Roll No.:

Name:

Indian Institute of Information Technology, Design and Manufacturing, Kancheepuram

End Semester – May 2024

Course Code: MA1002
Date of Examination: May 02, 2024
Duration: 3 hours

Course Title: Linear Algebra
Category: Core
Maximum Marks: 50

Instructions:

- Answer any 10 questions.

1. Consider the following matrix A and the elementary matrix E:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 3 & -1 & 3 \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Prove that if $B = AE$ then B is row equivalent to A^T, where A^T stands for the transpose of matrix A. [5]

2. (a) Let $AX = B$ be a given system of linear equations, where A is an $m \times n$ matrix, X is an $n \times 1$ matrix and B is an $m \times 1$ matrix. Discuss when this system has a solution. Further, discuss when this system has a unique solution and when it has infinitely many solutions. [2]

(b) Solve the following system of equations: [3]

$$\begin{aligned} 2y + 3z &= 7 \\ x + y - z &= -2 \\ -x + y - 5z &= 0 \ . \end{aligned}$$

3. Let v be a non-zero vector in $\mathbb{R}^n$, where $n \geq 2$ is a positive integer. Is it possible to find a basis of $\mathbb{R}^n$ that contains v? If so, how do you construct such a basis? [5]

4. An $n \times n$ matrix A is called lower triangular if all entries lying above the diagonal entries are zero, that is, $A_{ij} = 0$ if $i < j$. Show that the set of all lower triangular matrices forms a subspace of the space of all $n \times n$ matrices. What is the dimension of this subspace? Find a basis for this subspace. [2+1+2]

5. State and prove the Rank-Nullity-Dimension theorem. [1+4]

6. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a function defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$

Show that T is a linear operator on $\mathbb{R}^3$. Find the rank of T and the nullity of T. [1+2+2]

7. Let V, W be two vector spaces over the same field F and $T : V \longrightarrow W$ be a linear transformation. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W. [5]

8. Let $\mathbb{P}_3$ be the vector space of all real polynomials of degree at most three. Let D be the differentiation operator on $\mathbb{P}_3$ defined by $D(f(x)) = f'(x)$. Let $B = \{1, x, x^2, x^3\}$ and $B' = \{1, 2x, -3x^2, 2x^3\}$ be two ordered bases for $\mathbb{P}_3$. Find a matrix P such that [5]

$$[D]_{B'} = P^{-1}[D]_B P.$$

9. Find the eigenvalues and the corresponding eigenvectors of the matrix [5]

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix}.$$

10. (a) Define the characteristic polynomial of a matrix A. [1]
    (b) State the Cayley-Hamilton theorem. [1]
    (c) Find the inverse of the following matrix using the Cayley-Hamilton theorem: [3]

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

11. (a) Define an inner product on a vector space V. [2]
    (b) Prove that the function $\langle A, B\rangle = \operatorname{tr}(B^T A)$ is an inner product on $\mathbb{R}^{n\times n}$, where B^T denote the transpose of matrix B and "tr" denotes the trace. [3]

12. (a) Show that in any inner product space V over the field $\mathbb{R}$ the following **polar identity** holds:

$$\langle \alpha, \beta\rangle = \frac{1}{4}\|\alpha + \beta\|^2 - \frac{1}{4}\|\alpha - \beta\|^2.$$

[3]

    (b) Express the above identity involving an inner product and a norm on the vector space $C[0, 1]$ of all continuous functions from the interval $[0, 1]$ to $\mathbb{R}$. [2]