Signals and Systems

Dr. Rohini. P

Department of ECE,

IIITDM Kancheepuram

1.0 Introduction

- There is an analytical framework – that is, a language for describing signals and systems and an extremely powerful set of tools for analyzing them – that applies equally well to problems in many fields.

- We begin our development of the analytical framework for signals and systems by introducing their mathematical description and representation.

1.0 Introduction

- We build on this foundation in order to develop and describe additional concepts and methods that add considerably both to our understanding of signals and systems and to our ability to analyze and solve problems involving signals and systems that arise in a broad array of applications.

1.1.1 Examples and Mathematical Representation

- Signals may describe a wide variety of physical phenomena. Although signals can be represented in many ways, in all cases the information in a signal is contained in a pattern of variations of some form.

1.1.1 Examples and Mathematical Representation

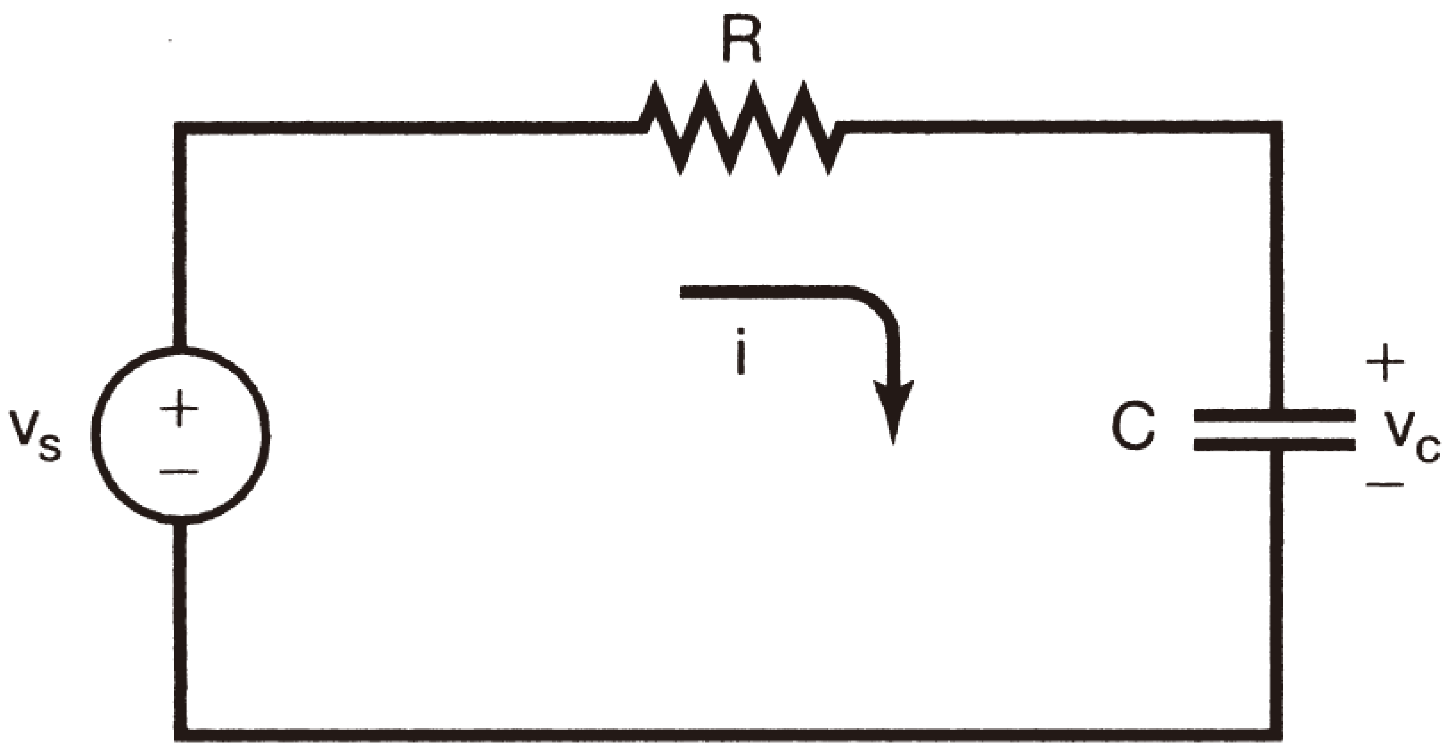


Figure 1.1 A simple *RC* circuit with source voltage v_s and capacitor voltage v_c.

1.1.1 Examples and Mathematical Representation

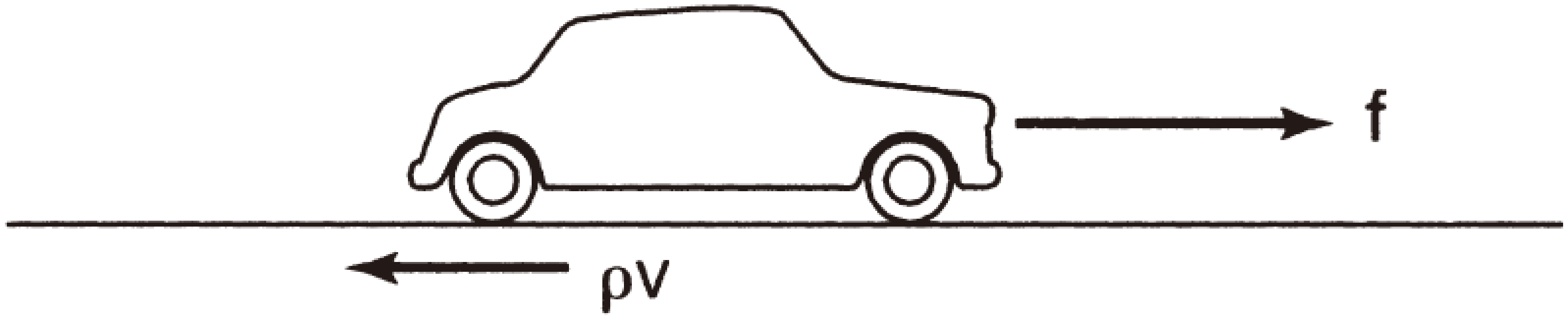


Figure 1.2 An automobile responding to an applied force f from the engine and to a retarding frictional force ρv proportional to the automobile's velocity v.

1.1.1 Examples and Mathematical Representation

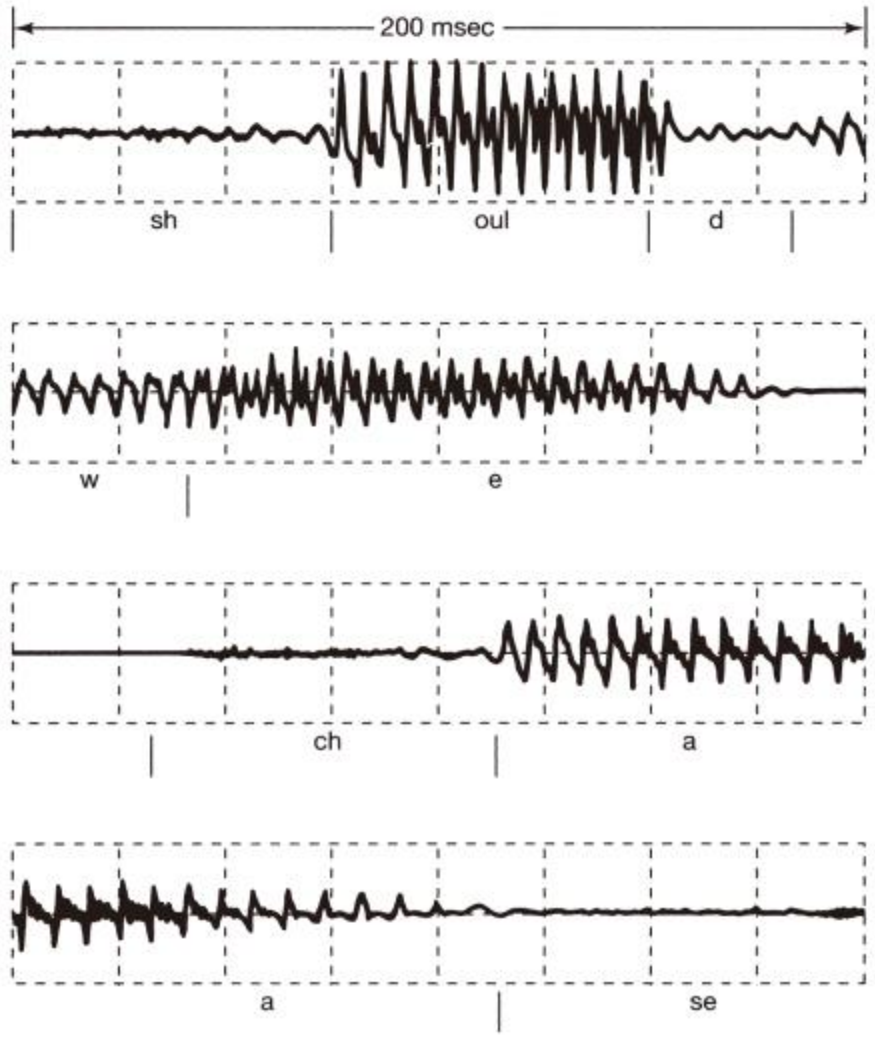


Figure 1.3 Example of a recording of speech. [Adapted from *Applications of Digital Signal Processing*, A.V. Oppenheim, ed. (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1978), p. 121.] The signal represents acoustic pressure variations as a function of time for the spoken words "should we chase." The top line of the figure corresponds to the word "should," the second line to the word "we," and the last two lines to the word "chase." (We have indicated the approximate beginnings and endings of each successive sound in each word.)

1.1.1 Examples and Mathematical Representation

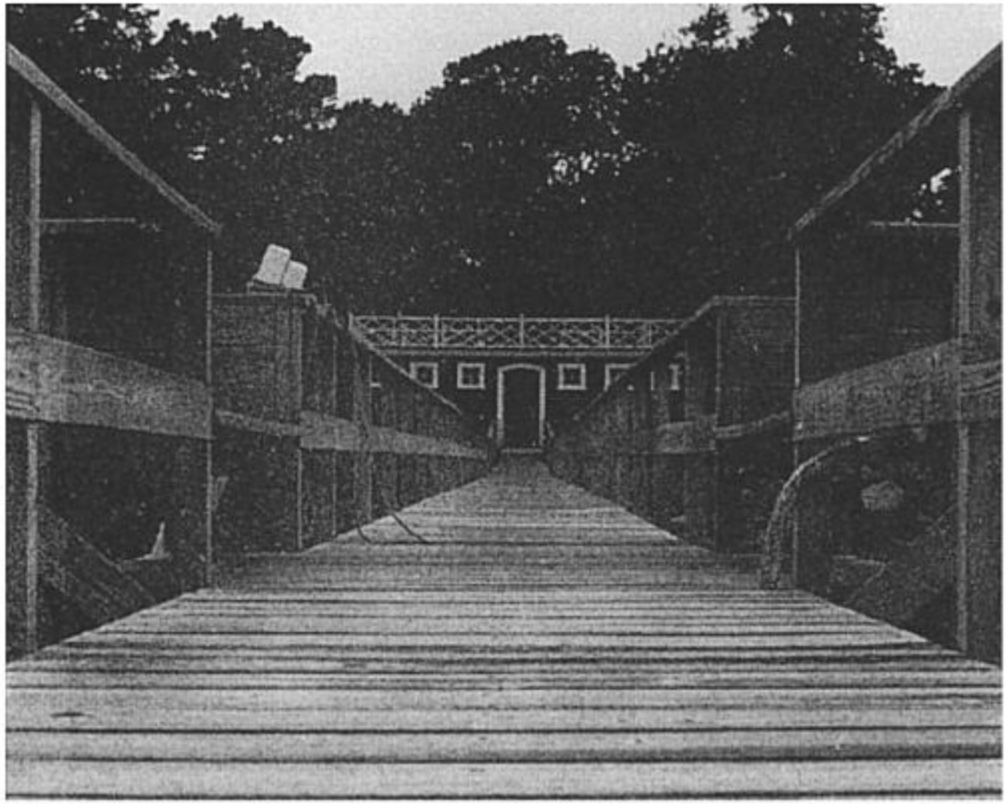


Figure 1.4 A monochromatic picture.

1.1.1 Examples and Mathematical Representation

- Signals are represented mathematically as functions of one or more independent variables.

- For example, a speech signal can be represented mathematically by acoustic pressure as a function of time, and a picture can be represented by the brightness as a function of two spatial variables.

1.1.1 Examples and Mathematical Representation

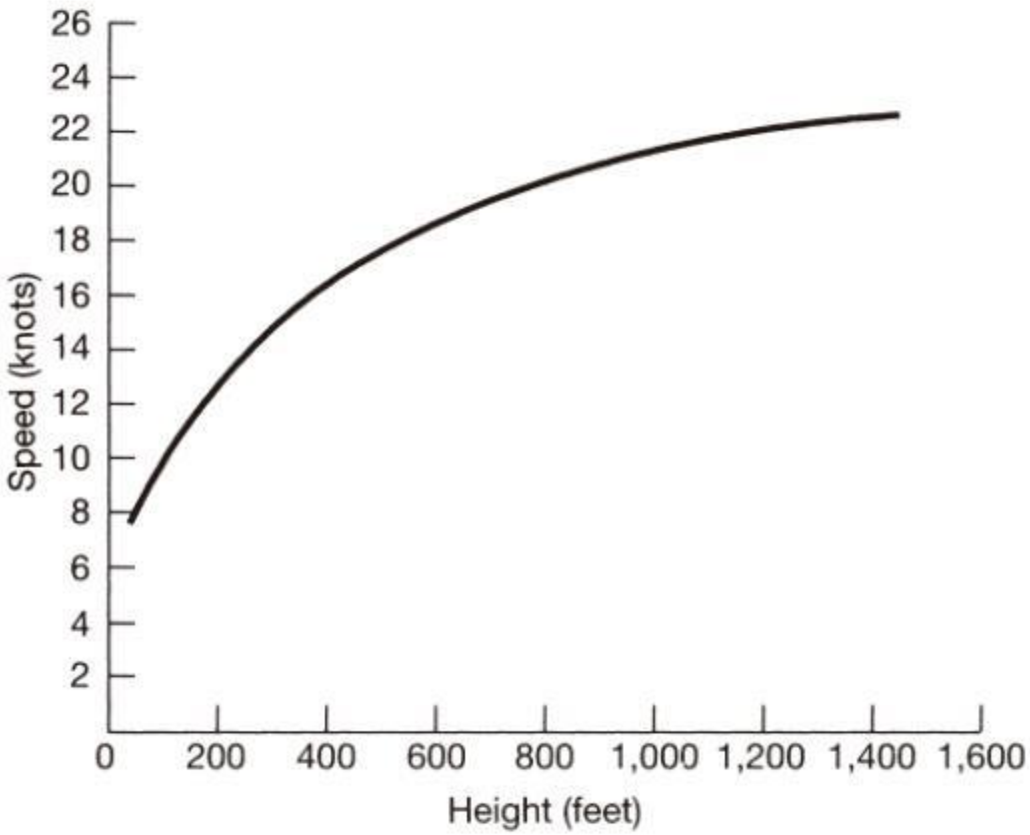


Figure 1.5 Typical annual vertical wind profile. (Adapted from Crawford and Hudson, National Severe Storms Laboratory Report, ESSA ERLTM-NSSL 48, August 1970.)

1.1.1 Examples and Mathematical Representation

- Two basic types of signals
  - Continuous-time signals
    - The independent variable is continuous, and thus these signals are defined for a continuum of values of the independent variable.
  - Discrete-time signals
    - Fore these signals, the independent variable takes on only a discrete set of values.

1.1.1 Examples and Mathematical Representation

Figure 1.6 An example of a discrete-time signal: The weekly Dow-Jones stock market index from January 5, 1929, to January 4, 1930.

1.1.1 Examples and Mathematical Representation

- To distinguish between continuous-time and discrete-time signals, we will use the symbol t to denote the continuous-time independent variable and n to denote the discrete-time independent variable.

1.1.1 Examples and Mathematical Representation

- For continuous-time signals we will enclose the independent variable in parentheses (….), whereas for discrete-time signals we will use brackets [….].
- A discrete-time signal x[n] may represent a phenomenon for which the independent variable is inherently discrete.

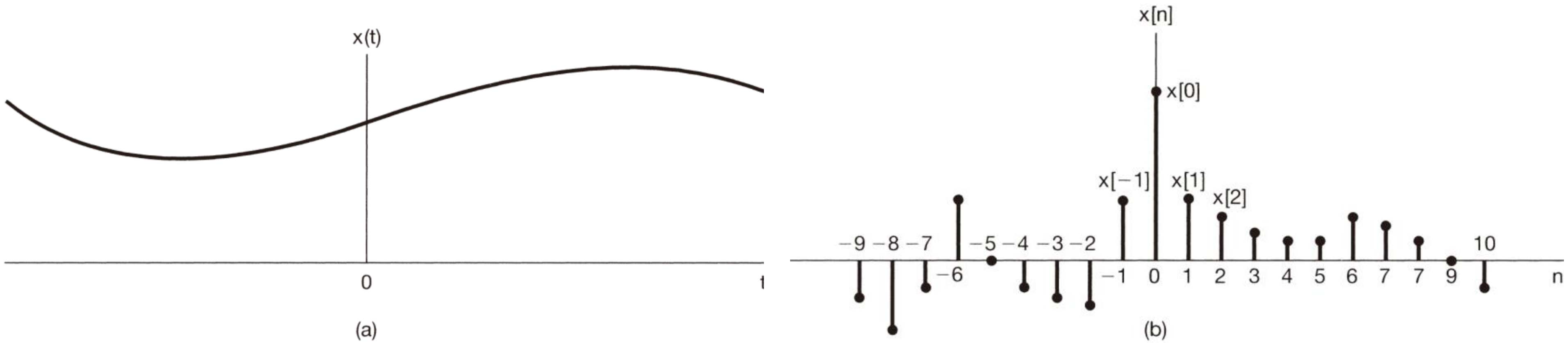


Figure 1.7 Graphical representations of (a) continuous-time and (b) discrete-time signals.

Classification of signals

l. Continuous-time and discrete-time signals

2. Analog and digital signals

3. Periodic and aperiodic signals

4. Energy and power signals

5. Deterministic and probabilistic signals

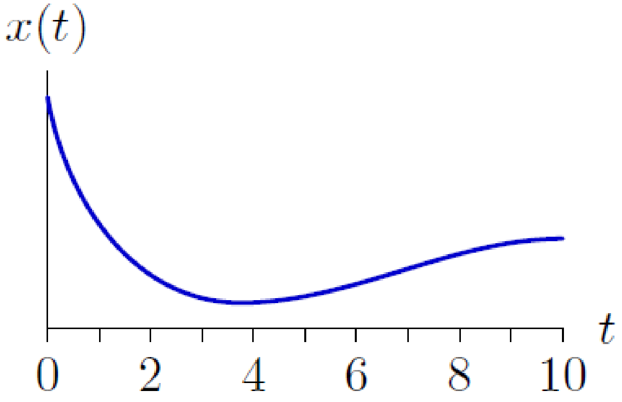
Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

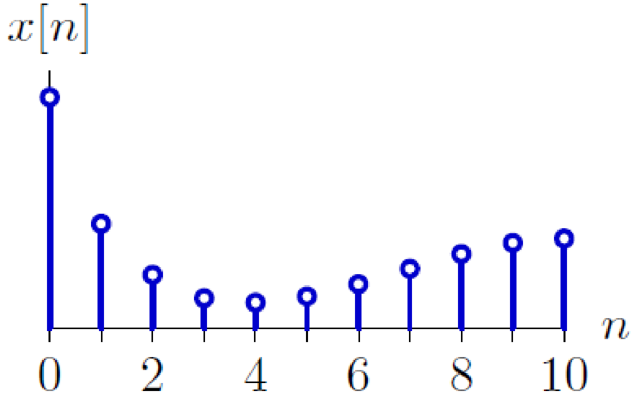
Classification of signals

Continuous-time and discrete-time signals

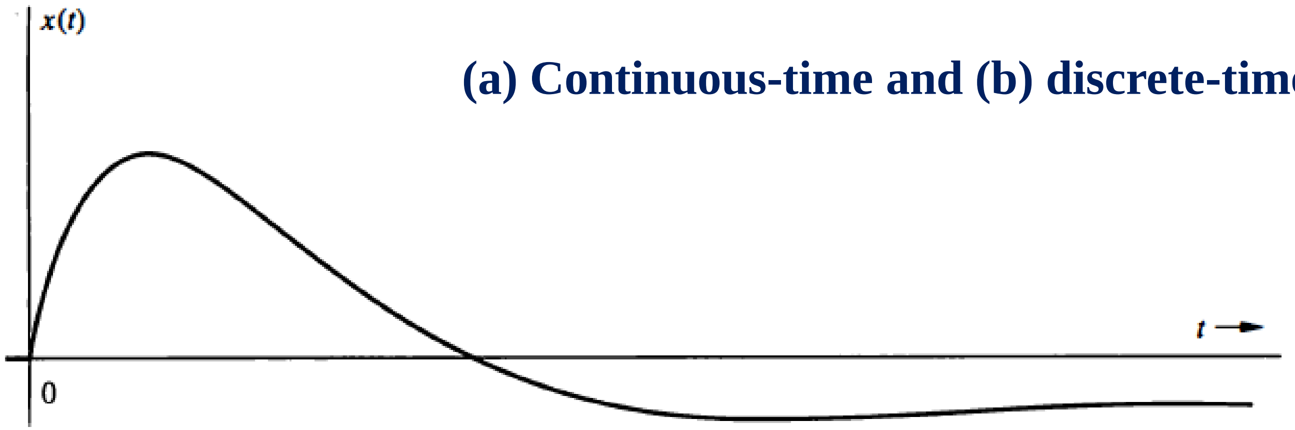
- A signal that is specified for a continuum of values of time t is a continuous-time signal.
  - Example: Telephone and video camera outputs

- A signal that is specified only at discrete values of t is a discrete-time signal.
  - Example: quarterly gross national product (GNP), monthly sales of a corporation, and stock market daily averages

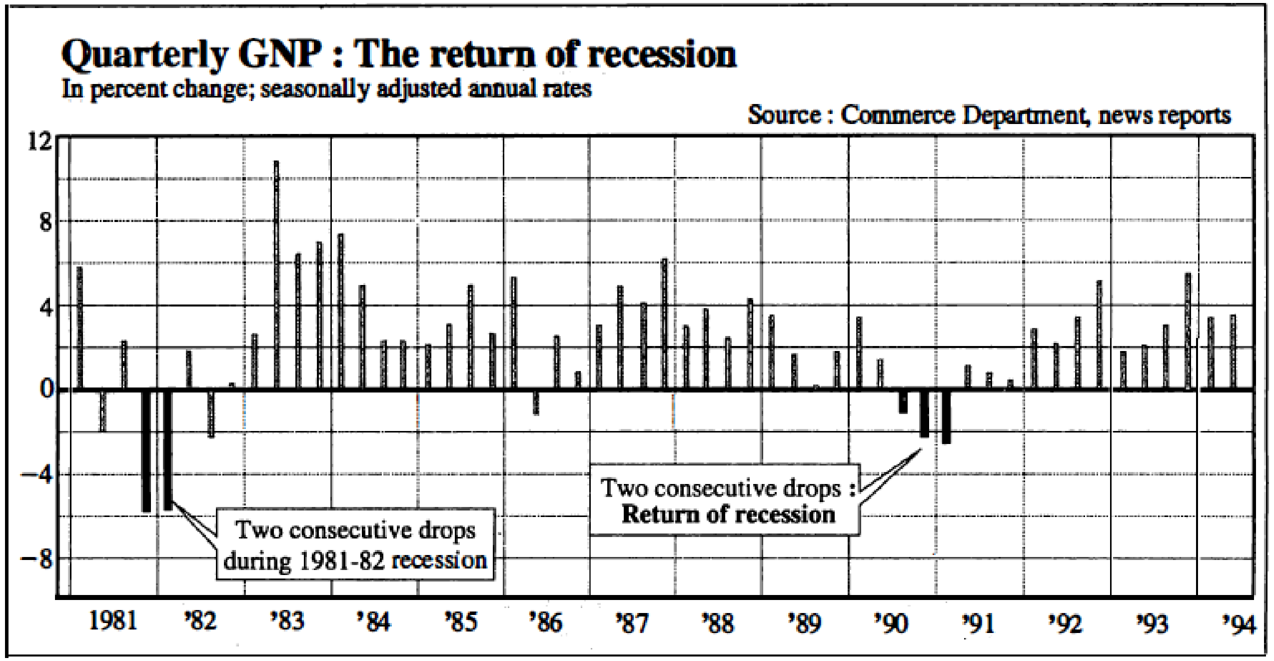




Classification of signals



(a) Continuous-time and (b) discrete-time signals.



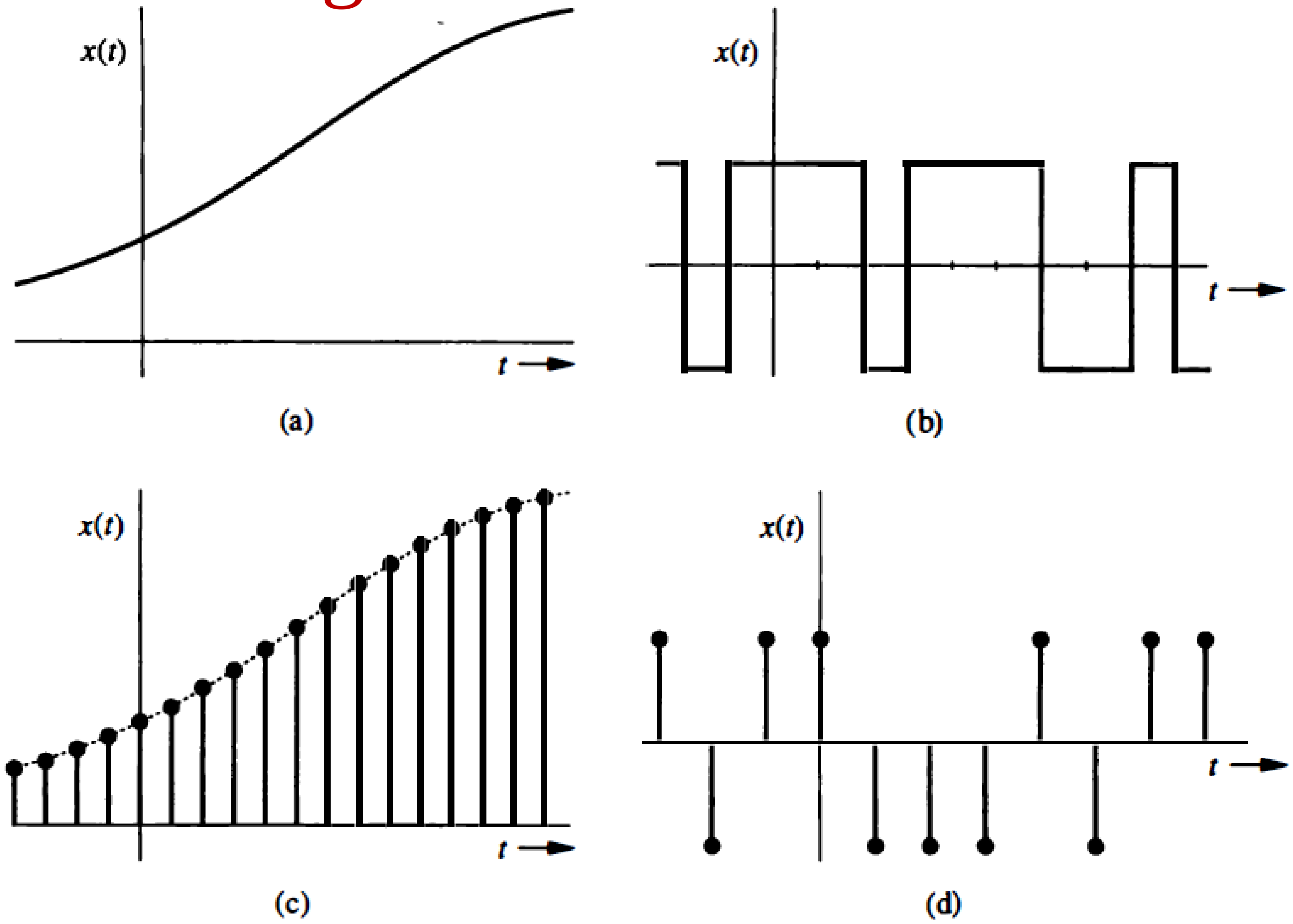
(b)

Classification of signals

Analog and Digital Signals

- A signal whose amplitude can take on any value in a continuous range is an *analog signal*. This means that an analog signal amplitude can take on an infinite number of values.
- A *digital signal*, on the other hand, is one whose amplitude can take on only a finite number of values. Signals associated with a digital computer are digital because they take on only two values (binary signals).
- The terms continuous time and discrete time qualify the nature of a signal along the time (horizontal) axis. The terms analog and digital, on the other hand, qualify the nature of the signal amplitude (vertical axis).

Classification of signals



Classification of signals

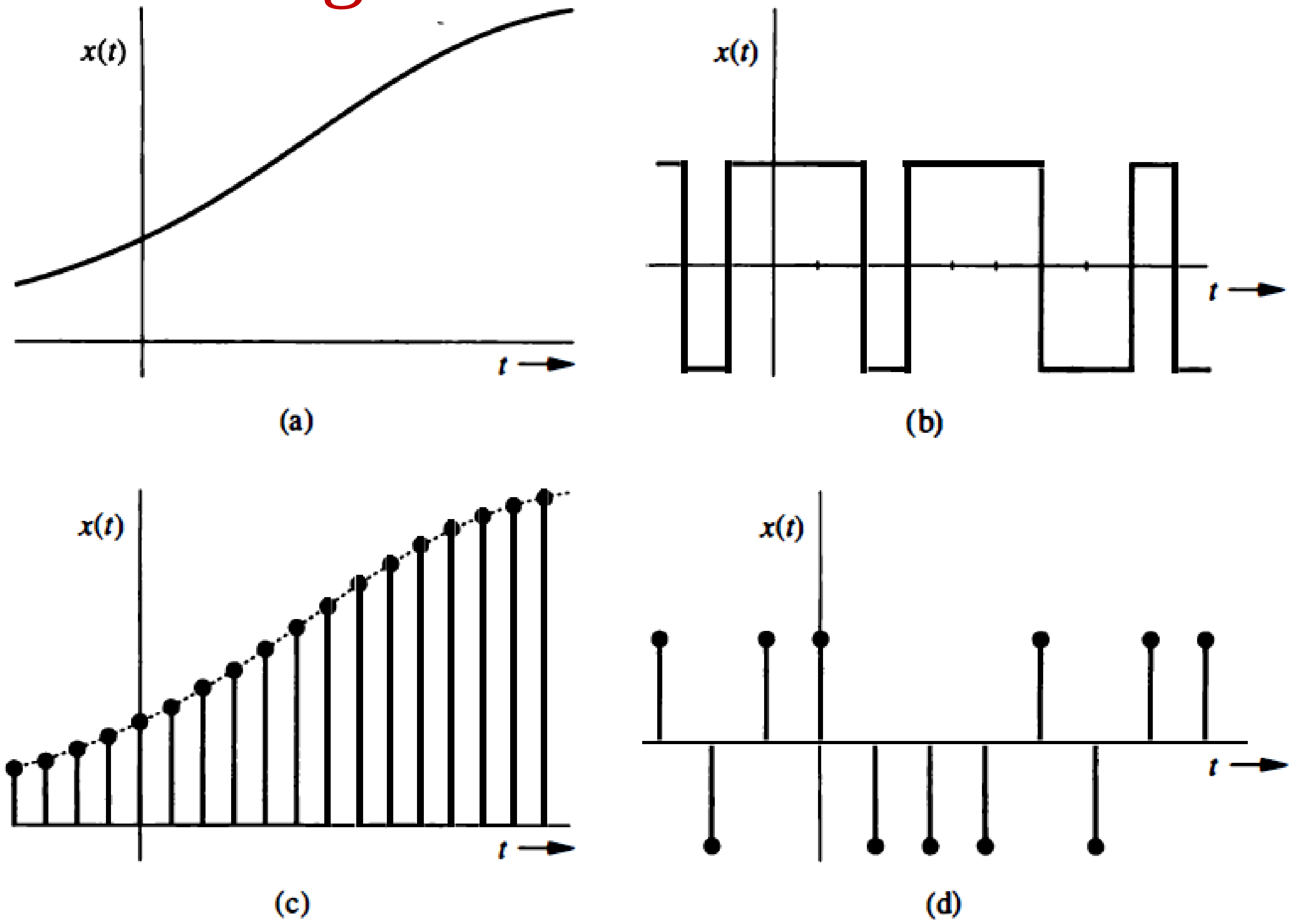


Figure 1.11 Examples of signals: **(a)** analog, continuous time, **(b)** digital, continuous time, **(c)** analog, discrete time, and **(d)** digital, discrete time.

Classification of signals

Periodic and Aperiodic Signals

- A signal $x(t)$ is said to be periodic if for some positive constant T_0.

$$x(t) = x(t + T_0) \quad \text{for all } t$$

The *smallest* value of T_0 that satisfies the periodicity condition of Eq is the *fundamental period* of $x(t)$.

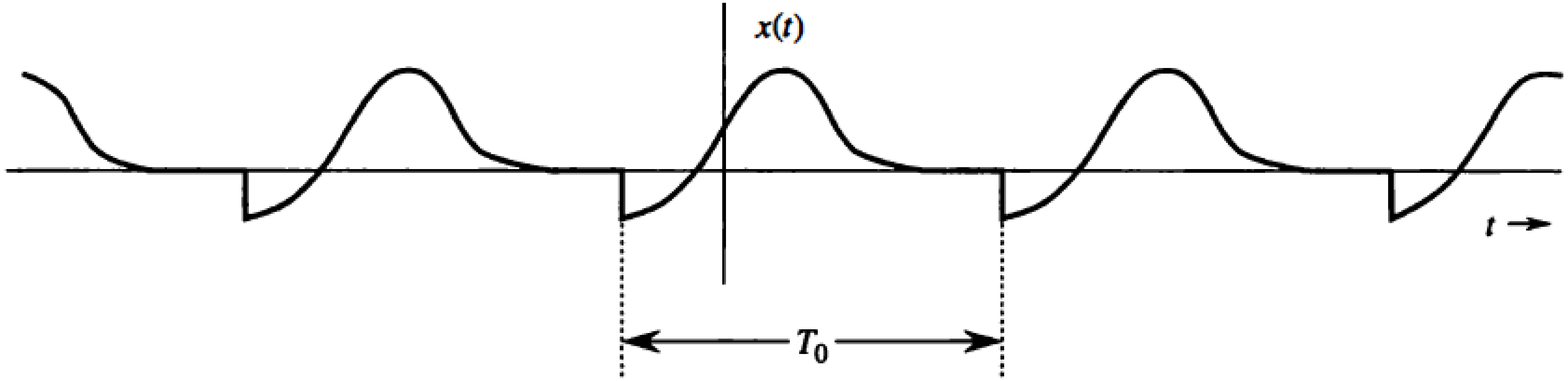
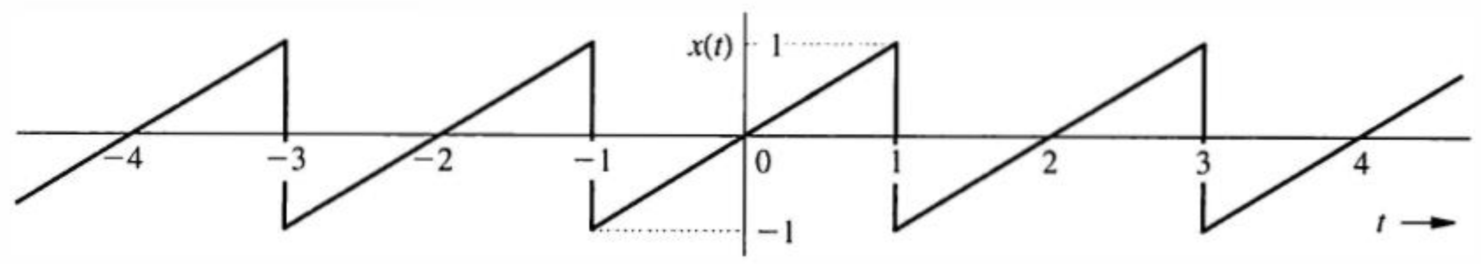


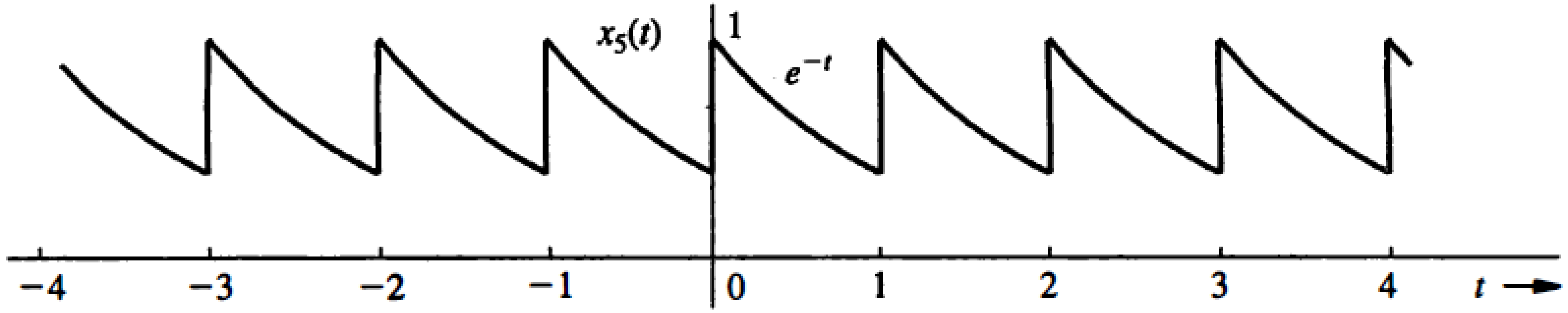
Figure 1.12 A periodic signal of period T_0.

Classification of signals

Periodic and Aperiodic Signals

- A signal is aperiodic if it is not periodic.

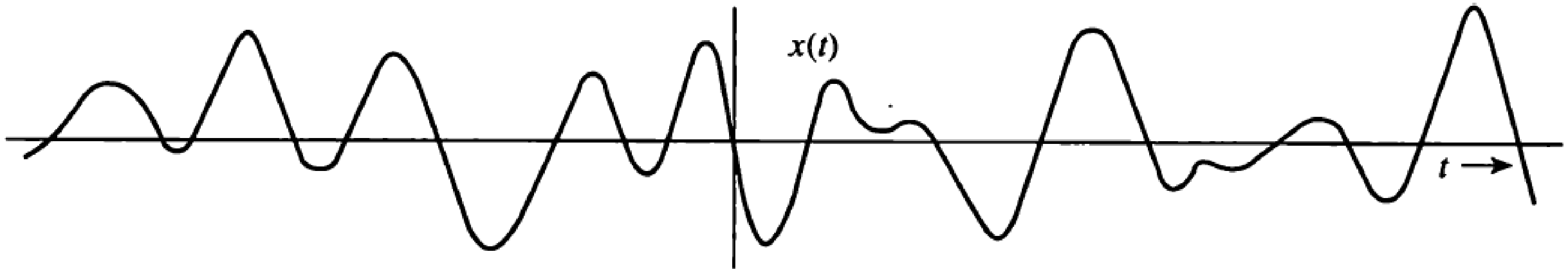




Classification of signals

Periodic and Aperiodic Signals

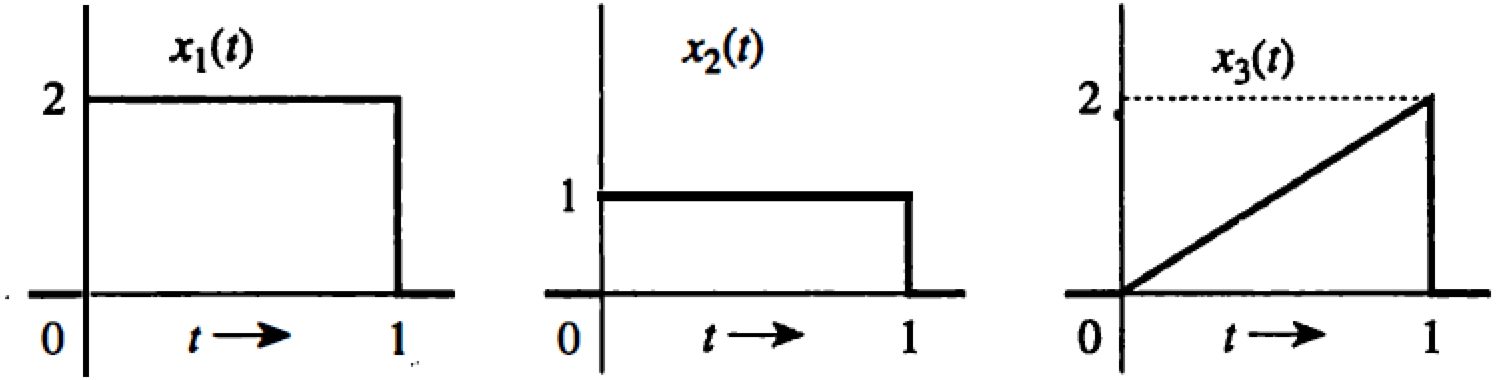
- A signal that starts at $t = -\infty$ and continues forever is an everlasting signal.
- Thus, an everlasting signal exists over the entire interval $-\infty < t < +\infty$.
- A periodic signal is an everlasting signal.



