CS 302.1 - Automata Theory

Lecture 07

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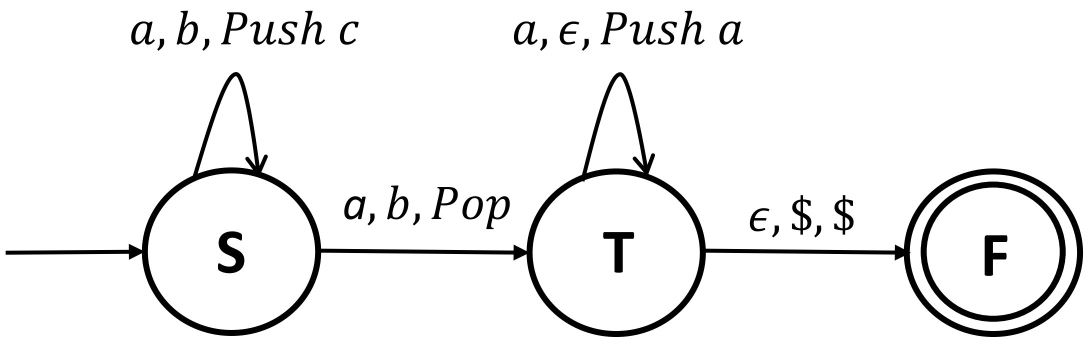
Quick Recap

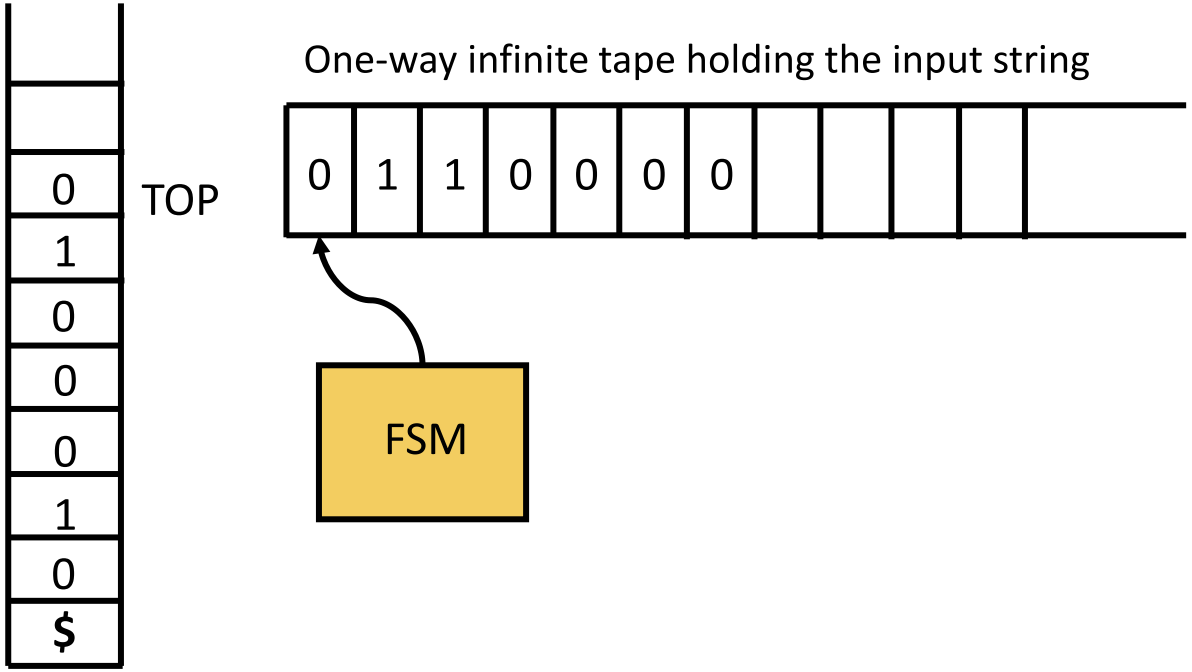
Pushdown Automata

- Automata that recognizes CFLs
- FSM + stack
- FSM transitions by reading an input symbol and by interacting with the stack

PDAs are **non-deterministic**.

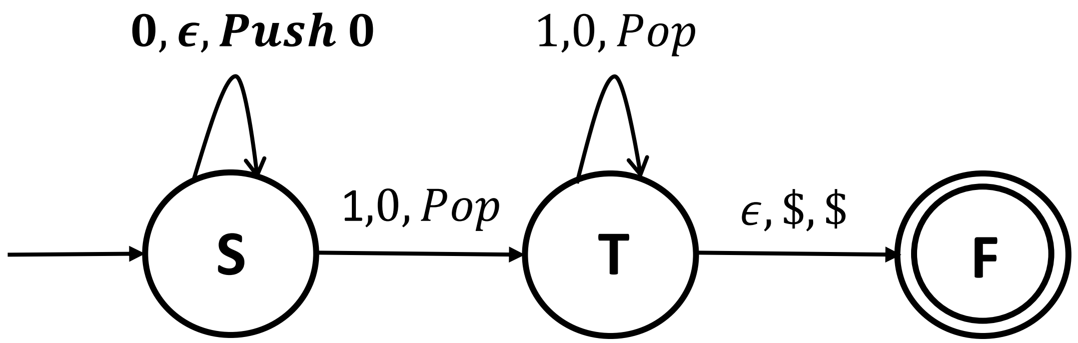
- Missing transitions
- ϵ-transitions
- Multiple transitions/input symbol possible





Pushdown Automata

What is the language recognized by this PDA?

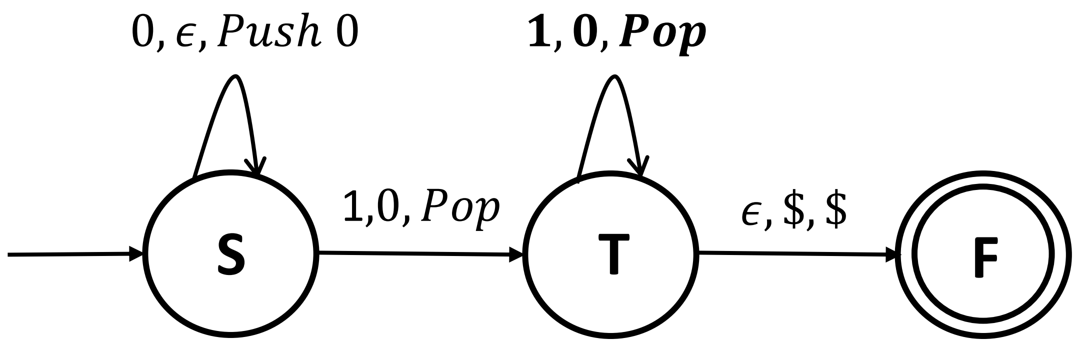


In some references (such as Sipser):

- The transitions of the PDA are labelled as "$\boldsymbol{a}, \boldsymbol{b} \rightarrow \boldsymbol{c}$", implying: If the input symbol read is $\boldsymbol{a}$, and the element at the top of the stack is $\boldsymbol{b}$ ($\boldsymbol{b}$ is popped), then push $\boldsymbol{c}$ on to the Stack.

Pushdown Automata

What is the language recognized by this PDA?



In some references (such as Sipser):

- The transitions of the PDA are labelled as "$\boldsymbol{a, b \to c}$", implying: If the input symbol read is $\boldsymbol{a}$, then pop $\boldsymbol{b}$ (the element at the top of the stack is $\boldsymbol{b}$) and push $\boldsymbol{c}$ on to the Stack.
- The label "$\boldsymbol{a, b \to \epsilon}$" implies that if the input symbol is $\boldsymbol{a}$ then **pop b**.

