# Engineering Electromagnetics

## Lecture 25

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*by*


**Debolina Misra**

Department of Physics
IIITDM Kancheepuram, Chennai, India

# Current and field due to line charge

Line charge λ travelling down a wire with velocity **v**.

$$I = \lambda \mathbf{v}$$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl$$

Inasmuch as **I** and $d\mathbf{l}$ both point in the same direction, we can just as well write this as

$$\boxed{\mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B}).}$$

Typically, the current is constant (in magnitude) along the wire, and in that case $I$ comes outside the integral:

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B}).$$

# Due to volume charge density ρ

Current density ***J*** = ρ **v**

**J** = current/area; Volume charge density = ρ and velocity is **v**

The magnetic force on a volume current is therefore

$$\boxed{\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\rho\, d\tau = \int (\mathbf{J} \times \mathbf{B})\, d\tau}$$

# Due to surface charge density σ

In words, $K$ is the *current per unit width*. In particular, if the (mobile) surface charge density is $\sigma$ and its velocity is $\mathbf{v}$, then

$$\mathbf{K} = \sigma \mathbf{v}. \tag{5.23}$$

In general, $\mathbf{K}$ will vary from point to point over the surface, reflecting variations in $\sigma$ and/or $\mathbf{v}$. The magnetic force on the surface current is

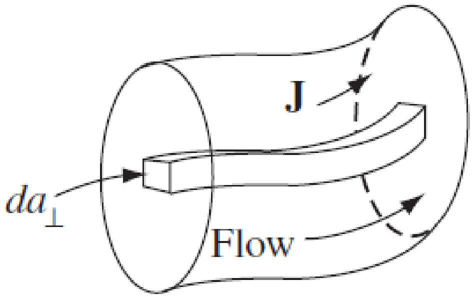
$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\sigma \, da = \int (\mathbf{K} \times \mathbf{B}) \, da. \tag{5.24}$$

# Continuity Equation

$$I = \int_S J \, da_\perp = \int_S \mathbf{J} \cdot d\mathbf{a}. \tag{5.28}$$

the charge per unit time leaving a volume $\mathcal{V}$ is

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau.$$




$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp}$$

*Because charge is conserved*, whatever flows out through the surface must come at the expense of what remains inside:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho \, d\tau = -\int_{\mathcal{V}} \left( \frac{\partial \rho}{\partial t} \right) d\tau.$$

(The minus sign reflects the fact that an *outward* flow *decreases* the charge left in $\mathcal{V}$.) Since this applies to *any* volume, we conclude that

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.} \tag{5.29}$$

This is the precise mathematical statement of local charge conservation; it is called the **continuity equation**.

# Steady Currents

*Stationary charges* produce electric fields that are constant in time; hence the term **electrostatics**.[8] *Steady currents* produce magnetic fields that are constant in time; the theory of steady currents is called **magnetostatics**.

| | | |
|---|---|---|
| **Stationary charges** | $\Rightarrow$ | **constant electric fields: electrostatics.** |
| **Steady currents** | $\Rightarrow$ | **constant magnetic fields: magnetostatics.** |

By **steady current** I mean a continuous flow that has been going on forever, without change and without charge piling up anywhere. (Some people call them "stationary currents"; to my ear, that's a contradiction in terms.) Formally, electro/magnetostatics is the régime

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0},$$

at all places and all times.

When a steady current flows in a wire, its magnitude $I$ must be the same all along the line; otherwise, charge would be piling up somewhere, and it wouldn't be a steady current. More generally, since $\partial \rho/\partial t = 0$ in magnetostatics, the continuity equation (5.29) becomes

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

$$\nabla \cdot \mathbf{J} = 0. \tag{5.33}$$

# Bio-savart’s law

The magnetic field of a steady line current is given by the **Biot-Savart law**:

