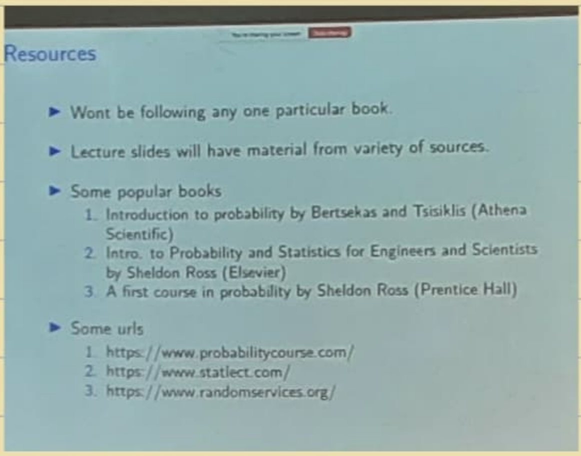
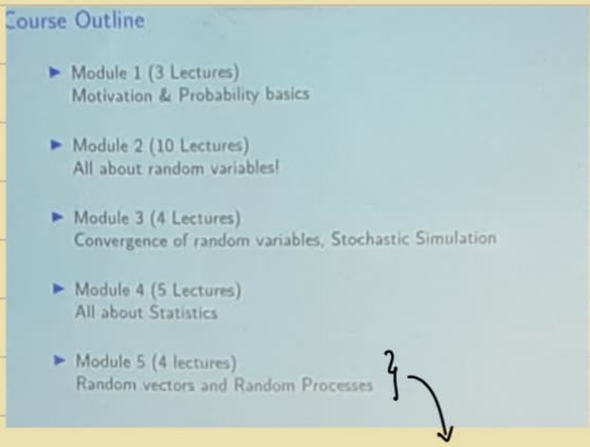
2/8

PnS





Finance

[3Blue1Brown]

→ **Random Experiments** :

Experiments involving randomness

- **Sample Space** (Ω) :

  Set of all possible outcomes of the random experiment.

  Can be finite or infinite.

e.g. $\Omega_c = \{H, T\}$

$$\Omega_{2c} = \{H,T\} \times \{H,T\}$$
$$= \{HH, HT, TH, TT\}$$

- **Sample point** / **Possible outcome** : $\omega \in \Omega$
- **Event** : $A \subseteq \Omega$

  $C_1 = \{T\}$ i.e. $C_1 \longrightarrow$ Event

- **Probability** of event A : $\mathbb{P}(A)$

  i.e. $\mathbb{P}(C_1) = 1/2$

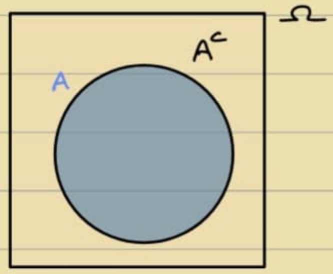
→ **Probability Theory** :

Probability is a measure.

↪ Set function

Set Theory :

• → H.W



ϕ : Empty set , $\phi \subseteq A, \forall A$

Union : $A \cup B$

Intersection : $A \cap B$

Difference : $A \setminus B \equiv A - B \equiv A \cap B^c$

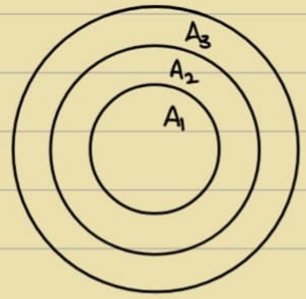
Symmetric difference : $A \Delta B \equiv (A \setminus B) \cup (B \setminus A)$

Disjoint events ≡ Mutually exclusive

Cardinality of A : $|A|$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(for n sets)



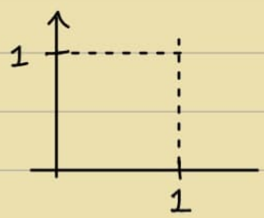
↳ Increasing Sequence

$A_1 \subseteq A_2 \subseteq A_3$

Cartesian Product : $\underbrace{A \times B}$

↳ $(a, b) \ \forall (a \in A \ \& \ b \in B)$

$$[0,1] \times [0,1] = \{(a,b) : a \in \underbrace{[0,1]}_{\Omega_1} \ \& \ b \in \underbrace{[0,1]}_{\Omega_2}\}$$



- Powerset of A : $P(A)$

$\Omega = \{1, 2, 3, 4, 5, 6\}$

1	2	3	4	5	6	
×	×	×	×	×	×	000000
×	×	×	×	×	✓	000001
×	×	×	×	✓	×	000010

} Binary Representation

i.e. $2^6 \longrightarrow |P(\Omega)|$ i.e. 2^{Ω}

$\underbrace{P([0,1])} = \ ? = \{(a,b) : a \leq b \ ; a, b \in [0,1]\}$ ⟵ COMES LATER (Sigma Algebra)

↳ Cardinality is uncountable infinite.

$\mathcal{B}(\mathbb{R})$??

→ **Functions** :

Domain D
Range R $\Big\} f : D \longrightarrow R$

(Injection
Surjection
Bijection)

e.g. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$

- **Set Functions** :

  Functions which act on sets

  D : Collection of sets

→ **Probability** : $\mathbb{P}$

↳ Set function

Axioms :

(1) $\mathbb{P}(\emptyset) = 0$

$\mathbb{P}(\Omega) = 1$

(2) For a set $A \subseteq \Omega$, we have $0 \le P(A) \le 1$

(3) For a disjoint collection of events $A_1, A_2, A_3, \ldots$, where $A_i \subseteq \Omega$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$ [Union ⟶ Countable sets]

Domain of $\mathbb{P}$ in general ⟶ Powerset of A , i.e. $P(A)$

$\mathbb{P}(\Omega) = 1$ $\qquad$ $P(\Omega) = \{A : A \subseteq \Omega\}$

6/8 **Counter - example** :

$\Omega = \mathbb{R} \Longrightarrow \mathbb{P}(\mathbb{R}) = 1$

Domain : $P(\mathbb{R}) \longrightarrow$ Complex

$\mathbb{P} : P(\mathbb{R}) \longrightarrow [0,1]$

$\mathbb{P}$ has the property that sets of equal 'length' have equal probability.

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n+1) \text{ , where } [n, n+1) \in P(\mathbb{R})$$

$$\mathbb{P}(\mathbb{R}) = 1 = \sum \mathbb{P}[n, n+1)$$

∴ Restrict domain to measurable sets

- Towards Sigma Algebra : Cantor's Theory

$\mathcal{F} = \{\phi, \Omega, R_-, R_+\}$

Event space or Sigma-Algebra $\mathcal{F}$ associated with a set Ω is a collection of subsets of Ω that satisfy

(1) $\phi \in \mathcal{F}$

$\Omega \in \mathcal{F}$

(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(3) $A_1, A_2, A_3, \ldots\ldots, A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

σ - Algebra is said to be closed under formation of complements & countable unions.

$A_1 \in \mathcal{F}$	$A_1^c \in \mathcal{F}$
$A_2 \in \mathcal{F}$	$A_2^c \in \mathcal{F}$

$(A_1^c \cup A_2^c)^c \Rightarrow$ Closed under the formation of countable intersections.

Note : When $\underbrace{\Omega}_{\text{Sample space}}$ is countable and finite $\Rightarrow \underbrace{P(\Omega)}_{\text{Sigma Algebra}}$ is the domain

$(\Omega, \mathcal{F}, \mathbb{P})$: Probability Space

$\mathbb{P}$ ↳ Probability measure on $(\Omega, \mathcal{F})$

$\mathbb{P} : \mathcal{F} \rightarrow [0,1]$

S.T. Axioms hold true

$|\Omega| < \infty$, $\mathcal{F} = 2^n$

- Probability space for $U[0,1]$:

$\Omega = [0,1]$

Suppose , $\mathcal{F} = \{\phi, [0,1], [0,.5), [.5,1]\}$

$P([.25,.75]) = .5$ but can we show?

$\mathcal{F}^+ = \{\phi, [0,1], [0,.5), [.5,1], [.25,.75]\}$

↳ Not a sigma algebra

$\mathcal{F}^{++} \rightarrow$ Not a sigma-algebra

Borel-Sigma Algebra $\mathcal{B}[0,1]$

$\mathcal{B}[0,1] \longrightarrow$ σ-Algebra generated by closed sets of form $[a, b]$

$a \leq b$ & $a, b \in [0,1]$

Can also be of form $(a, b]$ or $[a, b)$

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \tfrac{1}{n}, \, b - \tfrac{1}{n}\right]$$

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, \, b + \tfrac{1}{n}\right)$$

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a + \tfrac{1}{n}, \, b\right)$$

- $\mathcal{B}(\mathbb{R})$:

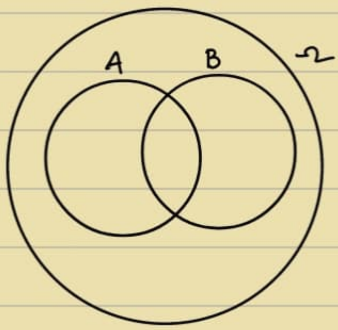
$\Omega = \mathbb{R}$

form $(a, b) \mid a \leq b \, ; \, a, b \in \mathbb{R}$

$\mathcal{B}(\mathbb{R})$ contains intervals of the form

$[a, b]$, $[a, b)$, $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b]$, $\{a\}$

- **Consequences of Probability Axioms** :
  - $P(A^c) = 1 - P(A)$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$A = (A \cap B) \cup (A \setminus B)$

$\underbrace{P(A) = P(A \cap B) + P(A \setminus B)}_{\text{H.W}}$

- $A \subseteq B$, Then prove that $P(A) \leq P(B)$

  ($A \subseteq B \Rightarrow$ Event A implies event B)

- $P(A \cup B \cup C) = ?$

  Hint : 3^{rd} Axiom

H.W : Inclusion – exclusion principle

- Impossible event v/s Zero prob. event :

In $U[0,1]$, $P(\omega = 0.5) = ?$

$$P(\omega = 0.5) = 0$$

$P(\omega \in [a,b]) = b-a$

Then, $P([.5,.5]) = P(\{.5\}) = 0$

$P(\phi) = 0 \Rightarrow \phi$ is impossible

e.g $C = [0,.5] \cap [.8,1]$ $\hookrightarrow$ Null Set

$$\Omega = \bigcup_{\omega \in \Omega} \{\omega\}$$

$$P(\Omega) = P\left(\bigcup_{\omega \in \Omega} \{\omega\}\right) = \sum_{\omega \in \Omega} P(\{\omega\}) = 0 \quad \times$$