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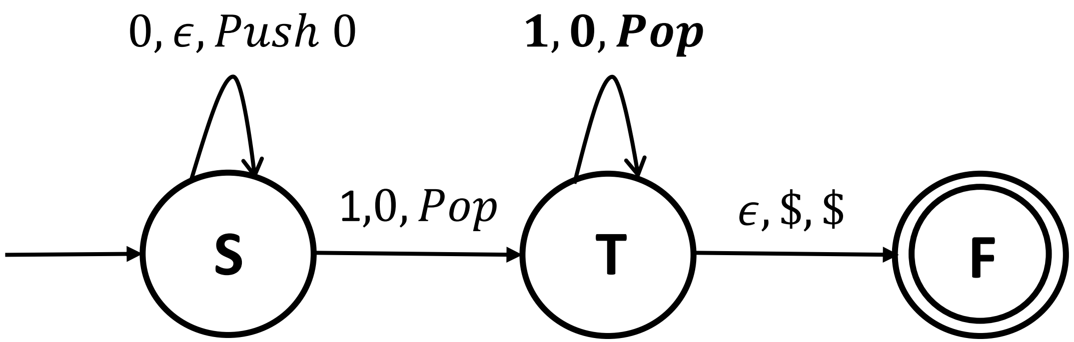

In some references (such as Sipser):

- The transitions of the PDA are labelled as "$\boldsymbol{a, b \rightarrow c}$", implying: If the input symbol read is $\boldsymbol{a}$, the element at the top of the stack is $\boldsymbol{b}$, then pop $\boldsymbol{b}$ and push $\boldsymbol{c}$ on to the Stack.

- The label "$\boldsymbol{a, b \rightarrow \epsilon}$" implies that if the input symbol is $\boldsymbol{a}$ and $\boldsymbol{b}$ is **popped**.

- The symbol signifying the bottom of the Stack $\$$ is pushed at the very beginning.

Pushdown Automata

Formally, a PDA M is a 6-tuple $(\mathrm{Q}, \Sigma, \Gamma, \delta, q_0, F\ )$ where

- Q is a finite set called the ***states.***
- Σ is the set of input ***alphabets.***
- Γ is the set of ***Stack alphabets***
- $\delta: Q \times \Sigma_\epsilon \times \Gamma_\epsilon \mapsto \mathcal{P}(Q \times \Gamma_\epsilon)$ is the ***transition function*** [$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon\ = \Gamma \cup \{\epsilon\}$]
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- It **reaches a final state** when the entire string is read.

OR

- The **stack is empty** when the entire string is read.

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[$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$]

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These two notions of acceptance are equivalent

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- If the input symbol read is a and a is popped, then Push a and remain at q_i : ?

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- The Language of the PDA P is the set of strings the PDA accepts, i.e.

$$L = \{w | P \text{ accepts } w\}$$

There exists an accepting run for $\boldsymbol{w}$ on $\boldsymbol{P}$

- If $\mathcal{L}(P) = L$, then the PDA P recognizes L

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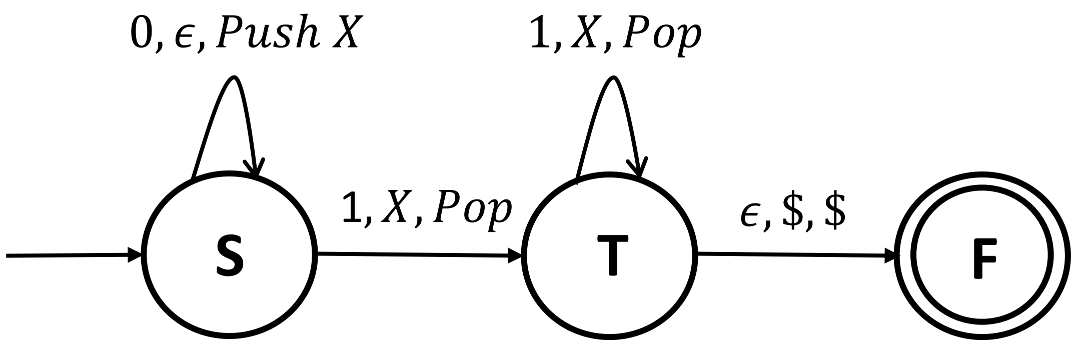
- Stack alphabet **can be different** from the input alphabet



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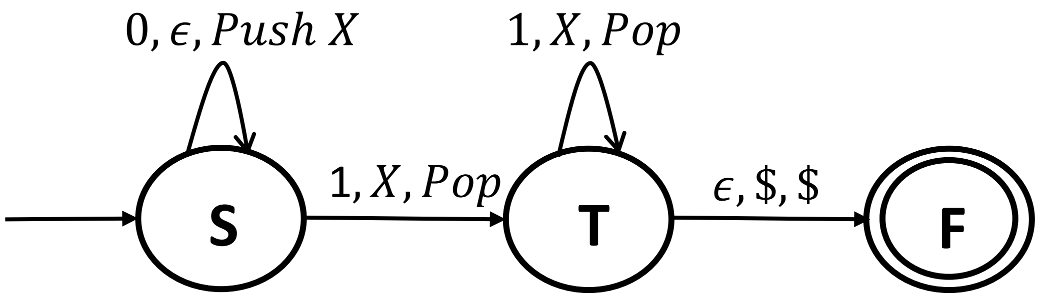

$\delta(S, 0, \epsilon) = (S, X)$

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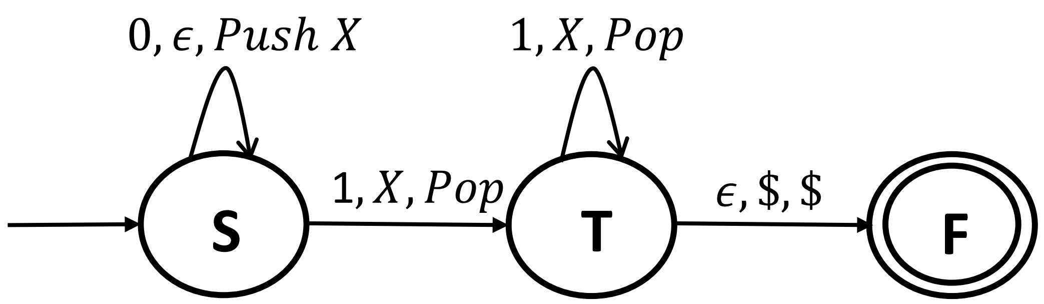

$$\delta(S, 0, \epsilon) = (S, X)$$
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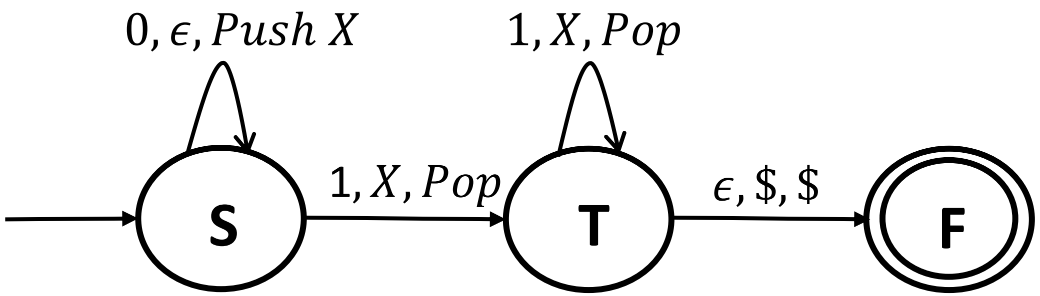

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Let $\Sigma = \{0,1\}$ consider the language $\boldsymbol{L} = \{\boldsymbol{w} \in \boldsymbol{\Sigma}^* \mid \boldsymbol{w}$ **is a Palindrome}**. Design a PDA P that recognizes L.

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- Push first half of the input string onto the stack.
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- The above intuition is applicable for even length palindromes of the form ww^R.

