# Invertible linear transformations

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# Invertible function

A function $f : X \longrightarrow Y$ is invertible if there exists a function
$g : Y \longrightarrow X$ such that
(i) $gof : X \longrightarrow X$ and
(ii) $fog : Y \longrightarrow Y$ are identity functions.

## Onto function

A function $f : X \longrightarrow Y$ is onto if range of $f$, $R(f) = Y$. That is every element in $Y$ has at least one pre-image under $f$.

# One to one (1:1) function

A function $f : X \longrightarrow Y$ is one to one if each element in $Y$ has at most one pre-image under $f$.

$$f(x) = f(y) \Longrightarrow x = y$$

# Lemma

A linear transformation : $V \longrightarrow W$ is invertible if and only if

(1) $T$ is $1:1$, that is, $T(\alpha) = T(\beta) \Longrightarrow \alpha = \beta$.

(2) $T$ is onto, that is, $R(T) = W$.

# Theorem 7 (Text book order)

Let $V$ and $W$ be two vector spaces over the field $F$ and let $T : V \longrightarrow W$ be a linear transformation. If $T$ is invertible, then the inverse function $T^{-1} : W \longrightarrow V$ is a linear transformation.

**Proof :** Suppose that $T : V \longrightarrow W$ is an invertible linear transformation. Then there exists a function $T^{-1} : W \longrightarrow V$ such that

(i) $TT^{-1} : W \longrightarrow W$ and
(ii) $T^{-1}T : V \longrightarrow V$ are identity functions.

It is enough to prove that

$T^{-1}(c\beta_1 + \beta_2) = cT^{-1}(\beta_1) + T^{-1}(\beta_2)$ for all $\beta_1, \beta_2 \in W, c \in F$

# Theorem 7 contd.

Let $\alpha_1 = T^{-1}(\beta_1)$ and $\alpha_2 = T^{-1}(\beta_2)$. Since $T$ is invertible, $\alpha_1, \alpha_2$ are unique vectors in $V$ such that $T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$. Since $T$ is a linear transformation,

$$T(c\alpha_1 + \alpha_2) = cT(\alpha_1) + T(\alpha_2) = c\beta_1 + \beta_2$$

$$\implies T^{-1}(c\beta_1 + \beta_2) = c\alpha_1 + \alpha_2 = cT^{-1}(\beta_1) + T^{-1}(\beta_2)$$

Hence $T^{-1}$ is a linear transformation.

# Non-singular linear transformation

A linear transformation $T : V \longrightarrow W$ is non-singular if

$$T(\alpha) = 0 \quad \Longrightarrow \quad \alpha = 0$$

# Lemma 1

If $T : V \longrightarrow W$ is a non-singular linear transformation, then $N(T) = \{0\}$.

**Proof :** Since $T$ is a linear transformation, $T(0) = 0$. Hence $0 \in N(T)$ and $\{0\} \subseteq N(T) - - - - - - -(1)$.

Let $\alpha \in N(T)$.$\Longrightarrow T(\alpha) = 0$. $\Longrightarrow \alpha = 0$, since $T$ is non-singular.

$\Longrightarrow \alpha \in \{0\}$ and thus $N(T) \subseteq \{0\} - - - -(2)$.

From (1) and (2)

$$N(T) = \{0\}$$

## Lemma 2

Let $T : V \longrightarrow W$ be a linear transformation. Then following statements are equivalent.

(1) $T$ is one to one.
(2) $T$ is non-singular.

**Proof :** (1) $\Longrightarrow$ (2).
Suppose that $T$ is one to one. Show that $T$ is non-singular.
Consider $T(\alpha) = 0 \quad = T(0)$ (Note that $T$ is a L.T.). Hence $T(\alpha) = T(0)$ and $T$ is one-one. $\Longrightarrow \alpha = 0$. $\Longrightarrow T$ is non-singular.
(2) $\Longrightarrow$ (1).
Suppose that $T$ is non-singular. Then $N(T) = \{0\}$, by previous Lemma. Consider $T(\alpha) = T(\beta)$ .
$\Longrightarrow T(\alpha - \beta) = T(\alpha) - T(\beta) = 0$. $\Longrightarrow \alpha - \beta \in N(T) = \{0\}$.
$\Longrightarrow \alpha - \beta = 0$. $\Longrightarrow \alpha = \beta$. $\Longrightarrow T$ is one to one.

## Theorem 8 (Non-singular linear transformations preserve linear independence)

Let $T : V \longrightarrow W$ be a linear transformation. Then $T$ is non-singular if and only if $T$ carries each linearly independent subset of $V$ onto a linearly independent subset of $W$.

