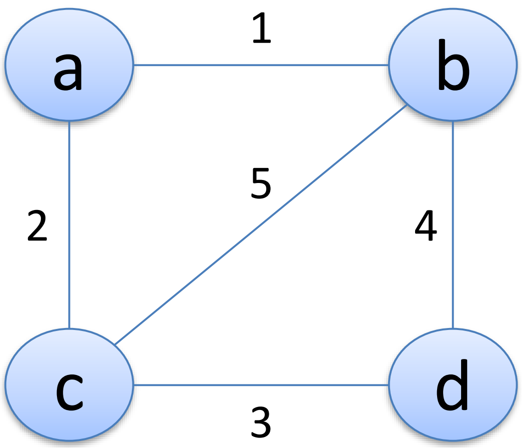
Minimum cost spanning tree (MCST)

- What is a minimum cost spanning tree?
  - Tree
    - No cycles; equivalently, for each pair of nodes u and v, there is only one path from u to v
  - Spanning
    - Contains every node in the graph
  - Minimum cost
    - Smallest possible total weight of any spanning tree

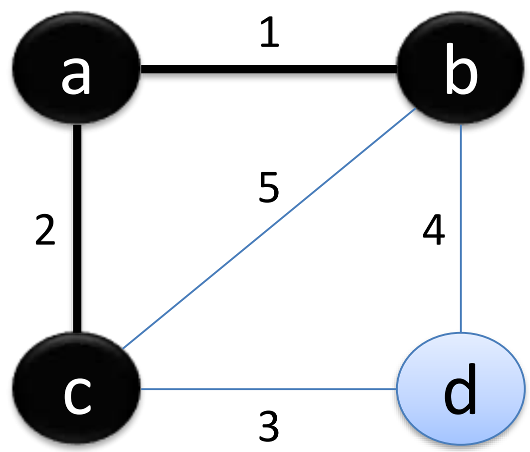
Minimum cost spanning tree (MCST)

- Let’s think about simple MCSTs on this graph:



Minimum cost spanning tree (MCST)

- Black edges and nodes are in T
- Is T a minimum cost spanning tree?



- Not spanning; d is not in T.

Minimum cost spanning tree (MCST)

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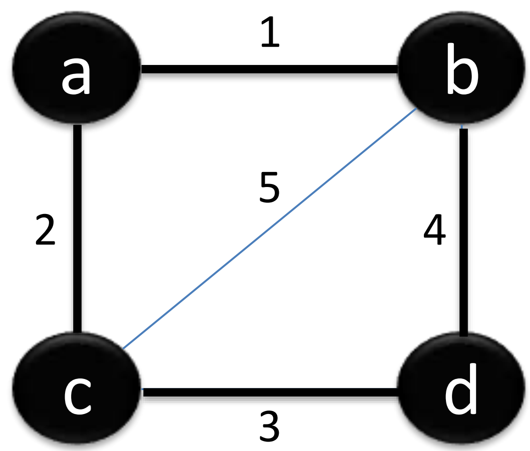

- Not a tree; has a cycle.

Minimum cost spanning tree (MCST)

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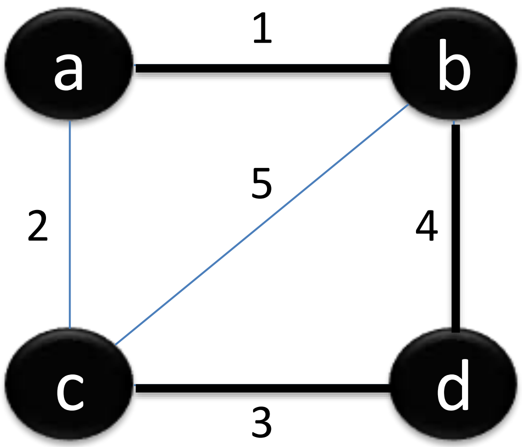
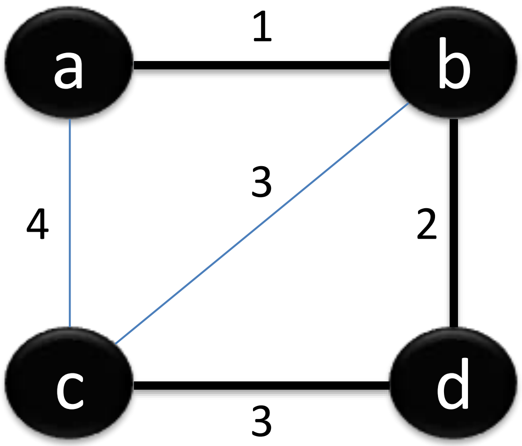
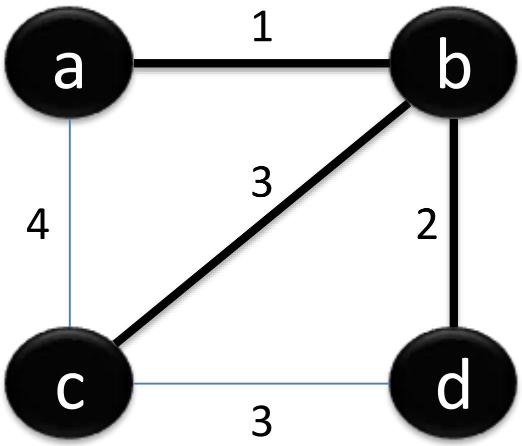


- Not minimum cost; can swap edges 4 and 2.

Minimum cost spanning tree (MCST)

- Which edges form a MCST?



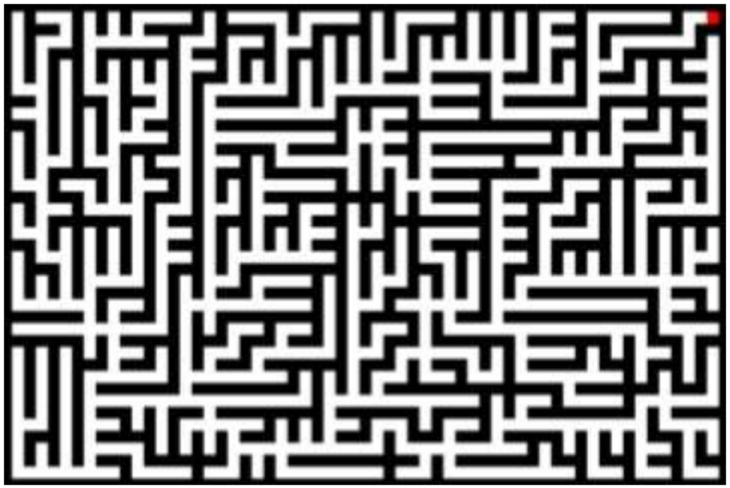


An application of MCSTs

- Electronic circuit designs (from Cormen et al.)
  - Circuits often need to wire together the pins of several components to make them electrically equivalent.
  - To connect *n* pins, we can use *n* - 1 wires, each connecting two pins.
  - Want to use the minimum amount of wire.
  - Model problem with a graph where each pin is a node, and every possible wire between a pair of pins is an edge.

A few other applications of MCSTs

- Planning how to lay network cable to connect several locations to the internet
- Planning how to efficiently bounce data from router to router to reach its internet destination
- Creating a 2D maze (to print on cereal boxes, etc.)



Building a MCST

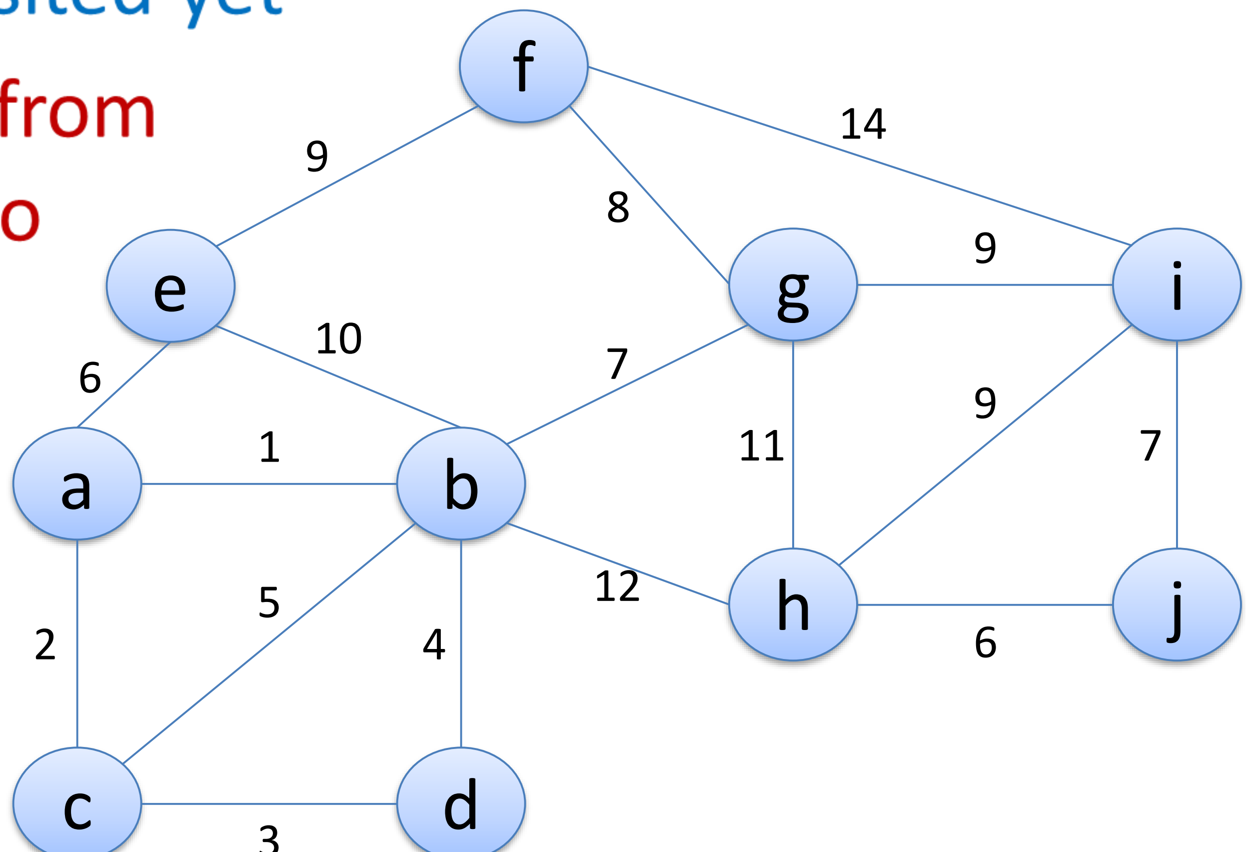
- Prim's algorithm takes a graph G = (V, E) and builds an MCST ***T***
- PrimMCST(V, E)
  - Pick an arbitrary node **r** from V
  - Add **r** to *T*
  - While *T* contains < |V| nodes
    - Find a **minimum weight edge (u, v)** where $\mathbf{u} \in \boldsymbol{T}$ and $\mathbf{v} \notin \boldsymbol{T}$
    - Add node v to *T*

In the book's terminology, we find a **light edge crossing the cut (T, V-T)**

The book proves that adding |V|-1 such edges will create a MCST

Running Prim's algorithm

- Start at an arbitrary node, say, h.
- **Blue:** not visited yet
- **Red:** edges from nodes $\in T$ to nodes $\notin T$
- **Black:** in T



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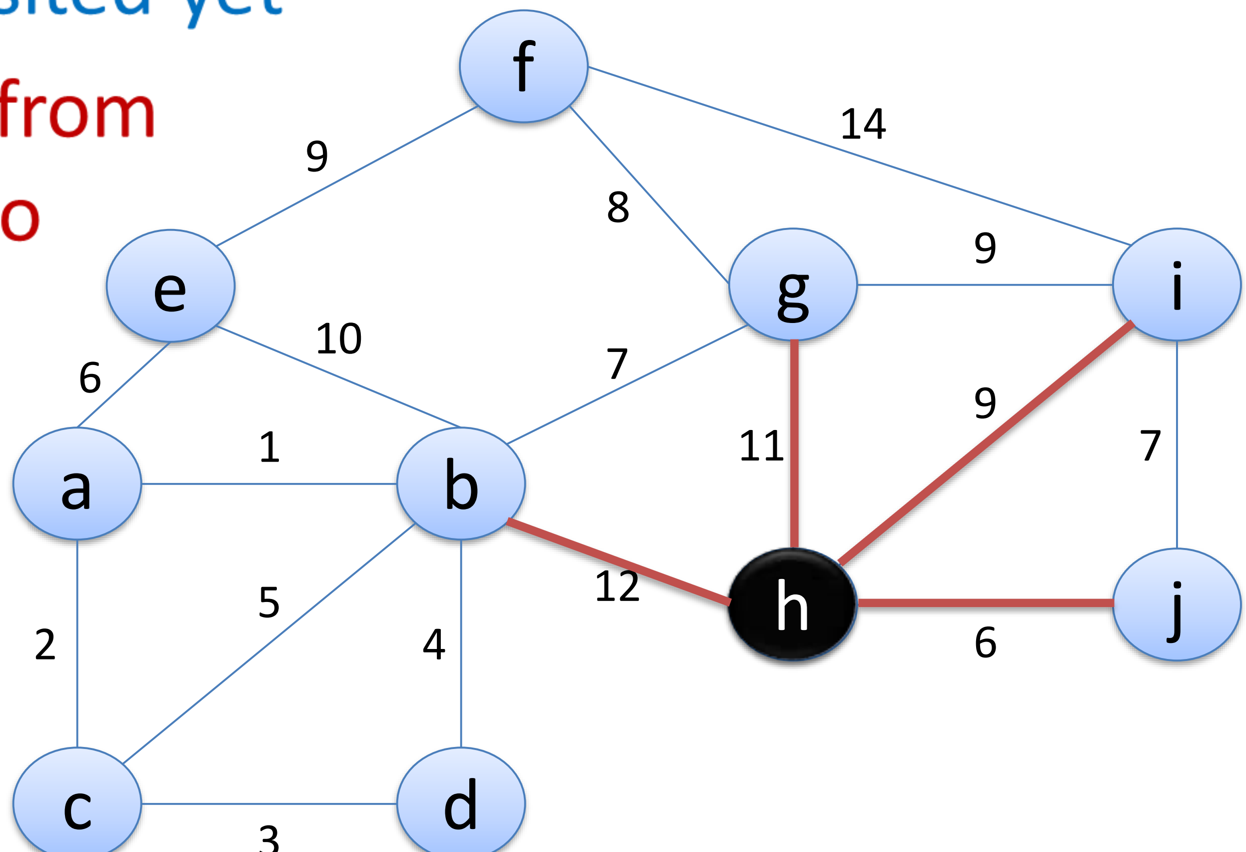

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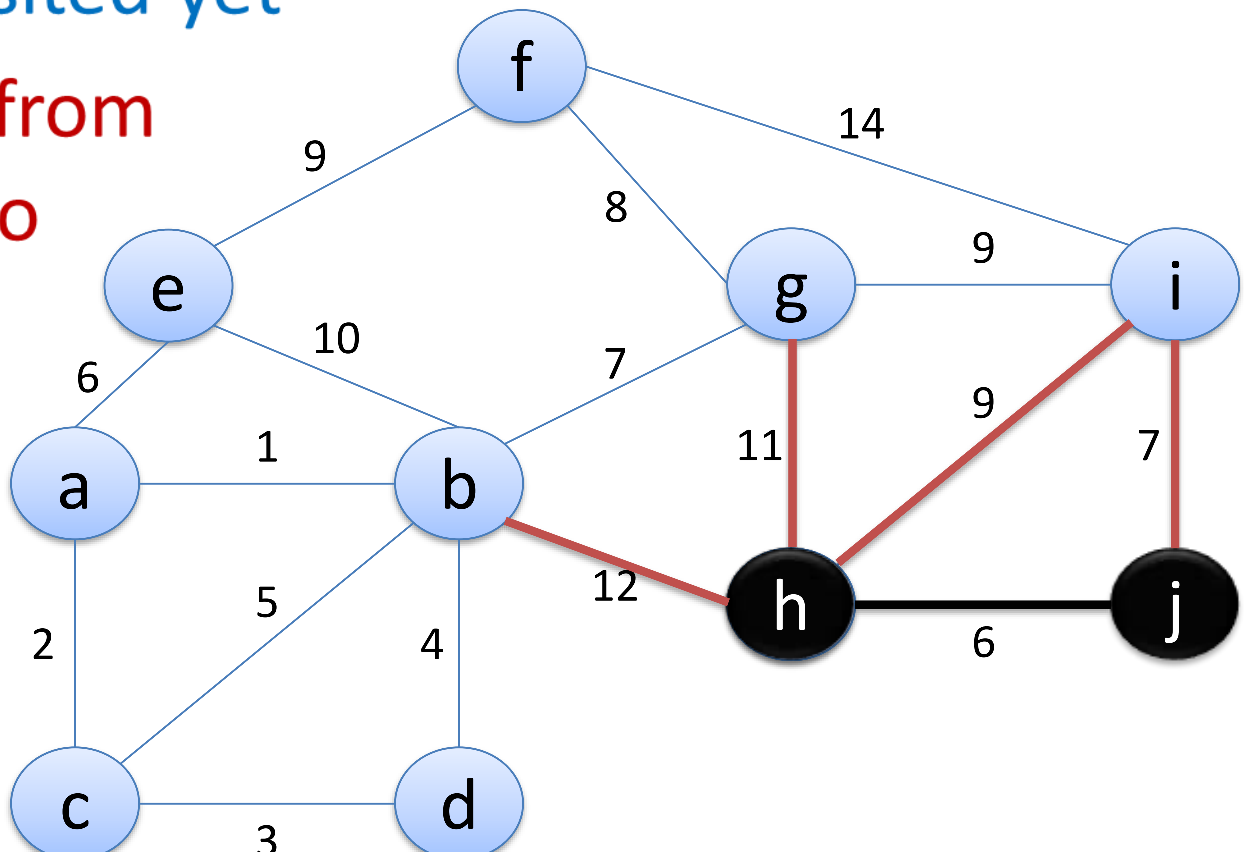