9/8

$a_1, a_2, \ldots\ldots, a_n$

$n > N_\varepsilon$

$|a_n - L| \leq \varepsilon$

Continuous : $x \to c \Rightarrow f(x) \to f(c)$

Continuous set function S, $A_m \longrightarrow A$.

$$S(A_m) \longrightarrow S(A)$$

Lemma :

$$A_n \uparrow A \text{ (or) } A_n \downarrow A \Longrightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$$

$$P(A) = \sum_{i=1}^{\infty} P(F_i) \qquad F_n = A_n - A_{n-1}$$

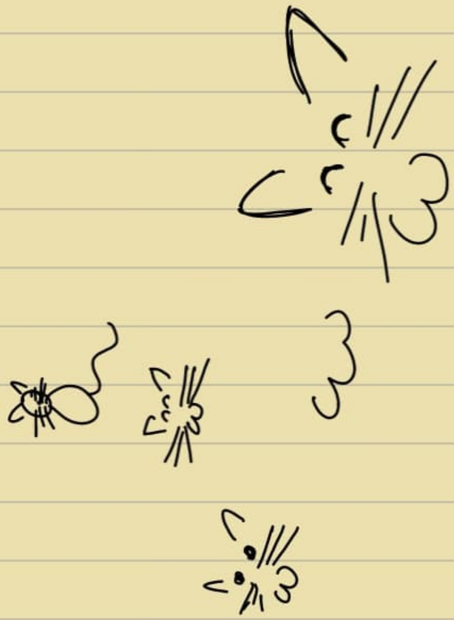
$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i)$$

$$= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} F_i\right)$$

$$= \lim_{n\to\infty} P(A_n)$$

- **Conditional Probability:**

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad , \quad P(A) > 1$$



Property:

$$P(A/B) \cdot P(B) = P(B/A) \cdot P(A)$$

H.W: $P(A/(B \cap C))$

$$P(A \cap B / C) = P(A/BC) \cdot P(B/C)$$

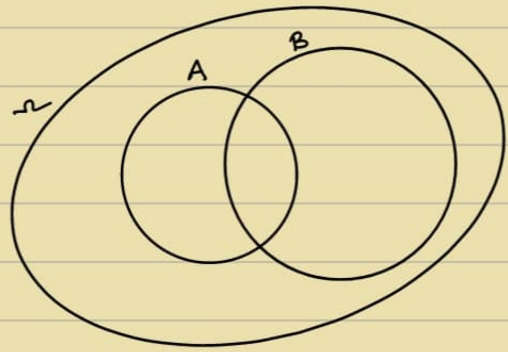
H.W: $P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/AB)$

H.W: $P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = P(A_1) \cdot P(A_2/A_1) \cdot \ldots\ldots$

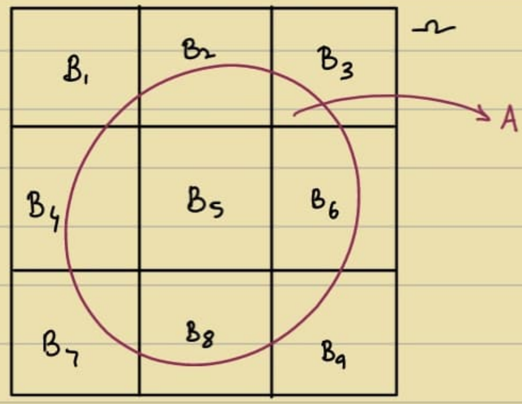
Ex. $A = (A \cap B) \cup (A \cap B^c)$

$P(A) = P(A \cap B) + P(A \cap B^c)$

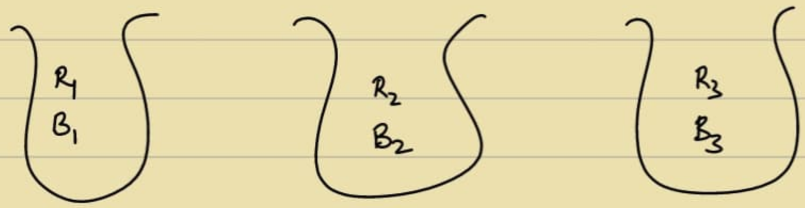
$= P(A/B) \cdot P(B) + P(A/B^c) \cdot P(B^c)$



$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A/B_i) \cdot P(B_i)$$



Q)



$R_1 + B_1 = M \qquad R_2 + B_2 = M \qquad R_3 + B_3 = M$

$$\frac{1}{3} \times \frac{R_1}{M} + \frac{1}{3} \times \frac{R_2}{M} + \frac{1}{3} \times \frac{R_3}{M}$$

13/8

- Independence:

Consider tossing a coin & rolling a dice simultaneously

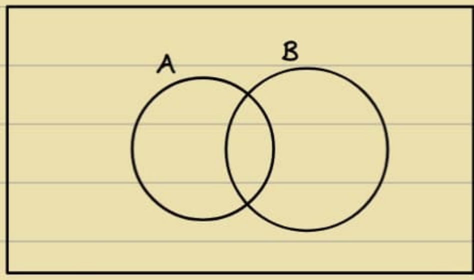
$$\Omega = \{\{H,1\}, \{H,2\}, \ldots\ldots, \{T,1\}, \ldots\}$$

$$P(\{H,6\}) = 1/12$$

$$\begin{aligned} P(\{T, odd\}) &= P(\{ \bigcup_{i=1,3,5} \{T, i\}\}) \\ &= 1/12 + 1/12 + 1/12 \\ &= 3/12 \\ &= 1/4 \end{aligned}$$

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A/B) = P(A)$$



To prove:

$$P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$$

$$\begin{aligned} P(A^c) \cdot P(B^c) &= (1 - P(A))(1 - P(B)) \\ &= 1 - P(A) - P(B) + P(A) \cdot P(B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= P(A^c \cap B^c) \end{aligned}$$

H.W: Are A and B^c independent?

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \prod_{i=1}^{n} \left[1 - P(A_i)\right] \quad \text{if } A_1, A_2, \ldots A_n \text{ are independent}$$

- Mutual Independence:

A_i, $i \in I$ ← collection of events

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j) \quad \text{for any subset } J \text{ of } I$$

- Pairwise Independence

MI ⇒ PI but vice-versa not always

All pairs are independent

H.W: find examples

Q) $\Omega = \{1, 2, 3, \ldots\ldots 10\}$

Event A : number < 7 $\Rightarrow P(A) = 3/5$

Event B : number < 8 $\Rightarrow P(B) = 7/10$

Event C : number is even $\Rightarrow P(C) = 1/2$

$A \subseteq B \Rightarrow P(A \cap B) = P(A) \Rightarrow$ A and B are not independent

$P(A \cap C) = 3/10 \Rightarrow$ A and C are independent

$\therefore$ No mutual independence

A and B are positively correlated iff $P(A/B) > P(A)$

A and B are negatively correlated iff $P(A/B) < P(A)$

A and B have same correlation as A^c and B^c

A and B have opp. correlation as A and B^c

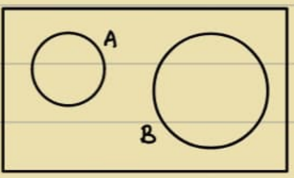
- Mutually exclusive :

One can happen $\Rightarrow$ Other event cannot occur

$P(A \cap B) = \phi$

$P(A/B) = 0$

$P(A/B^c) = \dfrac{P(A)}{P(B^c)}$



If $A \subseteq B$, then they are neither mutually exclusive nor independent.

Note : Zero probability events are always independent

Let $E \rightarrow$ Zero probability event, $P(E) = 0$

For any set, $P(E \cap F) = 0$

$$(E \cap F) \subseteq E$$

$$P(E \cap F) \leq P(E)$$

- Conditional Independence :

$P(A/B) = \dfrac{P(AB)}{B}$ | $P(A/BC) = \dfrac{P(AB/C)}{P(B/C)}$

$$P((AB)/C) = P(A/C) \cdot P(B/C) \quad , \quad P(C) > 0$$

$\therefore$ A and B are conditionally independent

$$\Rightarrow P(A/BC) = P(A/C)$$

Ex. 2 coins – 1 fair & 1 fake (Both heads)

Experiment $\longrightarrow$ Choose a coin uniformly & toss twice

Event A : First coin toss results in H

$$\tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot 1 = \tfrac{1}{4} + \tfrac{1}{2}$$
$$= \tfrac{3}{4}$$

Event B : Second coin toss results in H

$$= \tfrac{3}{4}$$

Event C : Coin 1 is chosen

$P(A \cap B)$

$\hookrightarrow$ Both coin tosses give heads

$$= P(A \cap B / C_F) \cdot P(C_F) + P(A \cap B / C_B) \cdot P(C_B)$$
$$= \tfrac{1}{8} + \tfrac{1}{2}$$

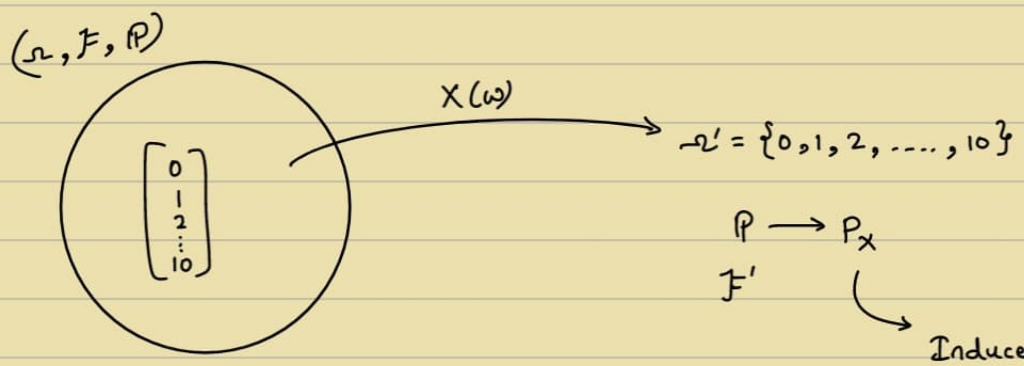
$P(A/C) = \tfrac{1}{2}$

$P(B/C) = \tfrac{1}{2}$

$P((A \cap B)/C) = \tfrac{1}{4}$

$\rightarrow$ Random variable :

