the end we get a basis S_K of V s.t. $S \subset S_K$.

$sp(S) \,!= V$

THEOREM : let W_1, W_2 be subspaces of a finite dimensional vector space V. Then,

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Proof : $W_1 \cap W_2 \rightarrow$ subspace of W_1 & W_2

* let $\{y_1, y_2, \cdots, y_\ell\}$ be a basis of $W_1 \cap W_2$
* let $\{y_1, y_2, \cdots, y_\ell, u_1, u_2, \cdots, u_s\}$ be a basis of W_1

* let $\{y_1, y_2, \cdots, y_\ell, v_1, v_2, \cdots, v_K\}$ be a basis of W_2

We can do this because $y_{1-\ell}$ are linearly indep. basis of W_1 & W_2 can hence be written like this {check prev proof - done same thing | proved it works}

* let $\{y_1, y_2, \cdots, y_\ell, u_1, u_2, \cdots, u_s, v_1, v_2, \cdots, v_s\}$
$= B$.

if B be basis of $W_1 + W_2$, then:

$$\dim(W_1 + W_2) = \ell + s + K = (\ell + s) + (\ell + K) - \ell$$
$$\dim(W_1) = \ell + s \qquad = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$$
$$\dim(W_2) = \ell + K$$
$$\dim(W_1 \cap W_2) = \ell$$

* show that $sp(B) = W_1 + W_2$ (done before as part of another proof)

* show that B is linearly indep.

$$\underbrace{(\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_\ell y_\ell)}_{y} + \underbrace{(\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_s u_s)}_{u} + \underbrace{(\gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_K v_K)}_{v} = 0$$

$y \in W_1 \cap W_2$ $\qquad$ $u \in W_1$ $\qquad$ $v \in W_2$

$\Rightarrow \quad y + u + v = 0$

$\Rightarrow \quad y + u = -v$

$y \in W_1, \; u \in W_1 \Rightarrow y + u \in W_1$

$v \in W_2, \; -v \in W_2 \longrightarrow v \in W_1 \cap W_2$

$v = \delta_1 y_1 + \cdots + \delta_\ell y_\ell$

$\Rightarrow \delta_1 y_1 + \cdots + \delta_\ell y_\ell - v = 0$

$\Rightarrow (\delta_1 y_1 + \cdots + \delta_\ell y_\ell) - (\gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_K v_K) = 0$

$$\underbrace{y_{1-\ell} \qquad\qquad v_{1-K}}$$

L.C. of basis of W_2

$\Rightarrow$ all coeffs $= 0$ for eqn. to be TRUE.

$\Rightarrow \gamma_{1-K} = 0$

$\Rightarrow v = 0$

$\Rightarrow y + u = 0$

$\Rightarrow \quad y = -u$

$u \in W_1$

$y \in W_1 \cap W_2$

$\Rightarrow (\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_\ell y_\ell) + (\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_s u_s) = 0$

$$\underbrace{y_{1-\ell} \qquad\qquad u_{1-s}}$$

BASIS of W_1

$\Rightarrow$ all coeffs are 0 for eqn. to be TRUE

$$\alpha_1 = \alpha_2 = \cdots = \alpha_\ell = \beta_1 = \beta_2 = \cdots = \beta_s = 0$$

$\Rightarrow$ B is linearly indep

$\therefore$, $Sp(B) = W_1 + W_2$.

LINEAR TRANSFORMATION : $T: V \to W$ is a function that satisfies -

* $T(u+v) = T(u) + T(v) \quad \forall u,v \in V$
* $T(\alpha u) = \alpha T(u) \quad \forall \alpha \in \mathbb{R}$ & $\forall u \in V$

1. $O: V \to W$ is defined as $\underset{\text{transformation}}{O}(u) = \underset{\text{vector}}{0} \quad \forall u \in V$

2. $I: V \to W$ is defined as $I(u) = u \quad \forall u \in V$

3. $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined as $T((x_1, x_2)) = (x_1, x_2, x_1 - x_2)$
$x \in \mathbb{R}^2$

$x = (x_1, x_2)$
$y = (y_1, y_2)$

$$T(x+y) = T((x_1, x_2) + (y_1, y_2)) = T((x_1+y_1), (x_2+y_2))$$

$T(x) + T(y)$
$= (x_1, x_2, x_1 - x_2) + (y_1 + y_2 + y_1 - y_2)$ $= (x_1+y_1, x_2+y_2, \ldots$

$x_1+y_1-x_2-y_2$

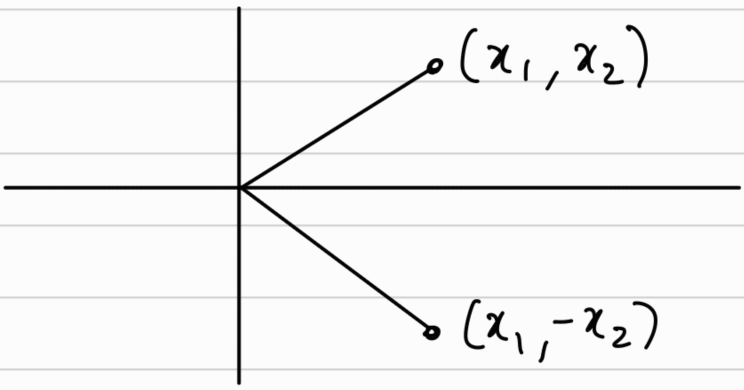
$= (x_1+y_1,\ x_2+y_2,\ x_1+y_1-x_2-y_2)$

$T(x+y) = T(x)+T(y)$ ✓

$$T(\alpha x) = T((\alpha x_1, \alpha x_2))$$
$$= (\alpha x_1, \alpha x_2, \alpha(x_1-x_2))$$
$$= \alpha(x_1, x_2, x_1-x_2)$$
$$= \alpha T((x_1, x_2, x_1-x_2))$$
$$= \alpha T(x)$$

$\therefore$, $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T(x) = (x_1, x_2, x-x_2)$
$x \in \mathbb{R}^2$ ↳ is a linear trfr.

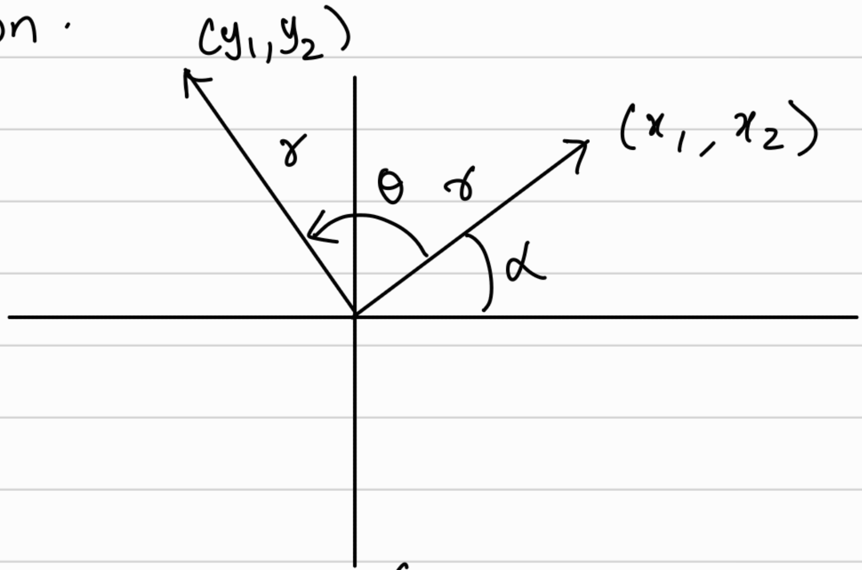
4. $T: \mathbb{R}^2 \to \mathbb{R}^2$; $T((x_1,x_2)) = (x_1, -x_2)$, $x \in \mathbb{R}^2$



the above $T \to$ is a linear transformation.
if u think abt it $\to$ $T(x+y)$ is like saying hey I have a resultant vector, reflect it abt x-axis. This is equivalent to saying hey I have 2 vectors x & y, reflect them & then add. as far as $T(\alpha x)$ is concerned, its magnification ; u can either magnify & then reflect / reflect & then magnify ...

reflect / reflect & then magnify, as the trfr. doesn't change magnitude of vector.

5. Rotation.



$$y_1 = r\cos(\theta+\alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$$
$$= x_1\cos\theta - x_2\sin\theta$$

$$y_2 = r\sin(\theta+\alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$= r\cos\alpha\sin\theta + r\sin\alpha\cos\theta$$
$$= x_1\sin\theta + x_2\cos\theta$$

$$\therefore \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_\theta x$$

$T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(x) = A_\theta x \ , \quad x \in \mathbb{R}^2$

↳ again, intuitively, T is a linear trfmn. $T(x+y)$ is like saying rotate after I add 2 vectors. $T(x)+T(y)$ is rotate each vector & then add. Both are equal logically → $\angle$'s b/w the 2 vectors won't change, whether I do trfmn. / not. Sim. for $\alpha T(x) / T(\alpha x)$

$$T(x+y) = T\left((x_1+y_1, x_2+y_2)\right)$$

$$= A_\theta(x_1+y_1, x_2+y_2)$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right]$$

$$= A_\theta x + A_\theta y$$

$$= T(x) + T(y)$$

$$T(\alpha x) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = \alpha A_\theta x$$

$$= \alpha T(x)$$

5. $T : \mathbb{R}^m \to \mathbb{R}^n$, $m \geq n$.

$$T((x_1, x_2, \ldots, x_m)) = (x_1, x_2, \ldots, x_n), \quad x \in \mathbb{R}^n$$

$T \to$ natural projection on $\mathbb{R}_m$.
$\hookrightarrow$ linear

6. $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$T((x_1, x_2, x_3)) = (x_1, x_2, 0), \quad x \in \mathbb{R}^3$$

$T \to$ projection operator (linear)

7. $T : \mathbb{R}^m \to \mathbb{R}^n$, $m \leq n$

$$T((x_1, x_2, \dots, x_m)) = (x_1, x_2, \dots, x_m, \underbrace{0, 0, \dots 0}_{n-m \text{ comps.}})$$

$x \in \mathbb{R}^m$.

$T \to$ natural inclusion (linear)

8. $T : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{n \times m}$: $T(A) = A^t$; $A \in \mathbb{R}^{m \times n}$

$$T(A+B) = (A+B)^t = A^t + B^t = T(A) + T(B)$$
$$T(\alpha A) = (\alpha A)^t = \alpha \cdot A^t = \alpha T(A)$$

9. $T : C'([0,1]) \to C([0,1])$: $T(f) = f'$, $f \in C([0,1])$

real space

$T \to$ differential operator (linear)

10. $T : C([0,1]) \to \mathbb{R}$

$$T(f) = \int_0^1 f(t)\,dt \; ; \quad f \in C([0,1])$$

$T \to$ integral transformation.

11. $T : \mathbb{R}^n \to \mathbb{R}^m$: $T(x) = Ax$, where $A \in \mathbb{R}^{m \times n}$ & $x \in \mathbb{R}^n$

examples $\to$ zero tsfr, identity, reflection, rotation, natural projection, inclusion, projection operator ARE ALL encapsulated in this ex. 11.

$T \to$ linear. (follows by matrix multipln.)

if $\exists$ a Linear trfx. T b/w finite dimensional vector spaces, then $\exists$ a matrix A which obeys:

$T(X) = AX$.

converse is TRUE

12. $T : \mathbb{R} \to \mathbb{R}$, $\quad y = T(x) = x + 1$, $\quad x \in \mathbb{R}$

$\hookrightarrow$ NOT a linear trfmn.

$$T(x+y) = T(x) + T(y)$$
$$T(mx + c + m'x + c')$$
$$= T((m+m')x + (c+c'))$$
$$= (m+m')x + (c+c') + 1$$
$$= mx + m'x + c + c' + 1$$
$$= (mx + c) + (m'x + c') + 1$$
$$\neq T(x) + T(y)$$

trfrmn. from line to line need not be linear.

if line thru origin, then yes.

$T(X) = mX \to$ would work.

THEOREM : let $T : V \to W$ be linear. Then,

a) $T(0) = 0$

b) $T(u \pm v) = T(u) \pm T(v)$, $u, v \in V$

c) $T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_K u_K)$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_K T(u_K)$$

$\alpha_1, \alpha_2, \ldots, \alpha_K \in \mathbb{R}$
$u_1, u_2, \ldots, u_K \in V$

<u>Proof</u> :

* let $u = T(0)$

$$= T(0+0)$$
$$u = T(0) + T(0)$$
$$u = u + u$$
$$\Rightarrow u = 0.$$
$$\Rightarrow T(0) = 0$$

* $T(u - v) = T(u + (-1)v)$

$$= T(u) + (-1)T(v)$$
$$= T(u) - T(v)$$

* $T(\alpha_1 u_1 + \underbrace{\alpha_2 u_2 + \cdots + \alpha_K u_K}_{W_1})$

$$= T(\alpha_1 u_1 + W_1)$$
$$= \alpha_1 T(u_1) + T(W_1)$$
$$= \alpha_1 T(u_1) + T(\alpha_2 u_2 + \cdots + \alpha_K u_K)$$
$$= \alpha_1 T(u_1) + T(\alpha_2 u_2 + W_2)$$
$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + T(W_2)$$

Keep repeating

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_{K-2} T(u_{K-2}) + T(\alpha_{K-1} u_{K-1} + \alpha_K u_K)$$
$$= \alpha_1 T(u_1) + \cdots + \alpha_{K-1} u_{K-1} + \alpha_K u_K$$

THEOREM : let $V \rightarrow$ finite dimensional vector space 4

$T_1, T_2 : V \to W$ be linear maps.

let $B = \{u_1, u_2, \dots, u_n\}$ be basis of V

if:

$$T_1(u_i) = T_2(u_i) \quad \forall i \in \{1, 2, \dots, n\}$$

then $T_1 = T_2$.

