Interval Of Halving

Algorithm:

Step-I Choose a lower bound 'a' and upper bound 'b'

Choose also a small Number ϵ

Let $x_m = \frac{a+b}{2}$, $L_0 = L = b - a$

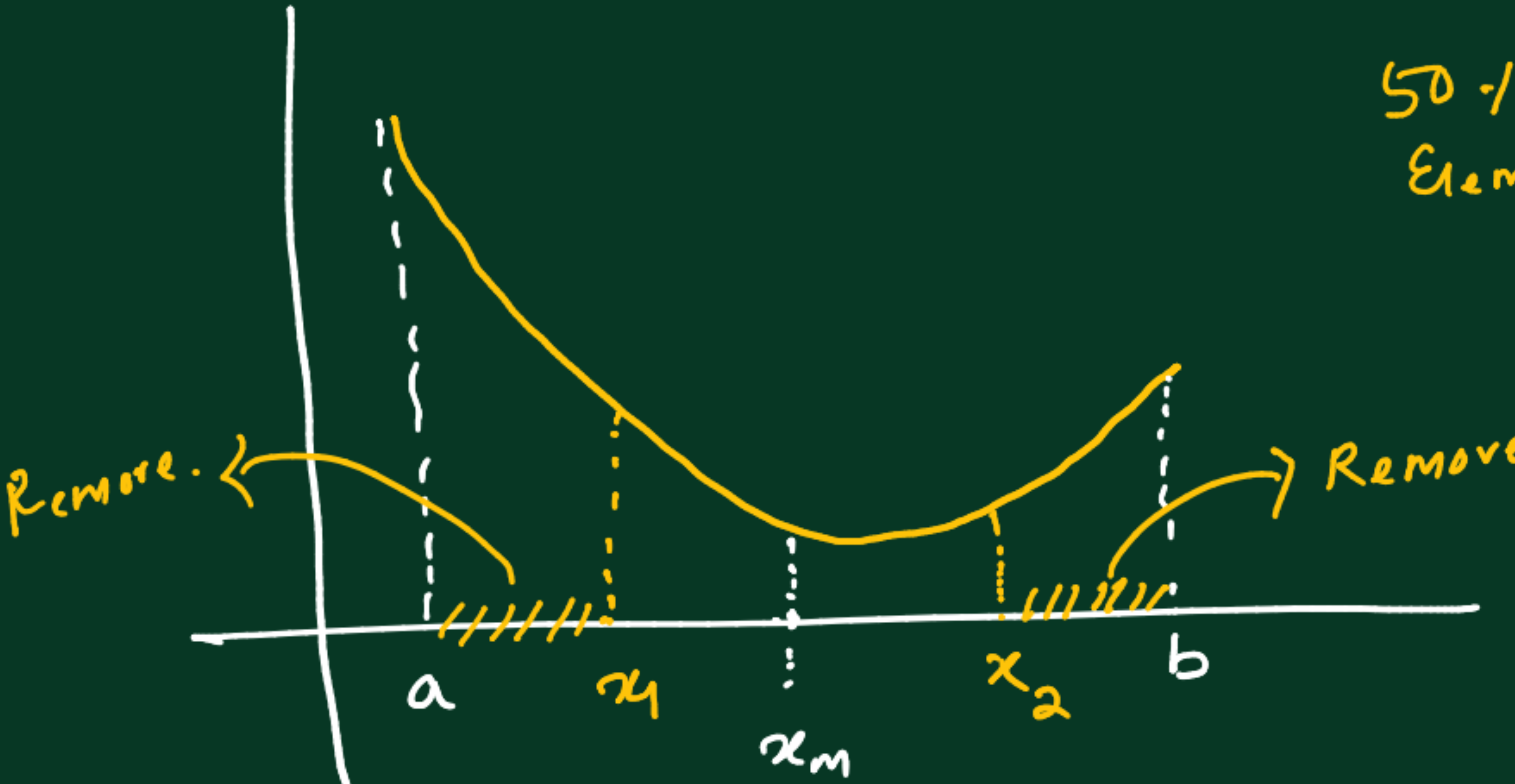
compute $f(x_m)$

$\boxed{\text{step-II}}$ Set $x_1 = a + L/4$,

$x_2 = b - L/4$

Compute $f(x_1)$ & $f(x_2)$

50 %
Eleminated



$$\left.\begin{aligned} f(x_1) &> f(x_m) \\ f(x_2) &> f(x_m) \end{aligned}\right\}$$