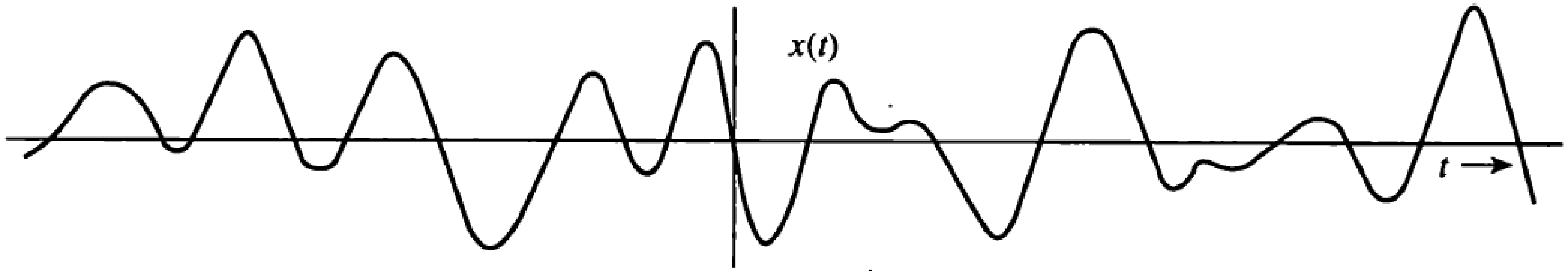
Classification of signals

Causal and Non Causal Signals

- A signal that does not start before t = 0 is a causal signal.
- A signal that starts before t = 0 is a noncausal signal.



causal

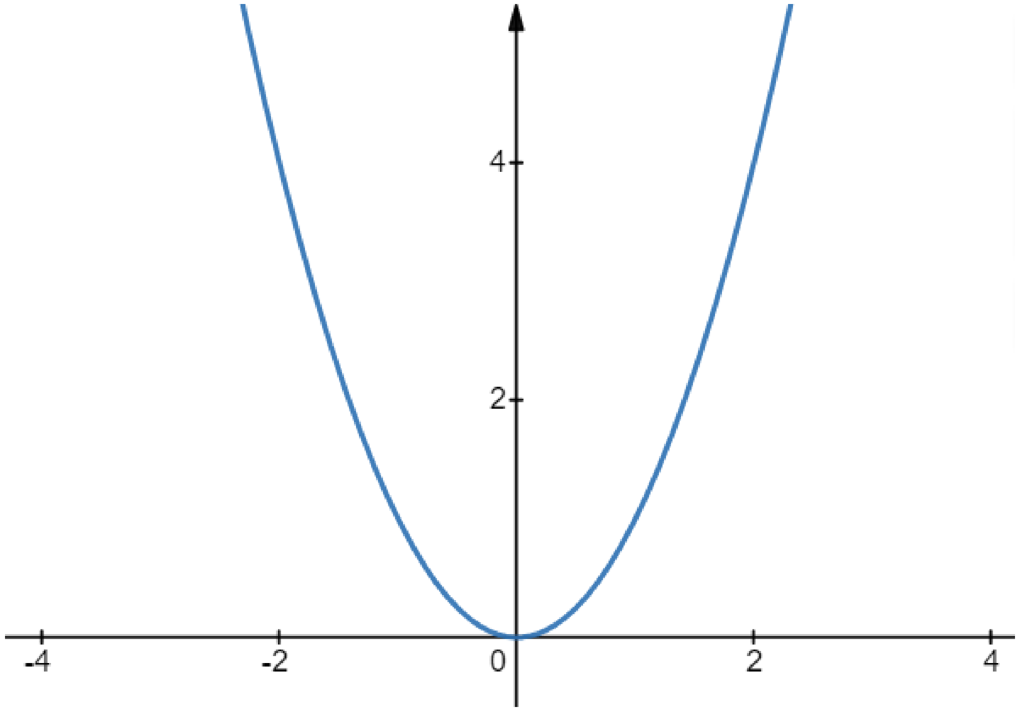


noncausal

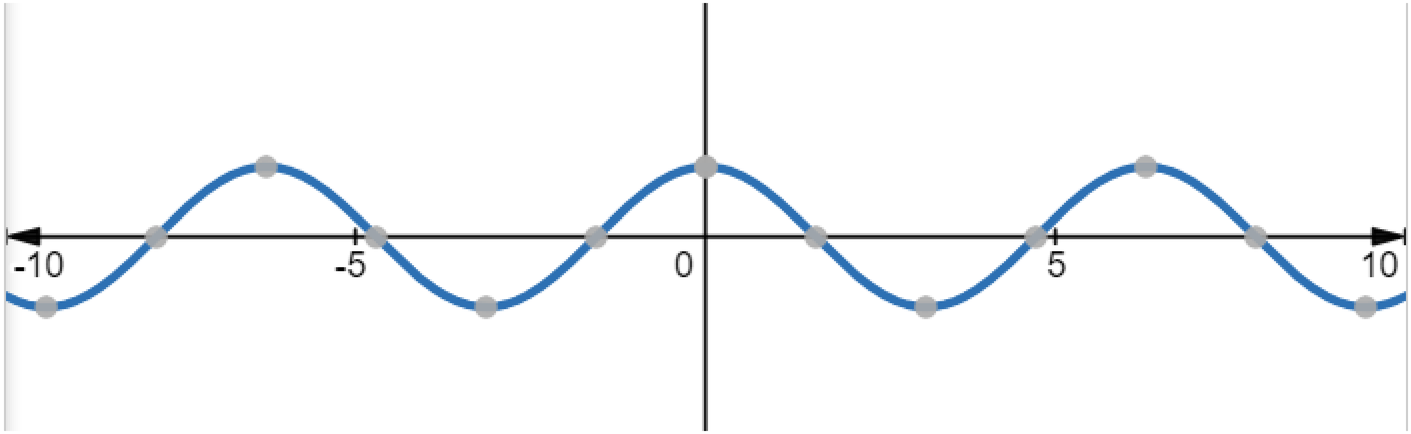
Classification of signals

Even and Odd signal

- If the signal f(t) = f(-t) for all time, then the signal is even signal
- Even signals are symmetric on Y-axis
- Ex: $X(t) = t^2 \rightarrow x(-t) = (-t)^2 \rightarrow x(-t) = t^2$



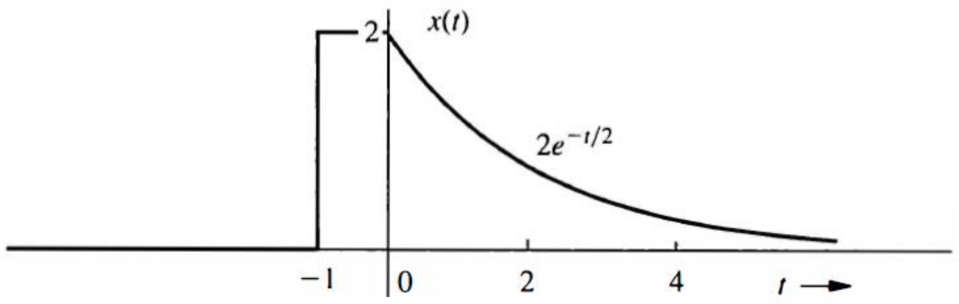
- X(t) = cos(t) → $x(-t)$ = cos(−t) → $x(-t)$ = cos(t)



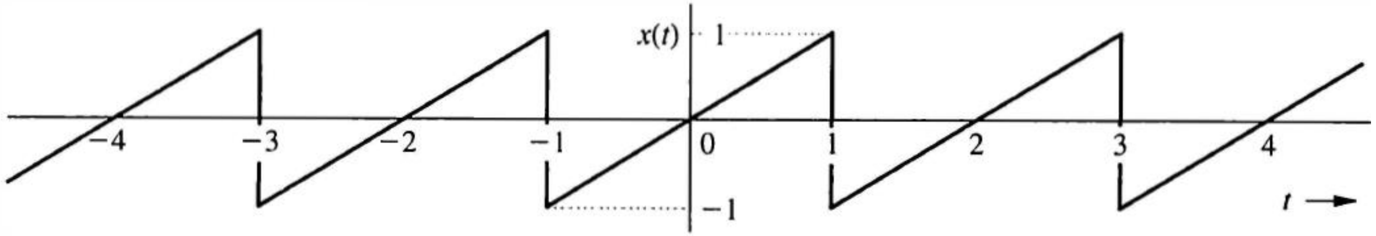
Classification of signals

Energy and Power Signals

- A signal with finite energy is an energy signal, and a signal with finite and nonzero power is a power signal.



energy



power

1.1.2 Signal Energy and Power

- The signals we consider are directly related to physical quantities capturing power and energy in a physical system.

- If $v(t)$ and $i(t)$ are, respectively, the voltage and current across a resistor with resistance R, then the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

- The total energy expended over the time interval $t_1 \le t \le t_2$ is

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

and the average power over this time interval is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

1.1.2 Signal Energy and Power

- The total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous-time signal x(t) is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 \quad dt$$

where $|x|$ denotes the magnitude of number x. Time averaged power is obtained by dividing the eq by t_2 - t_1

- The total energy in a discrete-time signal x[n] over the time interval $n_1 \leq n \leq n_2$ is defined as

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

and dividing by the number of points in the interval, $n_2 - n_1 + 1$, yields the average power over the interval.

1.1.2 Signal Energy and Power

- We define the total energy as the time interval increases without bound.

In continuous time,

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and in discrete time,

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

1.1.2 Signal Energy and Power

➢In an analogous fashion, we can define the time-averaged power over an infinite interval as:

In continuous time,

$$p_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 \, dt$$

and in discrete time,

$$p_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} |x[n]|^2$$

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

1.1.2 Signal Energy and Power

We see that

$$p_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0 \qquad 0 \le t \le 1$$

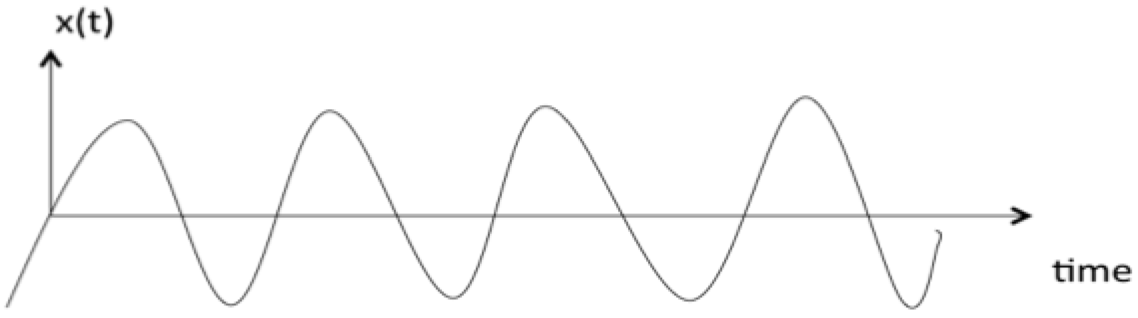
An example of finite-energy signal is a signal that takes on the value 1 for and 0 otherwise. In this case,

$$E_\infty = 1 \text{ and } P_\infty = 0.$$

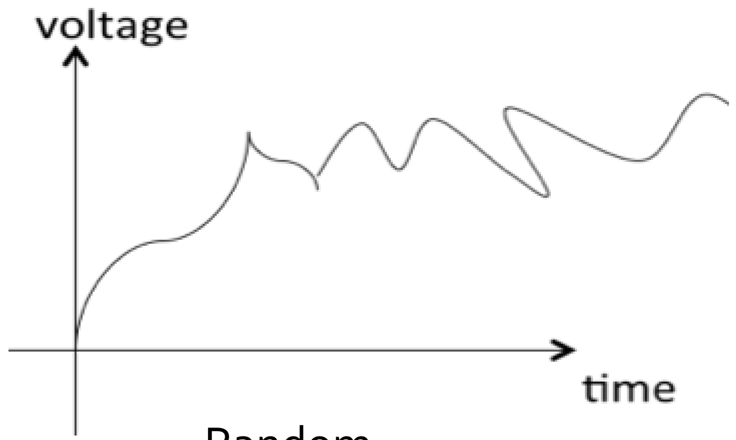
Classification of signals

Deterministic and Random Signals

- A signal whose physical description is known completely, either in a mathematical form or a graphical form, is a deterministic signal.
- A signal whose values cannot be predicted precisely but are known only in terms of probabilistic description, such as mean value or mean-squared value, is a random signal.



Deterministic



Random

1.2 Transformations of the Independent Variable

- Focus on a very limited but important class of elementary signal transformations that involve simple modification of independent variable
- These elementary transformations allow us to introduce several basic properties of signals and systems.
- We will find that they also play an important role in defining and characterizing far richer and important classes of systems.

1.2.1 Examples of Transformations of the Independent Variable

- A simple and very important example of transforming the independent variable of a signal
  - Time shift
  - Time reversal
  - Time scaling

1.2.1 Examples of Transformations of the Independent Variable

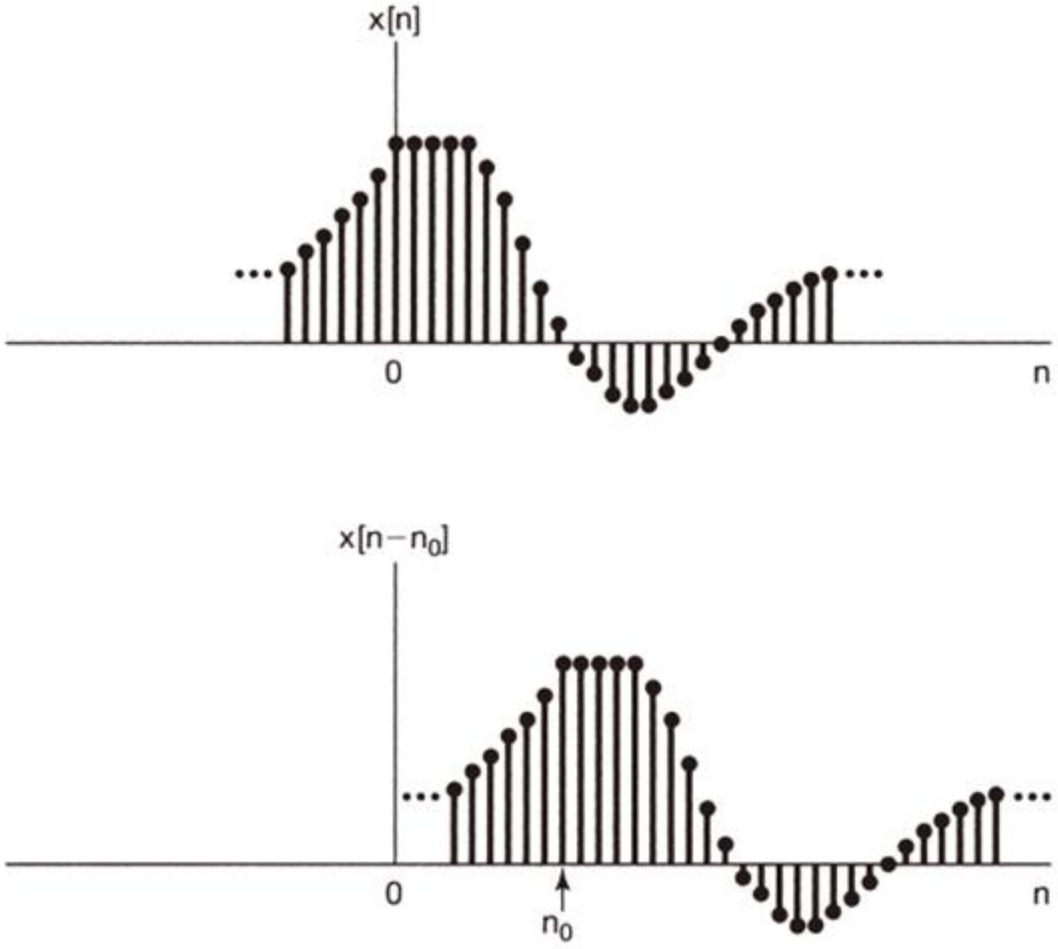


Figure 1.8 Discrete-time signals related by a time shift. In this figure $n_0 > 0$, so that $x[n-n_0]$ is a delayed verson of $x[n]$ (i.e., each point in $x[n]$ occurs later in $x[n-n_0]$).

1.2.1 Examples of Transformations of the Independent Variable

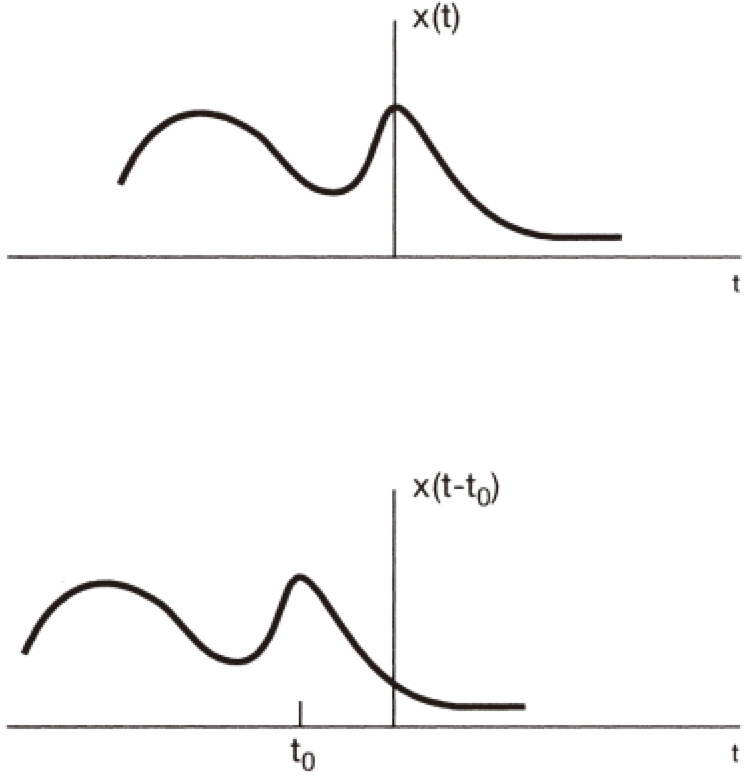


Figure 1.9 Continuous-time signals related by a time shift. In this figure $t_0 < 0$, so that $x(t - t_0)$ is an advanced version of $x(t)$ (i.e., each point in $x(t)$ occurs at an earlier time in $x(t - t_0)$).

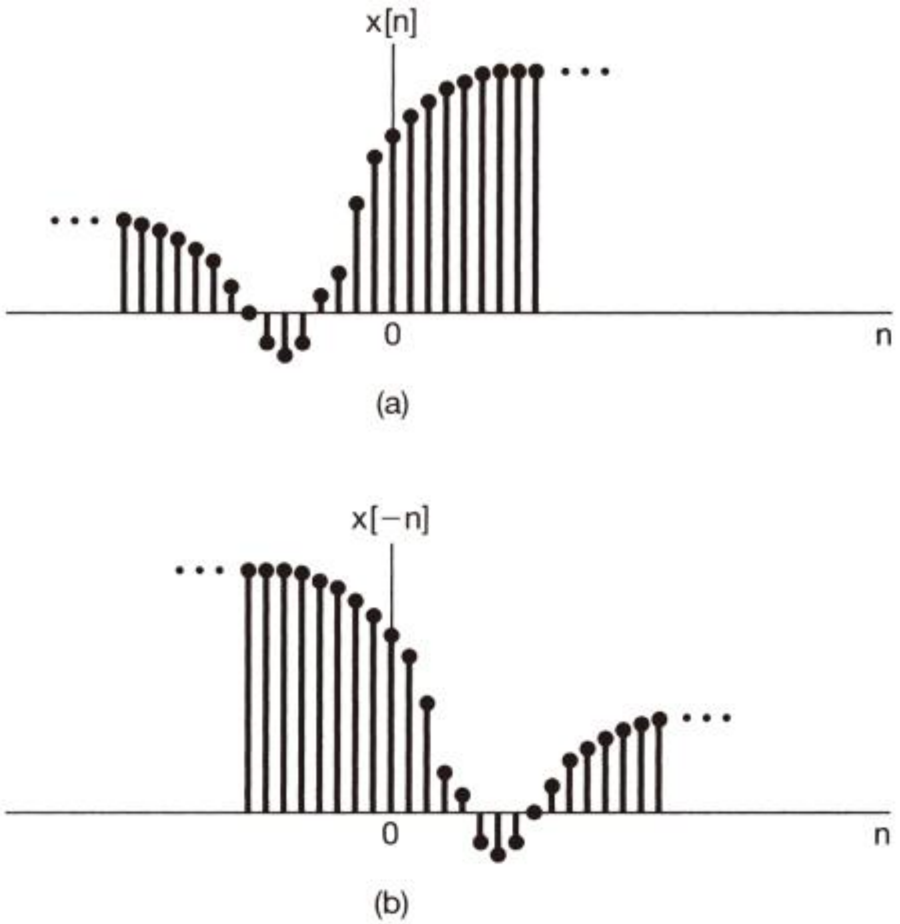


Figure 1.10 (a) A discrete-time signal $x[n]$; (b) its reflection $x[-n]$ about $n = 0$.

1.2.1 Examples of Transformations of the Independent Variable

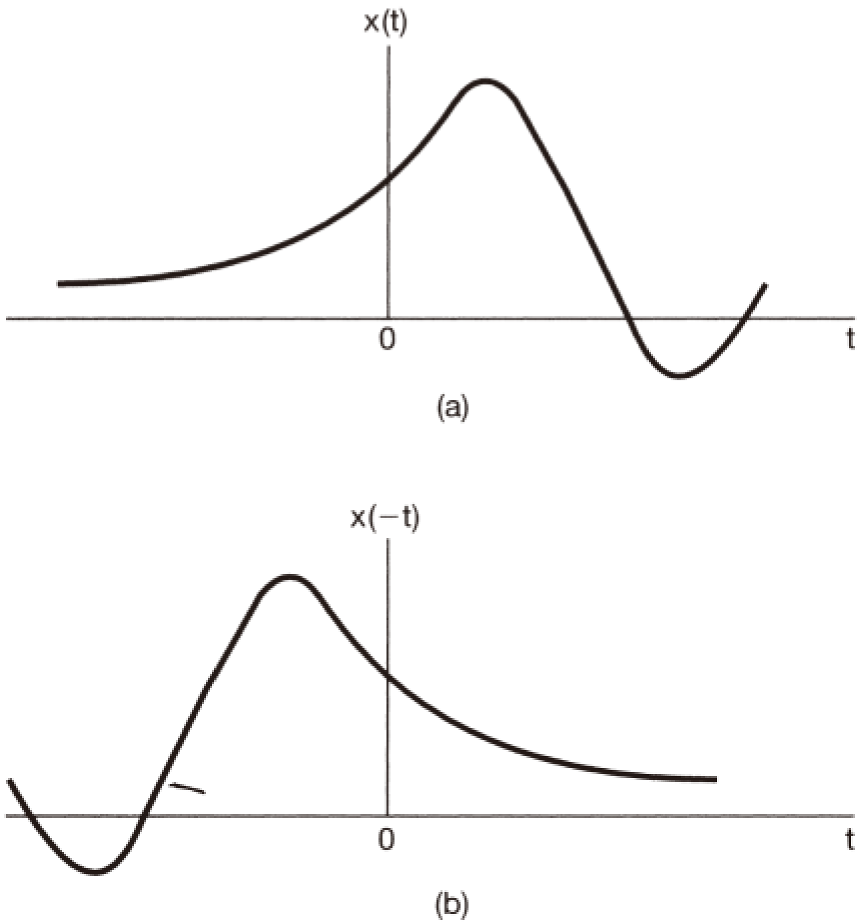


Figure 1.11 (a) A continuous-time signal $x(t)$; (b) its reflection $x(-t)$ about $t = 0$.

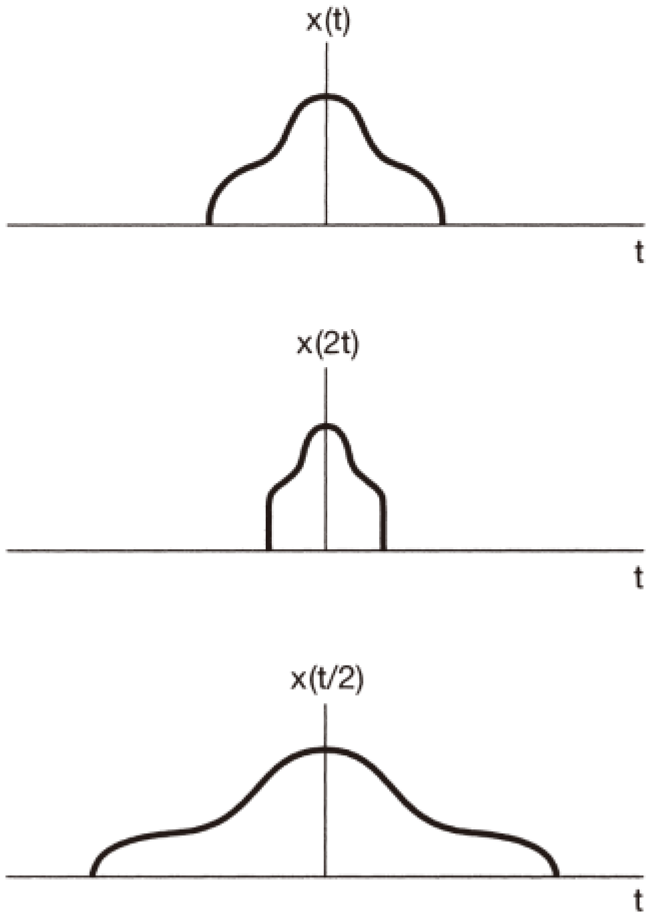
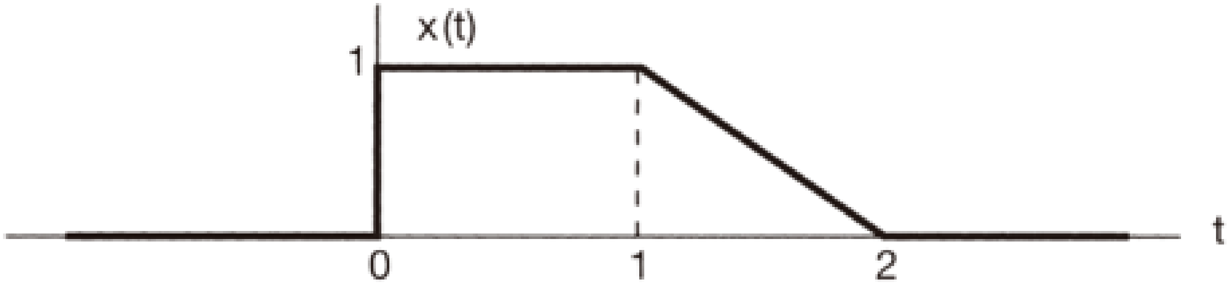


Figure 1.12 Continuous-time signals related by time scaling.

Example 1.1

Given the signal $x(t)$ shown in Figure 1.13(a), the signal $x(t+1)$ corresponds to an advance by one unit along the t axis as illustrated in Figure 1.13(b). Specifically, we note that the value of $x(t)$ at $t = t_0$ occurs in $x(t+1)$ at $t = t_0 - 1$. For example, the value of $x(t)$ at $t = 1$ is found in $x(t+1)$ at $t = 1-1=0$. Also, since $x(t)$ is zero for $t < 0$, we have $x(t+1)$ zero for $t < -1$. Similarly, since $x(t)$ is zero for $t > 2$, $x(t+1)$ is zero for $t > 1$.

Let us also consider the signal $x(-t+1)$, which may be obtained by replacing t with $-t$ in $x(t+1)$. That is, $x(-t+1)$ is the time reversed version of $x(t+1)$. Thus, $x(-t+1)$ may be obtained graphically by reflecting $x(t+1)$ about the t axis as shown in Figure 1.13(c).



(a)

Compute $x(\frac{3}{2}t+1)$

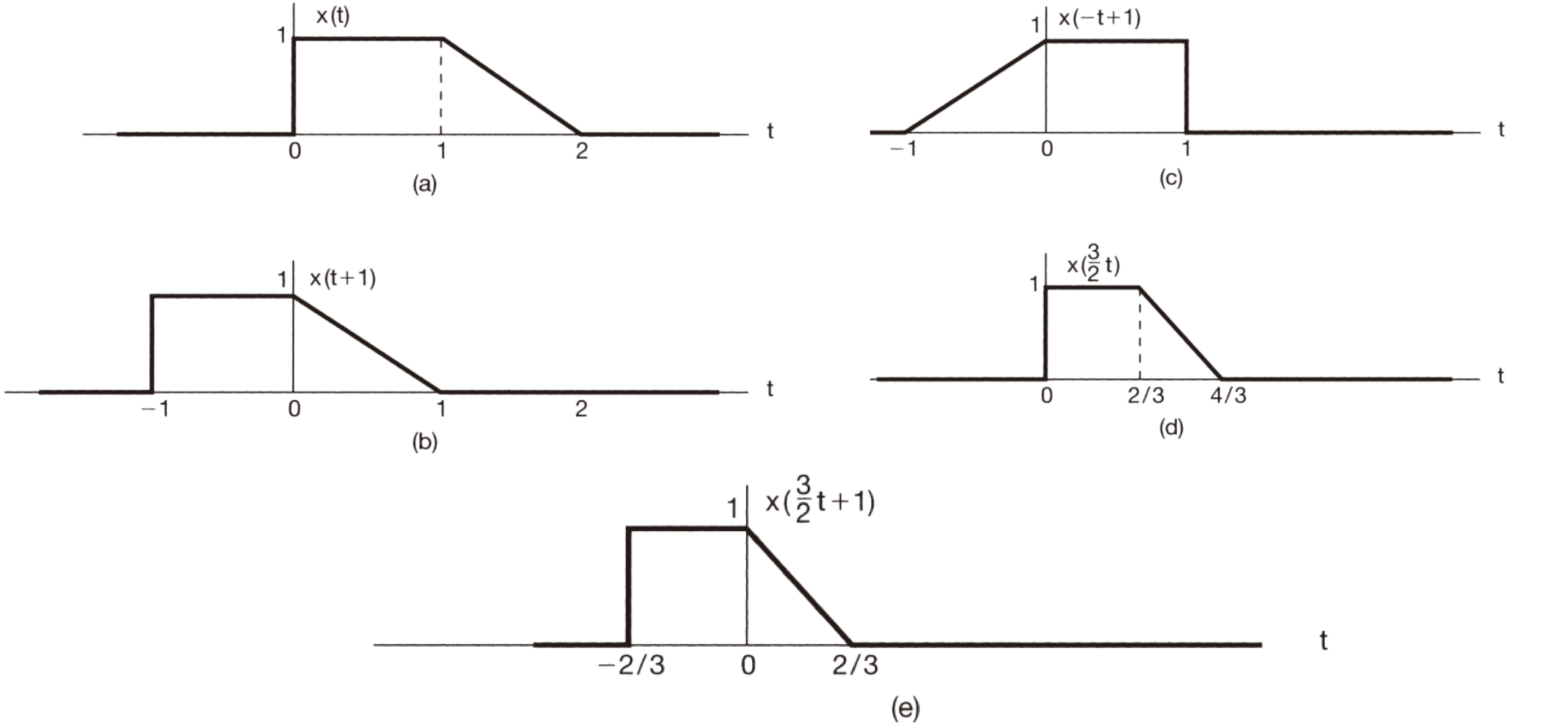


Figure 1.13 (a) The continuous-time signal $x(t)$ used in Examples 1.1–1.3 to illustrate transformations of the independent variable; (b) the time-shifted signal $x(t+1)$; (c) the signal $x(-t+1)$ obtained by a time shift and a time reversal; (d) the time-scaled signal $x(\frac{3}{2}t)$; and (e) the signal $x(\frac{3}{2}t+1)$ obtained by time-shifting and scaling.

1.2.2 Periodic Signals

A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which

$$x(t) = x(t + T) \quad (1.11)$$

for all values of t. In other words, a periodic signal has the property that it is unchanged by a time shift of T. We say that $x(t)$ is periodic with *period* T.

We can readily deduce that if $x(t)$ is periodic with period T, then $x(t) = x(t+mT)$ for all t and for any integer m. Thus, x(t) is also periodic with period $2T$, $3T$, $4T$,….The fundamental period T_0 of $x(t)$ is the smallest positive value of T for which eq. (1.11) holds.

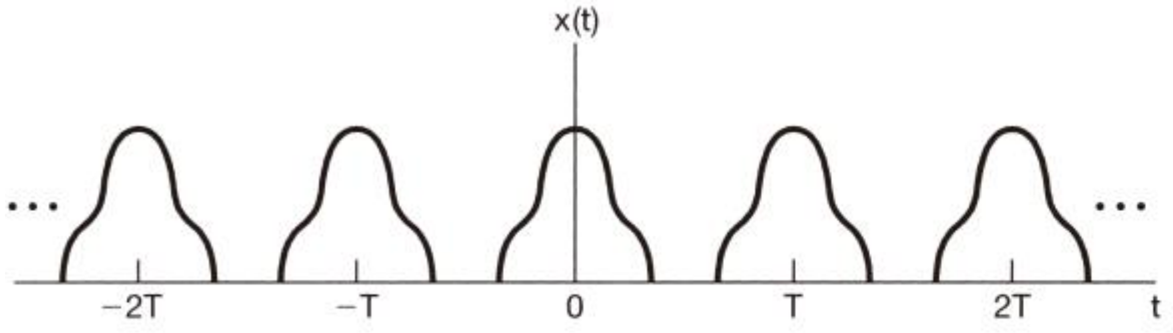


Figure 1.14 A continuous-time periodic signal.

1.2.2 Periodic Signals

- Periodic signals are defined analogously in discrete time. Specifically, a discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if it is unchanged by a time shift of N, i.e., if

$$x[n] = x[n+N] \qquad (1.12)$$

for all values of n.

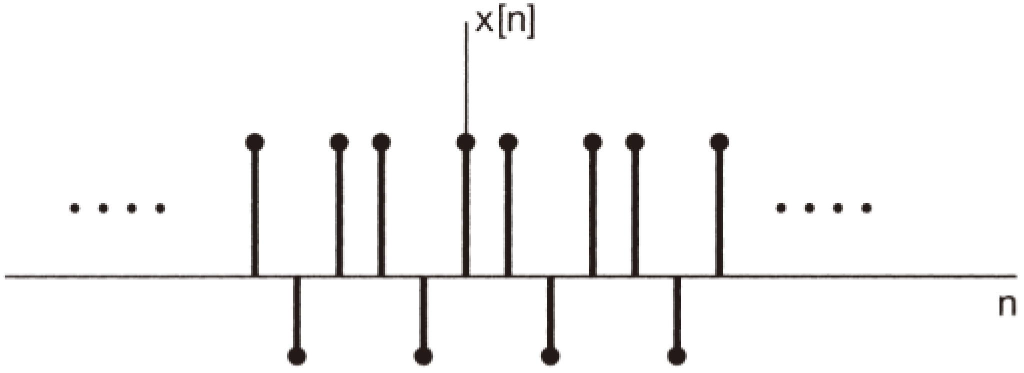


Figure 1.15 A discrete-time periodic signal with fundamental period $N_0 = 3$.

Example 1.4

Let us illustrate the type of problem solving that may be required in determining whether or not a given signal is periodic. The signal whose periodicity we wish to check is given by

$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases} \quad (1.13)$$

We know that $\cos(t+2\pi) = \cos(t)$ and $\sin(t+2\pi) = \sin(t)$. Thus, considering t >0 and t <0 separately, we see that $x(t)$ does repeat itself over every interval of length 2π. However, as illustrated in Figure 1.16, $x(t)$ also has a discontinuity at the time origin that does not recur at any other time. Since every feature in the shape of a periodic signal must recur periodically, we conclude that the signal $x(t)$ is not periodic.

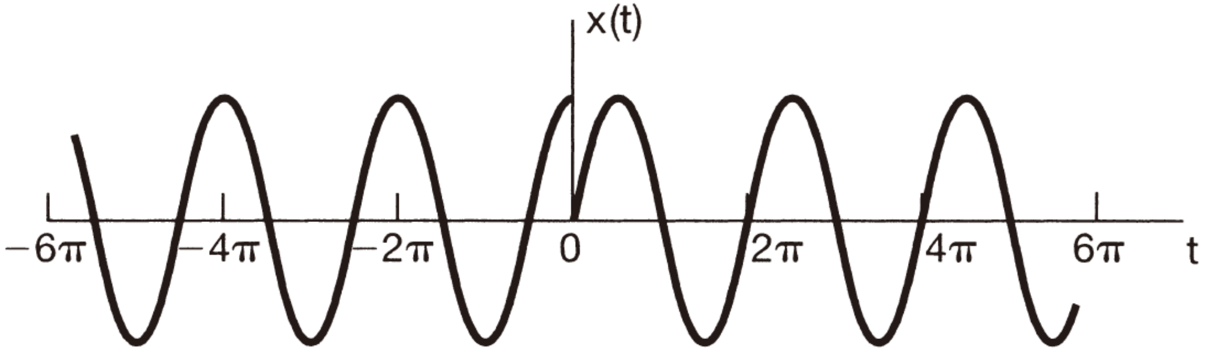
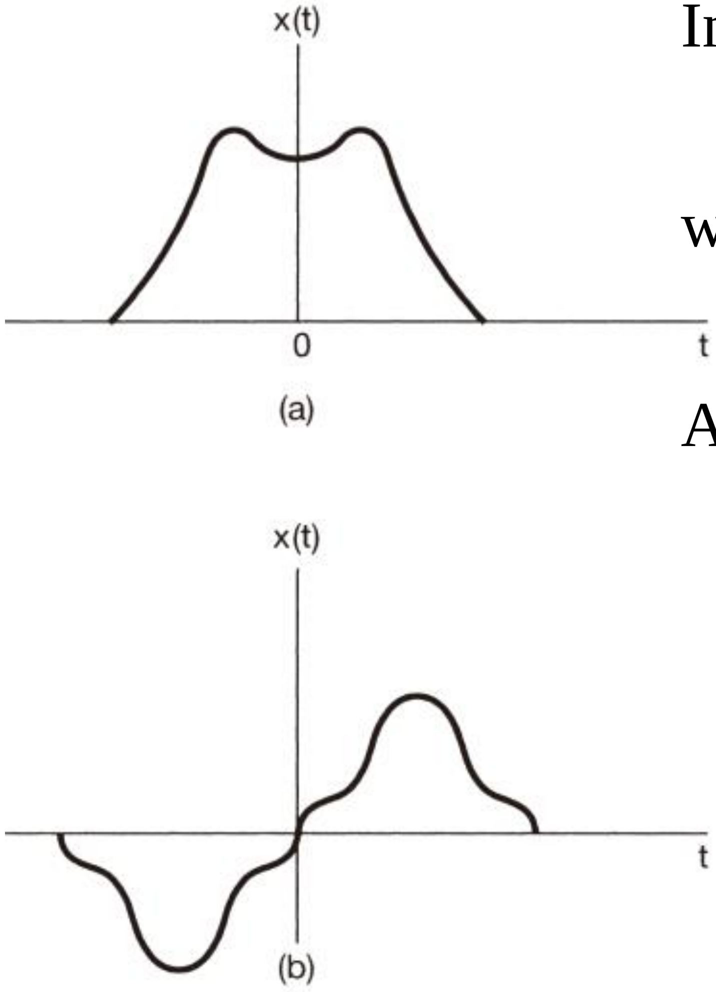


Figure 1.16 The signal $x(t)$ considered in Example 1.4.

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

1.2.3 Even and Odd Signals



In continuous time a signal is even if

$$x(-t) = x(t) \qquad (1.14)$$

while a discrete-time signal is even if

$$x[-n] = x[n] \qquad (1.15)$$

A signal is referred to as odd if

$$x(-t) = -x(t) \qquad (1.16)$$

$$x[-n] = -x[n] \qquad (1.17)$$

Figure 1.17 (a) An even continuous-time signal; (b) an odd continuous-time signal.

1.2.3 Even and Odd Signals

An important fact is that any signal can be broken into a sum of two signals, one of which is even and one of which is odd.

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the even part of x(t). Similarly, the odd part of x(t) is given by

$$\mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

1.2.3 Even and Odd Signals

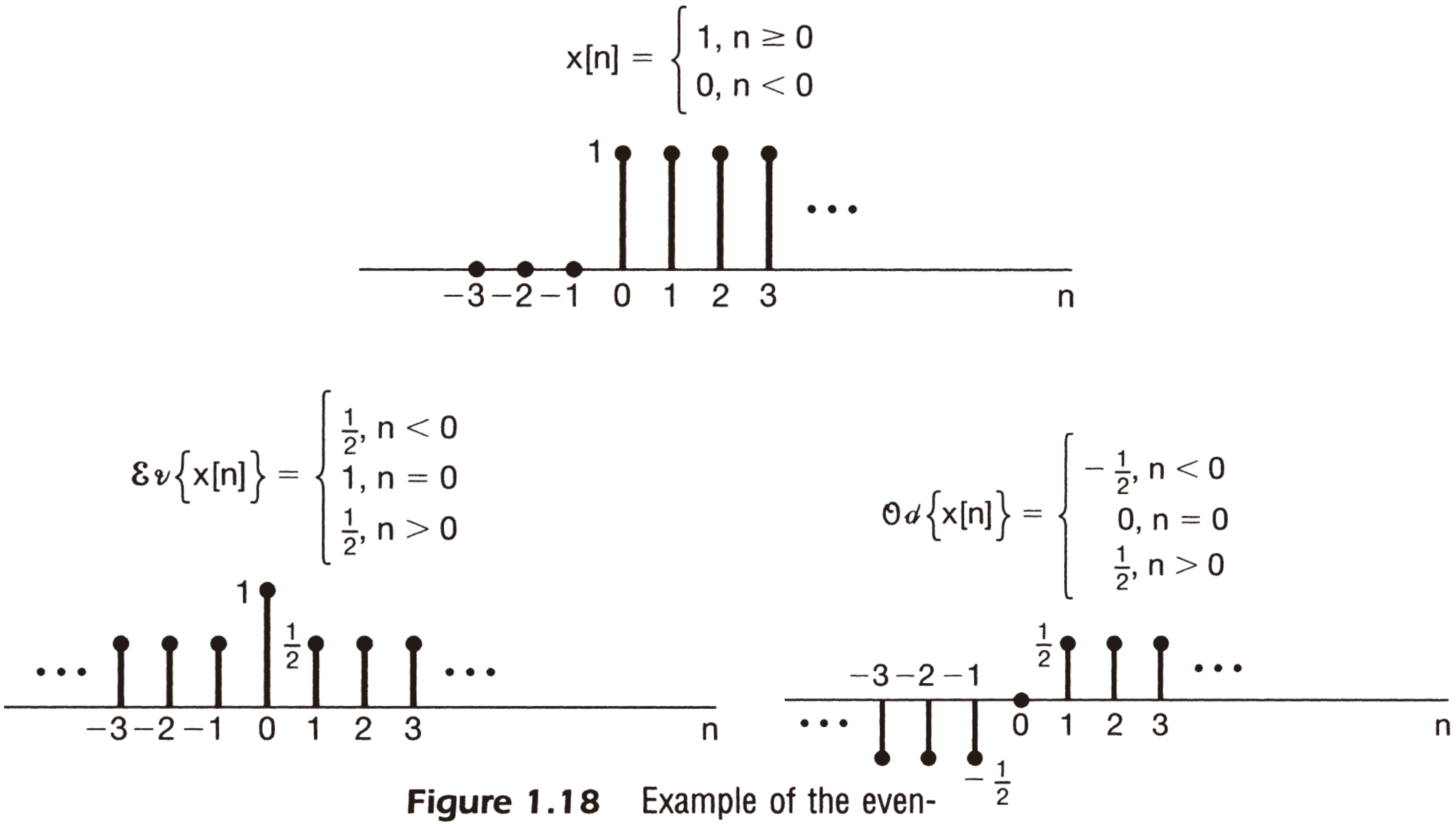


Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

The Discrete-Time Unit Impulse and Unit Step Sequences

- One of the simplest discrete-time signals is the unit impulse, which is defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \qquad (1.63)$$

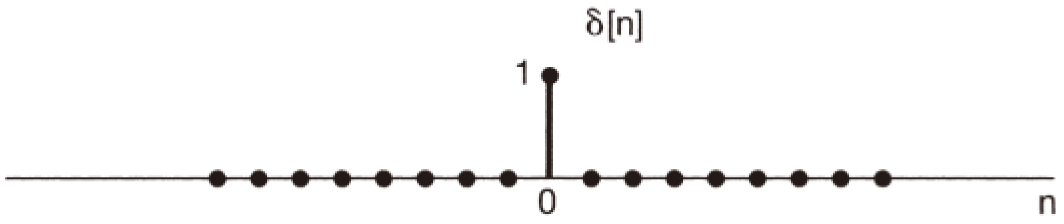


Figure 1.28 Discrete-time unit impulse (sample).

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

The Discrete-Time Unit Impulse and Unit Step Sequences

- A second basic discrete-time signal is the discrete-time unit step, denoted by $u[n]$ and defined by

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \quad (1.64)$$

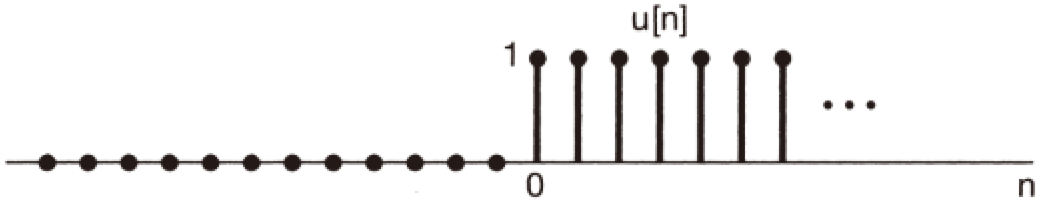


Figure 1.29 Discrete-time unit step sequence.

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

The Discrete-Time Unit Impulse and Unit Step Sequences

- In particular, the discrete-time unit impulse is the first difference of the discrete-time step

$$\delta[n] = u[n] - u[n-1] \qquad (1.65)$$

- Conversely, the discrete-time unit step is the running sum of the unit sample. That is,

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \qquad (1.66)$$

The Discrete-Time Unit Impulse and Unit Step Sequences

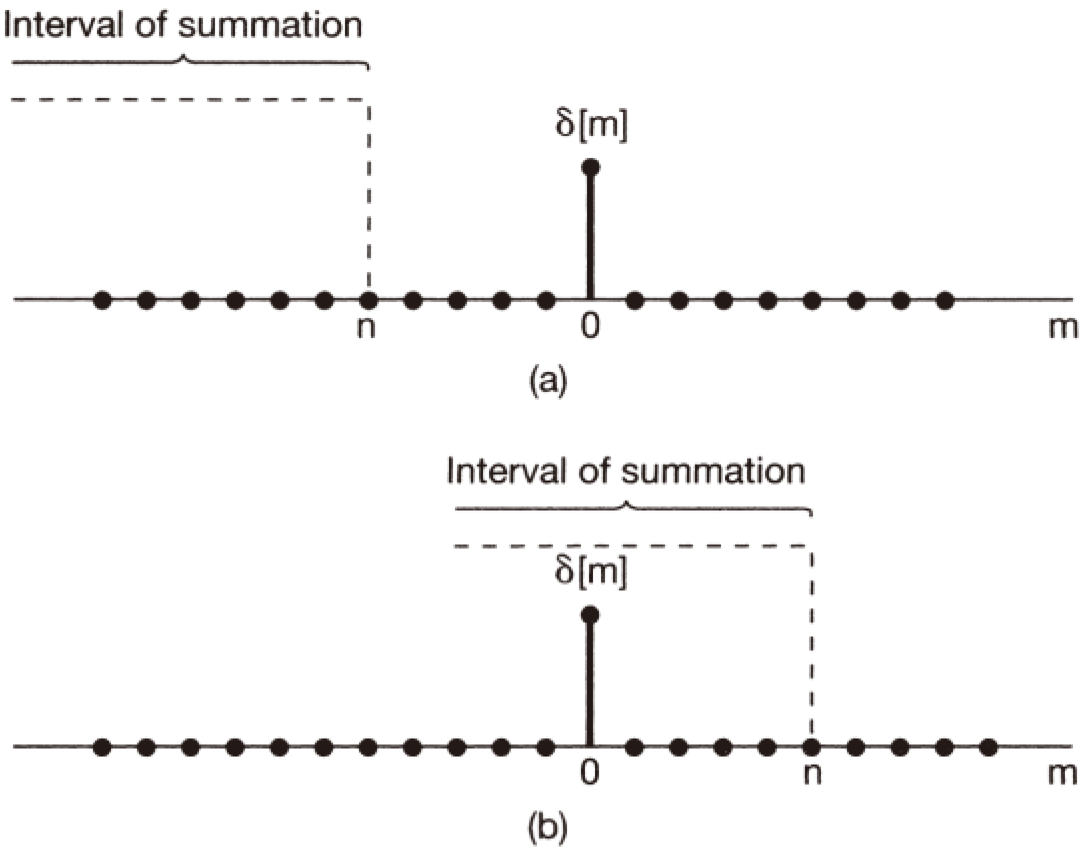


Figure 1.30 Running sum of eq. (1.66): (a) $n < 0$; (b) $n > 0$.

