Probability and Statistics: MA6.101

Tutorial 3

Topics Covered: Discrete Random Variables

Q1: Let X be a Poisson random variable with parameter $\lambda > 0$. The probability mass function (PMF) is given by:

$$p_X(k) = P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots$$

(a) Show that $p_X(k)$ is consistent.

(b) Derive the mean (expected value) of X.

(c) Derive the variance of X.

A:

(a) To show that $p_X(k)$ is a valid PMF, we must verify two conditions:

i. $p_X(k) \geq 0$ for all k.

ii. $\sum_{k=0}^{\infty} p_X(k) = 1$.

For the first condition, since $\lambda \geq 0$, we have $e^{-\lambda} \geq 0$, $\lambda^k \geq 0$, and $k! > 0$ for all $k \in \{0, 1, 2, \ldots\}$. Thus,

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \geq 0.$$

For the second condition, we sum the probabilities over all possible values of k:

$$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}.$$

We can factor out the constant term $e^{-\lambda}$:

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}.$$

The summation is just e^{λ}:

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots = e^{\lambda}.$$

Substituting this back, we get:

$$e^{-\lambda} \cdot e^{\lambda} = e^0 = 1.$$

Since both conditions are met, $p_X(k)$ is a valid PMF.

(b) The mean or expected value of X is given by

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot p_X(k).$$

So,

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!}.$$

The term for $k = 0$ is 0, so we can start the summation from $k = 1$:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!}.$$

Factor out $e^{-\lambda}$ and a λ term:

$$\mathbb{E}[X] = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}.$$

Let $j = k - 1$. As k goes from 1 to ∞, j goes from 0 to ∞:

$$\mathbb{E}[X] = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}.$$

The summation is again just e^{λ}:

$$\mathbb{E}[X] = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda.$$

Thus, the mean of the Poisson distribution is λ.

(c) The variance is given by the formula

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

We know from part (b) that $\mathbb{E}[X] = \lambda$. Now, we calculate $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2 \cdot p_X(k) = \sum_{k=0}^{\infty} k^2 \cdot \frac{e^{-\lambda}\lambda^k}{k!}.$$

The term for $k = 0$ is zero, so start from $k = 1$:

$$\mathbb{E}[X^2] = \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{(k-1)!}.$$

Rewrite $k = (k-1) + 1$:

$$\mathbb{E}[X^2] = \sum_{k=1}^{\infty} [(k-1)+1] \frac{e^{-\lambda}\lambda^k}{(k-1)!}.$$

This splits into two sums:

$$\mathbb{E}[X^2] = \sum_{k=1}^{\infty} (k-1) \frac{e^{-\lambda}\lambda^k}{(k-1)!} + \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!}.$$

For the first sum (starting from $k = 2$):

$$\sum_{k=2}^{\infty}(k-1)\frac{e^{-\lambda}\lambda^k}{(k-1)!} = \lambda^2 e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda^2.$$

For the second sum:

$$\sum_{k=1}^{\infty}\frac{e^{-\lambda}\lambda^k}{(k-1)!} = \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda.$$

Combining:

$$\mathbb{E}[X^2] = \lambda^2 + \lambda.$$

Thus,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda.$$

So, the variance of the Poisson distribution is also λ.

Q2: The probability distribution of the discrete random variable X is given by

x	2	3	4
$P(X = x)$	$0.4 - a$	$2a$	$0.6 - a$

where a is a constant.

(a) State the range of the possible values of a.

(b) Two independent observations of X, denoted by X_1 and X_2, are considered. Determine the range of possible values of $P(X_1 + X_2 = 6)$ as a varies over its allowable range.