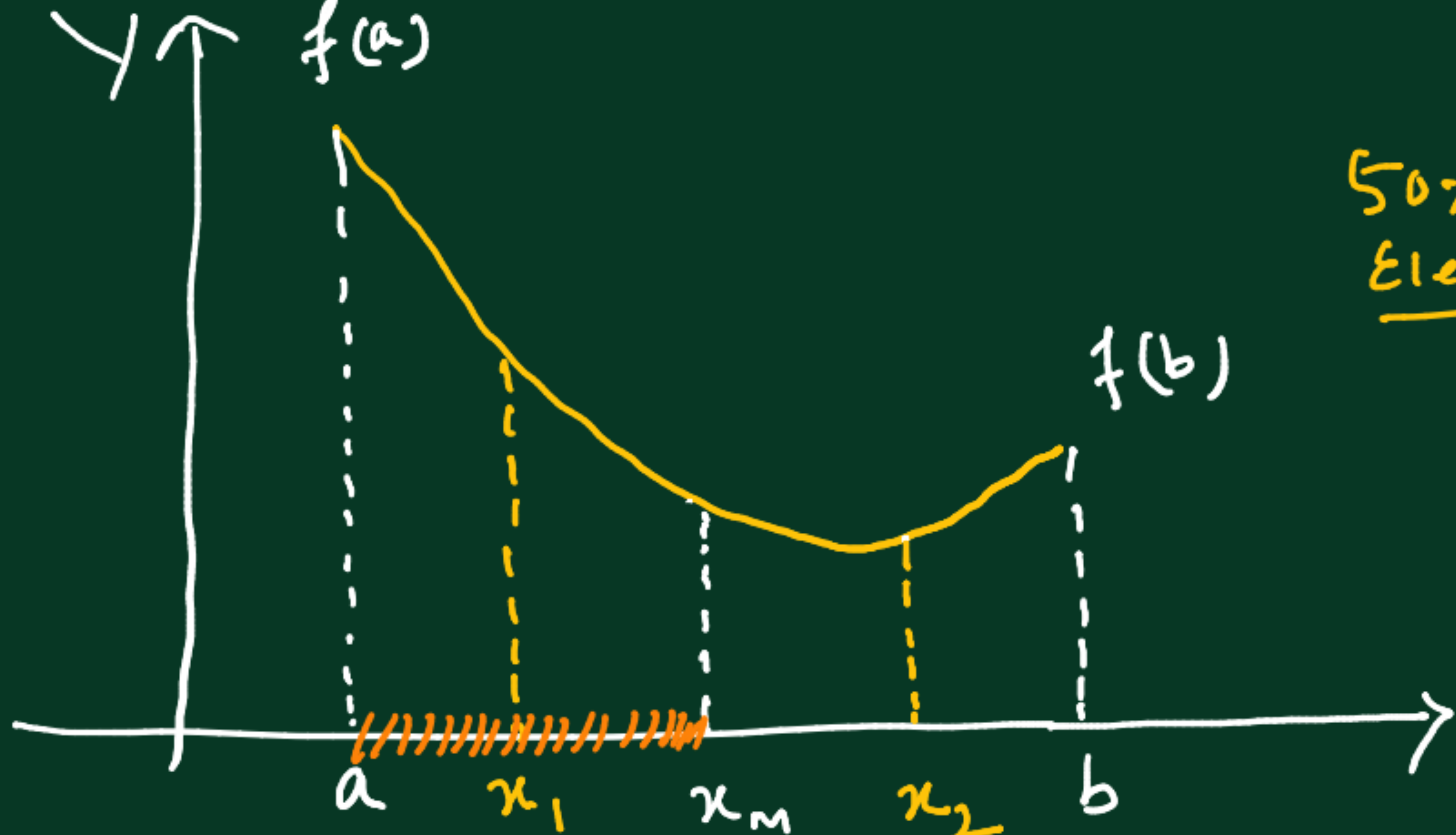
Then

$\boxed{a = x_1 \quad \& \quad b = x_2}$

$\boxed{\text{No change in } x_m}$

Next

$\boxed{\text{find } x_1 \ \& \ x_2}$

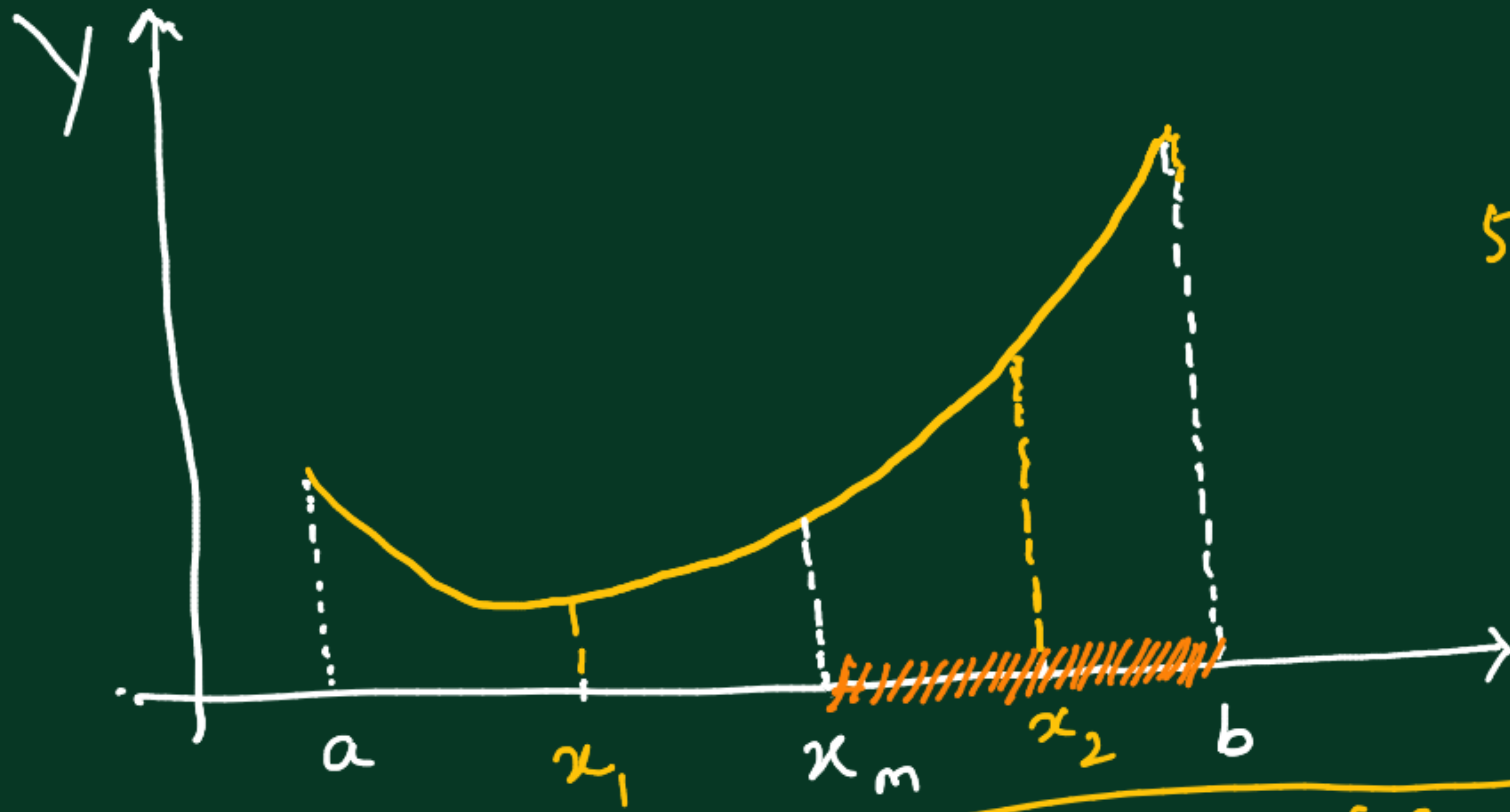


50%
Eleminated

$f(x_1) > f(x_m)$

$[a, x_m]$ is eleminated

For
Next step $\boxed{a = x_m}$ & $\boxed{x_m = x_2}$



50%
Eleminted

$\boxed{f(x_2) > f(x_m)}$

Remove $\rightarrow [x_m, b]$

For Next iteration:- $\boxed{b = x_m}$ & $\boxed{x_m = x_1}$

$\boxed{\text{Step-3}}$

If $f(x_1) < f(x_m)$

Set $b = x_m$, $x_m = x_1$

go to step (5) Else go to step (4)

$\boxed{\text{Step-4}}$

If $f(x_2) < f(x_m)$ set

$a = x_m$; $x_m = x_2$ go to Step (5)

Else set $a = x_1$, $b = x_2$ go to step (5)

$\boxed{\text{Step-5}}$

calculate $L = b - a$, If $|L| < \epsilon$

Terminate → Else go to Step-2.

