Conditioning with random variables

Today's class

▶ Conditioning X on an event $A \in \mathcal{F}$.

▶ Conditional Expectation $E[X|A]$.

▶ Conditioning X with disjoint partitions $\{A_i\}$ of Ω.

▶ Conditioning X on an event $\{X \in A\} \in \mathcal{F}'$

▶ Conditioning X on another random variable Y.

▶ Conditional expectation $E[X|Y = y]$.

A new running example

▶ Pick 2 integers from $\{1, 2, 3\}$ without replacement.

▶ $\Omega = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$

▶ $\mathbb{P}\{\omega\} = \frac{1}{6}$ for all $\omega \in \Omega$.

▶ Denote them by random variables X and Y.

▶ For $\omega = (1,3)$ $X(\omega) = 1$ and $Y(\omega) = 3$.

▶ Write down their joint PMF $p_{X,Y}(x,y)$.

▶ Write down their marginal PMFs p_X and p_Y?

▶ What is $E[X], E[Y]$ and $E[XY]$?

Conditioning on an event A

▶ Consider a discrete r.v. X with pmf $p_X(x)$. Suppose an event A has happened where $A \in \mathcal{F}$.

▶ Consider event $\{\omega \in \Omega : X(\omega) = x\}$. We will use shorthand $\{X = x\}$.

▶ What is $\mathbb{P}(X = x|A)$? $\mathbb{P}(X = x|A) = \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}$.

$$p_{X|A}(x) := \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}.$$

▶ $p_{X|A}(x)$ denotes the conditional PMF of X under event A.

▶ In the running example say A is the event that the first number is odd and second is even. $A = \{(1,2),(3,2)\}$. Compute $p_{X|A}(\cdot)$.

▶ How do we know that it is consistent, i.e., $\sum_x p_{X|A}(x) = 1$?

Consistency of conditional PMF

$$\sum_x p_{X|A}(x) = 1.$$

Proof:

▶ $\sum_x p_{X|A}(x) = \sum_x \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}$

▶ $\{\omega \in \Omega : X(\omega) = x\}$ are disjoint sets for different x.

▶ From theorem of total probability, this implies that $\{X = x\} \cap A$ are disjoint sets for all x.

▶ $\sum_x p_{X|A}(x) = \frac{\mathbb{P}(\bigcup_x \{\{X=x\} \cap A)\}}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1.$ $\qquad\square$

Another Example

▶ Lets X denote the outcome of a dice.

▶ Let A denote the event that the roll is odd.

▶ What is $p_{X|A}(x)$?

▶ Given that event A has happened, what is the average value of the dice, i.e., $E[X|A]$?

$$E[X/A] = \sum_x x p_{X|A}(x).$$

Using LOTUS,

$$E[g(X)/A] = \sum_x g(x) p_{X|A}(x).$$

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Conditioning with disjoint partitions

▶ Now let $\{A_i, i = 1, 2, \ldots, n\}$ be a disjoint partition of Ω.

▶ Prove the following using law of total probability

$$p_X(x) = \sum_{i=1}^{n} \mathbb{P}(A_i) p_{X|A_i}(x)$$

Proof:

▶ $\sum_{i=1}^{n} \mathbb{P}(A_i) \frac{\mathbb{P}(\{X=x\} \cap A_i)}{\mathbb{P}(A_i)} = \sum_{i=1}^{n} \mathbb{P}(\{X = x\} \cap A_i) = \mathbb{P}(\{X = x\}).$ ☐

▶ The last equality follows from the law of total probability.

▶ An important consequence is the following.

$$E[X] = \sum_{i=1}^{n} \mathbb{P}(A_i) E[X|A_i]$$

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Conditioning on event $X \in A$

▶ Consider a discrete r.v. X with pmf $p_X(x)$. Suppose an event $X \in A$ has happened where $A \in \mathcal{F}'$.

▶ $X \in A = \{\omega \in \Omega : X(\omega) \in A\}$ and $\mathbb{P}\{X \in A\} = \sum_{x \in A} p_X(x)$.

▶ We will use the same notation $p_{X|A}(x) := \frac{\mathbb{P}(\{X=x\} \cap \{X \in A\})}{\mathbb{P}(X \in A)}$.

▶ If $x \notin A$, we have $p_{X|A}(x) = 0$.

▶ Otherwise (when $x \in A$,), we have $p_{X|A}(x) = \frac{p_X(x)}{\mathbb{P}(X \in A)}$.

▶ Running example: Suppose we are given $X \in A$ where $A = \{2, 3\}$. What is $p_{X|A}(x)$?

Revisiting Geometric random variable

▶ Let N be a geometric random variable with parameter p.

▶ Its pmf is $p_N(k) = (1 - p)^{k-1} p$.

▶ Suppose we are given the event $A := N > n$. $P(A) = (1 - p)^n$.

▶ What is $p_{N|A}(k)$?

▶ For $k > n$, $p_{N|A}(k) = \frac{P\{(N>n) \cap N=k\}}{P(N>n)} = (1 - p)^{k-1-n} p$. For $k \le n$, we have $p_{N|A}(k) = 0$.