- What about odd length palindromes?
  - Non-determinism to the rescue once again

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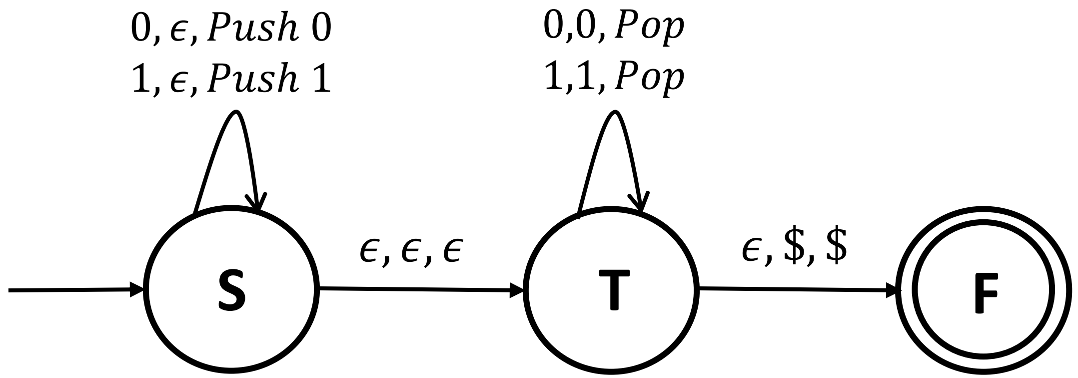

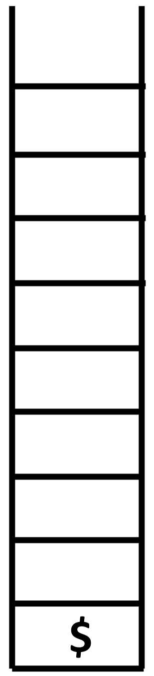

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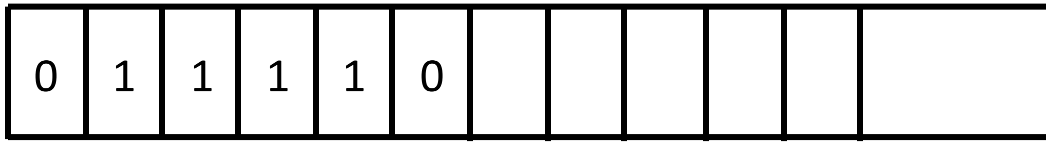

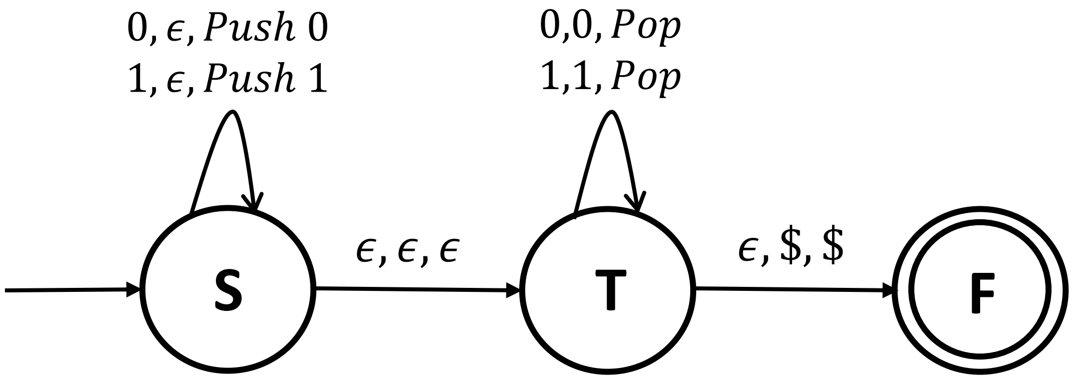

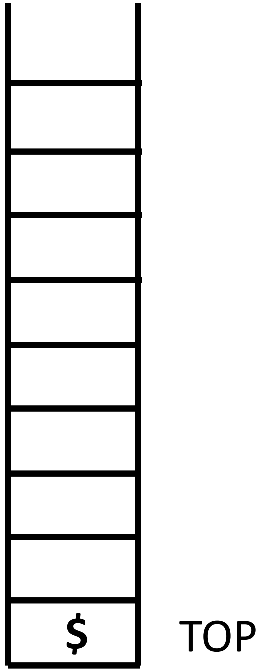

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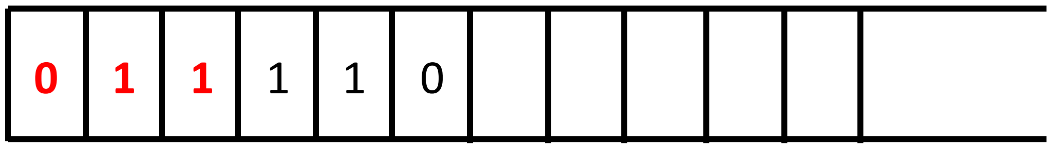

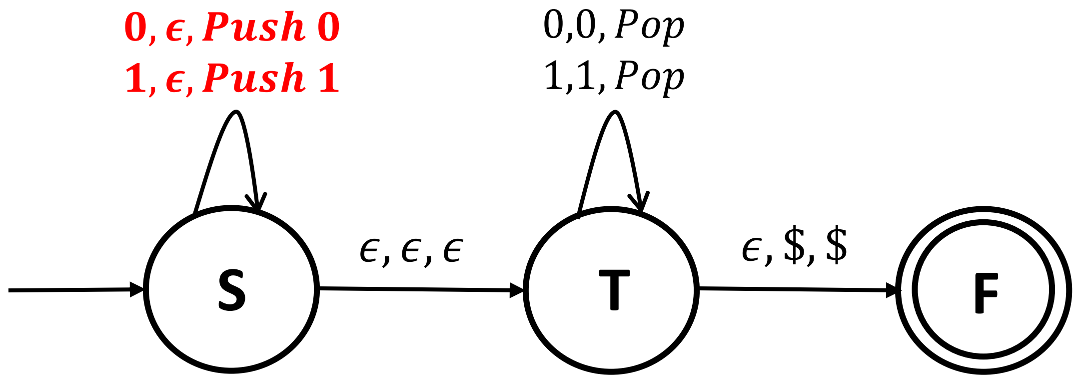

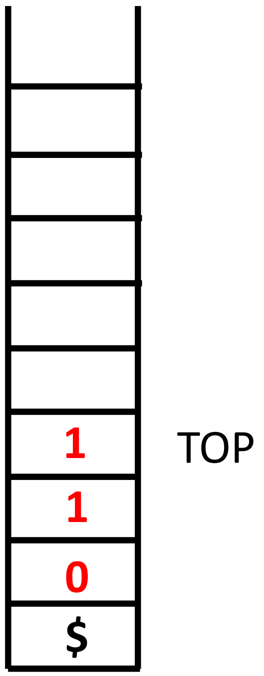

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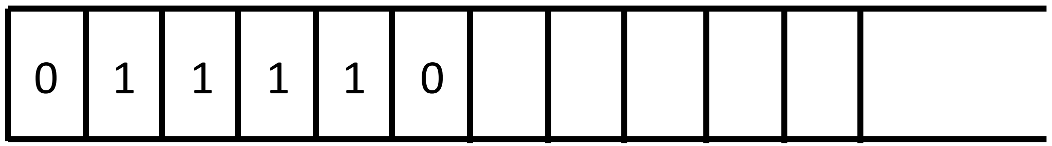

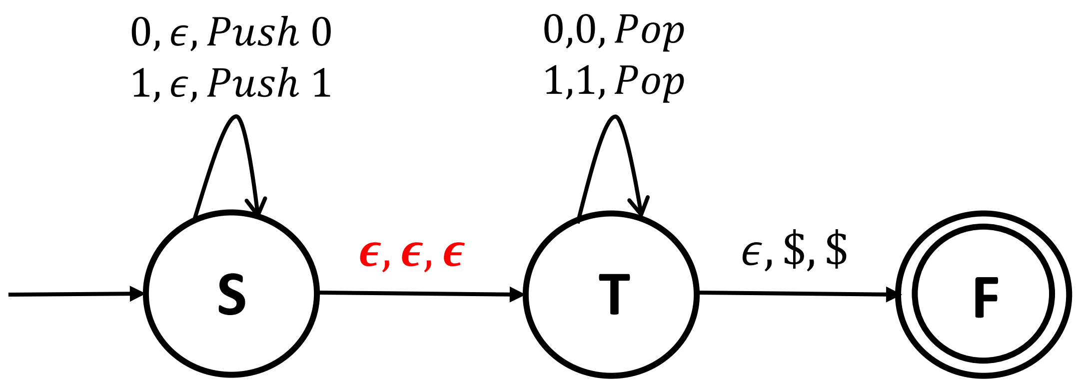

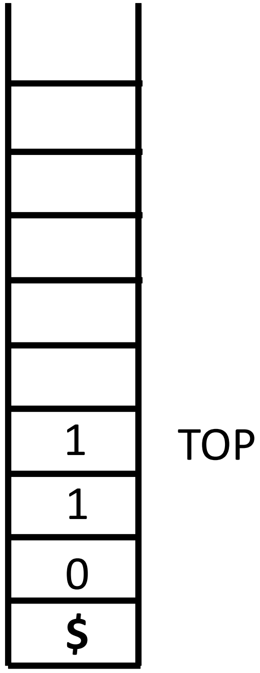

Pushdown Automata

Let $\Sigma = \{0,1\}$ consider the language $\boldsymbol{L} = \{\boldsymbol{w} \in \boldsymbol{\Sigma}^* \mid \boldsymbol{w}$ **is a Palindrome}**. Design a PDA P that recognizes L.