The Discrete-Time Unit Impulse and Unit Step Sequences

- We find that the discrete-time unit step can also be written in terms of the unit sample as

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

Or equivalently,

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \qquad (1.67)$$

The Discrete-Time Unit Impulse and Unit Step Sequences

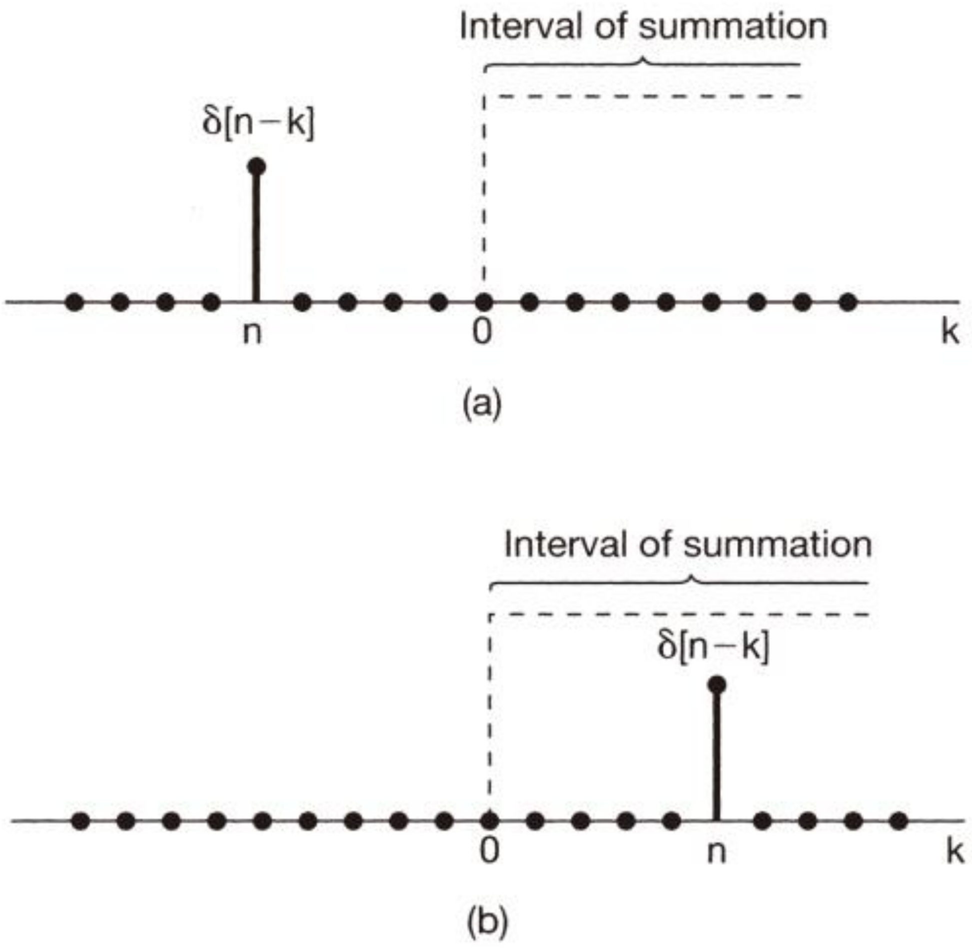


Figure 1.31 Relationship given in eq. (1.67): (a) $n < 0$; (b) $n > 0$.

The Discrete-Time Unit Impulse and Unit Step Sequences

- In particular, since $\delta[n]$ is nonzero only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n] \qquad (1.68)$$

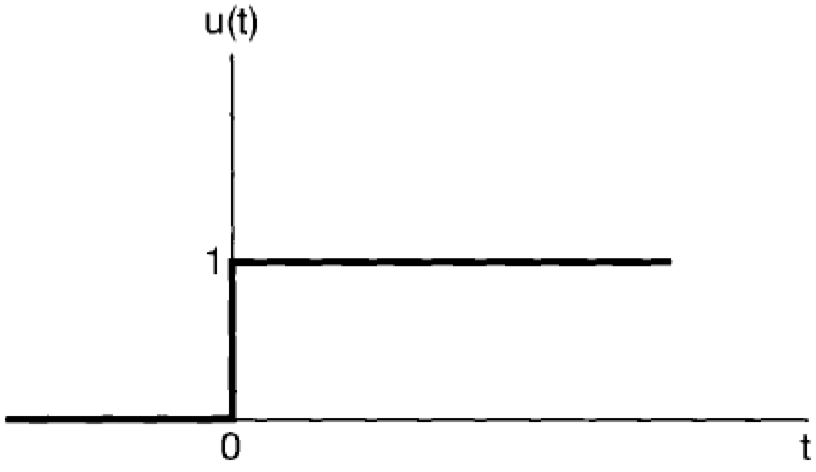
- More generally, if we consider a unit impulse $\delta[n-n_0]$ at $n = n_0$, then

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0] \qquad (1.69)$$

The Continuous-Time Unit Step and Unit Impulse Functions

- The continuous-time *unit step function u(t)* is defined in a manner similar to its discrete time

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



The Continuous-Time Unit Step and Unit Impulse Functions

The continuous-time unit step is the *running integral* of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau.$$

The continuous-time unit impulse can be thought of as the *first derivative* of the continuous-time unit step

$$\delta(t) = \frac{du(t)}{dt}.$$

The Continuous-Time Unit Step and Unit Impulse Functions

The continuous-time unit step is the *running integral* of the unit impulse

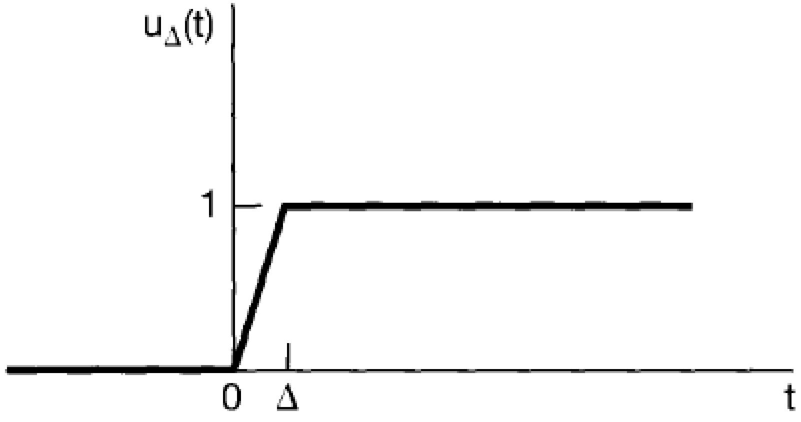
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau.$$

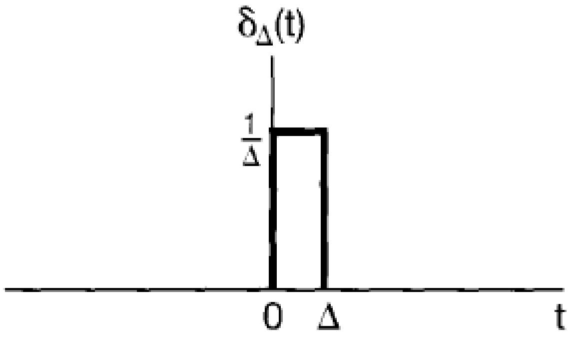
The continuous-time unit impulse can be thought of as the *first derivative* of the continuous-time unit step

$$\delta(t) = \frac{du(t)}{dt}$$

Since *u(t)* is discontinuous at *t* = 0 and consequently is formally not differentiable

- Considering an approximation to the unit step $u_\Delta(t)$





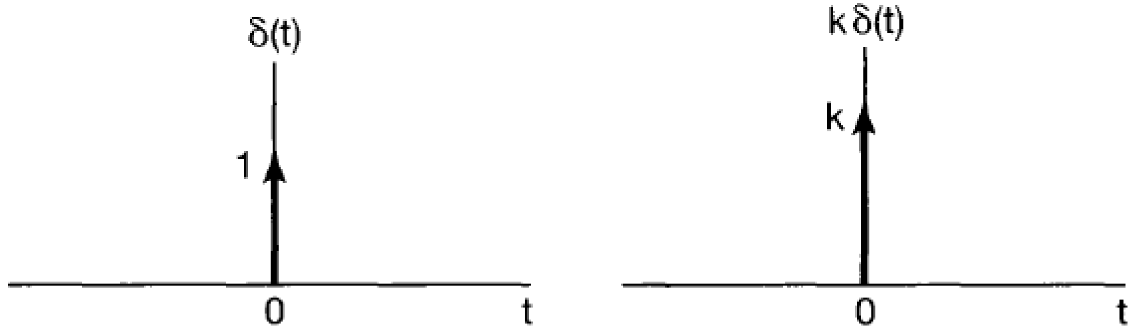
- Rises from the value 0 to the value 1 in a short time interval of length *Δ*.
- *u(t)* is the limit of $u_\Delta(t)$ as $\Delta \rightarrow 0$

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

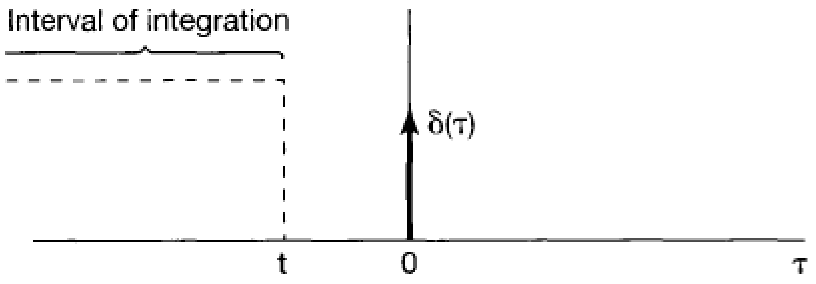
Note that $\delta_\Delta(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ. As $\Delta \to 0$, $\delta_\Delta(t)$ becomes narrower and higher, maintaining its unit area. Its limiting form,

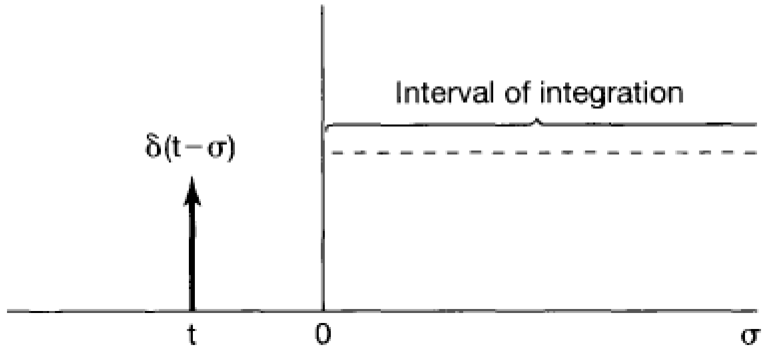
$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t),$$

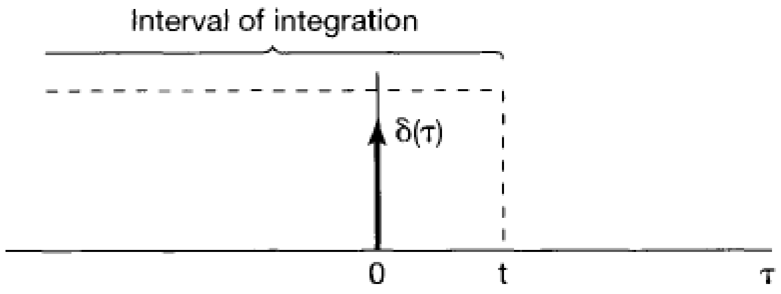


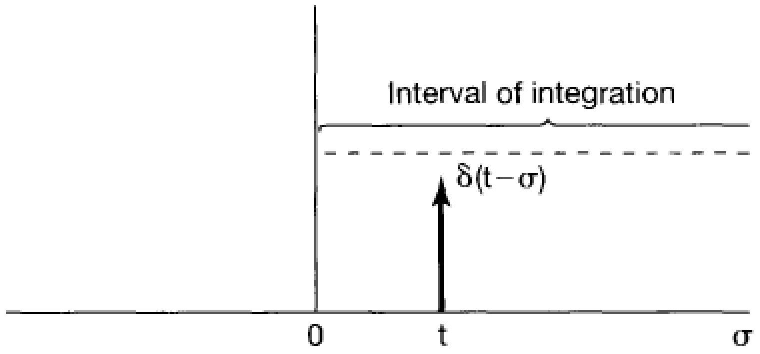
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma),$$

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma.$$

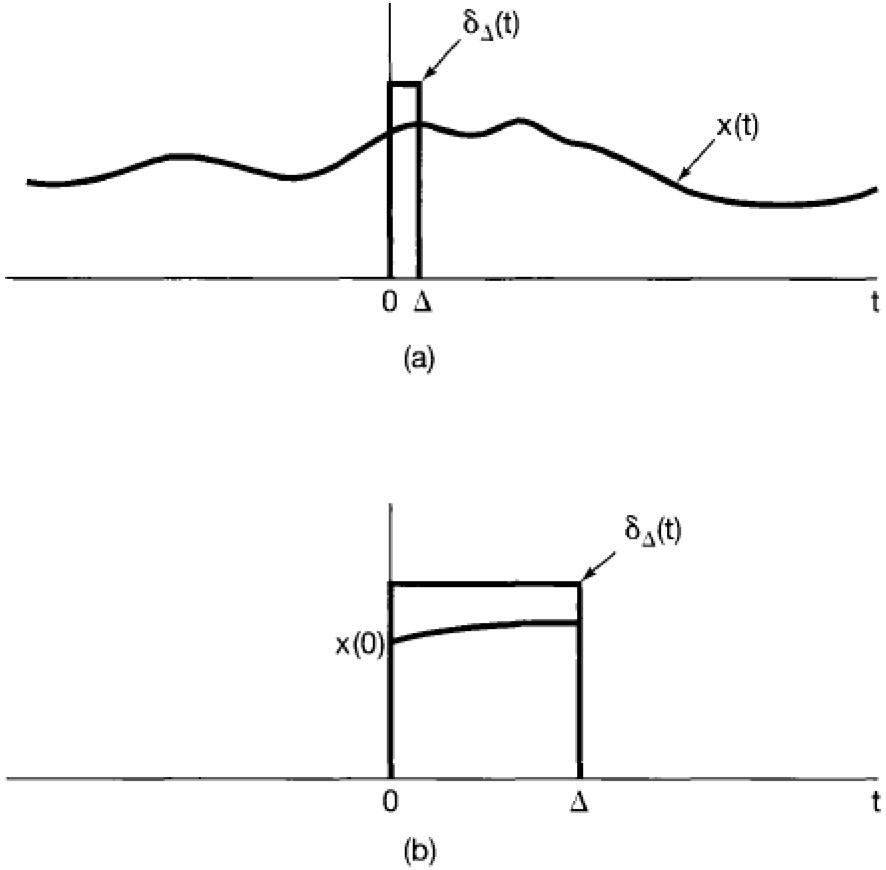








➢Consider $x_1(t) = x(t)\delta_\Delta(t).$



$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t).$$

Since $\delta(t)$ is the limit as $\Delta \to 0$ of $\delta_\Delta(t)$,

$$x(t)\delta(t) = x(0)\delta(t).$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

- The continuous-time complex exponential signal is of the form

$$x(t) = Ce^{at} \qquad (1.20)$$

where C and a are, in general, complex numbers.

- **Real Exponential Signals**

If C and a are real [in which case $x(t)$ is called a real exponential]

  - If a is positive, then as t increases $x(t)$ is a growing exponential
  - If a is negative, then $x(t)$ is a decaying exponential a=0, $x(t)$ is constant

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

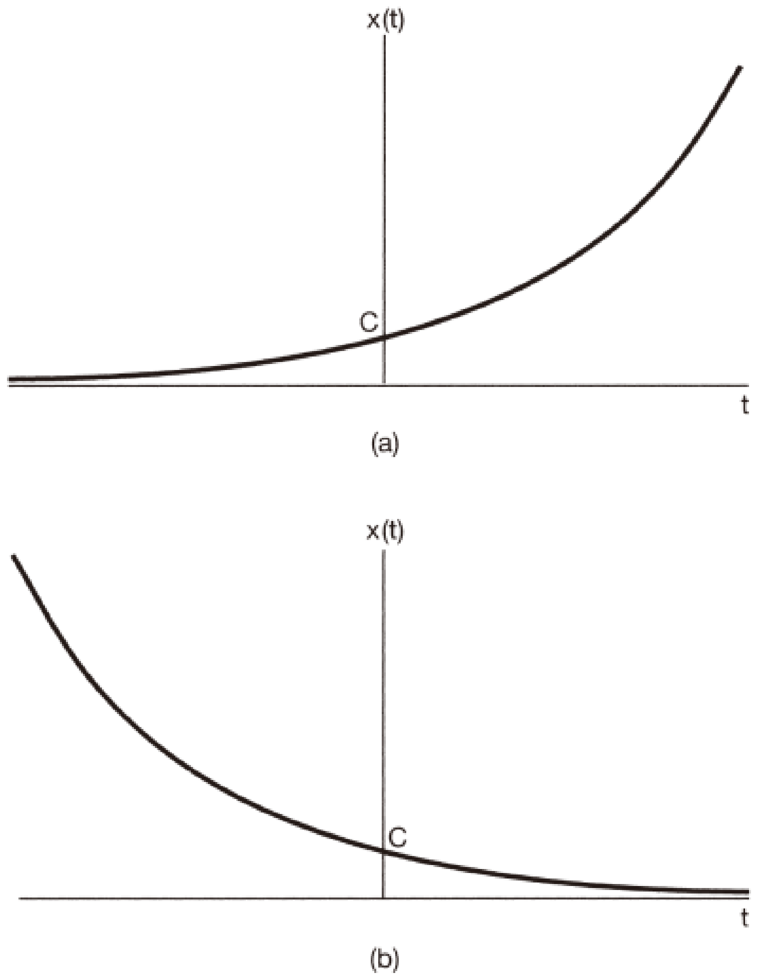


Figure 1.19 Continuous-time real exponential $x(t) = Ce^{at}$: (a) $a > 0$; (b) $a < 0$.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

Periodic Complex Exponential and Sinusoidal Signals

- A second important class of complex exponential is obtained by constraining a to be purely imaginary.

$$x(t) = e^{j\omega_0 t}$$

An important property of this signal is that it is **periodic**.

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)} \;---\rightarrow e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T}$$

it follows that for periodicity, we must have $e^{j\omega_0 T} = 1$

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T. If $\omega_0 \neq 0$, then the fundamental period T_0 of $x(t)$ is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

- A signal closely related to the periodic complex exponential is the *sinusoidal signal*

$$x(t) = A\cos(\omega_0 t + \varphi)$$

with seconds as the units of t, the units of φ and ω_0 are radians and radians per second, respectively. It is also common to write $\omega_0 = 2\pi f_0$, where f_0 has the units of cycles per second, or hertz (Hz).

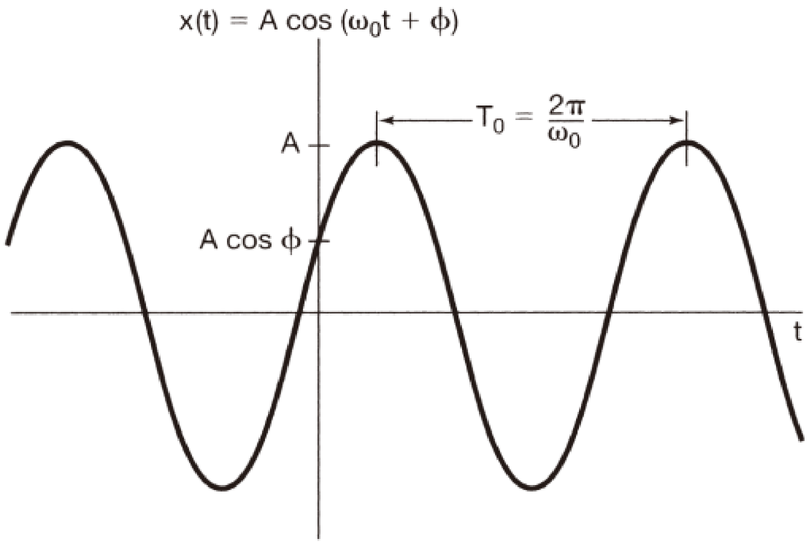


Figure 1.20 Continuous-time sinusoidal signal.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

- By using Euler's relation, the complex exponential can be written in terms of periodic complex exponentials, again with the same fundamental period:

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin\omega_0 t$$

- Similarly, the sinusoidal signal of can be written in terms of periodic complex exponentials

$$A\cos(\omega_0 t + \varphi) = \frac{A}{2} e^{j\varphi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\varphi} e^{-j\omega_0 t}$$

- We can express a sinusoid in terms of a complex exponential signal as

$$A cos(\omega_0 t + \varphi) = A\boldsymbol{Re}\left\{e^{j(\omega_0 t + \varphi)}\right\}$$

where, if c is a complex number, $\boldsymbol{Re}\{c\}$ denotes its real part.

Continuous-Time Complex Exponential and Sinusoidal Signals

- We see that the fundamental period T_0 of a continuous-time sinusoidal signal or a periodic complex exponential is inversely proportional to $|\omega_0|$, which we will refer to as the *fundamental frequency*.

- $\omega_0 = 0$, We mentioned earlier, *x(t)* is constant and therefore is periodic with period T for any positive value of *T.*

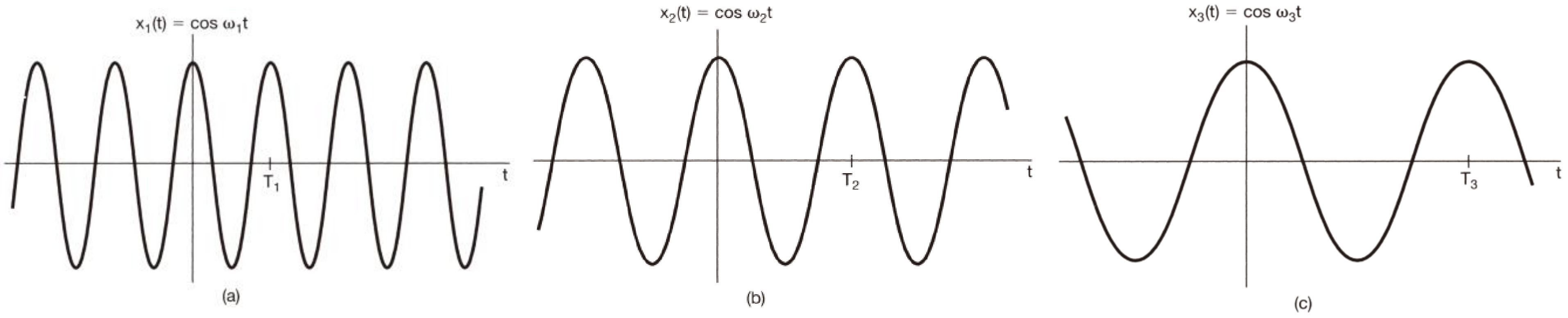


Figure 1.21 Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

$$E_{\text{period}} = \int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt$$

$$= \int_0^{T_0} 1 \cdot dt = T_0 \qquad (1.30)$$

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1 \qquad (1.31)$$

- The complex periodic exponential signal has finite average power equal to

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|e^{j\omega_0 t}\right|^2 dt = 1 \qquad (1.32)$$

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

- A necessary condition for a complex exponential $e^{j\omega t}$ to be periodic with period T_0 is that

$$e^{j\omega T_0} = 1 \qquad (1.33)$$

which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k,\ k = 0, \pm 1, \pm 2, \ldots. \qquad (1.34)$$

if we define

$$\omega_0 = \frac{2\pi}{T_0} \qquad (1.35)$$

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

- Harmonically related set of complex exponentials is a set of periodic exponentials with fundamental frequencies that are all multiples of a single positive frequency ω_0:

$$\varphi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots. \qquad (1.36)$$

- For $k = 0, \varphi_k(t)$ is a constant, while for any other of $k, \varphi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|} \qquad (1.37)$$

Example

- It is sometimes desirable to express the sum of two complex exponentials as the product of a single complex exponential and a single sinusoid. For example, suppose we wish to plot the magnitude of the signal $x(t) = e^{j2t} + e^{j3t}$
- We obtain, $x(t) = e^{j2.5t}\left(e^{-j0.5} + e^{j0.5t}\right)$
- Which, because of Euler's relation, can be rewritten as $x(t) = 2e^{j2.5t}\cos(0.5t)$
- We can directly obtain an expression for the magnitude of *x(t)* ----> $|x(t)| = 2|\cos(0.5t)|$

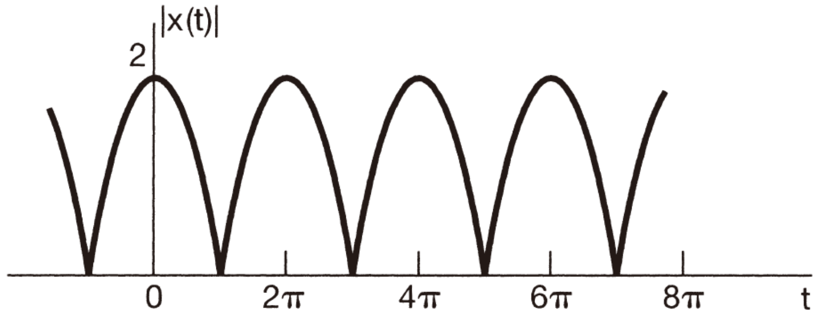


Figure 1.22 The full-wave rectified sinusoid of Example 1.5.

Continuous-Time Complex Exponential and Sinusoidal Signals

Consider a complex exponential Ce^{at}, where C is expressed in polar form and a in rectangular form.

$$C = |C|e^{j\theta}$$

and

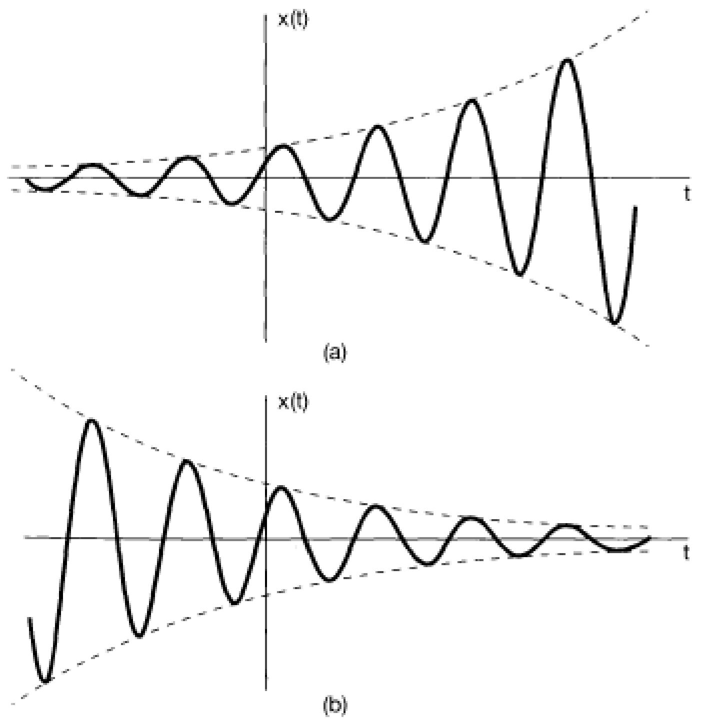
$$a = r + j\omega_0.$$

Then

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}.$$

Using Euler's relation, we can expand this further as

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta).$$



- Sinusoidal signals multiplied by decaying exponentials - damped sinusoids.
- Examples of damped sinusoids arise in the response of RLC circuits and in mechanical systems containing both damping and restoring forces, such as automotive suspension systems.

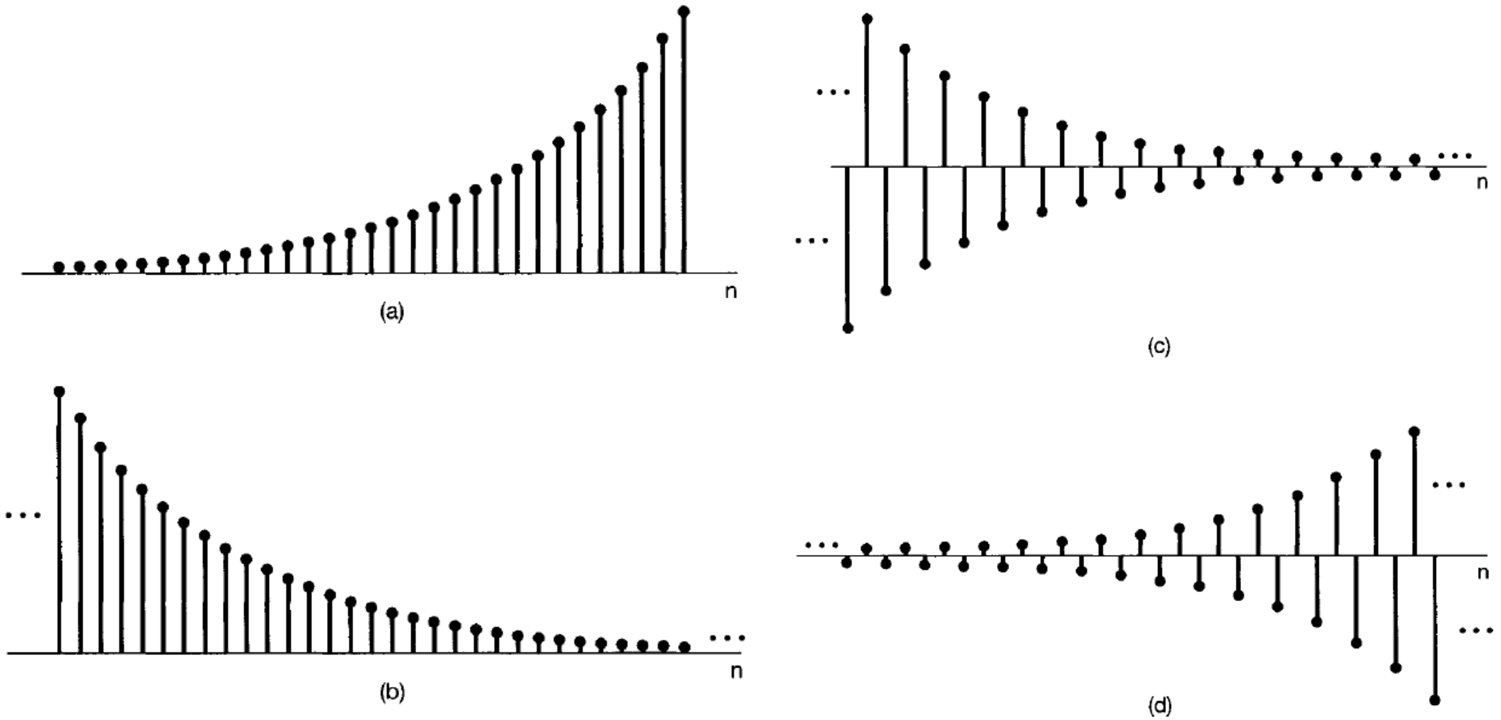
Discrete-Time Complex Exponential and Sinusoidal Signals

$$x[n] = C\alpha^n$$

- where C and a are, in general, complex numbers. This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}, \qquad \alpha = e^{\beta}.$$

- C and α are real
- (a) If | α |> 1 the magnitude of the signal grows exponentially with n
- (b) If | α | < 1 we have a decaying exponential.
- (c) α is positive, all the values of $C\alpha^n$ are of the same sign
- (d) α is negative then the sign of x[n] alternates.
- if α = 1 then x[n] is a constant,
- if α = -1, x[n] alternates in value between +C and -C.



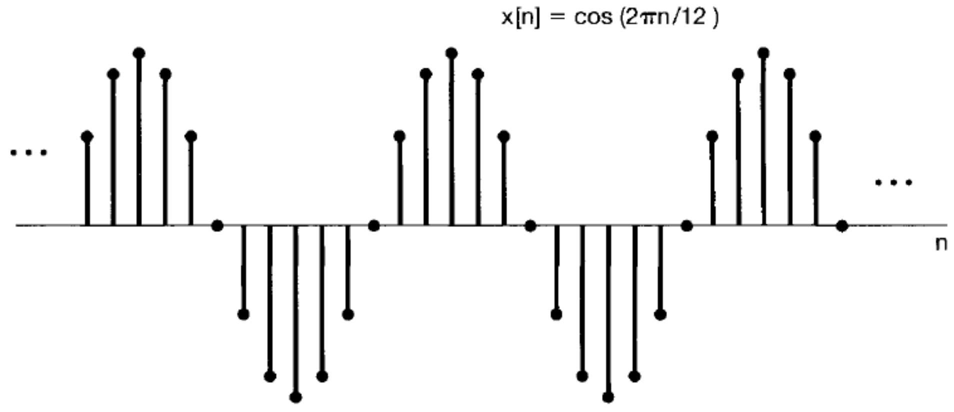
➢Another important complex exponential is obtained by constraining β to be purely imaginary (so that |a| = 1).

$$x[n] = Ce^{\beta n}, \quad x[n] = e^{j\omega_0 n}.$$

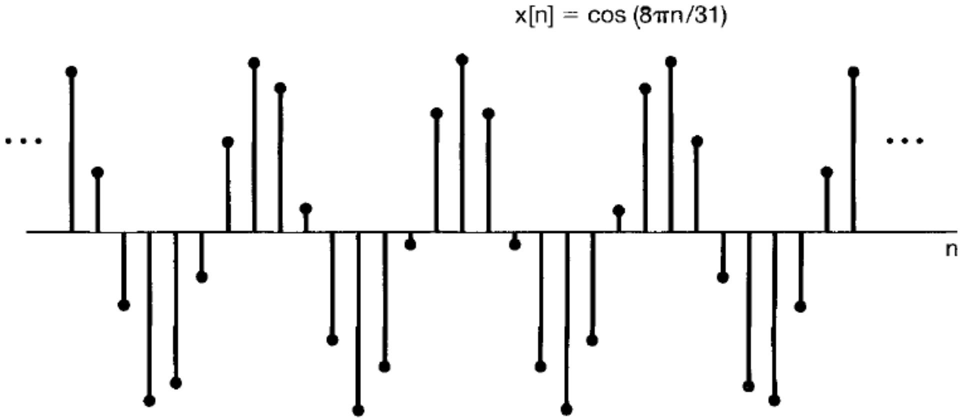
$$x[n] = A\cos(\omega_0 n + \phi).$$

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$$

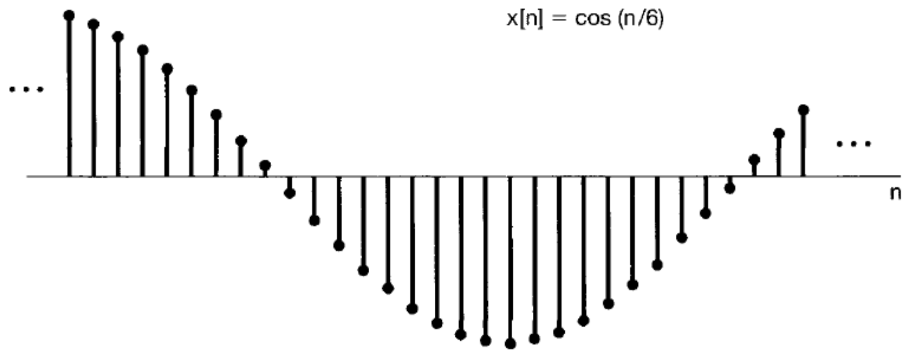
$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}.$$



(a)



(b)



General Complex Exponential Signals

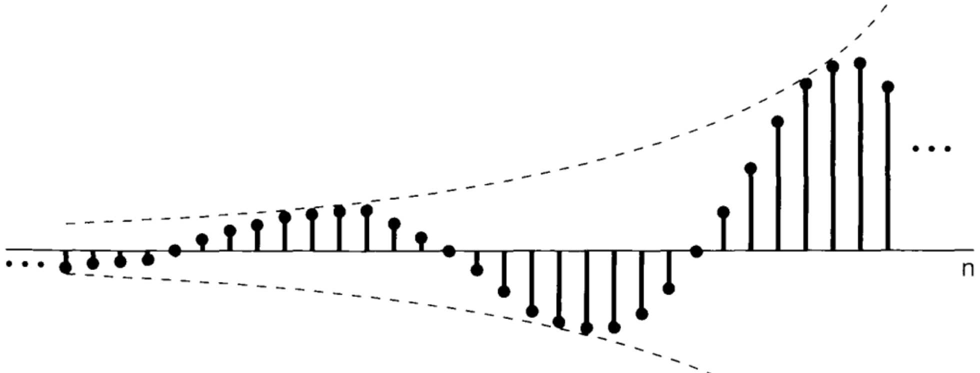
- The general discrete-time complex exponential can be written and interpreted in terms of real exponentials and sinusoidal signals.

$$C = |C|e^{j\theta}$$

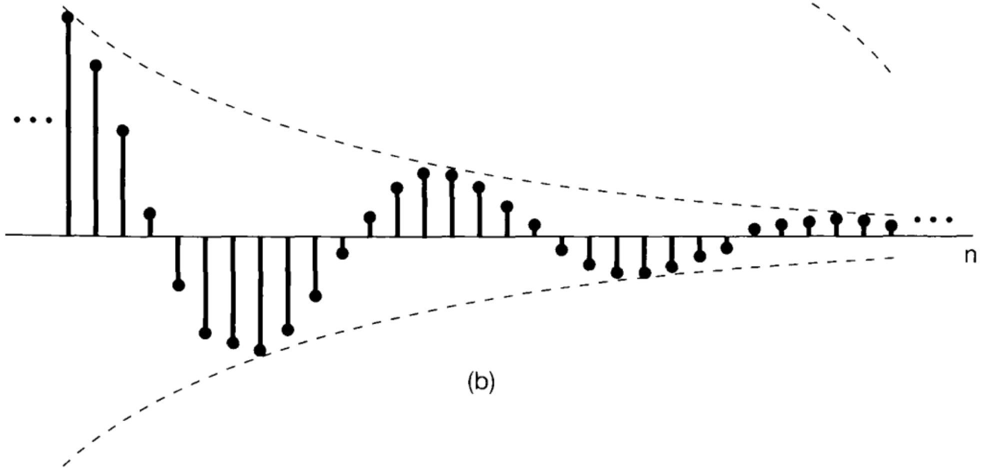
$$\alpha = |\alpha|e^{j\omega_0},$$

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta).$$

- For | α | = 1, the real and imaginary parts of a complex exponential sequence are sinusoidal.
- For | α | < 1 they correspond to sinusoidal sequences multiplied by a decaying exponential
- while for | α | > 1 they correspond to sinusoidal sequences multiplied by a growing exponential.



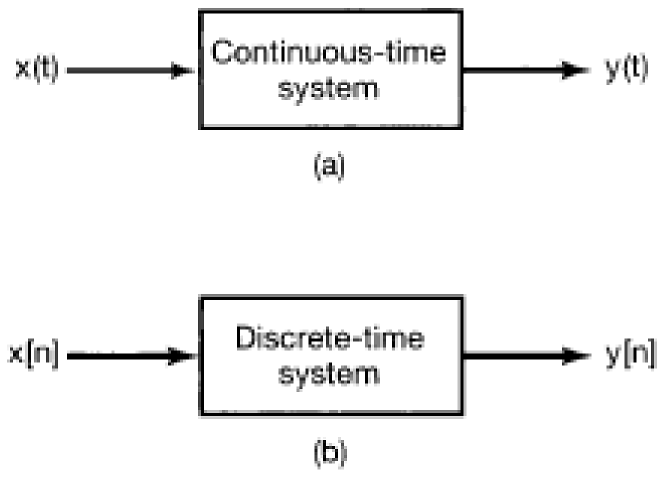
(a)



(b)

Continuous-time and Discrete-time Systems

- Input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs



Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

Simple Examples of Systems

- *RC* circuit
- If we regard *vs(t)* as the input signal and *vc(t)* as the output signal, then we can use simple circuit analysis to derive an equation describing the relationship between the input and output.

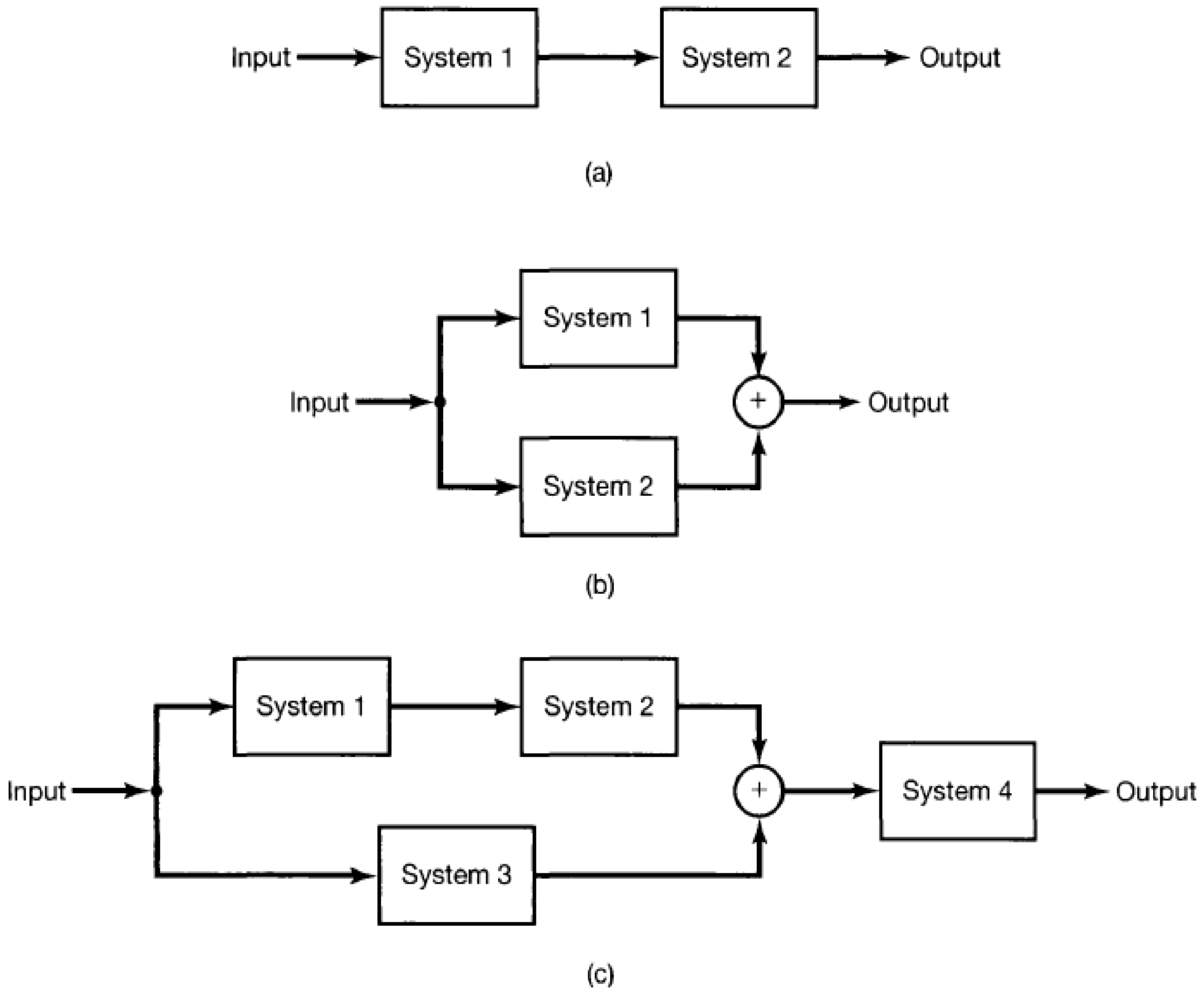
$$i(t) = \frac{v_s(t) - v_c(t)}{R}.$$

$$i(t) = C\frac{dv_c(t)}{dt}.$$

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t).$$

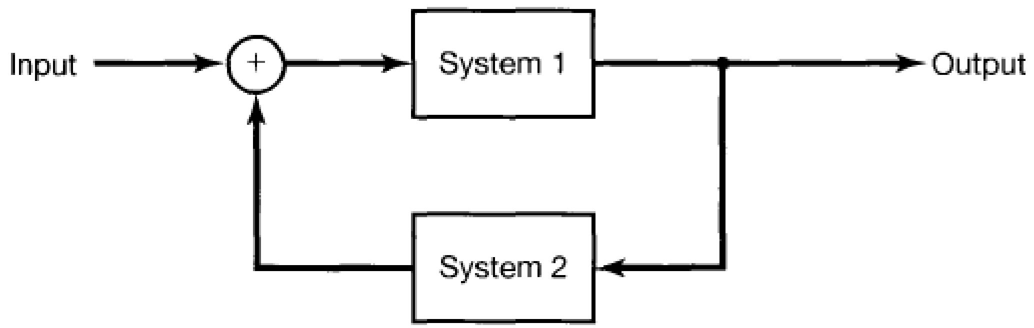
Interconnections of Systems

- Many real systems are built as interconnections of several subsystems.
- *Series/cascade interconnection* of two systems
  - Eg: radio receiver followed by an amplifier
- *Parallel interconnection* of two systems
  - simple audio system with several microphones feeding into a single amplifier and speaker system



Interconnections of Systems

- *Feedback interconnection*
  - Control system on an automobile senses the vehicle's velocity and adjusts the fuel flow in order to keep the speed at the desired level.