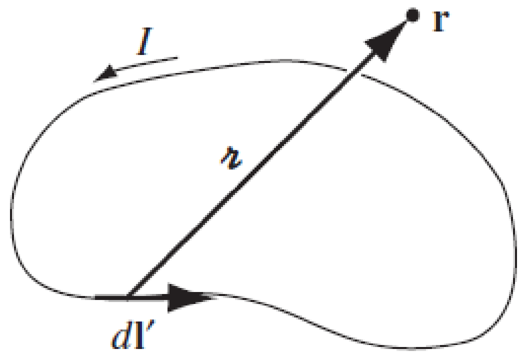
$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.} \tag{5.34}$$

The integration is along the current path, in the direction of the flow; $d\mathbf{l}'$ is an element of length along the wire, and $\boldsymbol{\mathscr{r}}$, as always, is the vector from the source to the point $\mathbf{r}$ (Fig. 5.17). The constant $\mu_0$ is called the **permeability of free space**:[9]

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}. \tag{5.35}$$

These units are such that $\mathbf{B}$ itself comes out in newtons per ampere-meter (as required by the Lorentz force law), or **teslas** (T):[10]

$$1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}. \tag{5.36}$$




From Griffith’s book

**Example 5.5.** Find the magnetic field a distance $s$ from a long straight wire carrying a steady current $I$ (Fig. 5.18).

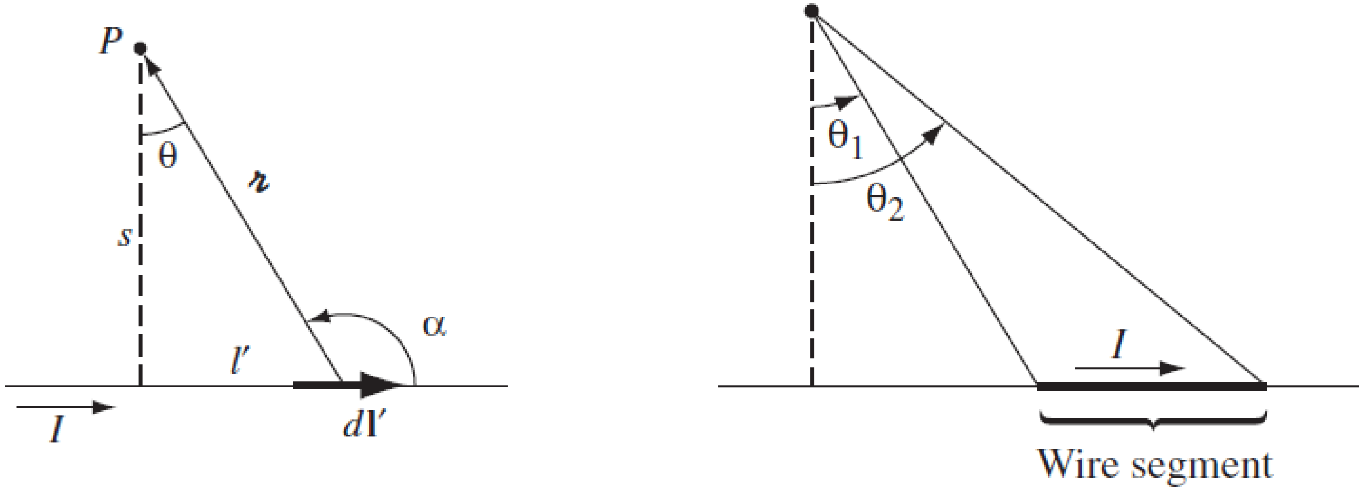
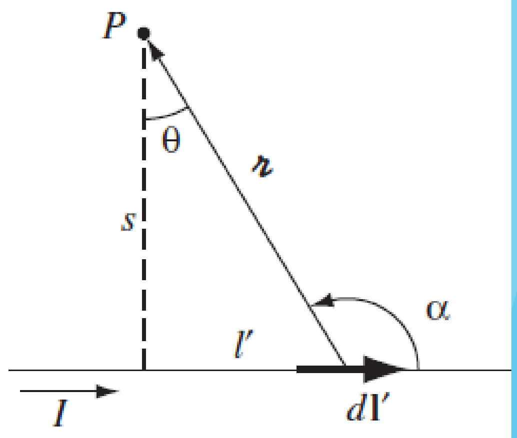



**FIGURE 5.18**

**Example 5.5.** Find the magnetic field a distance $s$ from a long straight wire carrying a steady current $I$ (Fig. 5.18).