$(\Omega, \mathcal{F}, \mathbb{P})$ to a simpler $(\Omega', \mathcal{F}', P_X)$



$\mathbb{P} \longrightarrow P_X$

$\mathcal{F}'$

$\hookrightarrow$ Induced Probability measure on Ω'

X is a function, $X : \Omega \rightarrow \Omega'$ that transforms the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega', \mathcal{F}', P_X)$ and is $(\mathcal{F}, \mathcal{F}')$ - measurable

$$X(\omega) \in \Omega' \quad \forall \, \omega \in \Omega$$

$$\forall B \in \mathcal{F}' \Rightarrow X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\} \quad , \text{ i.e. } X^{-1}(B) \in \mathcal{F}$$

16/8 & 20/8

Absent - Not well

23/8

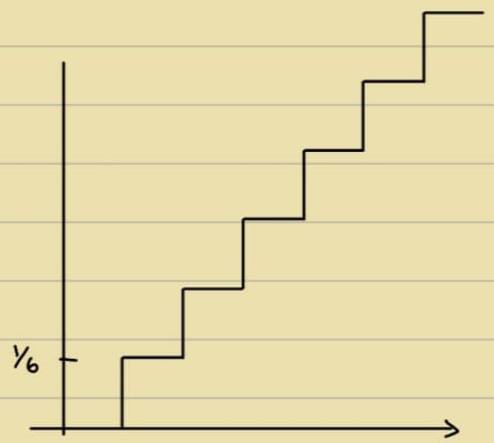
$$F_X(x) = \sum_{x \le x_1} P(x)$$

$P(X \le x_1)$

$\mathbb{P}(\omega \subset \Omega, X(\omega) \le x_1)$

- $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$

CDF : $F_X(x) = \mathbb{P}\{\omega \in \Omega : X(\omega) \le x\}$



$\Omega = \mathbb{R}$

CDF : Continuous $\longrightarrow$ Continuous variable

CDF : Discontinuous $\longrightarrow$ Discrete variable

PDF : $f_X(x) \quad \forall x \in \mathbb{R}$

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \le x + \Delta)}{\Delta}$$

$$= \lim_{\Delta \to 0^+} \frac{P(X \le x + \Delta) - P(X < x)}{(x + \Delta) - x}$$

$$= \lim_{\Delta \to 0^+} \frac{F_X(x+\Delta) - F_X(x)}{\Delta}$$

$P(A \cap B^c) = P(A) - P(B)$

$A : X \le x_1 + \Delta$

$B : X \le x_1$

$B^c : X > x_1$

$$\left(F_X(x_1) = P(X \le x_1)\right)$$

$$= \frac{d}{dx} F_X(x)$$

$$F_X(x) = \int_{u=-\infty}^{x} f_X(u)\, du$$

$$P_X(\mathbb{R}) = \int_{u=-\infty}^{\infty} f_X(u)\, du = 1$$

$$P_X([a,b]) = F_X(b) - F_X(a)$$

$$= \int_a^b f_X(u)\, du$$

For any $B \subseteq \mathbb{R}$, $P_X(B) = \int_{u \in B} f_X(u)\, du$

$$P_X(\{a\}) = 0$$

$$P_X([a,b]) = P_X((a,b]) = P_X([a,b)) = P_X((a,b))$$

- $E[X] = \int_{-\infty}^{\infty} u\, f_X(u)\, du$

$$E[X^n] = \int_{-\infty}^{\infty} u^n \cdot f_X(u)\, du$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(u) \cdot f_X(u)\, du$$

$Var[X] = E[g(X)]$ $\qquad g(X) = (x - E[X])^2$

For $Y = aX + b$,

$$E[Y] = a \cdot E[X] + b$$

$\underbrace{F_Y(y)}$ $= F_X\left(\frac{y-b}{a}\right)$ | $F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right)$ For $a < 0$

$$\left.\begin{array}{l} P(Y \le y) \\ P(aX + b \le y) \\ P(aX \le y - b) \\ P\left(X \le \frac{y-b}{a}\right) \\ F_X\left(\frac{y-b}{a}\right) \end{array}\right\} \underline{\underline{a > 0}}$$

$$\frac{d}{du} F_X(u) \cdot \frac{du}{dy} \qquad \left[u = \frac{y-b}{a}\right]$$

$$\frac{1}{a} P_X\left(\frac{y-b}{a}\right)$$

- Uniform Random variable:

$$f_X(x) = \frac{1}{b-a} \quad \forall x \in [a,b]$$

$$\text{CDF} : F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & x \in [a,b] \\ 1, & \text{otherwise} \end{cases}$$

$$E[X] = \frac{a+b}{2}$$

$$Var[X] = \frac{(b-a)^2}{12}$$

- Exponential Random variable:

Non-negative R.V with parameter λ

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0 \qquad E[X] = \frac{1}{\lambda}$$

$$F_X(x) = 1 - e^{-\lambda x} \text{ for } x \geq 0 \qquad Var(X) = 1/\lambda^2$$

Memoryless property

$$P(X > a+h \mid X > a) = \frac{e^{-\lambda(a+h)}}{e^{-\lambda(a)}} = e^{-\lambda h} = P(X > h)$$

Prep for Quiz - 1

Sample Space : Ω

Possible outcome : $\omega \in \Omega$

Event : $A \subseteq \Omega$

Probability : $\mathbb{P}(A)$

Powerset : $P(A)$

(1) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$

(2) $A \subseteq \Omega$, $0 \leq \mathbb{P}(A) \leq 1$

(3) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$, $A_i \rightarrow$ Disjoint events

$\rightarrow$ Sigma Algebra ?

$$\mathcal{F} = \{\phi, \Omega, \mathbb{R}_-, \mathbb{R}_+\}$$

(i) $\phi \in \mathcal{F}$ & $\Omega \in \mathcal{F}$

(ii) $B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$

(iii) $A_1 \in \mathcal{F}$ & $A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2 \in \mathcal{F}$ & $A_1 \cap A_2 \in \mathcal{F}$

Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$

Measurable space : $(\Omega, \mathcal{F})$

$\rightarrow$ Borel - Sigma Algebra ?

$\mathcal{B}[0,1]$

$\rightarrow$ LCD:

$\lim_{n \to \infty} a_n = L$ | $x \rightarrow c \Rightarrow f(x) \longrightarrow f(c)$

Q) Defective → 98% Acc.

Not defective → 99% Acc.

0.1% → Total Defective

$\mathbb{P}$(Robot says defective | Defective) = 0.98

$\mathbb{P}$(Robot says not def. | Not Def.) = 0.99

⇒ $\mathbb{P}$(Robot says def. | Not def.) = 0.01

$\mathbb{P}$(Defective) = 0.01

⇒ $\mathbb{P}$(Not def.) = 0.99

$P(A|B)$ occured

$$\mathbb{P}(\text{def.} \mid \text{Robot says def.}) = \frac{\mathbb{P}(\text{robot says def.} \mid \text{def}) \cdot \mathbb{P}(\text{def.})}{\mathbb{P}(\text{Robot says def.})}$$

$$= \frac{0.98 \times 0.01}{\frac{98}{100} \times \frac{0.1}{100} + \frac{99.9}{100} \times \frac{1}{100}}$$

$$= 8.9\%$$

Independence: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

+ve correlation: $\mathbb{P}(A|B) > \mathbb{P}(A)$

−ve correlation: $\mathbb{P}(A|B) < \mathbb{P}(A)$

$$\mathbb{P}(A/B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A/BC) = \frac{\mathbb{P}(AB/C)}{\mathbb{P}(B/C)}$$

$$\mathbb{P}(AB/C) = \mathbb{P}(A/BC) \cdot \mathbb{P}(B/C)$$

→ Binomial:

$$P(i) = P(X=i) = \binom{n}{i} p^i \cdot (1-p)^{n-i}$$

$$E[X] = \sum x P(x)$$

$$= \sum_{i=0}^{n} i \cdot P(i)$$

$$= \sum_{i=0}^{n} (i) \cdot \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i \cdot \frac{n!}{(n-i)!\,i!} \cdot p^i \cdot \frac{(1-p)^n}{(1-p)^i}$$

$$= n \cdot \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} \cdot p^i \cdot (1-p)^{n-i}$$

$$= np \cdot \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} \cdot (1-p)^{(n-1)-(i-1)}$$

$$= np \cdot \sum_{i=0}^{m} \binom{m}{j} p^j \cdot (1-p)^{m-j} = np$$

→ Geometric:

$$P(X=i) = (1-p)^{i-1} p$$

$$E[X] = \sum_{i=0}^{n} x \cdot P(x)$$

$$= \sum_{i=0}^{n} i \cdot (1-p)^{i-1} \cdot p$$

$$= \sum_{i=0}^{n} i \cdot (1-p)^{i} \cdot \frac{p}{1-p}$$

$$= \frac{p}{1-p} \cdot \sum_{i=1}^{n} i(1-p)^{i}$$

$$= \frac{p}{1-p} \cdot \frac{1-p}{p^2} \cdot (1-(1-p)^n)$$

$$= \frac{1}{p}$$

$$S = 1(1-p) + 2(1-p)^2 + 3(1-p)^3 + \ldots$$

$$(1-p) S = \quad 1(1-p)^2 + 2(1-p)^3 + \ldots$$

$$S \cdot p = (1-p) + (1-p)^2 + (1-p)^3 + \ldots \infty$$

$$= (1-p) \cdot \frac{1-(1-p)^n}{p}$$

$$\Rightarrow S = \frac{1-p}{p^2} \cdot (1-(1-p)^n)$$

→ Uniform:

$$P(X=x) = \frac{1}{b-a} \quad \forall x \in [a,b]$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot P(x) \, dx$$

$$= \int_{a}^{b} x \cdot \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \cdot \left(\frac{x^2}{2}\right]_a^b$$

$$= \frac{a+b}{2}$$

→ Exponential:

$$P(X=x) = \lambda e^{-\lambda x} \quad \forall x \geq 0$$

$$E[X] = \int_{0}^{\infty} \lambda e^{-\lambda x} = \left. e^{-\lambda x} \right]_0^{\infty} \cdot \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}(0+1) = \frac{1}{\lambda}$$

Q) 10 min break

$\mathbb{P}(\text{rain}) = 3/10$

$\mathbb{P}(\text{crowd}) = 1/2$

$\mathbb{P}(\text{late} \mid \text{rain}) = 1/2$

$\mathbb{P}(\text{Late} \mid \text{crowded}) = 3/10$

(a) $\mathbb{P}(\text{late}) = \mathbb{P}(\text{late} \mid \text{rain}) \cdot \mathbb{P}(\text{rain}) + \mathbb{P}(\text{late} \mid \text{crowded}) \cdot \mathbb{P}(\text{crowded})$