BASIC SYSTEM PROPERTIES

- Systems with and without Memory
- A system is said to be *memory less* if its output for each value of the independent variable at a given time is dependent only on the input at that same time. Eg: Resistor

$$y[n] = (2x[n] - x^2[n])^2$$

$$y[n] = x[n]$$

- A capacitor is an example of a continuous-time system with memory, since if the input is taken to be the current and the output is the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau,$$

- An example of a discrete-time system with memory is an *accumulator* or *summer*

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n - 1].$$

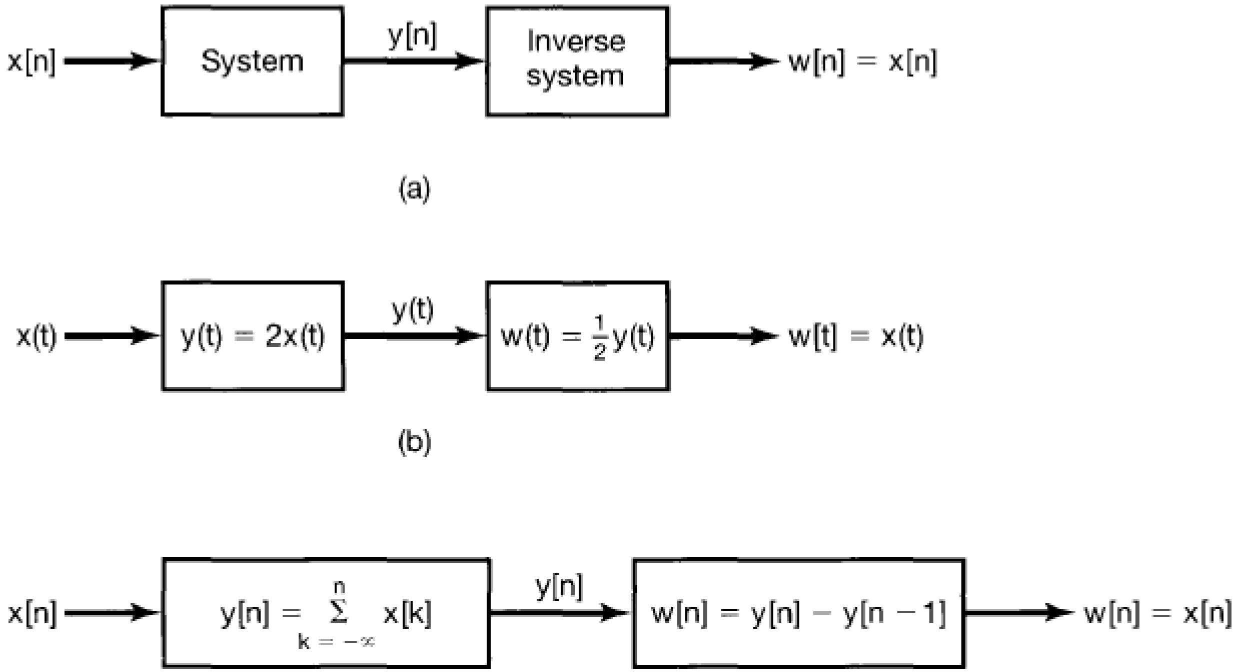
Invertibility and Inverse Systems

- A system is said to be *invertible* if distinct inputs lead to distinct outputs.
- if a system is invertible, then an *inverse system* exists that, when cascaded with the original system, yields an output *w[n]* equal to the input *x[n]* to the first system
- An example of an invertible continuous-time system is

$$y(t) = 2x(t)$$

- for which the inverse system is

$$w(t) = (½)y(t)$$



Invertibility and Inverse Systems

- Examples of noninvertible systems are

$$y[n] = 0$$

$$y(t) = x^2(t)$$

we cannot determine the sign of the input from knowledge of the output

- The concept of invertibility arises in systems for encoding used in a wide variety of communications applications.
- In such a system, a signal that we wish to transmit is first applied as the input to a system known as an encoder.
- There are many reasons for doing this, ranging from the desire to encrypt the original message for secure or private communication to the objective of providing some redundancy in the signal so that any errors that occur in transmission can be detected
- For lossless coding, the input to the encoder must be exactly recoverable from the output; i.e., the encoder must be invertible.

Causality

- A system is *causal* if the output at any time depends only on values of the input at the present time and in the past.
- Non-anticipative - the system output does not anticipate future values of the input

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n - 1]$$

All memory less systems are causal, since the output responds only to the current value of the input

Non-causal systems:

$$y[n] = x[n] - x[n + 1]$$

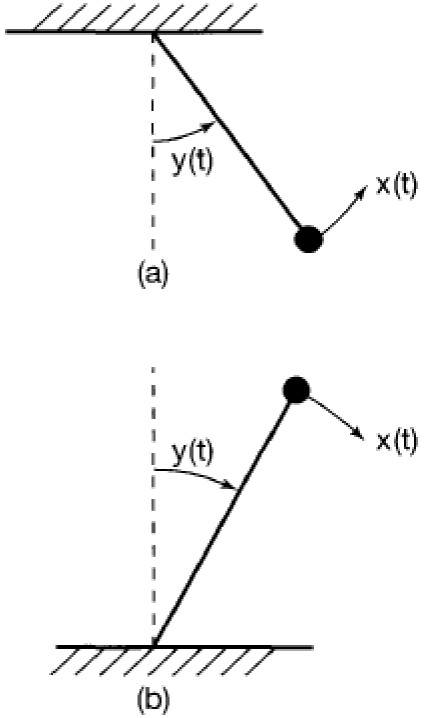
$$y(t) = x(t + 1)$$

Problem: Check for Causality: (a) $y[n] = x[-n]$ (b) $y(t) = x(t)\cos(t + 1)$.

Stability

➢*Stability* is another important system property

➢Stable system - small inputs lead to responses that do not diverge



Unstable systems

- Models for chain reactions or for population growth with unlimited food supplies and no predators - since the system response grows without bound in response to small inputs.
- Another example of an unstable system is the model for a bank account balance
- Stability of physical systems generally results from the presence of mechanisms that dissipate energy.
- Simple *RC* circuit - the resistor dissipates energy and this circuit is a stable system.
- **If the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge.**

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k].$$

- This system sums *all* of the past values of the input rather than just a finite set of values
- If the input to the accumulator is a unit step *u[n]*, the output will be

$$y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n].$$

- That is, *y[O]* = 1, y[l] = 2, y[2] = 3, and so on, and *y[n]* grows without bound

- If we suspect that a system is unstable, then a useful strategy to verify this is to look for a *specific* bounded input that leads to an unbounded output

$$S_1: y(t) = t x(t)$$

$$S_2: y(t) = e^{x(t)}.$$

- For system S1, a constant input *x(t)* = 1 yields *y(t)* = *t*, which is unbounded
- no matter what finite constant we pick, |y(t)| will exceed that constant for some *t*
- For system S2 , which happens to be stable, we would be unable to find a bounded input that results in an unbounded output -----------|x(t)| < *B* or -*B* < *x(t)* < *B*,

$$e^{-B} < |y(t)| < e^{B}.$$

- We conclude that if any input to S2 is bounded by an arbitrary positive number *B*, the corresponding output is guaranteed to be bounded by *e8* . Thus, S2 is stable.

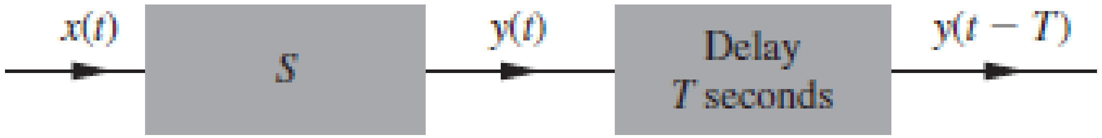
Time Invariance

- A system is time invariant if the behavior and characteristics of the system are fixed over time.
- *RC* circuit is time invariant if the resistance and capacitance values *R* and *C* are constant over time
- A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.

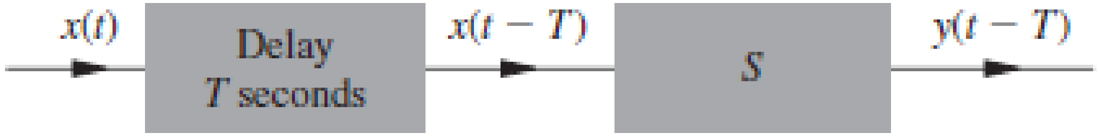
- if $y[n]$ is the output of a discrete-time, time-invariant system when $x[n]$ is the input, then

$y[n- n0]$ is the output when $x[n- n0]$ is applied

- $y(t)$ the output corresponding to the input $x(t)$, a time-invariant system will have $y(t- to)$ as the output when $x(t - to)$ is the input.



(a)



(b)

Example

y(t) = sin [x(t)]

y_1(t) = sin [$x_1(t)$]

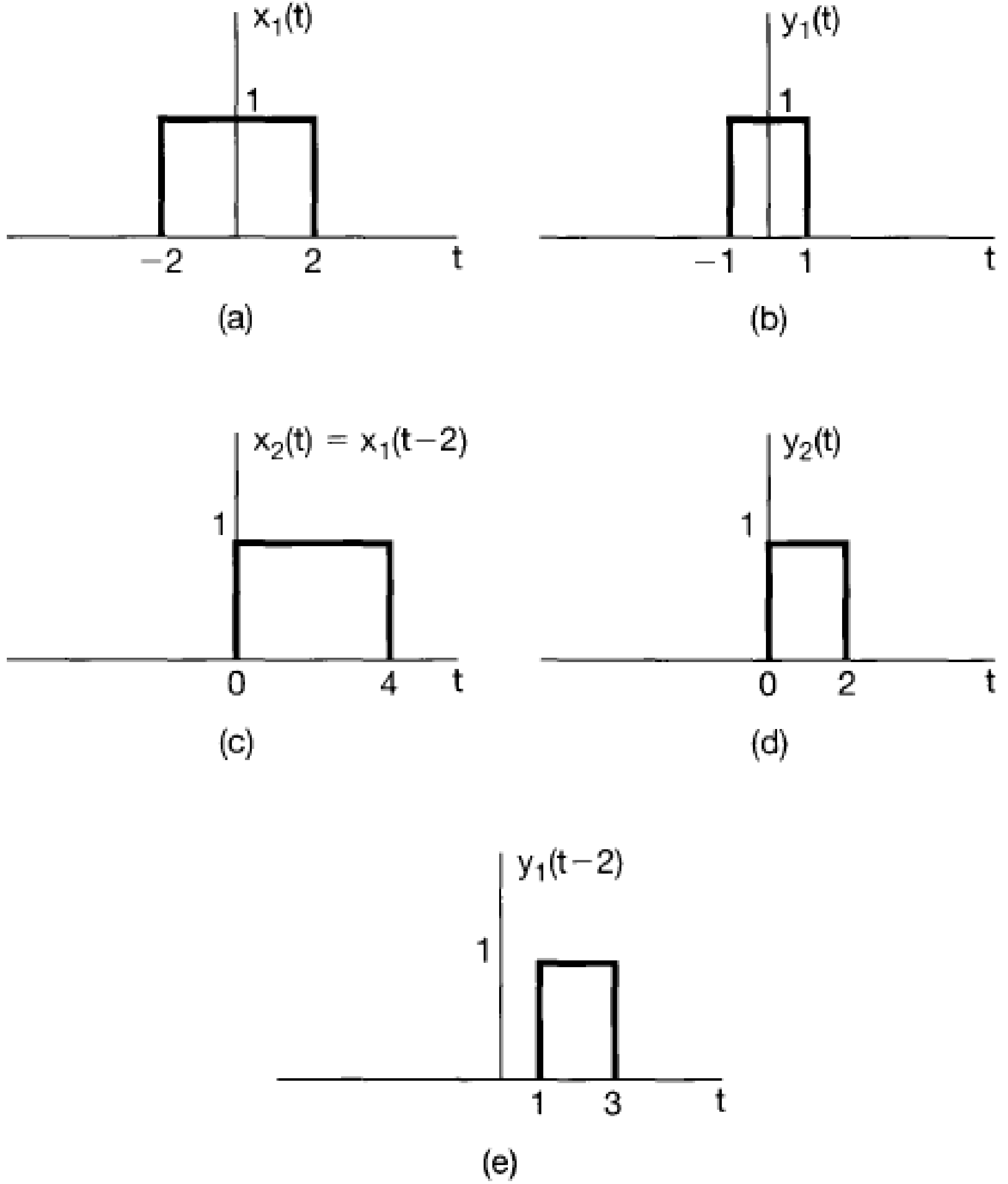
x_2(t) = x_1 (t - to)

y_2(t) = sin [x_2(t)] = sin [x_1(t - to)] ---------------------(1)

y_1(t- to) = sin [x_1(t - to)] --------------------------------(2)

- Comparing eqs. 1 and 2 , we see that *y2(t)* = *y 1* (t- *to)*, and therefore, this system is time invariant.
- Example 2: *y[n]* = *nx[n]*
- *Example 3:* y(t) = *x(2t).*

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)



$x_1(t)$
1
−2
2
t
(a)
$y_1(t)$
1
−1
1
t
(b)
$x_2(t) = x_1(t-2)$
1
0
4
t
(c)
$y_2(t)$
1
0
2
t
(d)
$y_1(t-2)$
1
1
3
t
(e)

Linearity

- A *linear system*, in continuous time or discrete time, is a system that possesses the important property of superposition

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant.

- The first is known as the *additivity* property; the second is known as the *scaling* or *homogeneity* property.

$$\text{continuous time: } ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t),$$
$$\text{discrete time: } ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n].$$

Example

- Consider a system S whose input $x(t)$ and output $y(t)$ are related by $y(t) = tx(t)$

To determine whether or not Sis linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$

$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$.

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to S, then the corresponding output may be expressed as

$$y_3(t) = tx_3(t) = t(ax_1(t) + bx_2(t))$$

$$= atx_1(t) + btx_2(t)$$

$$= ay_1(t) + by_2(t)$$

We conclude that the system S is linear.

Problem

1. $y(t) = x^2(t)$
2. $y[n] = (Re\{x[n]\}.$
3. $y[n] = 2x[n] + 3$

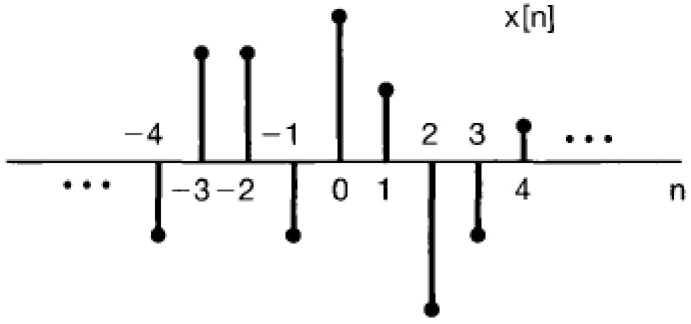
LINEAR TIME-INVARIANT SYSTEMS

- Linearity and time invariance, play a fundamental role in signal and system analysis
- **The Representation of Discrete-Time Signals in Terms of Impulses**

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases},$$

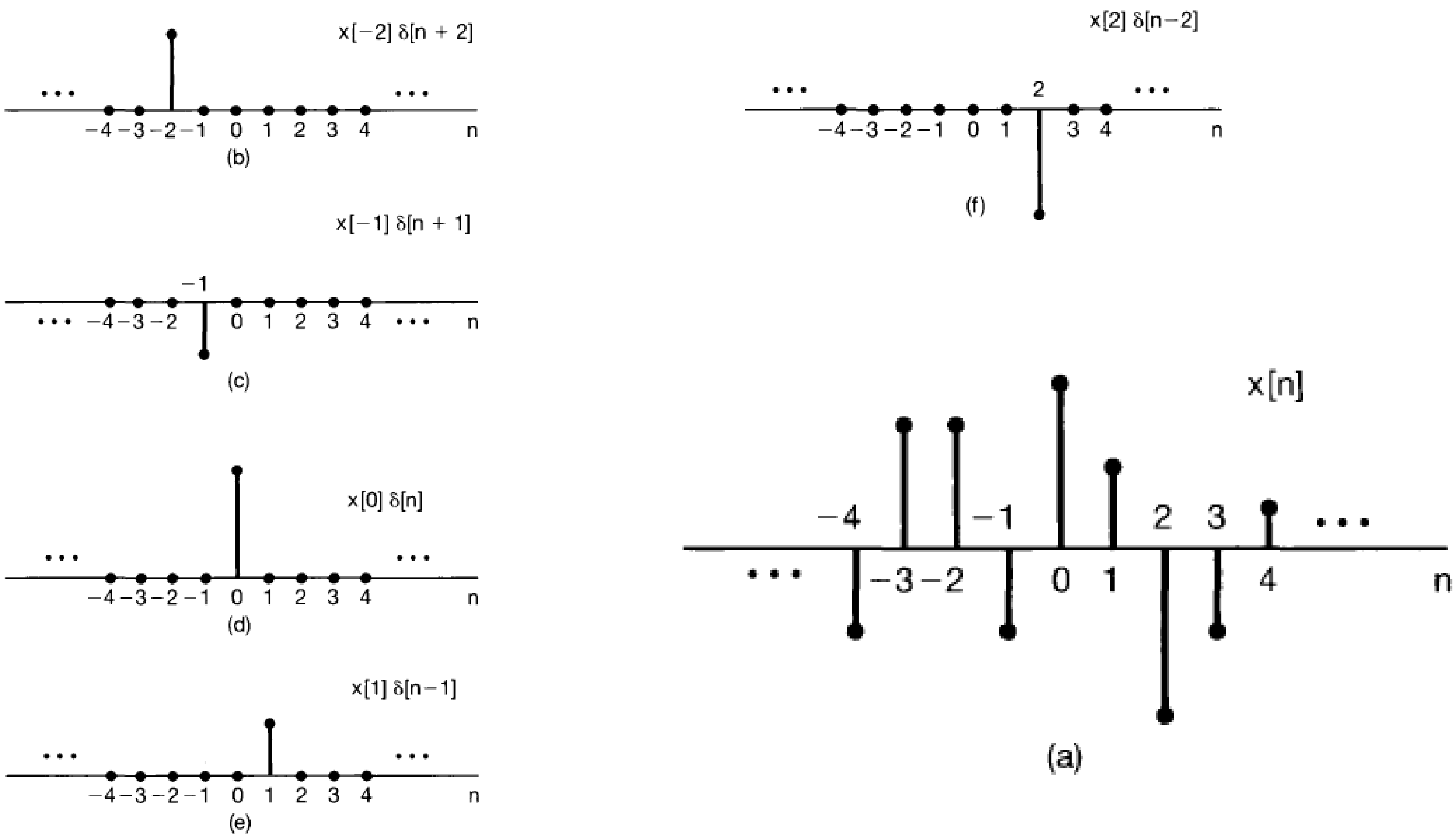
$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases},$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}.$$



(a)

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots.$$



Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

Sifting property

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k].$$

This corresponds to the representation of an arbitrary sequence as a linear combination of shifted unit impulses $\delta[n-k]$, where the weights in this linear combination are $x[k]$.

Consider $x[n] = u[n]$, the unit step. In this case, since $u[k] = 0$ for $k < 0$ and $u[k] = 1$ for $k >= 0$

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k],$$

Summation on the right hand side of eq "sifts" through the sequence of values $x[k]$ and preserves only the value corresponding to $k = n$.

The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

Linear:

- x[n] as a superposition of scaled versions of shifted unit impulses $\delta[n-k]$, each of which is specified by the corresponding value of k.
- The response of a linear system to x[n] will be the superposition of the scaled responses of the system to each of these shifted impulses.

Time invariance:

- The responses of a time-invariant system to the time-shifted unit impulses are simply time-shifted versions of one another.

The convolution -sum representation for discrete-time systems are linear and time invariant results from putting these two basic facts together.

➢Consider the response of a linear system to an arbitrary input x[n].

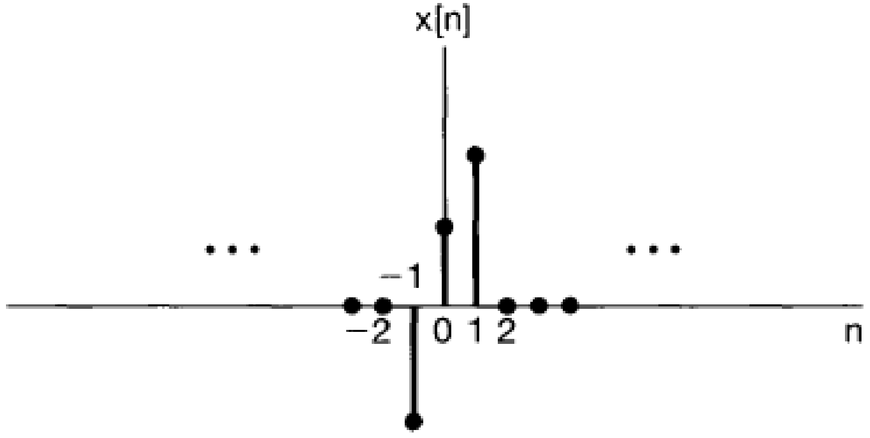
➢Let $h_k[n]$ denote the response of the linear system to the shifted unit impulse δ[n - k].

➢The response y[n] of the linear system to the input x[n] is simply the weighted linear combination of these basic responses.

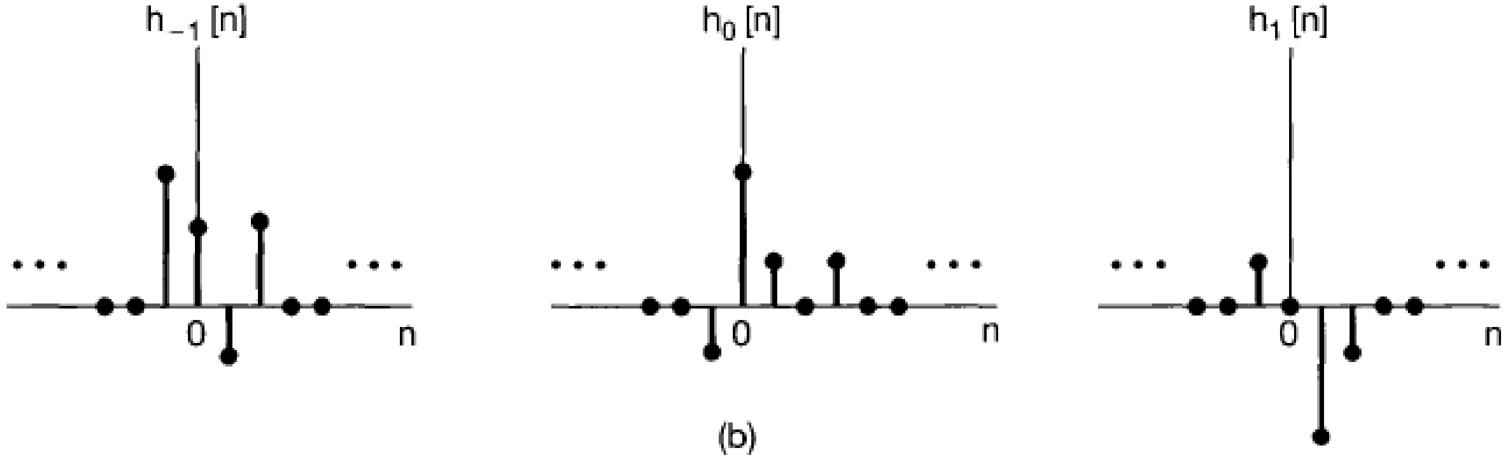
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n].$$

➢The signal x[n] is applied as the input to a linear system whose responses $h_{-1}[n]$, $h_0[n]$, and $h_1[n]$ to the signals δ[n + 1], δ[n], and δ[n- 1]

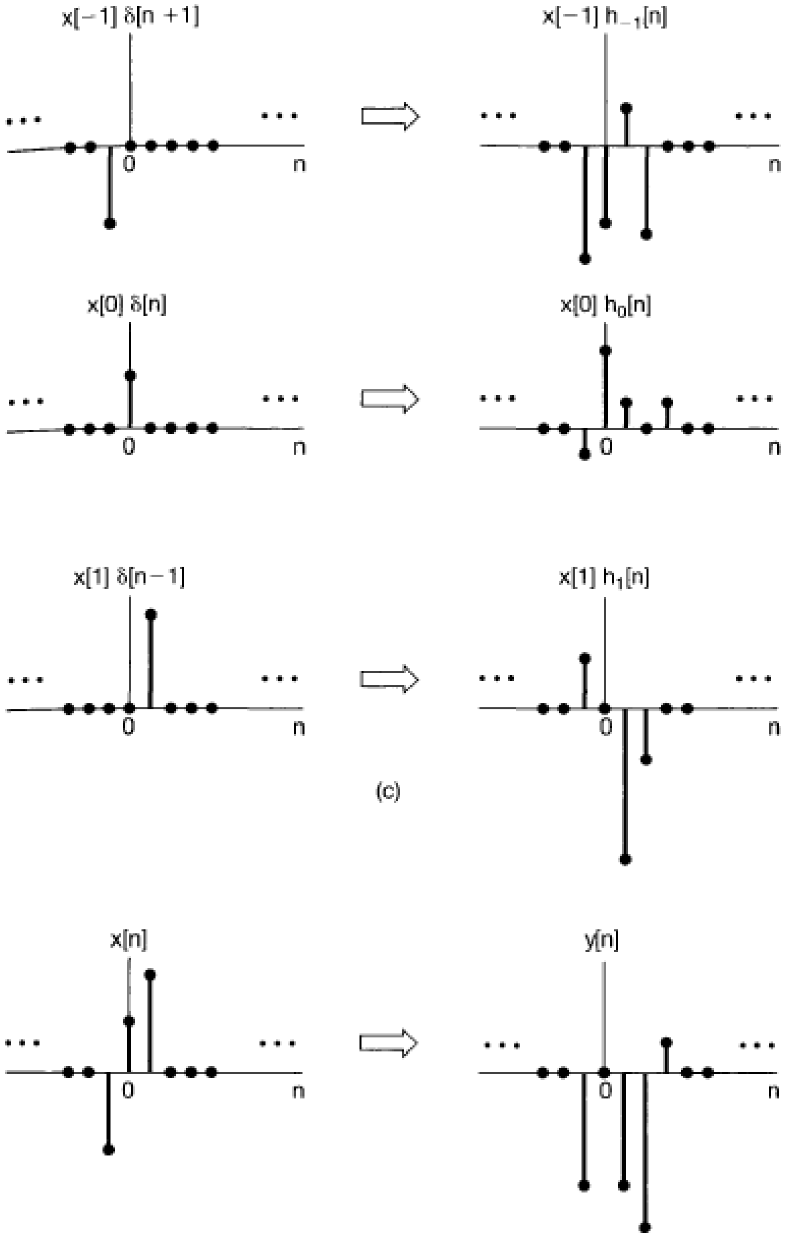
Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)



(a)



(b)



(c)

(d)

- If the system is *time invariant*, then these responses to time-shifted unit impulses are all time-shifted versions of each other
- Since δ*[n- k]* is a time-shifted version of δ*[n]*, the response $h_k[n]$ is a time-shifted version of *h0 [n]*

$$h_k[n] = h_0[n-k].$$

$$h[n] = h_0[n].$$

$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].}$$

$$y[n] = x[n] * h[n].$$

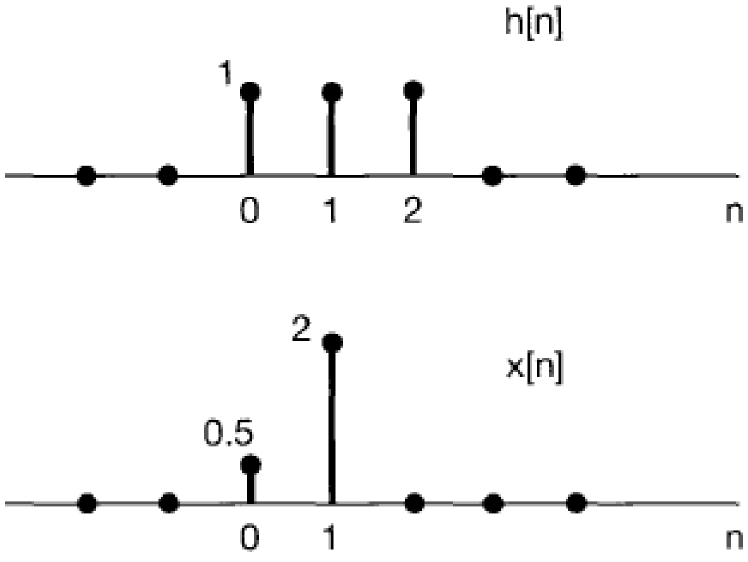
Example

➢Consider an LTI system with impulse response h[n] and input x[n]

➢For this case, since only x[O] and x[1] are nonzero

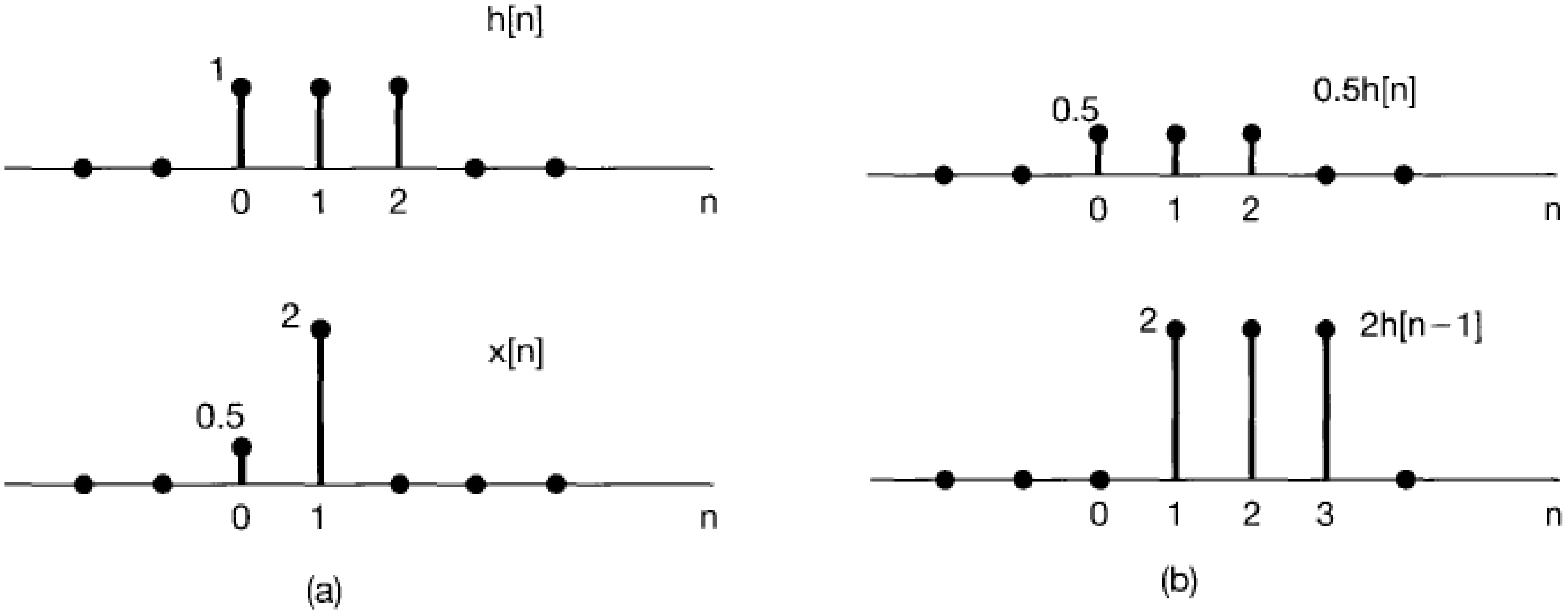
$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

$$= 0.5h[n] + 2h[n-1].$$

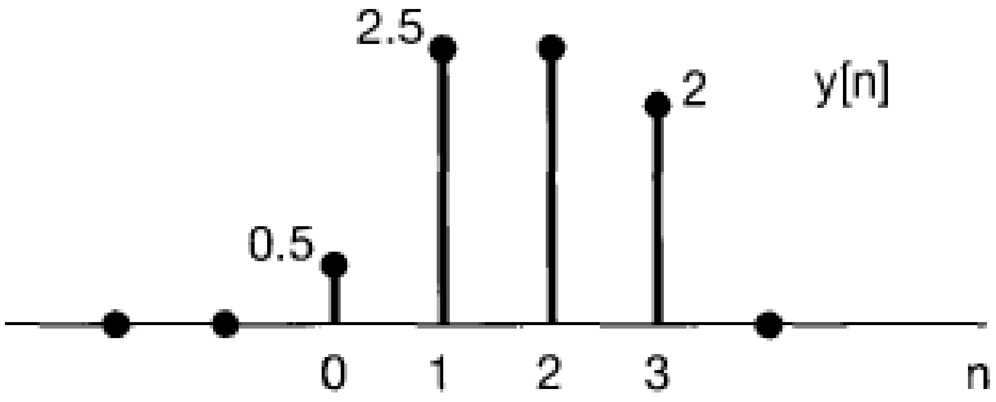


(a)

Method 1

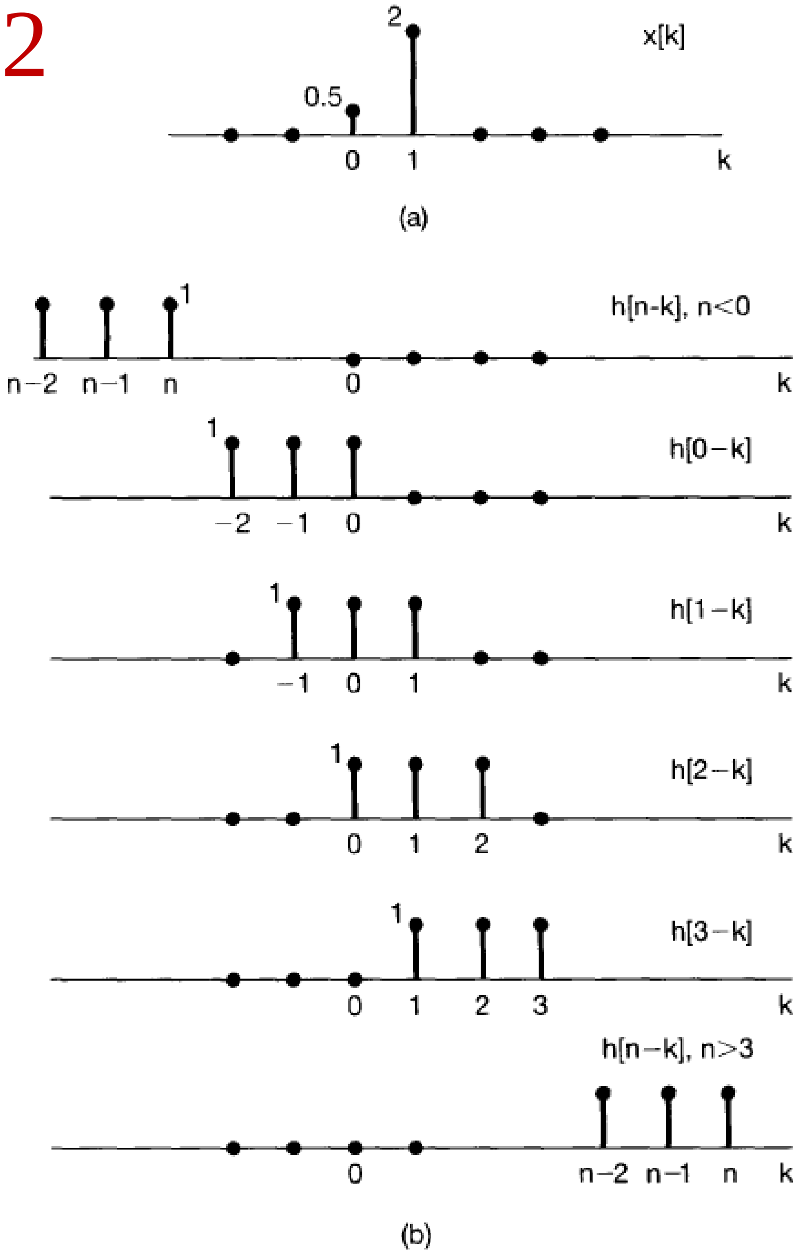


Method 1



(c)

Method 2



$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5.$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5.$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5,$$

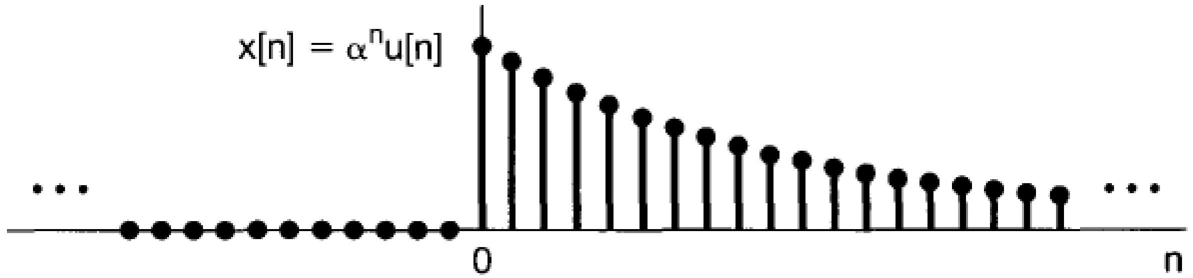
$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0.$$

Example

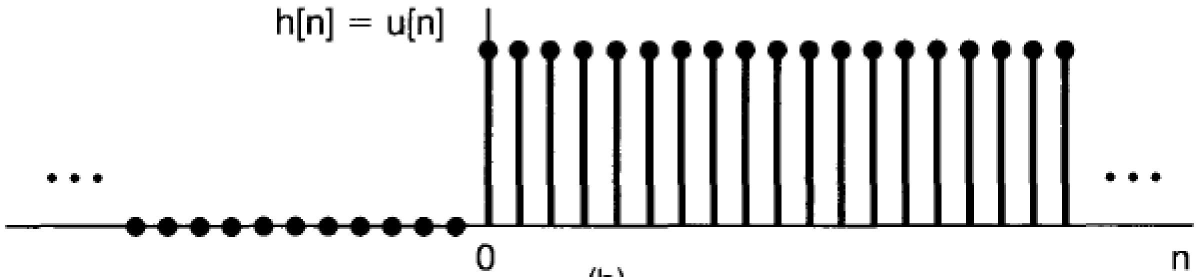
➢Consider an input *x[n]* and a unit impulse response *h[n]* given by *x[n]*, with $0 < a < 1$

$$x[n] = \alpha^n u[n],$$

$$h[n] = u[n],$$



(a)



(b)

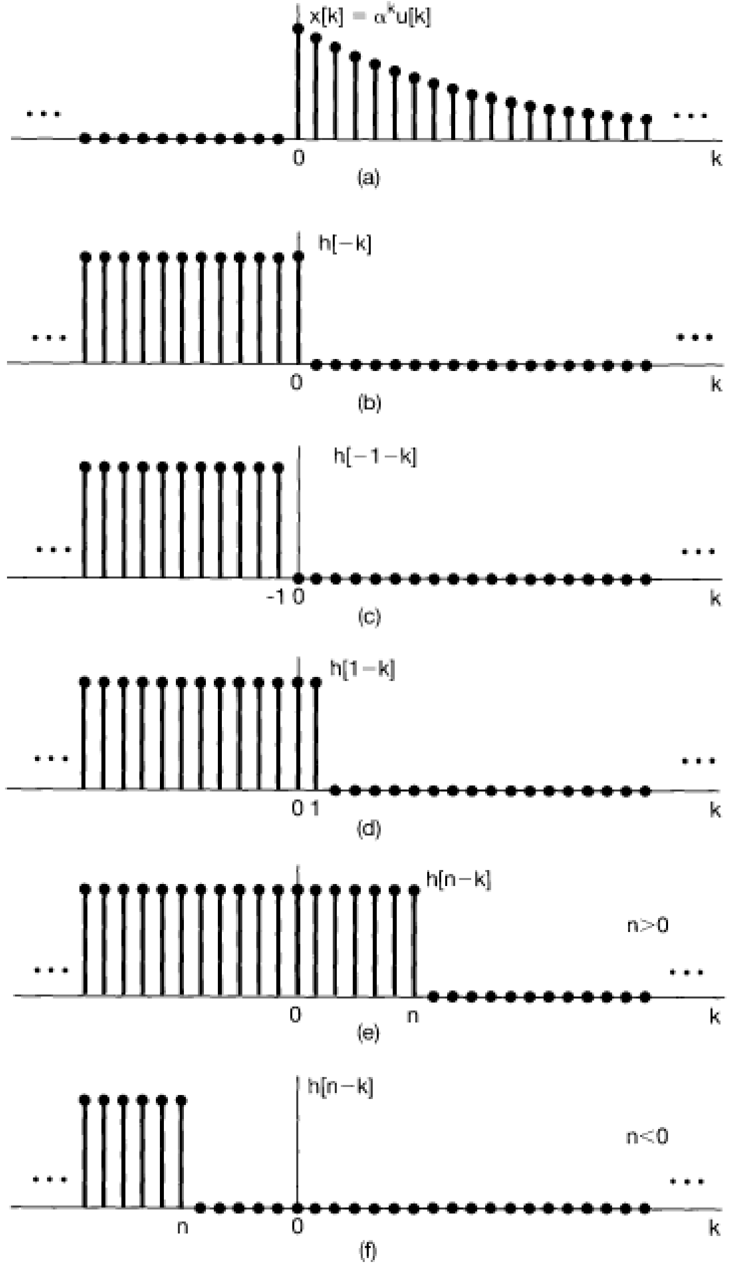
$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}.$$

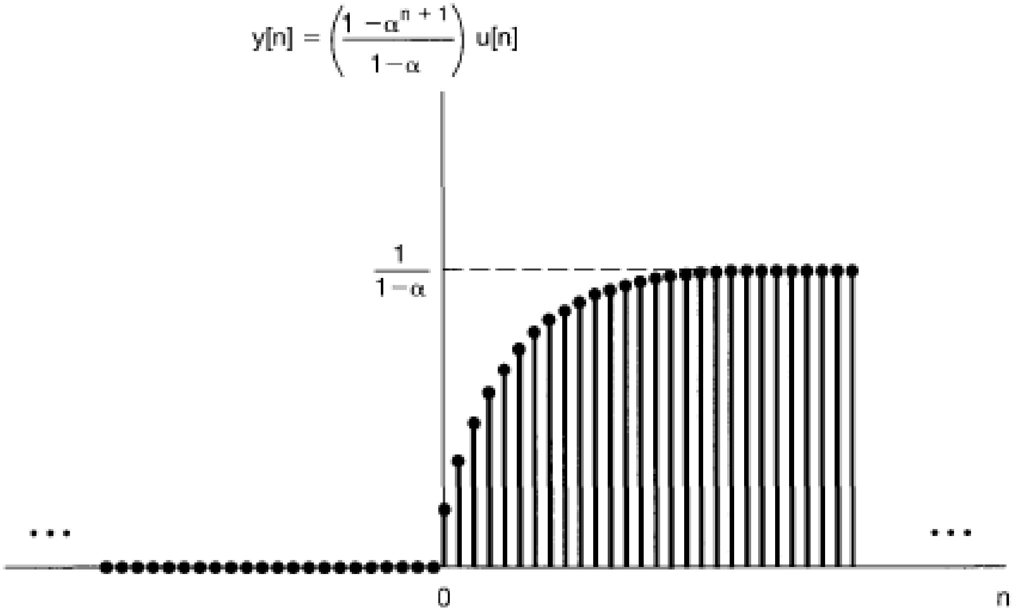
Thus, for $n \geq 0$,

$$y[n] = \sum_{k=0}^{n} \alpha^k,$$

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad \text{for } n \geq 0.$$

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u[n].$$


x[k] = α^k u[k]
0
k
(a)
h[−k]
0
k
(b)
h[−1−k]
-1 0
k
(c)
h[1−k]
0 1
k
(d)
h[n−k]
n>0
0
n
k
(e)
h[n−k]
n<0
n
0
k
(f)

