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MAT204T: Linear Algebra

Problem Set 7

1. Eigenvalues and eigenspace are invariant under similarity transformation.

2. If $A = PB$ for a invertible matrix P, then prove that $rank(A) = rank(B)$. Using this show that similarity matrices have same rank.
   Remark: Due to this property we say *Rank* is a *similarity invariant*

3. If A and B are similar, then the set of eigenvalues of A is equal to the set of eigenvalues of B. If the similarity transform is P (that is if $A = P^{-1}BP$) and (λ_0, X) is an eigenpair of A, then (λ_0, PX) is an eigenpair of B.
   Remark: Due to this property we say *eigenvalues* are *similarity invariants*

4. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the characteristic roots of A and let $f(\lambda)$ be a polynomial. Then show that $f(\lambda_1), f(\lambda_1), \cdots, f(\lambda_n)$ are the characteristic roots of $f(A)$.

5. If $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n = 0$ is a characteristic equation for $A_{n\times n}$. Then prove the following

   (a) Sum of all eigenvalues: $\lambda_1 + \lambda_2 + \cdots + \lambda_n = -c_1 = trace(A)$

   (b) Product of all eigenvalues: $\lambda_1\lambda_2\cdots\lambda_n = (-1)^n c_n = det(A)$

6. Compute

   (a) $\lim\limits_{n\to\infty} A^n$ for $A = \begin{pmatrix} 7/5 & 1/5 \\ -1 & 1/2 \end{pmatrix}$.

   (b) eigenvalue and eigenvectors of $A = \begin{pmatrix} n & 1 & 1 & 1 & 1 \\ 1 & n & 1 & 1 & 1 \\ 1 & 1 & n & 1 & 1 \\ 1 & 1 & 1 & n & 1 \\ 1 & 1 & 1 & 1 & n \end{pmatrix}$, for $n = 100$ and -4.

7. Show that a square matrix is orthogonally diagonalizable if and only if it is symmetric.
   Remark: In case of symmetric matrices the eigenvectors will make the matrix P as orthogonal matrix and the corresponding similarity transformation is $P^TAP = D$.

8. Let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \text{ then answer the following.}$$

(a) Show that the characteristic polynomial of A is $p(\lambda) = 4 - 4\lambda + \lambda^2$.

(b) Show that $p(A) = 0$

(c) Using the above result find the matrix A^6.

Remark: For every square matrix A, we always have $p(A) = 0$, where $p(\lambda)$ is the characteristic polynomial of A. This is by *Cayley-Hamilton Theorem.*

Definition :

(a) The set of all eigenvalues of T is called the eigenspectrum of T and is denoted by $\mathbf{eig}(T)$.

(b) The subspace $\mathbf{N}(T - \lambda I)$ of V is called the **eigenspace** of T with respect to the eigenvalue λ and is denoted by $\mathbf{E}(\lambda)$.

(c) For $\lambda \in \mathbf{eig}(T), \gamma(\lambda) := \mathbf{dimE}(\lambda)$ is called the geometric multiplicity of the eigenvalue λ.

9. Find $\mathbf{eig}(T)$, $\mathbf{E}(\lambda)$ and $\gamma(\lambda)$ of the following linear operators

(a) $T : \mathbb{F}^2 \to \mathbb{F}^2$ defined by $T(a, b) = (b, a)$.
Solution: Given that

$$T(a, b) = (b, a) \tag{1}$$

If λ is an eigenvalue of T with corresponding eigenvector (α, β), then

$$T(\alpha, \beta) = \lambda\,(\alpha, \beta) \tag{2}$$

Using (1) in (2), we get

$$\begin{aligned} & (\beta, \alpha) = \lambda\,(\alpha, \beta) \\ \Rightarrow \quad & \beta = \lambda\,\alpha \;\text{ and }\; \alpha = \lambda\,\beta. \end{aligned}$$

From the above two equation one can find the characteristic equation and characteristic roots as

$$(\lambda^2 - 1) = 0, \quad \Rightarrow \; \lambda = \pm 1.$$

Therefore, $\mathbf{eig}(T) = \{1, -1\}$.

Remark : $-$ To find the **eigenvector(v) corresponding to the eigenvalue(λ):**

Find $\mathrm{N}\,(T-\lambda\, I)$, where I is the identity map i.e the set of all such v so that $(T-\lambda\, I)\, v=\mathbf{0}$ and then find v.

For $\lambda = 1:$ we are looking for the null space of $(T-\ I)$

$$\begin{aligned}(T-\ I)\, v &= \mathbf{0}, \quad \text{where} \;\; v=(x,y)\in\mathbb{R}^2,\\ (y,x)-(x,y) &= (0,0)\\ \Rightarrow x &= y.\end{aligned}$$

$v=(x,y)=(x,x)=x(1,1)=\text{span}\{(1,1)\}$.

So, $\mathbf{E}(\lambda=1)=\text{span}\{(1,1)\}$ and $\gamma(1)=\mathbf{dim}\mathbf{E}(\lambda=1)=1$.

Similarly, for $\lambda=-1:$ we are looking for the null space of $(T+\ I)$

$$\begin{aligned}(T+\ I)\, w &= \mathbf{0}, \quad \text{where} \;\; w=(x,y)\in\mathbb{R}^2,\\ (y,x)+(x,y) &= (0,0)\\ \Rightarrow y &= -x.\end{aligned}$$

$w=(x,y)=(x,-x)=x(1,-1)=\text{span}\{(1,-1)\}$.

So, $\mathbf{E}(\lambda=-1)=\text{span}\{(1,-1)\}$ and $\gamma(-1)=1$.

