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let $f(x) = x$ be a fn. defined on $[a, b]$.

let $h = \dfrac{b-a}{n}$

let $P_n = \{a = x_0, x_1 = a+h, x_2 = a+2h, \dots, x_n = a+nh = b\}$

compute (i) $L(P_n, f)$ (ii) $U(P_n, f)$

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$$

$$= \inf\{x : a + (i-1)h \leq x \leq a + ih\}$$

$$m_i = a + (i-1)h$$

$$\therefore L(P_n, f) = \sum_{i=1}^{n} m_i \Delta x_i \quad ; \quad \boxed{\Delta x_i = h}$$

$$= \sum_{i=1}^{n} \big(a + (i-1)h\big)h$$

$$= ah \sum_{i=1}^{n} (1) + h^2 \sum_{i=1}^{n} (i-1)$$

$$= nah + h^2\left[\frac{n(n+1)}{2} - n\right]$$

$$= nah + h^2\left[\frac{n^2+n-2n}{2}\right] = nah + h^2\left[\frac{n^2-n}{2}\right]$$

$$\therefore,\ L(P_n, f) = nah + \frac{h^2 n(n-1)}{2}$$

$$= a(b-a) + \left(\frac{b-a}{n}\right)^2 \frac{n(n-1)}{2}$$

$$= a(b-a) + \frac{(b-a)^2}{2}\left[1 - \frac{1}{n}\right]$$

as $n \to \infty$, $\quad L(P_n, f) = a(b-a) + \frac{(b-a)^2}{2}$

$$= (b-a)\left(a + \frac{b-a}{2}\right)$$

$$= \frac{(b-a)(a+b)}{2}$$

$$= \frac{b^2 - a^2}{2}$$

$$U(P_n, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} (a+ih)h$$

$$= ah\sum_{i=1}^{n}(1) + h^2 \sum_{i=1}^{n} i$$

$$= nah + \frac{h^2 n(n+1)}{2}$$

$$= a(b-a) + \frac{(b-a)^2}{2}\left[1 + \frac{1}{n}\right]$$

as $n \to \infty$,
$U(P_n, f) = a(b-a) + \frac{(b-a)^2}{2}$
$= (b-a)\left[\frac{a+b}{2}\right] = \frac{b^2-a^2}{2}$

$\forall \varepsilon > 0$, there exists a N such that

$\Uparrow$ $\forall n \geq N$, $U(P_n,f) - L(P_n,f) < \varepsilon$

$\therefore$ $U(P_n, f) = \dfrac{b^2 - a^2}{2}$ $\to$ since $L(P,f)$ & $U(P,f)$ converge to same ℓ,

$\lim_{n\to\infty} \left(U(P_n,f) - L(P_n,f) \right) = 0$

TPT: $U(P,f) - L(P,f) < \varepsilon \quad \forall \varepsilon > 0$ implies

f is Riemann Integrable... $\Leftarrow$ * V.V. IMP *

$\therefore$ $L(P_n, f) \leq \int_a^b f(x)\,dx \leq U(P_n, f)$

$\Rightarrow \int_a^b f(x)\,dx = \int_a^b x\,dx = \dfrac{b^2 - a^2}{2}$ //

$f(x) = x^2$, $\quad h = \dfrac{b-a}{n}$ $\quad P_n = \{a = x_0, x_1, \dots, x_n = a + nh = b\}$

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REIMANN SUM:

let $f(x)$: bounded, real fn. on $[a,b]$.

let $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ be a partition on $[a,b]$. Let $c_i \in [x_{i-1} \cdot x_i]$, $1 \leq i \leq n$.

then,

$$S_p = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

Where S_p is called a REIMANN SUM for f corresponding to the partition P.

NOTE: $L(P,f) \leq S_p \leq U(P,f)$

$(m_i \leq f(c_i) \leq M_i)$

REIMANN INTEGRABILITY:

if f: continuous on $[a,b]$, then its Reimann integrable. if $h = \frac{b-a}{n}$, & $P_n = \{a = x_0, x_1, \cdots, x_n = b\}$ is a partition of $[a,b] \to$ equispaced.

$$\Rightarrow \lim_{n\to\infty} L(P_n,f) = \lim_{n\to\infty} U(P_n,f) = \int_a^b f(x)\,dx$$

Hence, if $S_{P_n} \to$ any Reinann sum $\equiv P_n$.

$$\lim_{n\to\infty} S_{P_n} = \int_a^b f(x)\,dx \quad \text{(follows by Sandwich Theorem)}$$

$f(x) = x + x^2$ on $[0,1]$ let $h = \frac{1}{n}$.

and $P_n = \{0 = x_0, x_1, x_2, \dots, x_n = nh = 1\}$

$\Rightarrow x_i = i/n$ and $\Delta x_i = 1/n$

$$S_{P_n} = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

$$= \sum_{i=1}^{n} \left(i/n + \frac{i^2}{n^2}\right)\frac{1}{n}$$

$$= 1/n^2 \sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$S_{P_n} = \frac{n(n+1)}{2n^2} + \frac{n(n+1)(2n+1)}{6n^3}$$

$$\lim_{n\to\infty} S_{P_n} = \frac{n^2}{2n^2} + \frac{2n^3}{6n^3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$\therefore \int_0^1 (x + x^2)\,dx = 5/6$$

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PROPERTIES of REIMANN INTEGRATION

(a) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(b) $$\int_a^a f(x)\,dx = 0$$

(c) $$\int_a^b K f(x)\,dx = K\int_a^b f(x)\,dx$$

(d) $$\int_a^b \left(f(x)+g(x)\right)dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b \left(f(x)-g(x)\right)dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

(e) $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

(f) m: minimum on $[a,b]$

M: maximum on $[a,b]$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

(g) if $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

FUNDAMENTAL THEOREM of CALCULUS

if f: Reimann Integrable on $[a,b]$, & if there is a diff.able fn. F on $[a,b]$ s.t. $F'(x) = f(x)$, then.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Let $\varepsilon > 0$ be given.

choose $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \varepsilon$.

The Mean Value Theorem implies that there is a t_i in $[x_{i-1}, x_i]$ s.t.

$$\frac{F(x_i) - F(x_{i-1})}{\Delta x_i} = F'(t_i) = f(t_i) \quad (1 \leq i \leq n)$$

(or)

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$$

A reimann sum!!

$$\Rightarrow \sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right) = F(x_n) - F(x_0)$$
$$= F(b) - F(a)$$

ALSO, $L(P,f) \leqslant \sum_{i=1}^{n} f(t_i)\Delta x_i \leqslant U(P,f)$

&

$$L(P,f) \leqslant \int_a^b f(x)\,dx \leqslant U(P,f)$$

$$-U(P,f) \leqslant -\int_a^b f(x)\,dx \leqslant -L(P,f)$$

$$\Rightarrow L(P,f) - U(P,f) \leqslant \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)\,dx \leqslant U(P,f) - L(P,f)$$

$$\Rightarrow \quad -\varepsilon \leqslant \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)\,dx \leqslant \varepsilon$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)\,dx \right| < \varepsilon$$

THEOREM: Let f: Reimann Integrable on $[a,b]$

for $a \leqslant x \leqslant b$, put

$$F(x) = \int_a^x f(x)\,dx \text{ , then}$$

$F \to$ continuous on $[a,b]$. Further, if f is continuous on $[a,b]$, then F is diff.able on $[a,b]$ & $F'(t) = f(t)$