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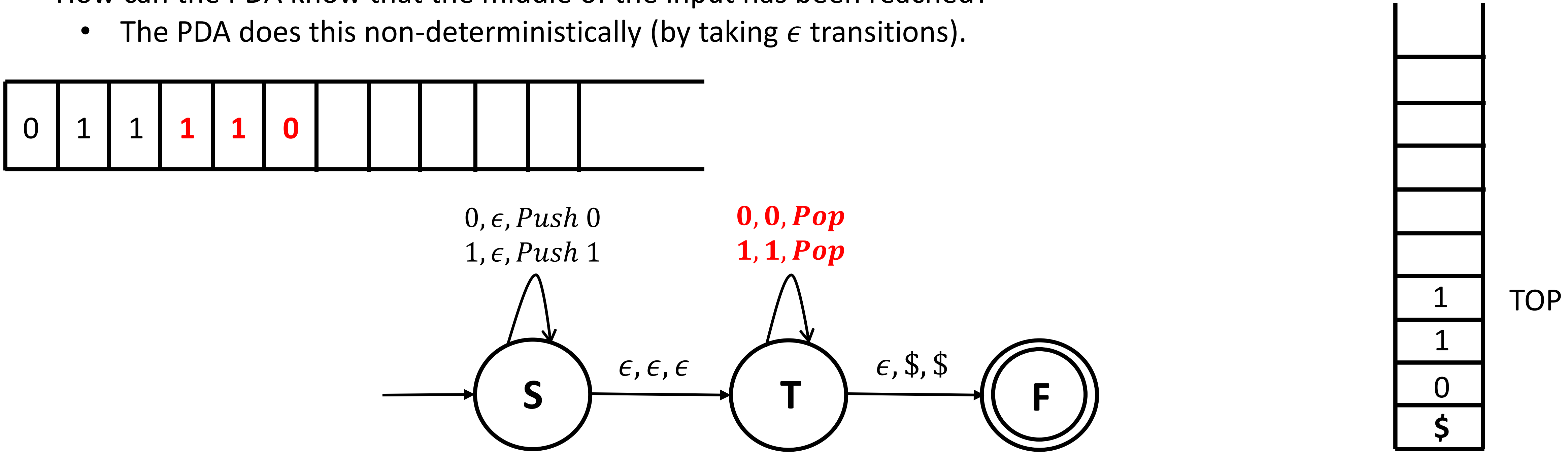

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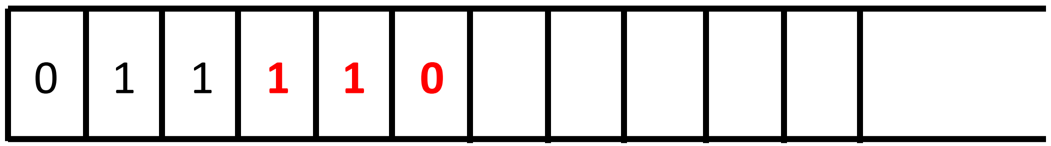

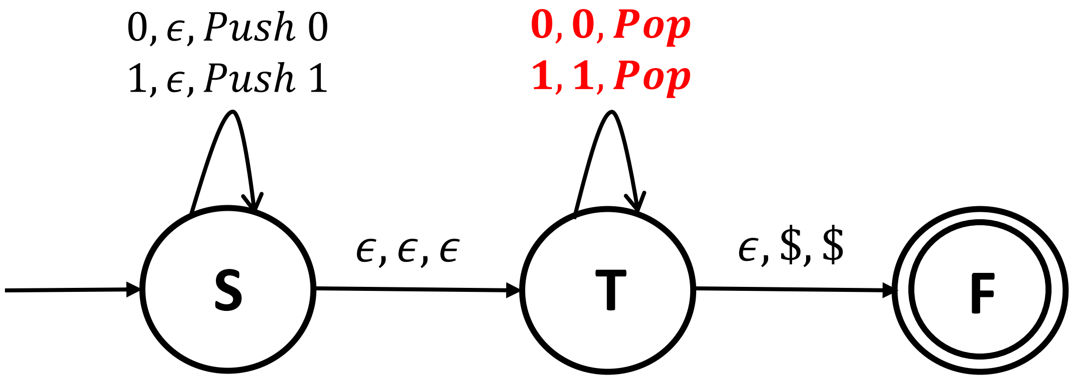

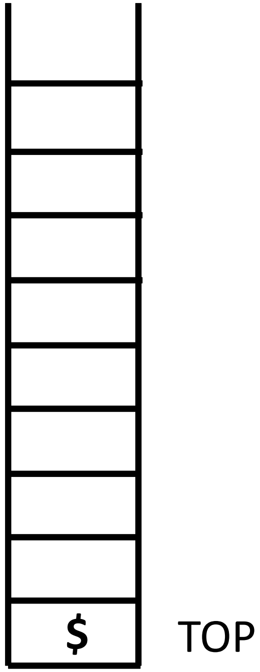

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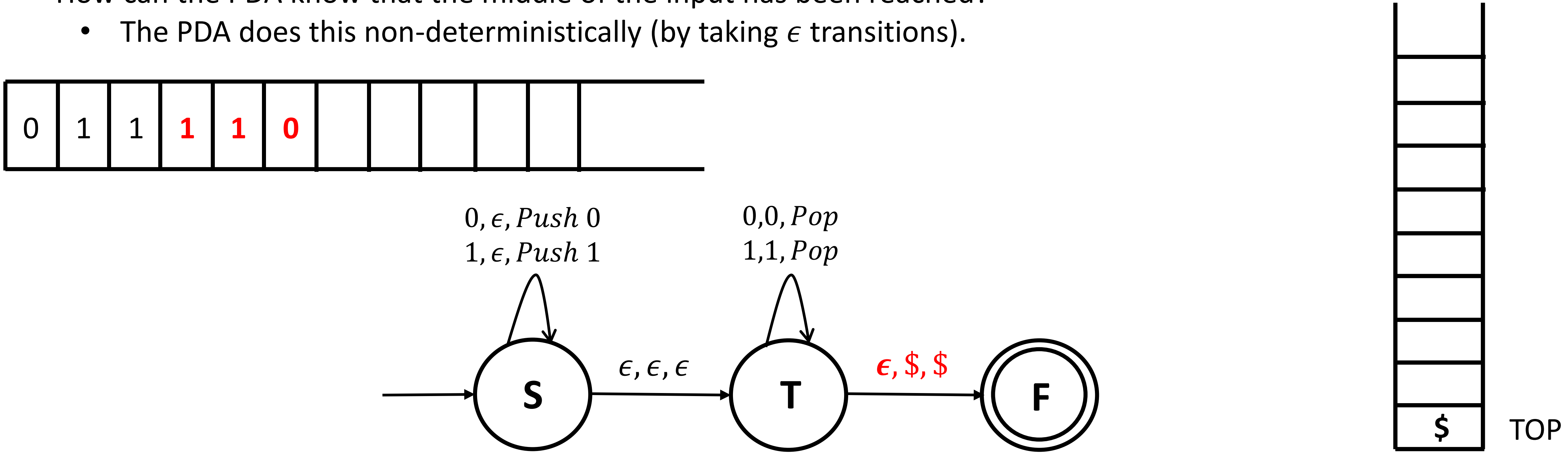