<u>proof</u>: let $x \in V$.

$$\Rightarrow x = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n.$$

$$\begin{aligned}
T_1(x) &= T_1(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) \\
&= \alpha_1 T_1(u_1) + \alpha_2 T_1(u_2) + \dots + \alpha_n T_1(u_n) \\
&= \alpha_1 T_2(u_1) + \alpha_2 T_2(u_2) + \dots + \alpha_n T_2(u_n) \\
&= T_2(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) \\
&= T_2(x)
\end{aligned}$$

$\therefore$, $T_1 = T_2$.

let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be s.t.

$$T((1,0,0)) = (-1,0)$$
$$T((0,1,0)) = (1,1)$$
$$T((0,0,1)) = (0,1)$$

let $x \in \mathbb{R}^3$. $x = (x_1, x_2, x_3)$

$$\begin{aligned}
&= x_1(1,0,0) + x_2(0,1,0) \\
&\quad + x_3(0,0,1) \\
&= x_1 e_1 + x_2 e_2 + x_3 e_3
\end{aligned}$$

$$\begin{aligned}
T(x) &= T(x_1 e_1 + x_2 e_2 + x_3 e_3) \\
&= x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) \\
&= x_1(-1,0) + x_2(1,1) + x_3(0,1) \\
&= (-x_1 + x_2, x_2 + x_3)
\end{aligned}$$

$\therefore$ action of L.T. on basis can determine L.T. completely.

<u>Question:</u>

$B = \{u_1, u_2, \cdots, u_n\}$ be basis of V.

$\{w_1, w_2, \cdots, w_n\}$ be a subset of W.

does there exist a linear map b/w $V \to W$ s.t.
$T(u_i) = w_i \quad , \quad 1 \leq i \leq n$.

take $x \in V$

$\Rightarrow x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$, $\alpha_i \in \mathbb{R}$
$u_i \in B$

the scalars α_is are UNIQUE for $x \in V$, in a FIXED basis

supp. that $x = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$, $\beta_i \in B$

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$

$\Rightarrow (\alpha_1 - \beta_1) u_1 + (\alpha_2 - \beta_2) u_2 + \cdots + (\alpha_n - \beta_n) u_n = 0$

u_is are linearly indep.

$\Rightarrow \quad \alpha_i - \beta_i = 0 \quad \forall i \in \{1, 2, \cdots, n\}$

$\Rightarrow \quad \alpha_i = \beta_i \quad \forall i \in \{1, 2, \cdots, n\}$

let $T: V \to W$ – $T(x) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$

$T \to$ well defined (α_is are unique for an x) – the image for x can't be 2 diff. elements

CAN'T BE 2 diff elements

if $x = y$,

then $f(x) = f(y)$

$T(x) = T(y)$

$x = y \rightarrow$ prove that α_i's are same for both. (same proof from before - linear indep)

$$y = \delta_1 u_1 + \delta_2 u_2 + \cdots + \delta_n u_n$$

$$T(y) = \delta_1 w_1 + \delta_2 w_2 + \cdots + \delta_n w_n$$

$$x + y = (\delta_1 + \alpha_1) u_1 + (\delta_2 + \alpha_2) u_2 + \cdots + (\delta_n + \alpha_n) u_n$$

$$\begin{aligned} T(x+y) &= (\delta_1 + \alpha_1) w_1 + (\delta_2 + \alpha_2) w_2 + \cdots + (\delta_n + \alpha_n) w_n \\ &= (\delta_1 w_1 + \delta_2 w_2 + \cdots + \delta_n w_n) \\ &\quad + (\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n) \\ &= T(y) + T(x) \\ &= T(x) + T(y). \end{aligned}$$

$\therefore$, $T(x+y) = T(x) + T(y)$

$$\begin{aligned} \alpha x &= \alpha(\alpha_1 u_1) + \alpha(\alpha_2 u_2) + \cdots + \alpha(\alpha_n u_n) \\ &= (\alpha\alpha_1) u_1 + (\alpha\alpha_2) u_2 + \cdots + (\alpha\alpha_n) u_n \\ \Rightarrow T(\alpha x) &= (\alpha\alpha_1) w_1 + (\alpha\alpha_2) w_2 + \cdots + (\alpha\alpha_n) w_n \\ &= \alpha(\alpha_1 w_1) + \alpha(\alpha_2 w_2) + \cdots + \alpha(\alpha_n w_n) \\ &= \alpha(\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n) \\ &= \alpha T(x) \end{aligned}$$

$T(u_i) = w_i \quad , 1 \leq i \leq n$

$$u_1 = 1 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n$$

$$\Rightarrow T(u_1) = 1 \cdot w_1 + 0 \cdot w_2 + \cdots + 0 \cdot w_n$$

} LIKE DOING ... PDT ?

$= w_1.$

$\hookrightarrow$ sim. done $\forall\, u_i$ & w_i.

$\therefore$, $T(u_i) = w_i \quad \forall\, i \in \{1, 2, \cdots, n\}$

{DO WE CHOOSE WHAT WE NEED?}

if $S: V \to W$ is linear & $S(u_i) = w_i$, $1 \leq i \leq n$, then $S = T$.

let $S(u_i) = w_i$. WKT $T(u_i) = w_i$

$\therefore$, $\forall\, i \in \{1, 2, \cdots, n\}$, $S(u_i) = w_i$ & $T(u_i) = w_i$

$\therefore$, $S = T$. (is that it?)

Note: $\{w_1, w_2, \cdots, w_n\} \subset W$ $\to$ needn't be a basis

$W \to$ needn't have finite dim

w_is $\to$ could all be equal (or) zero.

<u>DEFN</u>: let $T: V \to W$ be a linear map. The kernel of T (or) the null space of $T \to N(T)$ is defined as:

$$N(T) = \{x \in V : T(x) = 0\} \subseteq V$$

let $x, y \in N(T)$

$\Rightarrow T(x) = 0$ & $T(y) = 0$

$T(x+y) = T(x) + T(y) = 0$

$\Rightarrow T(x+y) = 0 \Rightarrow x+y \in N(T)$

let $x \in N(T)$

$T(\alpha x) = \alpha T(x) = \alpha \cdot 0 = 0.$

$\Rightarrow \alpha x \in N(T)$.

$\therefore$, $N(T) \to$ subspace of V.

DEFN: $T: V \to W$ is linear. The range space of T $R(T)$ is defined as:

$$R(T) = \{ w \in W : w = T(x) \text{ for some } x \in V \} \subseteq W$$

$R(T) \to$ subspace of W.

$w_1, w_2 \in R(T)$

$T(x) = w_1$ & $T(y) = w_2$

$T(x+y) = T(x) + T(y)$

$= w_1 + w_2$

we have shown that $T(x+y) = w_1 + w_2$ meaning $w_1 + w_2 \in R(T)$

Sim. for $\alpha w_1 \in R(T)$

$T(x) = w_1$

$\Rightarrow \alpha T(x) = \alpha w_1$

$\Rightarrow T(\alpha x) = \alpha w_1$

$\Rightarrow \alpha w_1 \in R(T)$

$\therefore$, $R(T) \to$ subspace of W.

1. $O: V \to W$

$N(O) = V$ $\quad \{ O(x) = 0 \ \forall x \in V$

$R(O) = \{0\}$ $\quad \therefore, \forall x \in V,$

$x \in N(O) \}$

2. $T: V \to V$

$$N(I) = \{0\}$$
$$R(I) = V$$

3. $T : \mathbb{R}^3 \to \mathbb{R}^3 \quad : \quad T(x) = (x_1, 0, x_3) \quad , \quad x \in \mathbb{R}^3$

$$N(T) = \{ x \in \mathbb{R}^3 : T(x) = 0 \}$$
$$= \{ x \in \mathbb{R}^3 : (x_1, x_2, x_3) = (0,0,0) \}$$
$$= \{ x \in \mathbb{R}^3 : x_1 = 0 \wedge x_3 = 0 \}$$

$x_2 \in \mathbb{R}$ (arbitrary)

$$= Sp(\{e_2\})$$

$$R(T) = \{ y \in \mathbb{R}^3 : T(x) = y, \text{ for some } x \in \mathbb{R}^3 \}$$
$$= \{ y \in \mathbb{R}^3 : (x_1, 0, x_3) = (y_1, y_2, y_3) \}$$
$$= \{ y \in \mathbb{R}^3 : y_2 = 0 \}$$
$$= Sp(\{e_1, e_3\})$$

$$\dim N(T) + \dim R(T) = \dim V = \dim \mathbb{R}^3$$

(here)

<u>DEFN</u> : $T : V \to W$ be a linear map. T is injective (or) one to one if $T(X) = T(Y) \implies X = Y$.

<u>DEFN</u> : $T : V \to W$ be a linear map. T is surjective (or)

onto if $\forall_{w \in W}, \exists_{x \in V}$ s.t. $T(x) = w$

Obs: 1. $T: V \to W$ is injective $\iff N(T) = \{0\}$

* by rule of linear trfrmn,

$T(0) = 0$.

$\Rightarrow 0 \in V \equiv T(0)$

$\Rightarrow$ NO OTHER $x \in V$ can take 0, by injective condn.

$\Rightarrow$ injection $\longrightarrow N(T) = \{0\}$

* let $T(x) = T(y)$

$\Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x-y) = 0$

$\Rightarrow x - y \in N(T)$

$N(T) = \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow \underline{\underline{x = y}}$. $\Rightarrow$ injection //

2. $T: V \to W$ is surjective $\iff R(T) = W$.

* let T be surjective

$\forall_{y \in W}\ \exists_{x \in V}$ s.t. $T(x) = y$

$\Rightarrow \forall_{y \in W}, y \in R(T)$

$\Rightarrow R(T) = W$.

* let $R(T) = W$

$\Rightarrow \forall_{y \in W}, y \in R(T)$

$\Rightarrow \forall_{y \in W}, \exists_{x \in V}$ s.t. $T(x) = y$

$\Rightarrow T \Rightarrow$ surjective

$\Rightarrow T \rightarrow$ surjective.

let $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x) = (x_1 + x_2, x_1 - x_2)$, $x \in \mathbb{R}^3$

$$N(T) = \{x \in \mathbb{R}^3 : T(x) = 0\}$$
$$= \{x \in \mathbb{R}^3 : (x_1 + x_2, x_1 - x_2) = (0,0)\}$$
$$= \{x \in \mathbb{R}^3 : x_1 = 0 \wedge x_2 = 0, \underbrace{x_3 \in \mathbb{R}}_{\hookrightarrow \text{arb.}}\}$$
$$= Sp(\{e^3\})$$

$N(T) \mathrel{!=} \{0\}$, hence $T \rightarrow$ not one-one.

$$R(T) = \{y \in \mathbb{R}^2 : T(x) = y \text{ for some } x$$
$$= \{y \in \mathbb{R}^2 : (x_1 - x_2, x_1 + x_2) = (y_1, y_2), x \in \mathbb{R}^3\}$$

if I can show that $e_1, e_2 \in W$ have pre-images, then I can prove that any vector $\in W$ will have its pre-image, thus the range becomes the whole space.
$\therefore$, $T \rightarrow$ onto.

$\star$ find if

$$\left.\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 0 \end{aligned}\right\} a) \rightarrow \text{can hold true. YES.} \quad x_1 = x_2 = 1/2$$

$$\left.\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 1 \end{aligned}\right\} b) \rightarrow \text{can hold true. YES.} \quad x_1 = -x_2 = 1/2$$

a) $\equiv e_1$ b) $\equiv e_2$. since $e_1, e_2 \in \mathbb{R}(T)$, any vector $\in W$ can belong to $R(T)$ - Why?

$T(x) = e_1$

$T(y) = e_2$

check if $u \in W \in R(T)$

$u = \alpha e_1 + \beta e_2$

$= \alpha T(x) + \beta T(y)$

$= T(\alpha x) + T(\beta y)$

$= T(\alpha x + \beta y)$

$= T(z)$

$\therefore$, $\exists\, z \in V$ s.t.

$T(z) = u$

$\Rightarrow u \in R(T)$.

$\Rightarrow \forall u \in W \ \exists z \in V$

s.t.