Exercise:

$$\boxed{f(x) = x^2 + 54/x}$$

Step-1 $\quad a = 0$, $b = 5$, $\epsilon = 10^{-3}$

The point $x_m = \frac{0+5}{2} = 2.5$

Initial Interval $L_0 = 5 - 0 = 5$

$f(x_m) = 27.85$

Step.2 $\quad$ set $x_1 = 0 + 5/4 = 1.25$

$x_2 = 5 - 5/4 = 3.75$

$f(x_1) = 44.76$, $f(x_2) = 28.46$

Step.3 $\quad f(x_1) > f(x_m)$ thus continue Step-4

$\boxed{\text{Step-4}}$

Again $f(x_2) > f(x_m)$

Thus Drop the intervals

$(0, 1.25)$ & $(3.75, 5)$

Now $a = 1.25$ & $b = 3.75$

$\boxed{\text{Step-5}}$

New interval

$$L = 3.75 - 1.25$$
$$= \underline{2.5}$$

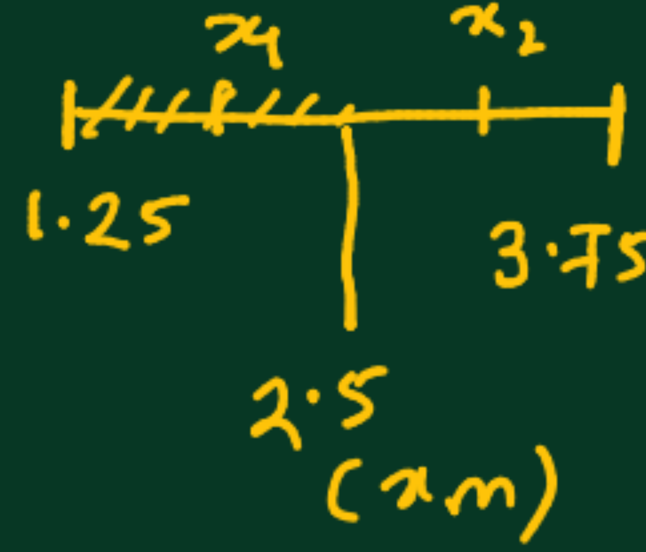
Exactly Half. of $L_0 = 5$

Stopping criteria → $|L| = 2.5 \nless \epsilon = 10^{-3}$

Go to $\boxed{\text{Step-2}}$

Step-2

Compute x_1 & x_2



$$\begin{cases} x_1 = 1.25 + \dfrac{2.5}{4} = 1.875 \\ x_2 = 3.75 - \dfrac{2.5}{4} = 3.125 \end{cases}$$

$f(x_1) = 32.32$ & $f(x_2) = 27.05$ respectively

Step-3

Test condition

$f(x_1) = 32.32 \geq$

$f(x_m) = 27.85$

Then go to step 4

Step-4 - Test condⁿ: $\rightarrow$ $f(x_2) = 27.05 < f(x_m) = 27.85$

Eleminate [1.25, 2.5]

