Mathematical structures in linear algebra

(1) **Field** (See Chapter 1)

(2) **Vector Space**

(3) **...........**

A vector space V over a field F

A vector space $\langle\ V,\ F,\ +,\ .\ \rangle$ consists of the following.

(1) a field F of scalars.

(2) a set V of objects, called vectors.

(3) an operation $\ +\ :\ V \times V \longrightarrow V$ (vector addition) which satisfies the following axioms.

  (a) addition is commutative.

$$\alpha + \beta = \beta + \alpha, \quad \text{for all } \alpha, \beta \in V$$

  (b) addition is associative.

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma, \quad \text{for all} \alpha, \beta, \gamma \in V$$

contd.

(c) there is a unique vector $0 \in V$ called the zero vector such that

$$\alpha + 0 = \alpha, \text{ for all } \alpha \in V$$

(d) for each vector $\alpha \in V$, there is a unique vector $-\alpha \in V$ such that $\alpha + (-\alpha) = 0$.

(4) an operation $. \ : \ F \times V \longrightarrow V$ (scalar multiplication), which satisfies the following axioms.

  (e) $1.\alpha = \alpha$, for all $\alpha \in V$
  (f) $(c_1 c_2)\alpha = c_1(c_2\alpha)$, for all $c_1, c_2 \in F, \alpha \in V$
  (g) $c(\alpha + \beta) = c\alpha + c\beta$, for all $\alpha, \beta \in V, c \in F$.
  (h) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$, for all $c_1, c_2 \in F, \alpha \in V$

Example 1 : The n-tuple space $\langle F^n, F, +, .\rangle$

Let F be a field.

$$V = F^n = \{(x_1, x_2, \ldots, x_n) \ : \ x_i \in F\}$$

Let $\alpha = (x_1, x_2, \ldots, x_n), \beta = (y_1, y_2, \ldots, y_n) \in V = F^n$
Let us define vector addition and scalar multiplication as follows:
Define $+ \ : \ V \times V \longrightarrow V$ **as (vector addition)**

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

and $. \ : \ F \times V \longrightarrow V$ **as**

$$c\alpha = (cx_1, cx_2, \ldots, cx_n)$$

Show that $\langle F^n, F, +, .\rangle$ is a vector space.

contd.

(a)

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$= (y_1 + x_1, y_2 + x_2, \ldots, y_n + x_n) = \beta + \alpha$$

Reason : F is a field and $x_i + y_i = y_i + x_i$

(b)

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

Verify

Reason : F is a field and $x_i + (y_i + z_i) = (x_i + y_i) + z_i$

(c) Let $0 = (0, 0, \ldots, 0) \in V = F^n$ such that
$\alpha + 0 = (x_1, \ldots, x_n) + (0, \ldots, 0) = (x_1, \ldots, x_n) = \alpha$
Reason : F is a filed and $x_i + 0 = x_i$

contd.

(d) For every $\alpha = (x_1, x_2, \dots, x_n)$, there exists $-\alpha = (-x_1, x_2, \dots, -x_n) \in V$ such that $\alpha + (-\alpha) = 0$

Reason : F is a field and $x_i + (-x_i) = 0$

(e)

$$1.\alpha = (1.x_1, 1.x_2, \dots, 1.x_n) = \alpha$$

Reason : F is a field and $1.x_i = x_i$

Please verify (f), (g) and (h)

Hence $\langle F^n, F, +, . \rangle$ is a vector space.

Note : (i) R^n is called the Euclidean vector space

Example 2 : The space of $m \times n$ matrices, $\langle F^{m\times n}, F, +, . \rangle$

Let F be a field and

$$V = F^{m\times n} = \{A = [a_{ij}]_{m\times n} \quad : \quad a_{ij} \in F\}$$

We define vector addition and scalar mulatiplication as follows, where $A, B \in V$ and $c \in F$

$$[A + B]_{ij} = a_{ij} + b_{ij}$$

and

$$[cA]_{ij} = ca_{ij}$$

Show that $F^{m\times n}$ is a vector space over F

Note that $F^{n\times n}$ is not a field

Example 3 : The set of all real valued continuous functions defined on $[0, 1]$

Let $V = \{f \quad : \ f : [0, 1] \longrightarrow \mathbb{R}$ and f is continuous on [0,1]$\}$
We define

$$+ \ : \ V \times V \longrightarrow V$$

$$\text{as} \ \ (f + g)(s) = f(s) + g(s), \quad s \in [0, 1]$$

$$. \ : \ \mathbb{R} \times V \longrightarrow V$$

$$\text{as} \ \ (cf)(s) = cf(s), \quad s \in [0, 1]$$

Show that $\langle V, \mathbb{R}, +, . \rangle$ is a a vector space.

Problem 1

Let $V = \{(x, y) \quad : \quad x, y \in \mathbb{R}\}$. We define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$c(x, y) = (cx, y)$$

Prove or disprove that $\langle V, \mathbb{R}, +, . \rangle$ is a vector space.

Solution of Problem 1

Suppose that $\langle V, \mathbb{R}, +, .\rangle$ is a vector space.

$$(0,2) = (0,1) + (0,1) = 2(0,1) = (0,1)$$

$\implies \quad 2 = 1$ **and** $1, 2 \in \mathbb{R}$**, a contradiction.**

Hence $\langle V, \mathbb{R}, +, .\rangle$ is not a vector space.