$T(z) = u$

$\Rightarrow \forall u \in W$, $u \in R(T)$

$\therefore$, $\underline{\underline{R(T) = W}}$

5. $T: \mathbb{R}^2 \to \mathbb{R}^3$; $T(x) = (x_1 + x_2, x_1 - x_2, 0)$, $x \in \mathbb{R}^2$

$T(x) = 0 \Rightarrow x_1 + x_2 = 0$, $x_1 - x_2 = 0$, $0 = 0$

$\Rightarrow \underline{\underline{x_1 = x_2 = 0}}$.

$\therefore$, $x = 0 \Rightarrow T \to$ injective

$\mathbb{R}^3$ contains (y_1, y_2, y_3) s.t. $y_3 \neq 0$, but $R(T)$ doesn't contain such eles. $\therefore$, $R(T) != W \Rightarrow T \to$ not surjective.

$e_3 \in R(T)$.

6. $T: \mathbb{R}^2 \to \mathbb{R}^2$; $T(x) = (x_1 + x_2, x_1 - x_2)$, $x \in \mathbb{R}^2$.

$\hookrightarrow$ BOTH INJECTIVE & SURJECTIVE (T is ONTO)

surjective proof:

RHS vector $= e_1$. $(x_1 + x_2, x_1 - x_2) = e_1$ has a soln.

LHS vector $= e_2$. $(x_1 + x_2, x_1 - x_2) = e_2$ has a soln.

$\left.\begin{array}{l} AX = e_1 \\ AX = e_2 \end{array}\right\} \to$ prove that solns. exist for each of e_1 & e_2. if they do, then $AX = b$ will have a soln. for any b.

T is one-one, meaning $AX = 0$ has 0 as its ONLY soln. A → square matrix

$\Rightarrow A \to$ invertible

$\Rightarrow AX = b$ for any b has a soln.

$AX = 0 \to X = 0$ is only soln → A is invertible
$\therefore$, $AX = b$ has soln $\forall$ b.

RANK - NULLITY DIMENSION THEOREM

DEFN :

let $T: V \to W$ with V being finite dimensional the rank of T is the dimensions of range of T (r)

$r = \dim R(T)$

the nullity of T (n) is the dimension of the null space of T.

$\eta = \dim N(T)$

RND Theorem : Let $V \to$ finite dimensional &

$T : V \to W$ (no constraints on W)

is linear. Then: $\gamma + \eta = \dim V$

Proof: $N(T) \subseteq V$ is of finite dimension.

Let $\{u_1, u_2, \ldots u_\eta\}$ be the basis of $N(T)$

the set mentioned is linearly indep., & can be extended to the basis of V (same proof as linearly indep set, spanning set proof

- keep including independent vectors from V into the set, until $Sp(\text{set}) = V$)

$\{u_1, u_2, \ldots, u_\eta, v_1, v_2, \ldots, v_\ell\}$

$\Rightarrow \dim V = \eta + \ell$

$\eta = \dim N(T) = \text{nullity}$

now, show that $\ell = \dim R(T) = \gamma = \text{rank}$

$\hookrightarrow$ use $\{v_1, v_2, \ldots, v_\ell\}$ to construct basis of $R(T) \Rightarrow \ell = \text{rank } T$.

* let $y \in R(T)$

$$\Rightarrow \exists_{x \in V} \text{ s.t. } y = T(x)$$

$$= T(\alpha_1 u_1 + \cdots + \alpha_\eta u_\eta + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_\ell v_\ell)$$

$$= \alpha_1 T(u_1) + \cdots + \alpha_\eta T(u_\eta) + \beta_1 T(v_1) + \cdots + \beta_\ell T(v_\ell)$$

$= u_1, u_2, \dots, u_\eta \in N(T)$

$\Rightarrow T(u_i) = 0$

$\Rightarrow y = \beta_1 T(v_1) + \beta_2 T(v_2) + \dots + \beta_\ell T(v_\ell)$

$\in Sp\{T(v_1), T(v_2), \dots, T(v_\ell)\}$

$\therefore, \{T(V_1), T(V_2), \dots, T(V_\ell)\}$

$\hookrightarrow$ spanning set of $\underline{\underline{R(T)}}$

Show that $\{T(V_1), T(V_2), \dots, T(V_\ell)\}$ is linearly indep.

$\alpha_1 T(V_1) + \alpha_2 T(V_2) + \dots + \alpha_\ell T(V_\ell) = 0$

$\Rightarrow T(\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_\ell V_\ell) = 0$

$V \in N(T)$

$\downarrow$

$\underline{\underline{V = 0}}$

$\hookrightarrow \Rightarrow \alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_\ell V_\ell = 0$

w.k.t $V_1, V_2, \dots, V_\ell$ are linearly indep, as they were added while extending basis of $N(T)$ to basis of V. Subset of linearly indep set is linearly indep.

WE CAN'T TELL THIS. PROVE ONCE MORE LATER

$\Rightarrow \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\ell = 0$ for it to be 0.

$\Rightarrow \alpha_1 T(V_1) + \alpha_2 T(V_2) + \dots + \alpha_\ell T(V_\ell) = 0$ if $\alpha_1, \alpha_2, \dots, \alpha_\ell = 0$

$\Rightarrow T(V_1), T(V_2), \dots, T(V_\ell) \rightarrow$ linearly indep.

$\therefore, \{T(V_1), T(V_2), \dots, T(V_\ell)\} \rightarrow$ basis of $R(T)$

$\Rightarrow \dim R(T) = \ell \Rightarrow \text{rank} = \ell$.

$\therefore, \dim V = \eta + \ell = \dim N(T) + \dim R(T)$

$\Rightarrow \boxed{\dim V = \text{nullity} + \text{rank}}$

COROLLARY 1 : $\dim V < \infty$, $\dim W = \dim V$

$T : V \to W$ is linear. Then,

T is injective $\Longleftrightarrow$ T is surjective.

proof :

* $T \to$ injective $\Rightarrow \dim N(T) = 0$

$\dim V = \dim N(T) + \dim R(T)$

$\dim V = 0 + \dim R(T)$

$\Rightarrow \dim V = \dim R(T)$

$\Rightarrow T \to$ surjective

* $T \to$ surjective $\Rightarrow \dim R(T) = \dim V$

$\Rightarrow \dim V = \dim N(T) + \dim R(T)$

$\Rightarrow \dim V - \dim R(T) = \dim N(T)$

$\Rightarrow \boxed{\dim N(T) = 0}$

COROLLARY 2 : $\dim V = n$, $\dim W = m$,

$T : V \to W$ is linear.

if $n > m$ then T is not injective.

$m > n$ then T is not surjective.

* PBC :

if $T \to$ injective, then WKT.

$N(T) = \{0\} \Rightarrow \eta = 0$

$\Rightarrow \gamma = \dim V = n$

$\Rightarrow \gamma = n$

$\Rightarrow \dim R(T) = n$

$\because R(T)$ is a subspace of W,

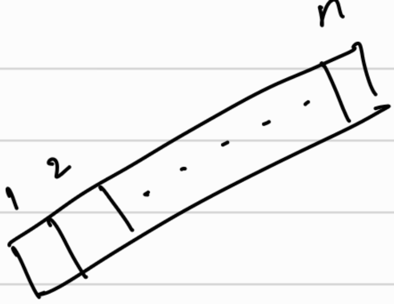
$\dim R(T) \leq \dim W$

$\Rightarrow n \leq m$

$\Rightarrow$ for T to be injective, $n \leq m$

So, if $n > m$, T is NOT injective.

if $T \to$ surjective, then :



$R(T) = W$

$\Rightarrow \dim R(T) = \dim W$

$\Rightarrow \dim R(T) = r = m.$

$\dim V = \dim N(T) + \dim R(T)$

$n = \dim N(T) + m$

$\Rightarrow n - m = \dim N(T) \geq 0$

$\Rightarrow n \geq m$

$\Rightarrow$ for $T \to$ surjective,

$n \geq m$.

$\therefore$, if $m > n$, $T \to$ not surjective.

<u>Lemma</u> : let $T : V \to W$ be linear & bijective.
Then $T^{-1} : V \to W$ exists. p.t. $T^{-1} \to$ linear.

<u>proof</u> :

* let $x, y \in W$

$T^{-1}(x+y) = z, \quad z \in V$

$\Rightarrow T(T^{-1}(x+y)) = T(z)$

$\Rightarrow x + y = T(z)$

$x = T(u), \quad y = T(v)$

$\Rightarrow T(u) + T(v) = T(z)$

$\Rightarrow T(u+v) = T(z)$

$\Rightarrow u + v = z$

$\Rightarrow T^{-1}(x) + T^{-1}(y) = z = T^{-1}(x+y)$

$\Rightarrow T^{-1}(x+y) = T^{-1}(x) + T^{-1}(y)$

* consider $T^{-1}(\alpha x)$

$T^{-1}(\alpha x) = z, \quad z \in V$

$\alpha x = T(z)$

$\Rightarrow \alpha T(u) = T(z)$; $\exists u \in V$ s.t. $x = T(u)$

$\Rightarrow T(\alpha u) = T(z)$

$\Rightarrow \alpha u = z$

$\Rightarrow \alpha T^{-1}(x) = z = T^{-1}(\alpha x)$

$\Rightarrow \alpha \cdot T^{-1}(x) = z = T^{-1}(\alpha x)$

$\Rightarrow T^{-1}(\alpha x) = \alpha \cdot T^{-1}(x)$

So, $T^{-1} \rightarrow$ linear trfrm.

DEFN: $T: U \rightarrow V$ is called an isomorphism if T is linear & bijective.

if $T \rightarrow$ isomorphism, T^{-1} is also isomorphism.

DEFN: if $T: U \rightarrow V$ is an isomorphism, then V is isomorphic to W $(V \cong W)$. if $V \cong W$, then $W \cong V$ as T^{-1} is also an isomorphism. So, V & W are isomorphic to each other.

$$(V \cong W,\ W \cong Z) \longrightarrow V \cong Z$$

THEOREM: $T: V \rightarrow W$ be an isomorphism.

let $\{u_1, u_2, \dots, u_n\}$ be a basis of V.

Then $\{Tu_1, Tu_2, \dots Tu_n\}$ is the basis of W.

So, $\dim V = \dim W$.

proof: let $y \in W$

$\Rightarrow \exists x$ s.t. $y = T(x)$

$= T(\alpha_1 u_1 + \dots + \alpha_n u_n)$

$= \alpha_1 T(u_1) + \dots + \alpha_n T(u_n)$

$\in Sp.\{T(u_1), \dots, T(u_n)\}$

let $\beta_1 T(u_1) + \dots + \beta_n T(u_n) = 0$

$\Rightarrow T(\beta_1 u_1) + \dots + T(\beta_n u_n) = 0$

$\Rightarrow T(\beta_1 u_1 + \dots + \beta_n u_n) = 0$

$\Rightarrow \beta_1 u_1 + \dots + \beta_n u_n = 0$, as, $N(T) = \{0\}$ due to isomorphism

u_{1-n} form the basis

{ bijective, hence

of V. injective }

$\therefore \beta_{1-n} = 0$

$\Rightarrow \{T(u_1), T(u_2), \dots, T(u_n)\}$ is linearly indep.

$\therefore \{T(u_1), T(u_2), \dots, T(u_n)\} \to$ basis of W.

$\dim W = n = \dim V$

$\therefore \dim V = \dim W$

<u>Converse:</u> $\dim V = \dim W \longrightarrow V$ & W are isomorphic

let $B_V = \{u_1, u_2, \dots, u_n\}$ be basis of V.

$B_W = \{w_1, w_2, \dots, w_n\}$ be basis of W.

define $T : V \to W$ by $T(u_i) = w_i$, $1 \leq i \leq n$.

tpt. $T \to$ injective & surjective { so that V is isomorphic to W }

$\Rightarrow$ show that $N(T) = \{0\} \to$ then, $\dim N(T) = 0$

$\dim R(T) = \dim V$

$= \dim W$

$\Rightarrow \dim R(T) = \dim W$

$\Rightarrow R(T) = W$

$\Rightarrow$ surjective.

let $x \in N(T)$

$\Rightarrow T(x) = 0$

$\Rightarrow T(\alpha_1 u_1 + \dots + \alpha_n u_n) = 0$

$\Rightarrow \alpha_1 T(u_1) + \dots + \alpha_n T(u_n) = 0$

$\Rightarrow \alpha_1 w_1 + \dots + \alpha_n w_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ w_is $\to$ BASIS for W

$\Rightarrow x = 0$

$\Rightarrow N(T) = \{0\}$

$\Rightarrow$ $T \rightarrow$ injective & surjective
$\Rightarrow$ $T \rightarrow$ isomorphism
$\Rightarrow$ $V \rightarrow$ isomorphic to W. $\quad \underline{\underline{V \cong W}}$

X ———————— X ———————— X

$$Sp(S) = \bigcap \left\{ W : S \subseteq W \ \& \ W \overset{*}{\subseteq} V \right\}$$

let $S = \{u_1, u_2, \ldots, u_s\}$

$$Sp(S) = \{ \underbrace{v_1, v_2, \ldots, v_s}_{S}, \underbrace{v_{s+1}, v_{s+2}, \ldots, v_{s+k}}_{\text{extras possibly due to intersection of } W\text{'s.}} \}$$

$Sp(S) \rightarrow$ subspace of V.

$$L(S) = \left\{ x \in V : x = \sum_{i=1}^{n} c_i \alpha_i, \ \alpha_i \in S, \ c_i \in \mathbb{F}, \ n \in \mathbb{N} \right\}$$

$L(S) \rightarrow$ subspace of V.

