Probability and Statistics: MA6.101

Tutorial 1

Topics Covered: Introductory Calculus, Maxima & Minima, Stationary Points, Sets, Basic Sequences and Series

Permutation and Combinations

Q1: **Basic Counting** For a trivia competition, 4-person teams are being formed from a pool of 15 interested students. How many different 4-person teams can be created?

A. 1345

B. 1260

C. 1365

D. 1512

E. 1780

Q2: **Pigeonhole Principle** In Bakul Hostel, a sudden power outage on prom night leaves Aanchik in complete darkness. His drawer contains:

- 100 red socks
- 80 green socks
- 60 blue socks
- 40 black socks

Desperate to match socks for the event, he grabs them one by one, unable to see their colors. For some weird reason, he wants 20 pairs of socks. What is the minimum number of socks Aanchik must pick to guarantee he has at least 10 pairs? (A pair consists of two socks of the same color, and no sock can be part of more than one pair.)

A. 20

B. 22

C. 23

D. 25

E. 30

Q3: **Combinatorics Applied to Probability** Two friends visit a restaurant randomly between 5-6 PM. Whoever arrives first waits 15 minutes and leaves. What is the probability they meet?

A. $\frac{1}{4}$

B. $\frac{3}{8}$

C. $\frac{7}{16}$

D. $\frac{15}{32}$

E. $\frac{1}{2}$

Linear Algebra

Q1: Let
$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x,y) = \big(u(x,y),\, v(x,y)\big) = \big(x^2y,\ e^{xy} + x + y\big).$$
Compute the Jacobian matrix $J_F(x,y)$. What is the geometrical significance of the Jacobian Matrix and its determinant?

Q2: Let $P \in \mathbb{R}^{n\times n}$ be a matrix for which each row sums to 1, i.e.,
$$\forall i \in \{1,\dots,n\}, \quad \sum_{j=1}^{n} P_{ij} = 1.$$
Prove that the equation
$$\pi = \pi P$$
always has a non-zero solution $\pi^\top \in \mathbb{R}^n$

Bonus: If the matrix P is irreducible and aperiodic, then there exists a unique stationary distribution π such that
$$\pi = \pi P \quad \text{and} \quad \sum_{i=1}^{n} \pi_i = 1.$$
Let
$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.2 & 0.3 & 0.5 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$
Find the solution π for the given P.

Introductory Calculus

Q1: Evaluate the limit:
$$\lim_{x\to 0} \frac{\int_0^x \tan t \, dt}{x}$$

Q2: Evaluate the integral:
$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

Q3: Let
$$I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$$
for any non-negative integer n.

(a) Show that $I_0 = \sqrt{\pi}$ and
$$I_1 = \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

(b) Using integration by parts, derive a recurrence relation of the form:

$$I_n = (2n-1)I_{n-1}$$

(c) [**Bonus**] Using mathematical induction and the recurrence above, prove that:

$$I_n = \frac{(2n-1)!! \cdot \sqrt{\pi}}{2^n}$$

where $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ denotes the **double factorial** of odd numbers.

Maxima / Minima and Stationary Points

Q1: Let

$$f(x) = \tan^{-1}(e^x - e^{-x}).$$

Find all stationary points and their nature.

Sets

Q1: **Bonferroni's inequality.**

(a) Prove that for any two events A and B, we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

(b) Generalize to the case of n events $A_1, A_2, \ldots, A_n$, by showing that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n) - (n-1).$$