$$x_2 \to x_m = 3.125$$

$$f(x_m) = 27.85$$

$$L = 3.75 - 2.5 = 1.25 \not< \epsilon = 10^{-3}$$

$\boxed{\text{Step-2}}$

$$x_1 = 2.5 + \frac{1.25}{4} = 2.5 + 0.3125 = 2.8125$$

$$x_2 = 3.4375$$

(*) End of each iteration interval Reduces half

After Iteration - 3 the interval $\left(\frac{1}{2}\right)^3 L_0 =$ 0.625

Per function evaluation

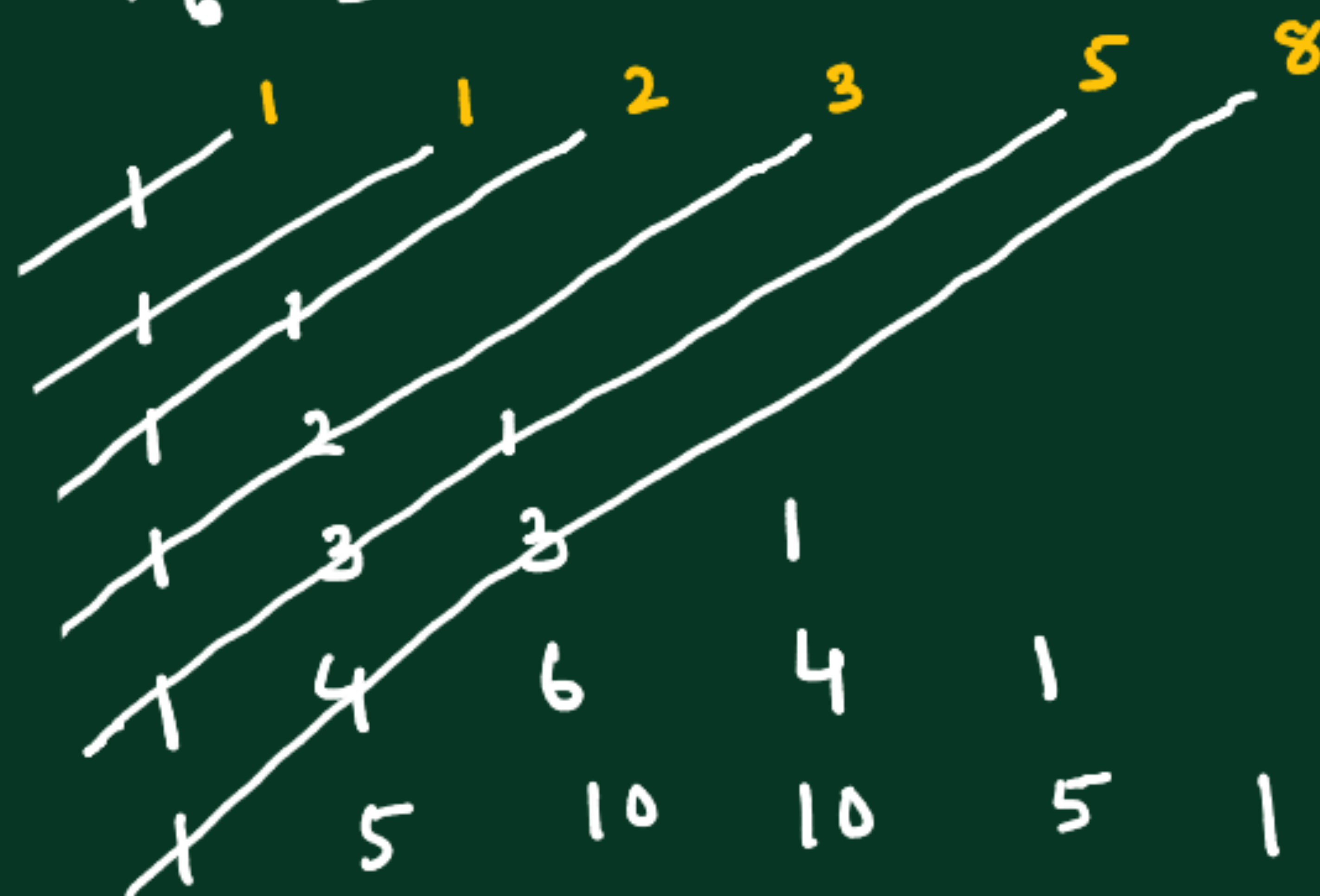
$$(0.5)^{n/2}(b-a) = \epsilon$$

Fibonacci Search Method

$$\boxed{F_n = F_{n-1} + F_{n-2}}$$

$F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$,

$F_5 = 8$, $F_6 = 13$ and so on.



Theorem:- $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1+\sqrt{5}}{2}$

Proof:-

$$u_{n+1} = u_n + u_{n-1}$$

$$\lim_{n\to\infty} \boxed{\frac{u_{n+1}}{u_n} = 1 + \frac{u_{n-1}}{u_n}} \Rightarrow L = 1 + \frac{1}{L}$$

$$\Rightarrow L^2 = L + 1 \Rightarrow L^2 - L - 1 = 0$$

$$L = \frac{1 \pm \sqrt{5}}{2}$$ (It can be only +ve)

$$L = \frac{1+\sqrt{5}}{2} \text{ only}$$

Fibonacci Search Method

Step-1

Choose lower bound (LB) a & UB 'b'

Set $L = b - a$

Assume n = Desired No of funⁿ evaluation

Step-2

Compute $L_k^* = \left(\frac{F_{n-k+1}}{F_{n+1}}\right) L$

Set $x_1 = a + L_k^*$ & $x_2 = b - L_k^*$

Step-3

Compute one $f(x_1)$ or $f(x_2)$ which was not been evaluated earlier.