$= 1/2 \cdot 3/10 + 3/10 \cdot 1/2$

$= 3/10$

(b) $\mathbb{P}(\text{rain} \mid \text{late}) = \dfrac{1/2 \cdot 3/10}{3/10} = 1/2$

$\mathbb{P}(\text{crowded} \mid \text{late}) = \dfrac{1/2 \cdot 3/10}{3/10} = 1/2$

(c) $\mathbb{P}(\text{rain} \cap \text{crowded}) = \mathbb{P}(\text{rain}) \cdot \mathbb{P}(\text{crowded})$

$= 3/10 \cdot 1/2$

$= 3/20$

Q) $E_{11}\ E_{12} \qquad E_{21}\ E_{22}$

$(E_{11}E_{21} + E_{11}E_{22} + E_{12}E_{21} + E_{12}E_{22} + \ldots)$

$P(\text{crash}) = \overline{E_{11}}\,\overline{E_{12}} + \overline{E_{21}}\,\overline{E_{22}} - \overline{E_{11}}\,\overline{E_{12}}\,\overline{E_{21}}\,\overline{E_{22}}$

$= 2/10 \cdot 2/10 + 2/10 \cdot 2/10 - 2/10 \cdot 2/10 \cdot 2/10 \cdot 2/10$

$= 8/100 - 16/10000 = 784/10000$

$\overline{P}(\text{crash}) = 1 - 784/10000 = \dfrac{9216}{10000}$

$= \underline{\underline{0.9216}}$

9) $P(X=k) = \dfrac{e^{-\lambda}\cdot\lambda^k}{k!}$ $(\lambda > 0)$, $k = 0, 1, 2, \ldots$

$$E[X] = \sum_{x=0}^{\infty} x\,P(x)$$

$$= \sum_{i=0}^{\infty} i \cdot \frac{e^{-\lambda}\cdot\lambda^i}{i!}$$

$$= e^{-\lambda} \cdot \sum_{i=0}^{\infty} i \cdot \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \cdot \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} \cdot \lambda$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

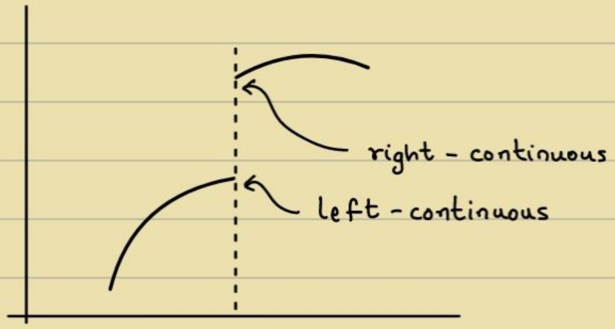
$$e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \cdots$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \underline{\lambda}$$

$$\mathrm{Var}[X] = \lambda(1 - \lambda/n)$$

$$\to E[X^2] - (E[X])^2$$

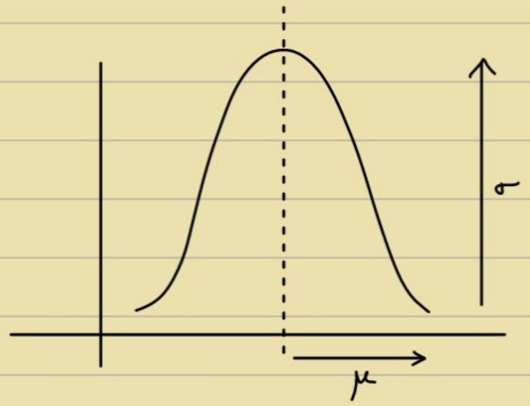
3/9



→ Gaussian Random variable ($N(\mu, \sigma^2)$) :

- Continuous RV

μ: mean, σ^2: variance



PDF : $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-\mu)^2}{2\sigma^2}} \quad \forall x \in \mathbb{R}$

H.W : $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$E[X] = \mu$

$Var[X] = \sigma^2$

- Standard Normal R.V : $N(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$F_X(x) : \Phi(x) \longleftarrow$ Table

- Normality preserved under Linear Transformations :

$X \sim N(\mu, \sigma^2)$

$Y = AX + B \longleftarrow N(A\mu + B,\ A^2\sigma^2)$

$E[Y] = A\mu + B$

$Var[Y] = A^2\sigma^2$

- Central Limit Theorem :

$\frac{1}{n}\sum_{i=1}^{n} X_i \approx N(\mu, \sigma^2/n)$ [X : Any RV with mean μ & variance σ^2]

- Multinomial Gaussian vector ← Countably finite

$\underbrace{\text{Gaussian Process}}$ ← Countably infinite

→ Used in Bayesian Optimization (Black-box optimization)

→ Brownian Motion

Note : Beta, Gamma, Erlang, Logistic

- $Y = aX + b$

$$F_Y(y) = P(Y \le y) = P(aX + b \le Y)$$
$$= F_X\left(\frac{y-b}{a}\right) \quad \text{if } a > 0$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \quad \text{when } a > 0$$

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \quad \text{if } a < 0$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{-1}{a} f_X\left(\frac{y-b}{a}\right) \quad \text{when } a < 0$$

$$\therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Generalising : [Intuition]

$$\left.Y = aX + b \implies X = \frac{Y-b}{a}\right\} \Rightarrow f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$
$$\Downarrow$$
$$\left.Y = g(X) \implies X = h(Y)\right\} \Rightarrow f_Y(y) = \left|\frac{d\,h(y)}{dy}\right| f_X(h(y))$$

Monotone function, continuous and differentiable

$$\therefore h(x) = g^{-1}(x)$$

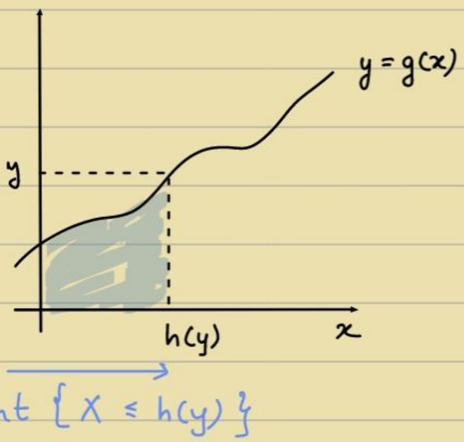
Proof :

$$Y = g(X) \implies X = h(y)$$

Monotone, continuous and differentiable function

$$F_Y(y) = P(g(X) \le y)$$
$$= P(X \le h(y))$$

$$\left[\begin{array}{l} g(X) \le y \;:\; A = \{\omega \in \Omega \mid g(X(\omega)) \le y\} \\ \Rightarrow h(g(X)) \le h(y) \\ \Rightarrow X \le h(y) \;:\; A = \{\omega \in \Omega \mid X(\omega) \le h(y)\} \end{array}\right]$$



As $g(x) \uparrow \Rightarrow h(x) \downarrow$

case - I :

$g(x) \longleftarrow$ Non-decreasing

$$\begin{aligned} F_Y(y) &= P(g(X) \le y) \\ &= P(X \le h(y)) \\ &= F_X(h(y)) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_X(h(y)) \\ &= \underbrace{f_X(h(y))}_{\text{non-dec}} \cdot \underbrace{\frac{dh}{dy}(y)}_{\ge 0} \end{aligned}$$

case - II :

$g(x) \longleftarrow$ Non-increasing

$$\begin{aligned} F_Y(y) &= P(g(X) \le y) \\ &= P(X \ge h(y)) \\ &= 1 - F_X(h(y)) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{-d}{dy} F_X(h(y)) \\ &= - \underbrace{f_X(h(y))}_{\text{non-inc.}} \cdot \underbrace{\frac{dh}{dy}(y)}_{\le 0} \end{aligned}$$

$$\Downarrow$$

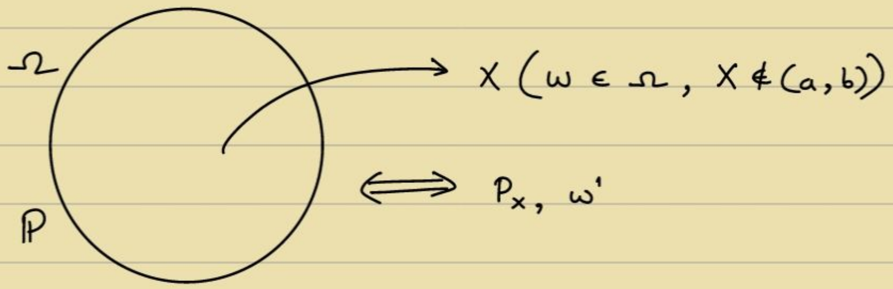
$$f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right|$$

Q) $Y = X^2$, $f_Y(y) = ?$ in terms of $f_X(x)$

6/9

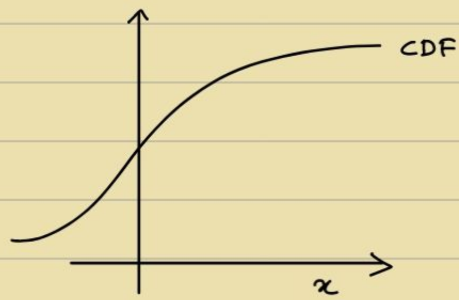
$\mathbb{P}$: Probability measure

P_X : Induced Probability measure



$$\begin{aligned} F_X(a) &= P_X(-\infty, a] \\ &= P(X \le a) \end{aligned}$$

Point mass $\rightarrow$ Discontinuity



- CDF:

(i) $F_X(\infty) = 1$

$F_X(-\infty) = 0$, when $P(-\infty < X < \infty) = 1$

(ii) $F_X : \mathbb{R} \rightarrow [0,1] \leftarrow$ Non-decreasing & right continuous

(iii) @ Points of discontinuity :

$$\left.\begin{aligned} F_X(x^+) &:= \lim_{\varepsilon \to 0} F_X(x+\varepsilon) \\ F_X(x^-) &:= \lim_{\varepsilon \to 0} F_X(x-\varepsilon) \end{aligned}\right\} \varepsilon \text{ decreases to } 0\text{, i.e. } \varepsilon \downarrow 0$$

[$\because \varepsilon$ is positive]

$$F_X(x^+) \ne F_X(x^-)$$

Right continuous : $F_X(x) = F_X(x^+)$

if $F_X(x)$ is continuous $\Rightarrow F_X(x^+) = F_X(x^-)$

<u>Th</u> :