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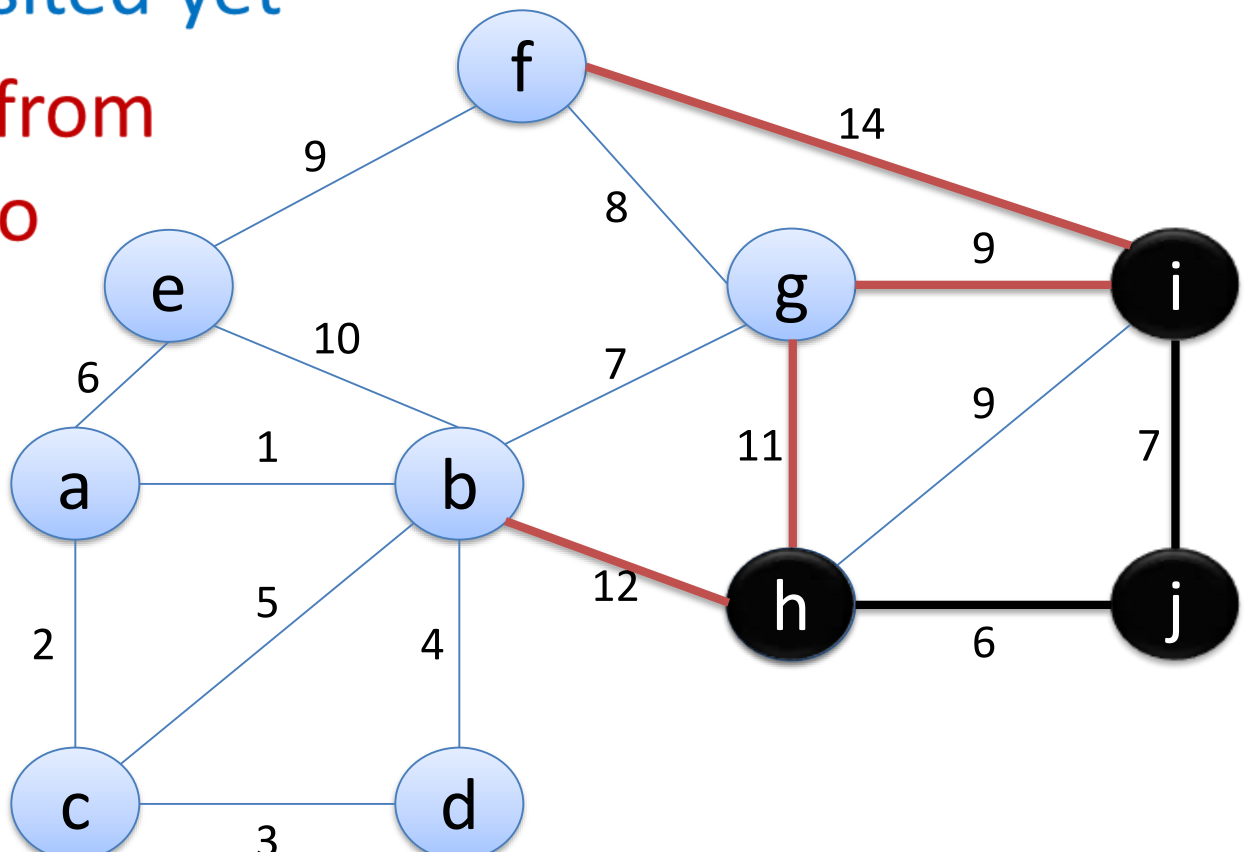

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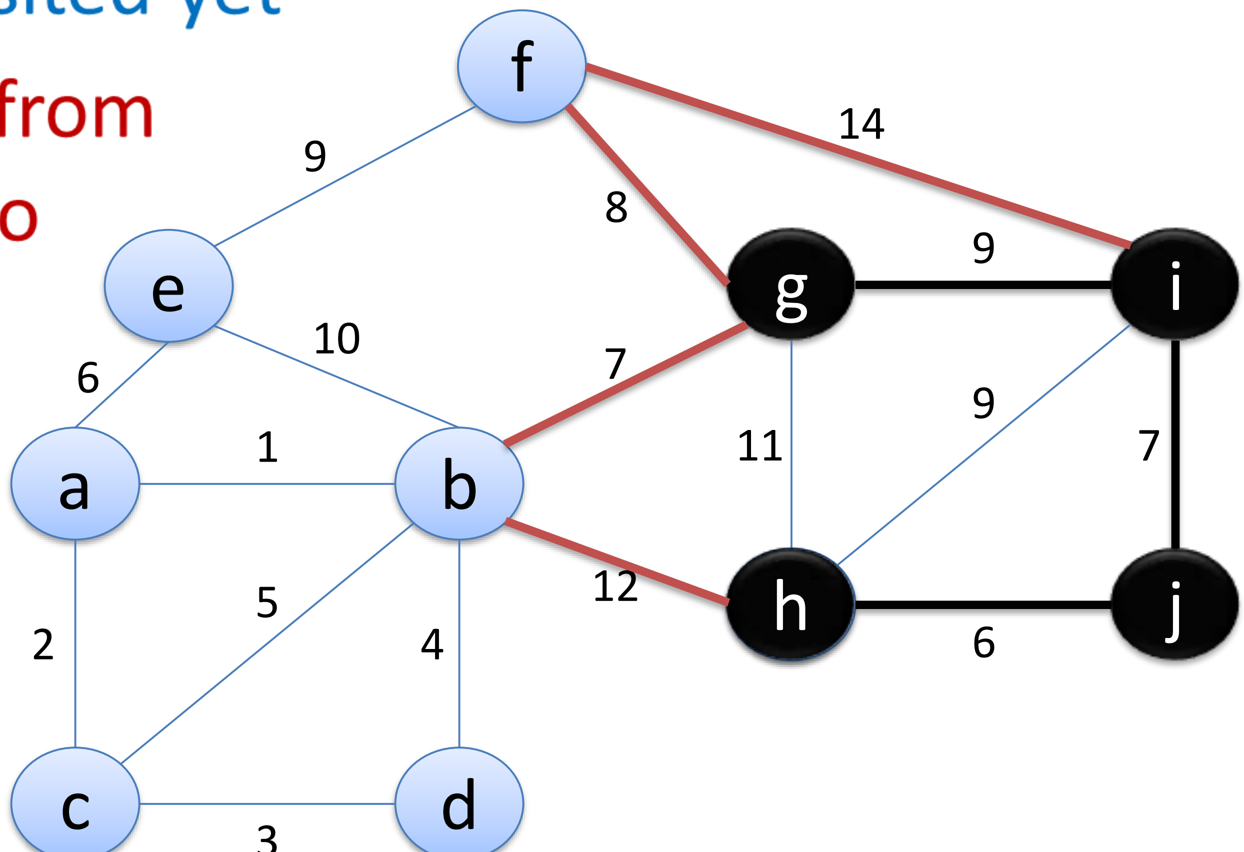

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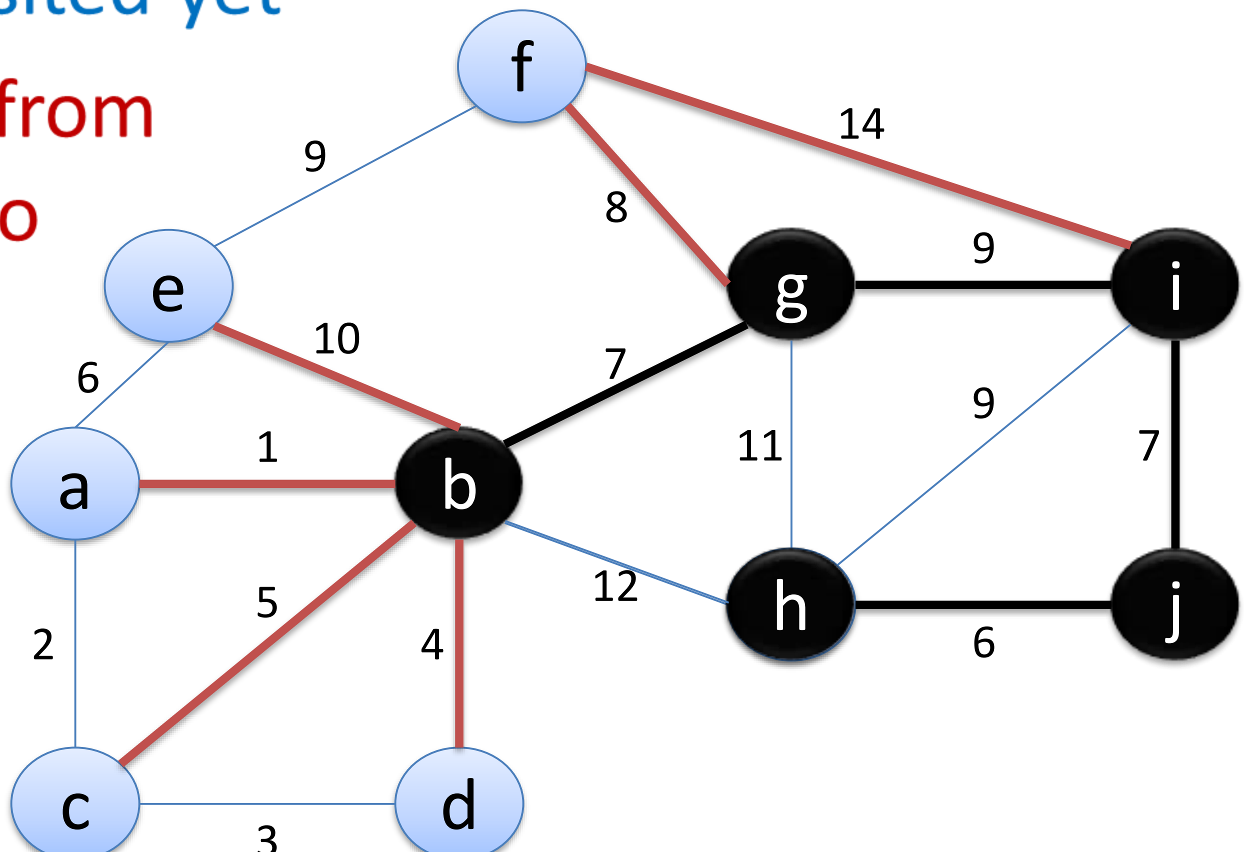

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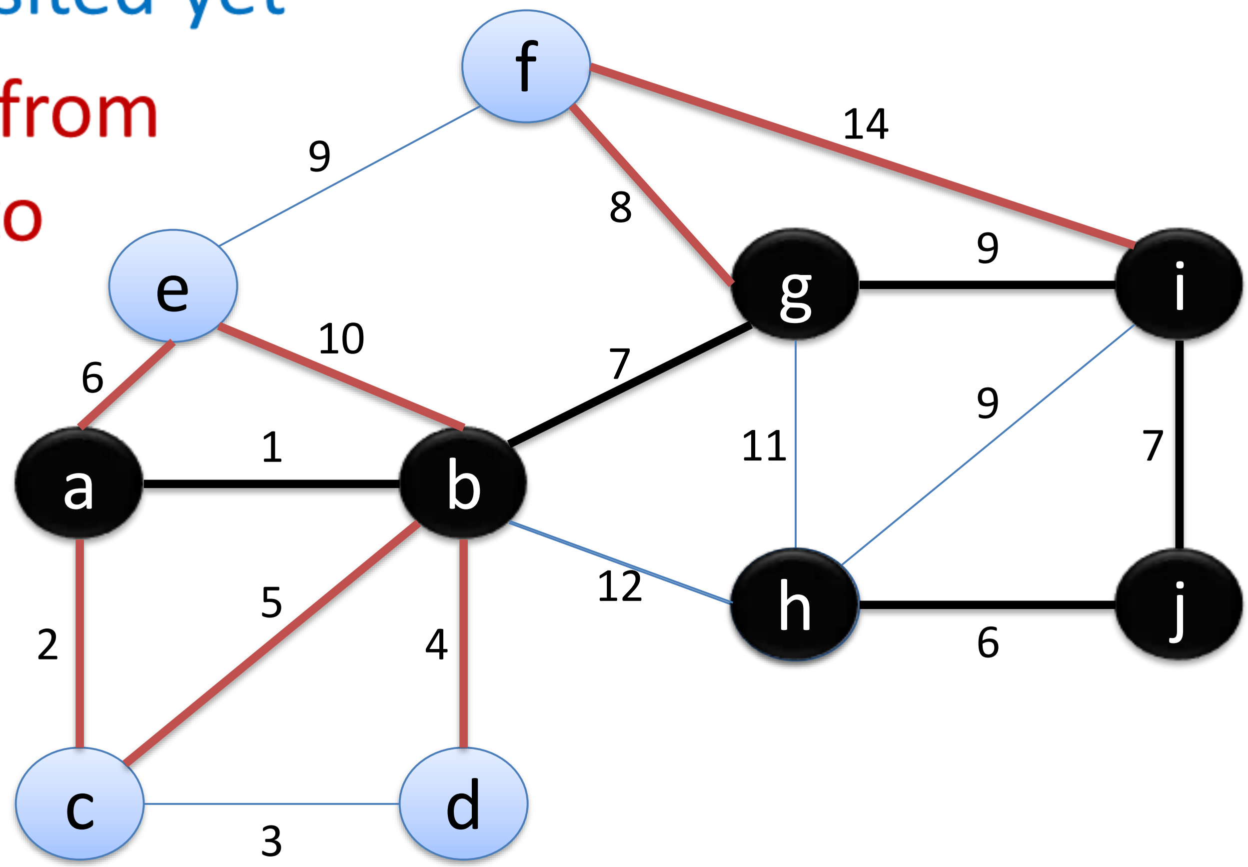

