# This Lecture ..

- Right-continuity of CDF
- Mixed random variables
- Multiple random variables

# For a r.v. $X$, its CDF satisfies the following

- $F_X(\infty) = 1$ and $F_X(-\infty) = 0$ when $P(-\infty < X < \infty) = 1$.
- $F_X : \mathbb{R} \to [0, 1]$ is non-decreasing and right continuous.
- At point of discontinuity $x$ we have
  1. right hand limit $F_X(x+) := \lim_{\epsilon \downarrow 0} F_X(x + \epsilon)$
  2. left hand limit $F_X(x-) := \lim_{\epsilon \uparrow 0} F_X(x - \epsilon)$
  3. $F_X(x-) \neq F_X(x+)$.
  4. $F_X(x)$ could be set to either of the two. Which one?
- Right continuity mandates that at point of discontinuity, we have $F_X(x) = F_X(x+)$.
- By default, $F_X(x) = F_X(x+) = F_X(x-)$ if $F_X(x)$ is continuous at $x$.

# Right-continuity

$F_X : \mathbb{R} \to [0,1]$ is non-decreasing and right continuous.

Proof

- Consider $a < b$ where $a$ and $b$ are arbitrary. We want to show that $F_X(a) \leq F_X(b)$.
- Define $A := \{\omega \in \Omega : X(\omega) \leq a\}$, $B := \{\omega \in \Omega : X(\omega) \leq b\}$.
- Easy to see that $A \subseteq B$ and hence $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- $F_X(a) = P_X((-\infty, a]) = \mathbb{P}(A) \leq \mathbb{P}(B) = F_X(b)$.
- This proves the non-decreasing part.

# Right-continuity

> $F_X : \mathbb{R} \to [0,1]$ is non-decreasing and right continuous.

Proof for right-continuity

- We want to prove that $F_X(x) = F_X(x+)$.
- Consider a sequence of numbers $\{x_n\}$ decreasing to $x$. In this case, we have $F_X(x+) = \lim_{x_n \downarrow x} F_X(x_n)$.
- Define $A_n := \{\omega : X(\omega) \leq x_n\}$ and $A := \{\omega : X(\omega) \leq x\}$.
- Is $A_n \uparrow A$ or $A_n \downarrow A$ ? Clearly, $A_n \downarrow A$.
- From continuity of probability, $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.
- This implies $\lim_{x_n \downarrow x} F_X(x_n) = F_X(x)$. $\square$
- You cannot prove the other way by considering $x_n \uparrow x$ because $\cup_n(-\infty, x_n] = (-\infty, x)$ and $P_X(-\infty, x) \neq F_X(x)$.

# Mixed random variables

# Mixed Random variables

- Random variables that are neither continuous nor discrete are called as mixed random variables.
- Their CDF is partly continuous and partly piece-wise continuous.
- Example: $X$ is a $U[0, 1]$ random variable and $Y = X$ if $X \leq 0.5$ and $Y = 0.5$ if $X > 0.5$.
- What is the CDF and PDF of $Y$?

# Mixed Random variables

- Let $F_Y(y) = C(y) + D(y)$ where $C(y)$ corresponds to the continuous part and $D(y)$ for the discontinuous part.
- $$E[Y] = \int_{-\infty}^{\infty} xc(x)dx + \sum_{y_k} y_k P(Y = y_k)$$

  where $\{y_1, y_2, \ldots\}$ are jump points of $D(y)$ where $P(Y = y_k) > 0$.
- See section 4.3.1 from probabilitycourse.com for more examples
- Amount of workload (pending) on a server! A server on a cluster may be idle with a finite probability. If busy, the pending work is a continuous random variable.

# Multiple random variables

# A running example

- Consider an experiment of tossing a coin and a dice together.
- $\Omega = \{0,1\} \times \{1,2,3,4,5,6\}$. $\mathcal{F} = 2^{\Omega}$. $\mathbb{P}(\omega) = \frac{1}{12}$.
- Let $X$ and $Y$ denote the random variables depicting outcome of a coin and dice respectively.
- For $\omega = (1,5)$ we have $X(\omega) = 1$ and $Y(\omega) = 5$.
- We are now interested in the joint PMF $p_{XY}(x,y)$ and joint CDF $F_{XY}(x,y)$ of $X$ and $Y$ together.

# An example

- We are now interested in the joint PMF $p_{XY}(x, y)$ and joint CDF $F_{XY}(x, y)$ of $X$ and $Y$ together.
- $p_{XY}(x, y) := \mathbb{P}\{\omega \in \Omega : X(w) = x \text{ and } Y(\omega) = y\}$.
- $F_{XY}(x, y) := \mathbb{P}\{\omega \in \Omega : X(w) \leq x \text{ and } Y(\omega) \leq y\}$.
- We can use PMF to calculate $P((X, Y) \in A)$.
- $P((X, Y) \in A) = \mathbb{P}\{\omega \in \Omega : (X(\omega), Y(\omega)) \in A\}$. Therefore $P((X, Y) \in A) = \sum_{(x,y)\in A} p_{XY}(x, y)$.
- Suppose $A$ is the event that you get a head and the roll is even. What is $P((X, Y) \in A)$?

# Marginals

- What is $p_{XY}(1, i)$? $(= \frac{1}{12})$.
- $\sum_i p_{XY}(1, i) = \mathbb{P}\{\omega \in \Omega : X(\omega) = 1\} = \frac{1}{2} = p_X(x)$.
- Similarly, $p_{XY}(1, i) + p_{XY}(0, i) = \frac{1}{6} = p_Y(i)$.

The marginal PMF's $p_X$ and $p_Y$ can be obtained from the joint PMF as follows:
$p_X(x) = \sum_y p_{XY}(x, y)$ and $p_Y(y) = \sum_x p_{XY}(x, y)$.

This is true in general, and requires a proof.