$\boxed{\text{Step-4}}$

Is $\boxed{k=n}$? If no, set $k = k+1$ and go to Step 2

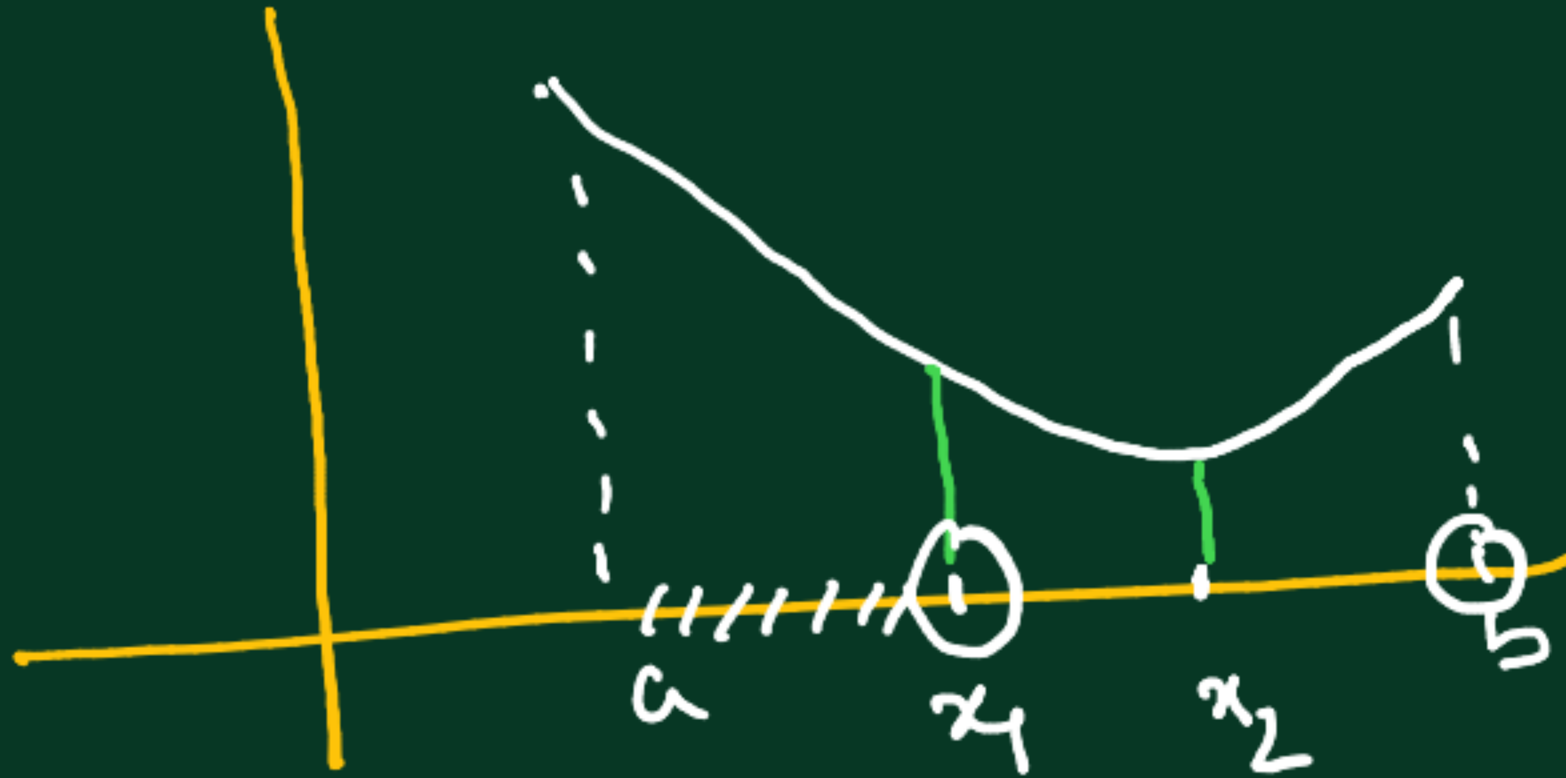
$$L - L_k^* = \underbrace{L - \left(\frac{F_{n-k+1}}{F_{n+1}}\right) L}$$