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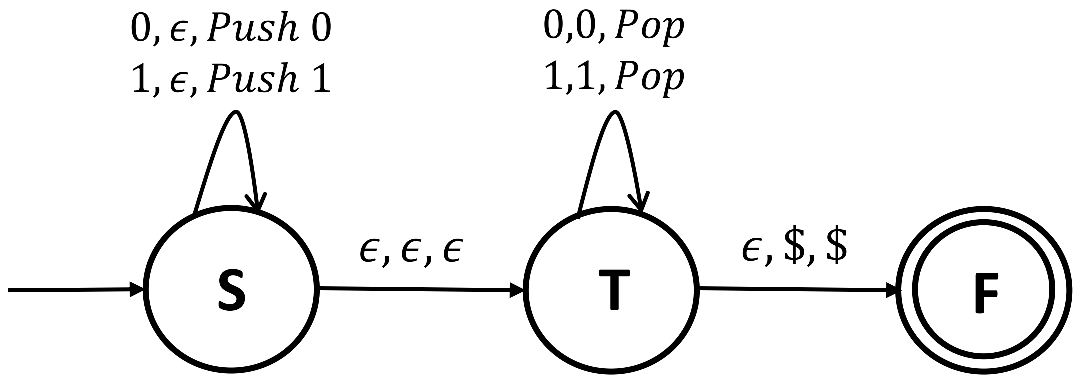

Recognizes even length palindromes of the form: $\boldsymbol{ww^R}$

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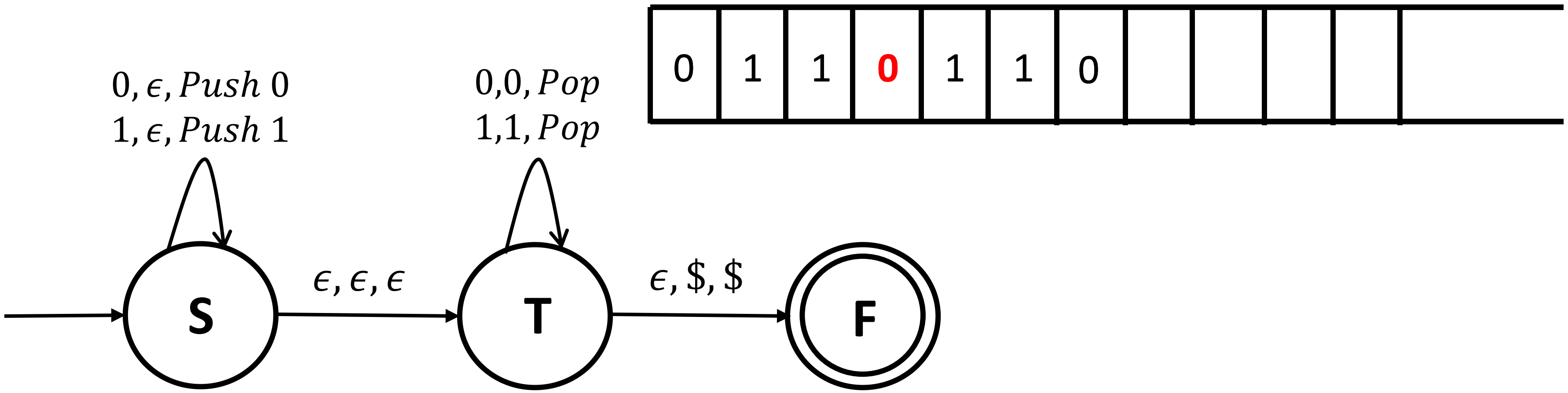
Odd length palindromes are of the form $\boldsymbol{wcw^R}$, such that $\boldsymbol{c \in \Sigma}$



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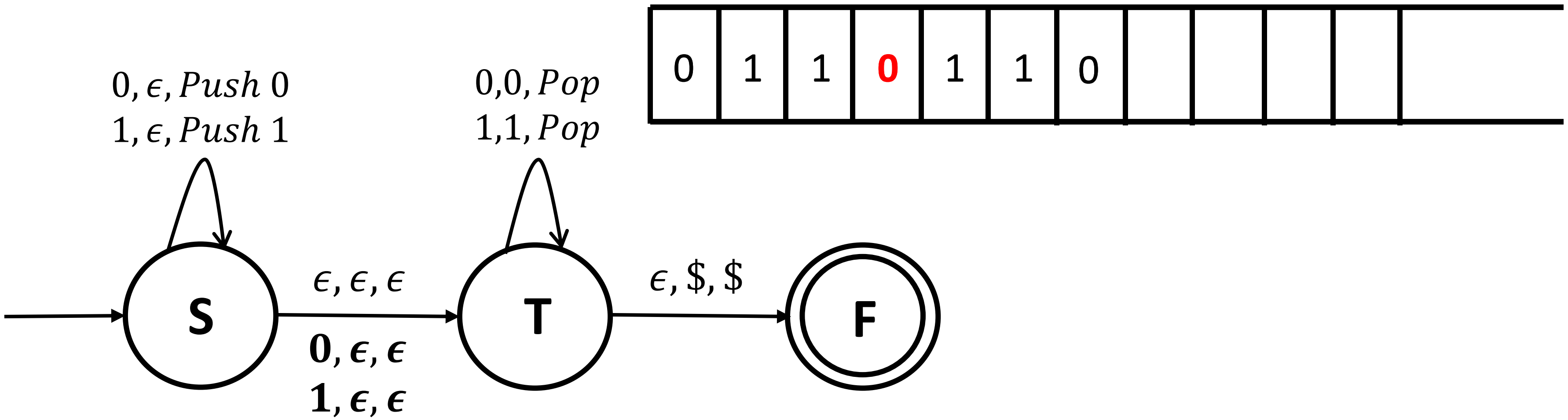

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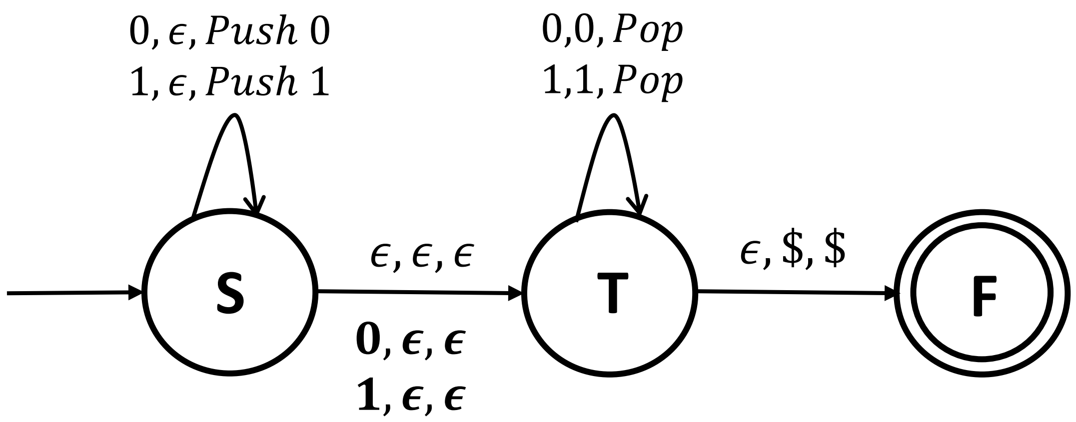

The transitions $\mathbf{0}, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}$ **and** $\mathbf{1}, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}$ **allow the PDA to consume one symbol** and then begin matching what it has encountered thus far.

