MA1000: Calculus

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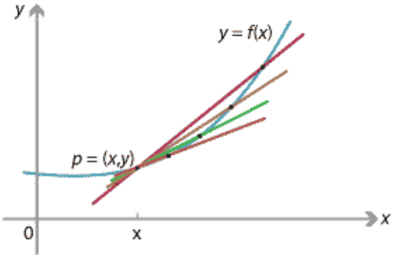
Differentiation

1. It is one of the main motivations for the concept of limit.
2. It tell us a how a curve bends at a point on the curve, by providing the slope of the curve at that point.
3. It gives us the rate of change of a function at a point.
4. It helps us to define many physical concepts such as velocity, acceleration and jerk with ease.

The Slope of a Curve, Informally

Consider a curve $y = f(x)$. Let P be a point on the curve. What is the slope of the curve at P? What is the tangent line to the curve at P?

1. Start with a neighbouring point Q on the curve.
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, take it as the **slope** of the curve at P.
4. And take the line through this P with the slope computed as the **tangent line** to the curve at P.



Note: The image is from the internet.

Example

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution: We begin with a secant line through $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$ nearby on the curive. (h may be positive or negative.)
The slope of this secant line is

$$\frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} = \frac{h^2 + 4h}{h} = h + 4.$$

Thus, as Q approaches P along the curve, h approaches 0 and the secant slope approaches 4:

$$\lim_{h \to 0} h + 4 = 4.$$

So, we take 4 as the parabola's slope at P. The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \quad \text{or} \quad y = 4x - 4.$$

Definition (Slope, Tangent Line)

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The tangent line to the curve at P is the line through P with this slope.

Example

Show that the line $y = mx + b$ is its own tangent at any point $(x_0, mx_0 + b)$.

Solution: Let $f(x) = mx + b$. We present the solution in three steps.

1. Here $f(x_0) = mx_0 + b$ and $f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b$.
2. And the slope at $(x_0, f(x_0))$ is

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h} = \lim_{h \to 0} \frac{mh}{h} = m.$$

3. Hence the tangent line is

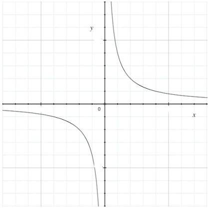
$$y = (mx_0 + b) + m(x - x_0) \quad \text{or} \quad y = mx_0 + b + mx - mx_0 \quad \text{or} \quad y = mx + b.$$

Example: Slope and Tangent to the Curve $y = 1/x, x \neq 0$

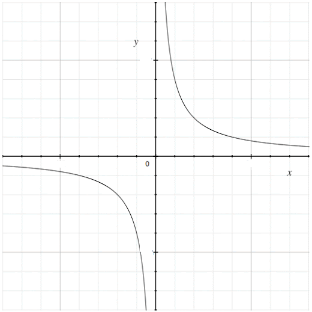
Find the slope of the curve $y = 1/x$ at $x = a \neq 0$. What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution: Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)} \\
&= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.
\end{aligned}$$



Example: Slope and Tangent to the Curve $y = 1/x, x \neq 0$



The slope at $x = a \neq 0$, $-\dfrac{1}{a^2}$, is always negative. As a approaches 0 from either direction, the slope approaches $-\infty$ and the tangent becomes increasingly steep.

As a moves away from the origin in either direction, the slope approaches 0^- and tangent becomes increasingly horizontal.

Definition (Derivative at a Point)

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is called the **difference quotient of** f **at** x_0 **with increment** h. If the difference quotient has a limit as h approaches 0, that limit is called the **derivative of** f **at** x_0.

Note:

1. If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point $x = x_0$.
2. If we interpret the difference quotient as an average rate of change, the derivative gives the function's rate of change with respect to x at the point $x = x_0$. (Recall the falling rock example.)