* $\underline{Sp(S) \subseteq L(S)}$

$$x \in Sp(S) \rightarrow x \in L(S)$$

$u_1, u_2, \ldots, u_s \in Sp(S)$
$\Rightarrow$ $c_1u_1, c_2u_2, \ldots, c_su_s \in Sp(S)$
$\Rightarrow$ $c_1u_1 + c_2u_2 \in Sp(S)$
$\Rightarrow$ $c_1u_1 + c_2u_2 + c_3u_3 \in Sp(S)$
$\vdots$
$\Rightarrow$ $c_1u_1 + c_2u_2 + \cdots + c_nu_n \in Sp(S)$
$\Rightarrow$ $\sum_{i=1}^{n} c_iu_i \in Sp(S)$

let $x = \sum_{i=1}^{n} c_i u_i \Rightarrow x \in Sp(S)$.

but! $\sum_{i=1}^{n} c_i u_i \rightarrow$ linear combo form.

$\therefore, x \in L(S)$

$\therefore, Sp(S) \subseteq L(S)$

x ———————— x ———————— x

<u>COROLLARY</u> : let $V \rightarrow$ real vector space of dimension n. Then, $V \rightarrow$ isomorphic to $\mathbb{R}^n$ ($V \cong \mathbb{R}^n$)

$\dim(\mathbb{R}^n) = \dim(V) = n$.

let $B_V = \{v_1, v_2, \ldots, v_n\}$

$B_{\mathbb{R}^n} = \{R_1, R_2, \ldots, R_n\}$

<u>define</u> $T : V \rightarrow \mathbb{R}^n$ s.t. $T(v_i) = R_i$

if $V, \mathbb{R}^n \rightarrow$ isomorphic, then $T \rightarrow$ injective & surjective.

injective $\longrightarrow N(T) = \{0\}$

surjective $\longrightarrow R(T) = \mathbb{R}^n$.

let $x \in N(T)$.

$\Rightarrow T(x) = 0$

$\Rightarrow T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) = 0$

$\Rightarrow \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(v_n) = 0$

$\Rightarrow \alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_n R_n = 0$

as $R_i \in$ Basis of $\mathbb{R}^n$, they are linearly indep.

$\Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$

$\Rightarrow \quad \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$

$\Rightarrow \quad x = 0.$

$\therefore$, if $x \in N(T)$, $x = 0$

$\Rightarrow \quad N(T) = \{0\}.$

$N(T) = \{0\} \Rightarrow \dim N(T) = 0$

$\Rightarrow \dim V = \dim N(T) + \dim R(T)$

$\Rightarrow \quad n = 0 + \dim R(T)$

$\Rightarrow \dim R(T) = n$

as $R(T) \subseteq \mathbb{R}^n$ & $\dim R(T) = \dim \mathbb{R}^n$,

$R(T) = \mathbb{R}^n.$

$\therefore$, $T \rightarrow$ surjective.

as $T \rightarrow$ injective & surjective, $T \rightarrow$ isomorphism.

$\therefore$, V & $\mathbb{R}^n \longrightarrow$ isomorphic to each other

$V \cong \mathbb{R}^n.$

Similar proof can be shown for $V \cong \mathbb{C}^n$, if $\dim V = n$.

if $\mathbb{C}^n \rightarrow$ v.s. over $\mathbb{R}$, then $\dim \mathbb{C}^n = 2n$, as any complex no $\rightarrow$ ordered pair of 2 Real nums.

$\mathbb{C}^n$ is ISOMORPHIC to $\mathbb{R}^{2n}$, not $\mathbb{R}$

RECALL: if $X \rightarrow$ finite set, $f : X \rightarrow X$ is injective iff $f \rightarrow$ surjective, iff $f \rightarrow$ bijective.

Compare this with the upcoming Theorem.

<u>THEOREM</u>: let $T: V \to V$ be linear, and $V \to$ finite dimensional. Then the following on T are equivalent:

REFER PREV NOTES.

a) $T \to$ injective.
b) $T \to$ surjective.
c) $N(T) = \{0\}$
d) $R(T) = V$.

CONSEQUENCE of RANK-NULLITY-DIMENSION THEOREM.

$4 \Leftrightarrow 1 \Leftrightarrow 2$, $1 \Updownarrow 3$

example: $V \to$ the real vector space of all polynomials over t, $t \in \mathbb{R}$.
$V \to$ infinite dimensional (check WHY)

$D \to$ LINEAR.

let $f(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n$, $t \in \mathbb{R}$
Define $D: V \to V$ by:
$(Df)(t) = \alpha_1 + 2\alpha_2 t + \cdots + n\alpha_n t^{n-1}$, $f \in V$.
$D \to$ NOT injective, as $N(T) \neq \{0\}$, but set of ALL constant polynomials.

$T \to$ LINEAR

let $g(t) = \beta_0 + \beta_1 t + \cdots + \beta_n t^n$, $t \in \mathbb{R}$
Define $T: V \to V$ by:
$$(Tg)(t) = \beta_0 t + \frac{\beta_1 t^2}{2} + \cdots + \frac{\beta_n t^{n+1}}{n+1}, \quad g \in V.$$
$T \to$ injective, as $g(t) = 0$ is the only case in which $(Tg)(t) = 0 \to N(T) = \{0\}$

what about:

DT $\rightarrow$ integrate & then differentiate

↳ get the same fn.

↳ identity.

$(DTf)(t) = f(t)$ //

TD $\rightarrow$ differentiate & then integrate

↳ if you diff. a constant & integrate, you WON'T get back the const, as the value of $(Df)(t)$ becomes 0.

↳ $\therefore$, $(TDf)(t) \neq f(t)$ //

↳ not identity.

$DT = I \longrightarrow$ * D has a right inverse

* then, D is surjective.

$TD \neq I \longrightarrow$ * T doesn't have a right inverse

* then T is not surjective.

right inverse $\longleftrightarrow$ surjectivity?

$\Rightarrow R(D) = V.S.$?

$$DT(f(t)) = D\left(\alpha_0 t + \frac{\alpha_1 t^2}{2} + \cdots + \frac{\alpha_n t^{n+1}}{n+1}\right)$$

$$\alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n = \underbrace{\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n}$$

↳ form of ANY arbitrary polynomial in real vector space.

$\therefore$, SURJECTIVE $\Longleftarrow$

$$TD(f(t)) = T(\alpha_1 + 2\alpha_2 t + \cdots + n\alpha_n t^{n-1})$$

$$= \underbrace{\alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n}$$

↳ doesn't represent all

$\to$ doesn't represent all the polynomials in real vector space {like constants}

$\therefore$!SURJECTIVE $\leftarrow$

let $T: V \to V$ be linear with $V \to$ finite dimensional. if $N(T) = R(T)$, then dimension of V is even. Give example of one such transformation.

$\dim V = \dim N(T) + \dim R(T) = 2K$

$\Rightarrow \dim V = 2K$

$\Rightarrow K = \dim V / 2$.

as $K \to \dim N(T)$ & $\dim R(T)$, K is a whole number.

$\Rightarrow \dim V \to$ must be an even number.

$N(T) = R(T)$

tpt. $T(T(x)) = 0$

$\Rightarrow \forall y \in R(T), \ y \in N(T)$

$y \in R(T) \Rightarrow \exists x \in V$ s.t. $T(x) = y$.

$y \in N(T) \Rightarrow T(y) = 0$

$\Rightarrow T(T(x)) = 0$

$\Rightarrow T^2 = 0$

$\forall x \ \exists y$ s.t. $T(x) = y$

$\because y \in R(T)$

& $R(T) = N(T)$,

$y \in N(T)$

$\Rightarrow \forall x \ T(T(x)) \in N(T)$, $T(T(x)) = 0$

$T(y) = Ay$

$T(T(x)) = A \cdot T(x) = A \cdot Ax = A^2 x$

$\Rightarrow T(T(x)) = A^2 x$

$\Rightarrow \forall x \; T(T(x)) = 0$ // $\qquad \Rightarrow N(T) = R(T)$?

$T(T(x)) = 0$

t.p.t. $N(T) = R(T)$

$N(T) \subseteq R(T)$.

$T(T(x)) = 0 \longrightarrow T(x) \in N(T)$

but, $T(x) \in R(T)$.

$\therefore$, $N(T) \subseteq R(T)$

$R(T) \subseteq N(T)$.

let $y \in R(T)$

$\Rightarrow \exists_{x \in V}$ s.t. $T(x) = y$.

$\Rightarrow T(T(x)) = T(y)$

$\Rightarrow T(y) = 0$

$\Rightarrow y \in N(T)$.

$\therefore$, $R(T) \subseteq N(T)$.

$\Rightarrow N(T) = R(T)$ //

<u>THEOREM</u> : let $A \in \mathbb{R}^{m \times n}$. Then :

the row-rank of A = the column-rank of A.

<u>Recall</u> - * row rank of a matrix is defined as the dimension of row space of A.

* col. rank is the dimension of col. space of A.

* row space of $A \subseteq \mathbb{R}$
* col. space of $A \subseteq \mathbb{R}^m$

let $A, B \in \mathbb{R}^{m \times n}$ be such that $B = PA$, where $P \in \mathbb{R}^{m \times m}$. then the row space of B is contained in the row space of A.

look at Ax (for any $x \in \mathbb{R}^n$). Ax is in the col. space of A:

cols. of A

$$Ax = (A_1, A_2, \cdots, A_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$\Rightarrow Ax = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$

$\Rightarrow Ax \rightarrow$ linear combination of cols. of A.

$\Rightarrow Ax \rightarrow$ contained in the col. space of A.

$B = PA$

$\Rightarrow B^T = (PA)^T = A^T P^T = A^T Q \quad ; \quad Q = P^T$

$P \rightarrow m \times m \Rightarrow Q \rightarrow m \times m$

$Q = (Q_1, Q_2, \cdots, Q_m)$

$B^T = (B_1, B_2, \cdots, B_m)$ $\quad B \rightarrow m \times n$, $B^T \rightarrow n \times m$

$B^T = A^T Q$

$\Rightarrow (B_1, B_2, \cdots, B_m) = A^T (Q_1, Q_2, \cdots, Q_m)$

$= (A^T Q_1, A^T Q_2, \cdots, A^T Q_m)$

$B_1 \rightarrow$ is in the col. space of B^T

$\therefore$, is in the row space of B.

$B_1 = A^T Q_1$.

$A^T Q_1$ is in the col. space of A^T (proved earlier for Ax) $\rightarrow$ (L.C. of cols. of A^T)

$\Rightarrow A^T Q_1$ is in the row space of A.

$\Rightarrow B_1$ is in the row space of A.

$\Rightarrow R.S.(B) \subseteq R.S.(A)$

$\therefore$, if $B = PA$,

* the row space of B is contained in row space of A. $(R.S.(B) \subseteq R.S.(A))$

if $P \rightarrow$ invertible,

$$B = PA$$
$$\Rightarrow P^{-1}B = P^{-1}(PA) = (P^{-1}P)A = A$$
$$\Rightarrow A = SB$$
$$\Rightarrow A^T = B^T S^T$$
$$\Rightarrow (A_1, A_2, \cdots, A_m) = B^T \cdot (S_1, S_2, \cdots, S_m)$$

same proof holds.

$A_1 \in$ col. space of A^T

$\Rightarrow A_1 \in$ row. space of A.

$B^T S_1 \in$ col. space of B^T

$\Rightarrow B^T S_1 \in$ row. space of B

$\Rightarrow A_1 \in$ row space of B

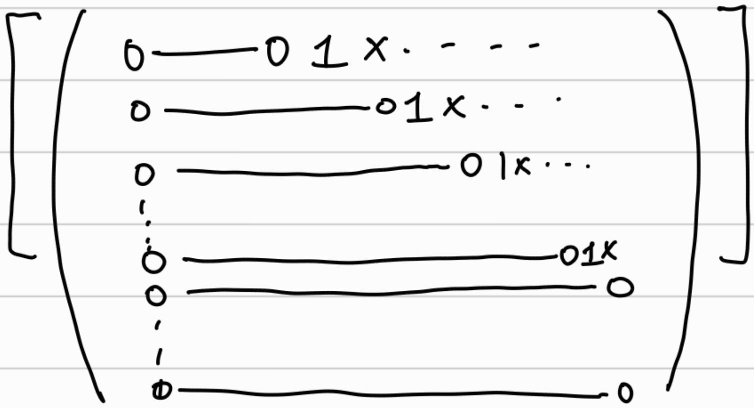
$\Rightarrow$ row space of $A \subseteq$ row space of B

$\Rightarrow$ $\boxed{\text{row space of } A = \text{row space of } B}$

if $A \rightarrow$ ROW EQUIVALENT TO B,
then $R.S.(A) = R.S.(B)$
as $A = PB$, $P \rightarrow$ invertible
↳ pdt. of ele. matrices.

determine basis for $R.S.(A)$:

* let $R \sim A$, $R \rightarrow$ RRE matrix.
* let $r \rightarrow$ no. of non-zero rows of R.
* row space of $R \rightarrow$ due to contribution / span of the vectors in r rows only (rest $n-r$ rows $\rightarrow$ ZERO).