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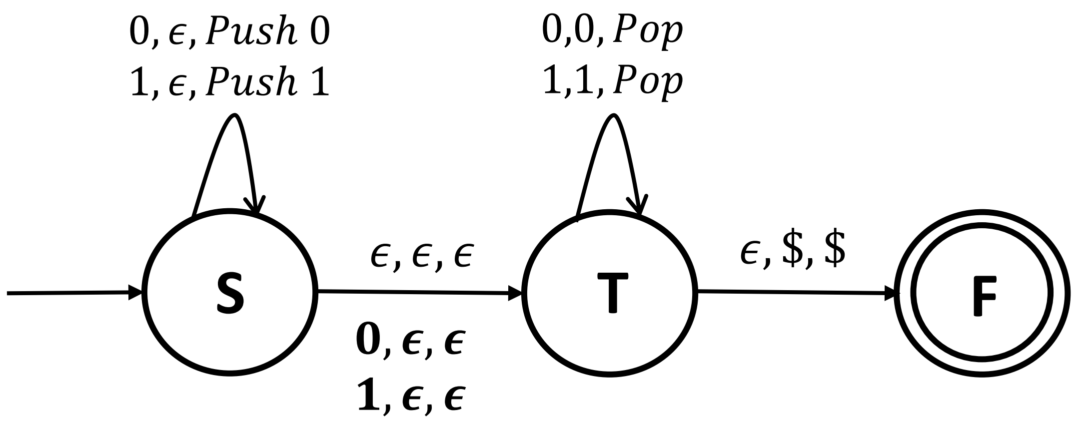

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This allows the PDA to **recognize strings of the form:** $\boldsymbol{\omega c w^R}$, where the aforementioned transitions non-deterministically guessed $c \in \{0,1\}$

Equivalence between PDA and CFL

- We already know that a language is Context-Free if and only if there exists a CFG that generates all the strings belonging to the CFL.

- It can be shown that a language is context free if and only if a PDA recognizes it.

  - If L is context free then there exists a PDA that recognizes L. (We'll prove this next)

  - If there exists a PDA for L, then L is context-free. (Won't prove this in class. Look up a standard text book)

Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

- Before formally proving this, we will use some examples in order to provide some intuition.
- For any L, we can write a context free grammar that can generate all strings that are in L.
- Any string w is generated by the CFG if there exists a derivation $S \overset{*}{\Rightarrow} w$.

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- Before formally proving this, we will use some examples in order to provide some intuition.
- For any L, we can write a context free grammar that can generate all strings that are in L.
- Any string w is generated by the CFG if there exists a derivation $S \overset{*}{\Rightarrow} w$.
- The proof consists of using the rules of the CFG to build a PDA so that it can simulate any derivation $S \overset{*}{\Rightarrow} w$.
    - The PDA accepts an input w if the CFG G generates w
    - It determines whether $\exists$ a derivation for w.
    - Takes advantage of non determinism

Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

Intuitions

- The PDA begins by pushing the start variable S onto the stack.

Pushdown Automata and CFL

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Example: Consider the grammar G with the rules: $S \to aTb|b$

$T \to Ta|\epsilon$

The string $\boldsymbol{w = aaab}$ can be generated by G. Derivation:

$$\boldsymbol{S \to aTb \to aTab \to aTaab \to aaab}$$

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Example: $S \rightarrow aTb|b$
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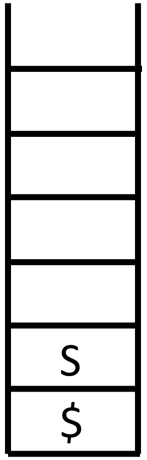

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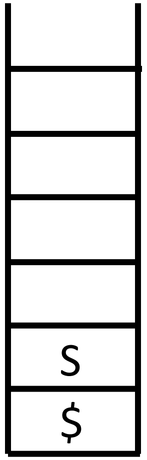

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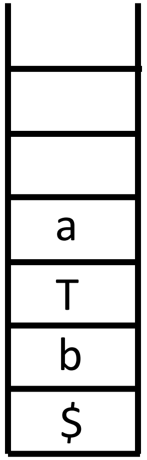

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Example: $S \rightarrow aTb|b$
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Derivation for input string $w = \boldsymbol{a}aab$ can be generated by G**:**

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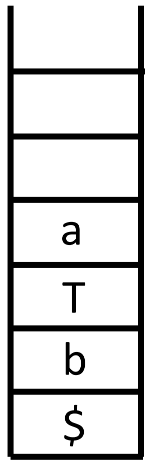

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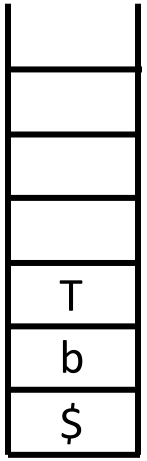

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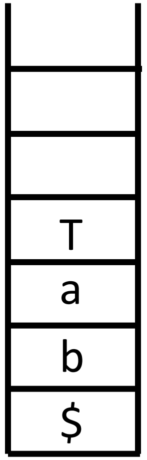

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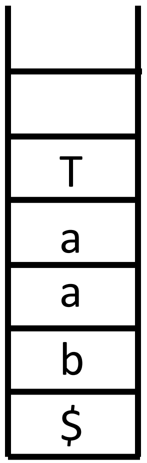

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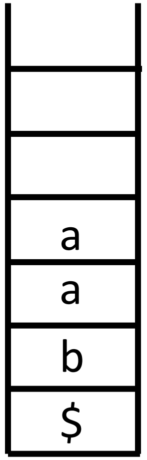

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Example: $S \to aTb|b$
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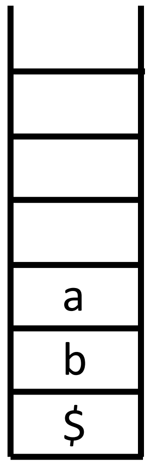

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Input to PDA: w $= \boldsymbol{aaab}$

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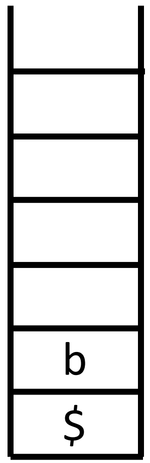

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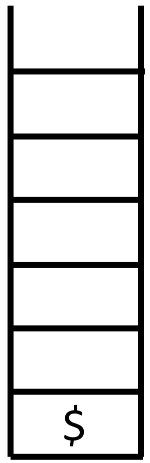
8. Read the input (a) (Pop a).
9. Read the input (b) (Pop b).
10. Since the stack is empty exactly when the input has been read, accept w.



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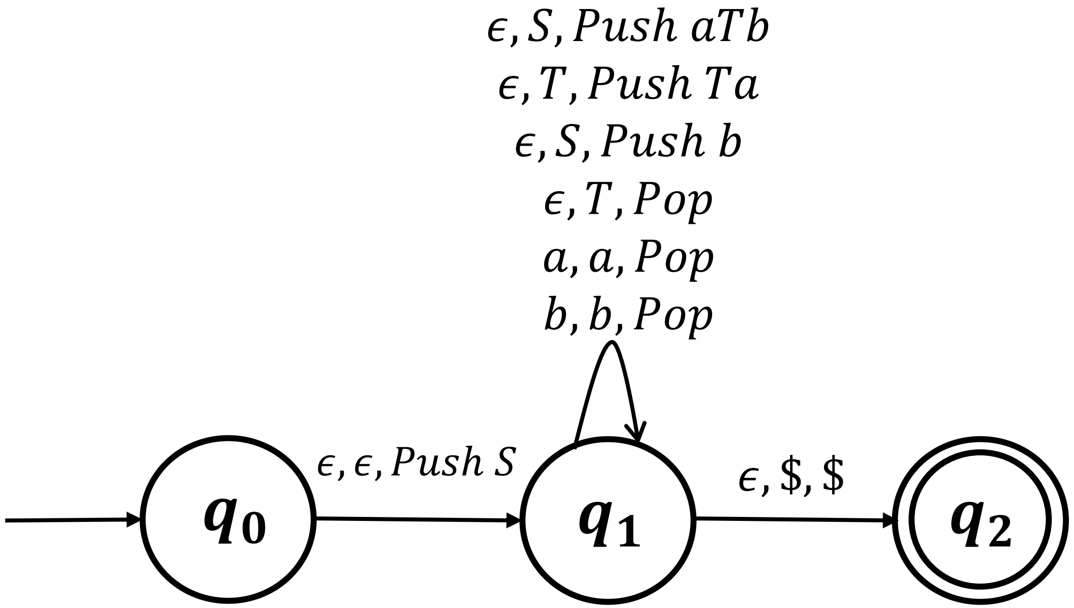
Example: $S \to aTb|b$
$T \to Ta|\epsilon$

