# *MISCELLANEOUS PRACTICE QUESTIONS –SET 1*

1. Prove: $\dfrac{(-3)^n}{n!} \longrightarrow 0.$ (3)

2. Let $a_1 = 3$ and $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{3}{a_n}\right)$ for $n \geq 1$. Prove: $a_n \longrightarrow \sqrt{3}$. (4)

3. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Prove that for every $\epsilon > 0$, there is an integer $N$ such that $|\sum_{n=N}^{\infty} a_n| < \epsilon$. (2)

4. State and prove the integral test. (4)

5. Prove that the terms of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

   can be rearranged to diverge or to converge to any preassigned value. (4)

6. Prove that if the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = c \neq 0$, then it converges absolutely for all $x$ with $|x| < |c|$. (3)

7. Prove: $\lim_{x \to 0} \dfrac{\sin x}{x} = 1.$ (3)

8. Prove that if $f'(x) = 0$ at each $x$ of an open interval $(a, b)$, then $f(x) = C$ for all $x$ in $(a, b)$, where $C$ is a constant. (3)

9. Prove: $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx.$ (3)

10. Prove that if $f(x)$ is monotonically increasing on $[a, b]$, then it is Riemann-integrable on $[a, b]$. (3)

11. State and prove the fundamental theorem of calculus. (4)

SET 2

1. Prove the following:
   (a) $\dfrac{2^n}{n!} \longrightarrow 0$;  (b) $n^{\frac{1}{n}} \longrightarrow 1$. (3 + 2)

2. State and prove a necessary and sufficient condition for the convergence of a monotonically increasing sequence. (4)

3. Prove that the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. (4)

4. Let $a_n = \begin{cases} n/2^n & \text{if } n \text{ is a prime number} \\ 1/2^n & \text{otherwise.} \end{cases}$
   Prove that the series $\sum a_n$ converges. (3)

5. Discuss the convergence (absolute/conditional) of $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(\ln n)^p}$, where $p$ is any constant. (4)

6. Show that if $\sum |a_n|$ converges, then $\sum a_n$ converges. (3)

7. Find the radius and interval of convergence of the following power series:
   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$;  (b) $\displaystyle\sum_{n=0}^{\infty} (\ln x)^n$. (3 + 2)

8. Prove that any polynomial function is continuous. (2)

9. Prove that if $f'(x) = 0$ for all $x$ in $[a, b]$, then $f(x)$ is a constant function. (3)

10. Find the absolute maximum and minimum values of $f(x) = 3x^{2/3}$ defined on the interval $-27 \leq x \leq 8$. (3)

11. Consider a bounded function $f : [a, b] \to \mathbb{R}$. Prove that its lower Riemann integral is less than or equal to its upper Riemann integral. (3)

12. Consider $f(x) = x^2$ on the interval $[1, 3]$. Find a sequence of partitions $P_n$ of $[1, 3]$ such that $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} R(f, P_n)$. Hence prove that the function is Riemann integrable. Prove also that this common limit equals the Riemann integral $\displaystyle\int_1^3 x^2 dx$. (7)