Alternate solution : Verify $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$.

Let $c_1 = c_2 = 1, \alpha = (1,1)$

Note 1

Let V be a vector space over a field F. We have

$$0 = 0 + 0, \quad \text{(additive identity)}$$

$$c0 = c(0+0), \qquad c \in F$$

$$c0 = c0 + c0, \quad (c(\alpha+\beta) = c\alpha + c\beta)$$

Add $-(c0) \in V$ on both sides

$$c0 + -(c0) = (c0 + c0) + -(c0),$$

$$0 = c0 + (c0 + -(c0)), \quad (\textit{Associative})$$

Note 1 contd.

$$0 = c0 + 0, \quad (Existence\ of\ \ inverse)$$

$$0 = c0 \quad (additive\ identity)$$

$$c0 = 0 \qquad \text{for all} \quad c \in F$$

Qn. Show that $0\alpha = 0$ for all $\alpha \in V$, where 0 is the additive identity in the field F and 0 is the zero vector in the vector space V

Note 2

$$
\begin{aligned}
0 &= 0\alpha, \quad \text{(see last question)} \\
&= (1-1)\alpha \\
&= (1+(-1))\,\alpha \\
&= 1.\alpha + (-1)\alpha \qquad (\text{ Reason: } \ (c_1+c_2)\alpha = c_1\alpha + c_2\alpha) \\
&= \alpha + (-1)\alpha \qquad (\mathit{Reason}: \ 1.\alpha = \alpha) \\
\Longrightarrow &\ \text{additive inverse of } \ \alpha, \qquad -\alpha = (-1)\alpha
\end{aligned}
$$

Note 3

Prove that if $c\alpha = 0$, then $c = 0$ or $\alpha = 0$.

Proof : Suppose that $c \neq 0$ (else $0\alpha = 0$).

Since $0 \neq c \in F$ and F is a field, $\quad c^{-1} \in F$.

$$c\alpha = 0 \Longrightarrow c^{-1}(c\alpha) = c^{-1}0 = 0$$

$$\Longrightarrow \quad (c^{-1}c)\alpha = 0 \qquad \text{Reason: } (c_1c_2)\alpha = c_1(c_2\alpha)$$

$$\Longrightarrow \qquad 1.\alpha = 0 \qquad \textit{Reason} \quad : \quad c^{-1}c = 1$$

$$\Longrightarrow \qquad \alpha = 0 \qquad \textit{Reason} \;\; : \quad 1.\alpha = \alpha$$

Linear combination

Let V be a vector space over a field F.
A vector $\beta \in V$ is said to be a linear combination of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in V provided there exist scalars $c_1, c_2, \ldots, c_n$ in F such that

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \sum_{i=1}^{n} c_i\alpha_i$$

Problem 2

Show that $(x, y, z) \in \mathbb{R}^3$ is a linear combination of vectors $\alpha = (1, 1, 1)$, $\beta = (0, 1, 1)$ and $\gamma = (0, 0, 1)$.

Solution : Find scalars(if exist) $a, b, c \in \mathbb{R}$ such that

$$(x, y, z) = a\alpha + b\beta + c\gamma$$

$$(x, y, z) = a(1, 1, 1) + b(0, 1, 1) + c(0, 0, 1)$$

$$(x, y, z) = (a, a + b, a + b + c)$$

$$(x, y, z) = x(1, 1, 1) + (y - x)(0, 1, 1) + (z - y)(0, 0, 1)$$

Problem 3

Prove or disprove that $(1, 2, 3)$ is a linear combination of $\alpha = (1, 1, 1)$ and $\beta = (0, 1, 1)$.

Ans. No.

$$(1, 2, 3) = a(1, 1, 1) + b(0, 1, 1)$$

$\Longrightarrow a + b = 2$ **and** $a + b = 3$**, lead us to a contradiction.**

Problem 3

Let $\mathbb{R}$ be the real field. Find all vectors in $\mathbb{R}^3$ that are linear combination of $(1, 0, -1), (0, 1, 1)$ and $(1, 1, 1)$.

Solution : Objective is to find all linear combinations of vectors $(1, 0, -1), (0, 1, 1)$ and $(1, 1, 1)$.

Find all (x, y, z) provided there exist $a, b, c \in \mathbb{R}$ such that

$$a(1, 0, -1) + b(0, 1, 1) + c(1, 1, 1) = (x, y, z)$$

Find a, b, c (if exist) such that

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\implies AX = Y$$

Problem 3 contd.

Find a row-reduced echelon matrix which is row-equivalent to A.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim I_3$$

By Theorem 12, A is invertible ($A \sim I$). By Theorem 13, the system $AX = Y$ has a solution X for all Y. Hence for every $Y^t = (x, y, z) \in \mathbb{R}^3$, there exists $X^t = (a, b, c)$ such that

$$a(1, 0, -1) + b(0, 1, 1) + c(1, 1, 1) = (x, y, z)$$

Matrix multiplication and linear combination

$$AX = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \end{bmatrix}$$

$$= x \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} + z \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$$

$$= xC_1 + yC_2 + zC_3 \quad (C_i \text{ is the ith column of } \quad A)$$

(1) AX is a linear combination of columns of the matrix A.

(2) Every column of AB is a linear combination of columns of A.

(3) Every row of AB is a linear combination of rows of B