We have defined the derivative of a function $f(x)$ at a point x_0. We now define the derivative function:

Definition (Derivative Function)

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provide the limit exists.
Alternatively,

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Definition (Differetiability, Differentiation)

1. For a function $f(x)$, if $f'(x)$ exists at a particular point x, we say that f is **differentiable (has a derivative)** at x.
2. If f' exists at every point in the domain of f, we say that the function f **is differentiable**.
3. The process of calculating a derivative is called **differentiation**.
4. The derivative is alternatively denoted as

$$f'(x) = \frac{d}{dx} f(x).$$

Examples

We have seen that the derivative of $y = mx + b$ at any point x is m. Thus

$$\frac{d}{dx}(mx + b) = m.$$

For instance,

$$\frac{d}{dx}\left(\frac{3}{2}x - 4\right) = \frac{3}{2}.$$

We have seen that the derivative of $y = 1/x, x \neq 0$, at any $x \neq 0$ is $-1/x^2$. Thus

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Example

Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution: Here $f(x) = \dfrac{x}{x-1}$ and $f(x+h) = \dfrac{x+h}{x+h-1}$. So,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{1}{h} \frac{-h}{(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.
\end{aligned}
$$

Example

(a) Find the derivative of $y = \sqrt{x}$ for $x > 0$. (b) Also find the tangent line to this curve at $x = 4$.
Solution: (b) We use the alternative formula to calculate derivative:

$$\begin{aligned}
f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\
&= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Example

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Thus the tangent at $x = 4$ is the line through the point $(4, 2)$ with slope $1/4$:

$$y = 2 + \frac{1}{4}(x - 4) \quad \text{or} \quad y = \frac{1}{4}x + 1.$$

Notation

The many notations for the derivative of a function $y = f(x)$ with respect to x are:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x(f(x)).$$

Definition (One-Sided Derivatives)

The **right-hand derivative** of a function $f(x)$ at a function $x = x_0$ is the limit

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \quad \text{(provide the limit exists).}$$

The **left-hand derivative** of a function $f(x)$ at a function $x = x_0$ is the limit

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{(provide the limit exists).}$$

Theorem

A function $f(x)$ has a derivative at x_0 if and only if it has left-hand and right-hand derivatives at x_0 and these one-sided derivatives are equal.

Definition (Differentiable on an Interval)

A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and has right-hand derivative at a and left-hand derivative at b.

Examples

Show that the function $f(x) = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Solution: For $x > 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1.$$

For $x < 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1.$$

We show that $|x|$ is not differentiable at $x = 0$ by showing that the one-sided derivatives differ there:

$$
\begin{aligned}
\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \\
&= \lim_{h\to 0^+} 1 = 1.
\end{aligned}
$$

$$
\begin{aligned}
\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} \\
&= \lim_{h\to 0^+} \frac{-h}{h} \\
&= \lim_{h\to 0^+} -1 = -1.
\end{aligned}
$$

Homework

Show that $y = \sqrt{x}$ is not differentiable at $x = 0$. (Show that the right-hand derivative does not exist at $x = 0$.)

Differentiable Functions are Continuous

Theorem

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof.

Suppose f is differentiable at $x = c$: i.e., $f'(c)$ exists. We must show that $\lim_{x \to c} f(x) = f(c)$ or, equivalently, that $\lim_{h \to 0} f(c + h) = f(c)$. For $h \neq 0$,

$$f(c + h) = f(c) + (f(c + h) - f(c)) = f(c) + \frac{f(c + h) - f(c)}{h} \cdot h.$$

Now take limit as $h \to 0$:

$$\lim_{h \to 0} f(c + h) = \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{h \to 0} h = f(c) + f'(c) \cdot 0 = f(c).$$

□

The Intermediate Value Property of Derivatives

Theorem

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Examples: Thus the unit step function cannot be the derivative of any function.

Differentiation: Some Simple Results

1. If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

2. If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof(2):

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{z\to x}\frac{z^n - x^n}{z - x} \\
&= \lim_{z\to x}\frac{(z-x)(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \ldots + x^{n-1})}{z-x} \\
&= \lim_{z\to x}(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \ldots + x^{n-1}) \\
&= nx^{n-1}
\end{aligned}$$

Differentiation Rules

1. **Constant Multiple Rule:** If u is a differentiable function of x and c is a constant, the

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

2. **Sum Rule:** If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where both u and v are differentiable. At such cases,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

3. **Sum Rule for More Than Two Functions:** If $u_1, u_2, \ldots, u_n$ are differentiable functions of x, then their sum $u_1 + u_2 + \ldots + u_n$ is differentiable at every point where all of $u_1, u_2, \ldots, u_n$ u are differentiable. At such cases,

$$\frac{d}{dx}(u_1 + \ldots + u_n) = \frac{du_1}{dx} + \ldots + \frac{du_n}{dx}.$$

Exmaple

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so where?
Solution: The horizontal tangents, if any, occur where the slope dy/dx is zero. Now

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4^3 - 4x.$$

Now, $\dfrac{dy}{dx} = 0$ implies that $4^3 - 4x = 0$ or $4x(x^2 - 1) = 0$ or $x(x^2 - 1) = 0$.
So $x = 0, 1, -1$. v
Thus the curve has horizontal tangents at $(0, 2), (1, 1)$ and $(-1, 2)$ on the curve.