A:

(a) Since probabilities must be non-negative:

$$0.4 - a \geq 0, \quad 2a \geq 0, \quad 0.6 - a \geq 0.$$

Hence,

$$a \leq 0.4, \quad a \geq 0, \quad a \leq 0.6.$$

Also, the total probability must equal 1:

$$(0.4 - a) + (2a) + (0.6 - a) = 1,$$

which is identically true for all a. Therefore the combined range is

$$\boxed{0 \leq a \leq 0.4}.$$

(b) We need $P(X_1 + X_2 = 6)$. The contributing pairs (X_1, X_2) are $(2,4), (4,2), (3,3)$. Since X_1 and X_2 are independent,

$$P(X_1 + X_2 = 6) = 2P(X = 2)P(X = 4) + \big(P(X = 3)\big)^2.$$

Substituting the probabilities,

$$P(a) := 2(0.4 - a)(0.6 - a) + (2a)^2.$$

Expanding and simplifying gives

$$P(a) := 6a^2 - 2a + 0.48.$$

We now find the range of $P(a)$ for $a \in [0, 0.4]$.
The quadratic $P(a) = 6a^2 - 2a + 0.48$ opens upwards (coefficient of a^2 is positive), so its minimum on $\mathbb{R}$ occurs at the vertex

$$a_{\mathrm{v}} = -\frac{b}{2a} = -\frac{-2}{2 \cdot 6} = \frac{1}{6}.$$

Note $a_{\mathrm{v}} = \frac{1}{6} \approx 0.1666$ lies inside the interval $[0, 0.4]$, so the minimum on $[0, 0.4]$ is attained at $a = \frac{1}{6}$, and the maximum is attained at one of the endpoints.
Compute the values exactly:

$$P\left(\tfrac{1}{6}\right) = 6 \cdot \frac{1}{36} - 2 \cdot \frac{1}{6} + 0.48 = \frac{1}{6} - \frac{1}{3} + \frac{12}{25} = \frac{47}{150}.$$

At the endpoints,

$$P(0) = 0.48 = \frac{12}{25}, \qquad P(0.4) = 6(0.4)^2 - 2(0.4) + 0.48 = \frac{16}{25} = 0.64.$$

Therefore the range of possible values is

$$\boxed{\frac{47}{150} \ \leq \ P(X_1 + X_2 = 6) \ \leq \ \frac{16}{25}}$$

or numerically

$$\boxed{0.313\overline{3} \ \leq \ P(X_1 + X_2 = 6) \ \leq \ 0.64}.$$

Q3: Let X be the number of students waiting for a Teaching Assistant (TA) during office hours. Based on past observations, we have the following information:

- At any time, there are at most 3 students waiting for the TA.
- The probability of finding two students waiting is the same as the probability of finding one student.
- The probability that no one is waiting is the same as the probability of finding three students waiting.
- The probability of finding either one or two students is half the probability of finding the office hours either completely empty or completely full (with 3 students).

Find the PMF of X.

A:
The random variable X can take values in the set $\{0, 1, 2, 3\}$ (from the first condition). Let's denote the probability of each outcome as $p_k = P(X = k)$. We can translate the given information into mathematical equations.

(a) From the second condition, we get:

$$p_2 = p_1$$

(b) From third, we get:

$$p_0 = p_3$$

(c) From fourth, we get:

$$P(X = 1 \text{ or } X = 2) = \frac{1}{2} P(X = 0 \text{ or } X = 3)$$

Since the events are mutually exclusive, this becomes:

$$p_1 + p_2 = \frac{1}{2}(p_0 + p_3)$$

Now, let's substitute the first two equations into the third one:

$$p_1 + p_1 = \frac{1}{2}(p_0 + p_0)$$

$$2p_1 = \frac{1}{2}(2p_0)$$

$$2p_1 = p_0$$

We can now express all probabilities in terms of p_1:

- $p_2 = p_1$
- $p_0 = 2p_1$
- $p_3 = p_0 = 2p_1$

The sum of all probabilities for a PMF must equal 1:

$$p_0 + p_1 + p_2 + p_3 = 1$$

Substituting our expressions in terms of p_1:

$$(2p_1) + p_1 + p_1 + (2p_1) = 1$$

$$6p_1 = 1$$

$$p_1 = \frac{1}{6}$$

Now we can find the other probabilities:

- $p_1 = \frac{1}{6}$
- $p_2 = p_1 = \frac{1}{6}$
- $p_0 = 2p_1 = 2\left(\frac{1}{6}\right) = \frac{2}{6} = \frac{1}{3}$
- $p_3 = p_0 = \frac{1}{3}$

Therefore, the PMF of X is:

$$p_X(k) = \begin{cases} \frac{1}{3} & \text{for } k = 0, \\ \frac{1}{6} & \text{for } k = 1, \\ \frac{1}{6} & \text{for } k = 2, \\ \frac{1}{3} & \text{for } k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Q4: The median of a random variable X is defined as any number m that satisfies both of the following conditions:

$$\mathbb{P}(X \geq m) \geq \frac{1}{2} \quad \text{and} \quad \mathbb{P}(X \leq m) \geq \frac{1}{2}.$$

Note that the median of X is not necessarily unique. Find the median of X if X is the result of rolling a fair die.

A:

$$\mathbb{P}(X \geq m) \geq \frac{1}{2} \quad \mathbb{P}(X \leq m) \geq \frac{1}{2}$$

$$\mathbb{P}(X > m) + p_X(X = m) \geq \frac{1}{2} \quad F_X(m) \geq \frac{1}{2}$$

$$1 - F_X(m) + p_X(X = m) \geq \frac{1}{2}$$

$$\boxed{\frac{1}{2} \leq F_X(m) \leq \frac{1}{2} + p_X(X = m)}$$

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{6} & 1 \leq x < 2 \\ \frac{2}{6} & 2 \leq x < 3 \\ \frac{3}{6} & 3 \leq x < 4 \\ \frac{4}{6} & 4 \leq x < 5 \\ \frac{5}{6} & 5 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$

For $m \geq 3$, we see that:

$$F_X(x) \geq \frac{1}{2}$$

For $m = 3$, we have:

$$F_X(3) \leq \frac{1}{2} + p_X(3)$$

$$\frac{1}{2} \leq \frac{1}{2} + \frac{1}{6} \quad \text{(satisfied)}$$

For $m = 4$, we have:

$$F_X(4) \leq \frac{1}{2} + p_X(4)$$

$$\frac{4}{6} \leq \frac{1}{2} + \frac{1}{6} = \frac{4}{6} \quad \text{(satisfied)}$$

For $m = 5$, we get:

$$F_X(5) \neq \frac{1}{2} + p_X(5)$$

$$\therefore \text{Discrete points satisfying the condition are} \{3, 4\}.$$

To check the interval $(3, 4)$:

$$F_X(x)\, x \in (3, 4) = \frac{1}{2}$$

$$\therefore \text{All points that are non-integer in } (3, 4) \text{ will satisfy the condition.}$$

$$\implies m \in (3, 4) \cup \{3, 4\} = m \in [3, 4]$$

Q5: The discrete random variable X has the probability mass function

$$p_X(X = x) = \begin{cases} kx & x = 2, 4, 6, \\ k(x-2) & x = 8, \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Show that $k = \frac{1}{18}$.
(c) Find the exact value of $\mathbb{E}(X)$.
(d) Find the exact value of $\mathbb{E}(X^2)$.

