Random variable as a measurable function

A random variable X is a function $X : \Omega \to \Omega'$ that transforms the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega', \mathcal{F}', P_X)$ and is '$(\mathcal{F}, \mathcal{F}')$-measurable'.

- The map $X : \Omega \to \Omega'$ implies $X(\omega) \in \Omega'$ for all $\omega \in \Omega$.
- For event $B \in \mathcal{F}'$, the pre-image $X^{-1}(B)$ is defined as $X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$

The '$(\mathcal{F}, \mathcal{F}')$-measurability' implies that for every $B \in \mathcal{F}'$, we have $X^{-1}(B) \in \mathcal{F}$.

Random variable as a measurable function

The '$(\mathcal{F}, \mathcal{F}')$-measurability' implies that for every $B \in \mathcal{F}'$, we have $X^{-1}(B) \in \mathcal{F}$.

- Since $X^{-1}(B) \in \mathcal{F}$, it can be measured using $\mathbb{P}$.
- What is $P_X(B)$?
- $P_X(B) := \mathbb{P}(X^{-1}(B))$ for all $B \in \mathcal{F}'$.
- $P_X(B)$ is therefore called as the induced probability measure.
- What if there is no $\omega \in \Omega$ such that $X(\omega) \in B$?

Random variables

- In general, the following convention is followed in most books:
  - Ω' will be the set of real numbers, denoted by $\mathbb{R}$.
  - $\mathcal{F}'$ as a result will be Borel σ-algebra, denoted by $\mathcal{B}(\mathbb{R})$.
  - Remember $\mathcal{B}(\mathbb{R})$?

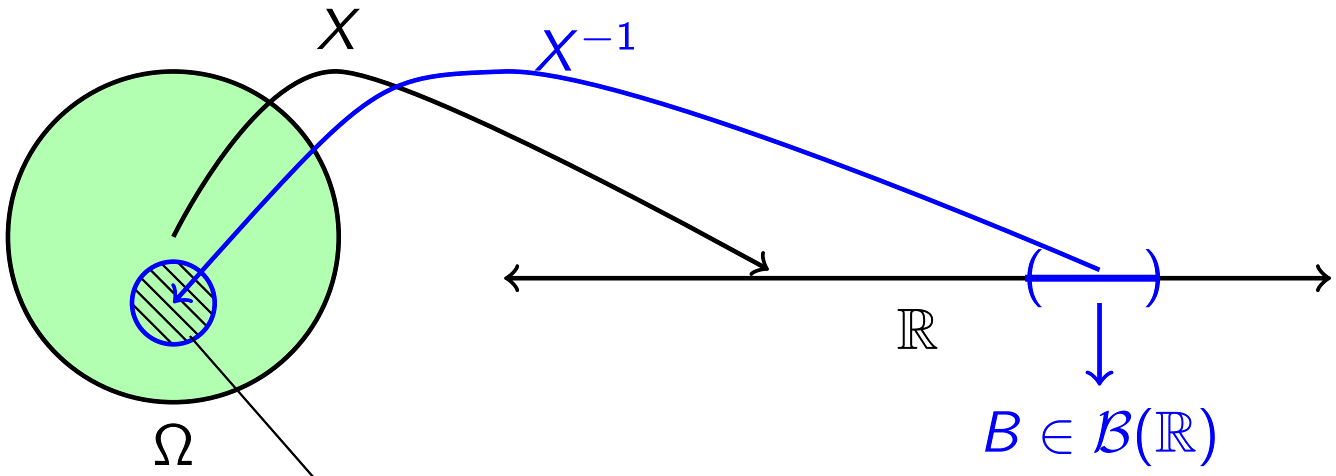
Borel σ-algebra

- Borel σ-algebra $\mathcal{B}(\mathbb{R})$:
  If $\Omega = \mathbb{R}$, then $\mathcal{B}(\mathbb{R})$ is the event set generated by open sets of the form (a, b) where $a \leq b$ and $a, b \in \mathbb{R}$.