In the diagram, $(d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}})$ points *out* of the page, and has the magnitude

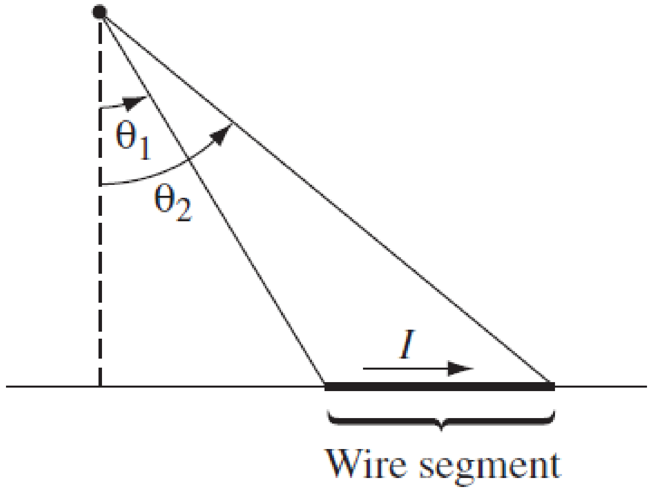
$$dl' \sin\alpha = dl' \cos\theta.$$

Also, $l' = s\tan\theta$, so
$$dl' = \frac{s}{\cos^2\theta}\,d\theta,$$

and $s = \mathscr{r}\cos\theta$, so
$$\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}.$$

Thus
$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\,d\theta$$

$$= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1). \quad (5.37)$$




Equation 5.37 gives the field of any straight segment of wire, in terms of the initial and final angles $\theta_1$ and $\theta_2$ (Fig. 5.19).



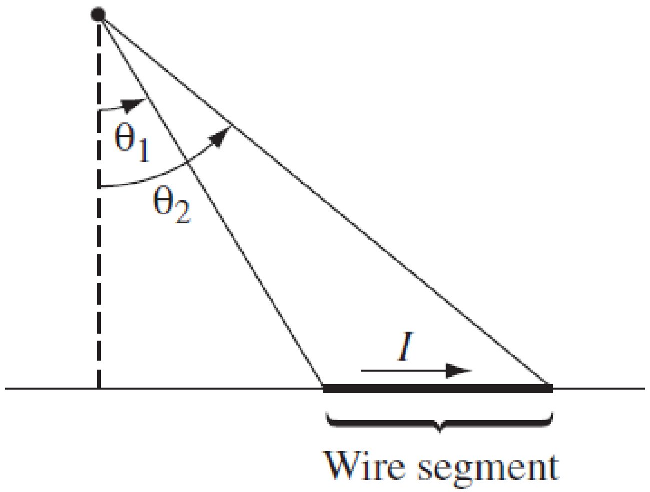
From Griffith’s book

# Infinite wire

In the case of an *infinite* wire,

$\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$, so we obtain

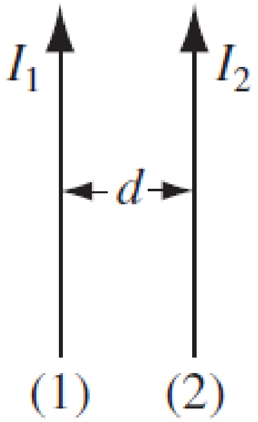
$$B = \frac{\mu_0 I}{2\pi s}.$$




From Griffith's book

# Force between two parallel wires

Force per unit length?