$\rightarrow$ first 'r' non-zero rows. the leading 1's occur at diff. columns. ($c_1 < c_2 \cdots < c_r$, where c_i rep. column in i'th row where leading 1 occurs)

these rows are linearly indep. because
for $\alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_r R_r = 0$,
$\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0 \rightarrow$ if not,
then $\exists$ columns in which non-zero entries occur (at least due to leading 1's, as they occur in UNIQUE columns).

columns).

these rows span the row space of R (as other rows are 0).

$\therefore$, $\dim(R\cdot S\cdot(R)) = r$.

but, $R\cdot S\cdot(R) = R\cdot S\cdot(A)$, as $R = PA$.

$\therefore$, $\dim(R\cdot S\cdot(R)) = \dim\cdot(R\cdot S(A)) = r$.

$\Rightarrow$ row rank of $A = r$

Column rank of A = dimension of col. space of A

= dimension of set of all linear combinations of cols. of A

$= \dim\{y : y = Ax, x \in \mathbb{R}^n\}$

consider $A \in \mathbb{R}^{m\times n}$.

let $S = \{x : Ax = 0\}$ be a subspace of $\mathbb{R}^n$

$= \{x : Rx = 0\}$, if $R \sim A$ (RREF)

let $J = \{1,2,\cdots,n\} \setminus \{c_1, c_2, \cdots, c_r\}$

consider

$$x_{c_1} + \sum_{j\in J} \alpha_{1j} x_j = 0$$
$$x_{c_2} + \sum_{j\in J} \alpha_{2j} x_j = 0$$
$$\vdots$$
$$x_{c_r} + \sum_{j\in J} \alpha_{rj} x_j = 0$$

let $S = \{x \in \mathbb{R}^n : Ax = 0\} \longrightarrow$ subspace of $\mathbb{R}^n$

$= \{x \in \mathbb{R}^n : Rx = 0\}$; $R \sim A$.

$$x_1, x_2, \cdots, x_n \begin{cases} \to x_{c_1}, x_{c_2}, \cdots, x_{c_r} \\ \to u_1, u_2, \cdots, u_{n-r} \end{cases}$$

- corresponds to leading non-zero entries of $i = 1, 2, \cdots, r$

$$RX = 0 \equiv \begin{cases} x_{c_1} + \sum_{j=1}^{n-r} \alpha_{1j} u_j = 0 \\ \vdots \\ x_{c_r} + \sum_{j=1}^{n-r} \alpha_{rj} u_j = 0 \end{cases}$$

let $J = \{1, 2, 3, \cdots, n\} / \{c_1, c_2, \cdots, c_r\}$

$\Rightarrow |J| = n-r$. so,

$$x_{c_1} + \sum_{j \in J} \alpha_{1j} x_j = 0$$
$$\vdots$$
$$x_{c_r} + \sum_{j \in J} \alpha_{rj} x_j = 0$$

let $S_j, j \in J$ s.t. $S_j \in \mathbb{R}^n$ $(n-r)$ S_j's.

$S_j \rightarrow$ vector with j'th coordinate $= 1$,
i.e., $x_j = 1$, $x_i = 0 \ \forall i \in J$ & $i != j$
{ its a specific assignment for the free variables. }

I make one free variable $x_j = 1$, $\forall i \in J$ & $i != j$, $x_i = 0$.
find $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$ for this particular assignment & include them in their corresponding rows in the S_j vector.

for e.g., take $RX = 0 \equiv \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$

here, free: x_1, x_3, x_5
pivots: x_2, x_4.

for each x_j s.t. $x_j \rightarrow$ free var, i'll form a set S_j s.t. $x_j = 1$ for $j \in J$ & $x_i = 0 \ \forall i \in J$ & $i != j$

take S_1. $S_1 = (1, x_2, 0, x_4, 0)$
i'll find x_2, x_4 s.t. $x_1 = 1$, x_3 & $x_5 = 0$.

$$x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0 \rightarrow x_2 = 0$$
$$x_4 - 2x_5 = 0 \longrightarrow x_4 = 0$$
$$\Rightarrow S_1 = (1, 0, 0, 0, 0)$$

$$S_3 = (0, x_2, 1, x_4, 0)$$

$$x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0 \rightarrow x_2 = 3$$
$$x_4 - 2x_5 = 0 \rightarrow x_4 = 0$$
$$\Rightarrow S_3 = (0, 3, 1, 0, 0)$$

$$S_5 = (0, x_2, 0, x_4, 1)$$

$$x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0 \Rightarrow x_2 = \tfrac{1}{2}.$$
$$x_4 - 2x_5 = 0 \Rightarrow x_4 = 1.$$
$$\Rightarrow S_5 = (0, \tfrac{1}{2}, 0, 1, 1)$$

its clear now that S_j's, hence, are part of the soln. set of $RX = 0$.
(S_j is that soln. set where my j'th FREE VAR is 1 & other free VAR's are ZERO, for which assignment I've found my pivot variables - that are also in my soln. set)

from this, its also clear that → there are $(n-r)$ S_js. ($n(S_j s)$ = no. of FREE vars, as in each S_j I can make one of my x_js $= 1$, $j \in J$ & others $i \,!= j \in J$ zero, & find pivots for that assignment)

we obtain $(n-r)$ S_j's s.t. $S_j \in S$. these S_js are linearly independent, as there is a unique the pivot variable indices (very similar argument as standard basis being L.I.)

So, for $\sum_{j \in J} c_j s_j = 0$, each c_j must be zero, as the entry corresponding to the j'th index will produce a non-zero entry if not.

goal: tpt. the $\dim(S) = n - r$.

* step 1: proved that subspace containing only S_j's are L.I.
* Step 2: prove that they span (S)

do the set of S_j's span S?

let $y \in S$. then $y = \{y_1, y_2, \dots, y_n\}$

$\Rightarrow Ay = 0$

look at entries corresponding to $j \in J$.

then I can write $y_j = y_j \cdot 1$. similarly I can do $\forall j \in J$, unless $y_j = 0 \rightarrow$ just leave it at that in that case.

<u>claim</u>: $(y_1, y_2, \dots, y_n) = \sum_{j \in J} y_j s_j$

each S_j will have $x_j = 1$ & $x_i = 0$ $\forall i \in J$ & $i != j$. Other entries are pivot variables, controlled only by $x_j = 1$.

in S_j, each pivot = linear combo of free vars.

$P_i = \sum c_{Ki} x_j = c_{Ki} \cdot 1 = c_{Kj}$

$j \in J$ $\swarrow$ $(x_j = 1)$
jth soln. $\leftarrow$ kth row.

$$S_j = \begin{cases} 1 \; ; & \text{if } i = j, j \in J \\ 0 \; ; & \text{if } i \in J \text{ \& } i != j \\ c_{kj} \; ; & \text{for other } k\text{'s. (pivots)} \end{cases}$$

$$y_j S_j = \begin{cases} y_j \cdot 1 \; ; & \text{if } i = j \\ 0 \; ; & i \in J, i != j \\ y_j c_{kj} \; ; & \text{for other } k\text{'s (pivots)} \end{cases}$$

$$\sum_{j \in J} y_j S_j = \begin{cases} y_j \cdot 1 \; ; & k = j, \forall j \in J \\ \sum_{j \in J} y_j c_{kj} \; ; & \text{for other } k\text{'s (pivots)} \end{cases}$$

for the y_j's that are chosen, $\exists$ EXACTLY 1 assignment for the pivots, and that is shown above. $\therefore$, if $y \in S$, y must be equal to $\sum_{j \in J} y_j S_j$. This means that, y is a linear combination of S_js $\forall j \in J$

$\therefore$, set of S_js SPAN the soln. set

$\Rightarrow$ dim (soln set) = dim $(S) = n - r$.

<u>define:</u> $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(X) = AX$, $X \in \mathbb{R}^n$

by RND theorem,

$$\text{rank}(T) + \text{nullity}(T) = n.$$

rank $(T) \rightarrow$ dimension of range space of T

rank(T) → dimension of range space of T

→ dim $\{y \in \mathbb{R}^m : y = T(x), x \in \mathbb{R}^n\}$

→ dim $\{y \in \mathbb{R}^m : y = Ax, x \in \mathbb{R}^n\}$

= col. rank of A.

nullity(T) → dim of null space of T.

→ dim $\{x \in \mathbb{R}^n : T(x) = 0\}$

→ dim $\{x \in \mathbb{R}^n : Ax = 0\}$

→ dim of soln. space of $Ax = 0$

→ $\underline{\underline{n - r}}$

dim V = rank(T) + nullity(T)

⇒ n = col. rank + n − r

⇒ Col rank = r.

w.k.t row rank = r.

∴, $\boxed{\text{row rank} = \text{col. rank}}$.

the reason why we found Ax = 0 earlier was to find dim of N(T) LATER !!

AMAZING proof, but less explanation by sir...

MATRIX of a LINEAR TRANSFORMATION

Recall :

$A \in \mathbb{R}^{m\times n}$, $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by -

$$T(x) = Ax, \quad x \in \mathbb{R}^n \quad \text{is Linear.}$$

Show that a certain converse is TRUE.

let $V, W \to$ finite dimensional ; $\dim V = n$, $\dim W = m$.

let $B_V = \{u_1, u_2, \cdots, u_n\}$ a basis of V
$B_W = \{v_1, v_2, \cdots, v_m\}$ a basis of W

let $x \in V$.

$\Rightarrow x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$;
$\alpha_i \in \mathbb{R}$ & unique for given x.

Define the matrix of x relative to the basis B_V by :

$$[x]_{B_V} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

→ changes with the ORDER in which I write the basis. ∴, the basis is ORDERED → the ele. of the basis forms a sequence → important wrt forming matrix for x.

$\mathbb{R}^3$: $B_1 = \{e_1, e_2, e_3\}$

if $x \in \mathbb{R}^3$; $x = (x_1, x_2, x_3)$

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

$$\Rightarrow [x]_{B_1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$B_2 = \{u_1, u_2, u_3\}$; $u_1 = (1,1,0)$
$u_2 = (1,-1,0)$
$u_3 = (0,0,1)$

$$x = (x_1, x_2, x_3) = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$$

$$\Rightarrow \underbrace{x_1 = \alpha_1 + \alpha_2 \ ; \ x_2 = \alpha_1 - \alpha_2}\ ; \ x_3 = \alpha_3$$

$\Rightarrow x_1 + x_2 = 2\alpha_1$

$\Rightarrow \alpha_1 = (x_1 + x_2)/2$ & $x_3 = \alpha_3$

$\Rightarrow \alpha_2 = (x_1 - x_2)/2$

$\therefore$, $x = (x_1 + x_2)/2 \, u_1 + (x_1 - x_2)/2 \, u_2 + x_3 u_3$

$$\Rightarrow [x]_{B_2} = \begin{pmatrix} (x_1 + x_2)/2 \\ (x_1 - x_2)/2 \\ x_3 \end{pmatrix}$$

$T : V \to W$; $B_V = \{u_1, u_2, \cdots, u_n\} \to \dim V = n$
$B_W = \{v_1, v_2, \cdots, v_m\} \to \dim W = m$.

T is completely determined by
$\hookrightarrow \{T(u_1), T(u_2), \cdots, T(u_n)\}$
$\hookrightarrow$ n vectors.