- $\mathcal{B}(\mathbb{R})$ contains intervals of the form

$$[a, b]$$
$$[a, b)$$
$$(a, \infty)$$
$$[a, \infty)$$
$$(-\infty, b]$$
$$(-\infty, b)$$
$$\{a\}$$

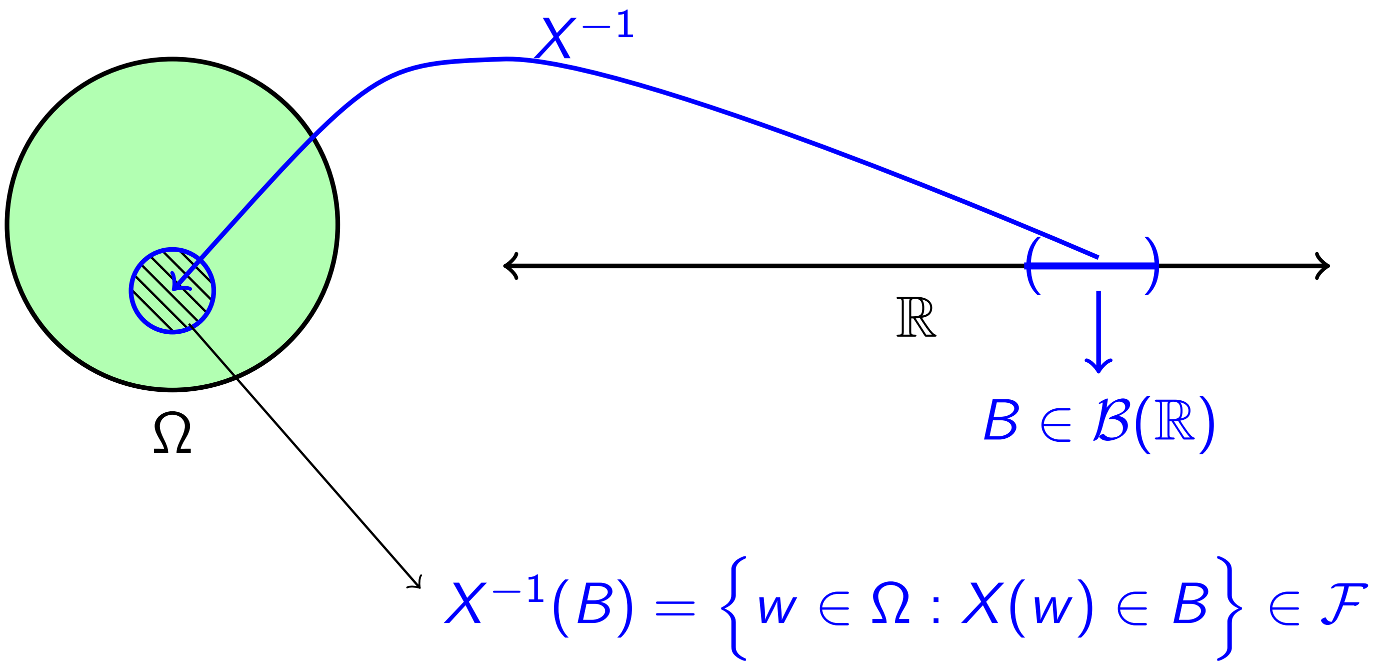
Random variables ($\Omega' = \mathbb{R}$)



$$X^{-1}(B) := \left\{ w \in \Omega : X(w) \in B \right\} \in \mathcal{F}$$

- $\Omega \xrightarrow{X} \mathbb{R}$, $\mathcal{F} \xrightarrow{X} \mathcal{B}(\mathbb{R})$, and $P(.) \xrightarrow{X} P_X(.)$
- Care must be taken such that the events you consider in the new event space $\mathcal{B}(\mathbb{R})$ are also valid events included in $\mathcal{F}$.
- $X^{-1}(B)$ is called as the preimage or the inverse image of B.

Definition of a random variables



A random variable X is a map $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ such that for each $B \in \mathcal{B}(\mathbb{R})$, the inverse image $X^{-1}(B) := \{w \in \Omega : X(w) \in B\}$ satisfies

$$X^{-1}(B) \in \mathcal{F} \text{ and}$$
$$P_X(B) = \mathbb{P}(w \in \Omega : X(w) \in B)$$

Random variable

- If Ω' is countable, then the random variable is called a discrete random variable.
- In this case it is convenient to use $\mathcal{F}'$ as power-set.
- If $\Omega' \subseteq \mathbb{R}$ or uncountable, then the random variable is a continuous random variable.
- In this case, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$ and the definition is a bit tricky. We will deal with it later.
- You can also use $\Omega' = \mathbb{R}$ for a discrete random variable and survive! Lets not get into that.
- Notation: Random variables denoted by capital letters like X, Y, Z etc. anf their realizations by small letters x, y, z..

