Inner product spaces

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Introducing (1) length and (2) angle (orthogonal) on vector spaces over R or C.

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$$\langle \alpha, \alpha\rangle = \sum_{j=1}^{n} x_j\overline{x_j} = \sum_{j=1}^{n} |x_j|^2 > 0, \quad \text{provided } \alpha \neq 0$$

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Let $V = R^2$. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ be two vectors in R^2.

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Prove that $\langle \rangle$ is an inner product. (Please do it now)

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Prove that $\langle \rangle$ is an inner product. (Please do it now)
Note that $\langle \alpha, \alpha \rangle = (x_1 - x_2)^2 + 3x_2^2$

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(2) Prove that $c\langle \alpha, \beta \rangle = \langle \alpha, \overline{c}\beta \rangle$

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$$\langle \alpha, \beta + \gamma \rangle = \overline{\langle \beta + \gamma, \alpha \rangle} = \overline{\langle \beta, \alpha \rangle + \langle \gamma, \alpha \rangle} = \overline{\langle \beta, \alpha \rangle} + \overline{\langle \gamma, \alpha \rangle}$$

$$\langle \alpha, \beta + \gamma \rangle = \langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle$$

Let $z = x + iy = \text{Re } z + i \text{ Im } z, \implies z + \bar{z} = 2 \text{ Re } z$

$-iz = -ix + y, \implies \text{Re } (-iz) = \text{Im } z$

$\langle \alpha, \beta \rangle = \text{Re } \langle \alpha, \beta \rangle + i \text{ Im } \langle \alpha, \beta \rangle$

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Note 2

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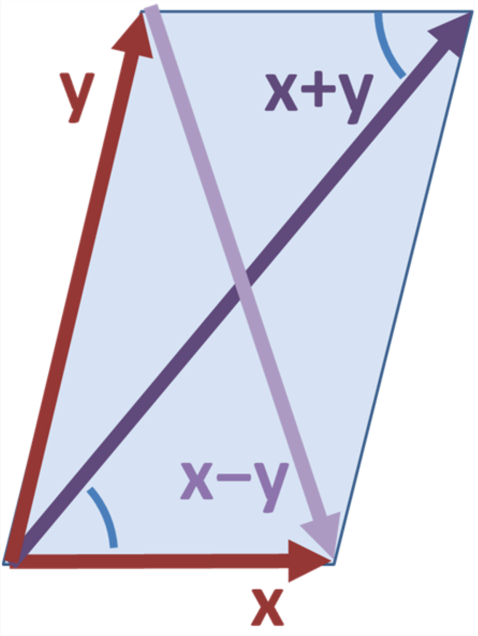
(Parallelogram law)

The sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals


y
x+y
x−y
x

Problem

Show that if $F = R$,

$$\langle \alpha, \beta \rangle = \frac{1}{4}\|\alpha + \beta\|^2 - \frac{1}{4}\|\alpha - \beta\|^2$$

Inner Product Spaces

An inner product space is a real or complex vector space, together with a specified inner product on that space.

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Theorem 1 contd.

(ii) Prove that $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \, \|\beta\|$

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So $\alpha \neq 0$. $\implies \|\alpha\|^2 > 0$.

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Let

$$\gamma = \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2} \alpha$$

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Let

$$\gamma = \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha = \beta - c\alpha$$

where

$$c = \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}$$

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Theorem 1 contd.

$$c\langle \alpha, \beta \rangle = \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2} \langle \alpha, \beta \rangle$$

Theorem 1 contd.

$$c\langle \alpha, \beta \rangle = \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\langle \alpha, \beta \rangle = \frac{\overline{\langle \alpha, \beta \rangle}}{\|\alpha\|^2}\langle \alpha, \beta \rangle$$

Theorem 1 contd.

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Theorem 1 contd.

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&\leq \|\beta\|^2 - \overline{c}\langle\beta,\alpha\rangle - c\langle\alpha,\beta\rangle + c\overline{c}\langle\alpha,\alpha\rangle
\end{aligned}$$

Theorem 1 contd.

$$0 \leq \|\beta\|^2 - 2\frac{|\langle \alpha, \beta \rangle|^2}{\|\alpha\|^2} + \frac{|\langle \alpha, \beta \rangle|^2}{\|\alpha\|^4}\|\alpha\|^2$$

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Orthogonal vectors

Let V be an inner product space and $\alpha, \beta \in V$.

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Orthogonal set and Orthonormal set

Let V be an inner product space. A set $S \subseteq V$ is called an orthogonal set if for all $\alpha, \beta \in S$ where $\alpha \neq \beta$, then $\langle \alpha, \beta \rangle = 0$.

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An orthogonal set S of V is orthonormal if $\|\alpha\| = 1$ for all $\alpha \in S$.

Gram - Schmidt orthogonalization process

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Problem 3

Find an orthogonal basis of R^3 with standard inner product from the basis $B = \{\beta_1 = (3, 0, 4), \beta_2 = (-1, 0, 7), \beta_3 = (2, 9, 11)\}$ using Gram-Schimdt process.