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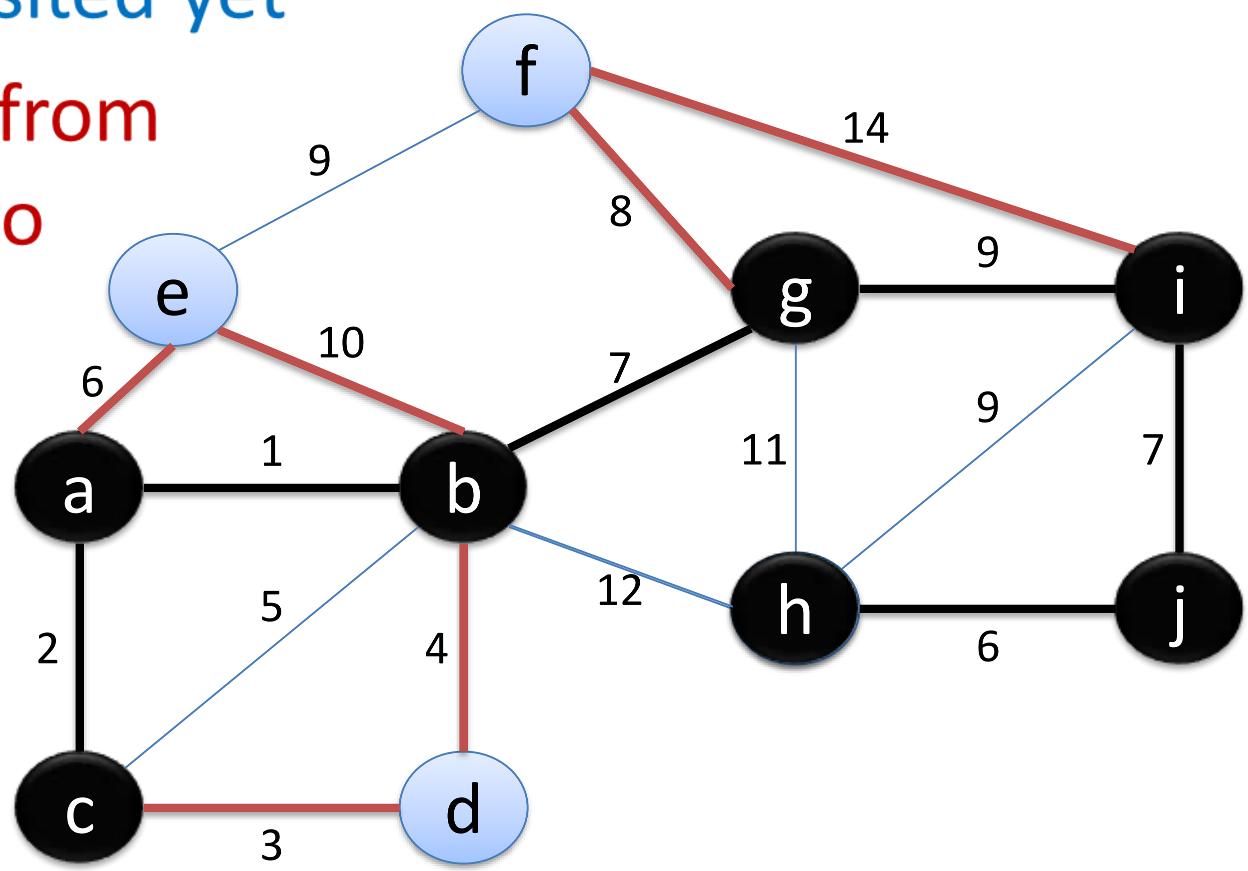

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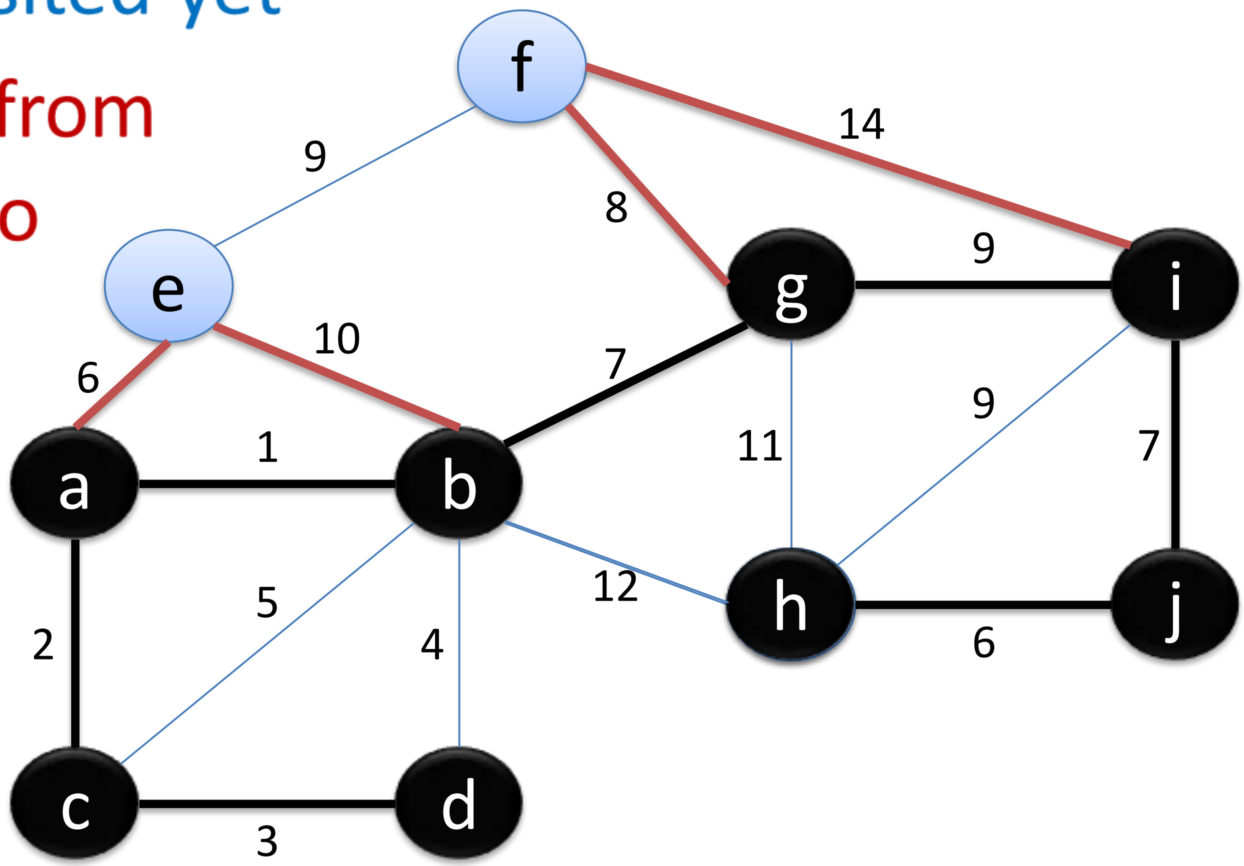

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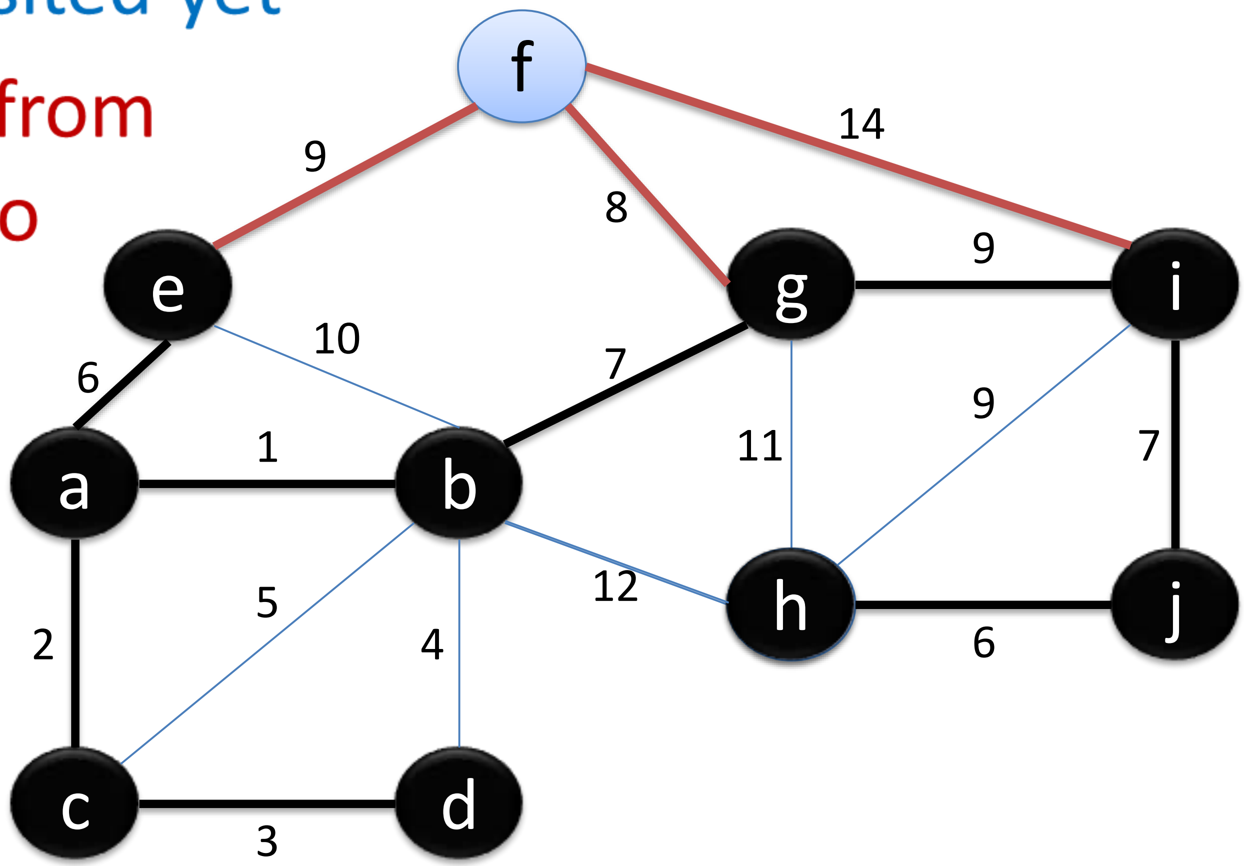

Running Prim’s algorithm

- Start at an arbitrary node, say, h.
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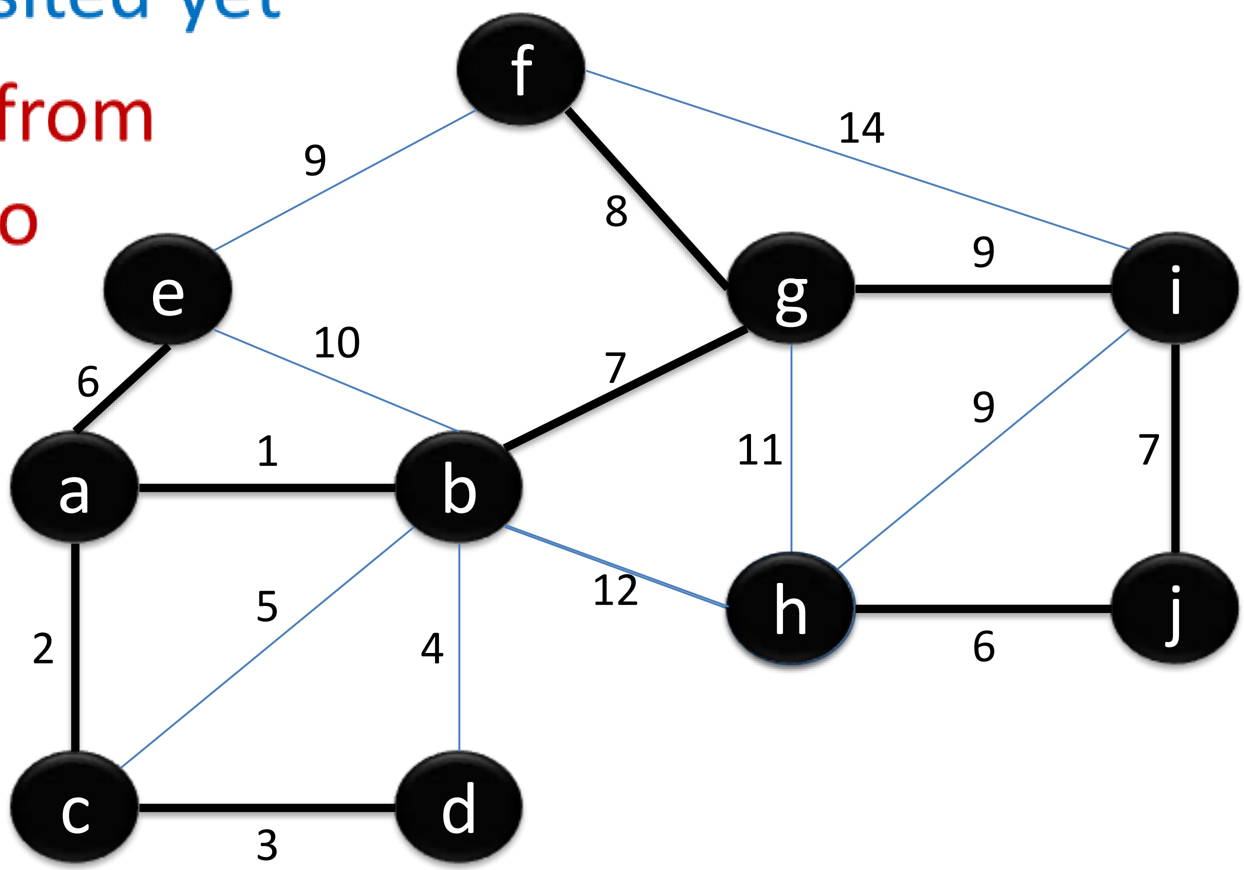
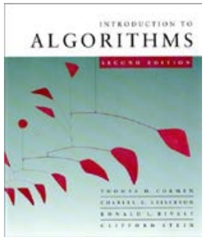
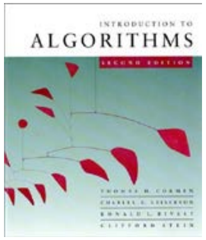
- Minimum Cost: 47





Analysis of Prim

$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        for each $v \in Adj[u]$
            do if $v \in Q$ and $w(u, v) < key[v]$
                then $key[v] \leftarrow w(u, v)$
                      $\pi[v] \leftarrow u$



Analysis of Prim

$\Theta(V)$ total:

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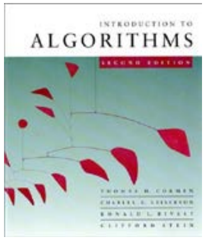

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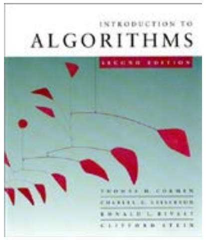

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$|V|$ times

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)

$degree(u)$ times

  for each $v \in Adj[u]$
    do if $v \in Q$ and $w(u, v) < key[v]$
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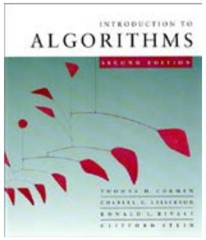

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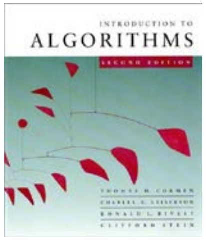
Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.