More Differentiation Rules

1. If u and v are differentiable at x, then so is their product uv and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

2. If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Proof: **Homework**

Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof.

Let $n = -m$, where m is a positive integer.
Then

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \frac{d}{dx}\left(\frac{1}{x^m}\right) \\
&= \frac{x^m \frac{d}{dx}(1) - 1\frac{d}{dx}(x^m)}{(x^m)^2} \\
&= \frac{0 - mx^{m-1}}{x^{2m}} \\
&= -mx^{-m-1} \\
&= nx^{n-1}
\end{aligned}$$

Homework

1. Find an equation for the tangent to the curve
$$y = x + \frac{2}{x}$$
at the point $(1, 3)$.
2. Find the derivative of
$$y = \frac{(x-1)(x^2-2x)}{x^4}.$$

Second and Higher Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function.
If f' is also differentiable, we can differentiate f' to get a new function of x denoted by f''.
So,

$$f'' = (f')'.$$

The function f'' is called the **second derivative** of f.
Notations:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

More generally, the ***n*th derivative** of $y = f(x)$ is the derivative of the $(n-1)$th derivative of f and is denoted by

$$f^{(n)} = y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y.$$

Homework

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Alternatively, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Homework

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Parametric Equations

Parametric Formula for dy/dx

If all three derivatives exist and $dy/dx \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Homework: Consider the curve described by a particle whose position $P(x, y)$ at time t is given by

$$x = a\cos t, \quad y = b\sin t, \quad 0 \leq t \leq 2\pi.$$

Find the line tangent to the curve at the point $(a/\sqrt{2}, b/\sqrt{2})$, where $t = \pi/4$.

Implicit Differentiation: Example

If p/q is a rational number, then $x^{p/q}$ is differentiable at every point where $x^{(p/q)-1}$ is defined and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

Proof: Let p and q be integers with $q > 0$ and suppose that $y = x^{p/q}$. Then

$$y^q = x^p.$$

Differentiating both sides with respect to x, we get

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

We divide both sides of the equation by qy^{q-1} and obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{px^{p-1}}{qy^{q-1}} \\
&= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}} \\
&= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} \\
&= \frac{p}{q} \cdot x^{(p-1)-(p-p/q)} \\
&= \frac{p}{q} \cdot x^{(p/q)-1}.
\end{aligned}$$

Definition (Linearization, Standard Linear Approximation)

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is called the **standard linear approximation** of f at a.

Example

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$.

Solution: Since

$$f'(x) = \frac{1}{2}(1+x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$. Thus the linearization at $x = 0$ is

$$L(x) = f(a) + f'(a)(x-a) = 1 + \frac{1}{2}(x-0) = 1 + \frac{x}{2}.$$

Definition (Differentials)

Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)dx.$$

Example: (a) Find dy if $y = x^5 + 37x$. (b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution: (a) $dy = (5x^4 + 37)dx$.

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we obtain

$$dy = (5 \cdot 1^4 + 37)(0.2) = 8.4.$$

Absolute Extrema

Definition (Absolute Maximum, Absolute Minimum)

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at a point c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Examples

Consider the function $y = x^2$. Absolute extrema depend on the domain D:

1. If $D = (-\infty, \infty)$, then the function has no absolute maximum. But it has absoute minimum of 0 at $x = 0$.
2. If $D = [0, 2]$, then the function has absolute maximum of 4 at $x = 2$ and absolute minimum of 0 at $x = 0$.
3. If $D = (0, 2]$, then the function has absolute maximum of 4 at $x = 2$ but it has no absolute minimum.
4. If If $D = (0, 2]$, then the function has no absolute extrema.

