Theorem: The Dual of the Dual is the primal.

Proof:-

Let the primal LPP be to determine $x^T \in \mathbb{R}^n$ so as to

$$\text{Max } f(x) = cx \quad , c \in \mathbb{R}^n$$

$$\text{STC} \quad Ax = b \text{ and } x \geq 0 \; , \; b^T \in \mathbb{R}^m$$

where A is an $m \times n$ real matrix.

The dual of this primal is the LPP of determining $w^T \in \mathbb{R}^n$ so as to

$$\text{Min } f(w) = b^T w \quad \in \mathbb{R}^m$$

STC

$A^T w \geq c^T$, w unrestricted, $c \in \mathbb{R}^n$

Now, introduce surplus variables $s \geq 0$ in the constraints of the dual and write

$w = w_1 - w_2$ where $w_1 \geq 0$ & $w_2 \geq 0$

The standard form of dual then is to

$$\text{Min} \quad g(w) = b^T(w_1 - w_2), \quad b^T \in \mathbb{R}^n$$

$$A^T(w_1 - w_2) - I_n s = c^T, \quad c \in \mathbb{R}^n$$

$$w_1, w_2 \text{ and } s \geq 0$$

Consider the LPP as our standard primal

the associated dual is

$$\max \quad h(y) = cy \quad , \quad c \in \mathbb{R}^n$$

STC

$$(A^T)^T y \leq (b^T)^T$$

$$\text{and} \quad -(A^T)^T y \leq -(b^T)^T$$

$$-y \leq 0 \quad (\Rightarrow \quad y \geq 0)$$

and y is unrestricted

Eliminating redundancy, the dual problem may be re-written as

$$\max \quad h(y) = cy \quad , \quad c \in \mathbb{R}^n$$

STC

$$\left.\begin{array}{l} Ay \leq b \\ Ay \geq b \end{array}\right\} \Rightarrow \quad Ay = b, \quad b^T \in \mathbb{R}^n$$

$$y \geq 0 \qquad \text{same as Primal}$$

$\square$

Theorem. Weak Duality Theorem

Let x_0 be a FS of Primal problem (PP)

$$\max f(x) = cx$$

$$\text{STC} : Ax \leq b, \quad x \geq 0$$

where x^T and $c \in \mathbb{R}^n$, $b^T \in \mathbb{R}^m$ and A is an $m \times n$ real matrix.

If w_0 be a FS to the Dual Problem (DP) of the primal, namely.

$$\min g(w) = b^T w$$

$$\text{STC} \quad A^T w \geq c^T, \quad w \geq 0$$

where $w^T \in \mathbb{R}^n$ then $cx_0 \leq b^T w_0$

Proof

Given x_0 & w_0 are FS to PP & DP respectively

Then

$$Ax_0 \leq b, \quad x_0 \geq 0$$

$$A^T w_0 \geq c^T, \quad w_0 \geq 0$$

Thus $c \leq w_0^T A$ or $cx_0 \leq w_0^T A x_0 \leq w_0^T b$

$$\Rightarrow cx_0 \leq b^T w_0 \quad (\because w_0^T b = b^T w_0)$$

Theorem: Let x_0 be a FS to the PP

$$\max f(x) = cx$$

STC $Ax \leq b$, $x \geq 0$

and w_0 be a FS to its dual

$$\text{Min } g(w) = b^T w$$

STC $\quad A^T w \geq c^T \quad , w \geq 0$

Where x^T and $c \in \mathbb{R}^n$, w^T and $b^T \in \mathbb{R}^n$ and A is an $n \times n$ real matrix.

If $cx_0 = b^T w_0$ then both x_0 and w_0 are optimal solution to the PP & DP respectively.

Proof:- Let x_0^* be any other feasible solution to primal problem then $\boxed{cx_0^* \leq b^T w_0}$

Then $\quad cx_0^* \leq cx_0 \left(= b^T w_0\right)$

Hence x_0 is an optimal solution of PP.

Similarly if w_0^* is any other F-S

then $$c x_0 \leq b^T w_0^*$$

but $c x_0 = b^T w_0$ then we have

$b^T w_0 \leq b^T w_0^*$ for any arbitrary solution w_0^* of DP.

$\Rightarrow$ w_0 is the optimal solution of DP.

Theorem - (Basic Duality theorem)

Let a PP be

max $f(x) = cx$, STC $Ax \leq b$, $x \geq 0$, $x^T, c \in \mathbb{R}^n$

The associated Dual problem be

min $g(w) = b^T w$, STC $A^T w \geq c^T$, $w \geq 0$, $w^T, b^T \in \mathbb{R}^m$

If x_0 (w_0) is an optimal solution of PP (DP) then there exists a FS to DP such that

$$\boxed{c x_0 = b^T w_0}$$

Proof :- Standard primal can be written as

$$\max \; Z = cX$$

STC $AX + I X_s = b$, where $X_s^T \in \mathbb{R}^m$

is a slack vector and I is the associated Identity matrix.

Let $X_0 = [X_B, 0]$ be an optimum solution to the primal, where $X_B^T \in \mathbb{R}^m$ be an optimal BFS given by $\boxed{X_B = B^{-1} b}$

B is an optimal basis of 'A'.

Then the optimal primal objective function is

$$Z = CX_0 = C_B X_B$$

where C_B is the cost vector associated with X_B

Now the Net Evaluation of the optimal simplex table are given by

$$z_j - c_j = c_B y_j - c_j = \begin{cases} c_B B^{-1} a_j - c_j, & \forall\, a_j \in A \\ c_B B^{-1} e_j - 0, & \forall\, e_j \in I \end{cases}$$

Since X_B is optimal, we must have $z_j - c_j \geq 0$ for all j. This gives

$$\begin{cases} c_B B^{-1} a_j \geq c_j \quad \text{and} \quad c_B B^{-1} e_j \geq 0 \\ c_B B^{-1} A \geq c \quad \text{and} \quad c_B B^{-1} \geq 0 \end{cases}$$

$$\boxed{A^T B^{-1} c_B^T \geq c^T} \quad \text{and} \quad \boxed{B^{-1} c_B^T \geq 0}$$

Now, if we let $B^{-1}c_B^T = w_0$ the above become

$$A^T w_0 \geq c^T \text{ and } w_0 \geq 0, \; w_0^T \in \mathbb{R}^m$$

This means that w_0 is a feasible solution to the dual problem. Moreover, the correspondg dual objective function value is

$$b^T w_0 = w_0^T b = c_B B^{-1} b = c_B x_B = c x_0$$

Thus given an optimal solution x_0 to the PP, there exis a FS w_0 to the dual such that

$$\boxed{c x_0 = b^T w_0}$$

(*) similarly, starting with w_0 the existence of x_0