$T(u_j) = a_{1j} v_1 + a_{2j} v_2 + \cdots + a_{mj} v_m$

$$= \sum_{i=1}^{m} a_{ij} v_i$$

$$[T(u_j)]_{B_W} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

then, the matrix of T wrt. B_V, B_W is defined by:

$$A = (a_{ij}) = [T]_{B_V}^{B_W}$$

where, the j'th col is: $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$

$$\begin{array}{ccccc} T(u_1) & & & & T(u_n) \\ \downarrow & - & - & - & \downarrow \end{array}$$

$$A = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$$

each col. is "like" the effect of T on u_js. $\therefore$ $\exists$ n u_js, <u>n cols</u>. each $T(u_j)$ can be rep. as L.C. of basis $\vec{v}$ in W, $\therefore$, <u>m rows</u>

$$\boxed{A \in \mathbb{F}^{\overset{m}{\downarrow}\times\overset{n}{\downarrow}}_{\;V\;\;W}}$$

<u>Examples:</u>

1) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T((x_1, x_2)) = (x_1+x_2, x_1-x_2, 2x_2)$

$[T]_{B_1}^{B_2}$; $B_1 = \{(1,0),(0,1)\}$
$B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(u_1) = T((1,0)) = (1,1,0) = 1\cdot(1,0,0) + 1(0,1,0) + 0\cdot(0,0,1)$
$T(u_2) = T((0,1)) = (1,-1,2) = 1\cdot(1,0,0) + (-1)(0,1,0) + 2(0,0,1)$

$\therefore$, $A = \left([T(u_1)]^{B_2}_{B_1} , [T(u_2)]^{B_2}_{B_1} \right)$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$$

let $B_3 = \{(1,1,0), (1,-1,0), (0,0,1)\}$

then: $T((1,0)) = (1,1,0) = \alpha(1,1,0) + \beta(1,-1,0) + \gamma(0,0,1)$
$= (\alpha+\beta, \alpha-\beta, \gamma)$

$\alpha+\beta = 1$
$\alpha-\beta = 1 \quad \Rightarrow \quad \alpha = 1$
$\gamma = 0 \quad \Rightarrow \quad \beta, \gamma = 0$

$T((0,1)) = (1,-1,2) = \alpha(1,1,0) + \beta(1,-1,0) + \gamma(0,0,1)$
$= (\alpha+\beta, \alpha-\beta, \gamma)$

$\left.\begin{array}{l} \alpha+\beta = 1 \\ \alpha-\beta = -1 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha = 0 \\ \beta = 1 \end{array}$
$\gamma = 2 \qquad \gamma = 2$

$\therefore$, $A = \left([T(u_1)]^{B_3}_{B_1} , [T(u_2)]^{B_3}_{B_1} \right)$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \longrightarrow \text{Changed } //$$

2) $D : \mathbb{P}_3 \to \mathbb{P}_2$; $D(p) = p'$; $p \in \mathbb{P}_3$.

$B_1 = \{1, t, t^2, t^3\}$; $B_2 = \{1, t, t^2\}$

$[D]_{B_1}^{B_2}$: a) $D(1) = 0 = \alpha + \beta t + \gamma t^2$
b) $D(t) = 1 = \alpha + \beta t + \gamma t^2$
c) $D(t^2) = 2t = \alpha + \beta t + \gamma t^2$
d) $D(t^3) = 3t^2 = \alpha + \beta t + \gamma t^2$

by obs : a) $\to$ $\alpha, \beta, \gamma = 0$
b) $\to$ $\alpha = 1$, $\beta, \gamma = 0$
c) $\to$ $\beta = 2$, $\alpha, \gamma = 0$
d) $\to$ $\gamma = 3$, $\alpha, \beta = 0$.

$\therefore$, $[T]_{B_1}^{B_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$T : V \to W$

$[x]_{B_V}$, $[T(x)]_{B_W}$, $[T]_{B_V}^{B_W}$

<u>THEOREM</u> :

$$[T(x)]_{B_W} = [T]_{B_V}^{B_W} \cdot [x]_{B_V}$$

<u>PROOF</u> :

recall – $A = [T] = (a_{ij})$, where :

$B_V = \{u_1, u_2, \cdots, u_n\}$ of V.
$B_W = \{v_1, v_2, \cdots, v_m\}$ of W.

$$T(u_j) = \sum_{i=1}^{m} a_{ij} v_i \quad \text{— } ①$$

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \quad (u_i \in B_V)$$

$$\Rightarrow [x]_{B_V} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$\begin{aligned} T(x) &= T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) \\ &= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n) \\ &= \sum_{j=1}^{n} \alpha_j T(u_j) \end{aligned}$$

from ①,

$$\begin{aligned} T(x) &= \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{m} a_{ij} v_i \right) \\ &= \alpha_1 \left((a_{11} v_1 + a_{21} v_2 + \cdots + a_{1m} v_m) \right) \\ &\quad + \alpha_2 \left((a_{12} v_1 + a_{22} v_2 + \cdots + a_{2m} v_m) \right) \\ &\quad + \cdots + \alpha_n \left((a_{1n} v_1 + \cdots + a_{mn} v_m) \right) \\ &= v_1 (a_{11} \alpha_1 + a_{12} \alpha_2 + \cdots + a_{1n} \alpha_n) \\ &\quad + v_2 (a_{21} \alpha_1 + a_{22} \alpha_2 + \cdots + a_{2n} \alpha_n) \\ &\quad + \cdots + v_m (a_{m1} \alpha_1 + \cdots + a_{mn} \alpha_n) \end{aligned}$$

$$\sum^{m} v_i \left(\sum^{n} a \ \alpha \right)$$

$$= \sum_{i=1} v_i \underbrace{\left(\sum_{j=1} a_{ij} \alpha_j\right)}$$

for every v_i, this is a number. call it β_i

$$T(x) = \underbrace{\sum_{i=1}^{m} \beta_i v_i}\ ; \quad \beta_i = \sum_{j=1}^{n} a_{ij}\alpha_j$$

matrix $T(x)$ represented as L.C. of vectors in B_W.
α_j fixed for $x \in V$ represented in B_V.

$$\therefore, \quad [T(x)]_{B_W} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots a_{1n}\alpha_n \\ \vdots \\ a_{m1}\alpha_1 + a_{m2}\alpha_2 + \cdots a_{mn}\alpha_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$[T(x)]_{B_W} = [T]_{B_V}^{B_W} [x]_{B_V}$$

$$[T(x)]_{B_W} = A\,[x]_{B_V}$$

define $\phi : L(V, W) \longrightarrow \mathbb{R}^{m \times n}$,

$$L(V, W) = \{ T : V \to W, \ T \text{ is linear} \}$$

$\phi \to$ Real vector space.

$*\ (S+T)(x) = S(x) + T(x)$

$*\ S(\alpha x) = \alpha S(x)$

$$\phi(T) = [T]_{B_V}^{B_W} \quad ; T \in L(V, W)$$

$\phi \to$ well defined, as for a linear transformation, $\exists!$ matrix of linear transformation.

$\underline{\phi \text{ is}}$:

$*$ linear $\to \phi(S+T) = \phi(S) + \phi(T)$.

$*\ \phi(\alpha S) = \alpha \phi(S)$.

$*\ \phi(T) = 0 \implies [T]_{B_V}^{B_W} = 0$

↳ j'th col is 0

↳ $T(u_j) = 0$

↳ L.C. of vectors in B_W

↳ Vectors in B_W are L.Indep

↳ $\therefore$, coeffs $= 0$

↳ $\therefore$, $T = 0$.

$(\forall j,\ T(u_j) = 0)$

$\therefore$, $N(\phi) = \{0\}$

$\forall A \in \mathbb{R}^{m \times n}$, $\exists\ T: V \to W$ s.t. $T(X) = AX$. $\therefore$, $\phi \to$ surjective.

$\Rightarrow \phi \to$ ISOMORPHISM.

take $A \in \mathbb{R}^{m \times n}$.

do I have $S: V \to W$ s.t.

$[S]_{B_V}^{B_W} = A$ (?).

let $S(x) = Ax$

if $S \to$ linear

$\downarrow$

* $S(x+y) = A(x+y) = A(x) + A(y) = S(x) + S(y)$.
* $S(cx) = A(cx) = cA(x) = cS(x)$
* $S(0) = A(0) = 0$.

$\therefore$, $S \to$ linear.

$\Rightarrow S \in L(V, W)$

$\Rightarrow \exists_{B \in \mathbb{R}^{m \times n}}$ s.t. $[S]_{B_V}^{B_W} = B$.

as $\phi \to$ injective, $\exists!$ such matrix, AND A is that matrix, since we showed that it satisfied ALL conditions for $S \to$ linear. $\therefore$, $[L]_{B_V}^{B_W} = A$ //

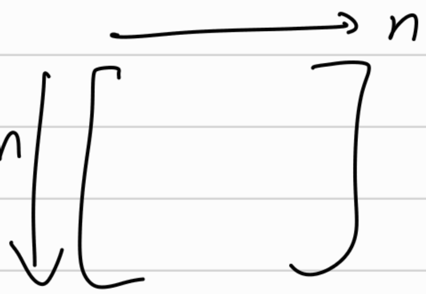
$\dim(L(V,W)) = mn$.

what are the basis ??

↳ 1 in 11 rest zero

↳ 1 in 22 rest zero

↳ & so on . . .



EIGEN VALUES AND VECTORS FOR LINEAR TRANSFORMATIONS

let T be a linear operator over V ($\dim V < \infty$)

can we obtain a basis B of V s.t.

$[T]_B$ is simple? ($T : V \to V$ is a linear operator)

simple $\longrightarrow$ if kI.

$\longrightarrow$ if diagonal matrix.

does there exist a basis B of V s.t.

$$[T]_B = D = \mathrm{diag}(\alpha_{11}, \alpha_{22}, \dots, \alpha_{nn})$$

$$= \begin{pmatrix} \alpha_{11} & & & \\ & \alpha_{22} & & 0 \\ & & \ddots & \\ 0 & & & \alpha_{nn} \end{pmatrix}$$

in this case, $T \longrightarrow$ diagonalizable.

Are ALL operators on finite dimensional vector spaces diagonalizable?

if T is diagonalizable, then:

$$T(u_i) = \alpha_{ii} u_i \quad ; \quad 1 \leq i \leq n$$
$$B = \{u_1, u_2, \dots, u_n\}$$

Then,

$$N(T) = \mathrm{Span}\{u_i : \alpha_{ii} \neq 0\}$$

$$R(T) = \mathrm{Span}\{u_i : \alpha_{ii} = 0\}$$

DIDN'T UNDERSTAND IF DIAGONALIZABLE, THEN $T(u_i) = \alpha_{ii} u_i$ PART (& the subsequent fellows)

<u>Defn</u>: let $T \in L(V)$. A number $\lambda \in \mathbb{F}$ is called an eigenvalue (or characteristic value), of T if, $\exists\, x \neq 0, x \in V$ s.t. $T(x) = \lambda x$.
Any such x is called an eigenvector corresponding to an eigenvalue λ.

<u>example</u>:

① define $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T((x_1, x_2)) = (-x_2, x_1) \; ; \quad x_1, x_2 \in \mathbb{R}$$

$$\lambda x = T(x) = (-x_2, x_1) \quad - ①$$
$$\lambda x = (\lambda x_1, \lambda x_2) \quad - ②$$
$$\Rightarrow (\lambda x_1, \lambda x_2) = (-x_2, x_1)$$

if $\lambda = 0$, $x = 0$. consider $\lambda \neq 0$

$$\Rightarrow \quad x_1 = \lambda x_2 = \lambda(-\lambda x_1) = -\lambda^2 x_1$$
$$\Rightarrow \quad (1+\lambda^2) x_1 = 0$$
$$\Rightarrow \quad x_1 = 0, \quad \text{as } \lambda \in \mathbb{R}.$$
$$\Rightarrow \quad x_2 = 0.$$
$$\Rightarrow \quad x = 0.$$

$\therefore$, $\not\exists\, x \neq 0$, $x \in V$ s.t. $T(x) = \lambda x$.

$\therefore$, $T \to$ DOESN'T have eigenvalue.

$\hookrightarrow$ due to deficiency of field (doesn't have a real value that satisfies $t^2+1=0$) $\to$ ONE POSSIBILITY.

identity ntr.

$T - \lambda I$ is NOT invertible iff $A - \lambda I$ is NOT invertible, where $A = [T]_B$

identity matrix

$$Tx = \lambda x, \quad x \neq 0$$
$$(T - \lambda I)x = 0, \quad x \neq 0$$
$$\Leftrightarrow (A - \lambda I)x = 0, \quad x \neq 0 \quad \text{(RREF has AT least ONE ZERO ROW)}$$
$$\Leftrightarrow \det(A - \lambda I) = 0,$$

$p(\lambda) = |A - \lambda I|$ is called the characteristic polynomial of A (a monic polynomial of degree n)

n)

X ———————————— X ———————————— X

ATTEMPT TO UNDERSTAND $E\vec{V}$, EV

DIAGONAL MATRICES:

$A = (a_{ij})$ is diagonal, if $a_{ij} = 0 \ \forall i != j$

$$A = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

let $\text{diag}(a_1, a_2, \cdots, a_n) = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}$

$\hookrightarrow$ $n \times n$ matrix.

* consider $\begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} + \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix}$

$$= \begin{pmatrix} a_1+b_1 & & & 0 \\ & a_2+b_2 & & \\ & & \ddots & \\ 0 & & & a_n+b_n \end{pmatrix}$$

$\Rightarrow \text{diag}(a_1, a_2, \cdots, a_n) + \text{diag}(b_1, b_2, \cdots, b_n)$

$= \text{diag}(a_1+b_1, \cdots, a_n+b_n)$

* for a scalar c, $c \cdot \text{diag}(a_1, a_2, \cdots, a_n)$

$= \text{diag}(ca_1, ca_2, \cdots, ca_n)$

↳ can be shown.