**Proof :**

Case 1 : Suppose that $T$ is non-singular. By Lemma 1, $N(T) = \{0\}$. Let $S$ be a linearly independent subset of $V$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be distinct vectors in $S$. It is enough to show that $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_k)\}$ is a linearly independent subset of $W$. Consider

$$c_1 T(\alpha_1) + c_2 T(\alpha_2) + \ldots + c_k T(\alpha_k) = 0$$

$$\Longrightarrow T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k) = 0$$

# Theorem 8 contd.

Since $N(T) = \{0\}$,

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k = 0$$

Since $S$ is linearly independent and $\alpha_1, \alpha_2, \ldots, \alpha_k$ be distinct vectors in $S$,

$$c_1 = c_2 = \ldots = c_k = 0.$$

Hence $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_k)\}$ is a linearly independent subset of $W$ and it completes Case 1.

# Theorem 8 contd.

Case 2 : Suppose that $T$ carries linearly independent subset onto linearly independent subset. Show that $T$ is non-singular.
Consider $T(\alpha) = 0$. If $\alpha \neq 0$, then $T$ carries a linearly independent set $\{\alpha\}$ onto a linearly dependent set $\{T(\alpha)\} = \{0\}$, a contradiction. $\implies \alpha = 0$. $\implies T$ is non-singular.

# Problem 1

Let $T(x_1, x_2) = (x_1 + x_2, x_1)$ be a linear operator defined on $F^2$.
Find $T^{-1}$ if exists.
**Solution :**

$$T(x_1, x_2) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow T(X) = AX$$

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] = \left[I|A^{-1}\right]$$

$$T^{-1}(Z) = A^{-1}Z = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \Longrightarrow T^{-1}(z_1, z_2) = (z_2, z_1 - z_2)$$

# Problem 2

Find the inverse of a linear operator $T$ on $R^3$ defined as

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

**Solution :**

$$T(x_1, x_2, x_3) = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Problem 2 contd.

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = [I|A^{-1}]$$

$$T^{-1}(Y) = A^{-1}Y$$

$$T^{-1}(y_1, y_2, y_3) = \left(\frac{1}{3}y_1, \frac{1}{3}y_1 - y_2, -y_1 + y_2 + y_3\right)$$

# Theorem 9

Let $V$ and $W$ be finite dimensional vector spaces over the field such that dim $V =$ dim $W$. If $T : V \longrightarrow W$ is a linear transformation, the following are equivalent.

(i) $T$ is invertible.

(ii) $T$ is non-singular.

(iii) $T$ is onto, that is, $R(T) = W$.

**Proof :** Let dim $V =$ dim $W = n$.
By Rank-Nullity-Dimension Theorem,

$$\text{rank } (T) + \text{ nullity } (T) = n - - - - (1)$$

# Theorem 9 contd.

First we prove that $(i) \implies (ii) \implies (iii)$.

$T$ is invertible. $\implies$ $T$ is one to one. $\implies$ $T$ is non-singular (by Lemma 2). $\implies N(T) = \{0\}$ (by Lemma 1). $\implies$ nullity $(T) = 0$.
$\implies$ rank $(T) = n$, see (1) . $\implies$ dim $R(T) =$ dim $W$.
$\implies R(T) = W$ (Reason: $R(T) \subseteq W$ and dim $R(T) =$ dim $W$).
$\implies$ $T$ is onto.

Next we prove that $(iii) \implies (i)$.

$T$ is onto. $\implies R(T) = W$. $\implies$ rank $(T) =$ dim $W = n$.
$\implies$ nullity $(T) = 0$, see (1). $\implies N(T) = \{0\}$.

**Claim :** $T$ is one to one.

Let $T(\alpha) = T(\beta)$. $\implies T(\alpha - \beta) = 0$. $\alpha - \beta \in N(T) = \{0\}$.
$\implies \alpha = \beta$. $\implies$ $T$ is one to one. Note that $T$ is onto (assumption) and $T$ is one to one (above Claim ). Hence $T$ is invertible.

# Theorem 9A (Assignment)

Let $V$ and $W$ be finite dimensional vector spaces over the field such that dim $V =$ dim $W$. If $T : V \longrightarrow W$ is a linear transformation, the following are equivalent.

(i) $T$ is invertible.

(ii) $T$ is non-singular.

(iii) $T$ is onto, that is, $R(T) = W$.

(iv) If $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for $V$, then $\{T(\alpha_1), \ldots, T(\alpha_n)\}$ is a basis for $W$.

(v) There is some basis $\{\alpha_1, \ldots, \alpha_n\}$ for $V$ such that $\{T(\alpha_1), \ldots, T(\alpha_n)\}$ is a basis for $W$.