A:

(a) The sum of the probabilities for a discrete random variable must equal 1. That is,

$$\Sigma P(X = x) = 1$$

In this case,

$$P(X = 2) + P(X = 4) + P(X = 6) + P(X = 8) = 1$$

(rest of the probabilities are 0)

$$\therefore (k \cdot 2) + (k \cdot 4) + (k \cdot 6) + (k \cdot (8 - 2)) = 1$$
$$\therefore 18 \cdot k = 1$$
$$\therefore k = \frac{1}{18}$$

(b) The expected value of X is given by:

$$\mathbb{E}(X) = \sum_x x \cdot P(X = x)$$

Hence,

$$\mathbb{E}(X) = 2 \cdot \frac{2}{18} + 4 \cdot \frac{4}{18} + 6 \cdot \frac{6}{18} + 8 \cdot \frac{(8-2)}{18}$$
$$\mathbb{E}(X) = \frac{1}{18}(4 + 16 + 36 + 48) = \frac{104}{18} = \frac{52}{9}$$

(c) The expected value of X^2 is given by:

$$\mathbb{E}(X^2) = \sum_x x^2 \cdot P(X = x)$$

Hence,

$$\mathbb{E}(X^2) = 2^2 \cdot \frac{2}{18} + 4^2 \cdot \frac{4}{18} + 6^2 \cdot \frac{6}{18} + 8^2 \cdot \frac{(8-2)}{18}$$

$$\mathbb{E}(X^2) = \frac{1}{18}(8 + 64 + 216 + 384) = \frac{672}{18} = \frac{112}{3}$$

Q6: Say there are 1000 households in Gachibowli. Specifically, there are 100 households with one member, 200 households with 2 members, 300 households with 3 members, 200 households with 4 members, 100 households with 5 members, and 100 households with 6 members. Thus, the total number of people living in Gachibowli is

$$N = 100 \cdot 1 + 200 \cdot 2 + 300 \cdot 3 + 200 \cdot 4 + 100 \cdot 5 + 100 \cdot 6 = 3300.$$

(a) We pick a household at random, and define the random variable X as the number of people in the chosen household. Find the PMF and the expected value of X.

(b) We pick a person in Gachibowli at random, and define the random variable Y as the number of people in the household where the chosen person lives. Find the PMF and the expected value of Y.

A:

(a) We pick a household at random X.
The sample space here is a 1000 households. Each household has an equal chance of getting selected. Hence, for instance:

$$P(X = 1) = \frac{100}{1000} = 0.1$$

Similarly, we get the PMF of X as:

$$p_X(k) = \begin{cases} 0.1 & \text{for } k = 1, \\ 0.2 & \text{for } k = 2, \\ 0.3 & \text{for } k = 3, \\ 0.2 & \text{for } k = 4, \\ 0.1 & \text{for } k = 5, \\ 0.1 & \text{for } k = 6, \\ 0 & \text{otherwise.} \end{cases}$$

The expected value of X is given by:

$$\mathbb{E}[X] = \sum_k k \cdot p_X(X = k)$$

Hence,

$$E[X] = (1\cdot 0.1)+(2\cdot 0.2)+(3\cdot 0.3)+(4\cdot 0.2)+(5\cdot 0.1)+(6\cdot 0.1) = 3.3$$

Alternatively, we can find the average by dividing the total number of people by the total number of households:

$$E[X] = \frac{3300}{1000} = 3.3$$

(b) We pick a person at random Y.
The sample space here is a 3300 people. Each person has an equal chance of getting selected. Hence, for instance,

$$P(Y=1) = \frac{100}{3300} = \frac{10}{330} = \frac{1}{33}$$

since a total of 100 people live in households with 1 member each. Similarly, we get the PMF of Y as:

$$p_Y(k) = \begin{cases} \frac{1}{33} & \text{for } k=1, \\ \frac{4}{33} & \text{for } k=2, \\ \frac{9}{33} & \text{for } k=3, \\ \frac{8}{33} & \text{for } k=4, \\ \frac{5}{33} & \text{for } k=5, \\ \frac{6}{33} & \text{for } k=6, \\ 0 & \text{otherwise.} \end{cases}$$

The expected value of Y is given by:

$$\mathbb{E}[Y] = \sum_k k \cdot p_Y(Y=k)$$

Hence,

$$E[Y] = (1\cdot\frac{1}{33})+(2\cdot\frac{4}{33})+(3\cdot\frac{9}{33})+(4\cdot\frac{8}{33})+(5\cdot\frac{5}{33})+(6\cdot\frac{6}{33}) \approx 3.91$$

Q7: Suppose that there are n different types of coupons. Each time you get a coupon, it is equally likely to be any of the n possible types. Let X be the number of coupons you will need to get before having observed each coupon at least once.