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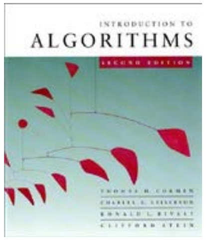
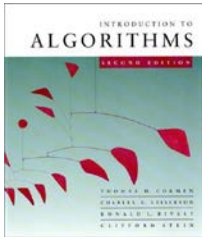
Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$



Analysis of Prim (continued)

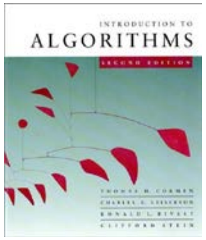
Time $= \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$



Analysis of Prim (continued)

Time = $\Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total



Analysis of Prim (continued)

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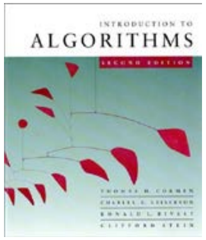
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$



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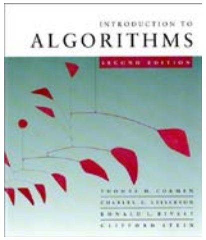
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$

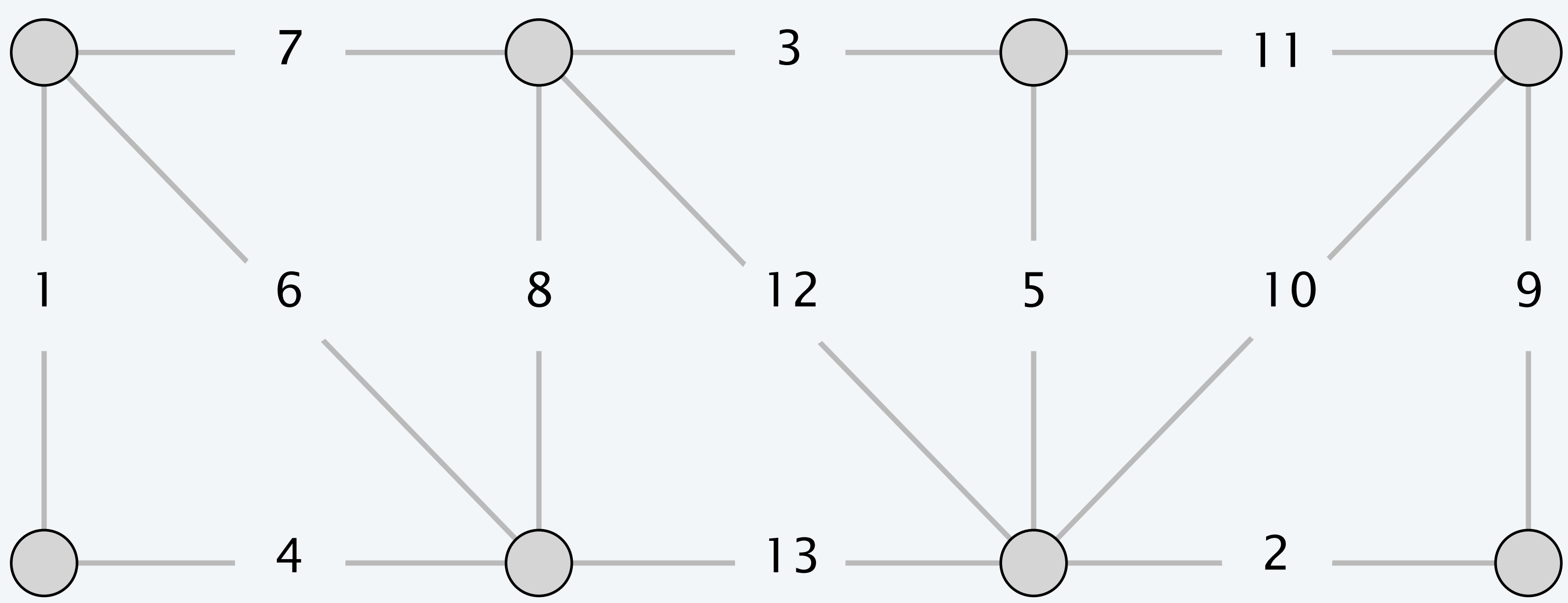


Analysis of Prim (continued)

Time = $\Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case

Prim's algorithm demo



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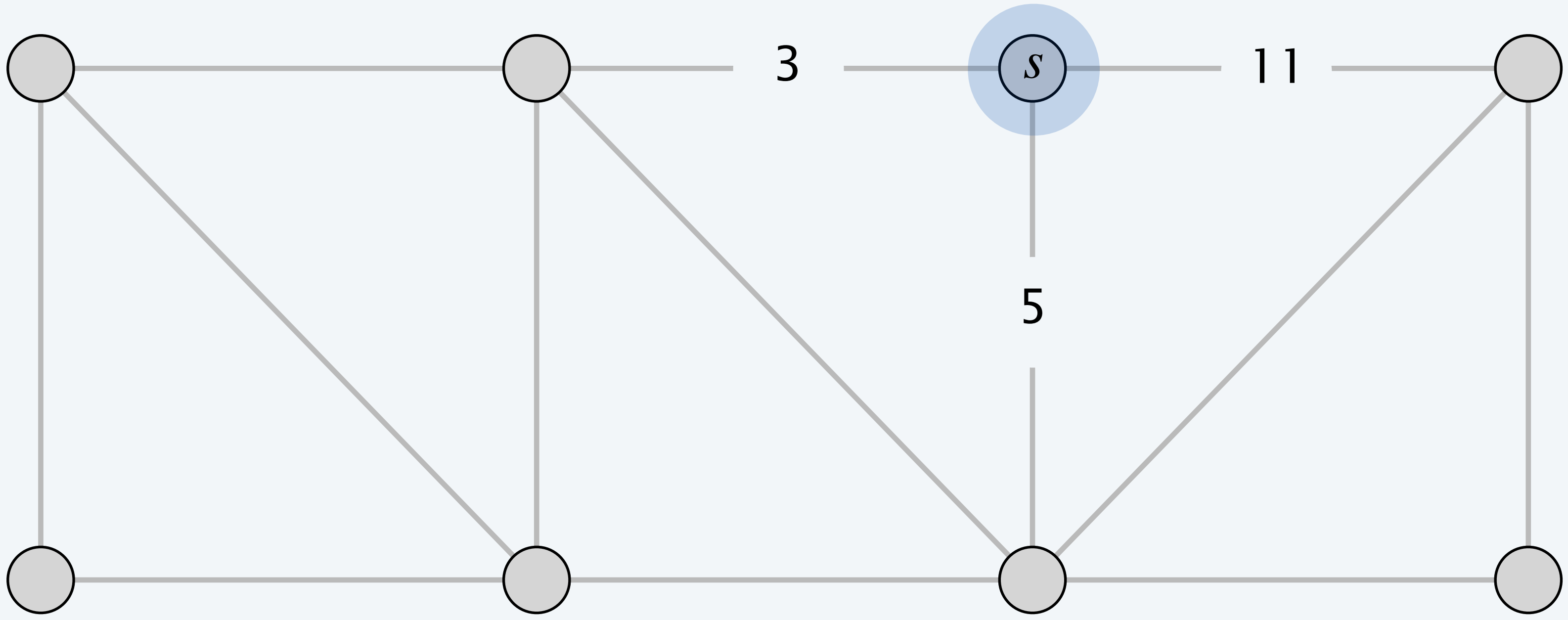

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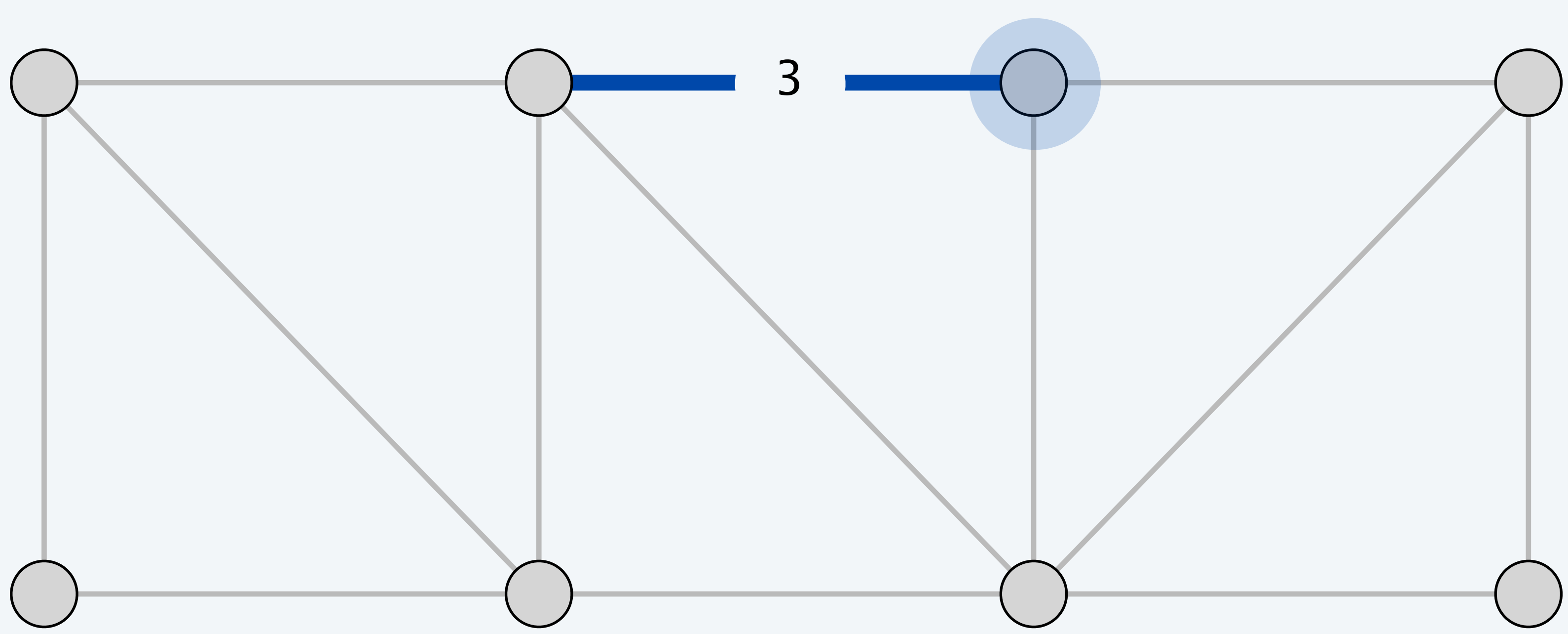

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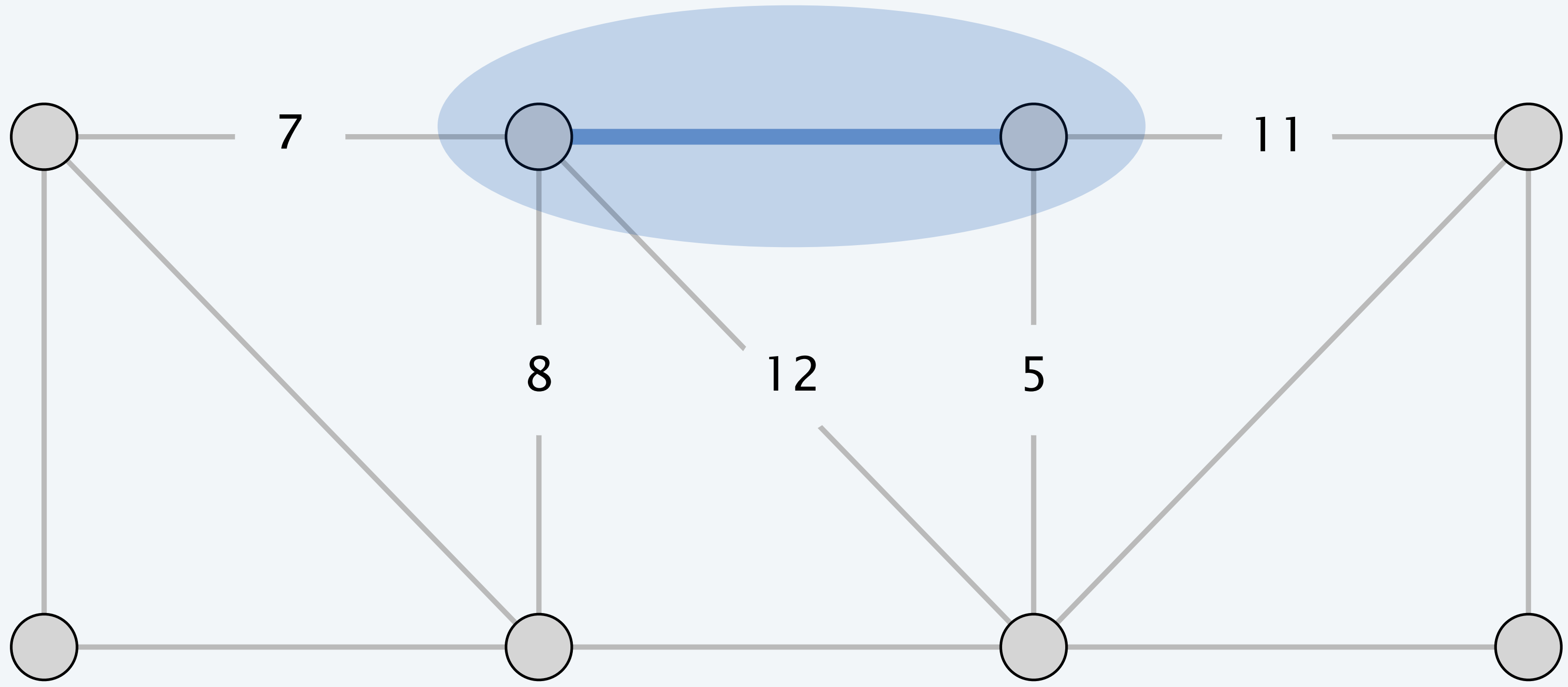

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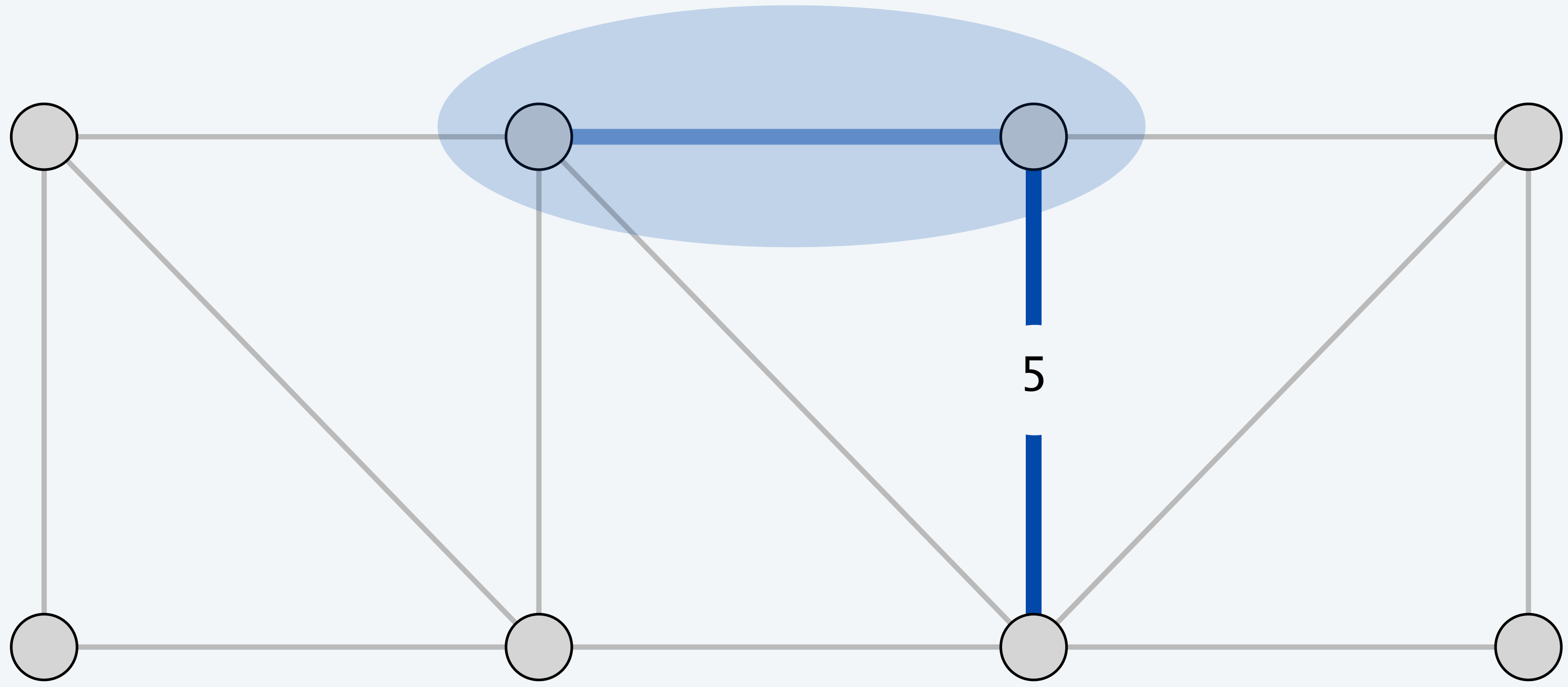