$F_X : \mathbb{R} \to [0,1]$ is non-decreasing and right continuous

(i) non-decreasing :

$\forall x_1, x_2 : x_1 \leq x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$ $\quad$ [x_1 & x_2 are arbitrary]

[From $a \leq b \Rightarrow P(A) < P(B)$]

Define $A := \{\omega \in \Omega : X(\omega) \leq x_1\}$

Define $B := \{\omega \in \Omega : X(\omega) \leq x_2\}$

$\therefore A \subseteq B$

$\therefore \mathbb{P}(A) \leq \mathbb{P}(B)$

$F_X(x_1) = P_X((-\infty, x_1]) = \mathbb{P}(A) < \mathbb{P}(B) = F_X(x_2)$

(ii) right-continuous :

Consider seq. of no. $\{x_n\}$ decreasing to x

$$F_X(x^+) = \lim_{x_n \to x} F_X(x_n)$$

Define $A_n := \{\omega : X(\omega) \leq x_n\}$

Define $A := \{\omega : X(\omega) \leq x\}$

[When $A_n \uparrow A$ or $A_n \downarrow A \Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$]

$$\left[\left. \begin{array}{c} (-\infty, x_n] = A_n \\ \downarrow \\ (-\infty, x] = A \end{array} \right\} \Rightarrow P_X \right]$$

$F_X(x) = P_X(-\infty, x]$

$\therefore$ RHS $= F_X(x)$

LHS $= F_X(x_n) = F_X(x)$

If $x_n \uparrow x \Rightarrow (-\infty, x_n]$

$\underbrace{\lim_{x_n \uparrow x} (-\infty, x_n]}_{\cup_n (-\infty, x_n]} = (-\infty, x)$ $\quad$ [$\because$ Going to x]

& $P_X(-\infty, x) \neq F_X(x)$ $\quad$ [By definition]

$\because$ There could be point mass at x.

[Ref : 4.3.1
probabilitycourse.com]

→ **Mixed RV** :

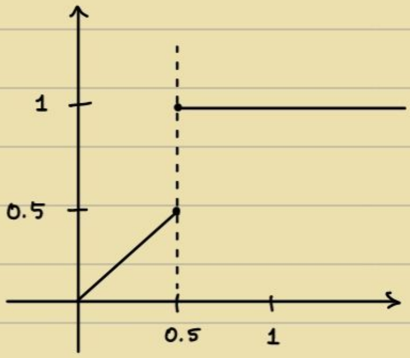
Neither discrete nor continuous , Partly both

CDF ⟶ Piece - wise continuous

e.g. $X = U[0,1]$ and $Y = X$ if $X \leq 0.5$

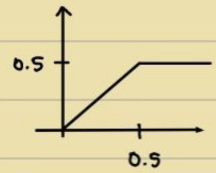
$Y = 0.5$ if $X \geq 0.5$

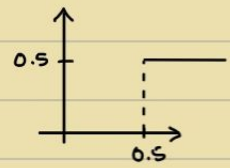
CDF :



$\therefore F_Y(y) = C(y) + D(y)$

Continuous   Discontinuous



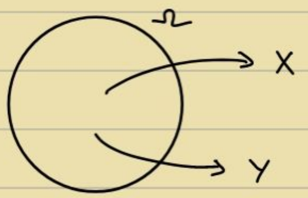


$$E[Y] = \int_{-\infty}^{\infty} x \cdot C(x)\, dx + \sum_{y_k} y_k P(Y = y_k)$$

where $\{y_1, y_2, \dots y_n\}$ are jump points of $D(y)$

where $P(Y = y_k) > 0$

- **Multiple R.V** :



$\Omega = \{0,1\} \times \{1,2,3,4,5,6\}$, $\mathcal{F} = 2^{\Omega}$, $P(\omega) = \frac{1}{12}$

$X_\omega \in \{0,1\}$ ⟵ Coin

$Y_\omega \in \{1,2,3,4,5,6\}$ ⟵ Dice

for $\omega = (1,5) \Rightarrow X(\omega) = 1$ and $Y(\omega) = 5$

Joint PMF : $P_{X,Y}(x,y)$

Joint CDF : $F_{X,Y}(x,y)$

$$P_{X,Y}(x,y) := \mathbb{P}\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = \omega\}$$

$$F_{X,Y}(x,y) := \mathbb{P}\{\omega \in \Omega : X(\omega) \leq x \text{ and } Y(\omega) \leq \omega\}$$

$$P((X,Y) \in A) = \mathbb{P}\{\omega \in \Omega : (X(\omega), Y(\omega)) \in A\}$$
$$= \sum_{(x,y) \in A} P_{X,Y}(x,y)$$

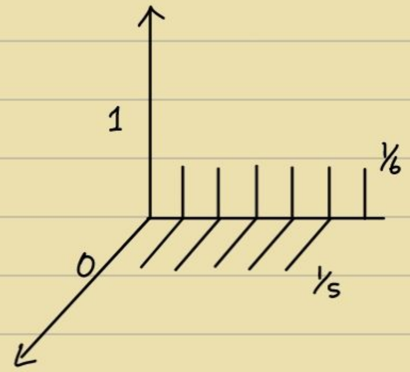
e.g. $P((X,Y) \in A)$; Given, A : coin $\rightarrow$ head & dice $\rightarrow$ 6

if $H \rightarrow \{1,2,3,4,5,6\}$

$T \rightarrow \{1,2,3,4,5\}$

$$\{\overset{1/12}{(1,1)}, (1,2), \ldots\ldots \overset{1/12}{(1,6)},$$
$$\underset{1/10}{(0,1)}, (0,2), \ldots\ldots \underset{1/10}{(0,5)}\}$$

$$\begin{bmatrix} 1/12 = 1/2 \times 1/6 \\ 1/10 = 1/2 \times 1/5 \end{bmatrix}$$



IMP

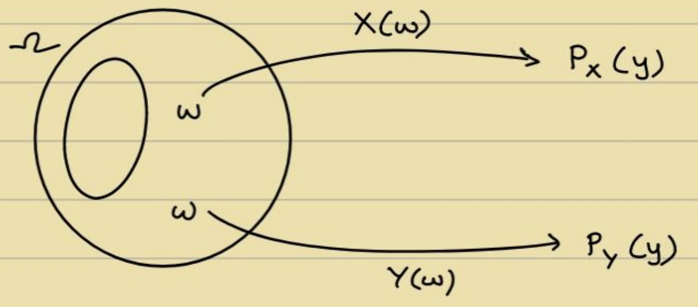
- <u>Marginals</u> :

$$\left.\begin{array}{l} P(Y=3) = \frac{1}{10} + \frac{1}{12} \\ P(Y=6) = \frac{1}{12} \\ P(X=1) = \frac{1}{2} \end{array}\right\} \begin{array}{l} P_{X,Y}(1,i) = 1/12 \\ \sum_i P_{X,Y}(1,i) = P\{X \in \Omega, X(\omega) = 1\} = 1/2 = P_X(x) \\ P_{X,Y}(1,i) + P_{X,Y}(0,i) = \frac{1}{6} = P_Y(i) \end{array}$$

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

<u>10/9</u>



$$\left.\begin{array}{l} P_X(y) \\ P_Y(y) \end{array}\right\} P_{XY}(x,y)$$

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

$$\sum_{\substack{x \le a \\ y \le b}} P_{XY}(x,y) = F_{XY}(a,b)$$
$$= P(X \le a, Y \le b)$$
$$= P(\omega \in \Omega, X(\omega) \le a, Y(\omega) \le b)$$

$$\sum_y P_{XY}(x,y) = P_X(x) = P(\omega \in \Omega, X(\omega) = x)$$
$$P_{XY}(x,y) = P(\omega \in \Omega, X(\omega) = x \ \& \ Y(\omega) = y)$$

$$\sum_y P_{XY}(x,y) = \mathbb{P}\left(\bigcup_y (\omega \in \Omega,\ X(\omega) = x \ \&\ Y(\omega) = y)\right)$$

$$= \sum_y \mathbb{P}\left((\omega \in \Omega,\ X(\omega) = x \ \&\ Y(\omega) = y)\right)$$

Note : $F_X(x) = \sum_y F_{XY}(x, y)$

- Independent :

$$P_{XY}(x, y) = P_X(x) \cdot P_Y(y)$$
$$F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$$

$$P_{XY}(x,y) = \mathbb{P}(\omega \in \Omega,\ X(\omega) = x \ \&\ Y(\omega) = y)$$

$$A = \{\omega \in \Omega,\ X(\omega) = x\}$$
$$B = \{\omega \in \Omega,\ Y(\omega) = y\}$$
$$\Longrightarrow A \cap B = \{\omega \in \Omega,\ X(\omega) = x \ \&\ Y(\omega) = y\}$$

$$P_X(x) = \mathbb{P}(A)$$
$$P_Y(y) = \mathbb{P}(B)$$

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$
$$= \mathbb{P}(\omega \in \Omega,\ X(\omega) \leq x \ \&\ Y(\omega) \leq y)$$

- Expectation : $E[XY]$

$$E[X] = \sum_x x \cdot P_X(x) = \sum_x \sum_y x \cdot P_{XY}(x, y)$$

$$E[Y] = \sum_y y \cdot P_Y(y) = \sum_y \sum_x y \cdot P_{XY}(x, y)$$

$$E[XY] = \sum_x \sum_y xy \cdot P_{XY}(x, y)$$

$E[XY] = E[X] \cdot E[Y]$ for independent events, X and Y given
(Converse may not be true)

e.g. When X and Y are dependent.
Roll a dice &

$$X = \begin{cases} 1 & \text{if outcome is odd} \\ 0, & \text{otherwise} \end{cases} \quad \Bigg| \quad Y = \begin{cases} 1 & \text{if outcome is even} \\ 0, & \text{otherwise} \end{cases}$$

$P_{XY}(0,0) = 0 \qquad P_{XY}(0,1) = \frac{1}{2}$

$P_{XY}(1,0) = \frac{1}{2} \qquad P_{XY}(1,1) = 0$

$$P_Y(0) = P_X(0) = \sum_y P_{XY}(0,y) = \frac{1}{2}$$

$$P_Y(0) = P_X(1) = \sum_y P_{XY}(1,y) = \frac{1}{2}$$

$$F_{XY}(x,y) \neq F_X(x) \cdot F_Y(y)$$

$$E[XY] \neq E[X] \cdot E[Y]$$

- Consistency Properties :