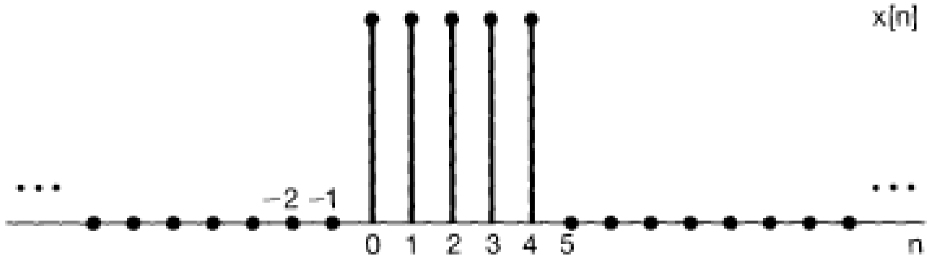

$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$
$\frac{1}{1-\alpha}$
0
n

Example

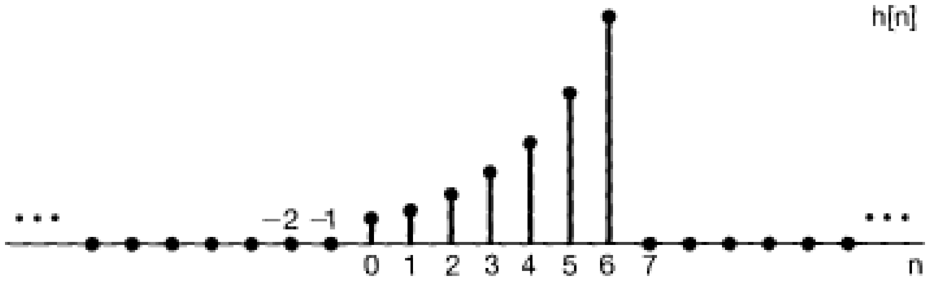
$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

- Consider two sequences

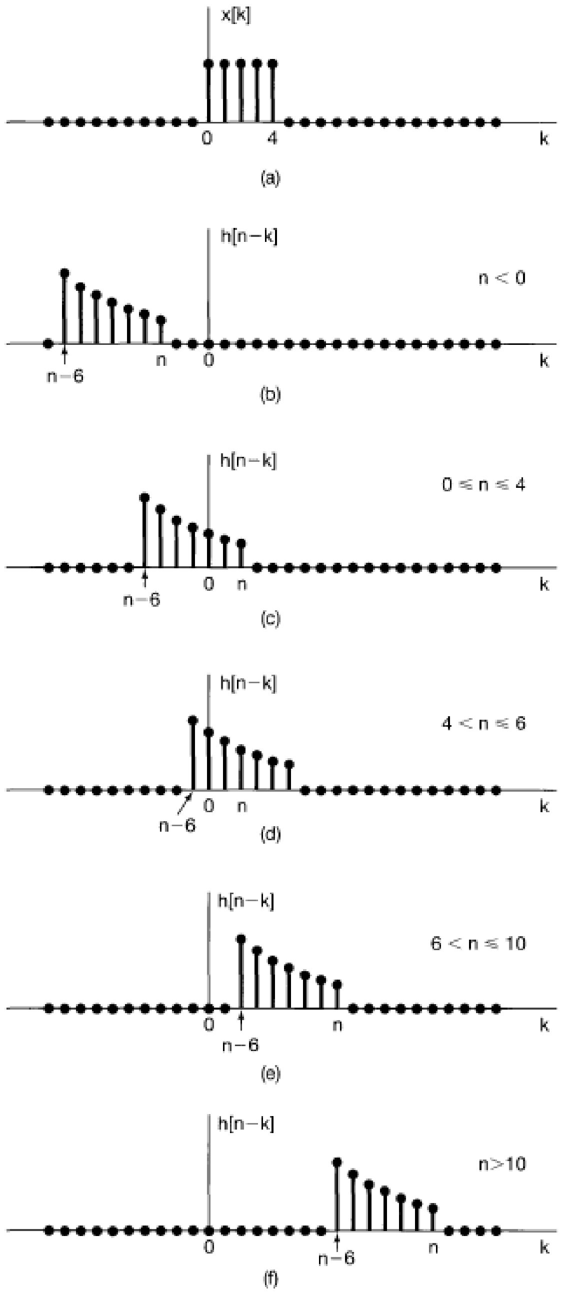
$$h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}.$$



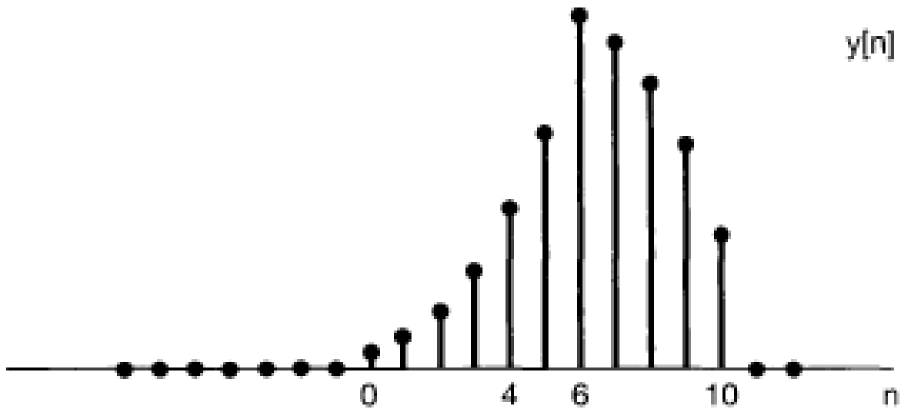
(a)



(b)

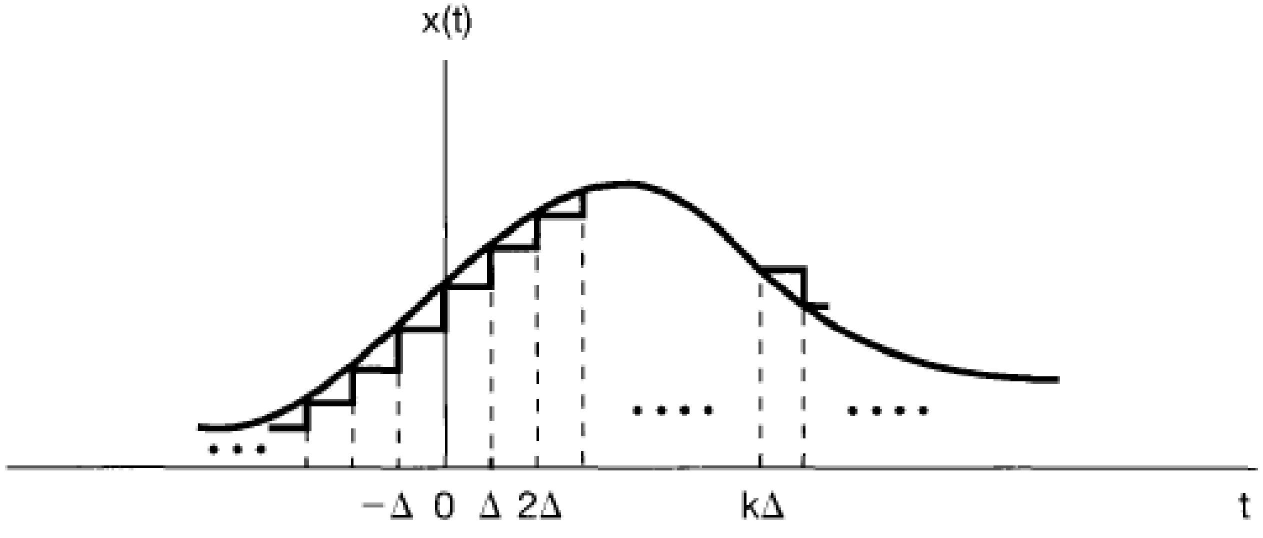


$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases},$$



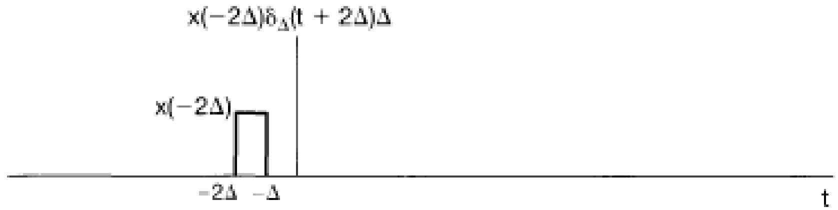
THE CONVOLUTION INTEGRAL

- Representation of Continuous-Time Signals in Terms of Impulses

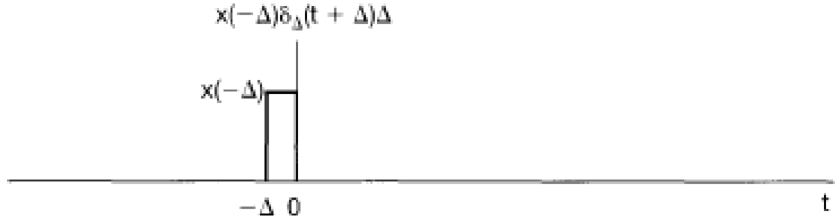


(a)

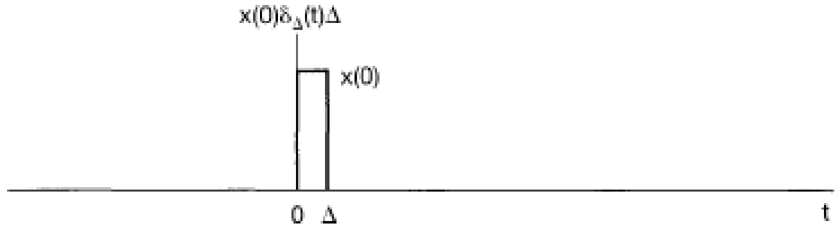
$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases},$$



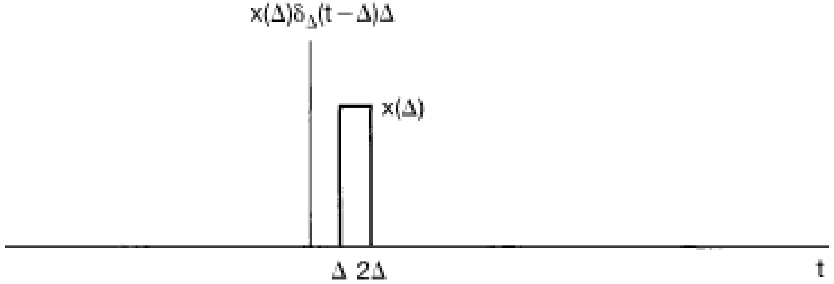
(b)



(c)



(d)



(e)

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta.$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta.$$

As Δ tends to 0, the summation approaches an integral

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau.$$

The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

- The response y(t) of a linear system to this signal x(t) is

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta.$$

- As $\Delta \rightarrow 0$, $y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta.$ $\rightarrow$ $y(t) = \int_{-\infty}^{+\infty} x(\tau)h_{\tau}(t)d\tau.$

- The system is also time invariant $\rightarrow$ $h_{\tau}(t) = h_0(t-\tau)$:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

Convolution integral

The convolution of two signals x(t) and h(t) will be represented as

$$y(t) = x(t) * h(t).$$

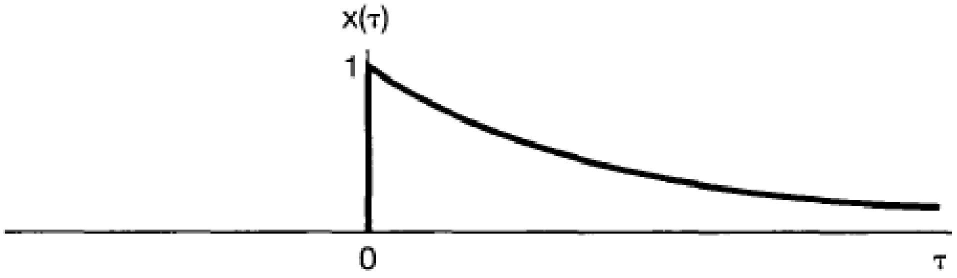
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

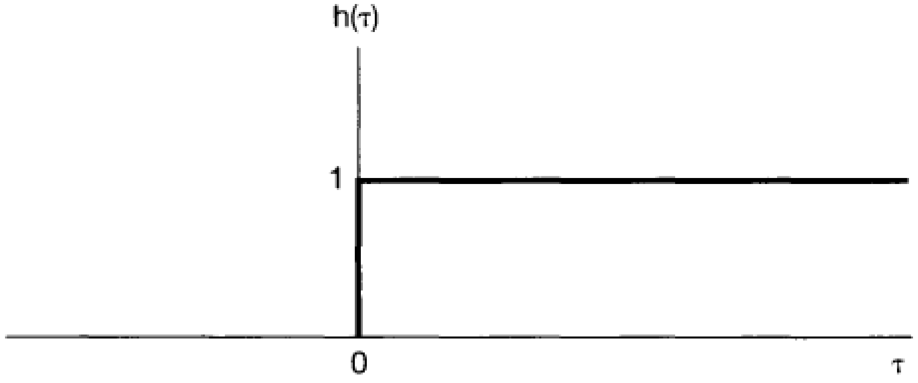
Example

➢Let *x(t)* be the input to an LTI system with unit impulse response h(t), where

$$x(t) = e^{-at}u(t), \quad a > 0$$

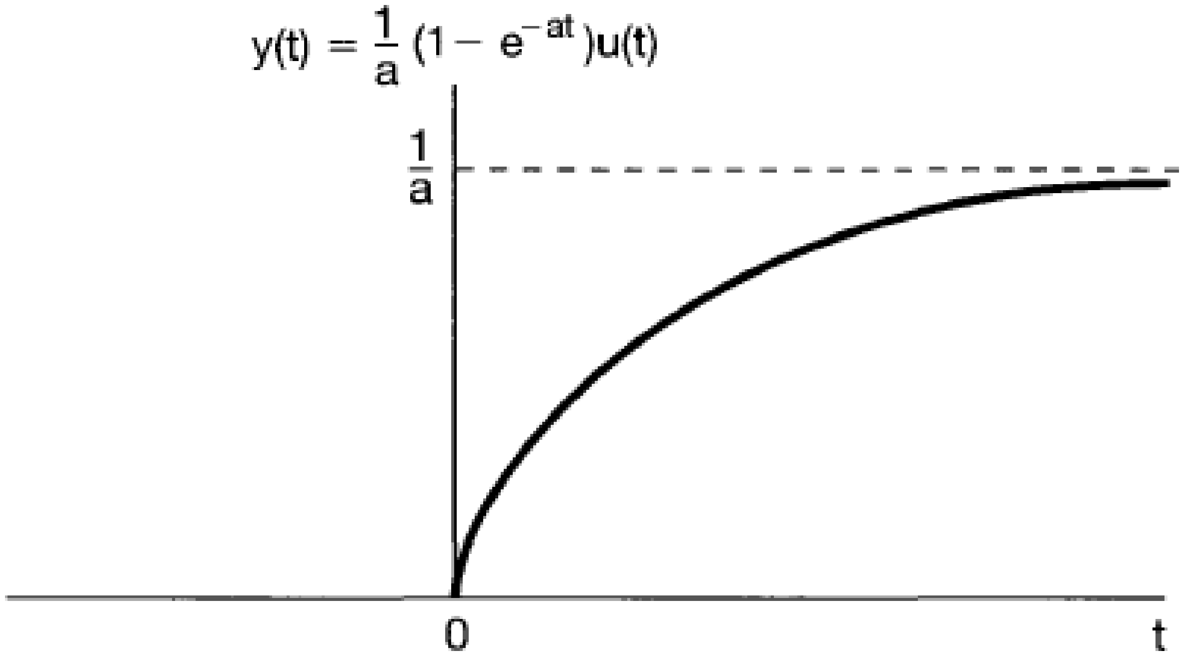
$$h(t) = u(t).$$





$$y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau}\Big|_0^t$$

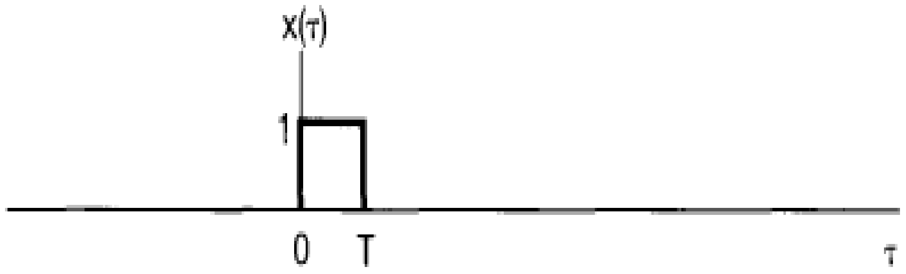
$$= \frac{1}{a}(1 - e^{-at}).$$

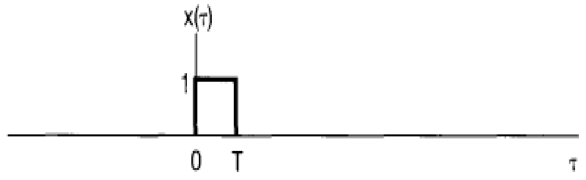


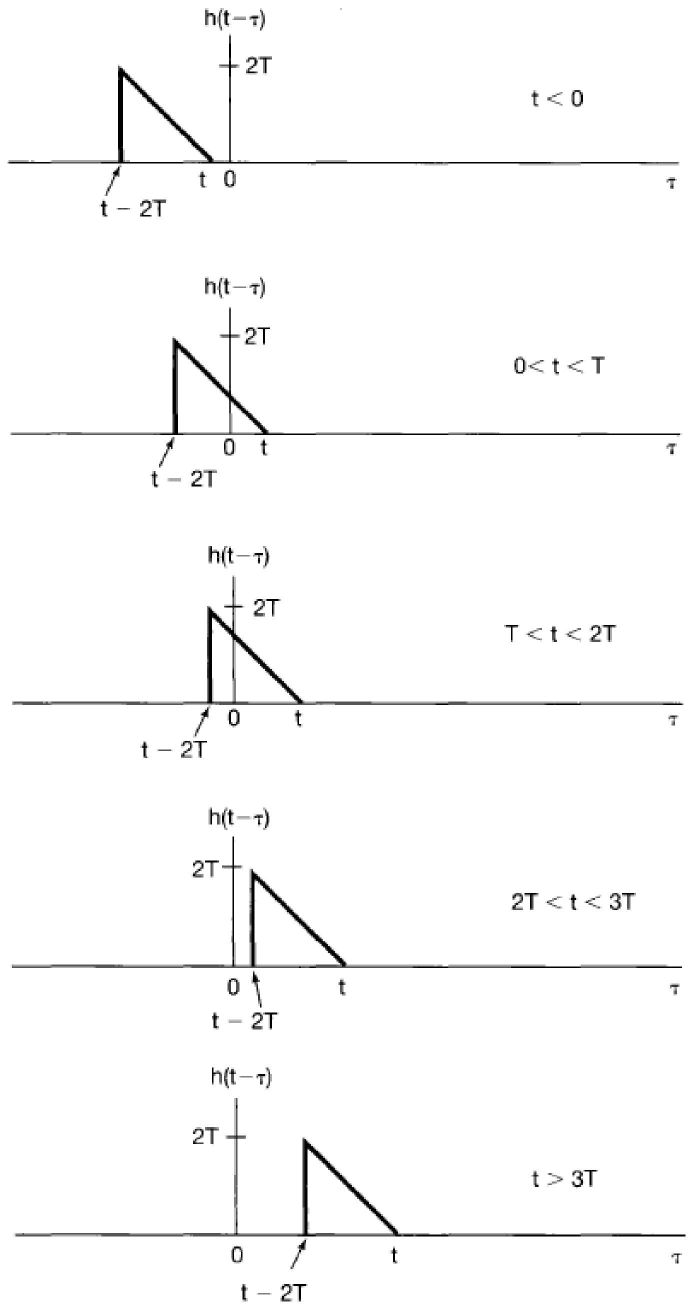
Example

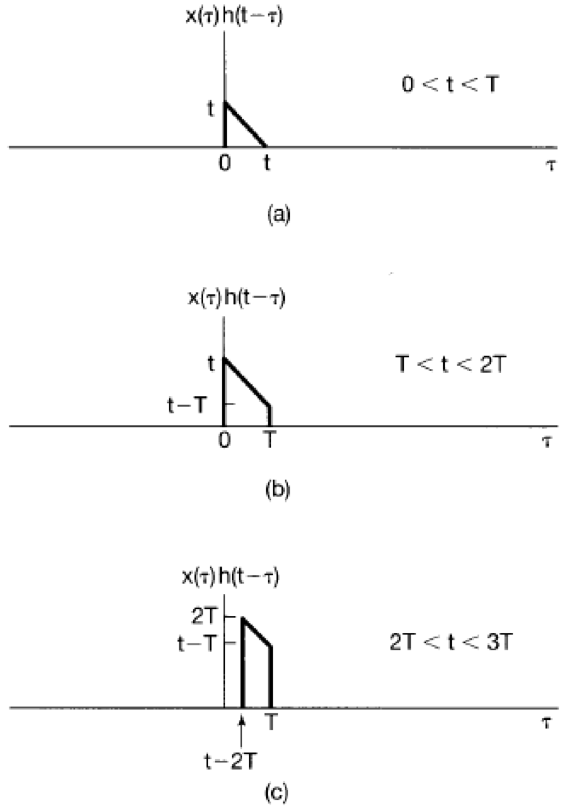
$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases},$$

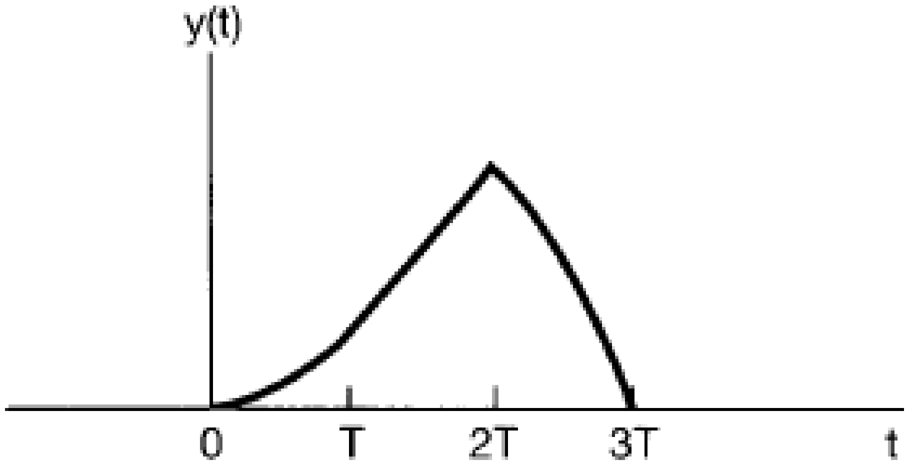
$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}.$$




x(τ)
1
0
T
τ


h(t−τ)
2T
t < 0
t − 2T
t 0
τ
h(t−τ)
2T
0< t < T
t − 2T
0 t
τ
h(t−τ)
2T
T < t < 2T
t − 2T
0 t
τ
h(t−τ)
2T
2T < t < 3T
0 t
t − 2T
τ
h(t−τ)
2T
t > 3T
0 t
t − 2T
τ


x(τ) h(t−τ)
t
0 < t < T
0 t
τ
(a)
x(τ) h(t−τ)
t
t−T
T < t < 2T
0 T
τ
(b)
x(τ) h(t−τ)
2T
t−T
2T < t < 3T
T
t−2T
τ
(c)



$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases},$$

PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

The Commutative Property

- A basic property of convolution in both continuous and discrete time is that it is a *commutative* operation.

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k],$$

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau.$$

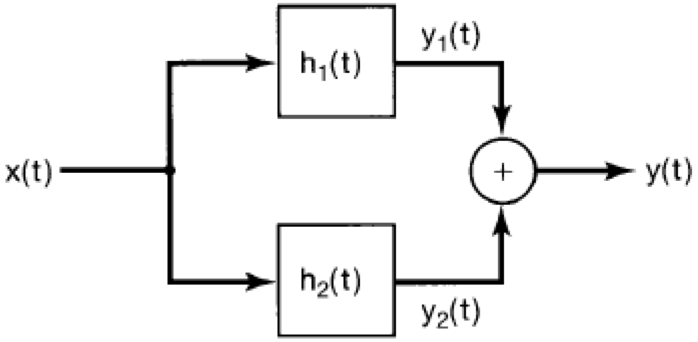
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=-\infty}^{+\infty} x[n-r]h[r] = h[n] * x[n].$$

The Distributive Property

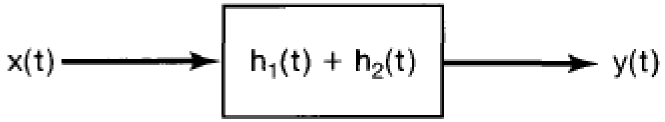
- Convolution distributes over addition

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n],$$

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$



(a)



(b)

➢The two systems, with impulse responses h1 (t) and h2(t), have identical inputs, and their outputs are added

$$y_1(t) = x(t) * h_1(t) \qquad y_2(t) = x(t) * h_2(t),$$

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t),$$

$$y(t) = x(t) * [h_1(t) + h_2(t)],$$

Let $y[n]$ denote the convolution of the following two sequences:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n],$$

$$h[n] = u[n].$$

The Associative Property

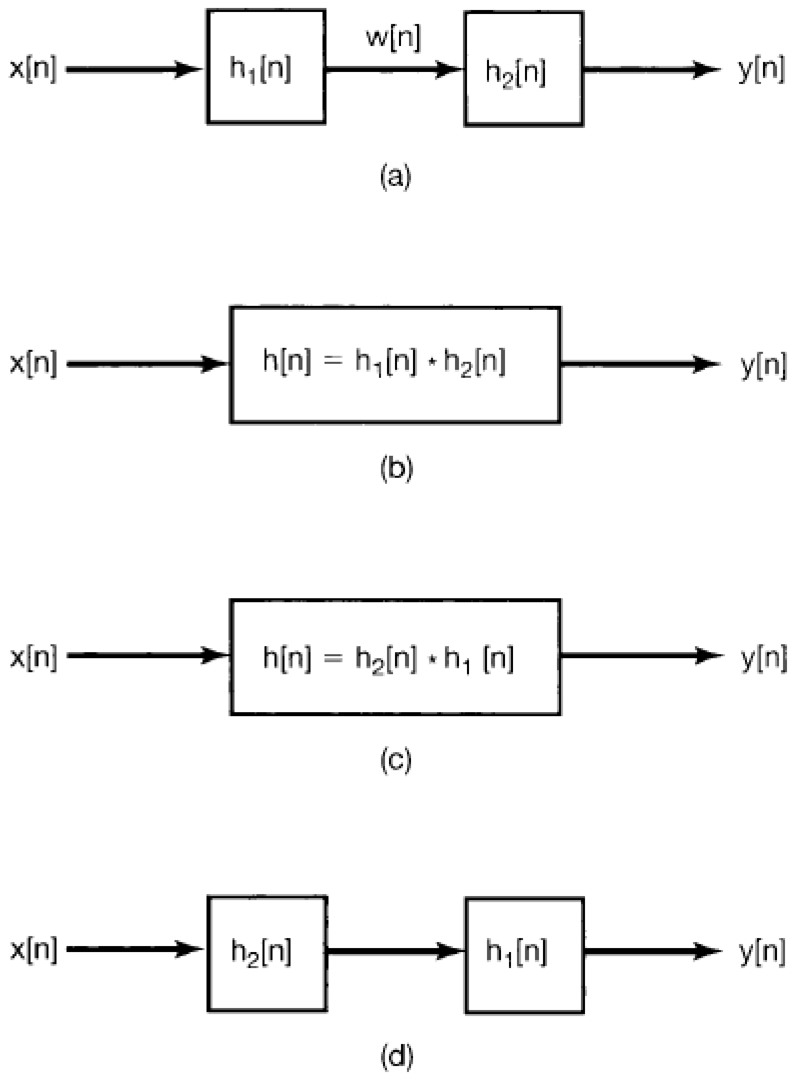
$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n],$$

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t).$$

As a consequence of the associative property,

$$y[n] = x[n] * h_1[n] * h_2[n]$$

$$y(t) = x(t) * h_1(t) * h_2(t)$$



LTI Systems with and without Memory

- A system is memory less if its output at any time depends only on the value of the input at that same time.

- This can be true for a discrete-time LTI system is if h[n] = 0 for n ≠ 0.

$$h[n] = K\delta[n],$$

$$y[n] = Kx[n].$$

If a discrete-time LTI system has an impulse response h[n] that is not identically zero for n ≠ 0, then the system has memory.

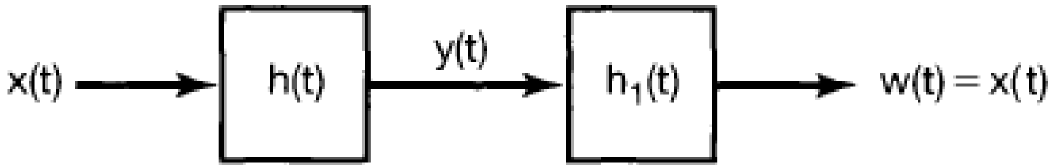
$$y(t) = Kx(t)$$

If K = 1, then these systems become identity systems, with output equal to the input and with unit impulse response equal to the unit impulse.

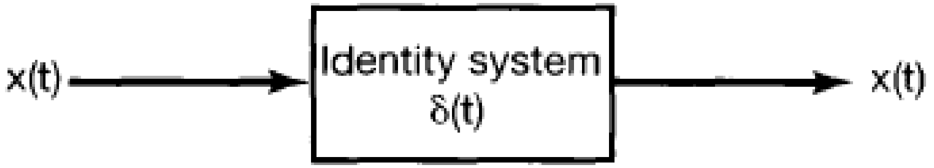
$$x[n] = x[n] * \delta[n] \qquad\qquad x(t) = x(t) * \delta(t),$$

Invertibility of LTI Systems

➢Inverse system with impulse response h1*(t)*, results in *w(t)* = x(t) ----- such that the series interconnection is identical to the



(a)



$$h(t) * h_1(t) = \delta(t).$$

- Consider the LTI system consisting of a pure time shift y(t) = x(t - to).
- The impulse response for the system can be obtained by taking the input equal to *δ(t)*, i.e.,
- *h(t)* = *δ*(*t*- to).

$$x(t - t_0) = x(t) * \delta(t - t_0).$$

- To recover the input from the output---- to invert the system ---- to shift the output back.
- The system with this compensating time shift is then the inverse

$$h_1(t) = \delta(t + t_0),$$

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t).$$

Causality for LTI Systems

- The output of a causal system depends only on the present and past values of the input to the system
- In order for a discrete-time LTI system to be causal, y[n] must not depend on x[k] for k > n
- All of the coefficients h[n- k] that multiply values of x[k] for k > n must be zero

$$h[n] = 0 \quad \text{for } n < 0.$$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k],$$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

Stability for LTI Systems

- A system is stable if every bounded input produces a bounded output.

$$|x[n]| < B \quad \text{for all } n.$$

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|.$$

The magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers

$$|y[n]| \le \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|.$$

$$|y[n]| \le B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n.$$

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty,$$

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty.$$

The Unit Step Response of an LTI System

- s[n] or s(t) ---- corresponding to the output when *x[n] = u[n]* or *x(t) = u(t)*.

$$s[n] = u[n] * h[n].$$

$$s[n] = \sum_{k=-\infty}^{n} h[k].$$

h[n] can be recovered from s[n] using the relation

$$h[n] = s[n] - s[n-1].$$

Unit step response of a continuous-time LTI system is the running integral of its impulse response

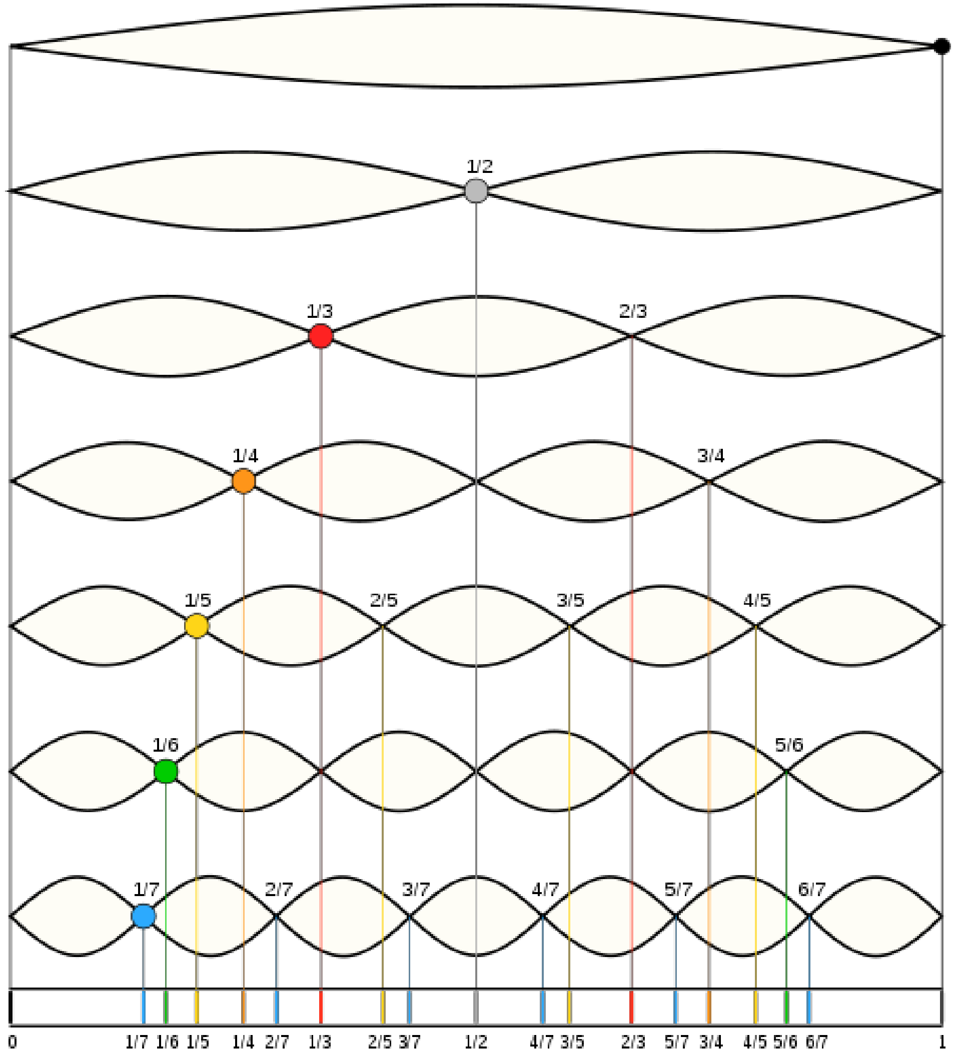
$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \qquad h(t) = \frac{ds(t)}{dt} = s'(t).$$

BIG IDEA: TRANSFORM ANALYSIS

- Make use of properties of LTI systems to simplify analysis
- Represent signals as a linear combination of basic signals with two properties
  - Simple response: easy to characterize LTI system response to basic signal
  - Representation power: the set of basic signals can be use to construct a broad/useful class of signals

NORMAL MODES OF VIBRATING STRING

- When plucking a string, length is divided into integer divisions or harmonics
  - Frequency of each harmonic is an integer multiple of a “fundamental frequency”
  - Also known as the normal modes
- Any string deflection could be built out of a linear combination of “modes”



FOURIER SERIES

- Fourier argued that periodic signals (like the single period from a plucked string) were actually useful
  - Represent complex periodic signals
- Examples of basic periodic signals
  - Sinusoid: $x(t) = cos\omega_0 t$
  - Complex exponential: $x(t) = e^{j\omega_0 t}$
  - Fundamental frequency: ω_0
  - Fundamental period: $T = \frac{2\pi}{\omega_0}$
- Harmonically related period signals form family
  - Integer multiple of fundamental frequency

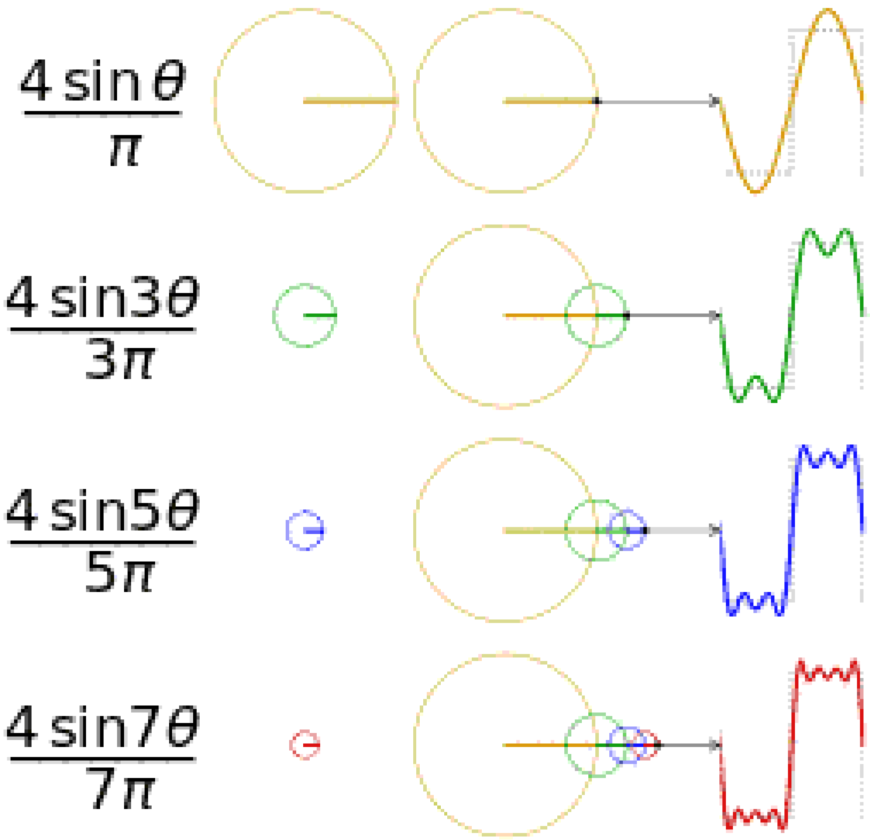
$$\phi_k(t) = e^{jk\omega_0 t} \text{ for } k = 0, \pm 1, \pm 2, \ldots$$

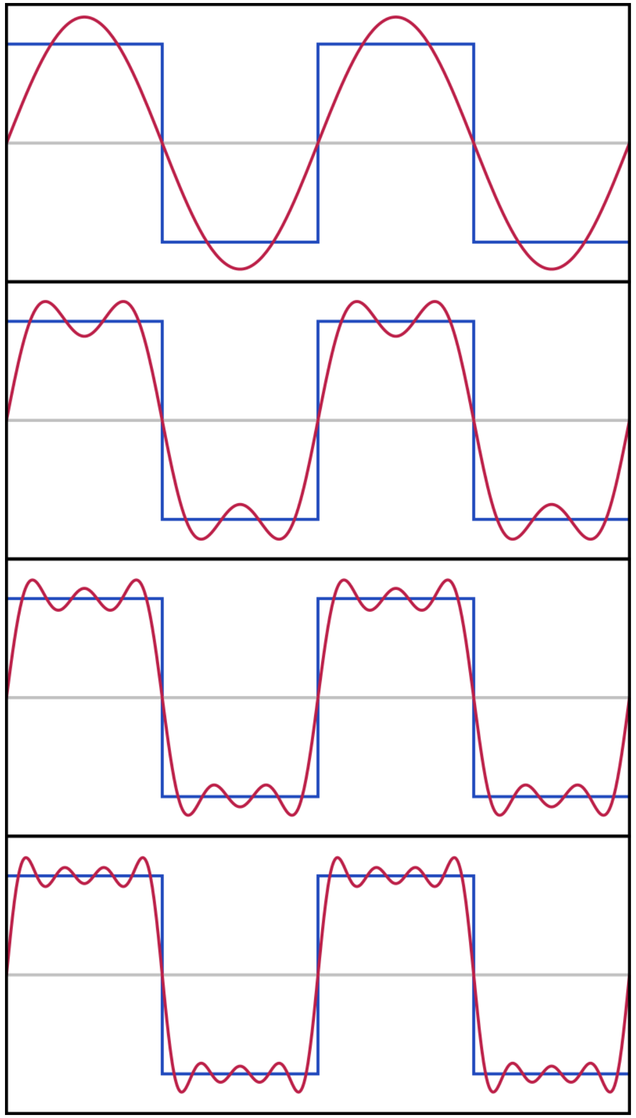
- Fourier Series is a way to represent a periodic signal as a linear combination of harmonics

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

a_k coefficient gives the contribution of a harmonic (periodic signal of k times frequency)

SQUARE WAVE EXAMPLE



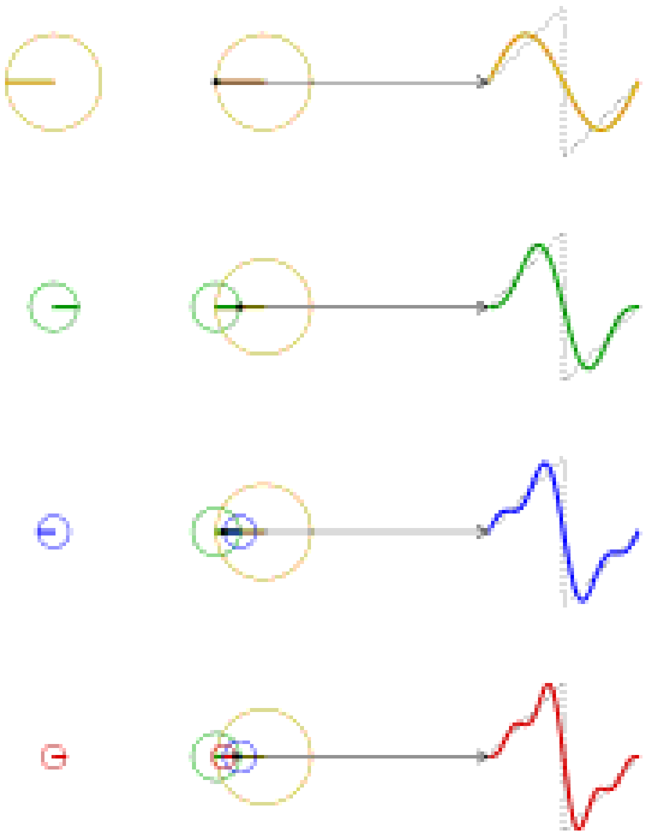


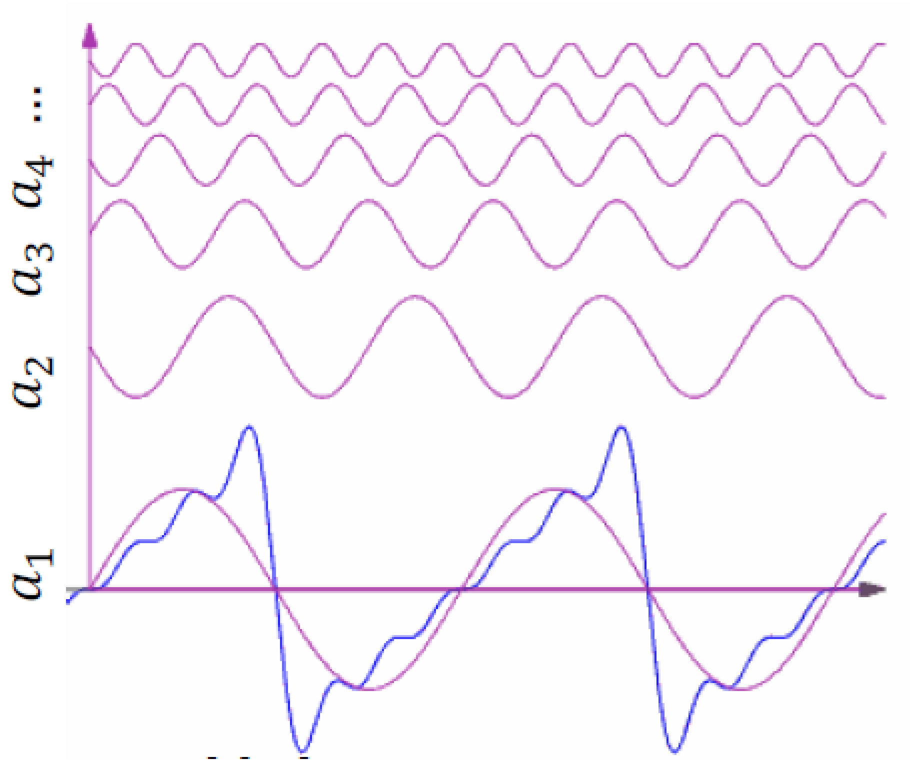
$$\frac{2\sin\theta}{-\pi}$$

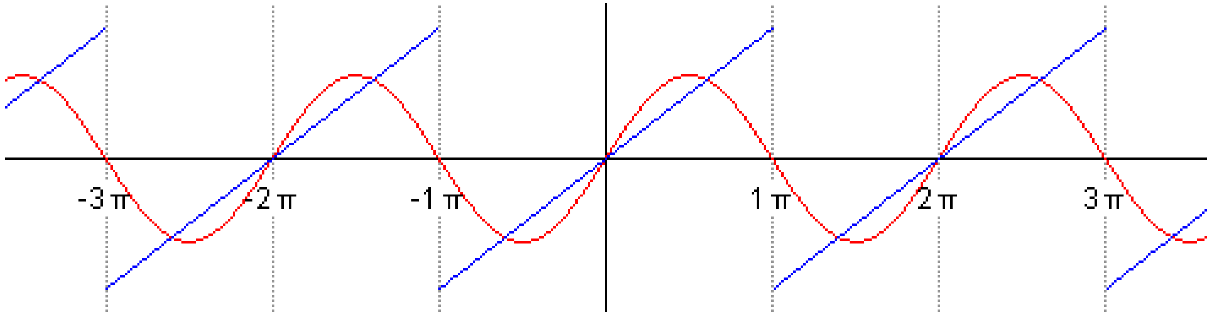
$$\frac{2\sin 2\theta}{2\pi}$$

$$\frac{2\sin 3\theta}{-3\pi}$$

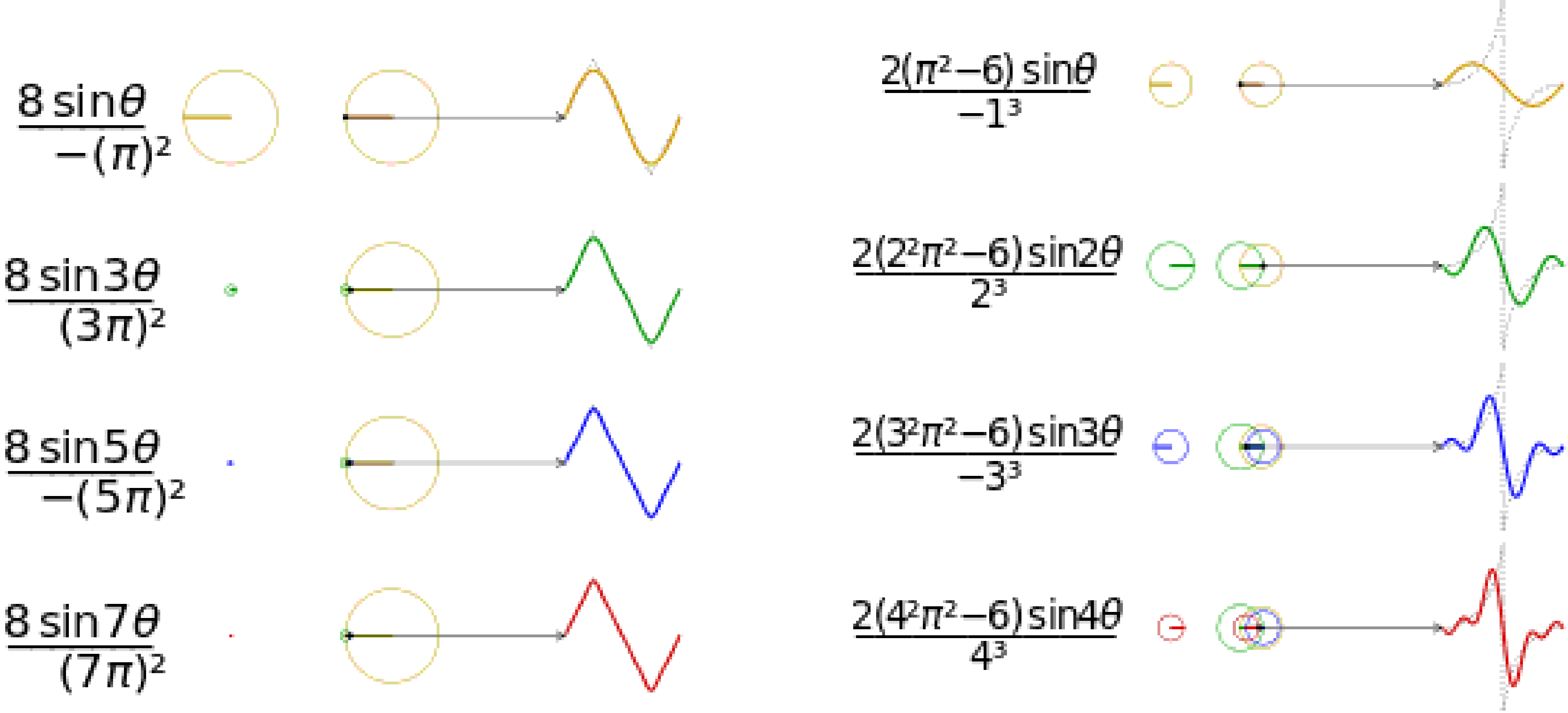
$$\frac{2\sin 4\theta}{4\pi}$$







Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)


$\frac{8\sin\theta}{-(\pi)^2}$
$\frac{8\sin 3\theta}{(3\pi)^2}$
$\frac{8\sin 5\theta}{-(5\pi)^2}$
$\frac{8\sin 7\theta}{(7\pi)^2}$
$\frac{2(\pi^2-6)\sin\theta}{-1^3}$
$\frac{2(2^2\pi^2-6)\sin 2\theta}{2^3}$
$\frac{2(3^2\pi^2-6)\sin 3\theta}{-3^3}$
$\frac{2(4^2\pi^2-6)\sin 4\theta}{4^3}$

Fourier Series Representation of Continuous-time Periodic Signals

- The concept of using "trigonometric sums"-that is, sums of harmonically related sines and cosines or periodic complex exponentials-to describe periodic phenomena

- A signal is periodic if, for some positive value of T, x(t) = x(t + T) for all t.