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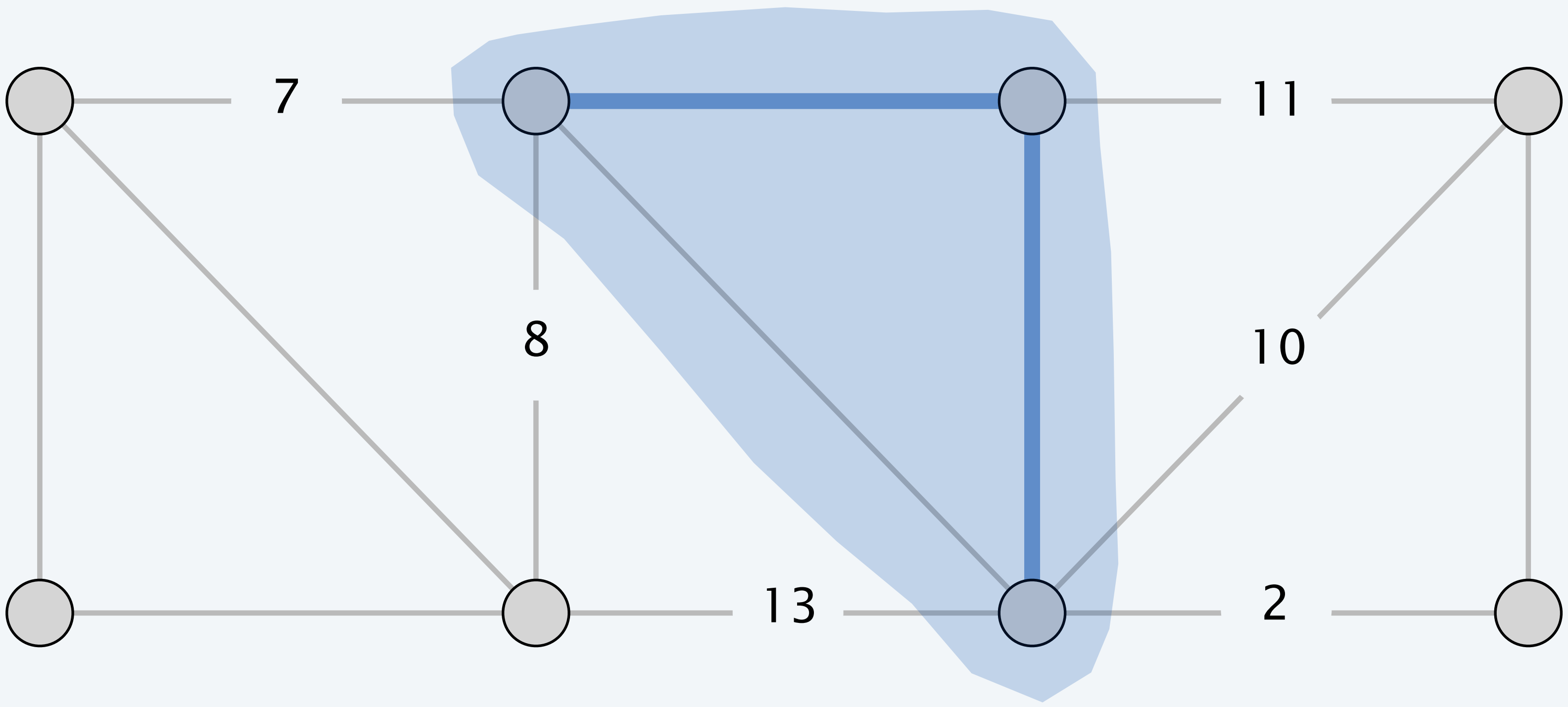

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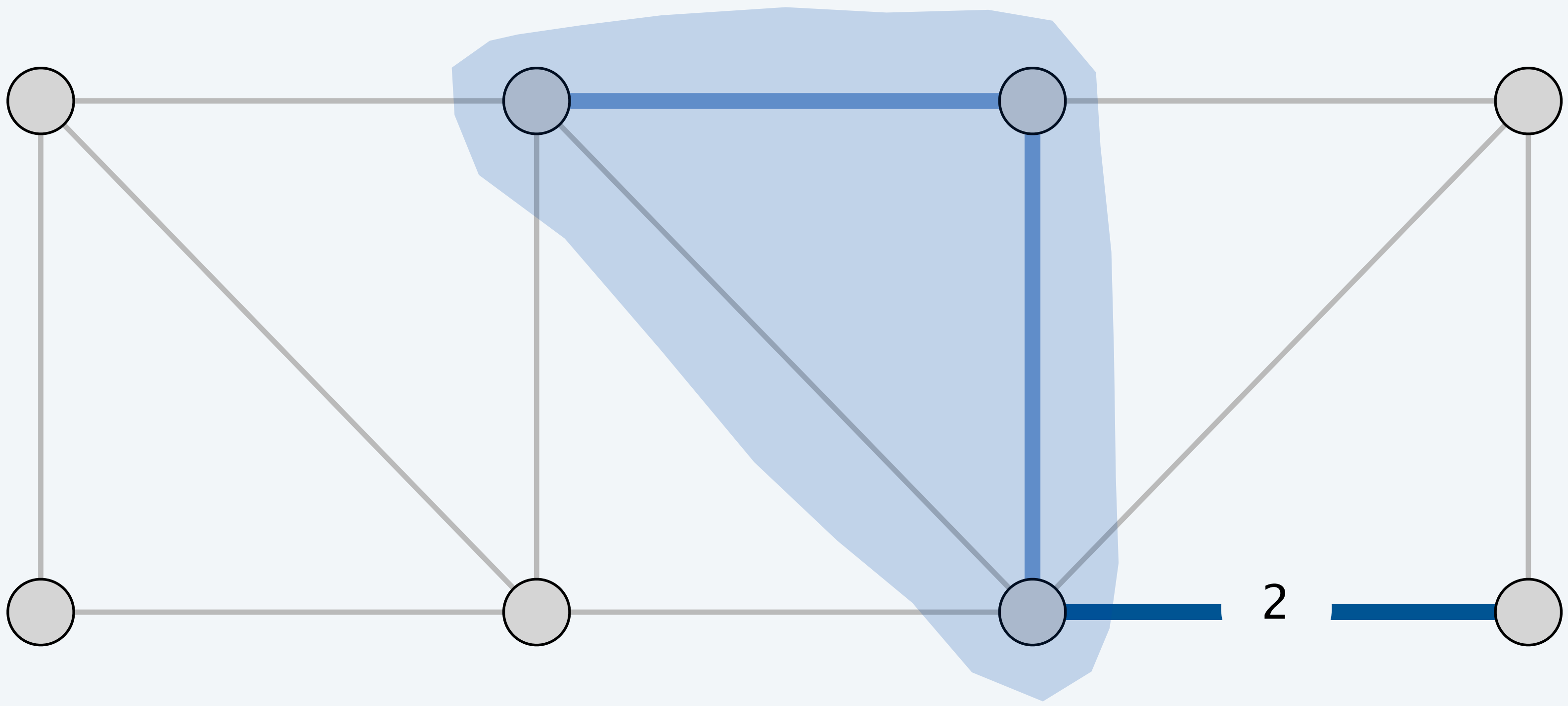

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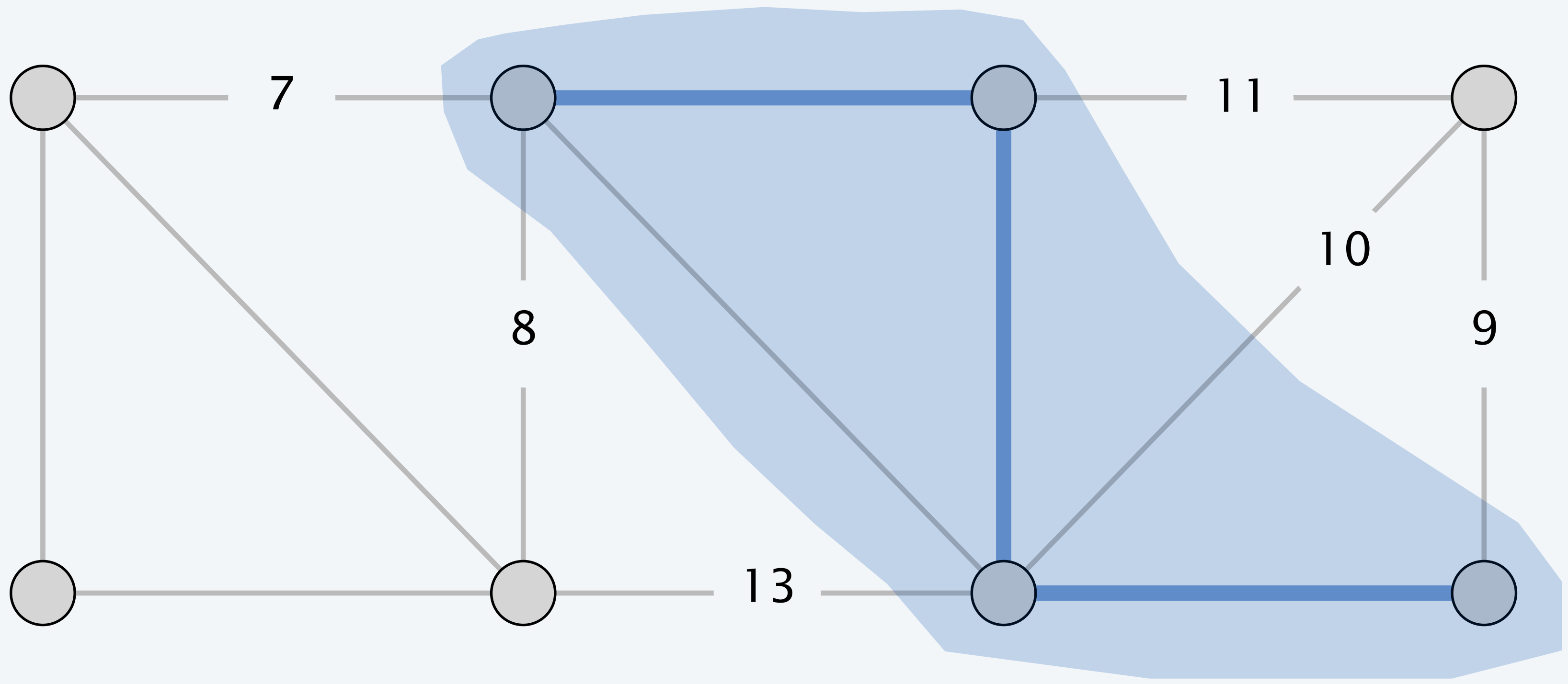

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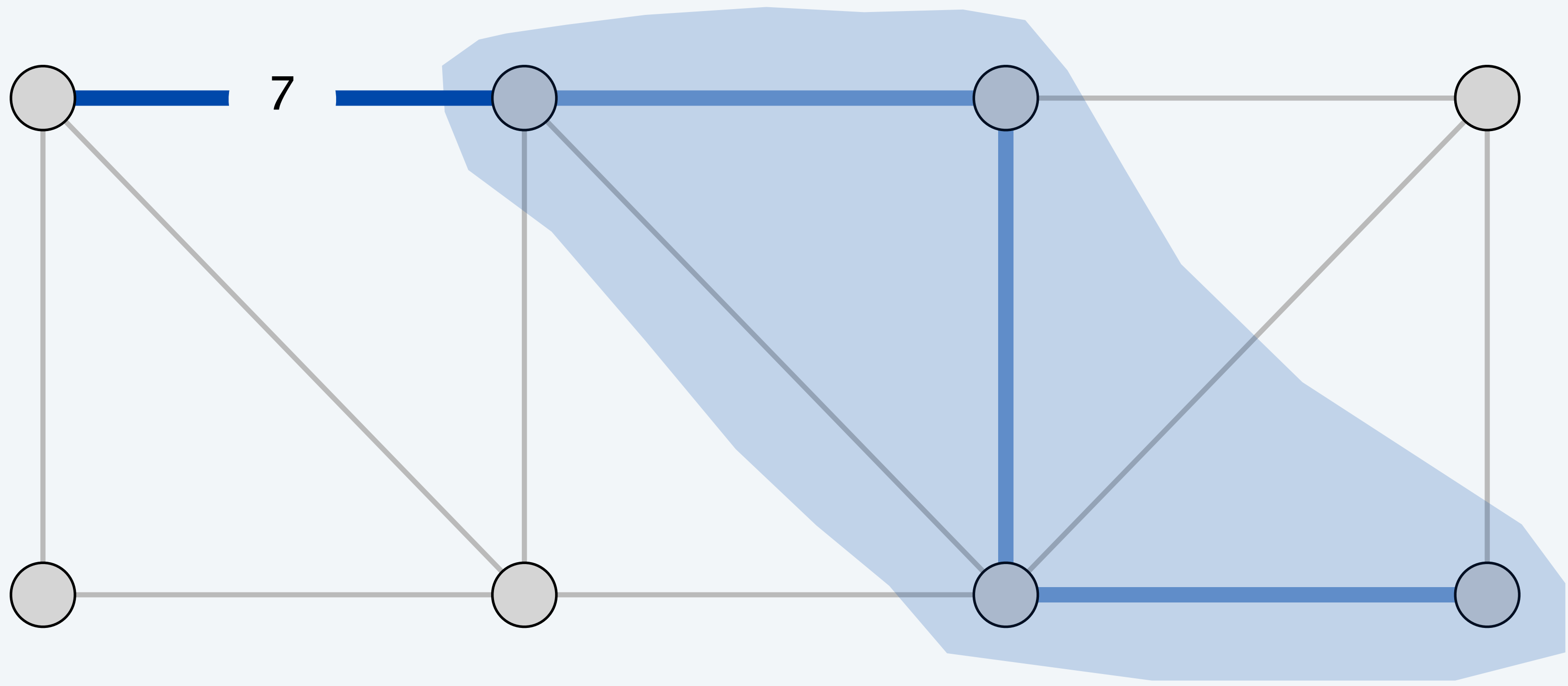

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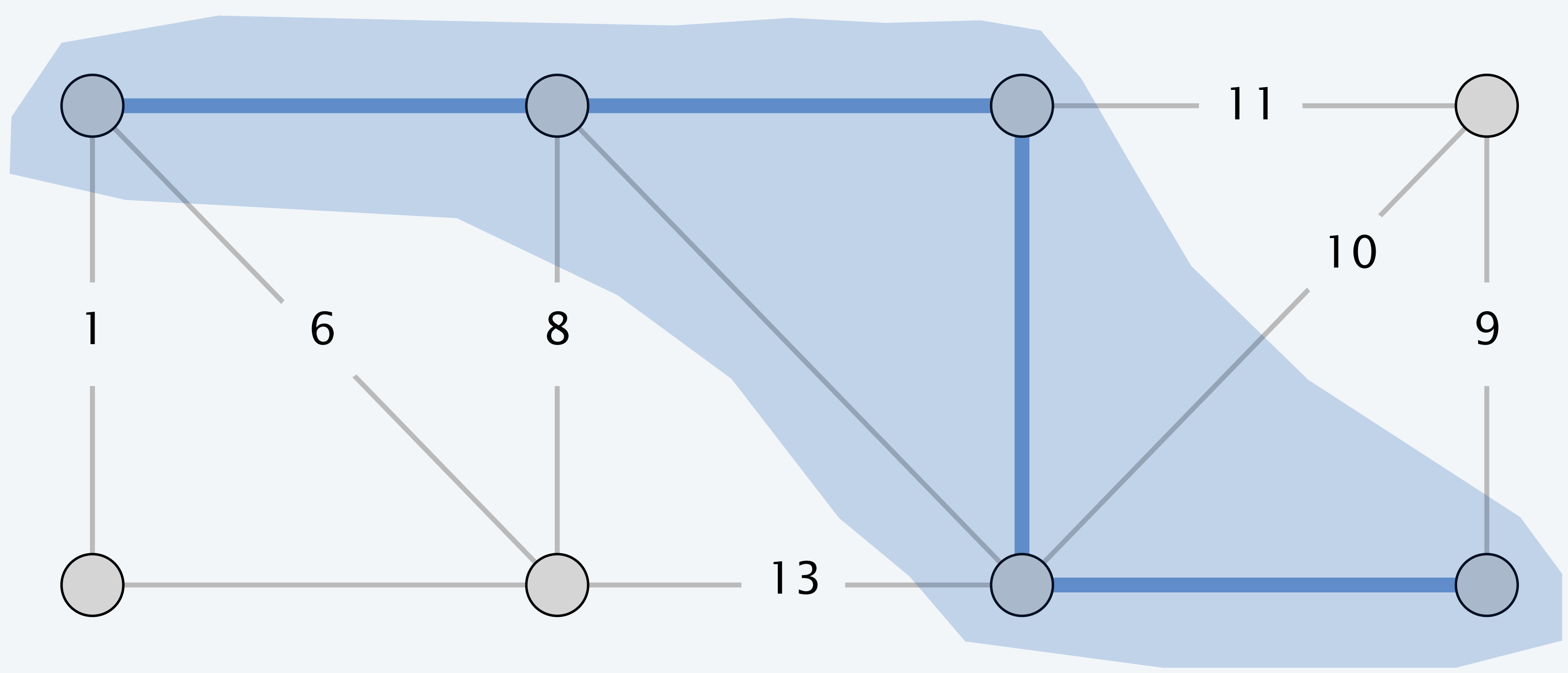