(b) $T:\mathbb{F}^3\to\mathbb{F}^3$ defined by $T(a,b,c)=(0,5a,2c)$.
Solution: Given that

$$T(a,b,c)=(0,5a,2c) \tag{3}$$

If λ is an eigenvalue of T with corresponding eigenvector (α,β,γ), then

$$T(\alpha,\beta,\gamma)=\lambda\,(\alpha,\beta,\gamma) \tag{4}$$

Using (3) in (4), we get

$$(0,5\alpha,2\gamma)=\lambda\,(\alpha,\beta,\gamma)$$

Comparing component-wise, $0=\lambda\,\alpha, 5\alpha=\lambda\,\beta$ and $2\gamma=\lambda\gamma$.

The above three equation are reduced as two equations

$$\lambda^2\left(\frac{\beta}{5}\right)=0, \quad \text{and} \quad (\lambda-2)\gamma=0$$

From the above two equations, if we will have $\beta=0$ and $\gamma=0$ simultaneously, that will contradict the fact of nonzero eigenvector, which means the product of λ^2 and $\lambda-2$ is always zero for all nonzero eigenvector. Hence, we have the equation

$$\lambda^2(\lambda-2)=0,$$

which is the characteristic equation of the linear operator T. Therefore, $\mathbf{eig}(T) = \{0, 2\}$, and the eigenvalue 0 has multiplicity two.

For $\lambda = 0:$ we are looking for the null space of $(T - 0\,I)$

$$\begin{aligned} (T - 0\,I)\,v &= \mathbf{0}, \quad \text{where} \quad v = (x, y, z) \in \mathbb{R}^3, \\ T(x, y, z) &= (0, 0, 0), \\ (0, 5x, 2z) &= (0, 0, 0), \\ \Rightarrow x = 0 \quad &\text{and} \quad z = 0. \end{aligned}$$

$v = (x, y, z) = (0, y, 0) = y(0, 1, 0) = \text{span}\{(0, 1, 0)\}$.

So, $\mathbf{E}(0) = \text{span}\{(0, 1, 0)\}$ and $\gamma(0) = 1$.

Similarly, for $\lambda = 2:$ we are looking for the null space of $(T - 2\,I)$

$$\begin{aligned} (T - 2\,I)\,w &= \mathbf{0}, \quad \text{where} \quad w = (x, y, z) \in \mathbb{R}^3, \\ T(x, y, z) - 2(x, y, z) &= (0, 0, 0), \\ (-2x, 5x - 2y, 0) &= (0, 0, 0), \\ \Rightarrow x = 0 \quad &\text{and} \quad y = 0. \end{aligned}$$

$w = (x, y, z) = (0, 0, z) = z(0, 0, 1) = \text{span}\{(0, 0, 1)\}$.

So, $\mathbf{E}(2) = \text{span}\{(0, 0, 1)\}$ and $\gamma(2) = 1$.

Remark:- The eigenvalue $\lambda = 0$ has algebraic multiplicity 2, geometric multiplicity 1, hence the dimension of eigenspace is 1.

(c) $T : \mathbb{F}^2 \to \mathbb{F}^2$ defined by $T(a, b) = (-b, a)$.

(d) $T : \mathbb{F}^4 \to \mathbb{F}^4$ defined by $T(a, b, c, d) = (b, -a, d, -c)$.

(e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a, b, c) = (a + b, 2b, 3c)$.

(f) $T : \mathcal{P}\big([a, b], \mathbb{R}\big) \to \mathcal{P}\big([a, b], \mathbb{R}\big)$, where $\big(Tp\big)(t) = \dfrac{d}{dt}p(t)$ for $p(t) \in \mathcal{P}\big([a, b], \mathbb{R}\big)$.

10. Suppose S and T are linear operators on V, λ is an eigenvalue of S, and μ is an eigenvalue of T. Is it necessary that $\mu\lambda$ is an eigenvalue of ST ?

11. Let A be an $n \times n$ matrix and α be a scalar such that each row (or each column) sums to α. Show that α is an eigenvalue of A.

12. T is a linear operator on a real inner product space, where $(T - \alpha I)^2 + \beta^2 I$ is not invertible for some real numbers α and β.

    (a) If $T : \mathbb{R}^n \to \mathbb{R}^n$ exists, then find 'n',

    (b) Give an example of such operator T, and find their eigenvalues.

13. Give a linear operator whose characteristic polynomial is $(t - 5)^2\,(t - 6)^2$.

14. Eigenvalues of the special matrices are given in the table(next page):

Let us consider, $A = (a_{i,j})_{n\times n}$ is a square matrix.

Table 1: Prove the following results

Matrix	**Definition**	**Eigenvalue**
Hermitian	$A^* = A$	Real
Symmetric	$A^T = A$	Real
Skew-hermitian	$A^* = -A$	Either zero or Purely imaginary
Skew-symmetric	$A^T = -A$	Either zero or Purely imaginary
Unitary	$A^*A = I$	modulus is equal to one
Orthogonal	$A^TA = I$	modulus is equal to one
Involutory	$A^2 = I$	± 1
Idempotent	$A^2 = A$	0 and 1
Nilpotent (index-k)	$A^r \neq 0$ for $r = 1\ldots(k-1)$ but $A^k = 0$.	0
Diagonal	$\begin{cases} 0 & \text{if } i \neq j \\ a_{ij} & \text{if } i = j \end{cases}$	a_{ii} for $i = 1, 2, ..., n$
Stochastic matrix	$a_{ij} \geq 0$ and $a_{i1} + a_{i2} + \cdots + a_{in} = 1$, for all $1 \leq i, j \leq n$	If λ is an eigenvalue then, $\lvert\lambda\rvert \leq 1$ and at least one eigenvalue is equal to 1