(a) Show that you can write $X = X_0+X_1+\cdots+X_{n-1}$ where $X_i \sim \text{Geometric}(\frac{n-i}{n})$

(b) Find $\mathbb{E}[X]$.

A:

(a) We can decompose the coupon collection process into phases. Let X_i be the number of coupons needed to go from having i distinct coupon types to having $i+1$ distinct coupon types.
Initially, we have 0 distinct coupon types. The total number of coupons needed is:

$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}$$

where:

- X_0 = number of coupons to go from 0 to 1 distinct type
- X_1 = number of coupons to go from 1 to 2 distinct types
- X_2 = number of coupons to go from 2 to 3 distinct types
- $\vdots$
- X_{n-1} = number of coupons to go from $n-1$ to n distinct types

Now, let's analyze X_i. When we already have i distinct coupon types, the probability that the next coupon we draw is a new type (i.e., one we haven't seen before) is:

$$p_i = \frac{\text{number of unseen types}}{\text{total number of types}} = \frac{n-i}{n}$$

This is because there are $n-i$ coupon types we haven't collected yet out of n total types.
The number of trials needed to get the first success (a new coupon type) in a sequence of independent Bernoulli trials with success probability p_i follows a geometric distribution. Therefore:

$$X_i \sim \text{Geometric}\left(\frac{n-i}{n}\right)$$

(b) Using the linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_0 + X_1 + \cdots + X_{n-1}] = \mathbb{E}[X_0] + \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_{n-1}]$$

For a geometric random variable $Y \sim \text{Geometric}(p)$, we have

$$\mathbb{E}[Y] = \frac{1}{p}$$

Therefore:

$$\mathbb{E}[X_i] = \frac{1}{\frac{n-i}{n}} = \frac{n}{n-i}$$

Substituting this into our expression for $\mathbb{E}[X]$:

$$\begin{aligned} \mathbb{E}[X] &= \sum_{i=0}^{n-1} \mathbb{E}[X_i] && (1) \\ &= \sum_{i=0}^{n-1} \frac{n}{n-i} && (2) \\ &= n \sum_{i=0}^{n-1} \frac{1}{n-i} && (3) \end{aligned}$$

Let's change the index of summation. Let $j = n - i$. When $i = 0$, $j = n$. When $i = n - 1$, $j = 1$. So:

$$\mathbb{E}[X] = n \sum_{j=1}^{n} \frac{1}{j} = n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right)$$

This can be written as:

$$\boxed{\mathbb{E}[X] = nH_n}$$

where $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is the n-th harmonic number.
For large n, we have the approximation $H_n \approx \ln(n) + \gamma$, where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. Therefore:

$$\mathbb{E}[X] \approx n(\ln(n) + \gamma) \approx n \ln(n)$$

This shows that the expected number of coupons needed to collect all n types grows like $n \ln(n)$.

Q8: You roll a single fair six-sided die, and you are paid an amount equal to the value shown. However, you have an option: you can refuse this payment and roll the die a second time. If you opt for the re-roll, you are paid an amount equal to the value of the second roll. What is the optimal strategy to maximize your winnings, and what is the expected payout if you follow this strategy?

Solution:
The core of this problem is to determine the best strategy for when to keep a roll versus when to re-roll. This strategy will depend on comparing the value of the first roll to the expected value of a second roll.