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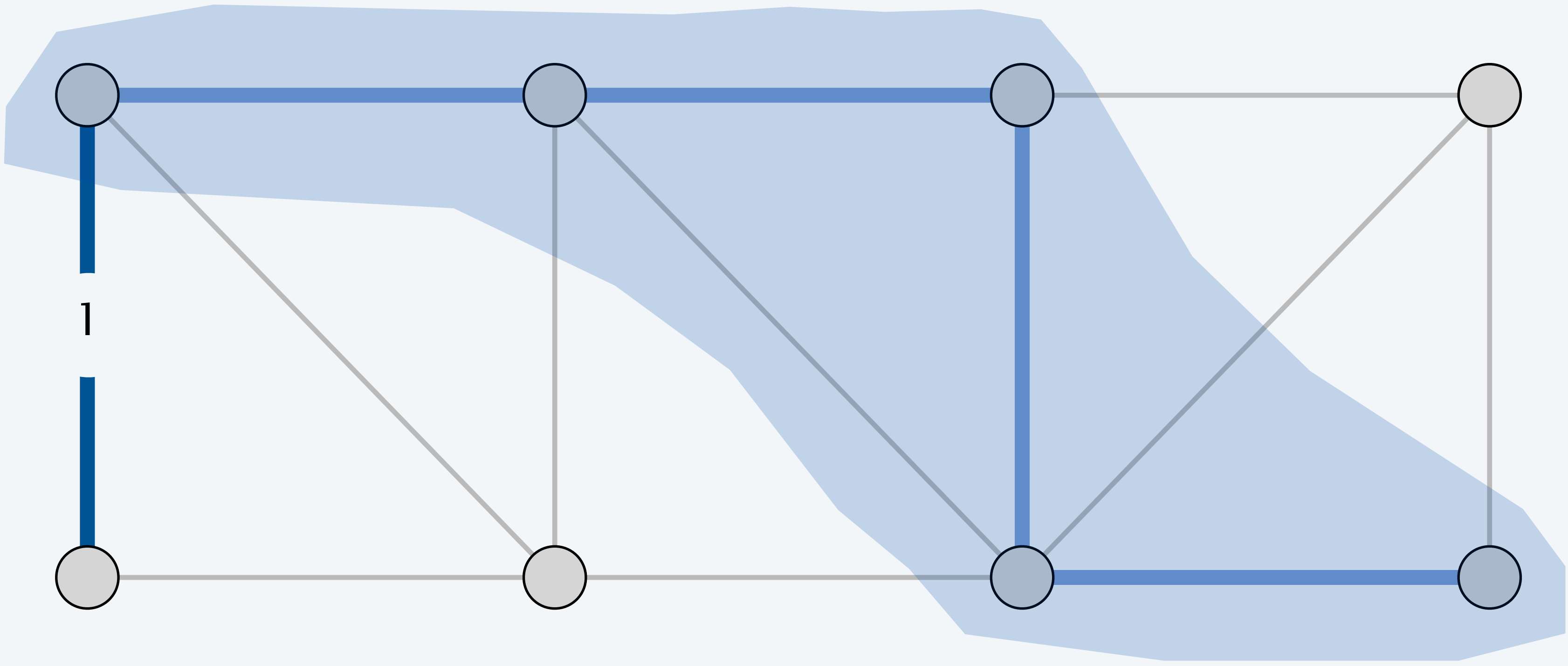

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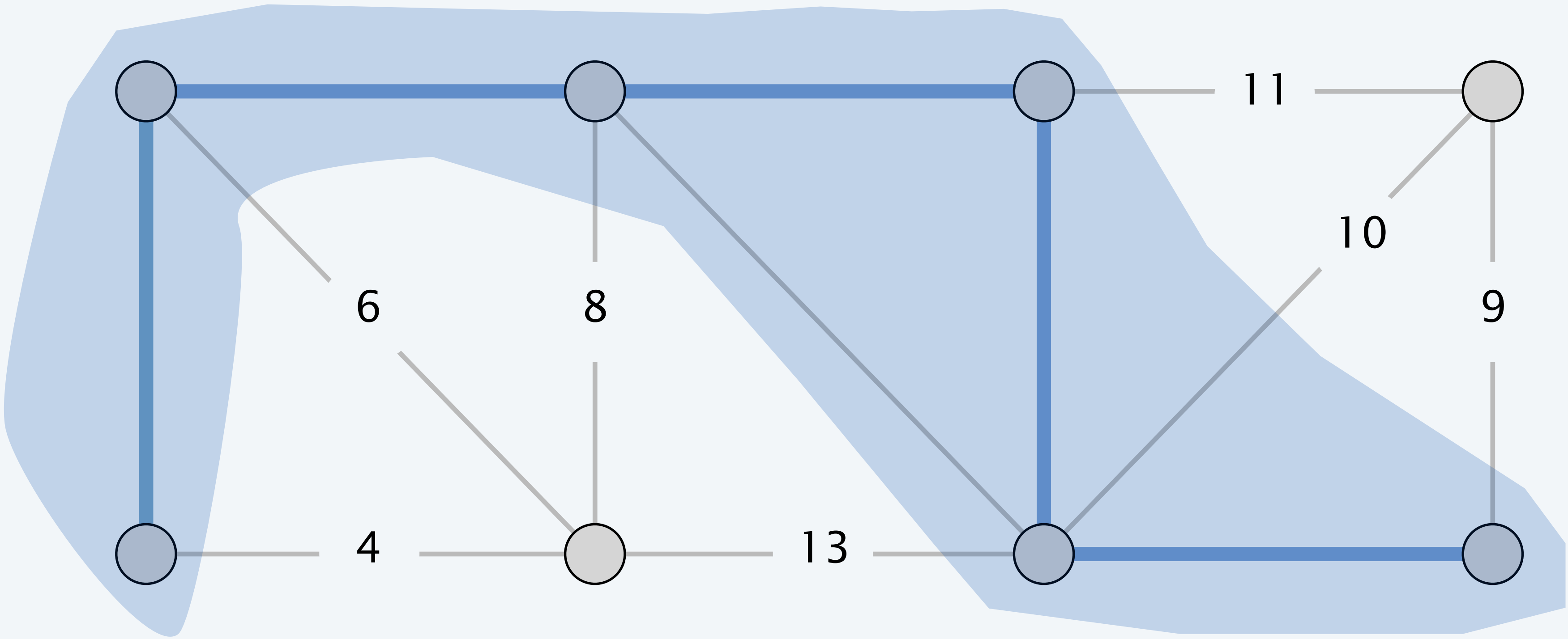

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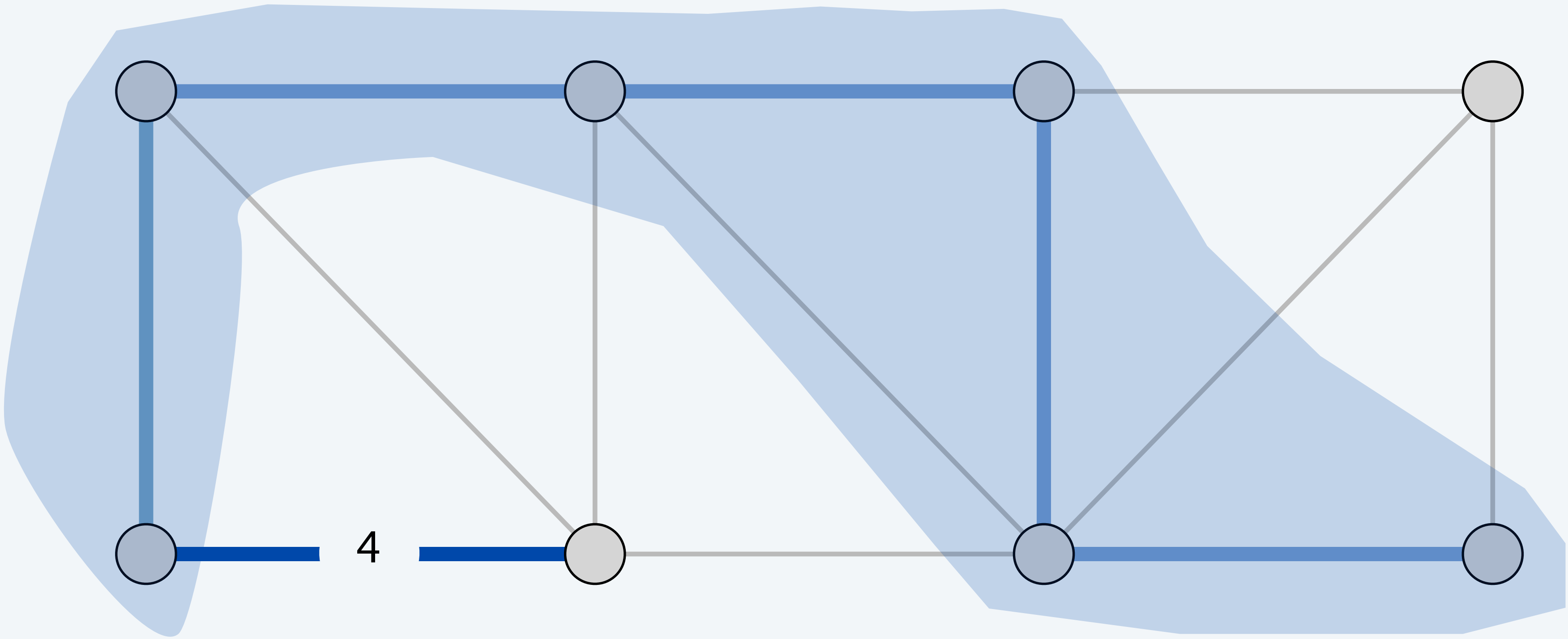

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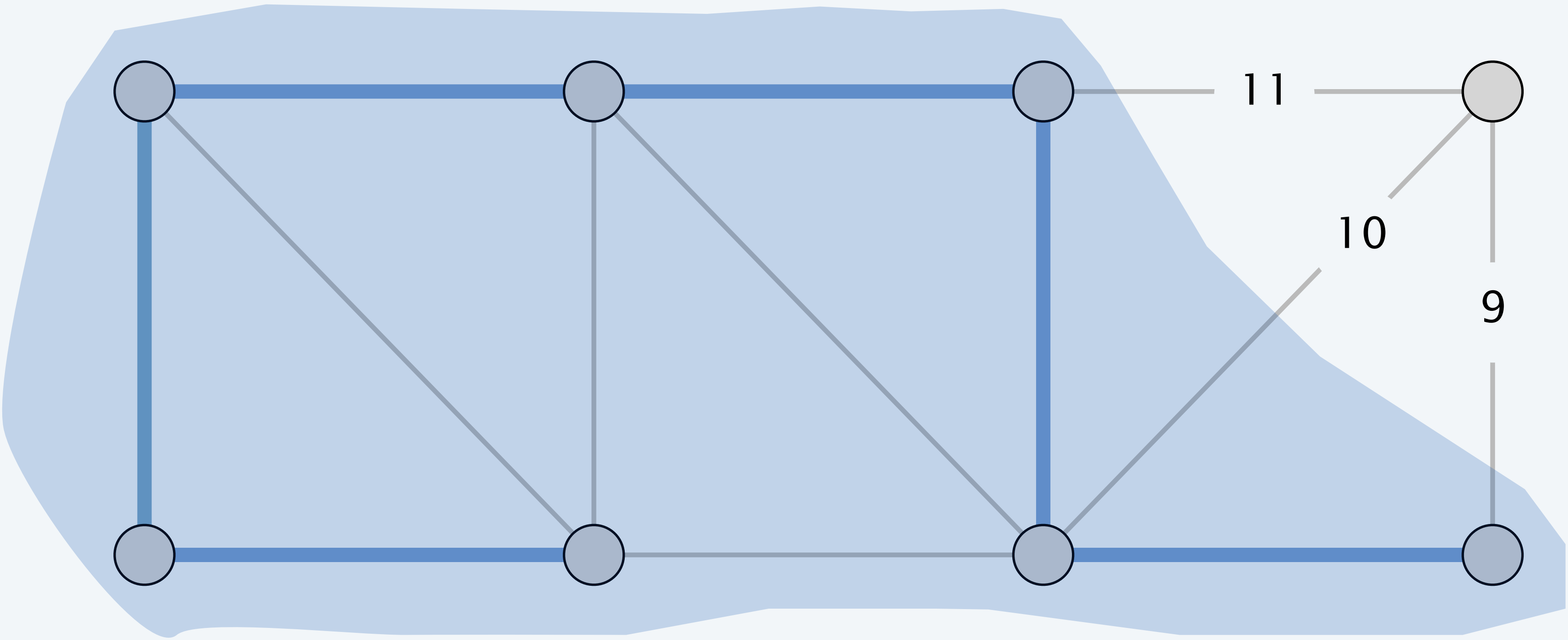
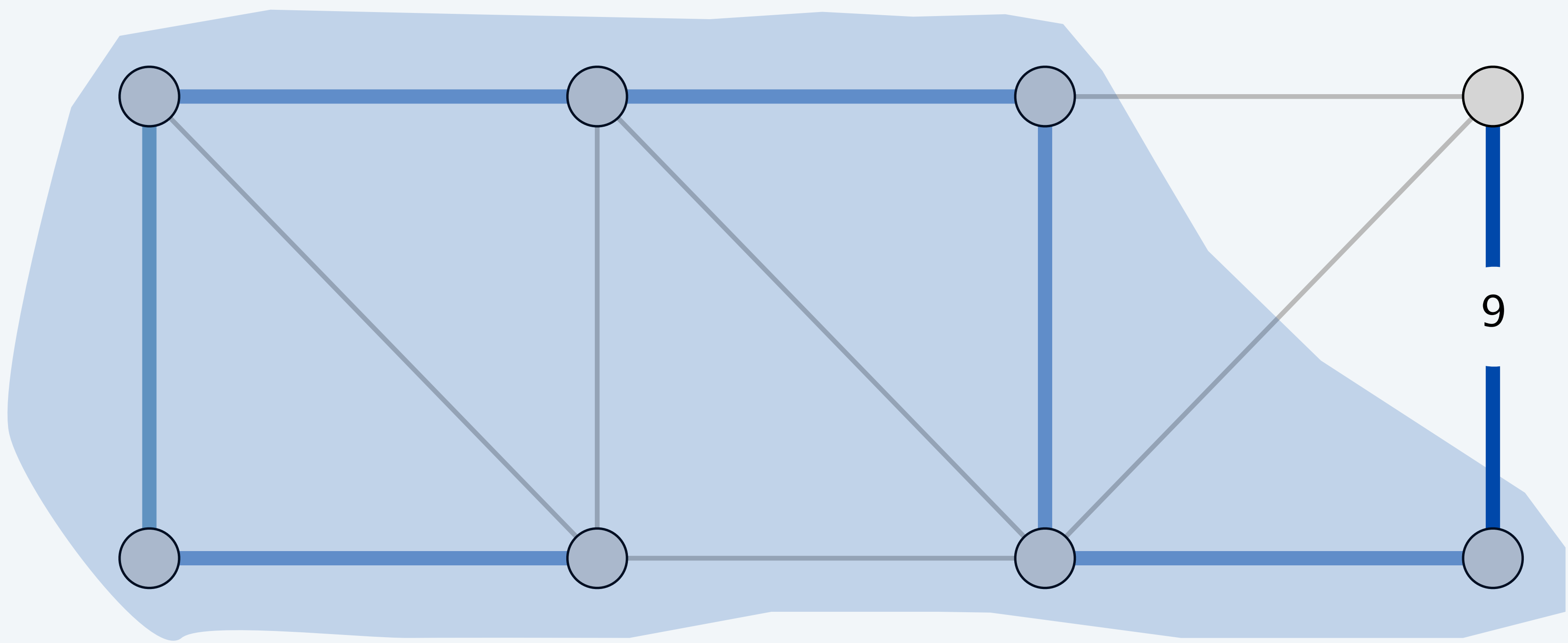