- The fundamental period of x(t) is the minimum positive, nonzero value of T for which w0 = 2π/T is referred to as the fundamental frequency.

- The sinusoidal signal x(t) = cos w_ot and the periodic complex exponential $e^{jw_o t}$ are periodic with fundamental frequency *w0* and fundamental period $T = 2\pi / wo$

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- Each of these signals has a fundamental frequency that is a multiple of *w0* , and therefore, each is periodic with period *T*
- Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- is also periodic with period *T.*
- The term for k = 0 is a constant.
- The terms for *k* = + 1 and *k* = - 1 both have fundamental frequency equal to *wo* and are collectively referred to as the *fundamental components* or the *first harmonic components*.
- The two terms fork = +2 and *k* = -2 are periodic with half the period of the fundamental components and are referred to as the *second harmonic components*.

➢Consider a periodic signal x(t), with fundamental frequency *2π*

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t},$$

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

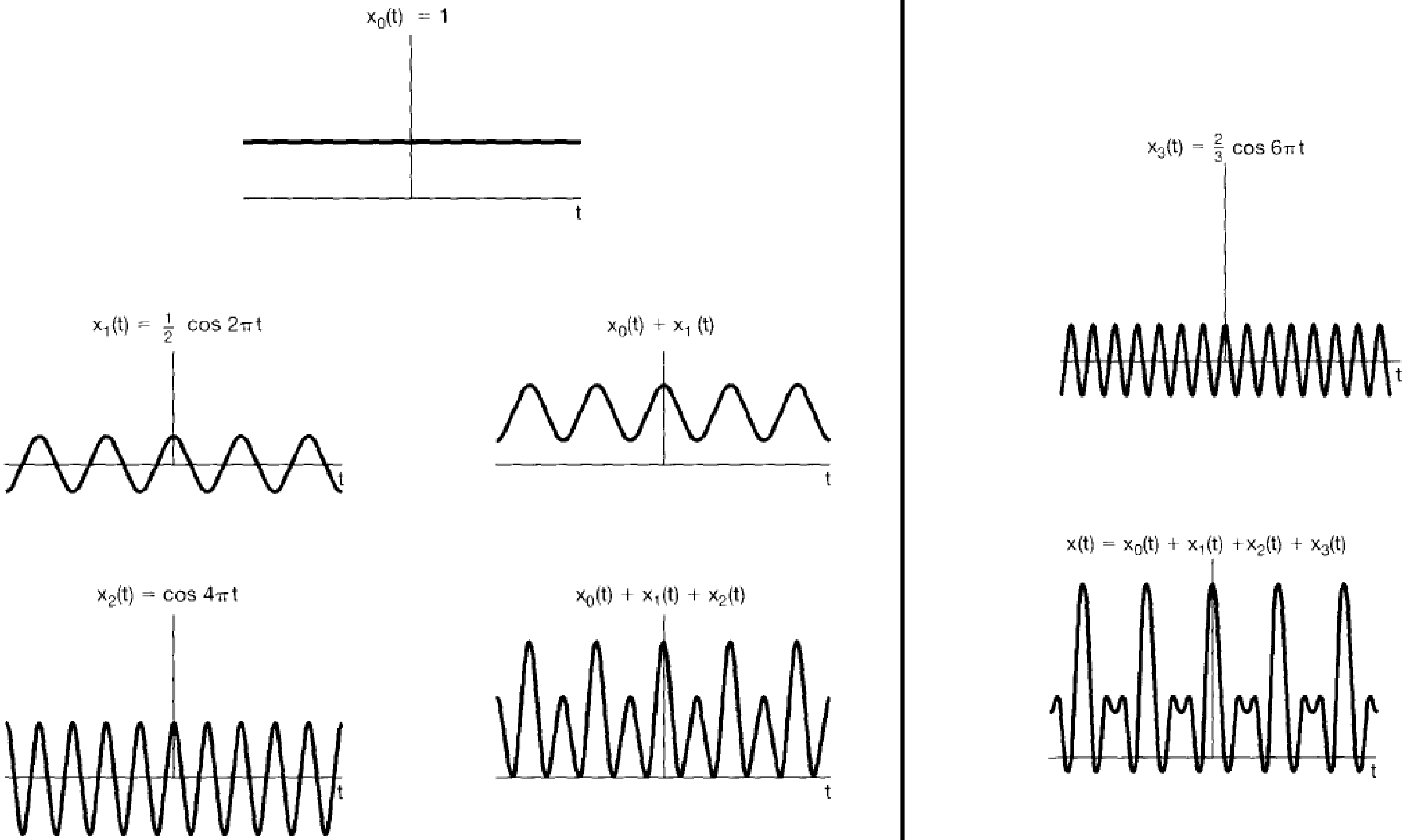
$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}).$$

Equivalently, using Euler's relation, we can write x(t) in the form

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$


x0(t) = 1
t
x1(t) = 1/2 cos 2πt
t
x0(t) + x1(t)
t
x2(t) = cos 4πt
t
x0(t) + x1(t) + x2(t)
t
x3(t) = 2/3 cos 6πt
t
x(t) = x0(t) + x1(t) + x2(t) + x3(t)
t

- Specifically, suppose that *x(t)* is real -----> $x^*(t) = x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}.$$

- Replacing *k* by - *k* in the summation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}, \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- which, by comparison requires that $a_k = a_{-k}^*$,------> $a_k^* = a_{-k}$

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}].$$

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\{a_k e^{jk\omega_0 t}\}.$$

If a_k is expressed in polar form as $$a_k = A_k e^{j\theta_k},$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\{A_k e^{j(k\omega_0 t + \theta_k)}\}.$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k).$$

Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

- We need a procedure for determining the coefficients *ak*

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}.$$

- Integrating both sides from 0 to $T = 2\pi/w0$, we have

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}\,dt.$$

- Here, we are integrating over one period. Interchanging the order of integration and summation yields

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}\,dt\right].$$

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

➢Evaluating the integral

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt,$$

➢we will obtain the same result if we integrate over any interval of length *T.*

$$\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt.$$

- if *x(t)* has a Fourier series representation as a linear combination of harmonically related complex exponentials
- The pair of equations, defines the Fourier series of a periodic continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t},$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt.$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt,$$

Example

$$x(t) = \sin \omega_0 t,$$

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

CONVERGENCE OF THE FOURIER SERIES

- x(t) by a linear combination of a finite number of harmonically related complex exponential

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$

- Let *eN(t)* denote the approximation error

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

- In order to determine how good any particular approximation is, we need to specify a quantitative measure of the size of the approximation error.
- The criterion that we will use is the energy in the error over one period:

$$E_N = \int_T |e_N(t)|^2 \, dt.$$

The Dirichlet conditions

Condition 1

➢Over any period, *x(t)* must be *absolutely integrable;* that is,

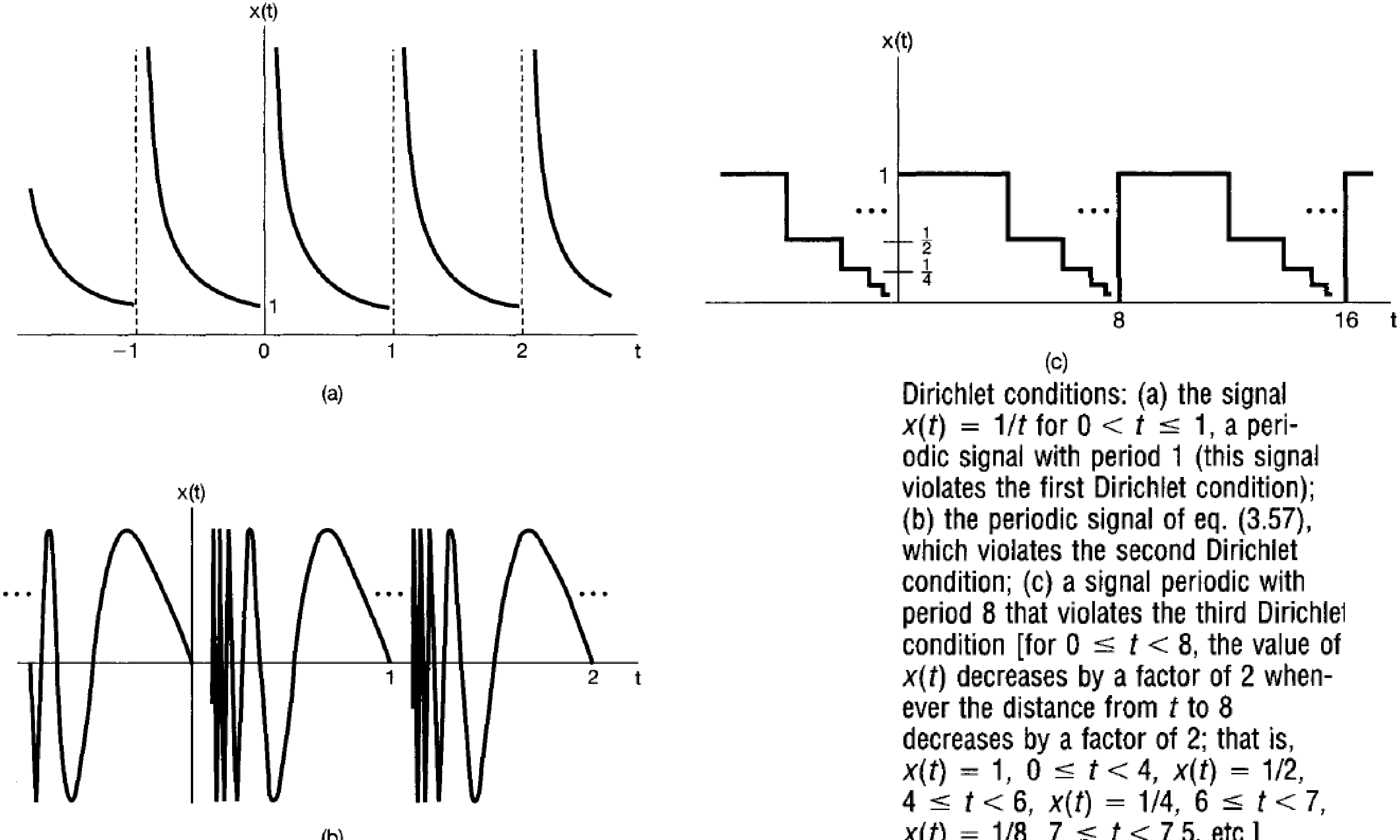
$$\int_T |x(t)|\,dt < \infty. \qquad |a_k| < \infty.$$

Condition 2

In any finite interval of time, *x(t)* is of bounded variation, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3.

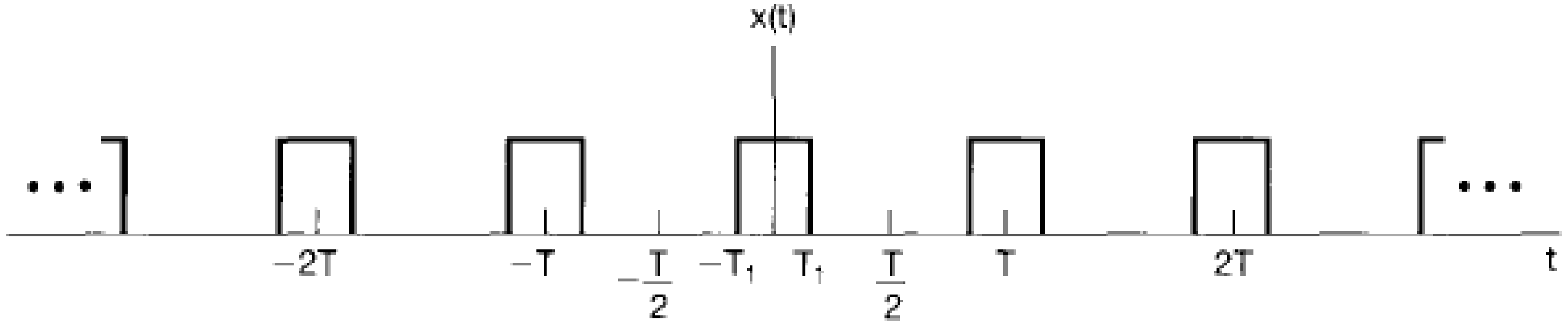
In any finite interval of time, there are only a finite number of discontinuities.



Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \le 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \le t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \le t < 4$, $x(t) = 1/2$, $4 \le t < 6$, $x(t) = 1/4$, $6 \le t < 7$, $x(t) = 1/8$, $7 \le t < 7.5$, etc.].

Example

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

a0 is interpreted to be the average value of x(t), which in this case equals the fraction of each period during which x(t) = 1.

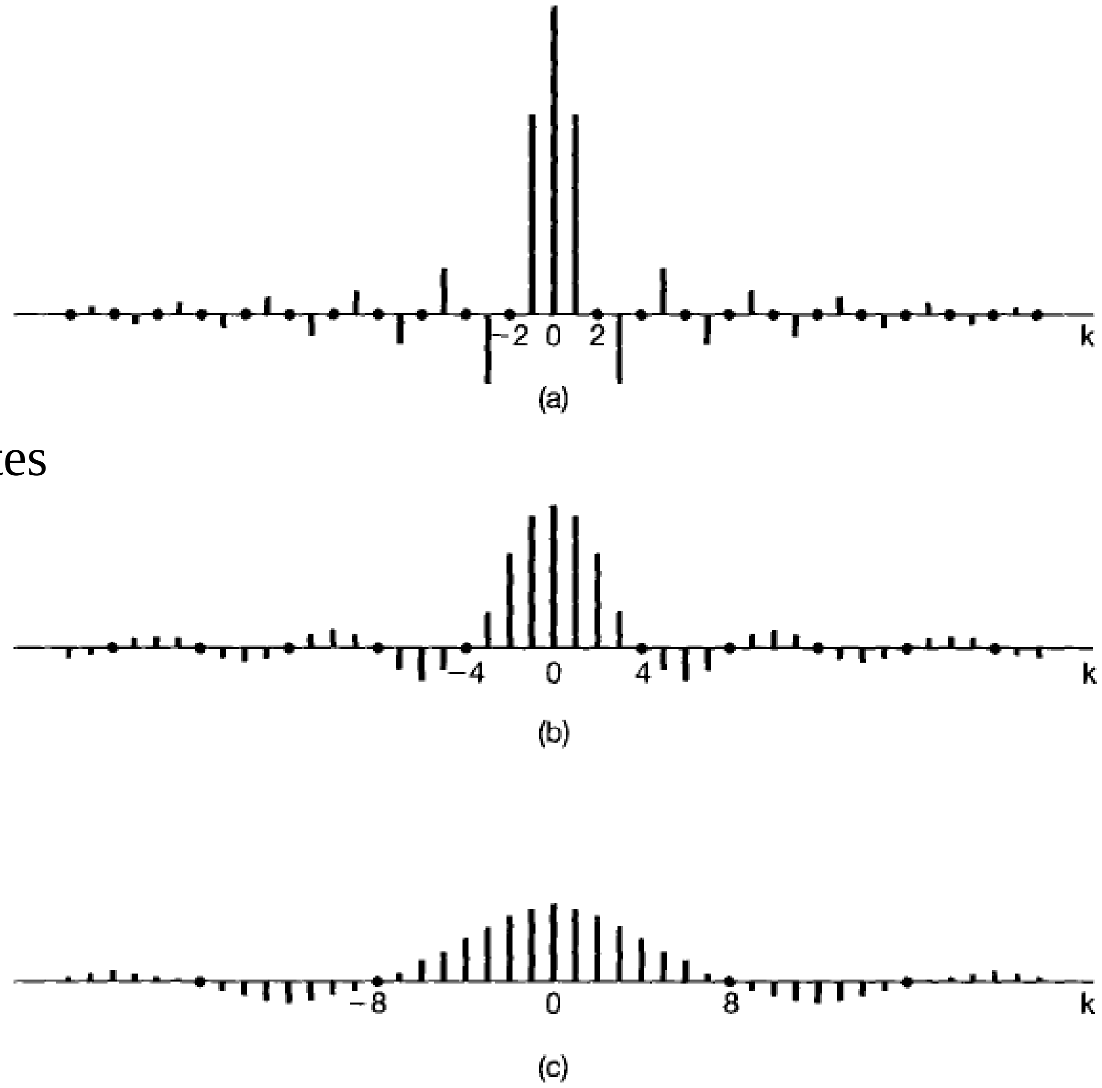
$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0.$$

For T = 4 T1, x(t) is a square wave that is unity for half the period and zero for half the period.

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \quad k \neq 0,$$



ak = 0 for k even and nonzero. Also, sin(πk/2) alternates between ± 1 for successive odd values of k. Therefore

$$a_0 = \frac{1}{2}.$$

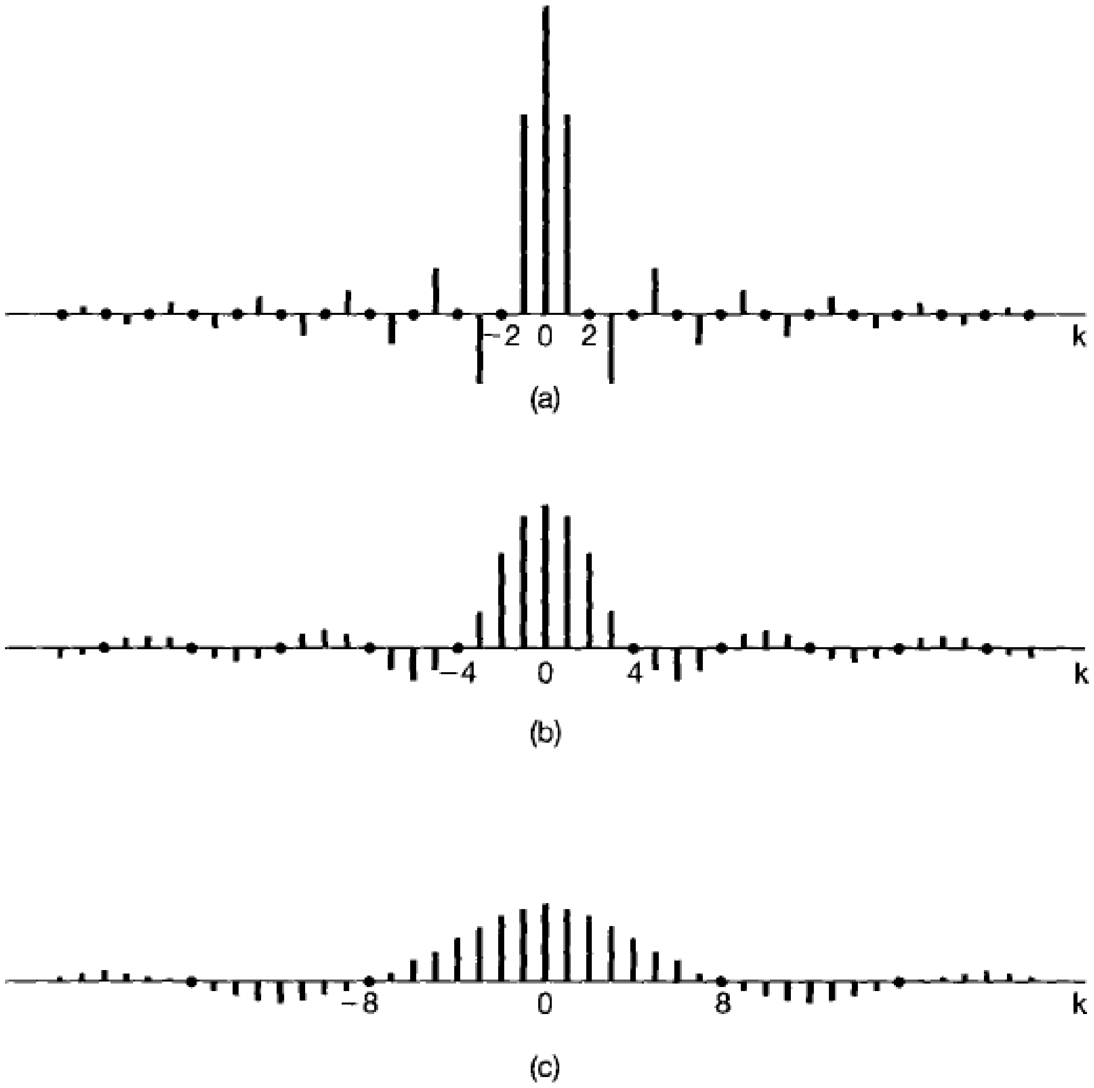
$$a_1 = a_{-1} = \frac{1}{\pi},$$

$$a_3 = a_{-3} = -\frac{1}{3\pi},$$

$$a_5 = a_{-5} = \frac{1}{5\pi},$$

Plots of the scaled Fourier series coefficients for the periodic square wave with T1 fixed and for several values of T: (a) T = 4 T;(b) T = 8 T1; (c) T = 16 T;. The coefficients are regularly spaced samples of the envelope (2 sin wT1)/w, where the spacing between samples, 2π/T, decreases as T increases.

- Consider different “duty cycle” for the rectangle wave
  - $T = 4T_1$ 50% (square wave)
  - $T = 8T_1$ 25%
  - $T = 16T_1$ 12.5%



- Note all plots are still a sinc shape
- Difference is how the sync is sampled
- Longer in time (larger T) smaller spacing in frequency -> more samples between zero crossings

PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

- To indicate the relationship between a periodic signal and its Fourier series coefficients.
- Suppose that x(t) is a periodic signal with period T and fundamental frequency w0, if the Fourier series coefficients of x(t) are denoted by a_k we will use the notation

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k$$

to signify the pairing of a periodic signal with its Fourier series coefficients.

Linearity:

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k, \qquad z(t) = Ax(t) + By(t) \overset{\mathfrak{F}S}{\longleftrightarrow} c_k = Aa_k + Bb_k.$$

$$y(t) \overset{\mathfrak{F}S}{\longleftrightarrow} b_k.$$

If Time period T1 and T2 are different, then is x1+x2 is periodic?

Time Shifting

- When a time shift is applied to a periodic signal x(t), the period T of the signal is preserved.
- The Fourier series coefficients bk of the resulting signal y(t) = x(t- to)

$$b_k = \frac{1}{T}\int_T x(t - t_0)e^{-jk\omega_0 t}dt.$$

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau$$

$$= e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k,$$

$$x(t) \overset{\mathfrak{FS}}{\longleftrightarrow} a_k,$$

$$x(t - t_0) \overset{\mathfrak{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k.$$

Time Reversal

- The period T of a periodic signal $x(t)$ also remains unchanged when the signal undergoes time reversal. To determine the Fourier series coefficients of $y(t)$ = x(- t),

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}.$$

- Making the substitution k = - m,

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}$$

$$b_k = a_{-k}.$$

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

$$x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{-k}.$$

Time Scaling

- Time scaling is an operation that in general changes the period of the underlying signal
- If x(t) is periodic with period T, then x(at), where a is a positive real number, is periodic with period T/a or aw_o

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

- The time-scaling operation applies directly to each of the harmonic components of *x(t)*,
- The Fourier coefficients for each of those components remain the same.

Multiplication

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k.$$

$$x(t)y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Conjugation and Conjugate Symmetry

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k, \qquad x^*(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}.$$

When x(t) real-that is, x(t) = x*(t).

$$a_{-k} = a_k^*,$$

- Some interesting consequences of this property may be derived for x(t) real-that is, when $x(t) = x^*(t)$

- The Fourier series coefficients will be conjugate symmetric

$$a_{-k} = a_k^*$$

- If x(t) is real, then a0 is real and $|a_k| = |a_{-k}|$

- If x(t) is real and even, $a_k = a_{-k}$

- we know that $a_k^* = a_{-k}$ so $a_k = a_k^*$

- If x(t) is real and even, then so are its Fourier series coefficients

- It can be shown that if x(t) is real and odd, then its Fourier series coefficients are purely imaginary and odd

Parseval's Relation for Continuous-Time Periodic Signals

- Parseval's relation for continuous-time periodic signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2,$$

- Left-hand side of eq is the average power (i.e., energy per unit time) in one period of the periodic signal x(t).

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2,$$

- $|a_k^2|$ is the average power in the kth harmonic component of x(t).

- Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

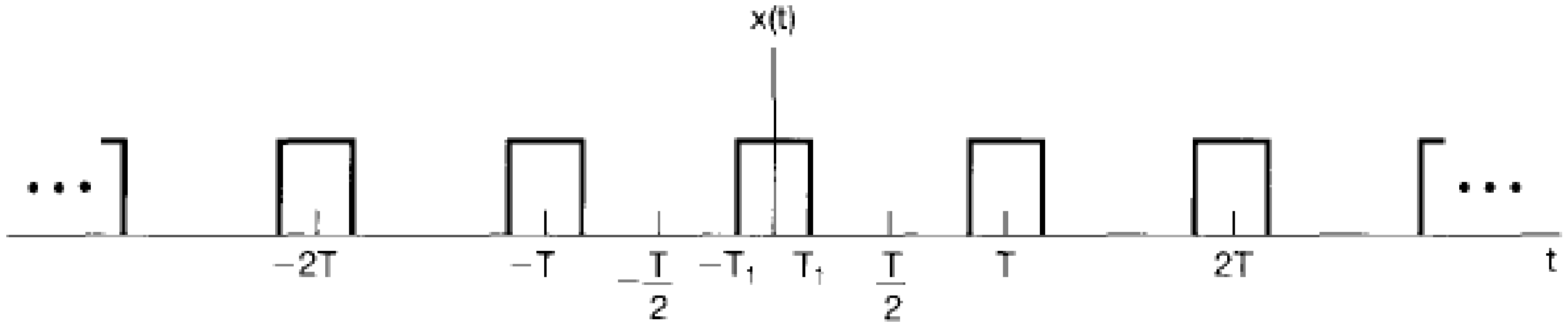
TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

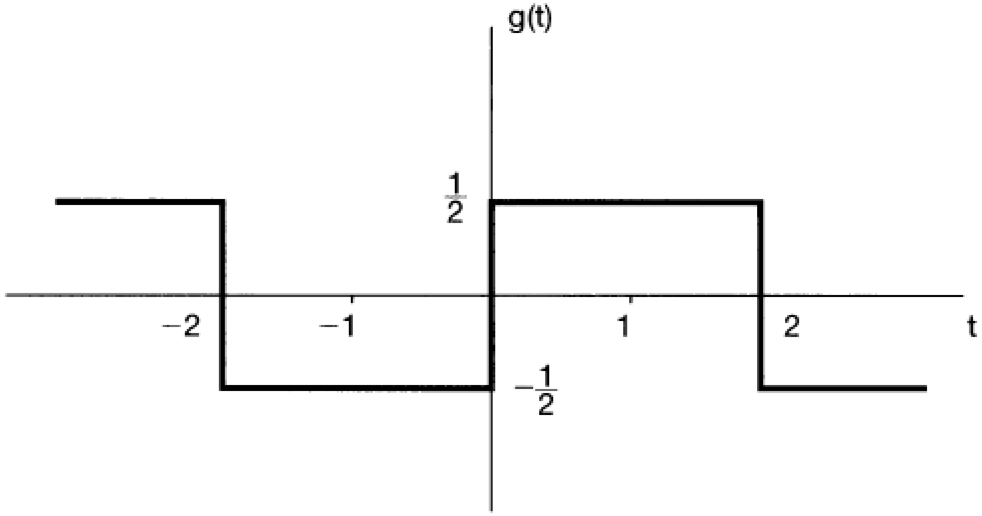
Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & T_1<|t|<T/2\end{cases}$$





$$g(t)=x(t-1)-1/2.$$

➢The time-shift property indicates that, if the Fourier Series coefficients of x(t) are denoted by a_k the Fourier coefficients of x(t – 1)

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k,$$

$$b_k = a_k e^{-jk\pi/2}$$

$$x(t-t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

➢The Fourier coefficients of the *dc offset* in g(t)-i.e., the term -1/2

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}.$$

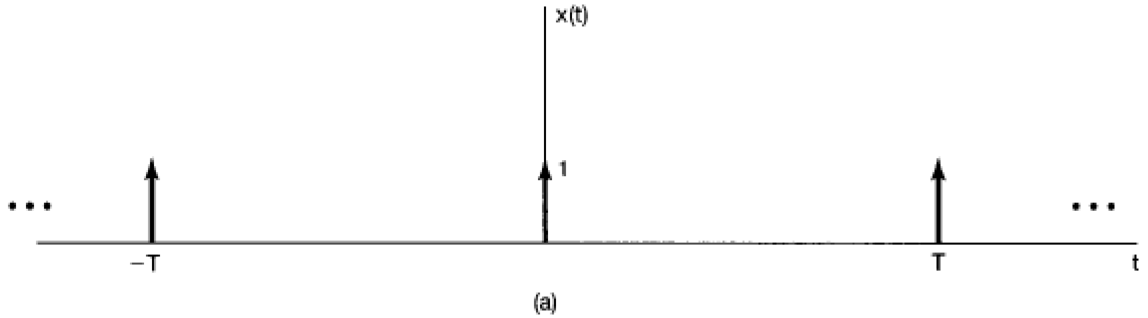
➢Applying the linearity property, the coefficients for *g(t)* may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases}$$

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

Fourier series representation of a periodic train of impulses, or impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT);$$



To determine the Fourier series coefficients a_k , we use eq and select the interval of integration to be -T/2 <= t <= T/2, avoiding the placement of impulses at the integration limits.

Within this interval, x(t) is the same as δ(t)

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

FOURIER SERIES AND LTI SYSTEMS

- If $x(t) = e^{jwt}$ is the input to a continuous-time LTI system, then the output is given by $y(t) = H(jw)\, e^{jwt}$
- The input with a complex exponential at frequency w
- $H(jw)$ is referred to as the *frequency response* of the system and is given by

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt.$$

- let $x(t)$ be a periodic signal with a Fourier series representation given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}.$$

- Suppose that we apply this signal as the input to an LTI system with impulse response $h(t)$.

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}.$$

- Thus, y(t) is also periodic with the same fundamental frequency as x(t).
- Furthermore, if $\{a_k\}$ is the set of Fourier series coefficients for the input x(t), then $\{a_k H(jkwo)\}$ is the set of coefficients for the output y(t).

The effect of the LTI system is to modify individually each of the Fourier coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

Example

➢Suppose that the periodic signal *x(t)* is the input signal to an LTI system with impulse response

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t},$$

$$h(t) = e^{-t}u(t).$$

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

➢To calculate the Fourier series coefficients of the output *y(t)*, we first compute the frequency response:

$$H(j\omega) = \int_0^{\infty} e^{-\tau} e^{-j\omega\tau}\, d\tau$$

$$= -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau}\Big|_0^{\infty}$$

$$= \frac{1}{1+j\omega}.$$

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t},$$

with $b_k = a_k H(jk2\pi)$, so that

$$b_0 = 1,$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right), \qquad b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right),$$

$$b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right), \qquad b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right),$$

$$b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right), \qquad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right).$$

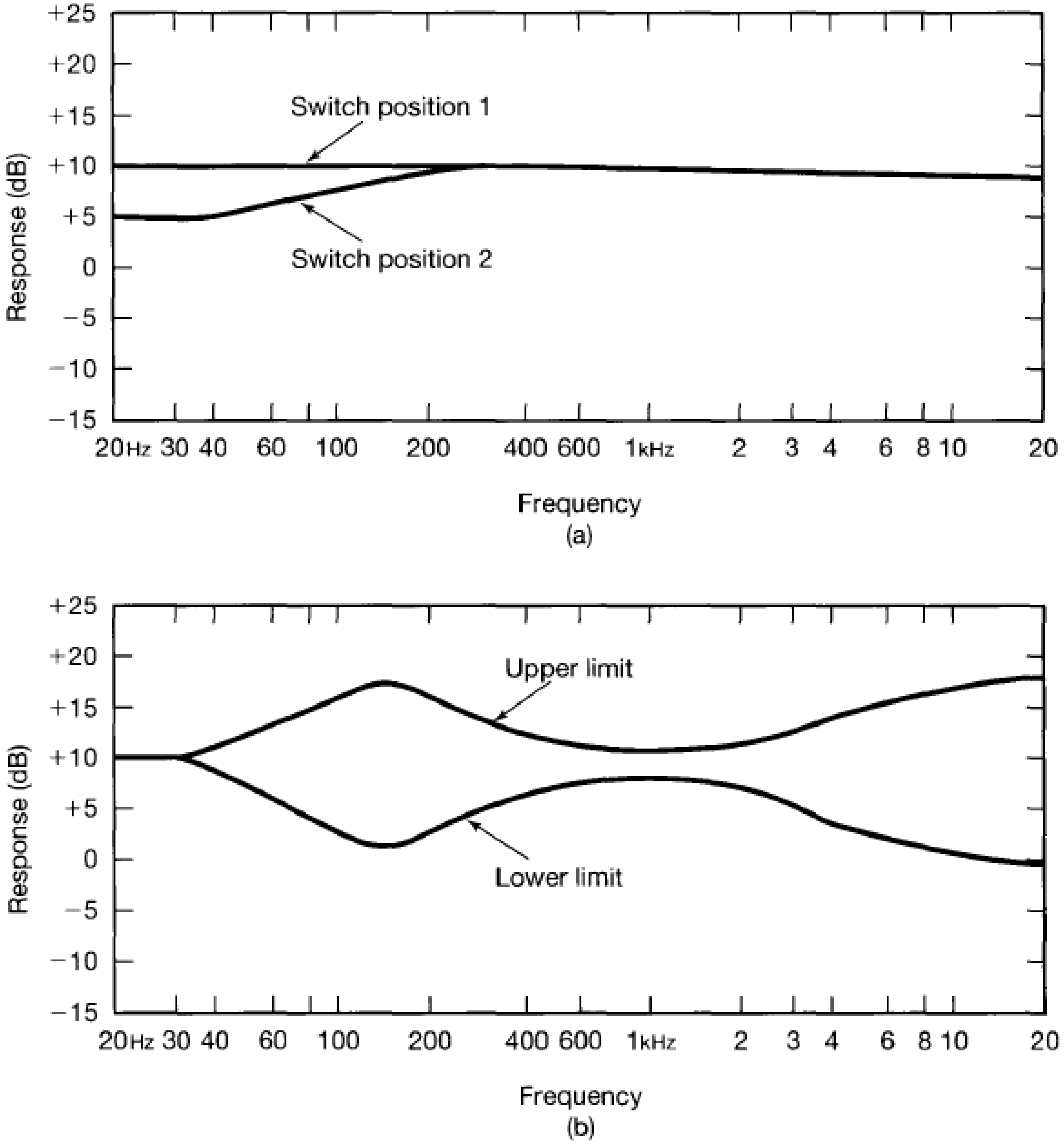
FILTERING

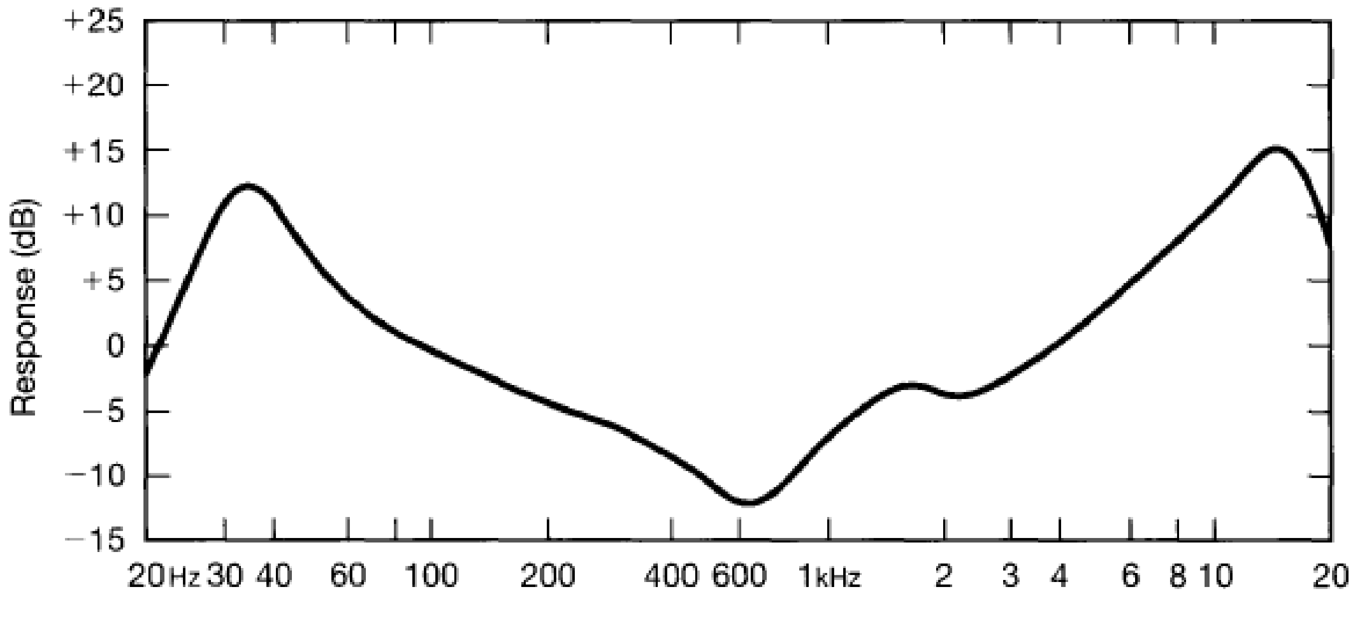
- To change the relative amplitudes of the frequency components in a signal or eliminate some frequency components entirely, a process referred to as filtering.
- Linear time-invariant systems that change the shape of the spectrum are often referred to as ***frequency-shaping filters***.
- Systems that are designed to pass some frequencies and significantly attenuate others are referred to as ***frequency-selective filters***.

Frequency-Shaping Filters

- Audio systems - Modify relative amounts of low-frequency energy (bass) and high-frequency energy (treble)

- Equalizing filter is often included in the preamplifier to compensate for the frequency response characteristics of the speakers.