From Griffith’s book

# Force between two parallel wires

As an application, let's find the force of attraction between two long, parallel wires a distance $d$ apart, carrying currents $I_1$ and $I_2$ (Fig. 5.20). The field at (2) due to (1) is

$$B = \frac{\mu_0 I_1}{2\pi d},$$

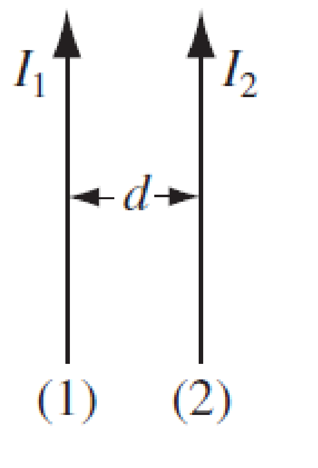
and it points into the page. The Lorentz force law (in the form appropriate to line currents, Eq. 5.17) predicts a force directed towards (1), of magnitude

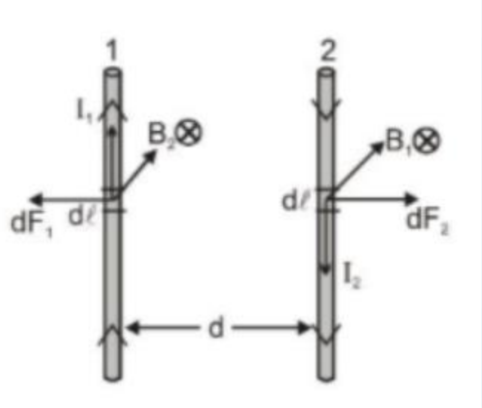
$$F = I_2 \left(\frac{\mu_0 I_1}{2\pi d}\right) \int dl.$$

The *total* force, not surprisingly, is infinite, but the force per unit length is

$$f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}. \tag{5.40}$$

If the currents are antiparallel (one up, one down), the force is repulsive—consistent again with the qualitative observations in Sect. 5.1.1.




From Griffith's book

**Example 5.6.** Find the magnetic field a distance $z$ above the center of a circular loop of radius $R$, which carries a steady current $I$ (Fig. 5.21).

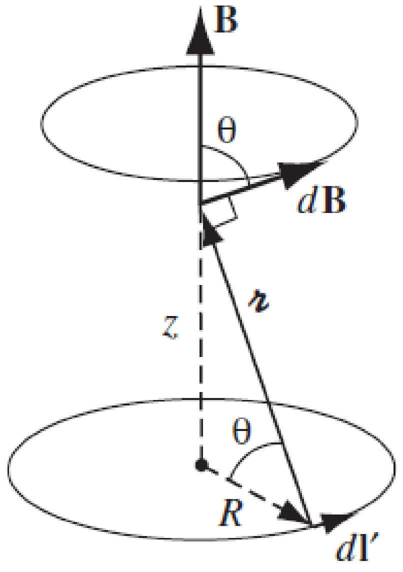



**FIGURE 5.21**

**Solution**

The field $d\mathbf{B}$ attributable to the segment $d\mathbf{l}'$ points as shown. As we integrate $d\mathbf{l}'$ around the loop, $d\mathbf{B}$ sweeps out a cone. The horizontal components cancel, and the vertical components combine, to give

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{\mathscr{r}^2} \cos\theta.$$

(Notice that $d\mathbf{l}'$ and $\boldsymbol{\mathscr{r}}$ are perpendicular, in this case; the factor of $\cos\theta$ projects out the vertical component.) Now, $\cos\theta$ and $\mathscr{r}^2$ are constants, and $\int dl'$ is simply the circumference, $2\pi R$, so

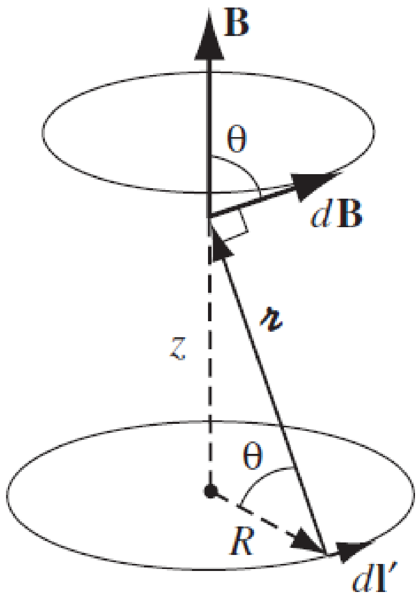
$$B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{\mathscr{r}^2}\right) 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}. \tag{5.41}$$

