Limits and Continuity

- How do we define limit of a sequence $\{a_1, a_2, \ldots, \}$?
- Notation: $\lim_{n \to \infty} a_n = L$.
- How do you define limit of a function at a point c?
- Notation: $\lim_{x \to c} f(x) = L$
- How do you define continuity of a function $f(x)$ at c ?
- When do you say a function is continuous ?
- (ϵ, δ)-definition of limits and continuity?

Limits and Continuity

Definition in terms of limits of sequences.

For a continuous function $f(\cdot)$, as $x \to c$, we have $f(x) \to f(c)$

For a continuous set-function S, as $A_n \to A$, we have $S(A_n) \to S(A)$

- Recall that $\mathbb{P}$ is a set-function. Is it continuous?
- We will see the proof shortly.

Sequence of sets

- Given $(\Omega, \mathcal{F})$, If $A_1 \subset A_2 \ldots$ is an increasing sequence of events defined on $\mathcal{F}$ and $\cup_{n=1}^{\infty} A_n = A \in \mathcal{F}$, then we say thats the sequence of sets A_n are increasing to A ($A_n \uparrow A$).

- Similarly when $A_1 \supset A_2 \ldots$ is a decreasing sequence of events and $\cap_{n=1}^{\infty} A_n = A$, then we have $A_n \downarrow A$.

- Alternative notation: For an increasing sequence of sets A_n we often write $\lim\limits_{n \to \infty} A_n$ for $\cup_{n=1}^{\infty} A_n$ and for a decreasing sequence of sets A_n that $\lim\limits_{n \to \infty} A_n = \cap_{n=1}^{\infty} A_n$.

Continuity of set-function $\mathbb{P}$

Lemma

For sequence of events of the type $A_n \uparrow A$ or $A_n \downarrow A$, we have $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Proof

- Consider increasing sequence first. Similar arguments follow for decreasing seq.
- Define $F_n = A_n - A_{n-1}$
- $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} F_n$.
- $\mathbb{P}(A) = \mathbb{P}(\cup_{n=1}^{\infty} A_n) = \mathbb{P}(\cup_{n=1}^{\infty} F_n)$
- But $\mathbb{P}(\cup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i) = \lim_{n\to\infty} \mathbb{P}(A_n)$. $\square$

Equivalently if $An \to \emptyset$, then $\mathbb{P}(A_n) \to 0$.

Conditional probability

- Given/If dice rolls odd, what is the probability that the outcome is 1?
- Given/If $\bar{\omega} \in [0, 0.5]$ what is the probability that $\bar{\omega} \in [0, 0.25]$?

The conditional probability of event B given event A is defined as $\mathbb{P}(B/A) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$ when $\mathbb{P}(A) > 0$.

Conditional probability

- Show that $P(A/B)P(B) = P(B/A)P(A)$.

- Bayes rule: $P(B/A) = \frac{P(A/B)P(B)}{P(A)}$.

- What is $P(A/(B \cap C))$?.This is also denoted as $P(A/BC)$

- Prove the chain rule
$P(A \cap B \cap C) = P(A)P(B/A)P(C/(AB))$.

HW: Prove the chain rule for conditional probability given by

$P(A_1 \cap A_2 \ldots A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1A_2)\ldots P(A_n/A_{n-1}\ldots A_1)$.

Conditional probability – Examples

- Suppose you draw 4 cards from a deck at random without replacement. What is the probability that (in order) these cards are 9 of club, 8 of diamond, king of spade and king of club?

- What if you do the above with replacement?

- Consider a finite sample space Ω where each outcome is equally likely. Then what is $P(B/A)$?

- $P(B/A) = \frac{|A \cap B|}{|A|}$.

Law of total probability

- $A = (A \cap B) \cup (A \cap B^c)$. What is $P(A)$?
- $P(A) = P(A \cap B) + P(A \cap B^c)$.
- This is same as $P(A) = P(A/B)P(B) + P(A/B^c)P(B^c)$.
- This formula is useful when $P(A)$ is not given or is difficult to find but $P(B)$ or $P(A/B)$ is readily available.

Let $B_1, B_2, \ldots B_n$ be the partition of the sample space Ω. Then for any event A we have

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A/B_i)P(B_i).$$

Example 1

- I have 3 bags that contain M marbles. Bag i has R_i red and B_i blue marbles respectively (for $i = 1, 2, 3$).
- I choose a bag at random and then draw a marble. What is the probability that the chosen marble is red ?
- Solution: $P(Red) = \sum_i P(Red/B_i)P(B_i)$

Example 2

1. If an item is defective, a robot can spot it with 98% accuracy.

2. If an item is not defective, a robot will declare it so with 99% accuracy.

3. A total of 0.1% items are defective.

4. If the robot says that the item you drew at random is defective, what is the probability that the robot is correct?

- $P(\text{defective}/\text{robot says defective}) = \frac{P(\text{robot says defective}/\text{defective})P(\text{defective})}{P(\text{robot says defective})}$

- What is $P(\text{robot says defective})$?

Bayes rule revisited

Let $B_1, B_2, \ldots B_n$ be the partition of the sample space Ω. Then for any event A with $P(A) > 0$ we have

$$P(B_j/A) = \frac{P(A/B_j)P(B_j)}{\sum_{i=1}^{n} P(A/B_i)P(B_i)}.$$

In the marble example, given that the marble drawn is red, what is the probability that bag 1 was chosen ?