$K = 2, 3, 4, 5 \ldots$

for $k=3$: $\quad L_1 - (L_1) \dfrac{F_{n-2}}{F_{n+1}}$

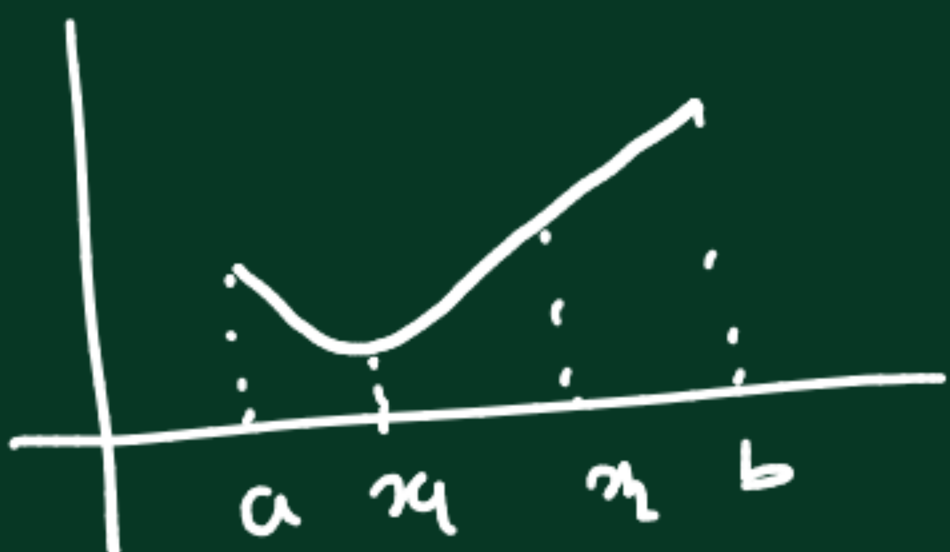
$$= L\left[\underline{\left(1 - \frac{F_{n-1}}{F_{n+1}}\right)} - \underline{\left(1 - \frac{F_{n-1}}{F_{n+1}}\right)}\left(\frac{F_{n-2}}{F_{n+1}}\right)\right]$$