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$$\alpha_1 = \beta_1 = (3,0,4); \quad \|\alpha_1\|^2 = 3^2 + 0^2 + 4^2 = 25$$

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$$\alpha_2 = (-1, 0, 7) - \frac{\langle (-1, 0, 7), (3, 0, 4) \rangle}{25} (3, 0, 4)$$

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$$\alpha_2 = (-1,0,7) - \frac{(-1 \times 3 + 0 \times 0 + 7 \times 4)}{25}(3,0,4)$$

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$$\alpha_1 = \beta_1 = (3,0,4); \quad \|\alpha_1\|^2 = 3^2 + 0^2 + 4^2 = 25$$

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$$\alpha_2 = (-4,0,3);$$

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$$\alpha_1 = \beta_1 = (3,0,4); \quad \|\alpha_1\|^2 = 3^2 + 0^2 + 4^2 = 25$$

$$\alpha_2 = \beta_2 - \frac{\langle \beta_2, \alpha_1 \rangle}{\|\alpha_1\|^2} \alpha_1$$

$$\alpha_2 = (-1,0,7) - \frac{\langle (-1,0,7),(3,0,4) \rangle}{25}(3,0,4)$$

$$\alpha_2 = (-1,0,7) - \frac{(-1 \times 3 + 0 \times 0 + 7 \times 4)}{25}(3,0,4)$$

$$\alpha_2 = (-4,0,3); \quad \|\alpha_2\|^2 = (-4)^2 + 0^2 + 3^2 = 25$$

Problem 3 contd.

$$\alpha_3 = \beta_3 - \frac{\langle \beta_3, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle \beta_3, \alpha_2 \rangle}{\|\alpha_2\|^2}\alpha_2$$

Problem 3 contd.

$$\alpha_3 = \beta_3 - \frac{\langle \beta_3, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle \beta_3, \alpha_2 \rangle}{\|\alpha_2\|^2}\alpha_2$$

$$\alpha_3 = (2,9,11) - \frac{\langle (2,9,11),(3,0,4) \rangle}{25}(3,0,4) - \frac{\langle (2,9,11),(-4,0,3) \rangle}{25}(-4,0$$

Problem 3 contd.

$$\alpha_3 = \beta_3 - \frac{\langle \beta_3, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle \beta_3, \alpha_2 \rangle}{\|\alpha_2\|^2}\alpha_2$$

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$$\alpha_3 = (2, 9, 11) - 2(3, 0, 4) - (-4, 0, 3) = (0, 9, 0);$$

Problem 3 contd.

$$\alpha_3 = \beta_3 - \frac{\langle \beta_3, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle \beta_3, \alpha_2 \rangle}{\|\alpha_2\|^2}\alpha_2$$

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$$\alpha_3 = (2, 9, 11) - 2(3, 0, 4) - (-4, 0, 3) = (0, 9, 0); \quad \|\alpha_3\|^2 = 81$$

Problem 3 contd.

$$B' = \{\alpha_1 = (3,0,4), \alpha_2 = (-4,0,3), \alpha_3 = (0,9,0)\}$$

is an orthogonal basis of R^3.

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Verification

$$\langle \alpha_1, \alpha_2 \rangle = 3 \times (-4) + 0 \times 0 + 4 \times 3 = 0$$

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$$\langle \alpha_1, \alpha_2 \rangle = 3 \times (-4) + 0 \times 0 + 4 \times 3 = 0$$

$$\langle \alpha_1, \alpha_3 \rangle = \langle \alpha_2, \alpha_3 \rangle = 0$$

Note that B' is a L.I. subset of R^3 and its an orthogonal basis of R^3.

Problem 3 contd.

$$B'' = \left\{ \frac{\alpha_1}{\|\alpha_1\|}, \frac{\alpha_2}{\|\alpha_2\|}, \frac{\alpha_3}{\|\alpha_3\|} \right\}$$

is an orthonormal basis of R^3.

Problem 3 contd.

$$B'' = \left\{ \frac{\alpha_1}{\|\alpha_1\|}, \frac{\alpha_2}{\|\alpha_2\|}, \frac{\alpha_3}{\|\alpha_3\|} \right\}$$

is an orthonormal basis of R^3.

$$B'' = \left\{ \frac{1}{5}(3,0,4), \frac{1}{5}(-4,0,3), (0,1,0) \right\}$$

Problem 4

Using Gram-Schimdt process, find an orthonormal basis for the Euclidean space R^3 from a given ordered basis $B = \{\beta_1 = (1, 1, 1), \beta_2 = (0, 1, 1), \beta_3 = (0, 0, 1)\}$.

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$$\alpha_1 = \beta_1 = (1,1,1,),$$

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$$\alpha_1 = \beta_1 = (1, 1, 1,), \quad \|\alpha_1\|^2 = 3$$

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Using Gram-Schimdt process, find an orthonormal basis for the Euclidean space R^3 from a given ordered basis $B = \{\beta_1 = (1,1,1), \beta_2 = (0,1,1), \beta_3 = (0,0,1)\}$.

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The orthonormal basis :

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$