Prim's algorithm demo



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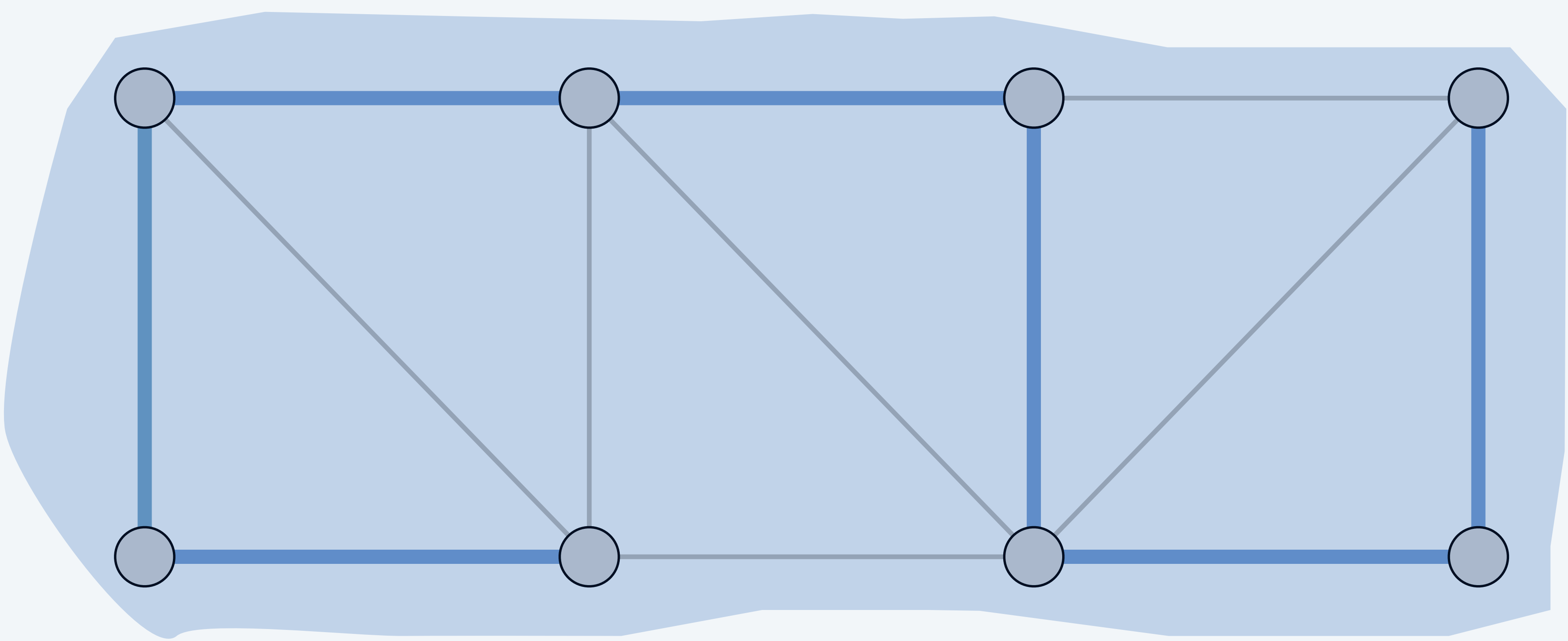

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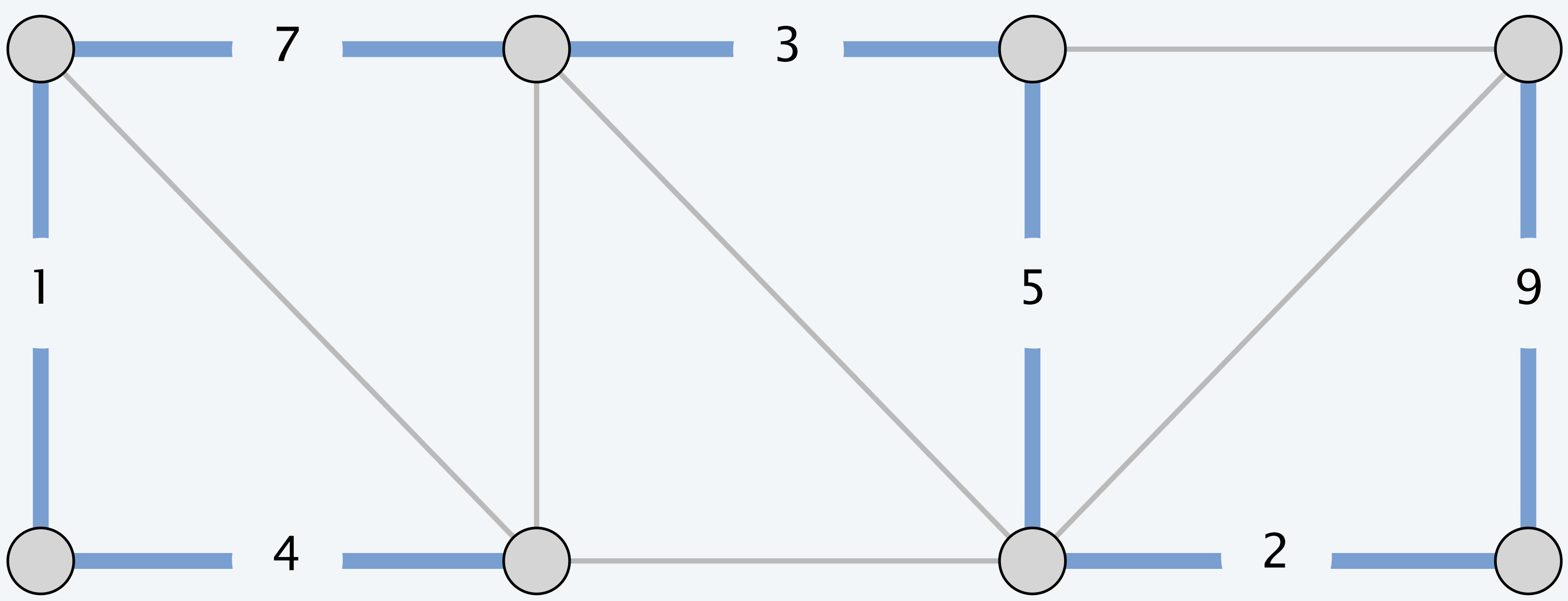
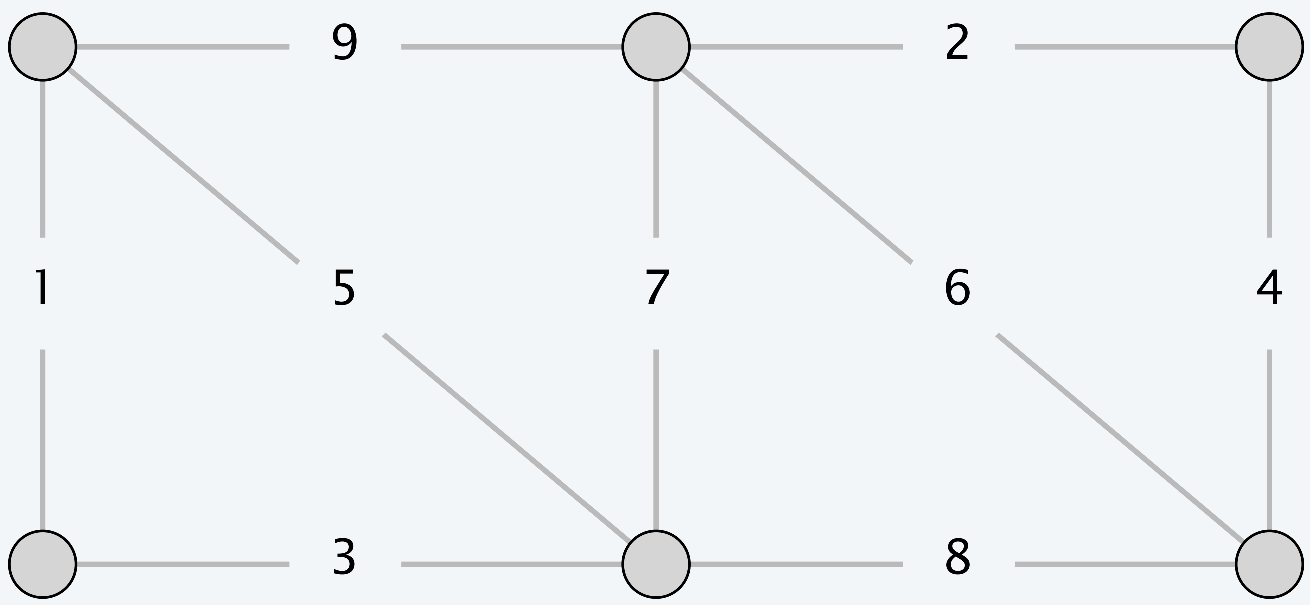


Kruskal's algorithm demo

Consider edges in ascending order of weight:

- Add to T unless it would create a cycle.



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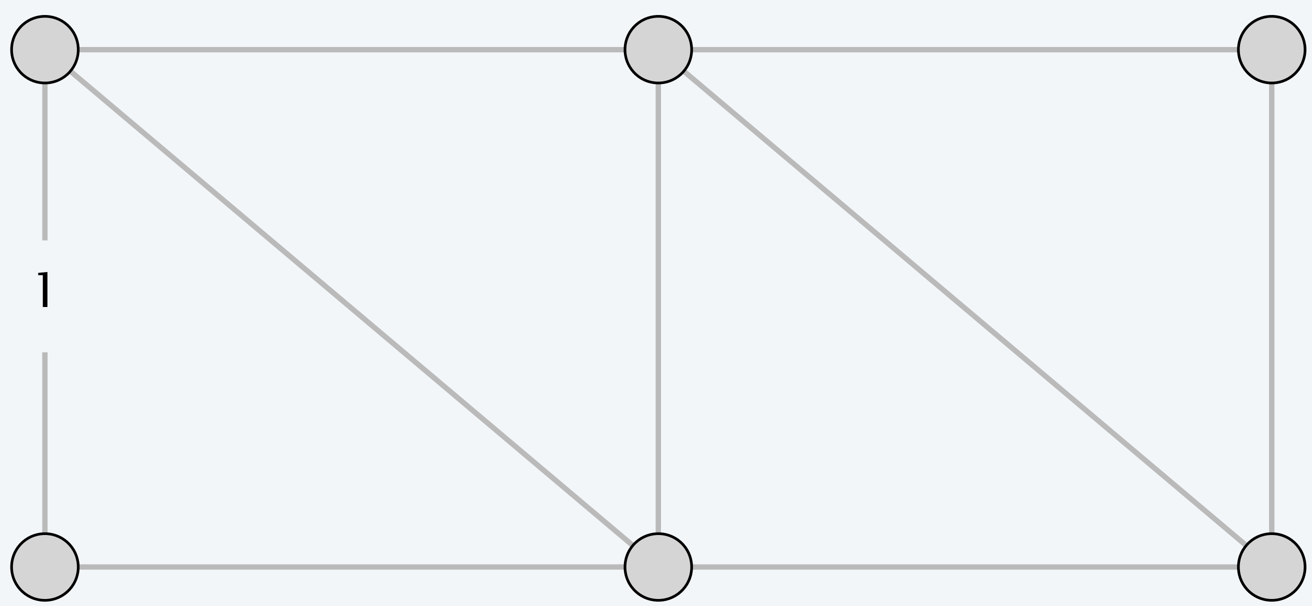

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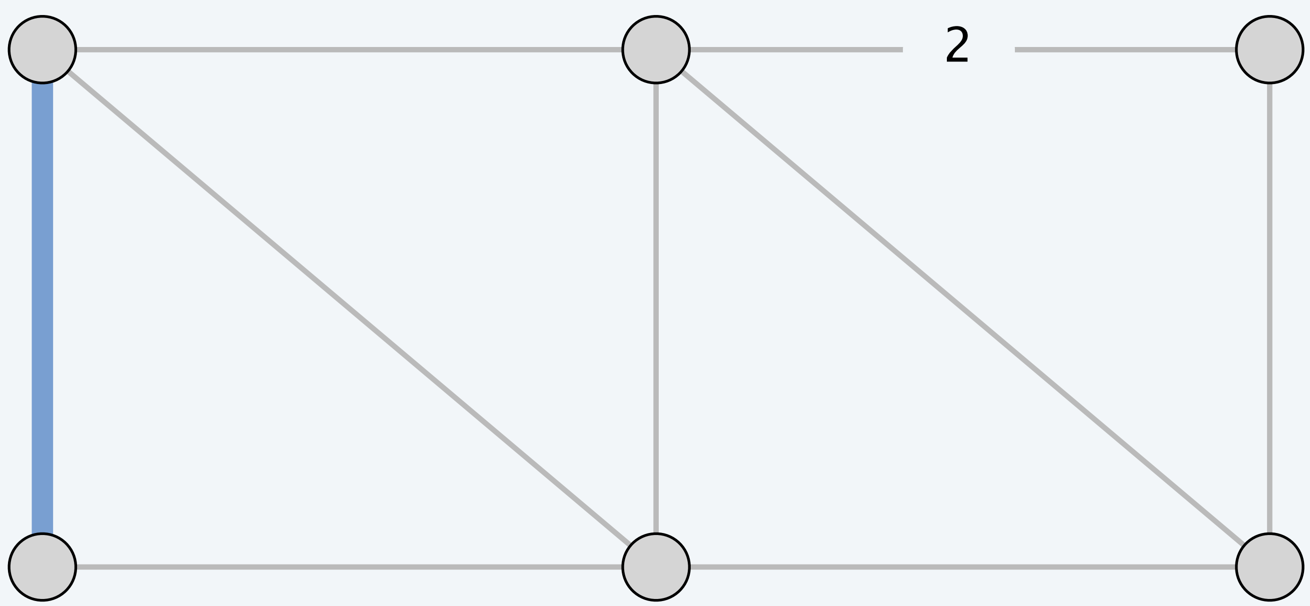

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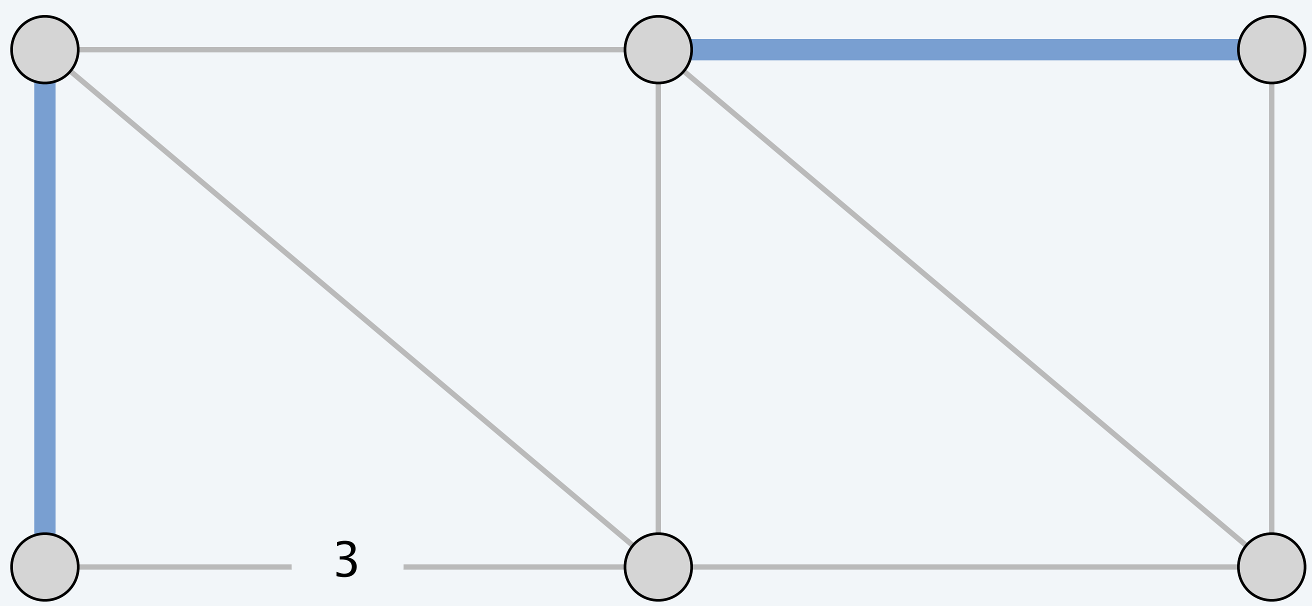