# Magnetic dipole moment

$$\boxed{B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{\mathscr{r}^2}\right)2\pi R = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}.}$$

magnetic flux density at the center of the loop as

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2R}\vec{\mathbf{a}}_z$$




When the point of observation is far away from the loop, we can approximate the term in the denominator of (5.7) as

$$(R^2+z^2)^{3/2} \approx z^3$$

and obtain the expression for the magnetic flux density as

$$\vec{\mathbf{B}} = \frac{\mu_0 I R^2}{2z^3}\vec{\mathbf{a}}_z$$

When the point of observation is far away from the loop, the size of the loop is very small in comparison with the distance $z$. In this case, we refer to the current-carrying loop as a *magnetic dipole*. If we define the *magnetic dipole moment* as

$$\boxed{\mathbf{m} = \mathrm{I}\pi R^2\overrightarrow{a_z}}$$

Q: If there are N turns?

$$\boxed{\vec{\mathbf{B}} = \frac{\mu_0\vec{\mathbf{m}}}{2\pi z^3}}$$

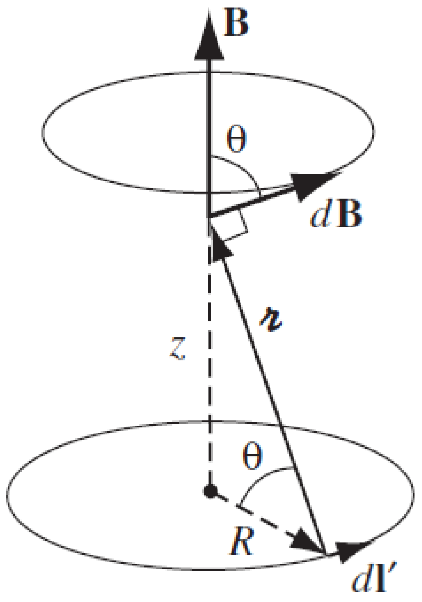
From Griffith's and BhagGuru books

# Magnetic dipole moment

$$B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{\imath^2}\right) 2\pi R = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}.$$

magnetic flux density at the center of the loop as

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2R}\vec{\mathbf{a}}_z$$




When the point of observation is far away from the loop, the size of the loop is very small in comparison with the distance $z$. In this case, we refer to the current-carrying loop as a *magnetic dipole*. If we define the *magnetic dipole moment* as

$$\mathbf{m} = I\pi R^2 \overrightarrow{a_z}$$

$$\vec{\mathbf{B}} = \frac{\mu_0 \vec{\mathbf{m}}}{2\pi z^3}$$

Q: If there are N turns?

Torque = ??



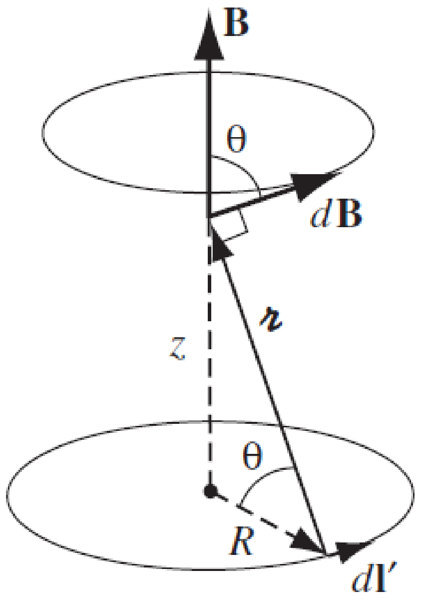
From Griffith's and BhagGuru books

# Magnetic dipole moment

$$\boxed{B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{\imath^2}\right)2\pi R = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}.}$$

magnetic flux density at the center of the loop as

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2R}\vec{\mathbf{a}}_z$$




When the point of observation is far away from the loop, the size of the loop is very small in comparison with the distance $z$. In this case, we refer to the current-carrying loop as a *magnetic dipole*. If we define the *magnetic dipole moment* as

$$\boxed{\mathbf{m} = I\pi R^2\overrightarrow{a_z}}$$

$$\boxed{\vec{\mathbf{B}} = \frac{\mu_0\vec{\mathbf{m}}}{2\pi z^3}}$$

Q: If there are N turns?

