MEZ2B2012

Indian Institute of Information Technology,
Design and Manufacturing Kancheepuram
MA1002 Linear Algebra

Date : 17/11/2023
Time : 09.30–12.30

End Semester Examination
Marks : 50

1. Show that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 8 & 10 \end{bmatrix}$ are row-equivalent. Find a 3×3 matrix P as product of elementary matrices such that $B = PA$. [4]

2. Let S be a non-empty subset of a vector space V over the field F. Then prove that the subspace spanned by the set S is the set of all linear combinations of vectors in S [4]

3. Show that the vectors

$$\alpha_1 = (1,1,0,0), \quad \alpha_2 = (0,0,1,1)$$
$$\alpha_3 = (1,0,0,4), \quad \alpha_4 = (0,0,0,2)$$

form a basis for R^4. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. [4]

4. Let $T : V \longrightarrow W$ be a linear transformation. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W. [4]

5. Let $B = \{\alpha_1 = (1,0,-1), \alpha_2 = (1,1,1), \alpha_3 = (1,0,0)\}$ and $B_1 = \{\beta_1 = (0,1), \beta_2 = (1,0)\}$ be ordered bases of R^3 and R^2 respectively. Let $T : R^3 \longrightarrow R^2$ be a linear transformation defined as $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$. Find the matrix A of T relative to the pair B and B_1. [3]

6. State and prove rank nullity theorem. [4]

7. Let $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for a fixed $\theta \in (0, \pi)$. (i) Find all eigen values of A_θ. (ii) Find all eigen spaces of A_θ corresponding to its eigen values. (iii) Find (if exists) an orthogonal 2×2 matrix P and a digonal 2×2 matrix D such that $A_\theta = PDP^{-1}$. Justify your answer. [7]

8. Let $\lambda_1, \lambda_2, \lambda_3$ be eigen values of a 3×3 real matrix A such that (i) $\lambda_i \neq \lambda_j$ for $1 \leq i < j \leq 3$, and (ii) $AX_i = \lambda_i X_i$ for some $X_i \neq 0$ for $1 \leq i \leq 3$. Prove or disprove that $\{X_1, X_2, X_3\}$ is a linearly independent set. [3]

9. State and prove Cauchy-Schwarz inequality. [4]

10. Let V be a finite dimensional real inner product space and let $T_\alpha : V \longrightarrow R$ be a function defined as $T_\alpha(\beta) = \langle \beta, \alpha \rangle$, where α is a fixed non-zero vector in V. (i) Show that T_α is a linear transformation, (ii) Find the dimension of the null space of T_α, and (iii) Find a basis of the null space of T_α when $V = R^3$ with standard inner product and $\alpha = (1, 1, 1)$ [6]

11. Show that all eigen values of real symmetric matrix are real. [2]

12. Let V be the vector space of all polynomials of degree at most three with real coefficients. Find an orthonormal basis of V from the basis $\{1, x, x^2, x^3\}$ using the inner product defined by [5]

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$$