then

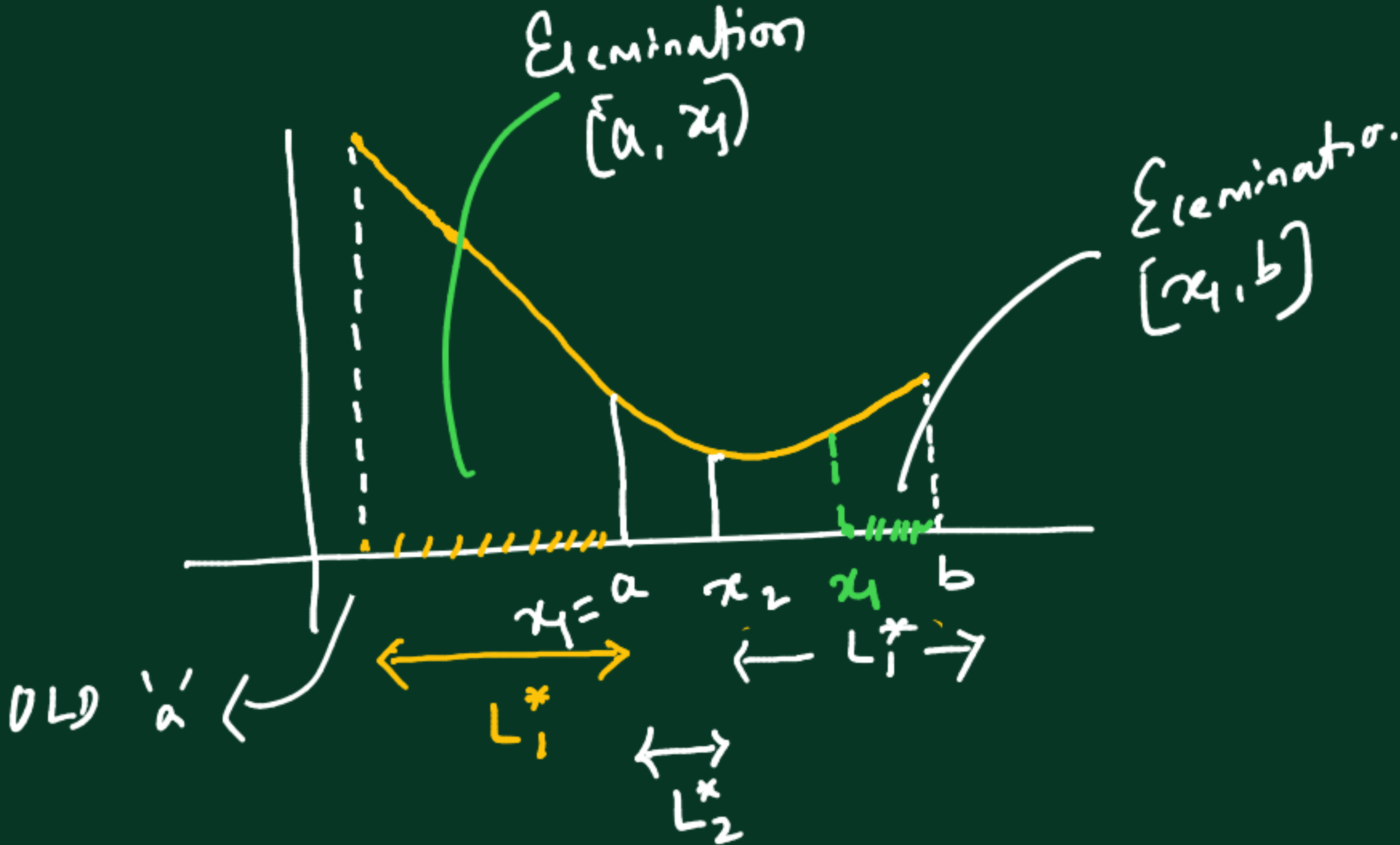


$f(x_1) > f(x_2) \rightarrow$ Eleminate $[a, x_1]$

$f(x_1) < f(x_2) \rightarrow$ Eleminate $[x_2 . b]$



L^* reduced



Elemination
[a, x1)
Elemination
[x1, b]
OLD 'a'
x1 = a
x2
x1
b
L1*
L1*
L2*

Example:

Minimize the fun^n

$$f(x) = x^2 + \frac{54}{x}$$

Step-1

We choose $a = 0$ and $b = 5$

Thus, the initial is $L = 5$.

Let $n = 3$ is the No. of function evaluation

Start with $\boxed{k=2}$

Step-2

We compute L_2^* as follows:

$$L_2^* = \left(F_{3-2+1} / F_{3+1}\right) L = \left(F_2 / F_4\right) \cdot 5$$

$$= 2/5 \cdot 5 = 2$$

Thus, we calculate $x_1 = 0+2 = 2$ —— (1)

and $x_2 = 5-2 = 3$ —— (2)

Step - 3

$f(x_1) = 31$ and $f(x_2) = 27$

$f(x_1) > f(x_2)$, we eliminate the region $[0, x_1]$ or $[0, 2]$.

For Next Level $a = 2$ & $b = 5$

interval : $(2, 5)$

Step-4

Since $k = 2 \neq n = 3$, we set $k = 3$ and go to step-2. This completes one iteration of the Fibonacci search method.

Step-2

$$L_3^* = \left(\frac{F_1}{F_4}\right)L = \frac{1}{5}\cdot 5 = 1$$

$x_1 = 2+1 = 3$

and $x_2 = 5-1 = 4$

Step-3

$x_1 = 3$ is already evaluated in previous iteration

Only evaluation point $x_2 = 4$, $f(x_2) = 29.5$

by comparing function values at $x_1 = 3$ and $x_2 = 4$, we observe that

$$\boxed{27 = f(2) = f(x_1) < f(x_2) = f(4) = 29.5}$$

Hence the eleminated part is [4.5]

For the next Level: $a = 2$ & $b = 4$

$\boxed{\text{Step-4}}$

At this iteration, $k = n = 3$
and we terminate the algorithm
The final interval is $(2,4)$

X

Interval reduces to $\left(\frac{2}{F_4}\right) L$ or $\left(\frac{2}{5} \times 5\right) = 2$

At iteration k, a proportion of F_{n-k} / F_{n-k+2} of search space of previous iteration is eleminated.

For large value of 'n' : $\boxed{38.2}$ %.

Which is better than 25%.

HW. – Find How F_{n-k} / F_{n-k+2} ?

$$L_k^* = \left(\frac{F_{n-k+1}}{F_{n+1}}\right) L$$ Eleminated part

& $L_k = L\left(\frac{F_{n-k+2}}{F_{n+1}}\right)$, is the remaing search space

$$L_k - L_k^*$$
$$= \left(\frac{F_{n-k+2} - F_{n-k+1}}{F_{n+1}}\right) L = \left(\frac{F_{n-k}}{F_{n+1}}\right) L = \frac{F_{n-(k+1)+1}}{F_{n+1}}$$
$$= L_{k+1}^*$$

x_1 & x_2 of k^{th} iteration are at distance L_{k+1}^*.