(i) $\sum_{x,y} P_{XY}(x,y) = 1$

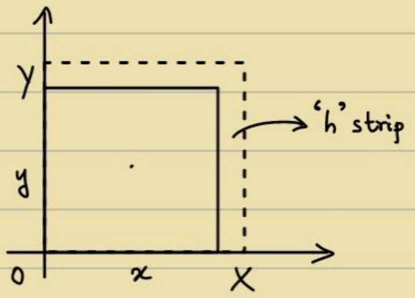
(ii) $F_{XY}(\infty, \infty) = 1$

(iii) $F_{XY}(-\infty, -\infty) = 0$

(iv) $F_{XY}(-\infty, \infty) = F_{XY}(\infty, -\infty) = 0$

(v) $\left.\begin{array}{l} F_{XY}(x, \infty) = F_X(x) \\ F_{XY}(\infty, y) = F_Y(y) \end{array}\right\} \longleftarrow$ Marginal

$F_{XY}(x,y)$:



$F_{XY}(x,y) = xy \quad \& \quad f_{XY}(x,y) = 1$

$F_{XY}(x+h, y) - F_{XY}(x,y)$

$$\frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \longleftarrow \left\{ \begin{array}{l} \dfrac{\partial F_{XY}(x,y)}{\partial x} = \lim_{h \to 0} \dfrac{F_{XY}(x+h,y) - F_{XY}(x,y)}{h} \\ \dfrac{\partial F_{XY}(x,y)}{\partial y} = \lim_{h \to 0} \dfrac{F_{XY}(x,y+h) - F_{XY}(x,y)}{h} \end{array} \right.$$

$\downarrow$

Joint PDF

$f_{X,Y}(x,y)$

$$F_{XY}(x,y) := \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(a,b) \, da \cdot db$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) \, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) \, dx$$

Apply for ≥ 2 R.Vs

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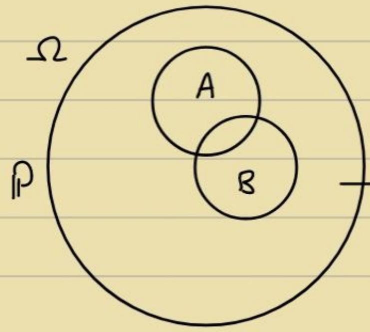
- 

$P(A|B)$

$\xrightarrow{X(\omega)} \Omega' = \mathbb{R}$

$P_X(\cdot)$

$F_X(x_1) = \sum_{x \le x_1} P_X(x)$

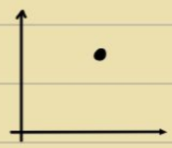
$P_X(x) = \mathbb{P}\{\omega \in \Omega,\ X(\omega) = x\}$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$P_{XY}(x, y) \leftarrow$ Joint PDF

$$F_{XY}(x_1, y_1) = \sum_{x \le x_1} \sum_{y \le y_1} P_{XY}(x, y)$$

$$f_{XY}(x, y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} F_{XY}(x, y)$$

$\underbrace{f_{XY}(x, y)}\ dx.dy \longrightarrow$ 

Probability per unit length / area / vol
i.e. Probability Density Function

- $X \longrightarrow p_X(x)$

$$E[\underbrace{g(X)}_{z}] = \sum_x g(x)\, P_X(x)$$

$$P_Z(z) = \sum_{\substack{x \\ z = g(x)}} P_X(x)$$

If $z = g(x, y)$

$P_Z(z) = ?$

$$P_Z(z) = \sum_{\substack{xy \\ g(x,y) = z}} P_{XY}(x, y)$$

$$E[aX + bY + c] = a\,E[X] + b\,E[Y] + c$$

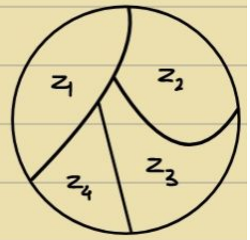
Proof : $\sum_{xy} g(x, y) \cdot P_{XY}(x, y)$

$$\left\{ \sum_{xy} (ax + by + c)\, P_{XY}(x, y) \right\}$$

$$\sum_x \sum_y P_{XY}(x, y) = 1$$

$$E[Z] = \sum_{z} \sum_{\substack{x,y \\ g(x,y)=z}} z \cdot p_{XY}(x,y) \qquad \because p_Z(z) = \sum_{\substack{x,y \\ g(x,y)=z}} p_{XY}(x,y)$$

$$E[g(X,Y)] = \sum_{x,y} g(x,y)\, p_{XY}(x,y) \qquad E[Z] = \sum_{z} z\, p_Z(z)$$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, dx \cdot dy$$

- X and Y $\rightarrow$ continuous independent R.V

$W = \max(X, Y)$

$Z = \min(X, Y)$

$$\left\{ \begin{array}{l} \text{i.e. } P_{XY}(x,y) = P_X(x) \cdot P_Y(y) \\ F_{XY}(x,y) = F_X(x) \cdot F_Y(y) \\ E[XY] = E[X] \cdot E[Y] \end{array} \right\}$$

(i) $F_W(\omega_1) = P(W \leq \omega_1)$

$= P(X \leq \omega_1 \ \& \ Y \leq \omega_1)$

$= P(X \leq \omega_1) \cdot P(Y \leq \omega_1)$

$= F_X(\omega_1) \cdot F_Y(\omega_1)$

$F_W(\omega_1) = F_{XY}(\omega_1, \omega_1)$, i.e. $W \leq \omega_1 \Leftrightarrow X, Y \leq \omega_1$

$$p_W(\omega) = \frac{d}{d\omega} F_W(\omega)$$

$$= F_X(\omega) \cdot f_Y(\omega) + f_X(\omega) \cdot F_Y(\omega)$$

(ii) $F_Z(z_1) = P(Z \leq z_1)$

$Z \geq z_1 \Leftrightarrow X \geq z_1$ and $Y \geq z_1$

$\overline{F_Z}(z_1) = P(Z > z_1)$

$= \overline{F_{XY}}(z_1, z_1)$

$= 1 - F_{XY}(z_1, z_2)$

$= 1 - F_X(z_1) \cdot F(z_2)$

e.g. if $\underset{(\lambda_1)}{X}$ & $\underset{(\lambda_2)}{Y}$ $\rightarrow$ exponential , Then $\underset{(\lambda_1+\lambda_2)}{Z}$ $\rightarrow$ exponential

$$\left\{ \text{Hint : } e^{-\lambda_1 \omega} \cdot e^{-\lambda_2 \omega} = e^{-(\lambda_1+\lambda_2)\omega} \right\}$$

9) $Z = X + Y$

$$P_Z(z_1) = \sum_{\substack{x,y \\ x+y=z_1}} P_{XY}(x,y)$$
$$= \sum_{x,y} P_X(x) \cdot P_Y(y)$$
$$= \sum_x P_X(x) \cdot \sum_y P_Y(y)$$
$$= \sum_x P_X(x) \cdot P_Y(z_1 - x)$$

$$\begin{bmatrix} x + y = z \\ \Rightarrow y = z - x \end{bmatrix}$$
for satisfying (x,y)

- Covariance of X and Y:
  (Joint Variation)

$$\mathrm{Cov}(X,Y) = E\Big[(X - E[X])(Y - E[Y])\Big]$$
$$= E[XY] - E[X] \cdot E[Y]$$

$$\left\{ \text{What is } \begin{bmatrix} \mathrm{cov}(X,X) & \mathrm{cov}(X,Y) \\ \mathrm{cov}(Y,X) & \mathrm{cov}(Y,Y) \end{bmatrix} ? \right\} \leftarrow \text{Covariance Matrix } \mathrm{cov}[X,Y]$$

Gives correlation

$$P_X(x) = \sum_y P_{XY}(x,y)$$

$$P_Y(y) = \sum_x P_{XY}(x,y)$$

$$g(x,y) = (x - E[X]) \cdot (y - E[Y])$$

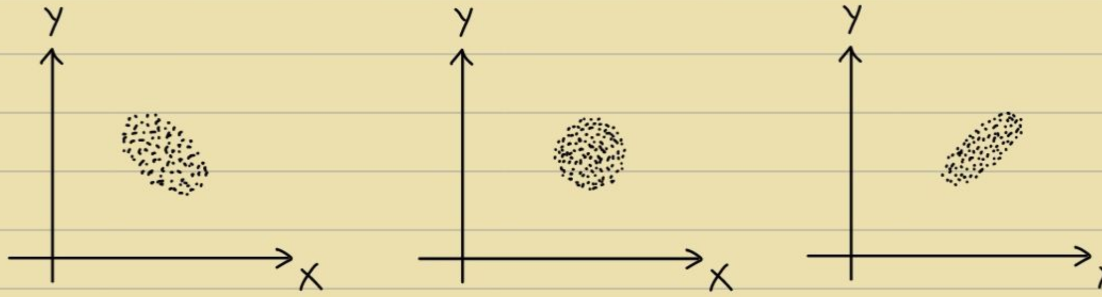
$$\underbrace{E}_{E_{P_{XY}(x,y)}}[g(X,Y)] = \sum_{xy} g(x,y) \cdot P_{XY}(x,y)$$

$$E_{P_{XY}(x,y)}\Big[(X - E_{P_X(x)}[X])(Y - E_{P_Y(y)}[Y])\Big] = \sum_{xy} g(x,y) \cdot P_{XY}(x,y)$$

<u>Note</u>: $E[XY] = E[X] \cdot E[Y]$ only when X and Y are uncorrelated & doesn't necessarily imply independence

Independence $\Rightarrow$ Uncorrelated

Uncorrelated $\not\Rightarrow$ Independence (Not always)



$\mathrm{cov}(X, Y) < 0$ $\qquad$ $\mathrm{cov}(X, Y) \approx 0$ $\qquad$ $\mathrm{cov}(X, Y) > 0$

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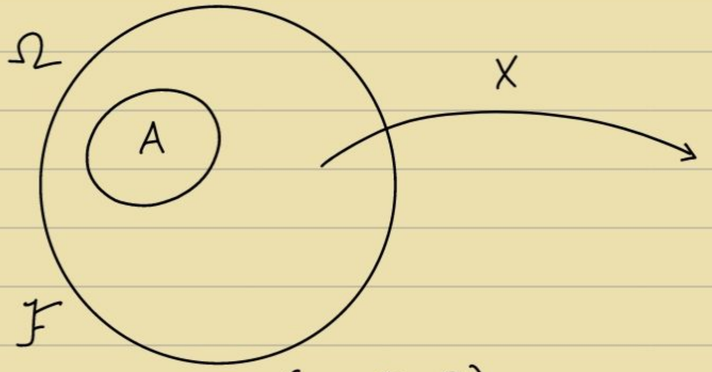