+25
+20
+15
+10
+5
0
−5
−10
−15
Response (dB)
20Hz 30 40 60 100 200 400 600 1kHz 2 3 4 6 8 10 20
Frequency
(c)

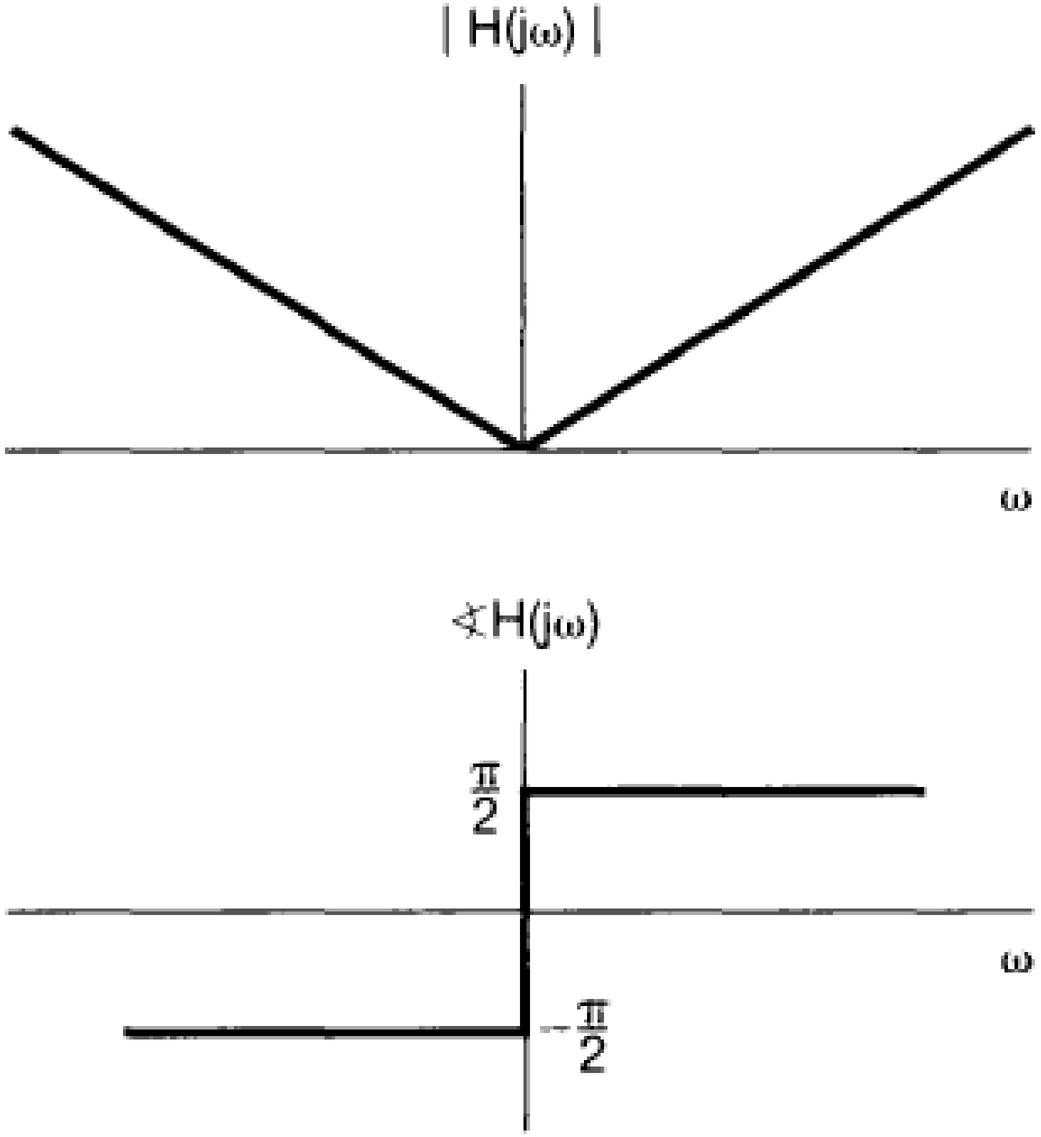
- A class of frequency-shaping filters often encountered is that for which the filter output is the derivative of the filter input

$$y(t) = dx(t)/dt$$

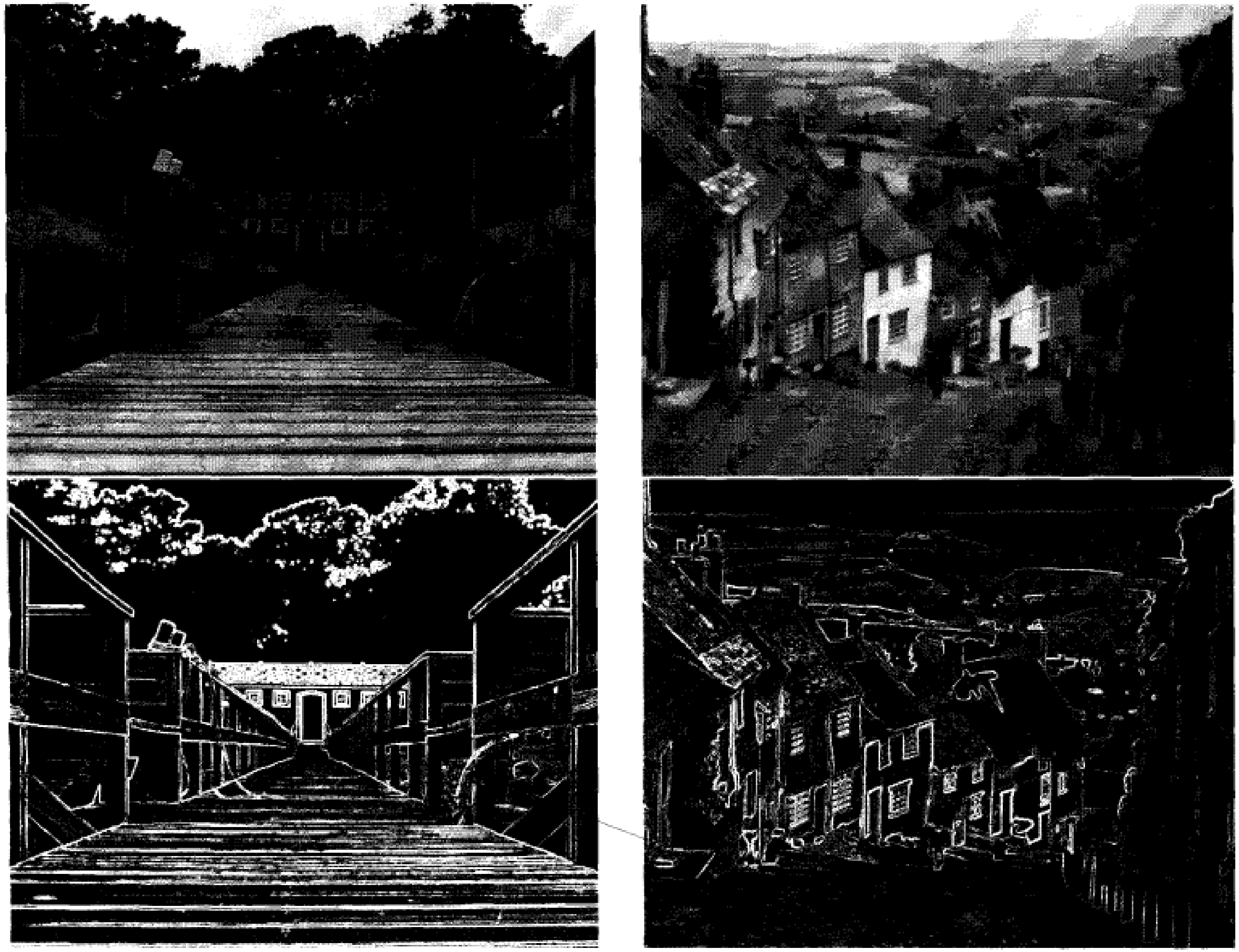
- With x(t) of the form x(t) = e^{jwt}, *y(t)* will be *y(t)* = jwe^{jwt}, from which it follows that the frequency response is

$$H(j\omega) = j\omega$$

- Differentiating filters are useful in enhancing rapid variations or transitions in a signal.



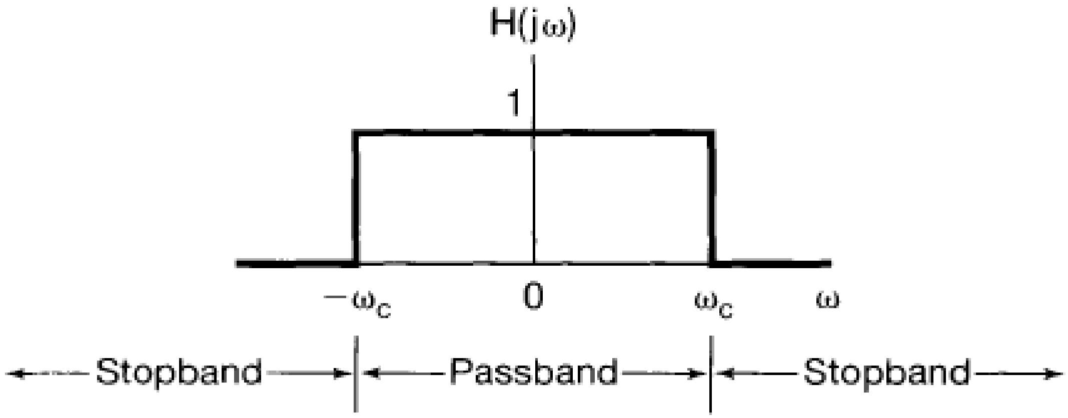
| H(jω) |
ω
∢H(jω)
π/2
ω
−π/2

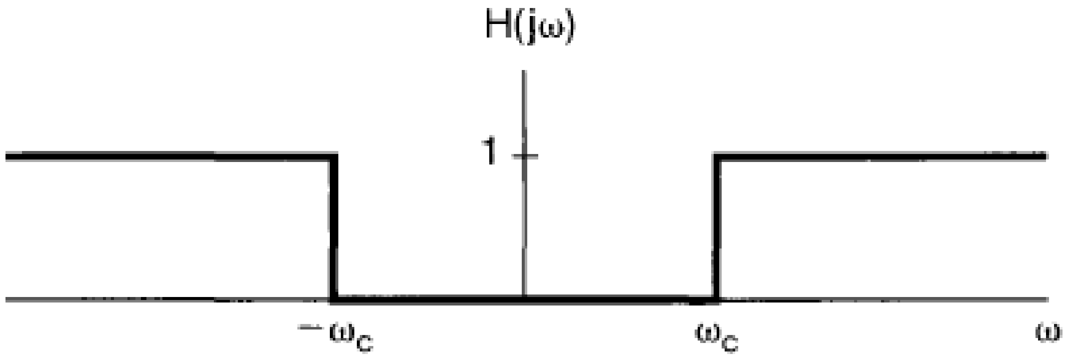


Frequency-Selective Filters

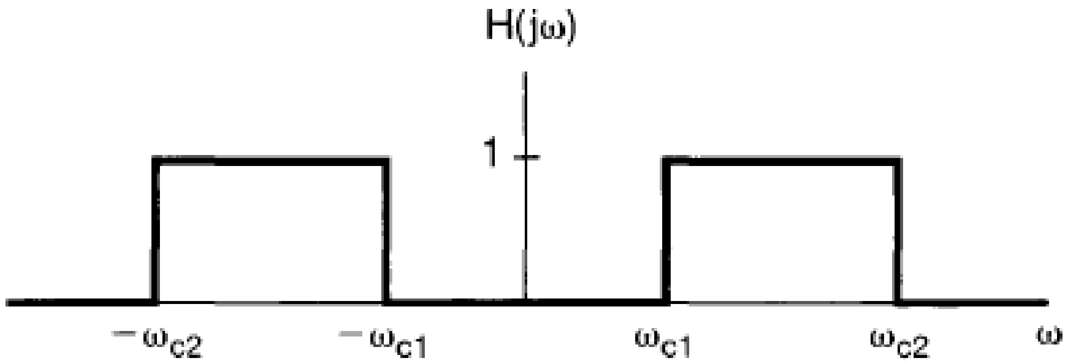
- Select some bands of frequencies and reject others

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$





(a)



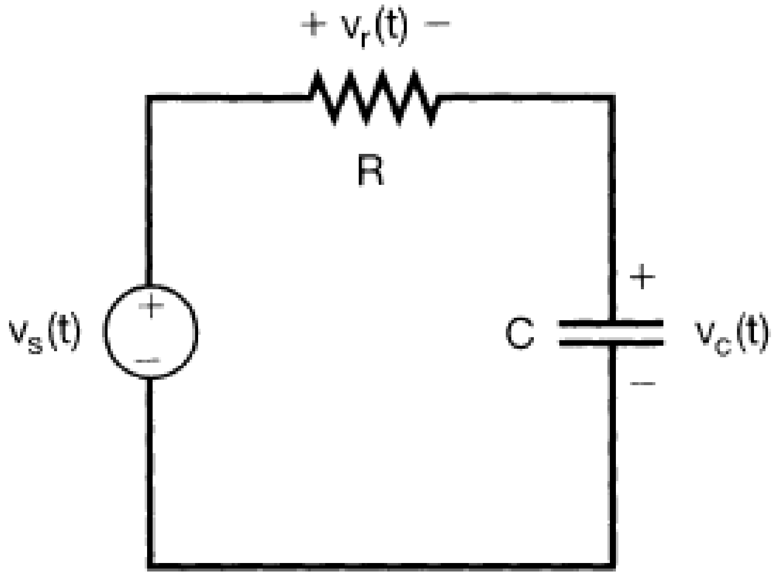
(b)

EXAMPLES OF CONTINUOUS-TIME FILTERS DESCRIBED BY DIFFERENTIAL EQUATIONS

- Frequency-selective filtering is accomplished through the use of LTI systems described by linear constant-coefficient differential equations.

- Electrical circuits are used for continuous-time filtering operations

- First-order *RC* circuit can be used to perform either a lowpass or highpass filtering operation



$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

➢To determine its frequency response *H(jw*), with input voltage *v,(t)* = e^{jwt}, the output voltage *vc(t)* = *H(jw)* e^{jwt}

$$RC\frac{d}{dt}[H(j\omega)e^{j\omega t}] + H(j\omega)e^{j\omega t} = e^{j\omega t},$$

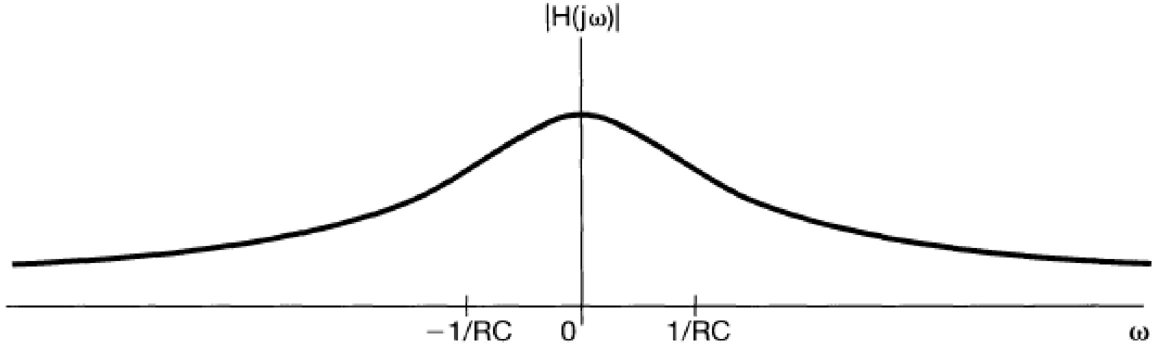
$$RC\,j\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t},$$

$$H(j\omega)e^{j\omega t} = \frac{1}{1 + RC\,j\omega}e^{j\omega t}$$

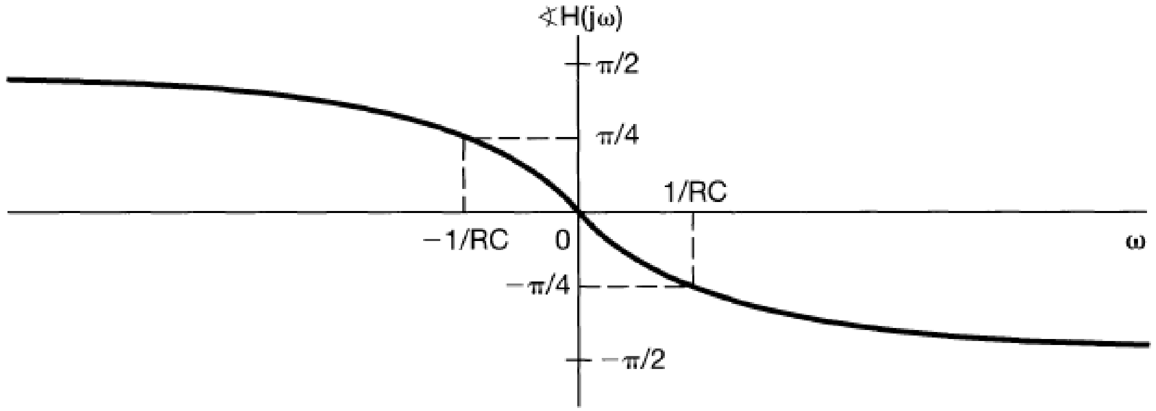
$$H(j\omega) = \frac{1}{1 + RC\,j\omega}$$

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

➢The magnitude and phase of the frequency response *H(jw)*



(a)

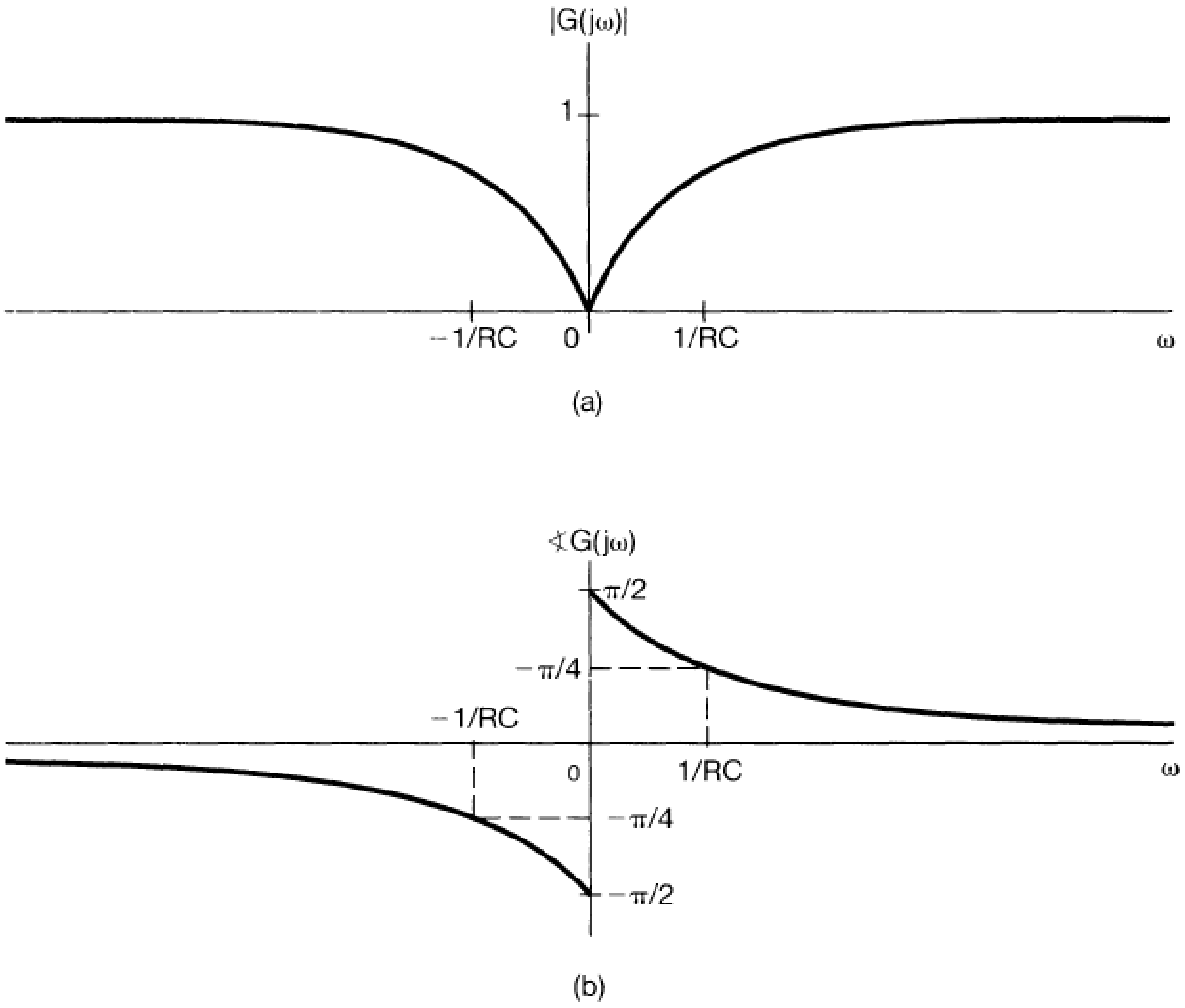


(b)

Simple RC Highpass Filter

$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}.$$

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}.$$



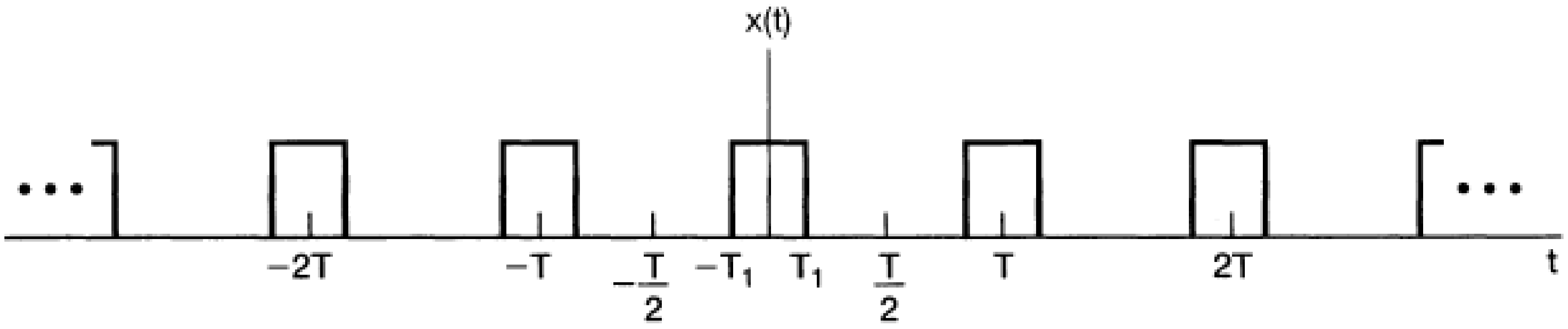
REPRESENTATION OF APERIODIC SIGNALS

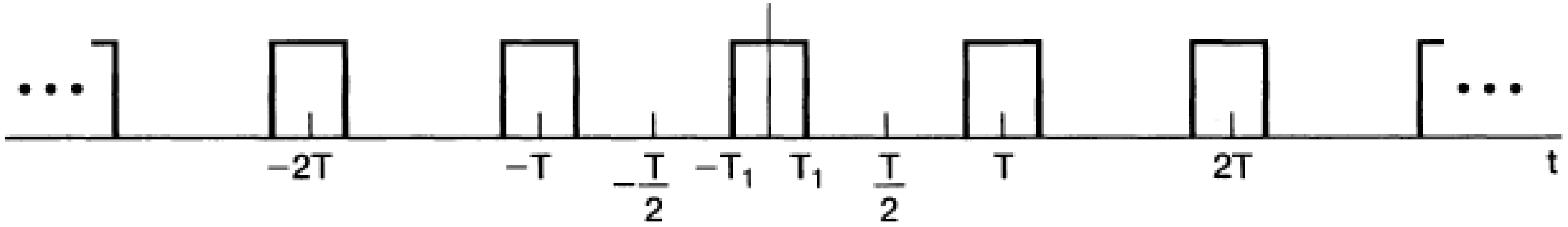
- Revisiting the Fourier series representation for the continuous-time periodic square wave

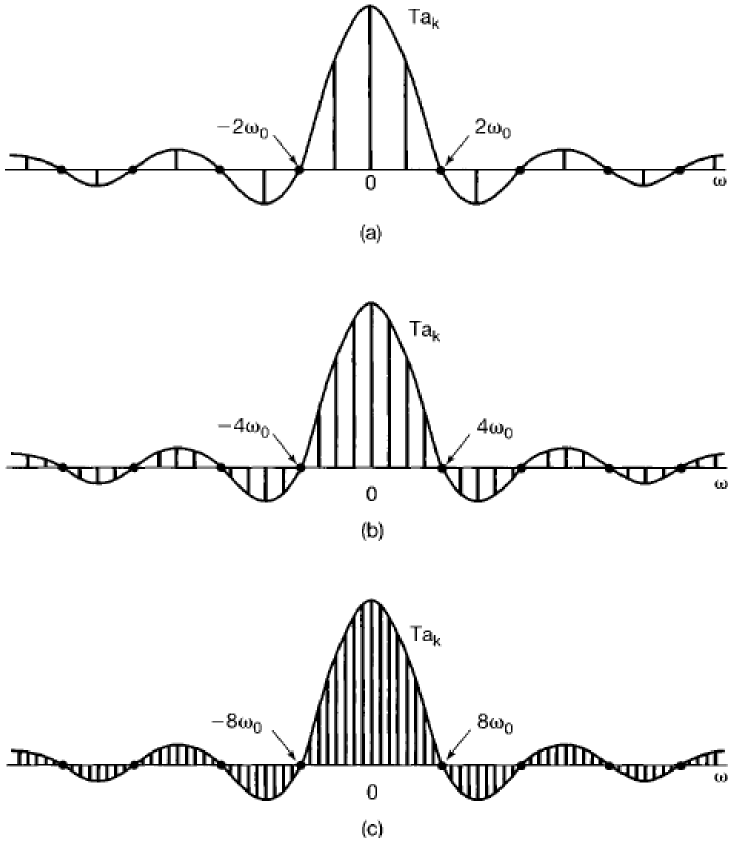
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Fourier series coefficients

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}, \qquad T a_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}$$

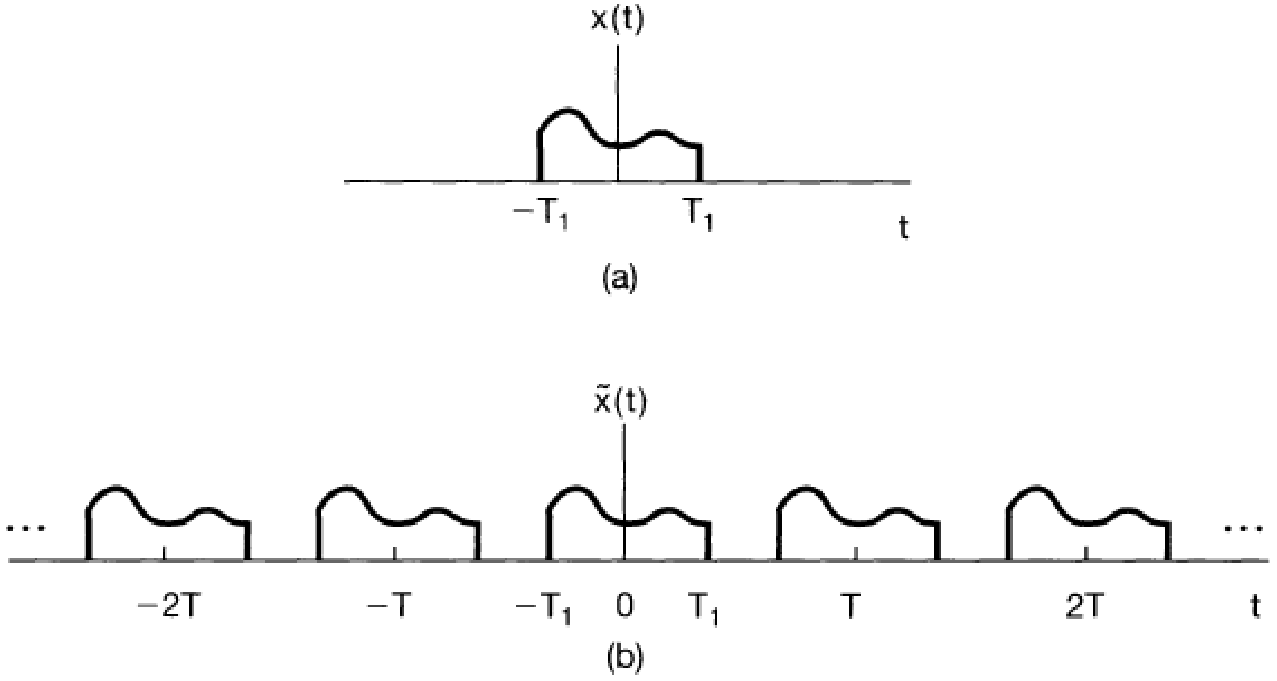






(a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

THE CONTINUOUS-TIME FOURIER TRANSFORM



$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\,dt,$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

Therefore, defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt,$$

we have, for the coefficients a_k,

$$a_k = \frac{1}{T}X(jk\omega_0).$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t},$$

or equivalently, since $2\pi/T = \omega_0$,

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0.$$

As $\omega_0 \rightarrow 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$

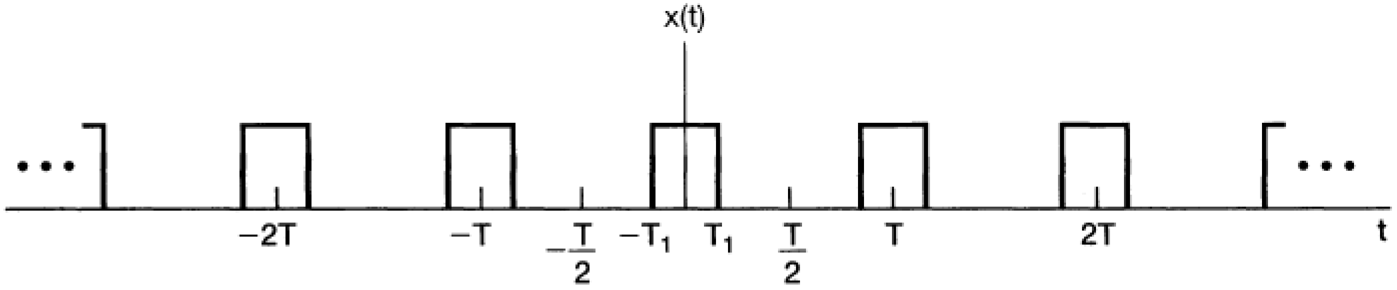
$\tilde{x}(t) \rightarrow x(t)$ as $T \rightarrow \infty$ $\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega}$

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt.}$$

THE CONTINUOUS-TIME FOURIER TRANSFORM

- Fourier series representation for the continuous-time periodic square wave

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



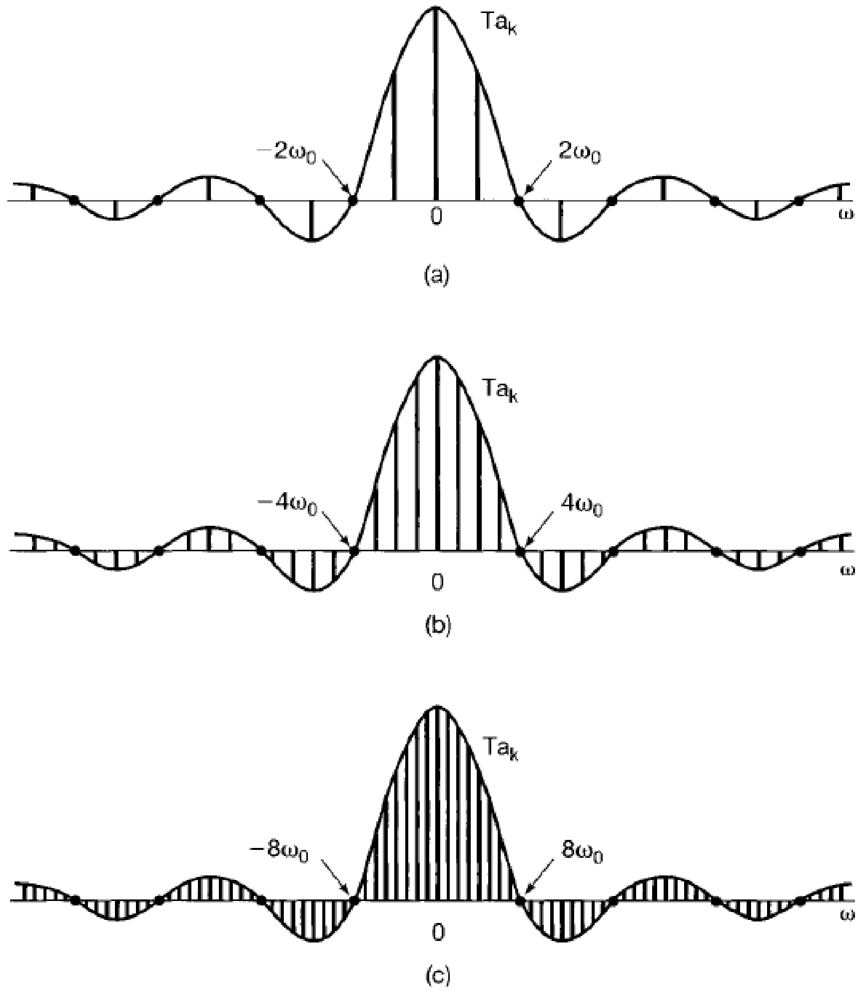
- Fourier series coefficients for this square wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

THE CONTINUOUS-TIME FOURIER TRANSFORM

- An alternative way of interpreting equation is as samples of an envelope function

$$T a_k = \left. \frac{2 \sin \omega T_1}{\omega} \right|_{\omega = k\omega_0}$$



Convergence of Fourier Transforms

1. $x(t)$ be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty.$$

2. $x(t)$ have a finite number of maxima and minima within any finite interval.
3. $x(t)$ have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

Examples of Continuous-Time Fourier Transforms

$$x(t) = e^{-at}u(t) \quad a > 0.$$

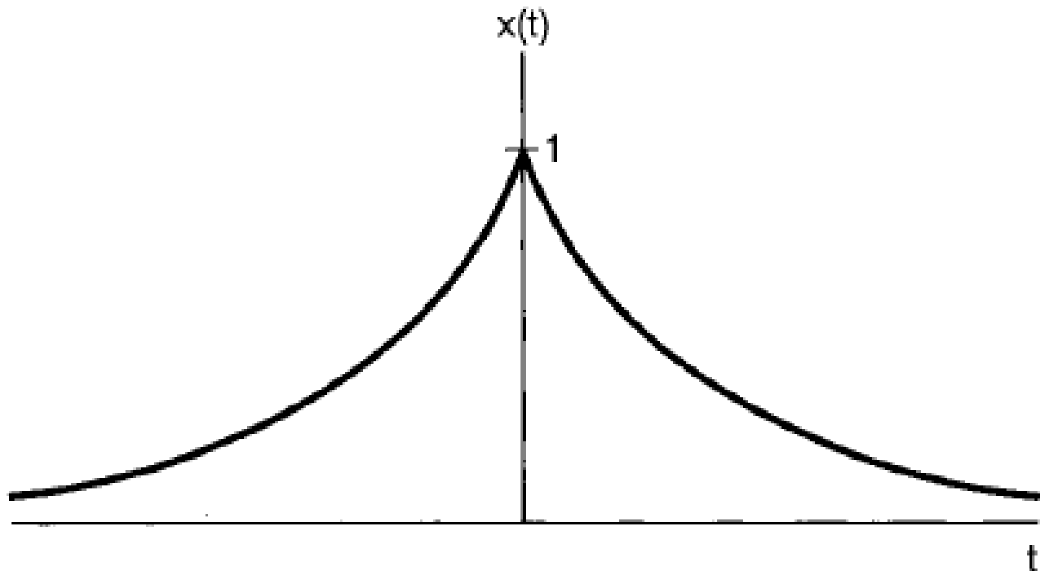
$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}.$$

$$X(j\omega) = \frac{1}{a + j\omega}, \quad a > 0.$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \quad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Example

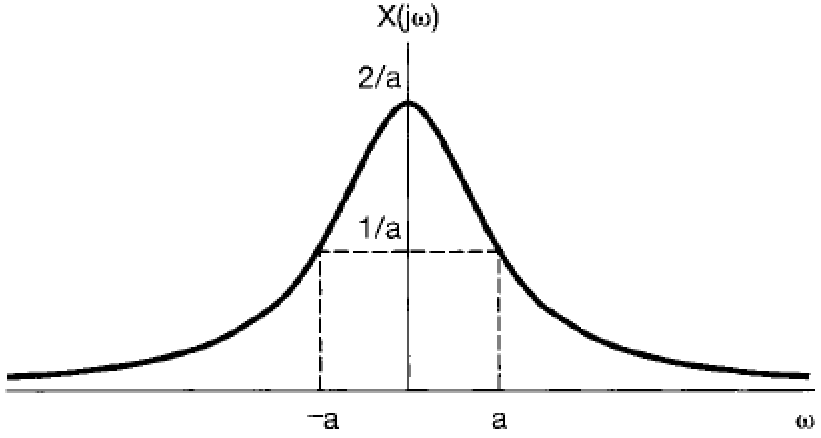
$$x(t) = e^{-a|t|}, \quad a > 0.$$



$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}.$$



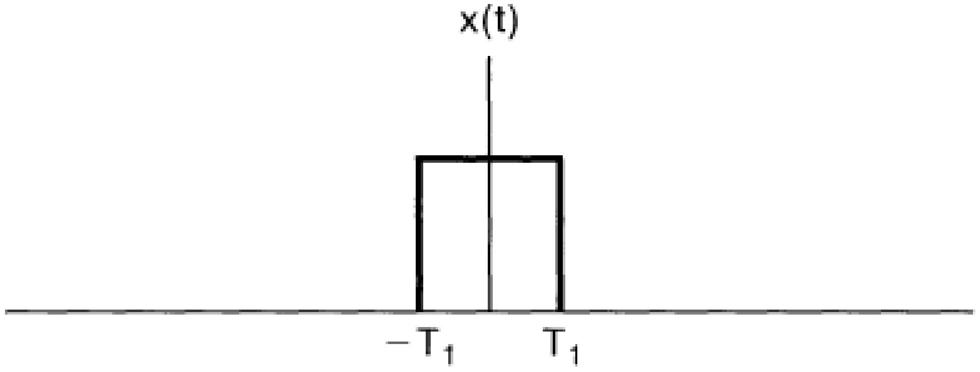
Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

Example

$$x(t) = \delta(t).$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}\,dt = 1.$$

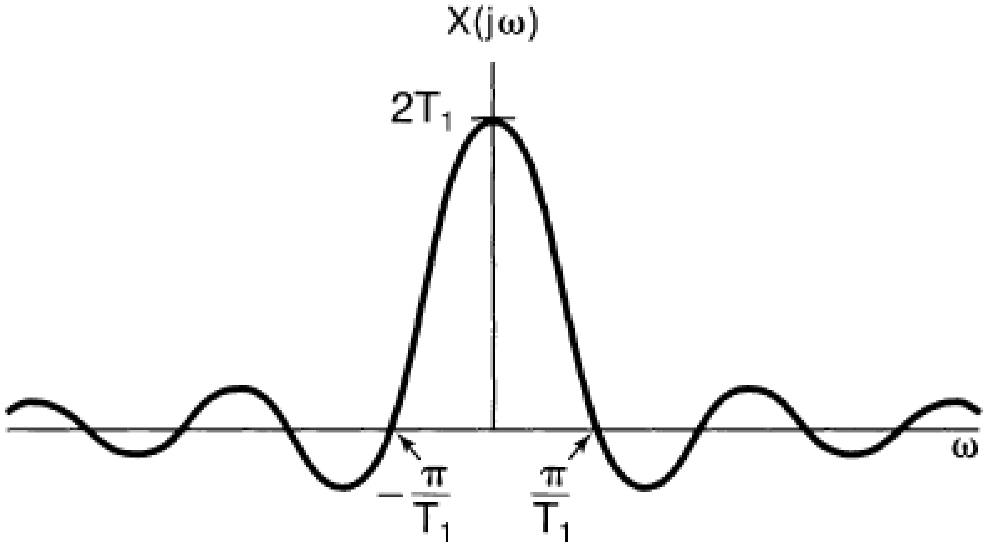
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



(a)

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = 2\frac{\sin \omega T_1}{\omega},$$

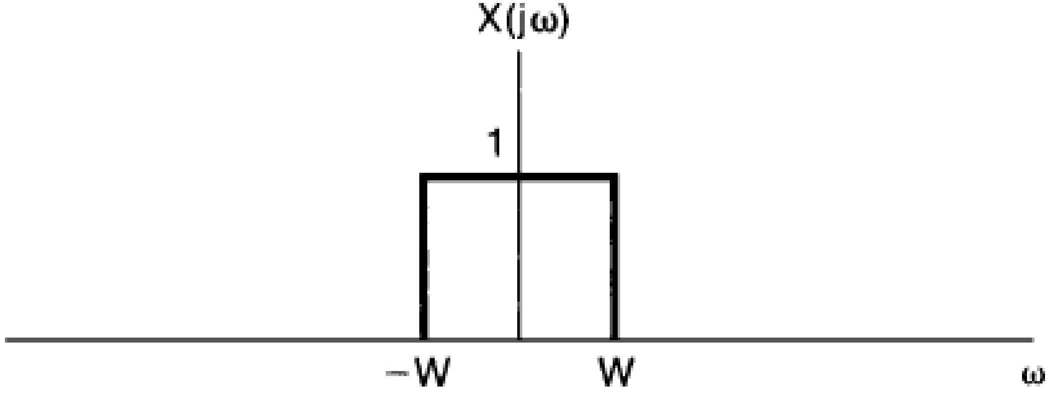
$$\frac{2\sin\omega T_1}{\omega} = 2T_1\,\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$$



(b)

Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

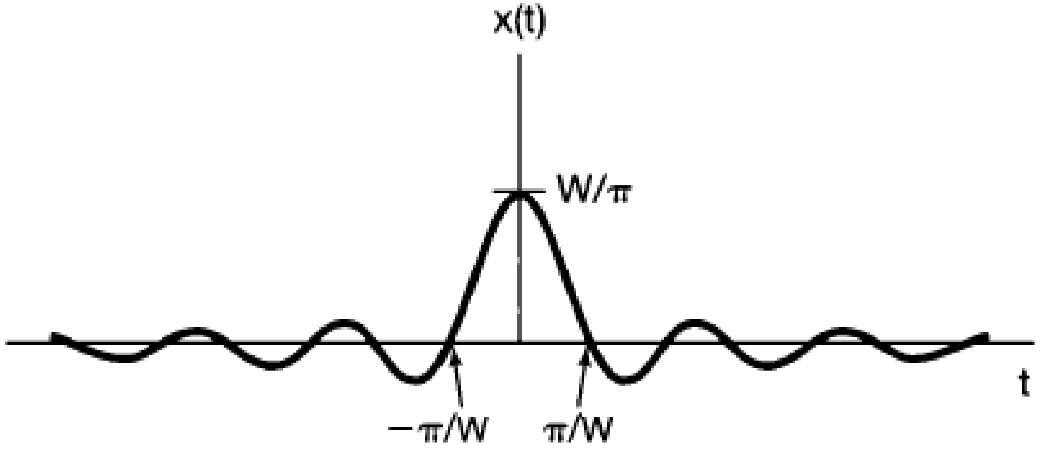
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}.$$



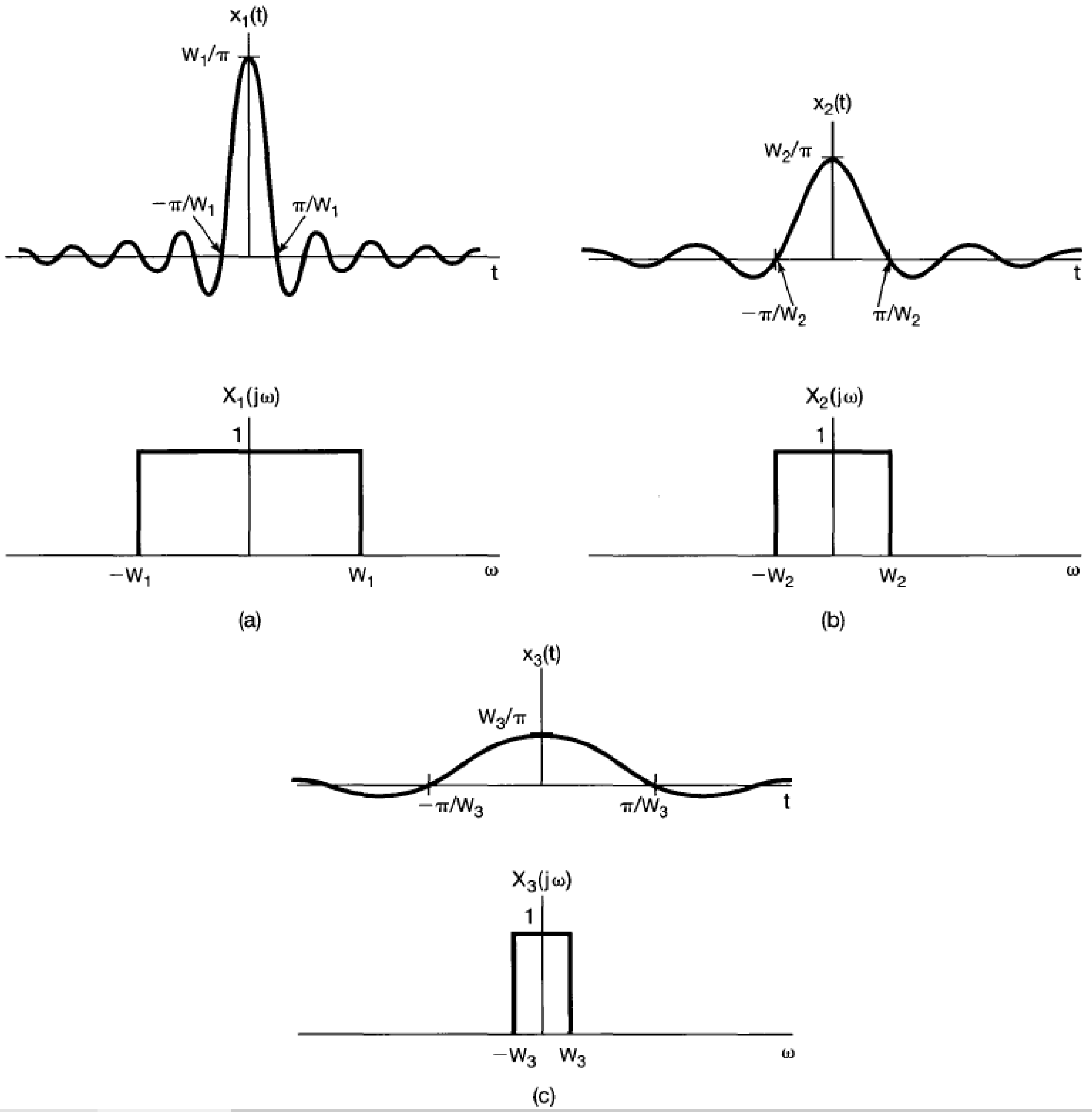
(a)

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin Wt}{\pi t}$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$



(b)



$x_1(t)$
W_1/π
$-\pi/W_1$
π/W_1
t
$X_1(j\omega)$
1
$-W_1$
W_1
ω
(a)
$x_2(t)$
W_2/π
$-\pi/W_2$
π/W_2
t
$X_2(j\omega)$
1
$-W_2$
W_2
ω
(b)
$x_3(t)$
W_3/π
$-\pi/W_3$
π/W_3
t
$X_3(j\omega)$
1
$-W_3$
W_3
ω
(c)

FT of Periodic signals

- Important property
  - $x(t) = e^{jk\omega_0 t} \leftrightarrow X(j\omega) = 2\pi\delta(\omega - k\omega_0)$