Discrete random variables

Example of rolling two dice

- Example of rolling two dice where we are interested in the sum of two dice.
- Suppose X = sum of two dice. Then we have

$$\begin{aligned}\Omega = \Big\{&(1,1),(1,2),\ldots,(1,6)\\ &(2,1),(2,2),\ldots,(2,6)\\ &\qquad\qquad\vdots \qquad\qquad\qquad \xrightarrow{X} \qquad \Omega' = \big\{2,3,\ldots,12\big\}\\ &(6,1),(6,2),\ldots,(6,6)\Big\}\end{aligned}$$

- $\mathcal{F}$ and $\mathcal{F}'$ are power sets of Ω and Ω' respectively.
- Is X $(\mathcal{F},\mathcal{F}')$-measurable?

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- Suppose X = sum of two dice. Then we have

$$\Omega = \Big\{ (1,1),(1,2),\dots,(1,6) \\ (2,1),(2,2),\dots,(2,6) \\ \vdots \\ (6,1),(6,2),\dots,(6,6) \Big\} \xrightarrow{X} \Omega' = \big\{2,3,\dots,12\big\}$$

- $\{X = 3\}$ is an event in $\mathcal{F}'$. What is its probability $P_X(\{3\})$?
- $P_X(\{3\}) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = 3\}) = \mathbb{P}(\{(1,2),(2,1)\})$.

In general for $x \in \Omega'$, $P_X(\{x\}) := \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$.
Find $P_X(\{x\})$ for all $x \in \Omega'$?

Sum of two dice

- $\Omega' = \{2, 3, \ldots, 12\}$
- $\mathcal{F}' = \mathcal{P}(\Omega)$
- $P_X(\{x\}) = \begin{cases} \frac{x-1}{36} & \text{for } x \in \{2, 3, \ldots, 7\} \\ \frac{13-x}{36} & \text{for } x \in \{8, 9, \ldots, 12\}. \end{cases}$
- $Z =$Sum of 4 rolls ? Ω for 4 rolls is even complicated.
- This is where X is useful. $P(Z = 4) = P(X_1 = 2, X_2 = 2)$
- Here X_1 and X_2 are independent copies of random variable X.

PMF and CDF

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> The function $p_X(x) := P_X(\{x\})$ for $x \in \Omega'$ is called as a probability mass function (PMF) of random variable X.

- What is the PMF for a random variable corresponding to coin toss or roll a dice ?

-

> The cumulative distribution function (CDF) is defined as $F_X(x_1) := \sum_{x \leq x_1} p_X(x) = \mathbb{P}\{\omega \in \Omega : X(\omega) \leq x_1\}$.

- What is the CDF for the random variable corresponding to the coin toss or dice experiment?

Expectation and Moments

- How do you define the mean of a collection of numbers?
-

The mean or expectation of a random variable X is denoted by $E[X]$ and is given by $E[X] = \sum_{x \in \Omega'} x p_X(x)$.

- What is $E[X]$ for the random variable X that corresponds to the outcome of coin toss or dice experiment?
-

The n^{th} moment of a random variable X is denoted by $E[X^n]$ and is given by $E[X^n] = \sum_{x \in \Omega'} x^n p_X(x)$.

- For a function $g(\cdot)$ of a random variable X, its expectation is given by $E[g(X)] := \sum_{x \in \Omega'} g(x) p_X(x)$

Consistency of the PMF

- PMF: $p_X(x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$ for $x \in \Omega'$.
- How do you check if p_X is legitimate PMF?
- $\sum_{x \in \Omega'} p_X(x) = 1$. Can you prove this?

$$
\begin{aligned}
\sum_{x \in \Omega'} p_X(x) &= \sum_{x \in \Omega'} \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\}) \\
&= \mathbb{P}(\cup_{x \in \Omega'} \{\omega \in \Omega : X(\omega) = x\}) \\
&= \mathbb{P}(\Omega) \quad \square
\end{aligned}
$$

Linearity of Expectation

- Recall that $E[X] = \sum_{x \in \Omega'} x p_X(x)$.
- Functions of random variables are random variables.
- Furtermore, $E[g(X)] := \sum_{x \in \Omega'} g(x) p_X(x)$
- For $Y = aX + b$, what is $E[Y]$?

$$\begin{aligned} E[Y] &= \sum_{x \in \Omega'} (ax + b) p_X(x) \\ &= a \sum_{x \in \Omega'} x p_X(x) + b \\ &= aE[X] + b. \end{aligned}$$

- What is the PMF of Y?