$(\Omega, \mathcal{F}, \mathbb{P})$

$x_1 \quad x_2 \; \ldots\ldots \; x_{10}$

$P_1 \quad P_2 \; \ldots\ldots \; P_{10}$

$(\Omega', \mathcal{F}', P_X) \qquad \Omega' = \mathbb{R}$

$(a, b) \qquad \mathcal{F} = \mathcal{B}(\mathbb{R})$

$$P_X(x) \equiv P_{X/A}(x)$$

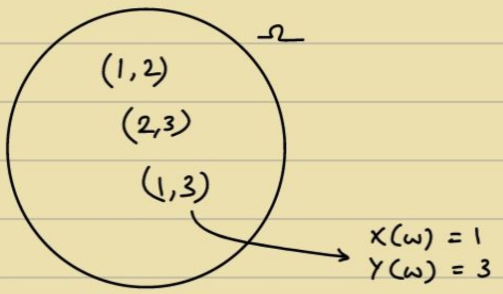
$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

$$\sum P(A_i) \cdot P_{X/A_i}(x) = P_X(x)$$

Ex. Pick 2 integers from $\{1, 2, 3\}$ without replacement

$$\Omega = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$$

$$\mathbb{P}\{\omega\} = 1/6 \quad \forall \omega \in \Omega$$



$$P_{XY}(x, y) = 1/6$$

$$P_X(x) = \sum_{\substack{y \\ \{1,2,3\}}} P_{XY}(x, y) = 1/6 + 1/6 = 1/3$$

$$P_Y(y) = \sum_{\substack{x \\ \{1,2,3\}}} P_{XY}(x, y) = 1/6 + 1/6 = 1/3$$

$E[X] = 2$

$E[Y] = 2$

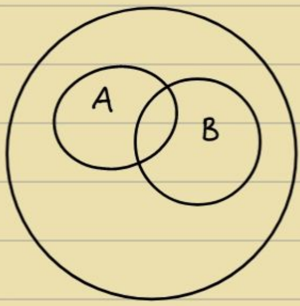
$$\begin{aligned} E[XY] &= \sum g(x, y) \cdot P_{XY}(x, y) \\ &= 2 \cdot P_{XY}(1, 2) + \ldots \\ &= 11/3 \end{aligned}$$

$$P_X(x) \cdot P_Y(y) = 1/9$$

→ R.V. X with $P_X(x)$

Event A, $A \in \mathcal{F}$

Consider event $\{\omega \in \Omega : X(\omega) = x\} \equiv \{X = x\}$



$$\mathbb{P}(X = x \mid A) = \frac{\mathbb{P}(\{X = x\} \cap A)}{\mathbb{P}(A)} = P_{X|A}(x)$$

e.g. first no. → odd
second no. → even $\Big\}$ $A = \{(1,2), (3,2)\}$

$$P_{X|A}(1) = \frac{\mathbb{P}(X=1 \cap A)}{\mathbb{P}(A)} = \frac{1/6}{1/3} = \frac{1}{2}$$

Similarly, $P_{X|A}(3) = 1/2$

[Prev. $P_X(x) = 1/3$ for $x = 1, 2, 3$]

$$\sum_x P_{X|A}(x) = \sum_x \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}$$

$$= \frac{\sum_x \mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1$$

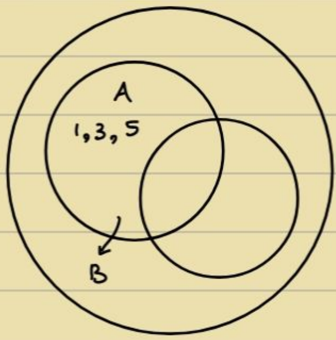
Ex. Roll a dice

A : roll is odd

$P_{X|A}(x) = ?$ $\qquad \Omega = \{1, 2, 3, 4, 5, 6\}$

$$P_{X|A}(x) = \frac{1/6}{1/2} = 1/3$$

$\{1/3, 0, 1/3, 0, 1/3, 0\}$



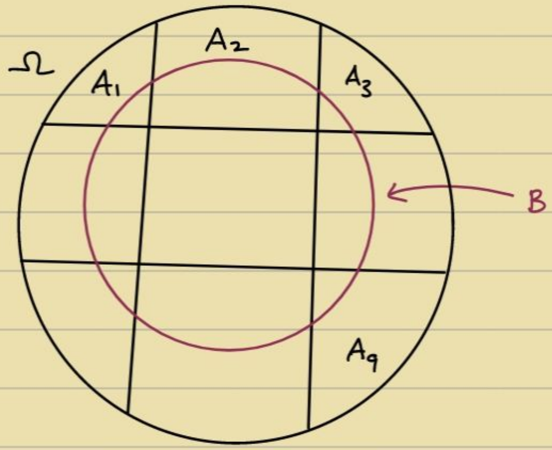
$E[X|A] = 3$

$E[X|A] = 4$ (if $A \to$ even)

$$E[X|A] = \sum_x x \, P_{X|A}(x)$$

Using LOTUS, $E[g(X)|A] = \sum_x g(x) \cdot P_{X|A}(x)$

- Law of Total Probability :



$$P(B) = \frac{\sum P(B \cap A_i)}{P(A_i)} \cdot P(A_i)$$

$$= \sum_i P(B|A_i) \cdot P(A_i)$$

$$P_X(x) = \sum_i P(A_i) \cdot P_{X|A_i}(x)$$

Proof : $B = \{\omega \in \Omega, X(\omega) = x\}$

$$P_X(x) = \sum_i P(A_i) \cdot \frac{P(\{X=x\} \cap A_i)}{P(A_i)}$$

$$= \sum_i P(\{X=x\} \cap A_i)$$

$$= P(\{X=x\})$$

$B = \bigsqcup (B \cap A_i)$

Disjoint Union

$$P(B) = \sum_i \frac{P(B \cap A_i)}{P(A_i)} \cdot P(A_i)$$

Now, $E[X] = \sum_i P(A_i) \cdot E[X|A_i]$

- e.g. $A = (X > 5) = \{6\}$

$A \in \mathcal{F}'$

$X \in A = \{\omega \in \Omega : X(\omega) = A\}$

$P\{X \in A\} = \sum_{x \in A} P_X(x)$

$\Omega' = \{x_1, x_2, \ldots\ldots, x_{10}\}$

$\mathcal{F} = 2^{\Omega'}$

$A \in \mathcal{F}$

$x_1 \quad x_2 \quad x_{10}$

Let $A = \{x_1, x_9, x_{10}\}$

$$P_{X|A}(x_1) = ? = 1/3 \qquad P_{X|A}(x_1) = \frac{\mathbb{P}(\{X = x_1\} \cap A)}{\mathbb{P}(A)} = \frac{P(A \cap \overset{\{x_1\}}{B})}{P_X(A)} = \frac{P_X(x_1)}{P_X(A)}$$

$$P_{X|A}(x_2) = ? = 0$$

$\vdots$

$$P_{X|A}(x_{10}) = ? = 1/3$$

$$P_X(A) = \mathbb{P}(A^{-1}) = \{\omega \in \Omega, X(\omega) \in A\}$$

i.e. if $x \notin A \Rightarrow P_{X|A}(x) = 0$

if $x \in A \Rightarrow P_{X|A}(x) = \frac{P_X(x)}{\mathbb{P}(X \in A)}$

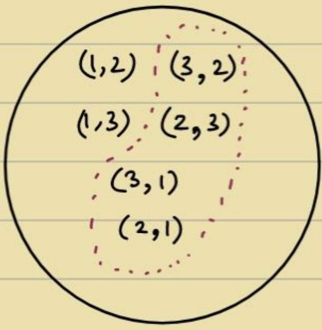
Ex. Rolling a dice, $X > 2$

i.e. $\{3, 4, 5, 6\} = A$

$$P_{X|A}(x_1) = 0$$
$$P_{X|A}(x_2) = 0$$
$$P_{X|A}(x_3) = 1/4 = \frac{1/6}{4/6}$$
$$P_{X|A}(x_4) = 1/4$$
$$P_{X|A}(x_5) = 1/4$$
$$P_{X|A}(x_6) = 1/4$$

e.g. $X \in A$ where $A = \{2, 3\}$

$P_{X|A}(x) = ?$



$P_A = 4/6$

$P_X(x) = 1/3$

- Geometric R.V :

$$P_N(k) = (1-p)^{k-1} p \quad \leftarrow \text{PMF}$$

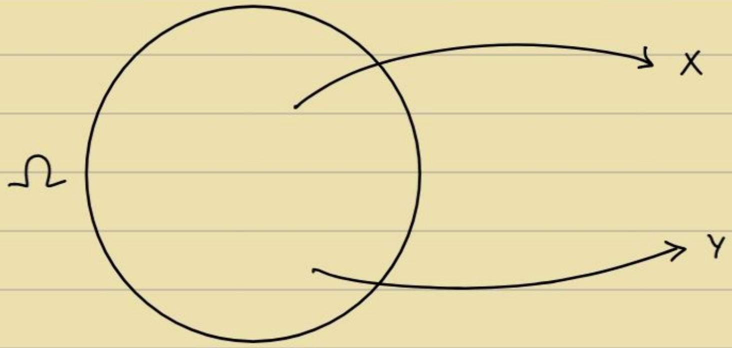
If $A := N > n$ & $B := N + j$

$P_{N|A}(k) = ?$

$\rightarrow P_{N|A}(k) = 0$ if $k < n-1$

$$P_{N|A}(k) = \frac{\overset{\to P\{N=k\}}{P\{(N > n) \cap N = k\}}}{\underset{\to (1-p)^n}{P(N > n)}} = (1-p)^{k-1-n} \cdot p$$

20/9



$P_{XY}(x, y)$ $\qquad$ @ $Y = y$

$f_{XY}(x, y)$ $\qquad$ $P_{X|A}(x)$

$P_{X|Y}(x|y) \longrightarrow f_{X|Y}(x, y)$

$\hookrightarrow P(A|B) = P(X=x | Y=y) / P(Y=y)$

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}$$

$$\Rightarrow P_{XY}(x,y) = P_{X|Y}(x|y) \cdot P_Y(y)$$

$$[P(A \cap B) = P(A|B) \cdot P(B)]$$

- Independence :