- Transform pair
- $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$
  - Each a_k coefficient gets turned into a delta at the harmonic frequency

FT of Periodic signals

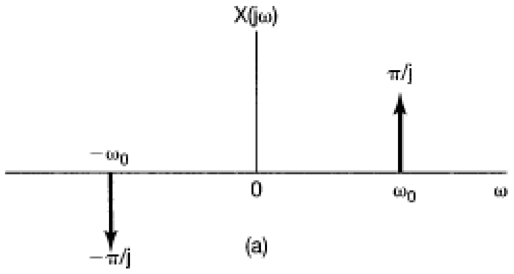
- FT of periodic signals is important because of sinusoidal signals (cannot solve using FT integral)
  - Can use insight of complex exponential ↔ shifted delta from periodic FT derivation
- Important examples

$$x(t) = \sin\omega_0 t \xrightarrow{\text{FS}} a_1 = \frac{1}{2j} \qquad \Rightarrow \quad X(j\omega) = \frac{2\pi}{-2j}\delta(\omega+\omega_0) + \frac{2\pi}{2j}\delta(\omega-\omega_0)$$

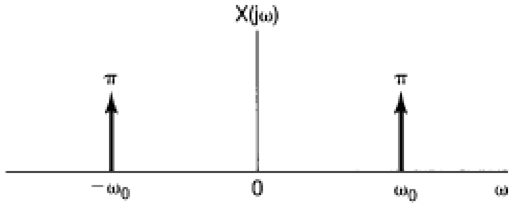
$$a_{-1} = -\frac{1}{2j} \qquad = -\frac{\pi}{j}\delta(\omega+\omega_0) + \frac{\pi}{j}\delta(\omega-\omega_0)$$

$$x(t) = \cos\omega_0 t \xrightarrow{\text{FS}} a_1 = \frac{1}{2} \qquad \Rightarrow \quad X(j\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$$

$$a_{-1} = \frac{1}{2}$$



(a)



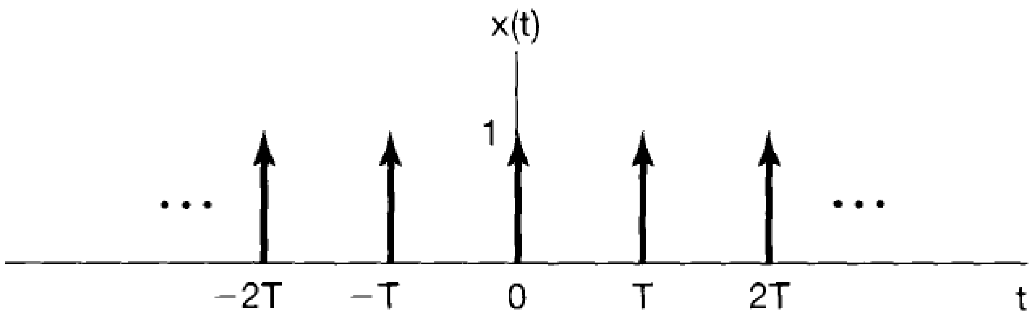
(b)

FT of impulse train

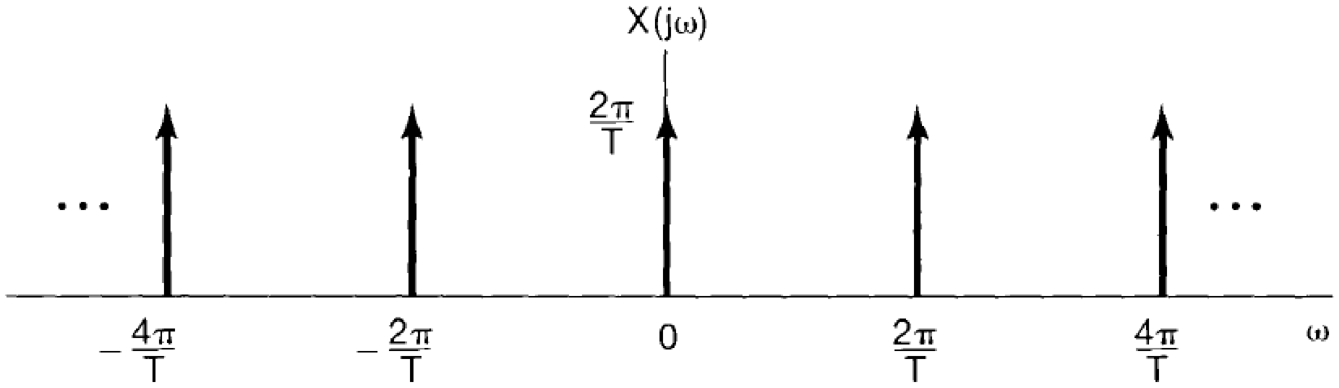
$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}.$$

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$



(a)



(b)

Properties

- **Linearity**

$$x(t) \xleftrightarrow{\mathfrak{F}} X(j\omega)$$

$$y(t) \xleftrightarrow{\mathfrak{F}} Y(j\omega),$$

$$\boxed{ax(t) + by(t) \xleftrightarrow{\mathfrak{F}} aX(j\omega) + bY(j\omega).}$$

Time Shifting

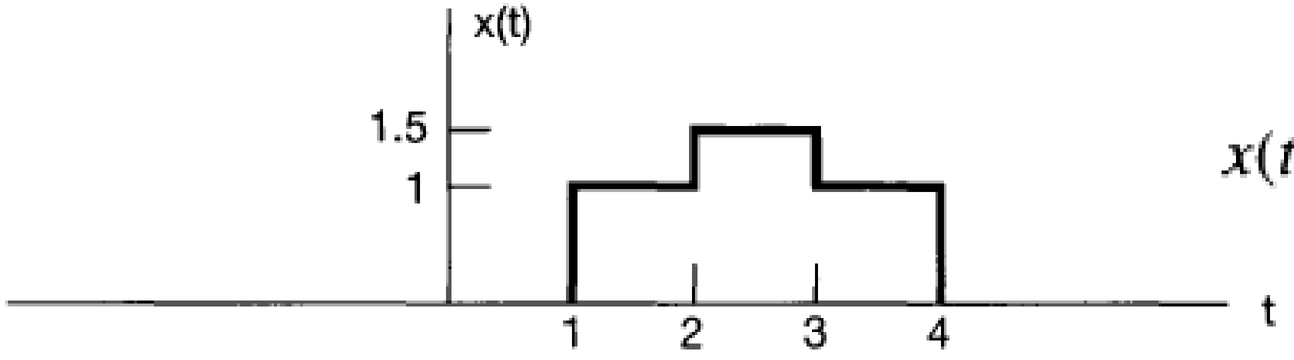
$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\boxed{x(t - t_0) \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega).}$$

$$x(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega(t - t_0)} d\omega$$

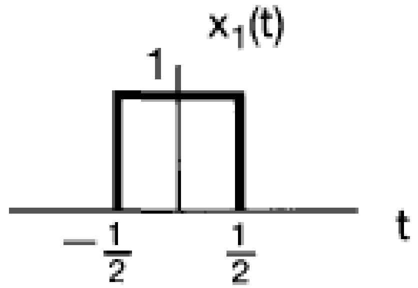
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(e^{-j\omega t_0} X(j\omega)\right) e^{j\omega t} d\omega.$$

$$\mathfrak{F}\{x(t - t_0)\} = e^{-j\omega t_0} X(j\omega).$$



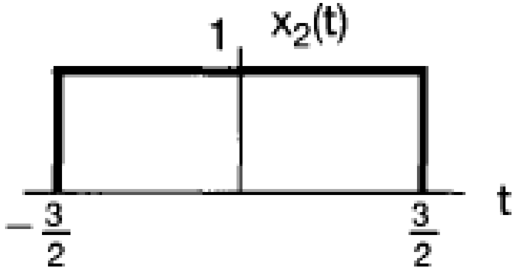
(a)

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5),$$



(b)

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \quad \text{and} \quad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}.$$



(c)

Conjugation and Conjugate Symmetry

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\boxed{x^*(t) \overset{\mathfrak{F}}{\longleftrightarrow} X^*(-j\omega).}$$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^*$$

$$= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt.$$

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}\,dt.$$

Conjugation and Conjugate Symmetry

- if x(t) is real so that x*(t) = x(t),

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}\,dt = X(j\omega),$$

- If *X(jw)* is expressed in rectangular form as

$$X(j\omega) = \mathcal{R}e\{X(j\omega)\} + j\mathcal{I}m\{X(j\omega)\},$$

- if x(t) is real

$$\mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\}$$

$$\mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\}.$$

The real part of the Fourier transform is an *even* function of frequency, and the imaginary part is an *odd* function of frequency

Conjugation and Conjugate Symmetry

- Similar results can be obtained in the polar form

$$X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)},$$

|X(jw)| is an even function of *w* and *angle X(jw)* is an odd function of *w*

- if *x(t)* is both real and even, then *X(jw)* will also be real and even

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}\,dt,$$

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(-\tau)e^{-j\omega\tau}\,d\tau. \qquad X(-j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega t}\,d\tau$$

$$= X(j\omega).$$

- In a similar manner, it can be shown that if *x(t)* is a real and odd function of time, so that *x(t)* = - *x(* -t), then X(jw) is purely imaginary and odd

$$x(t) = x_e(t) + x_o(t).$$

$$\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\},$$

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\mathcal{E}v\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \mathcal{R}e\{X(j\omega)\},$$

$$\mathcal{O}d\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} j\mathcal{I}m\{X(j\omega)\}.$$

Laplace Transform

- $x(t) = e^{st} \rightarrow y(t) = H(s)e^{st}$
  - Derived from convolution integral directly
- Transfer function: $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$

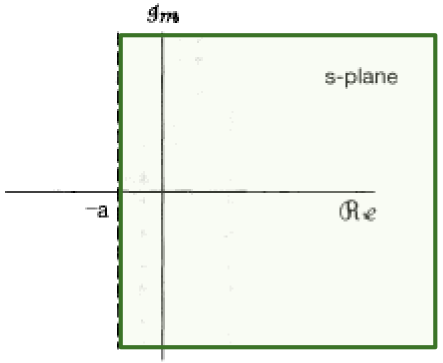
- Let $s = j\omega$ for Fourier Transform
- $H(s)|_{s=j\omega} = H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$

- Define LP for $s = \sigma + j\omega \in \mathbb{C}$
- $X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$

Laplace Transform

- $x(t) = e^{-at}u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$
$$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$
$$= -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^{\infty} = \frac{1}{s+a}[1-0]$$
$$= \frac{1}{s+a}$$



(a)

- However, consider $s = \sigma + j\omega$

$$X(s) = X(\sigma + j\omega) = \int_0^{\infty} e^{-(\sigma+j\omega+a)t}dt$$
$$= \int_0^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt$$
$$= \mathcal{F}\left\{e^{-(\sigma+a)t}u(t)\right\} = \frac{1}{(\sigma+a)+j\omega}$$
$$\sigma + a > 0 \quad \text{from Table 4.2}$$
$$= \frac{1}{(\sigma+a)+j\omega} = \underbrace{\frac{1}{s+a}}_{\text{algebraic expression}}$$

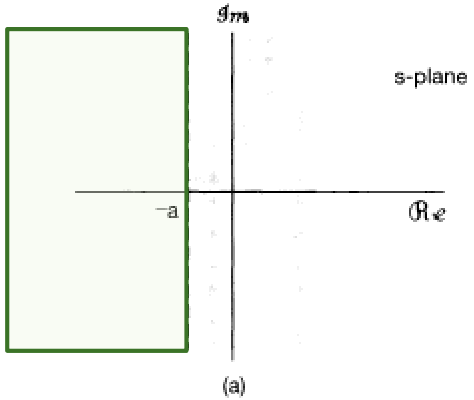
$$Re\{s\} + a > 0$$
$$\underbrace{Re\{s\} > -a}_{\text{region of convergence (ROC)}}$$

Laplace Transform

- $x(t) = -e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$
$$= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt = \int_{-\infty}^{0} -e^{-(s+a)t}dt$$
$$= -\int_{-\infty}^{0} e^{-\sigma+a}te^{-j\omega t}dt = -\int_{-\infty}^{0} e^{-(a+\sigma+j\omega)t}dt$$
$$= -\frac{1}{a+\sigma+j\omega}\left[e^{-(a+\sigma+j\omega}\right]_{-\infty}^{0}$$
$$= \frac{1}{a+\sigma+j\omega} = \underbrace{\frac{1}{s+a}}_{\text{same as in previous example}}$$

$$a+\sigma<0$$
$$Re\{s\} < -a$$



- $\Rightarrow -e^{-at}u(-t) \leftrightarrow \frac{1}{s+a}, Re\{x\} < -a$

THE REGION OF CONVERGENCE

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

the ROC of the Laplace transform of $x(t)$ consists of those values of s for which $x(t)e^{-\sigma t}$ absolutely integrable:[2]

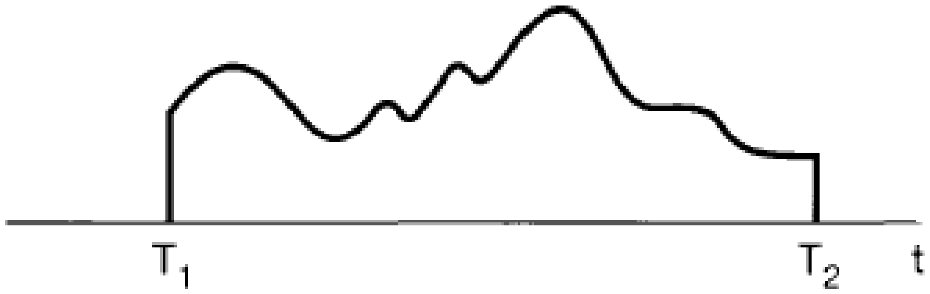
$$\int_{-\infty}^{+\infty} |x(t)|e^{-\sigma t}\,dt < \infty.$$

Property 1 then follows, since this condition depends only on σ, the real part of s.

Property 2: For rational Laplace transforms, the ROC does not contain any poles.

➢ Since X (s) is infinite at a pole, the integral clearly does not converge at a pole, and thus the ROC cannot contain values of s that are poles.

Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

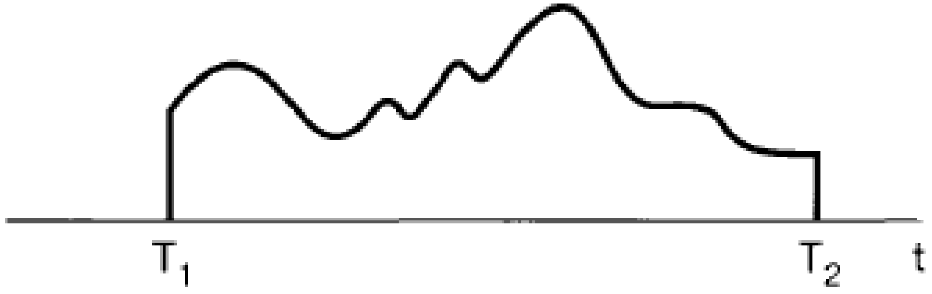


$$\int_{T_1}^{T_2} |x(t)|\, dt < \infty.$$

For $s = \sigma + j\omega$ to be in the ROC, we require that $x(t)e^{-\sigma t}$ be absolutely integrable,

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t}\, dt < \infty. \quad (9$$

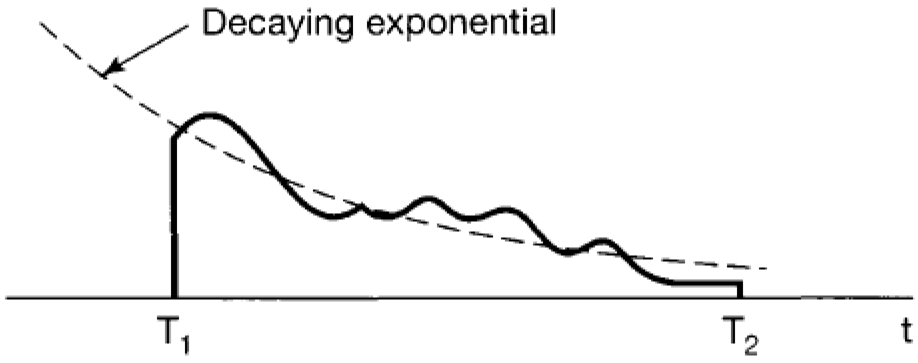
Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

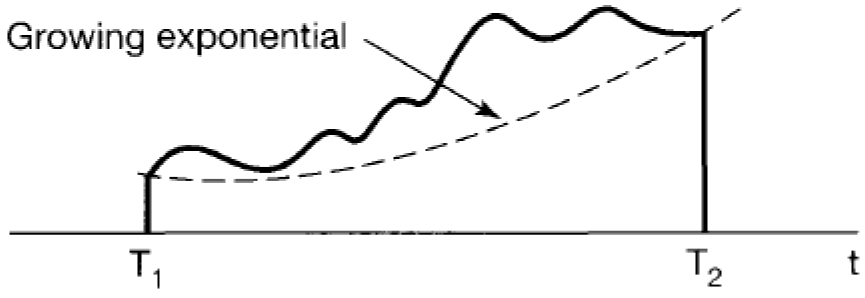


$$\int_{T_1}^{T_2} |x(t)|\, dt < \infty.$$

For $s = \sigma + j\omega$ to be in the ROC, we require that $x(t)e^{-\sigma t}$ be absolutely integrable,

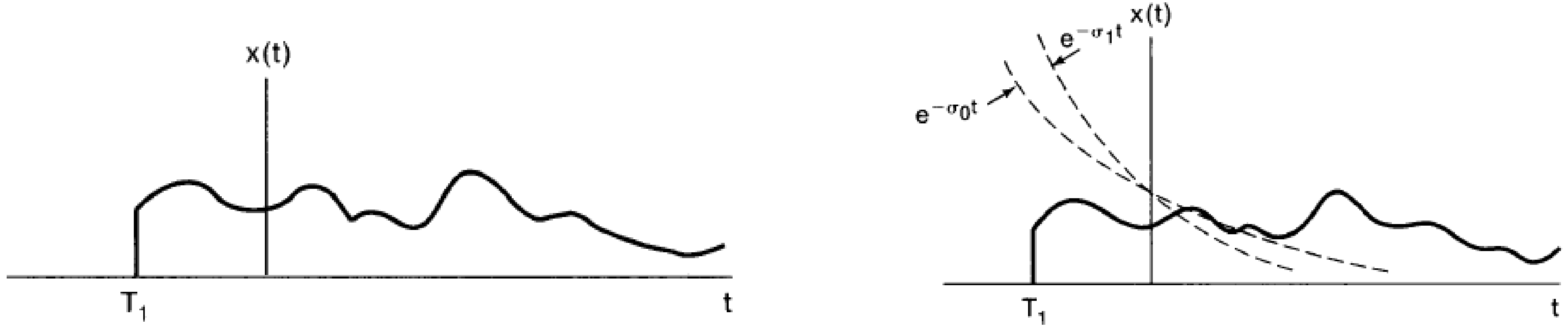
$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t}\, dt < \infty.$$





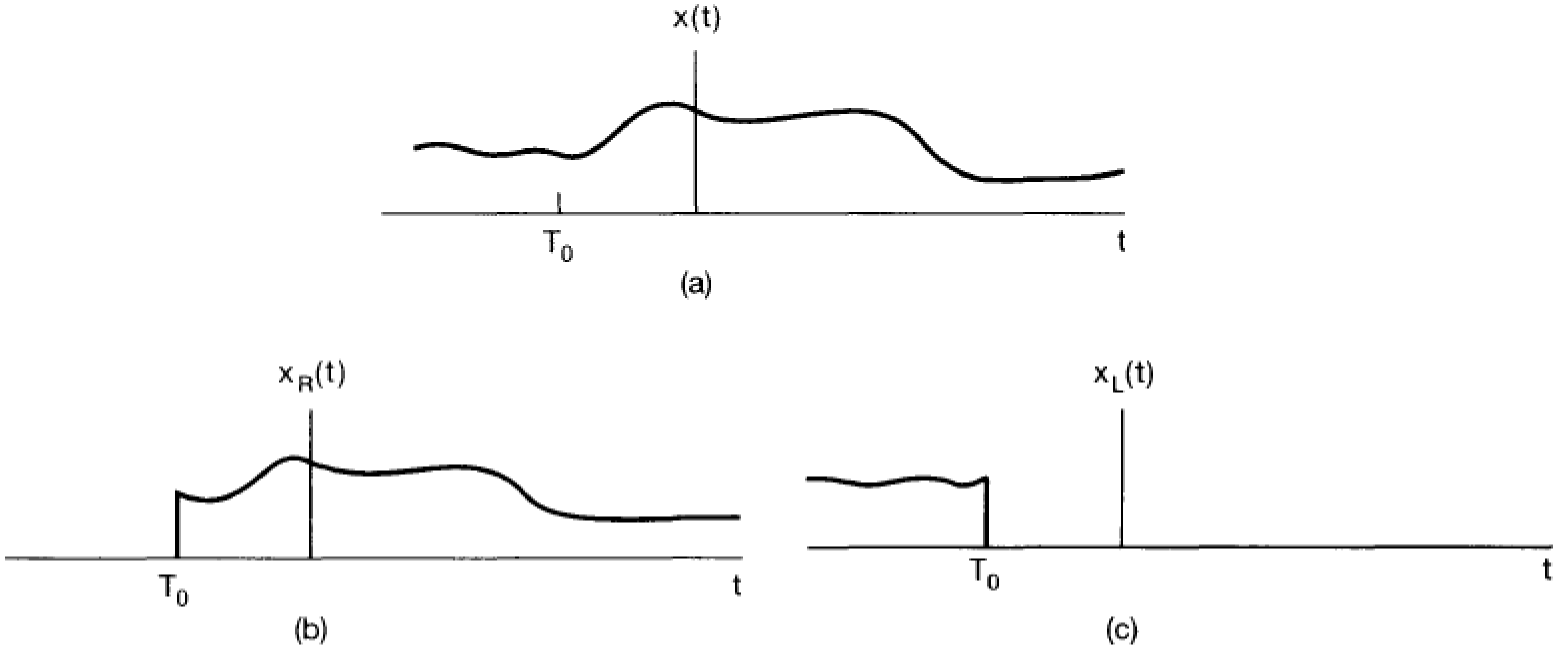
Dr Rohini P, Department of ECE, IIITDM Kancheepuram. (Email: rohinip@iiitdm.ac.in)

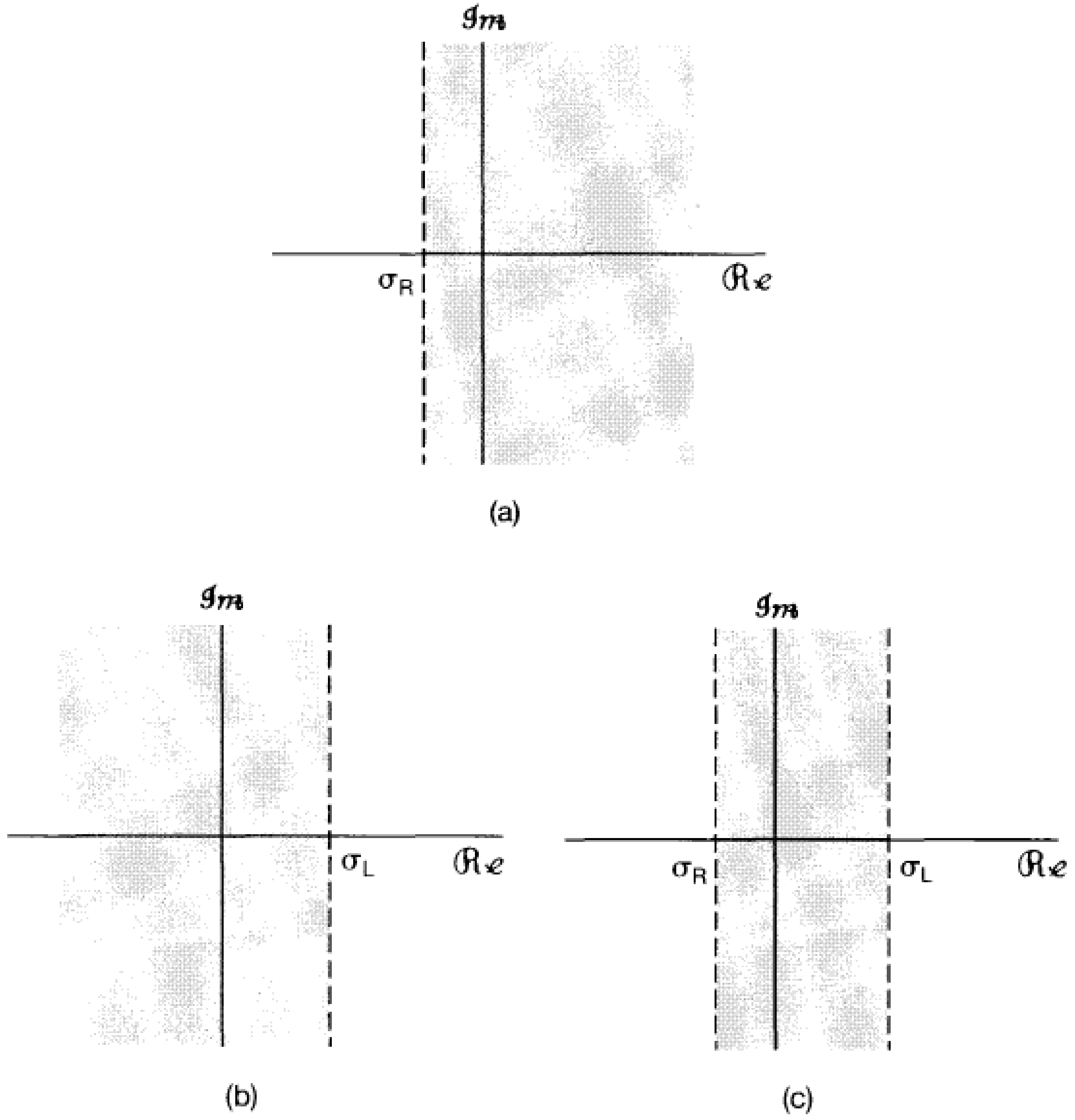
Property 4: If $x(t)$ is right sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e\{s\} > \sigma_0$ will also be in the ROC.



Property 5: If $x(t)$ is left sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e\{s\} < \sigma_0$ will also be in the ROC.

Property 6: If $x(t)$ is two sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\mathcal{Re}\{s\} = \sigma_0$.





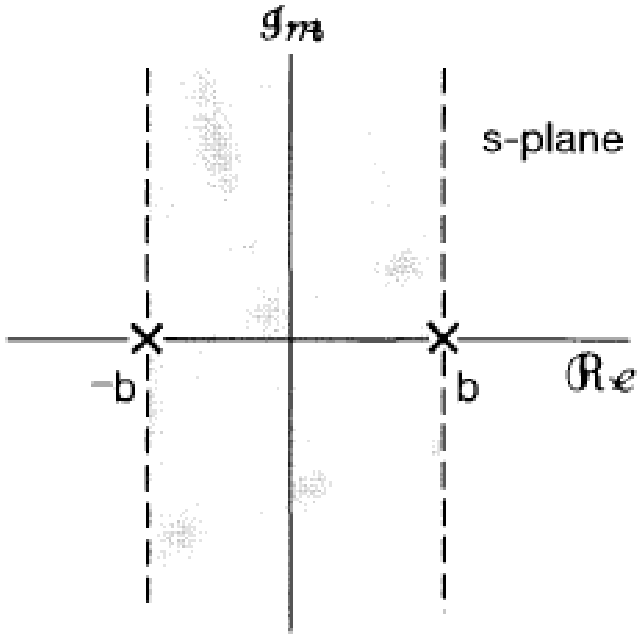
$\mathcal{Im}$
σ_R
$\mathcal{Re}$
(a)
$\mathcal{Im}$
σ_L
$\mathcal{Re}$
(b)
$\mathcal{Im}$
σ_R
σ_L
$\mathcal{Re}$
(c)

Example

$$x(t) = e^{-b|t|},$$

$$e^{-bt}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+b}, \quad \mathcal{Re}\{s\} > -b,$$

$$e^{+bt}u(-t) \xleftrightarrow{\mathcal{L}} \frac{-1}{s-b}, \quad \mathcal{Re}\{s\} < +b.$$

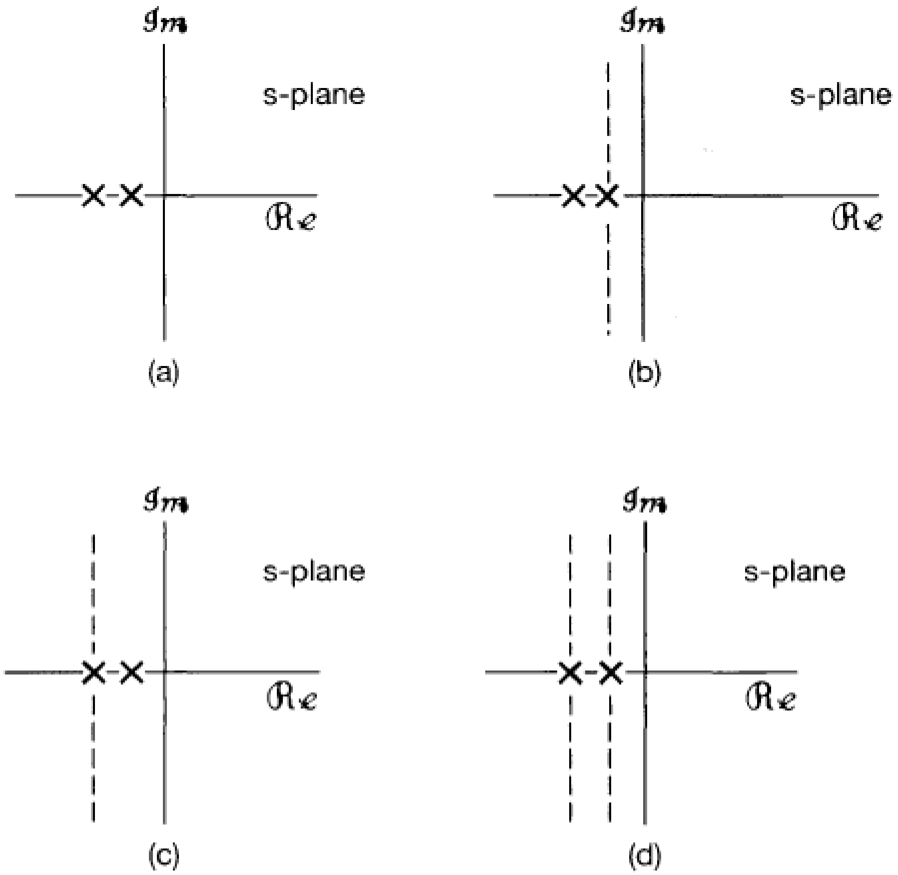


$$e^{-b|t|} \xleftrightarrow{\mathcal{L}} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2-b^2}, \quad -b < \mathcal{Re}\{s\} < +b.$$

Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

$$X(s) = \frac{1}{(s+1)(s+2)},$$



Properties of Laplace Transform

Linearity of the laplace Transform

$$x_1(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} X_1(s)$$ with a region of convergence that will be denoted as R_1

$$x_2(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} X_2(s)$$ with a region of convergence that will be denoted as R_2,

$$\boxed{ax_1(t) + bx_2(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} aX_1(s) + bX_2(s), \quad \text{with ROC containing } R_1 \cap R_2.}$$

Time Shifting

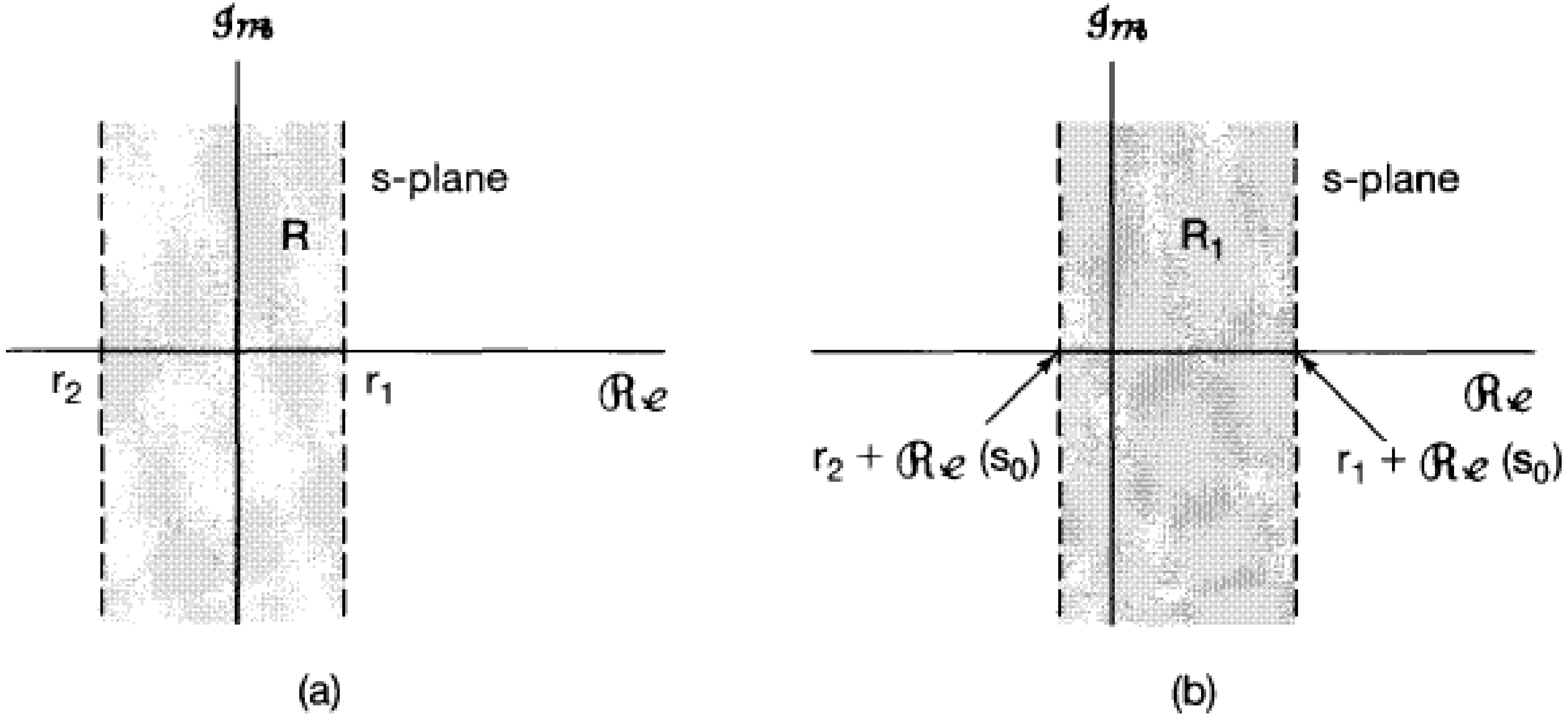
$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \qquad \text{with ROC} = R,$$

$$\boxed{x(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}X(s), \quad \text{with ROC} = R.}$$

Shifting in the s-Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \qquad \text{with ROC} = R,$$

$$\boxed{e^{s_0 t}x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0), \quad \text{with ROC} = R + \Re e\{s_0\}.}$$

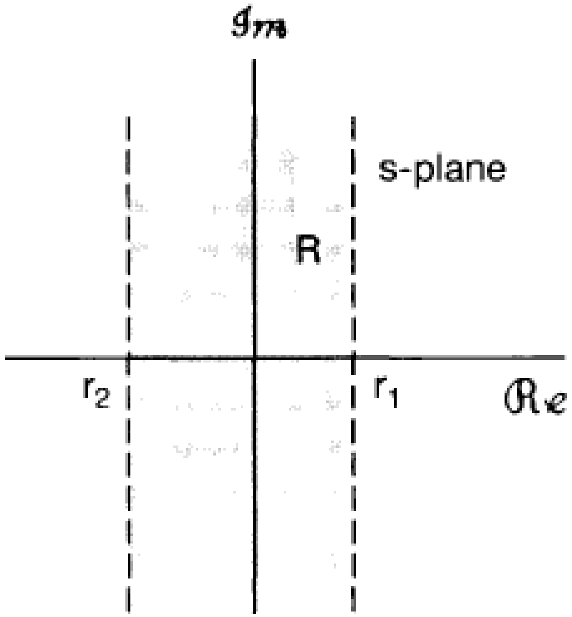


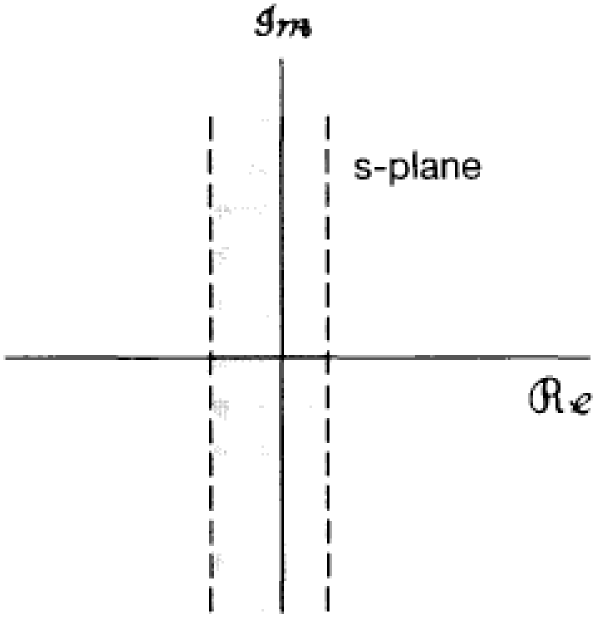
Im
s-plane
R
r2
r1
Re
(a)
Im
s-plane
R1
r2 + Re (s0)
r1 + Re (s0)
Re
(b)

Time Scaling

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$x(at) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad \text{with ROC } R_1 = aR.$$





Conjugation

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

$$\boxed{x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*), \quad \text{with ROC} = R.}$$

$$X(s) = X^*(s^*) \quad \text{when } x(t) \text{ is real.}$$

Convolution Property

$$x_1(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} X_1(s), \qquad \text{with } ROC = R_1,$$

$$x_2(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} X_2(s), \qquad \text{with } ROC = R_2,$$

$$\boxed{x_1(t) * x_2(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} X_1(s)X_2(s), \quad \text{with ROC containing } R_1 \cap R_2.}$$

Differentiation in the Time Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \qquad \text{with ROC} = R,$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s), \qquad \text{with ROC containing } R.}$$

Differentiation in the s-Domain

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt,$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} (-t)x(t)e^{-st}dt.$$

$$x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

$$\boxed{-tx(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{dX(s)}{ds}, \qquad \text{with ROC} = R.}$$

Example

$$x(t) = te^{-at}u(t).$$

- Find x(t)

$$X(s) = \frac{2s^2 + 5s + 5}{(s+1)^2(s+2)}, \quad \Re e\{s\} > -1.$$

Integration in the Time Domain

$$x(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s}X(s), \quad \begin{array}{l}\text{with ROC containing} \\ R \cap \{\mathcal{R}e\{s\} > 0\}.\end{array}}$$

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t)$$

$$u(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s}, \qquad \mathcal{R}e\{s\} > 0,$$

$$u(t) * x(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s}X(s),$$

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

The Initial- and Final-Value Theorems

$x(t) = 0$ for $t < 0$ and that $x(t)$ contains no impulses or higher order singularities at the origin

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

$x(t) = 0$ for $t < 0$ and, in addition, $x(t)$ has a finite limit as $t \to \infty$,

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s).$$

Causality for LTI Systems

- ➢The output of a causal system depends only on the present and past values of the input to the system
- ➢In order for a discrete-time LTI system to be causal, y[n] must not depend on x[k] for k > n
- ➢All of the coefficients h[n- k] that multiply values of x[k] for k > n must be zero

$$h[n] = 0 \quad \text{for } n < 0.$$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k],$$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

Causality

- For a causal LTI system, the impulse response is zero for $t < 0$ and thus is right sided.

The ROC associated with the system function for a causal system is a right-half plane.

- Converse of this statement is not necessarily true

- If H (s) is *rational*, we can determine whether the system is causal simply by checking to see if its ROC is a right-half plane.

For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

Example

- Consider a system with impulse response

$$h(t) = e^{-t}u(t).$$

- Since $h(t) = 0$ fort < 0, this system is causal.

$$H(s) = \frac{1}{s+1}, \qquad \Re e\{s\} > -1.$$

- In this case, the system function is rational and the ROC is to the right of the rightmost pole

Example

$$h(t) = e^{-|t|}.$$

$$H(s) = \frac{-2}{s^2 - 1}, \qquad -1 < \Re e\{s\} < +1.$$

$$H(s) = \frac{e^s}{s + 1}, \qquad \Re e\{s\} > -1.$$

Stability

- Stability of an LTI system is equivalent to its impulse response being absolutely integrable

An LTI system is stable if and only if the ROC of its system function H(s) includes the entire $j\omega$-axis [i.e., $\mathcal{Re}\{s\} = 0$].

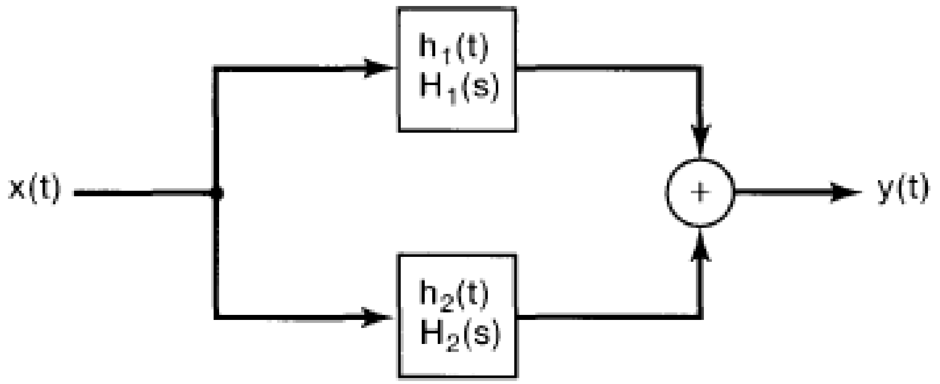
- Example

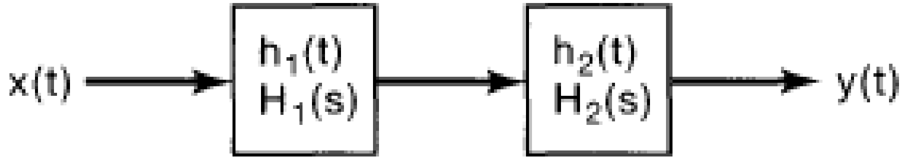
$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane—i.e., all of the poles have negative real parts.

BLOCK DIAGRAM REPRESENTATIONS

- System Functions for Interconnections of LTI Systems



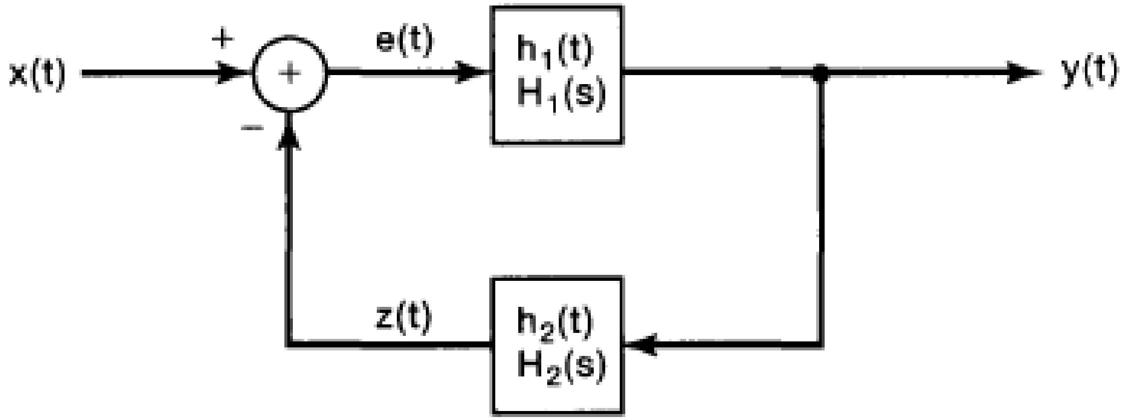


$$h(t) = h_1(t) + h_2(t).$$

$$h(t) = h_1(t) * h_2(t).$$

$$H(s) = H_1(s) + H_2(s).$$

$$H(s) = H_1(s)H_2(s).$$



$$Y(s) = H_1(s)E(s),$$
$$E(s) = X(s) - Z(s),$$

$$Z(s) = H_2(s)Y(s),$$

$$Y(s) = H_1(s)[X(s) - H_2(s)Y(s)],$$

$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}.$$

Differential Equations

$$\frac{dy(t)}{dt} + 3y(t) = x(t),$$

$$sY(s) + 3Y(s) = X(s).$$

$$H(s) = \frac{Y(s)}{X(s)},$$

$$H(s) = \frac{1}{s+3}.$$

The corresponding impulse response in the causal case is $h(t) = e^{-3t}u(t),$

The corresponding impulse response in the anticausal case is $h(t) = -e^{-3t}u(-t).$