* $\begin{pmatrix} a_1 & & & \\ & a_2 & & 0 \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} \times \begin{pmatrix} b_1 & & & \\ & b_2 & & 0 \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & & & \\ & a_2b_2 & & 0 \\ & & \ddots & \\ 0 & & & a_nb_n \end{pmatrix}$

$\Rightarrow \text{diag}(a_1, a_2, \cdots, a_n) \times \text{diag}(b_1, b_2, \cdots, b_n) = \text{diag}(a_1b_1, \cdots, a_nb_n)$

* $[\text{diag}(a_1, a_2, \cdots, a_n)]^m = \text{diag}(a_1^m, a_2^m, \cdots, a_n^m)$

* let $p(x) = x^3 + 2x + 3$

$p(A) := A^3 + 2A + 3I_n \; ; \; A \in \mathbb{F}^{n \times n}$.

if $A = \text{diag}(a_1, a_2, \cdots, a_n)$, then:

$p(A) = \text{diag}(a_1^3, a_2^3, \cdots, a_n^3) + \text{diag}(2a_1, \cdots, 2a_n) + \text{diag}(\underbrace{3, 3, \cdots, 3}_{\hookrightarrow n \ 3's})$

$= \text{diag}(a_1^3 + 2a_1 + 3, \cdots, a_n^3 + 2a_n + 3)$

$p(A) = \text{diag}(p(a_1), \cdots, p(a_n))$

LEMMA : let $A \rightarrow$ diagonal matrix. Then the rank of $A \rightarrow$ no. of non-zero entries.

proof : rank of A = row rank of A

$= \dim.$ of $R\cdot S\cdot(A)$

$R\cdot S\cdot(A)$ ↳ Rowspace of A

$= \dim.$ of $R\cdot S\cdot(R)$ $\{R \sim A\}$

$= r =$ no. of non-zero rows.

if $A \to$ diagonal matrix, then:

$A = \text{diag}(a_1, a_2, \ldots, a_n)$, $A \in \mathbb{R}^{n\times n}$.

if any one of a_is $= 0$, they will be pushed to the end of the RREF. of matrix. Hence its clear that: no. of non zero rows = no. of non-zero entries $= r$. $\therefore r =$ rank of R = rank of A, rank of A is the no. of non-zero entries in A. //

AND in RREF, u just divide a_{ii} by itself. ↳ ∵ its diagonal, no need to worry about other elements in same column.

(we proved earlier, that $R\cdot S(A) = R\cdot S\cdot(R)$, by saying that $R = PA$ (recall that proof, row-rank = col-rank), and $\dim R\cdot S\cdot(A) =$ no. of non-zero rows, hence $\dim R\cdot S\cdot(A) =$ no. of non-zero rows).

let $A = \text{diag}(a_1, a_2, \ldots, a_n)$ be a diagonal matrix. Consider linear transformation L_A. Then:

$$L_A v = \begin{pmatrix} a_1 & & \\ & a_2 & 0 \\ 0 & & \ddots \ a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \end{pmatrix}$$

↳ DILATES the 1st coordinate by a_1, 2nd by a_2 and so on.

let $\beta = \{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$.

Then,

$$L_A e_1 = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & \ddots & a_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= a_1 e_1$$

$$\Rightarrow L_A e_1 = a_1 e_1, \ L_A e_2 = a_2 e_2, \ \dots, \ L_A e_n = a_n e_n$$

here, $a_i \to$ eigen values, e_is $\to$ eigen vectors

EIGEN VALUES & EIGEN VECTORS

Let $T: V \to V$ be a linear operator on V (i.e., T is a linear transformation from V to itself).

(e.g.) * $T = I_V$. Then, $I_V v = v$, $\forall v \in V$
* $T = \lambda I_V$. Then, $T_v = \lambda v$, $\forall v \in V$

every v is dilated by λ.
every v is dilated by 1.

the "special" vectors that get dilated by an arbitrary linear transformation are the eigenvectors for that linear transformation.

DEFN: let $T: V \to V$ be a linear operator on V. We say that a non-zero vector $v \in V$ is an eigen vector of T if $T_v = \lambda v$, for some scalar λ. The scalar λ is called the eigenvalue corresponding to the eigenvector v.

(e.g). * $T: \mathbb{R}^2 \to \mathbb{R}^2$. given by:

$$T(x, y) = (2x, 3y)$$

let $v_1 = (1,0)$. then : $Tv_1 = T(1,0) = (2,0)$

$= 2\cdot(1,0)$

$= 2v_1$.

$v_2 = (0,1)$. then $Tv_2 = T(0,1) = (0,3)$

$= 3(0,1)$

$= 3v_2$.

$\Rightarrow$ v_1, v_2 (STANDARD BASIS of $\mathbb{R}^2$) are examples of eigenvectors of T.

* if $T = I_V$, then every NON-ZERO $v \in V$ is an eigenvector with eigenvalue 1.
  sim. for $T = \lambda I_V$, but eigenvalue λ.

* if $T: V \to V$ is NOT injective. let $v \in N(T)$ s.t. $v \neq 0$. Then, $Tv = \underset{\text{vector}}{\underset{\downarrow}{0}} = \underset{\text{scalar}}{\underset{\downarrow}{0}}v$.

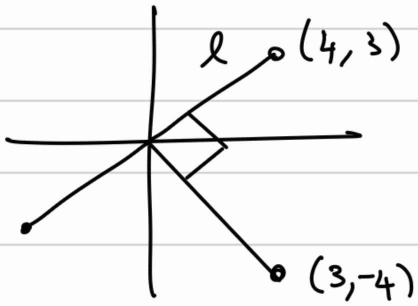
  $\therefore$, $v \to$ eigenvector, with eigenvalue $= 0$ { constraint only on vector, not value }.

* show that $T \to$ invertible, if all eigenvalues are non-zero.

  if all eigenvalues are non zero, then:
  $Tv = \lambda v$, iff. $v = 0$.
  $\Rightarrow$ $T(v) = 0 \to$ possible only if $v = 0$
  $\Rightarrow$ $N(T) = \{0\} \to$ INJECTIVE.
  $\Rightarrow$ nullity $= 0$
  $\Rightarrow$ rank $= \dim V \to$ SURJECTIVE.
  $\Rightarrow$ $T \to$ invertible.

{ recall why $N(T) = \{0\}$ means isomorphism }

* let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T \to$ reflection along l.



if $v = (4,3)$, then:

$$T_{v_1} = v_1$$

(reflecting v_1 along $(0, v_1)$ doesn't change it) $\hookrightarrow l$

if $v_2 = (3,-4)$. then:

$$T_{v_2} = (-3,4) = -1(3,-4) = -v_2.$$

Then T has eigenvector v_1 with eigenvalue $=1$ & v_2 with eigenvalue $= -1$.

Let $A \in \mathbb{R}^{m \times n}$. We say that a vector $v \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue λ if v is an eigenvector of L_A with eigenvalue λ.

(e.g.) let $A = \text{diag}(a_1, a_2, \cdots, a_n)$ be a diagonal matrix. Then, e_i is an eigenvector of A.

$L_A e_i = a_i e_i$ (seen earlier)

$\therefore$, $e_i \to$ eigenvector of L_A, with eigenvalue a_i.

DEFN: let $T: V \to V$ be a linear operator on V. Then the eigenspace of a scalar λ is the set of all vectors s.t.

$$T_v = \lambda v \quad ; v \in V, \lambda \in \mathbb{R}$$

$$T_v = \lambda v \iff T_v = \lambda I_V v$$

$$\iff T_v - \lambda I_V v = 0$$

$$\iff (T - \lambda I_V) v = 0$$

$$\iff v \in \text{Null}(T - \lambda I_V)$$

$$\{[T]_V^V - \lambda I_V\}$$

$\Rightarrow$ eigenspace of $\lambda \rightarrow$ Null space of $T-\lambda I_V$, which is a subspace of V.

Observe that $\lambda \rightarrow$ eigenvalue iff $\exists$ a non zero vector in the null space of $T-\lambda I_V$.

$\Leftrightarrow$ $T-\lambda I_V \rightarrow$ NOT injective.

$[T]_V^V$

$*$ $T(x,y) = (2x, 3y) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$*$ $(T-2I_V)(x,y) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (0,y)$

$\Rightarrow$ $\forall_{x,y \in \mathbb{R}}$ s.t. $y=0$; $(x,y) \in N(T-2I_V)$

$\hookrightarrow$ NOT INJECTIVE.

$\Rightarrow$ $\forall_{x,y \in \mathbb{R}}$ s.t. $x=0$; $(x,y) \in N(T-3I_V)$

exercise: show that $T-\lambda I_V$ is invertible $\forall$ $\lambda \neq 2,3$, for above e.g.

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$$

$$(T-\lambda I_V)(x,y) = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow (T-\lambda I_V)(x,y) = \begin{bmatrix} (2-\lambda)x \\ (3-\lambda)y \end{bmatrix}$$

for $(T-\lambda I_V)(x,y) = 0$, $(x,y)=(0,0)$ is the ONLY soln., as $2-\lambda \neq 0$

$4 \quad 3-\lambda \neq 0.$

$\Rightarrow N(T) = \{0\} \rightarrow$ INJECTIVE

↳ HENCE

SURJECTIVE

↳ HENCE INVERTIBLE

PROPOSITION : let $T: V \rightarrow V$ be a linear operator on V with $\dim V < \infty$ (say, n). If $\beta = (v_1, v_2, \cdots, v_n) \rightarrow$ ordered basis of V consisting of eigenvectors of T, then $[T]_\beta^\beta \rightarrow$ diagonal matrix. Converse is ALSO true. (if $[T]_\beta^\beta \rightarrow$ diag, where $\beta = (v_1, v_2, \cdots, v_n)$, then v_is $\rightarrow$ eigenvectors of T)

proof : we have a basis $\beta = (v_1, v_2, \cdots, v_n)$ with eigenvectors of T. Then,

$T(v_j) = \lambda_j v_j$, where $\lambda_j \rightarrow$ eigenvalue of eigenvector v_j

$$\Rightarrow [T(v_j)]^\beta = \begin{pmatrix} 0 \\ \vdots \\ \lambda_j \\ \vdots \\ 0 \end{pmatrix} \rightarrow \text{in } j\text{'th row}$$

$\rightarrow$ in j'th col. of matrix $[T]_\beta^\beta$

$\Rightarrow [T]_\beta^\beta = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\lambda_i \rightarrow$ eigenvalue of eigenvector v_i $(1 \leq i \leq n)$

let $\beta = (v_1, v_2, \cdots, v_n)$ s.t.

$[T]_\beta^\beta = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$\Rightarrow [T(v_j)]_\beta = \begin{pmatrix} 0 \\ \vdots \\ \lambda_j \\ \vdots \\ 0 \end{pmatrix} \rightarrow$ coeffs. corresponding to the basis vectors contributing to $T(v_j)$ //

$\Rightarrow T(v_j) = 0 \cdot u_1 + 0 \cdot u_2 + \cdots + \lambda_j v_j + \cdots 0 \cdot v_n$

$\Rightarrow T(v_j) = \lambda_j v_j \rightarrow \forall j \in \{1, 2, \cdots, n\}$

$\therefore$, $(v_1, v_2, \cdots, v_n) \rightarrow$ eigenvectors corresponding to λ_j.

<u>DEFINITION</u> : A linear transformation $T: V \rightarrow V$ is diagonalizable, if $\exists$ basis β s.t. $[T]_\beta \rightarrow$ diagonal.

* $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = (2x, 3y)$
$\hookrightarrow$ diagonalizable

* let $A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$; $A \in \mathbb{R}^{n \times n}$

$\Rightarrow L_A \rightarrow$ diagonalizable wrt standard basis of $\mathbb{R}^n$.

* <u>revisiting</u> : $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by reflection along $l \rightarrow$ line joining $(0,0)$ and $(4,3)$.

<u>recall</u> : $(4,3)$ & $(3,-4) \rightarrow$ ARE eigen vectors, with eigenvalues 1 &

-1 respectively.

- $(4,3)$ & $(3,-4) \rightarrow$ linearly indep:

$*\ c_1(4,3) + c_2(3,-4) = 0$

$4c_1 + 3c_2 = 0 \quad \times 4$

$3c_1 + 4c_2 = 0 \quad \times 3$

$16c_1 + 12c_2 = 0$

$9c_1 + 12c_2 = 0 \quad \Rightarrow \boxed{c_1, c_2 = 0}$

2 linearly indep vectors $\in \mathbb{R}^2 \rightarrow$ form basis of $\mathbb{R}^2$.

let $\beta = \{(4,3), (3,-4)\}$

$$\Rightarrow [T]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow [T^2]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I_V]_\beta^\beta$$

2 conflicts until now: ① $(T - 2I_V)(x,y)$
② $[T^2]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [I_V]_\beta^\beta$

let $A \in \mathbb{R}^{n \times n}$. A is diagonalizable if the linear transformation L_A is diagonalizable.