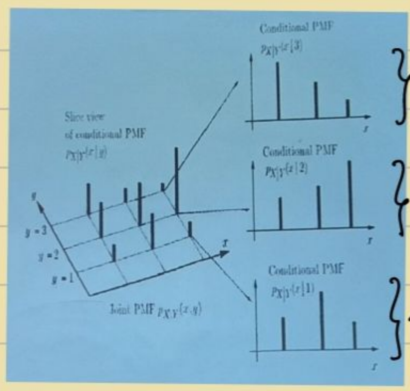
$$P_{XY}(x,y) = P_{X|Y}(x|y) \cdot P_Y(y)$$
$$\Rightarrow P_X(x) \cdot P_Y(y) = P_{X|Y}(x|y) \cdot P_Y(y)$$
$$\Rightarrow P_X(x) = P_{X|Y}(x|y)$$
$$\Rightarrow P_{X|Y}(x|y) = P_X(x)$$

$$\therefore E[XY] = E[X] \cdot E[Y]$$
$$\& \ \mathrm{cov}(X,Y) = 0$$

- $P_{XY}(x,y) = P_{X|Y}(x|y) \cdot P_Y(y)$

$$\Rightarrow P_X(x) = \sum_y P_{X|Y}(x|y) \cdot P_Y(y)$$

$\hookrightarrow$ Law of Total Probability



Normalised

Summing over x instead of y :

$$P_Y(y) = \sum_x P_{X|Y}(x|y) \cdot P_X(x)$$

$$E[X|Y=y] = \sum_x x \, P_{X|Y}(x|y)$$

$$\begin{aligned} E[X] &= \sum_x x \, P_X(x) \qquad \text{(PS-34)} \\ &= \sum_x x \cdot \sum_y P_{X|Y}(x|y) \cdot P_Y(y) \\ &= \sum_y \sum_x x \, P_{X|Y}(x|y) \cdot P_Y(y) \\ &= \sum_y P_Y(y) \cdot E[X|Y=y] \end{aligned}$$

Summary :

① $$P_{X|A}(x) = \frac{P_X(x)}{P(A)} \quad (\text{if } x \in A) = \frac{f_X(x)}{P(A)} \quad (\text{if } x \in A)$$

② $$E[X|A] = \sum_x x\, P_{X|A}(x) = \int_{-\infty}^{\infty} x \cdot f_{X|A}(x)$$

③ $$P_X(x) = \sum_{i=1}^{n} P(A_i)\, P_{X|A_i}(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)$$

④ $$E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]$$

⑤ $$P_{XY}(x,y) = P_{X|Y}(x|y) \cdot P_Y(y)$$
$$f_{XY}(x,y) = f_{X|Y}(x,y) \cdot f_Y(y)$$

⑥ $$P_X(x) = \sum_y P_{X|Y}(x|y) \cdot P_Y(y)$$
$$f_X(x) = \int_y f_{X|Y}(x|y) \cdot f_Y(y)\, dy$$

⑦ $$E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y)$$
$$E[X|Y=y] = \int_x f_{X|Y}(x|y)\, dx$$

⑧ $$E[X] = \sum_y P_Y(y) \cdot E[X|Y=y]$$
$$E[X] = \int_y f_Y(y) \cdot E[X|Y=y] \cdot dy$$

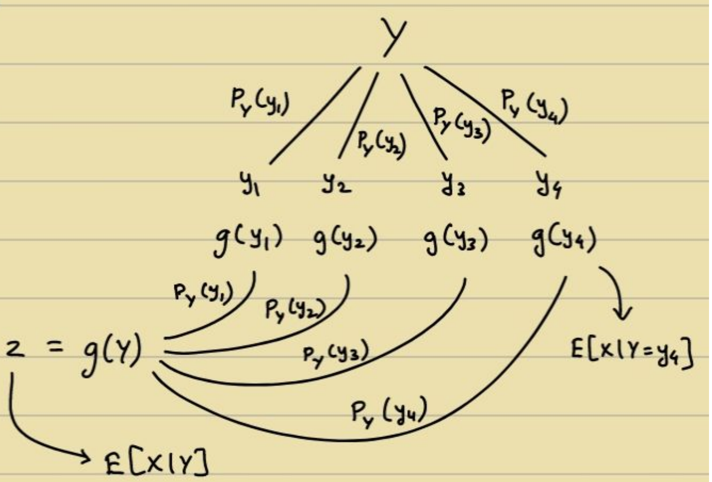
→ Conditional Expectation : $E[X|Y]$

$$E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y)$$

$$g(y) = E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y)$$
$$Y = y_1, y_2, y_3, \dots y_{10}$$

$$E[Z] = E[g(Y)]$$
$$= \sum_y P_Y(y) \cdot g(y)$$

$$E[E[X|Y]] = \sum_y P_Y(y) \cdot E[X|Y=y]$$
$$= E[X]$$



- Ex. $Y = \begin{cases} \lambda_1 & \text{with prob } p \\ \lambda_2 & \text{with prob } 1-p \end{cases}$

$X \sim \text{Exponential}(Y)$

$$\left[\begin{array}{l} \because f_X(x) = \lambda e^{-\lambda x} \\ \quad X \rightsquigarrow \text{Exp}(\lambda) \\ \quad E[X] = 1/\lambda \end{array}\right]$$

$$\begin{tikzcd}[column sep=small] & E[X|Y] \arrow[dl, dash, "P_Y(y_1)"'] \arrow[d, dash, "P_Y(y_2)"] \arrow[dr, dash, "P_Y(y_3)"] & \\ E[X|Y=y_1] & E[X|Y=y_2] & E[X|Y=y_3] \\ g(y_1) & g(y_2) & g(y_3) \end{tikzcd}$$

$$E[E[X|Y]] = E[X]$$

$E[X] = ?$

$$E[X|Y=\lambda_1] = 1/\lambda_1$$
$$E[X|Y=\lambda_2] = 1/\lambda_2$$

$$\begin{aligned} E[X] &= \sum_{i=1}^{2} E[X|Y=y_i] \cdot P_Y(y_i) \\ &= 1/\lambda_1 \cdot (p) + 1/\lambda_2 (1-p) \\ &= p(1/\lambda_1 - 1/\lambda_2) + 1/\lambda_2 \end{aligned}$$

$$f_{X|Y}(x|\lambda_1) = \lambda_1 e^{-\lambda_1 x}$$
$$f_{X|Y}(x|\lambda_2) = \lambda_2 e^{-\lambda_2 x}$$

$$\therefore E[X] = (1/\lambda_1 - 1/\lambda_2)\, p + 1/\lambda_2$$

$$E[X] = E[\underbrace{E[X|Y]}]$$

Misuse of Notation, but intuitively.

Ex. $Y = X_1 + X_2 + \dots + X_N$, $N > 0$

X_i's are independent & identically distributed with mean $E[X]$

i.i.d $\rightarrow$ PMF of all X_i are same

$E[Y] = ?$

$$\begin{aligned} E[Y|N=n] &= E[X_1 + X_2 \dots + X_n] \\ &= E[X_1] + E[X_2] + \dots + E[X_n] \\ &= n\,E[X] \end{aligned}$$

$$E[Y|N] = \sum_{i=1}^{N} E[X_i|N] = N \cdot E[X]$$

$$\begin{aligned} E[Y] &= \sum_n E[Y|N=n] \cdot P_N(n) \\ &= E[X] \cdot E[N] \end{aligned}$$

(OR) $E[E[Y|N]] = E[N \cdot E[X]]$

$\Rightarrow E[Y] = E[N] \cdot E[X]$

$Var[Y] = ?$ $\longleftarrow$ H.W

$$Var[Y] = E[Y^2] - E[Y]^2$$

As $Y = \sum X_i$

& $\because E[Y] = E[X] \cdot E[N]$

$$E[Y^2] = E\left[\left(\sum X_i\right)^2\right]$$
$$= E\left[\sum (X_i)^2 + 2 \cdot \sum_{1} \sum_{i+1} X_i X_j\right]$$
$$= E\left[\sum (X_i)^2\right] + 2 \cdot E\left[\sum_{1} \sum_{i+1} X_i X_j\right]$$
$$\therefore \; E[Y^2] = \sum E[(X_i)^2] + 2 \sum_{1} \sum_{i+1} E[X_i X_j]$$

$$Var[Y] = E[Y^2] - E[Y]^2$$
$$= \sum E[(X_i)^2] + 2 \sum_{1} \sum_{i+1} E[X_i] \cdot E[X_j] - (E[Y])^2$$
$$= \sum E[(X_i)^2] + 2 \sum_{1} \sum_{i+1} E[X_i] \cdot E[X_j] - \left(E\left[\sum X_i\right]\right)^2$$
$$= \sum E[(X_i)^2] + 2 \sum_{1} \sum_{i+1} E[X_i] \cdot E[X_j] - \left(\sum E[X_i]\right)^2$$
$$= \sum E[(X_i)^2] + 2 \sum_{1} \sum_{i+1} E[X_i] \cdot E[X_j] - \sum (E[X_i])^2 - 2 \sum_{1} \sum_{i+1} E[X_i] E[X_j]$$
$$= \sum E[(X_i)^2] - \sum (E[X_i])^2$$
$$= \sum Var[X_i]$$
$$= Var[X_1] + Var[X_2] + \dots + Var[X_N]$$

$$\left\{ \begin{array}{l} \therefore \text{ For Independent R.Vs:} \\ Var[X+Y] = Var[X] + Var[Y] \end{array} \right\}$$

$$Var[Y \mid N=n] = Var[X_1 + X_2 \dots + X_n]$$
$$= Var[X_1] + Var[X_2] + \dots + Var[X_n]$$
$$= n \cdot Var[X]$$
$$Var[Y \mid N] = \sum_{i=1}^{N} Var(X_i \mid N) + 2 \cdot \sum_{i<j} cov(X_i, X_j \mid N)$$

$$Var[Y] = \sum_{n} Var[Y \mid N=n] \cdot P_N(n)$$

$$E[Var[Y \mid N]] + Var\left[\sum_{i=1}^{N} E[X_i \mid N]\right]$$
$$= E[N] \cdot Var[X] + Var[N] \cdot (E[X])^2$$

$$E[Var[Y \mid N]] = E[N \cdot Var[X]]$$
$$= E[N] \cdot Var[X]$$
$$Var[E[Y \mid N]] = Var[N \cdot E[X]]$$
$$= Var[N] \cdot (E[X])^2$$