Torque = **m x B**



From Griffith's and BhagGuru books

# Infinite wire and integral of B along a path

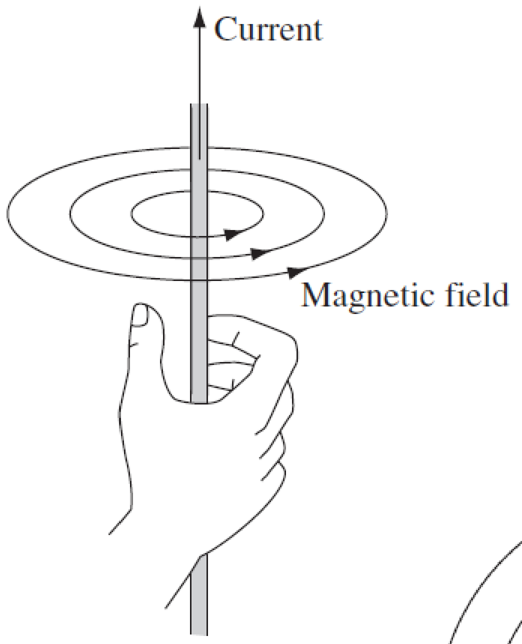
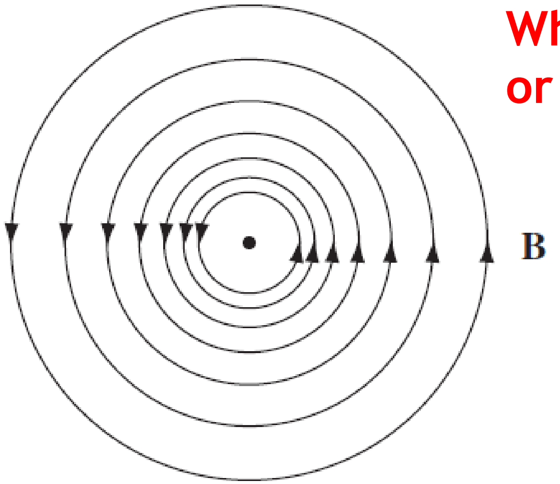



FIGURE 5.3

The magnetic field of an infinite straight wire is shown in Fig. 5.27 (the current is coming *out* of the page).

According to Eq. 5.38, the integral of **B** around a circular path of radius $s$, centered at the wire, is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\, dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I.$$

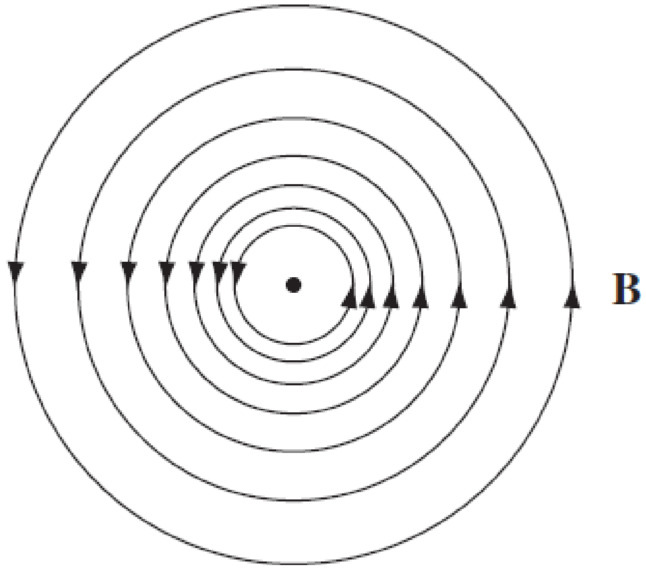



What do you think? B is rotational or irrotational?

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a},$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

# THE DIVERGENCE OF B




**B** is solenoidal or non-solenoidal?

$$\boldsymbol{\nabla} . \boldsymbol{B} = ?$$

# Thank You