Theorem (The Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum valued m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$ and $f(x_2) = M$ and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

Local Extreme Values

Definition (Local Maximum, Local Minimum)

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

Theorem (The First Derivative Theorem for Local Extreme Values)

If f has a local maximum or minimum value at an interior point c of its domain and if f' is defined at c, then

$$f'(c) = 0.$$

Proof:

1. Suppose f is a local maximum at an interior point $x = c$ so that $f(x) - f(c) \leq 0$ for all values of x close enough to c.
2. Suppose $f'(c)$ is defined. Since c is an interior point, it then follows this derivative is defined by the two-sided limit
$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$
3. This means that the left-hand and right-hand derivatives exist at c and both equal $f'(c)$.
4. So, we have
$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$
and
$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$
5. These inequalities imply that $f'(c) = 0$.

Note: The converse of this theorem is not true: $y = x^3$ has derivative 0 at $x = 0$ but it is not a point of local extremum.

Definition (Critical Point)

An interior point of the domain of a function f where f' is zero or undefined is a critical point.

How to Find the Absolute Extrema of a Continuous Function f on a finite closed interval:

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Example

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution: The function is differentiable over the entire domain. So the only critical point is where $f'(x) = 2x = 0$; i.e, $x = 0$. Thus we must check the values at $x = 0$ and at the end points $x = -2$ and $x = 1$:
Critical point value: $f(0) = 0$
Endpoint values: $f(-2) = 4$ and $f(1) = 1$.
The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Homework

1. Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.
2. Find the absolute extrema values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Theorem (Rolle's Theorem)

Suppose that $y = f(x)$ *is continuous at every point of the closed inetrval* $[a, b]$ *and differentiable at every point of its interior* (a, b)*. If*

$$f(a) = f(b),$$

then there is at least one number c *in* (a, b) *at which*

$$f'(c) = 0.$$

Proof:

1. Since f is continuous on $[a, b]$, it assumes absolute maximum and minimum values on $[a, b]$.
2. By theorem these can occur (i) at interior points where f' is zero; (ii) at interior points where f' does not exist; (iii) at the endpoints a and b.
3. By hypothesis, f' exists at every interior point. So, the second possibility is ruled out.
4. If the minimum or maximum occur at an interior point c, then $f'(c) = 0$ (by theorem) and the Rolle's theorem follows.
5. If both the absolute maximum and absolute minimum occur at the end points, then since $f(a) = f(b)$ it follows that $f(x) = f(a) = f(b)$ for all x in $[a, b]$. So, it follows that $f'(x) = 0$ and that we can take any c from the interior (a, b).

Example

The function

$$f(x) = \frac{x^3}{3} - 3x$$

is continuous at every point of $[-3, 3]$ and is differentiable at every point of $(-3, 3)$. Also $f(-3) = f(3) = 0$.
Thus by Rolle's theorem there is at least one point c in $(-3, 3)$ where f' is zero.
In fact, there are two points, namely $x = -\sqrt{3}$ and $x = \sqrt{3}$ for which f' is zero.

Example

Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution: Let

$$f(x) = x^3 + 3x + 1.$$

Then the derivative

$$f'(x) = 3x^2 + 3$$

is never zero because it is always positive.
If the equation has two points $x = a$ and $x = b$ for which $f(x)$ is zero, then by Rolle's theorem, there should be a c between a and b such that $f'(c) = 0$.
But it must have a real root by the intermediate value theorem as $f(-1) = -3$ and $f(0) = 1$.

Theorem (The Mean Value Theorem)

Suppose $y = f(x)$ *is continuous on a closed interval* $[a, b]$ *and differentiable on the open interval* (a, b)*. Then there is at least one point* c *in* (a, b) *at which*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof:

1. Consider the line through the points $A(a, f(a))$ and $B(b, f(b))$. It is the graph of the function
$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$
2. The vertical difference between the graphs f and g at x is
$$h(x) = f(x) - g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$
3. The function $h(x)$ satisfies the hypothesis of Rolle's theorem on $[a, b]$ (?) and so there is a point c in (a, b) where $h'(c) = 0$
4. But
$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$
5. So, $h'(c) = 0$ implies that $f'(c) - \dfrac{f(b) - f(a)}{b - a} = 0$. That is
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollaries

Corollary

If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all x in $(a,)$, where C is a constant.

Corollary

If $f'(x) = g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all x in $(a,)$. That is, $f - g$ is a constant.