First, let's find the expected value of a single roll of a fair six-sided die:

$$E[\text{Roll}] = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = \frac{21}{6} = 3.5$$

To maximize our winnings, we should only re-roll if the amount we expect to get from the re-roll (3.5) is greater than the amount we already have from the first roll. Thus, the optimal strategy is:

- **Re-roll** if the first roll is a 1, 2, or 3 (since these are all less than 3.5).
- **Keep** the result if the first roll is a 4, 5, or 6 (since these are all greater than or equal to 3.5).

Now, we can calculate the overall expected payout by considering the six equally likely outcomes of the first roll and applying our strategy:

- If the first roll is a 1: We re-roll. The expected payout in this case is 3.5.
- If the first roll is a 2: We re-roll. The expected payout in this case is 3.5.
- If the first roll is a 3: We re-roll. The expected payout in this case is 3.5.

- If the first roll is a 4: We keep it. The payout is 4.
- If the first roll is a 5: We keep it. The payout is 5.
- If the first roll is a 6: We keep it. The payout is 6.

Since each of these starting outcomes has a probability of 1/6, the overall expected payout is the average of the payouts for each case:

$$E[\text{Overall Payout}] = \frac{3.5 + 3.5 + 3.5 + 4 + 5 + 6}{6} = \frac{25.5}{6} = 4.25$$

By following the optimal strategy, the expected payout of the game is \$4.25.

Q9: Suppose that n people attend a party, each wearing wigs. As part of a game, their wigs are mixed up, and each person randomly selects one. Let X be the number of people who select their own wig. Find the expected value and the variance of X.

Solution:
This problem can be solved using indicator random variables. Let X_i be an indicator variable for the i-th person selecting their own wig, for $i = 1, 2, \ldots, n$.

$$X_i = \begin{cases} 1 & \text{if person } i \text{ selects their own wig} \\ 0 & \text{otherwise} \end{cases}$$

The total number of people who select their own wig is the sum of these indicator variables:

$$X = \sum_{i=1}^{n} X_i$$

.

To find the **expected value** of X, use the linearity of expectation. First, let's find the expectation of a single indicator, X_i.

$$E[X_i] = 1 \cdot p_{X_i}(1) + 0 \cdot p_{X_i}(0) = p_{X_i}(1)$$

The probability that person i selects their own wig is $\frac{1}{n}$, since there is one correct wig for them out of n randomly distributed wigs. So, $E[X_i] = 1/n$. The expected value of X is then:

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

Thus, the expected number of people who get their own wig is 1, regardless of the size of n.

To find the **variance** of X, we use the formula $Var(X) = E[X^2] - (E[X])^2$. We already know $E[X] = 1$, so we just need to find $E[X^2]$.

$$E[X^2] = E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = E\left[\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j\right]$$

By linearity of expectation, this becomes:

$$E[X^2] = \sum_{i=1}^{n} E[X_i^2] + \sum_{i \neq j} E[X_i X_j]$$

For an indicator variable, $X_i^2 = X_i$, so $E[X_i^2] = E[X_i] = 1/n$. The first sum is $\sum_{i=1}^{n}(1/n) = 1$.

For the second term, we need $E[X_i X_j]$ for $i \neq j$. The product $X_i X_j = 1$ only if both person i and person j select their own wigs. The probability of this can be computed by counting: there are $n!$ equally likely wig assignments, and in $(n-2)!$ of these both i and j get their own wigs (since their wigs are fixed and the remaining $n-2$ wigs can be permuted freely). Hence

$$\mathbb{P}(X_i = 1 \text{ and } X_j = 1) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

So, $E[X_i X_j] = \frac{1}{n(n-1)}$. There are $n(n-1)$ ordered pairs of (i, j) with $i \neq j$, so

$$\sum_{i \neq j} E[X_i X_j] = n(n-1) \cdot \frac{1}{n(n-1)} = 1.$$

Substituting back, we get $E[X^2] = 1 + 1 = 2$. Finally, the variance is:

$$Var(X) = E[X^2] - (E[X])^2 = 2 - 1^2 = 1$$

Thus, both the mean and the variance of the number of correct matches are 1.