Input to PDA: $\mathbf{w} = \boldsymbol{aaab}$

Derivation for input string $w = \boldsymbol{aaab}$ can be generated by G:

$$S \to aTb \to aTab \to aTaab \to aaab$$



Pushdown Automata and CFL

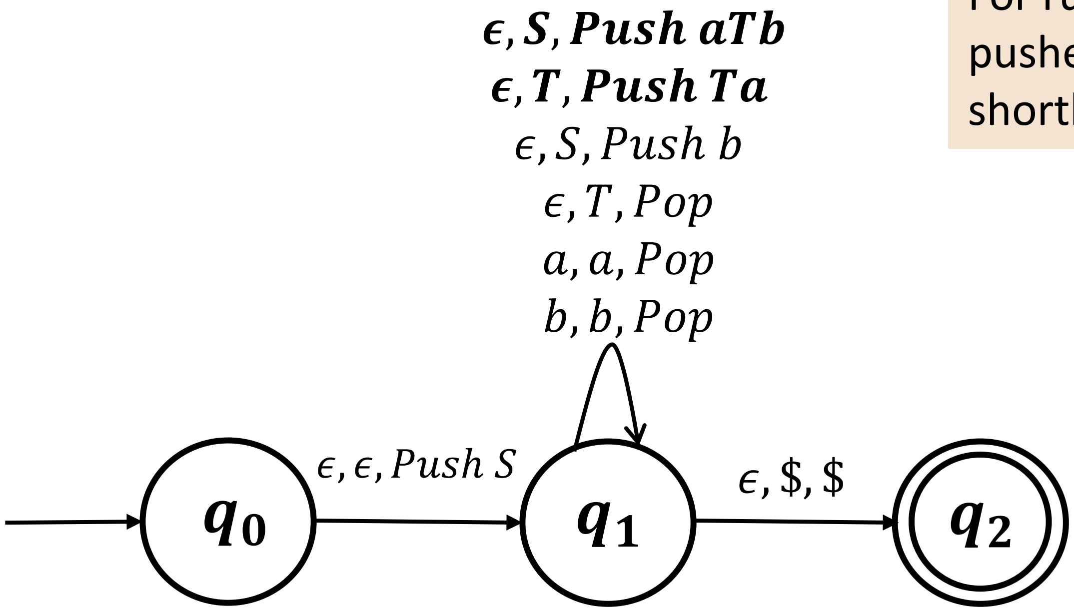
Example: $S \to aTb|b$
$T \to Ta|\epsilon$

Input to PDA: $\mathbf{w} = \boldsymbol{aaab}$

Derivation for input string $w = \boldsymbol{aaab}$ can be generated by G:

$$S \to aTb \to aTab \to aTaab \to aaab$$

For rules where several elements need to be pushed, new states are introduced. This is only a shorthand for that.



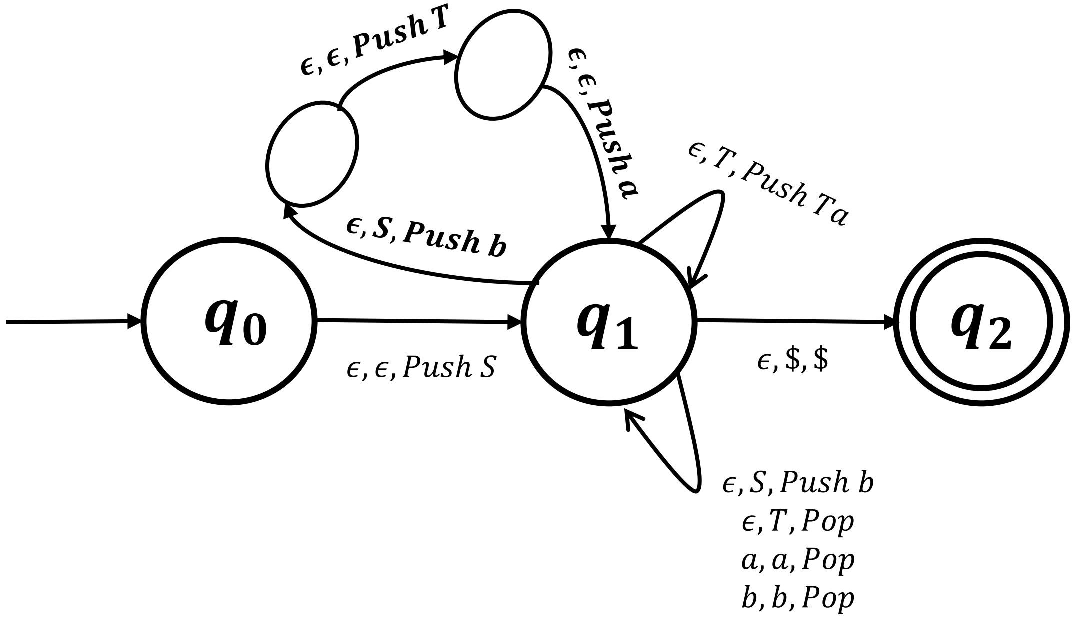
Pushdown Automata and CFL

Example: $S \rightarrow aTb|b$
$T \rightarrow Ta|\epsilon$

Input to PDA: $\mathbf{w} = \boldsymbol{aaab}$

Derivation for input string $w = \boldsymbol{aaab}$ can be generated by G:

$$S \rightarrow aTb \rightarrow aTab \rightarrow aTaab \rightarrow aaab$$



Pushdown Automata and CFL

Example: $S \to aTb|b$
$T \to Ta|\epsilon$

Input to PDA: w $= \boldsymbol{aaab}$

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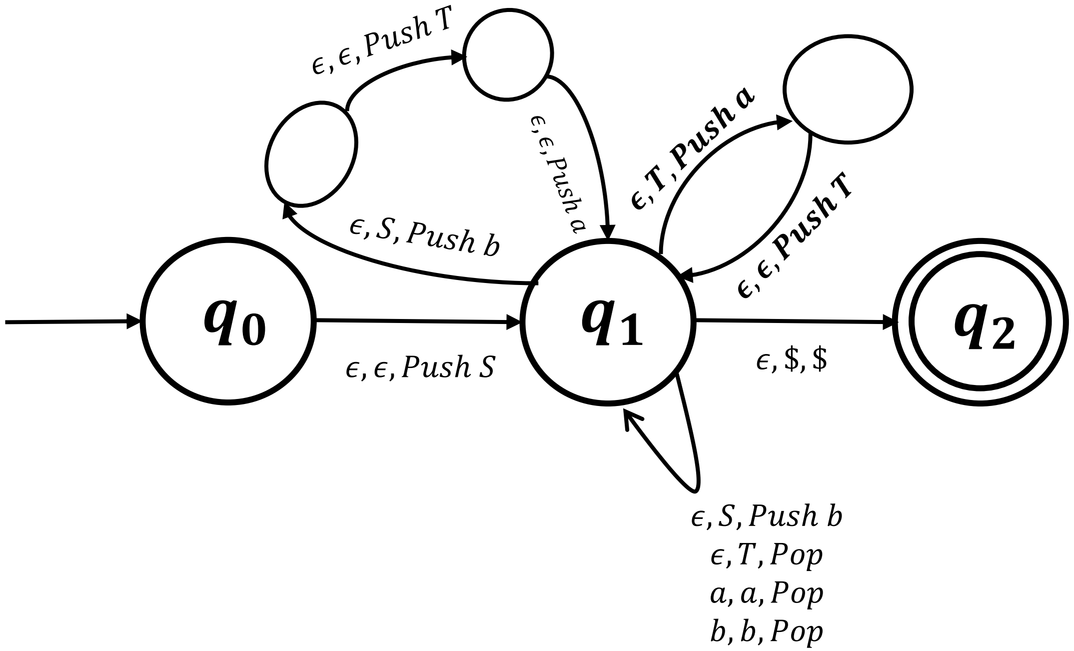


Summary

Given the rules of a CFG G, the equivalent PDA either non deterministically chooses which rule to use or matches part of the input symbol.

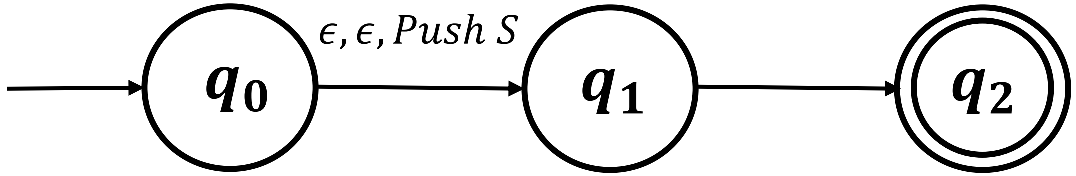
Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

Proof: For convenience, we shall be using the shorthand notation.