Kruskal's algorithm demo

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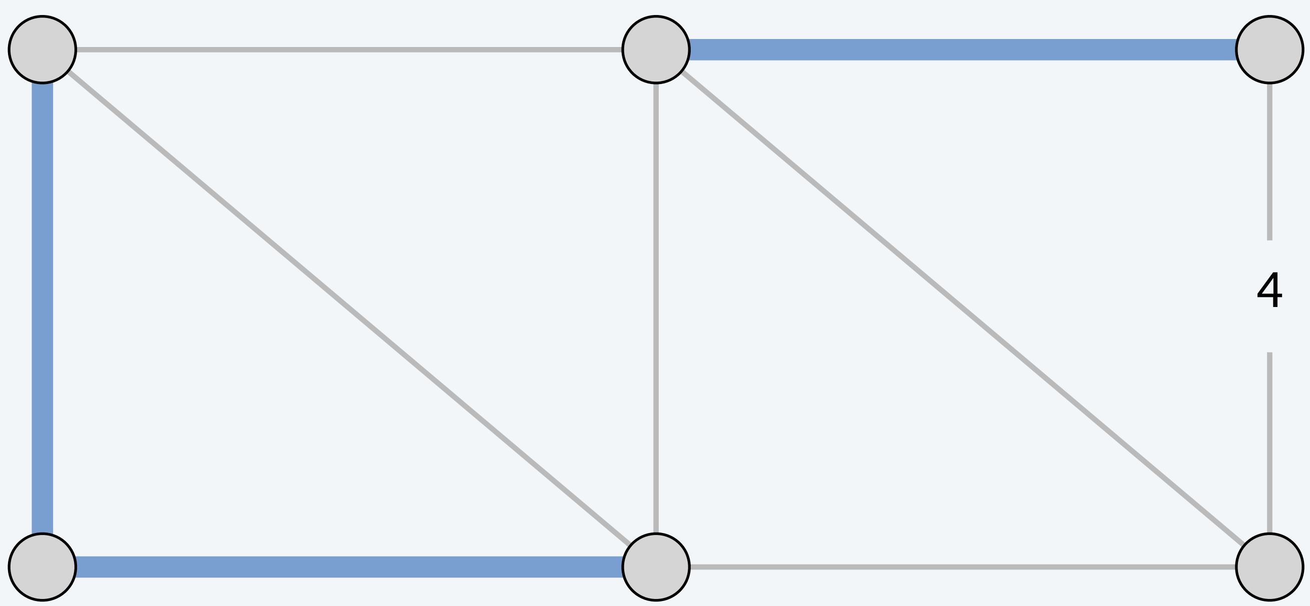

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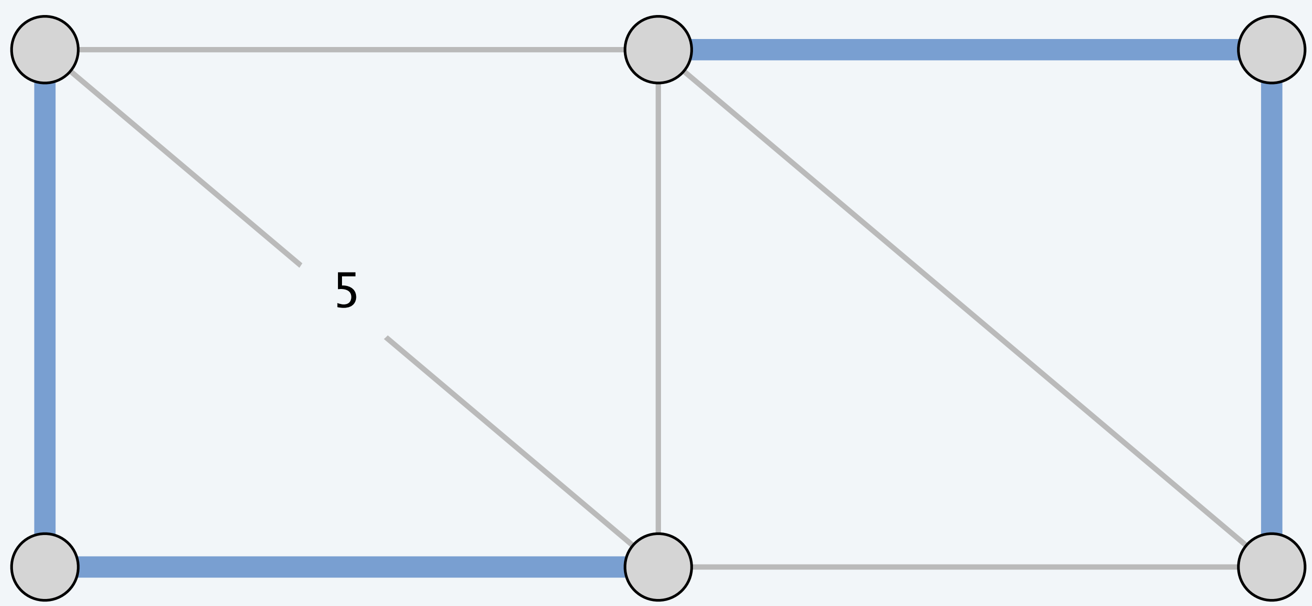

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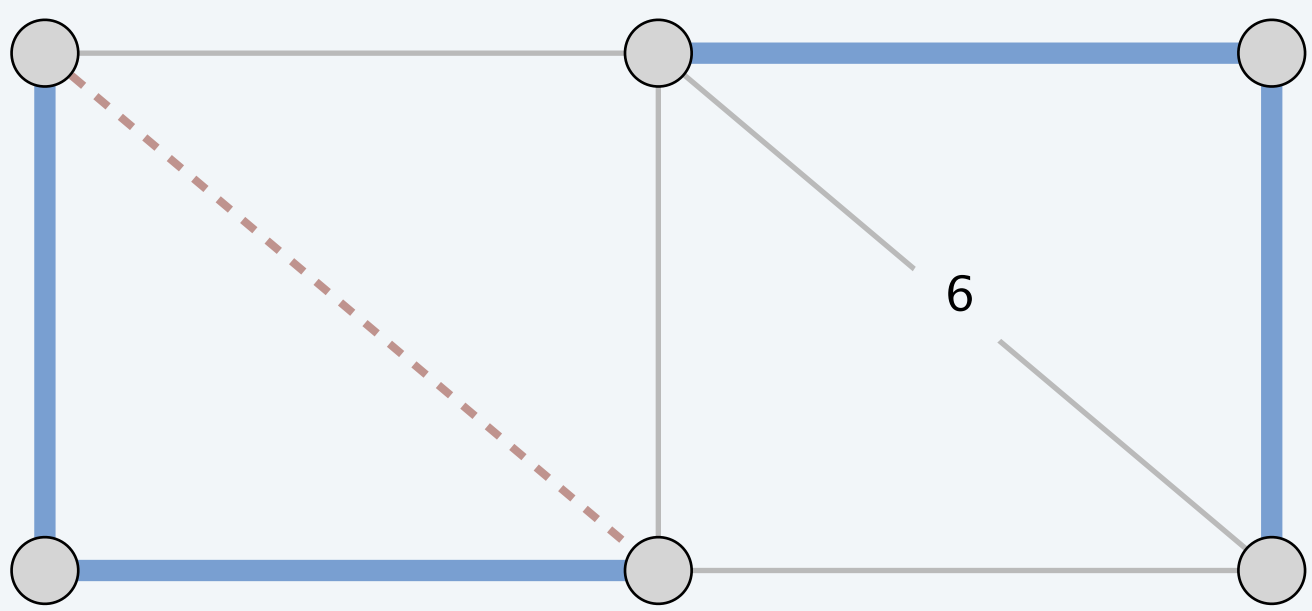

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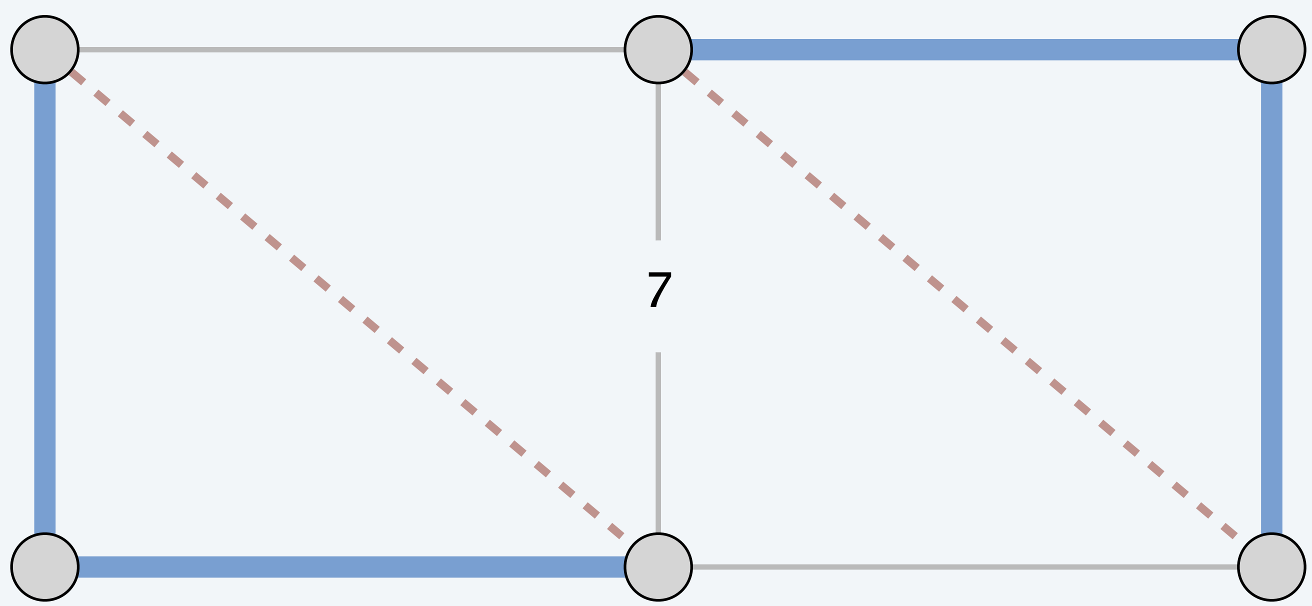

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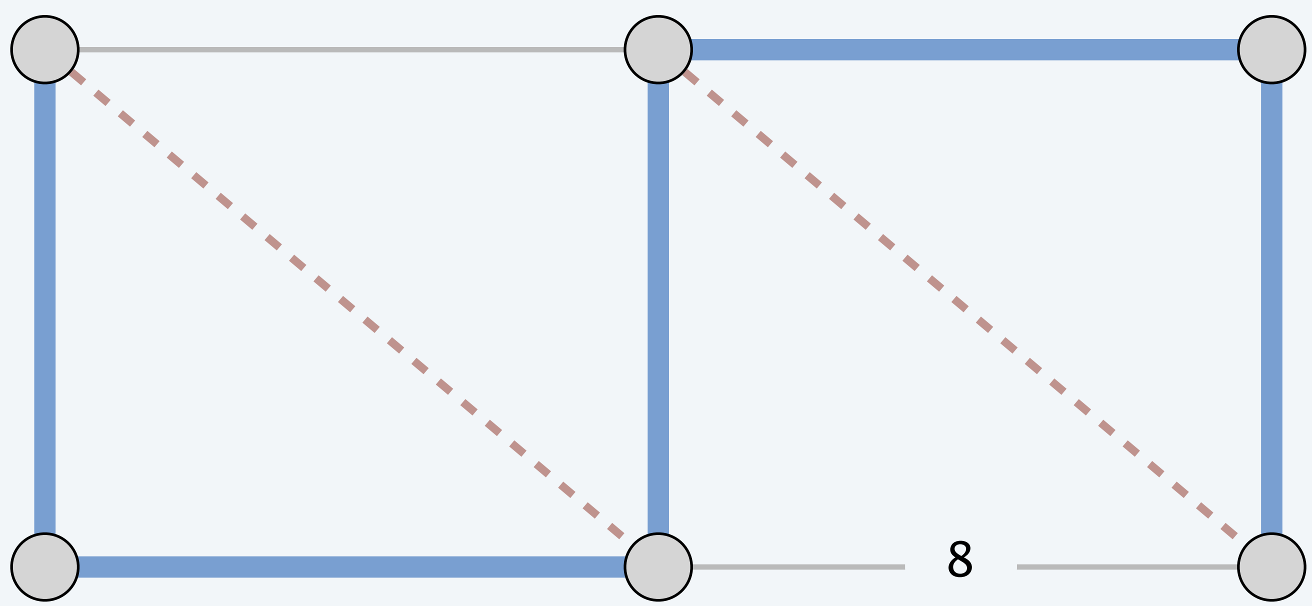

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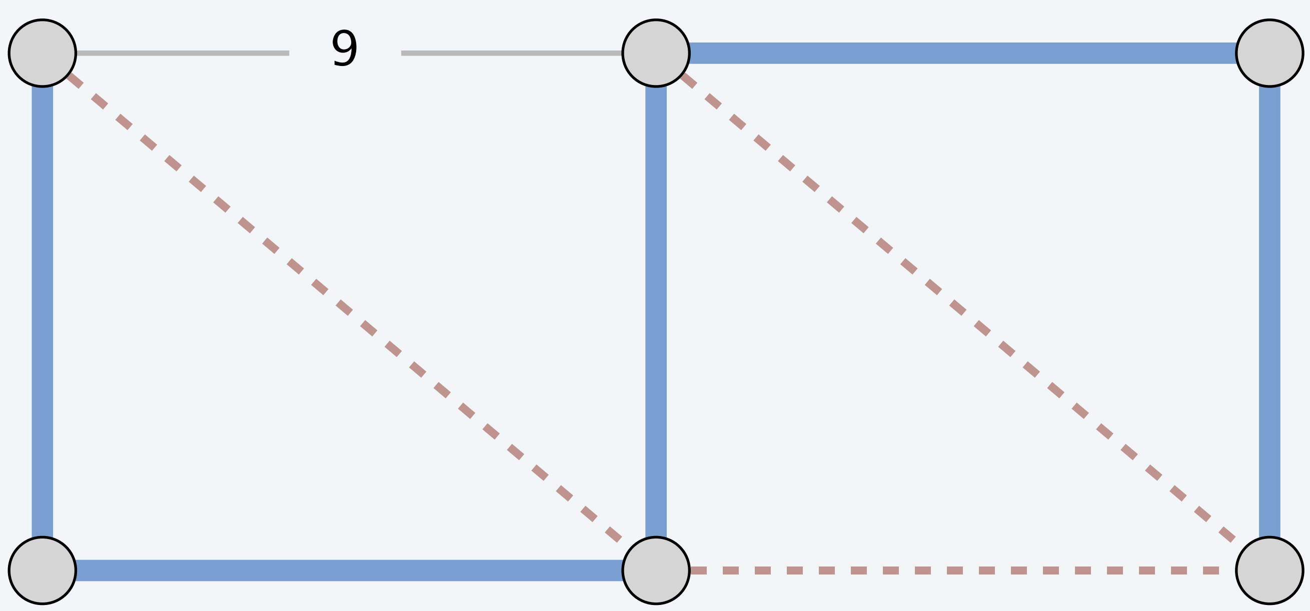

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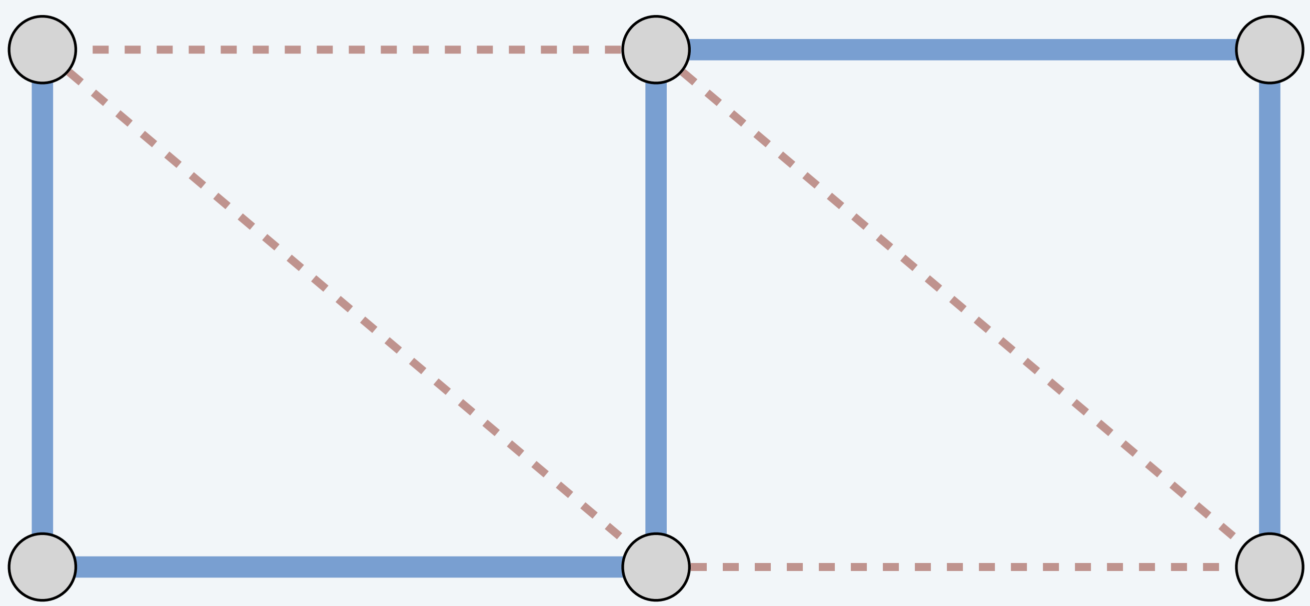

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