<u>example</u>: All diagonal matrices $\longrightarrow$ diagonalizable

<u>proposition</u>: let A be an $n \times n$ matrix. Then A is diagonalizable iff. $\exists$ a diagonal matrix D and an invertible matrix Q s.t.

$$A = QDQ^{-1}$$

(diagonalizable if its similar to a diagonal matrix)

proof : let $A \to$ be diagonalizable.

let $\beta' = (v_1, v_2, \dots, v_n)$ be a basis of $\mathbb{R}^n$ s.t.

$$[L_A]_{\beta'}^{\beta'} = \text{diag}(a_1, a_2, \dots, a_n) = D.$$

$L_A = I_{\mathbb{R}^n} \cdot L_A \cdot I_{\mathbb{R}^n}$; where $I_{\mathbb{R}^n} \to$ identity linear trfr. in $\mathbb{R}^n$.

let $\beta \to$ standard basis of $\mathbb{R}^n$

$$\Rightarrow A = [L_A]_{\beta}^{\beta} = [I \, L_A \, I]_{\beta}^{\beta}$$

$$= [I]_{\beta'}^{\beta} \cdot [L_A]_{\beta}^{\beta} \cdot [I]_{\beta}^{\beta'}$$

let $[I]_{\beta}^{\beta'} = Q$

$$\Rightarrow Q^{-1} = [I]_{\beta}^{\beta'}$$

$$\Rightarrow A = QDQ^{-1}.$$

prove the converse :

let $A = QDQ^{-1}$, where $D \to$ diag matrix

& $Q \to$ invertible matrix.

let $\beta = (e_1, e_2, \dots, e_n)$ be std. basis.

then, $De_j = \lambda_j e_j$ (where $D = \text{diag}(\lambda_1, \dots \lambda_n)$)

consider $\beta' = (Qe_1, \cdots, Qe_n)$

$$(DQ^{-1})(Qe_j) = De_j = \lambda_j e_j$$

then $(QDQ^{-1})(Qe_j) = Q(\lambda_j e_j) = \lambda_j Qe_j$

i.e. $Qe_j \to$ eigenvector of QDQ^{-1}

$\beta' \to$ set of eigenvectors of QDQ^{-1}.

<u>claim</u>: $\beta' \to$ basis of $\mathbb{R}^n$.

<u>exercise</u>: image of invertible L.T. $\to$ turns out to be a basis.

(refer NM/JB slides)

X ———————— X ———————— X

INNER PRODUCT SPACES

<u>DEFN</u>: let $V = \mathbb{C}^n$ (or any complex vector space). An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ which satisfies

i) $\langle x, x \rangle \geq 0 \quad \forall x \in V$

$\langle x, x \rangle = 0 \iff x = 0, \forall x \in V$

(POSITIVE DEFINITENESS)

ii) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad \forall x, y, z \in V$

iii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle, \quad \forall \lambda \in \mathbb{C} \quad \forall x, y \in V$

(LINEARITY wrt FIRST ARGUMENT)

iv) $\langle y, x \rangle = \overline{\langle x, y \rangle}, \quad \forall x, y \in V.$

(CONJUGATE SYMMETRY)

A vector space V together with an inner product is called an inner product space $\rightarrow (V, \langle \cdot, \cdot \rangle)$

examples :

1) let $V = \mathbb{R}^n$. let $x, y \in \mathbb{R}^n$.

$$x = (x_1, x_2, \ldots, x_n) \; ; \; y = (y_1, y_2, \ldots, y_n)$$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

$$\langle x, x \rangle = \sum_{i=1}^{n} x_i^2 = x_1^2 + \cdots + x_n^2$$

$x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$, as sums of squares are positive / equal to 0.

for $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$, each $x_i = 0$ as $x_i^2 \geq 0$.

$\therefore$, (i) $\langle x, x \rangle \geq 0$, AND
(ii) $\langle x, x \rangle = 0$ iff. $x = 0$. $\Big\} \rightarrow$ TRUE.

(iii) $\langle x+y, z \rangle = \sum (x_i + y_i) \cdot z_i$
$= \sum x_i z_i + \sum y_i z_i$
$= \langle x, z \rangle + \langle y, z \rangle$

(iv) $\langle \lambda x, y \rangle = \sum \lambda x_i y_i = \lambda \sum x_i y_i$

$= \lambda \langle x, y \rangle$

(v) $\langle y, x \rangle = \sum y_i x_i$
$= \sum x_i y_i$
$= \langle x, y \rangle \rightarrow$ SATISFIED without complement / conjugate.

2) $V = \mathbb{C}^n$. let $x, y \in V$.

$x = (x_1, x_2, \cdots, x_n)$; $y = (y_1, y_2, \cdots, y_n)$

$\langle x, y \rangle = \sum x_i \overline{y_i}$; $x_i, y_i \in \mathbb{C}$, $\therefore$, $\overline{x_i}$ is complex conjuagate of x_i

i) $\langle x, x \rangle = \sum x_i \overline{x_i}$
$= \sum (a_i + jb_i)(a_i - jb_i)$
$= \sum (a_i^2 + b_i^2)$
≥ 0.

$\langle x, x \rangle = 0 \implies \sum (a_i^2 + b_i^2) = 0$
$\Rightarrow a_i^2 + b_i^2 = 0$
$\Rightarrow a_i, b_i = 0$
$\Rightarrow |x_i| = 0 \longrightarrow x_i = 0$ //

ii) $\langle x+y, z \rangle = \sum (x_i + y_i) \overline{z_i}$
$= \sum x_i \overline{z_i} + \sum y_i \overline{z_i}$
$= \langle x, \overline{z} \rangle + \langle y, \overline{z} \rangle$

$\langle \lambda x, y \rangle \rightarrow$ DIY.

iv) $\langle x, y \rangle = \sum x_i \overline{y_i}$

$= \sum (a_i + jb_i)\cdot(c_i - jd_i)$

$= \sum (a_ic_i - ja_id_i + jb_ic_i + b_id_i)$

$= \sum (a_ic_i + b_id_i + j(b_ic_i - a_id_i))$

$\langle y, x\rangle = \sum (c_i + jd_i)\cdot(a_i - jb_i)$

$= \sum (c_ia_i - jc_ib_i + ja_ia_i + b_id_i)$

$\underline{\qquad\quad} = \sum (c_ia_i + b_id_i + j(a_id_i - c_ib_i))$

$\Rightarrow \langle y, x\rangle = \sum (c_ia_i + b_id_i - j(a_id_i - c_ib_i))$

$= \sum (c_ia_i + b_id_i + j(c_ib_i - a_id_i))$

$= \langle x, y\rangle.$

$\therefore \overline{\langle y, x\rangle} = \langle x, y\rangle$ //

The above inner product is referred to as the "usual inner product"/ standard inner product on $\mathbb{C}^n$.

(iii) let $w = (w_1, w_2, \ldots, w_n) \subseteq \mathbb{R}^+$.

$x, y \in \mathbb{C}^n$. $\langle x, y\rangle_w = \sum_{i=1}^{n} w_i x_i \overline{y_i}$

$w_i \longrightarrow$ weights. ($w_i > 0 \longrightarrow$ positive definiteness)

iv) $V = \mathbb{C}^{n\times n}$

let $A, B \in \mathbb{C}^{n\times n}$. let trace of A be defined as:

$$tr(A) = \sum_{i=1}^{n} a_{ii}, \text{ where } A = (a_{ij})$$

$A^* = \overline{A}^t$

define $<A, B> = tr(AB^*)$, $A, B \in V$.

$C = (c_{ij}) \in V$
$D = (d_{ij}) \in V$.

$$(CD)_{rs} = \sum_{k=1}^{n} C_{rk} D_{ks}$$

$$\Rightarrow (AB^*)_{rs} = \sum_{k=1}^{n} a_{rk} b^*_{ks} = \sum_{k=1} a_{rk} \bar{b}_{sk}$$

$$\Rightarrow (AB^*)_{ss} = \sum_{k=1}^{n} a_{sk} \bar{b}_{sk}$$

$$tr(AB^*) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \overline{b_{ij}}$$

show that $<A, B> = tr(AB^*)$ is an inner product.

(v) $V = C([0,1]) \rightarrow$ vector space of all complex valued continuous fns. on $[0,1]$.

$x, y \in V$. $<x, y> = \int_0^1 f(t) \overline{g(t)} dt$

$(V, <\cdot,\cdot>) \longrightarrow$ is AN inner product space.

DEFN: let $(V, <\cdot,\cdot>)$ be an inner pdt. space. let $x \in V$. the norm of x denoted by $||x||$ is defined by :

$$\|x\| = +\sqrt{\langle x, x\rangle} \qquad \text{(non-negative, real)}$$

$$\|x\| \geq 0 \quad \forall\, x \in V$$

$$\|x\| = 0$$
$$\Leftrightarrow \|x\|^2 = 0$$
$$\Leftrightarrow \langle x, x\rangle = 0$$
$$\Leftrightarrow x = 0.$$

<u>Theorem</u>: let $(V, \langle \cdot, \cdot \rangle)$ be an inner pdt. space. then:

(a) $\|\alpha x\| = |\alpha|\,\|x\|$, $\forall\, \alpha \in \mathbb{C}$ & $x \in V$

(b) $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$ $\quad \forall\, x, y \in V$
(Cauchy-Schwarz inequality)

(c) $\|x+y\| \leq \|x\| + \|y\|$ $\quad \forall\, x, y \in V$.
(triangle inequality)

(a)
$$\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle$$
$$= \alpha \langle x, \alpha x\rangle$$
$$= \alpha \bar{\alpha} \langle x, x\rangle$$
$$\Rightarrow \|\alpha x\|^2 = |\alpha|^2 \langle x, x\rangle$$
$$\Rightarrow \|\alpha x\| = |\alpha|\,\|x\|.$$

(b) we have $\forall\, \lambda \in \mathbb{C}$

$$\langle x - \lambda y, x - \lambda y\rangle \geq 0 \qquad \forall\, x, y \in V$$

$$\Rightarrow \langle x, x\rangle + \langle x, -\lambda y\rangle + \langle -\lambda y, x\rangle + \langle -\lambda y, -\lambda y\rangle \geq 0$$
$$\Rightarrow \langle x, x\rangle - \bar{\lambda}\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \geq 0$$
$$\Rightarrow \|x\|^2 - \bar{\lambda}\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\|y\|^2 \geq 0$$

$\Rightarrow \quad \|x\|^2 - \lambda\langle y,x\rangle - \bar{\lambda}\langle x,y\rangle + |\lambda|^2\|y\|^2 \geq 0$

let $\alpha = \bar{\lambda}\langle x, y\rangle$

$\Rightarrow \bar{\alpha} = \lambda\overline{\langle x, y\rangle} = \lambda\langle y, x\rangle$.

$\Rightarrow \quad \|x\|^2 - \alpha - \bar{\alpha} + |\lambda|^2\|y\|^2 \geq 0$

$\Rightarrow \quad \|x\|^2 + \|y\|^2 - (\alpha + \bar{\alpha}) \geq 0$

$\Rightarrow \quad \|x\|^2 + |\lambda|^2\|y\|^2 - 2\,\mathrm{Re}(\alpha) \geq 0 \quad - \quad (1)$

set $\lambda = \dfrac{\langle x, y\rangle}{\|y\|^2}$, if $y \neq 0$

$$\Rightarrow \quad 0 \leq \|x\|^2 + \frac{|\langle x, y\rangle|^2}{\|y\|^4}\cdot\|y\|^2 - 2\,\mathrm{Re}(\alpha)$$

$\alpha = \bar{\lambda}\langle x, y\rangle$

$= \dfrac{|\langle x, y\rangle|^2}{\|y\|^2}$

$$\Rightarrow 0 \leq \|x\|^2 + \frac{|\langle x, y\rangle|^2}{\|y\|^2} - 2\frac{|\langle x, y\rangle|^2}{\|y\|^2}$$

$$\Rightarrow \quad 0 \leq \|x\|^2 \cdot \|y\|^2 - |\langle x, y\rangle|^2$$

$$\Rightarrow \quad |\langle x, y\rangle| \leq \|x\|^2 \cdot \|y\|^2 \quad //$$

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle$ $\quad \overline{\langle x, y\rangle}$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle)$

$\Rightarrow \|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle|$

$z = \alpha + i\beta$

$z\bar{z} = \alpha^2 + \beta^2$

$\Rightarrow \| x+y \|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\|$

(BY CAUCHY SCHWARZ)

$|z|^2 = \alpha^2 + \beta^2$

$\Rightarrow |z|^2 \geq \alpha^2$

&

$|z|^2 \geq \beta^2$

$\Rightarrow \|x+y\|^2 \leq (\|x\| + \|y\|)^2$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

in CAUCHY SCHWARZ INEQUALITY,

$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

if $y = 0$, $|\langle x, 0 \rangle| = 0$ & $\|y\| = 0$

$\Rightarrow \|y\| \, \|x\| = 0$

$\therefore$, trivially proven.

$$\begin{aligned} \langle x, 0 \rangle &= \langle x, y - y \rangle \\ &= \langle x, y \rangle + \langle x, -y \rangle \\ &= \langle x, y \rangle - \langle x, y \rangle = 0. \end{aligned}$$

$\|$