Let G be a CFG with a set of rules R, then the equivalent PDA P will have three kind of states $\{q_0, q_1, q_2\}$.

The PDA P first pushes the start symbol S into the stack, irrespective of the input symbol and transitions from the initial state q_0 to q_1, i.e. $\delta(q_0, \epsilon, \epsilon) = (q_1, S)$.



Pushdown Automata and CFL

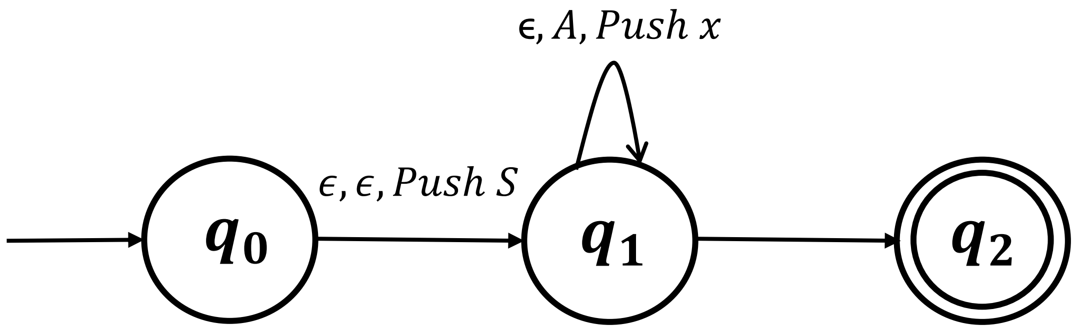
Prove that if L is context free then there exists an equivalent PDA that recognizes L.

Proof: Let G be a CFG with a set of rules R, then the equivalent PDA P will have three states $\{q_0, q_1, q_2\}$.

The PDA P first pushes the start symbol S into the stack, irrespective of the input symbol and transitions from the initial state q_0 to q_1, i.e. $\delta(q_0, \epsilon, \epsilon) = (q_1, S)$.

At q_1, the PDA P implements the rules R of G.

- Pop A and push x onto the stack, where $A \rightarrow x$ is a rule in R and return back to q_1, i.e. let $\delta(q_1, \epsilon, A) = (q_1, x)$.



For rule $A \rightarrow x$ in R

Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

Proof: Let G be a CFG with a set of rules R, then the equivalent PDA P will have three states $\{q_0, q_1, q_2\}$.

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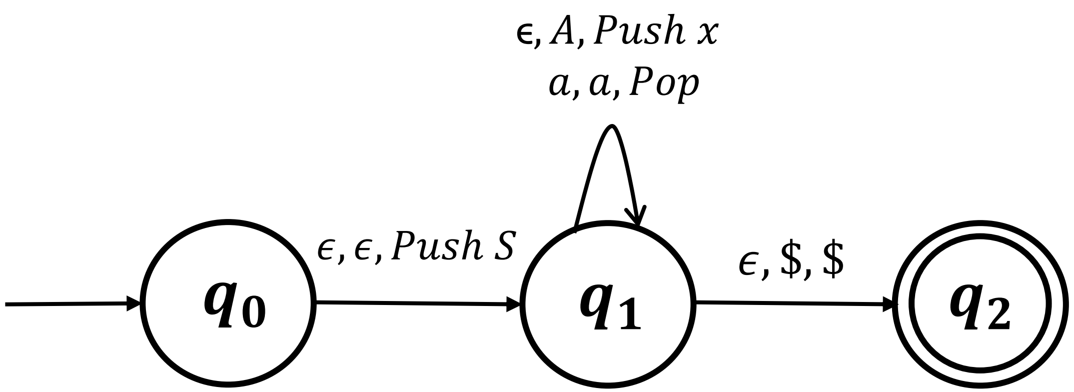
At q_1, the PDA P implements the rules R of G.

- Pop A and push x onto the stack, where $A \rightarrow x$ is a rule in R and return back to q_1, i.e. let $\delta(q_1, \epsilon, A) = (q_1, x)$.
- Pop a, i.e. let $\delta(q_1, a, a) = (q_1, \epsilon)$. Matching the input string with the terminals in stack.



For terminal a

Pushdown Automata and CFL

Prove that if L is context free then there exists an equivalent PDA that recognizes L.

Proof: Let G be a CFG with a set of rules R, then the equivalent PDA P will have three states $\{q_0, q_1, q_2\}$.

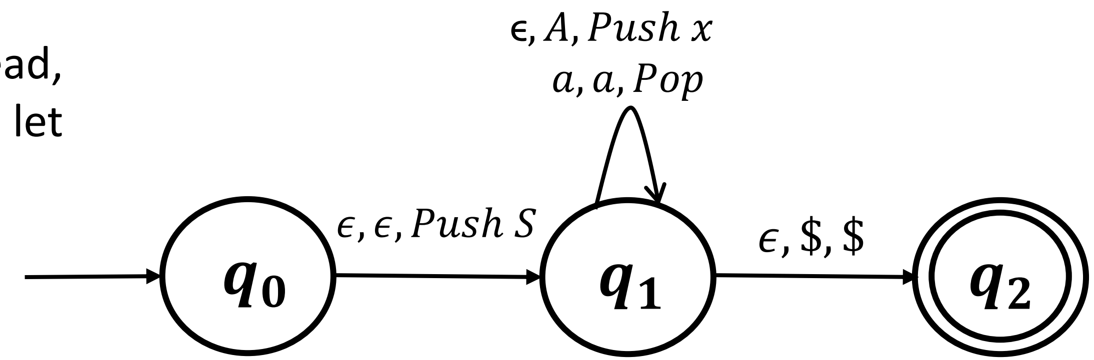
The PDA P first pushes the start symbol S into the stack, irrespective of the input symbol and transitions from the initial state q_0 to q_1, i.e. $\delta(q_0, \epsilon, \epsilon) = (q_1, S)$.

At q_1, the PDA P implements the rules R of G.

- Pop A and push x onto the stack, where $A \rightarrow x$ is a rule in R and return back to q_1, i.e. let $\delta(q_1, \epsilon, A) = (q_1, x)$.

- Pop a, i.e. let $\delta(q_1, a, a) = (q_1, \epsilon)$. Matching the input string with the terminals in stack.

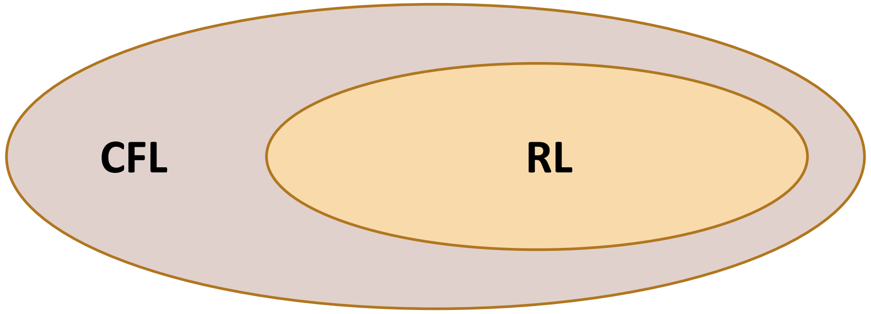
- If the stack is empty, when all the input symbols are read, transition from q_1 to the accepting state q_2, i.e. let $\delta(q_1, \epsilon, \$) = (q_2, \$)$



Equivalence between PDA and CFL

- It can be shown that a language is context free **if and only if** a PDA recognizes it.
  - If L is context free then there exists a PDA that recognizes L. (We proved this)
  - The proof for the other direction (Constructing a CFG that generates L given a PDA that recognizes L) is quite elaborate
  - We won't be covering it in class. But the proof itself is quite easy to understand.
  - Refer to a standard text book (e.g. Sipser)

$$(\boldsymbol{RL \equiv Regular\ Grammar \equiv Regular\ Expressions \equiv NFA \equiv DFA}) \subseteq (\boldsymbol{CFL \equiv CFG \equiv PDA})$$



Thank You!