Induced measure P_X and CDF

- Consider a random variable X that maps $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$
- We define the cumulative distribution function (CDF) by $F_X(x) = \mathbb{P}\{\omega \in \Omega : X(\omega) \leq x\}$.
- $F_X(x)$ can also be expressed using induced measure P_X.
- Since the domain of P_X is $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R})$ is made up of sets of the form $(-\infty, a]$ for $a \in \mathbb{R}$.
- $F_X(x) = P_X((-\infty, x]) = \mathbb{P}\{\omega \in \Omega : X(\omega) \leq x\}$.
- This is a general definition of CDF (applicable for both continuous or discrete).

Continuous random variables

Continuous random variables

- A random variable defined on $\mathbb{R}$ is discrete, if $F_X(\cdot)$ is piecewise constant.
- A random variable defined on $\mathbb{R}$ is continuous, if $F_X(\cdot)$ is a continuous function.
- Note that in general, a continuous function say $F_X(\cdot)$ need not be differentiable.
- But in this course, we will only consider continuous random variables for which $F_X(\cdot)$ is continuous as well as differentiable.
- Examples of Continuous random variables
  1. Pick a number uniformly from $[a, b]$.
  2. Level of water in a dam or pending workload on a server.

Continuous random variables

- Associated with a continuous random variable is a probability density function (pdf) $f_X(x)$ for all $x \in \mathbb{R}$. Its unit is probability per unit length and is defined as

$$
\begin{aligned}
f_X(x) &:= \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta} \\
&= \lim_{\Delta \to 0^+} \frac{F_X(x + \Delta) - F_X(x)}{\Delta} \\
&= \frac{dF_X(x)}{dx} (\text{ if derivative exists}).
\end{aligned}
$$

- Equivalently we have $F_X(x) = \int_{u=-\infty}^{x} f_X(u) du$.

Properties of pdf

- $P_X(\mathbb{R}) = \int_{u=-\infty}^{\infty} f_X(u)du = 1.$
- $P_X([a,b]) = F_X(b) - F_X(a) = \int_a^b f_X(u)du.$ (Area under the curve)
- In general, for any $B \subseteq \mathbb{R}$, $P_X(B) = \int_{u \in B} f_X(u)du.$
- $P_X(\{a\}) = 0.$ (no mass at any point)
- $P_X([a,b]) = P_X((a,b)) = P_X([a,b)) = P_X((a,b])$

Mean, Variance, Moments

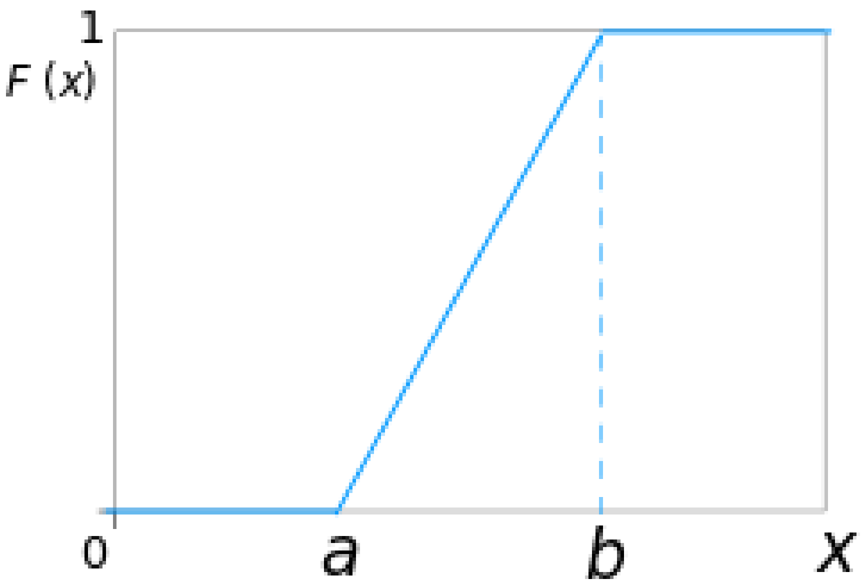
- $E[X] = \int_{-\infty}^{\infty} uf_X(u)du$
- $E[X^n] = \int_{-\infty}^{\infty} u^n f_X(u)du$
- $E[g(X)] = \int_{-\infty}^{\infty} g(u)f_X(u)du$
- $Var[X] = E[g(X)]$ where $g(x) = (x - E[X])^2$.
- For $Y = aX + b$, $E[Y] = aE[X] + b$.
- For $Y = aX + b$, $F_Y(y) = F_X(\frac{y-b}{a})$ when $a \geq 0$.
- For $Y = aX + b$ and $a < 0$, $F_Y(y) = 1 - F_X(\frac{y-b}{a})$.

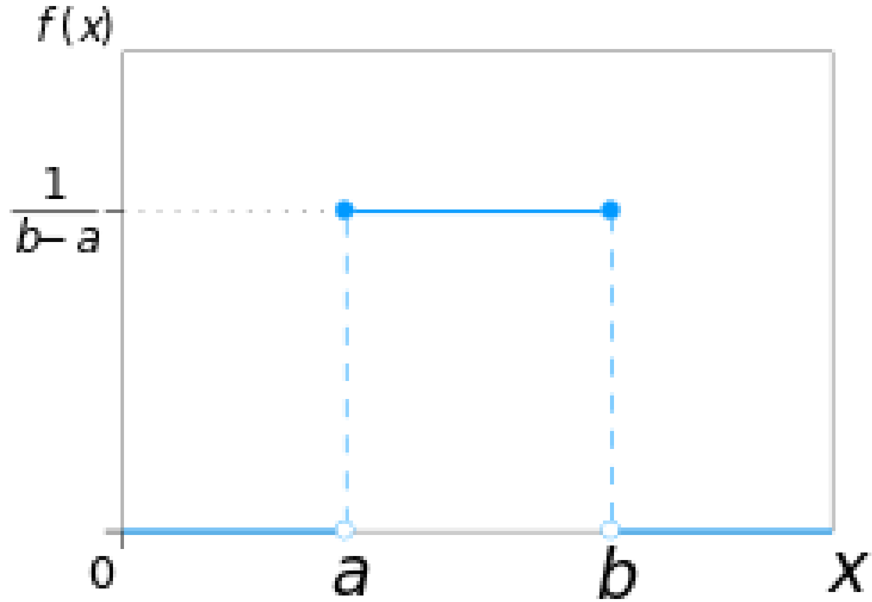
Standard Examples

Uniform random variable ($U[a, b]$)

- This is a real valued r.v.
- Its pdf $f_X(x) = \frac{1}{b-a}$ for all $x \in [a, b]$.
- Its CDF is given by $F_X(x) = \begin{cases} 0 \text{ for } x < a. \\ \frac{x-a}{b-a} \text{ for } x \in [a, b] \\ 1 \text{ otherwise.} \end{cases}$
- HW: Verify $E[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$

$U[a, b]$

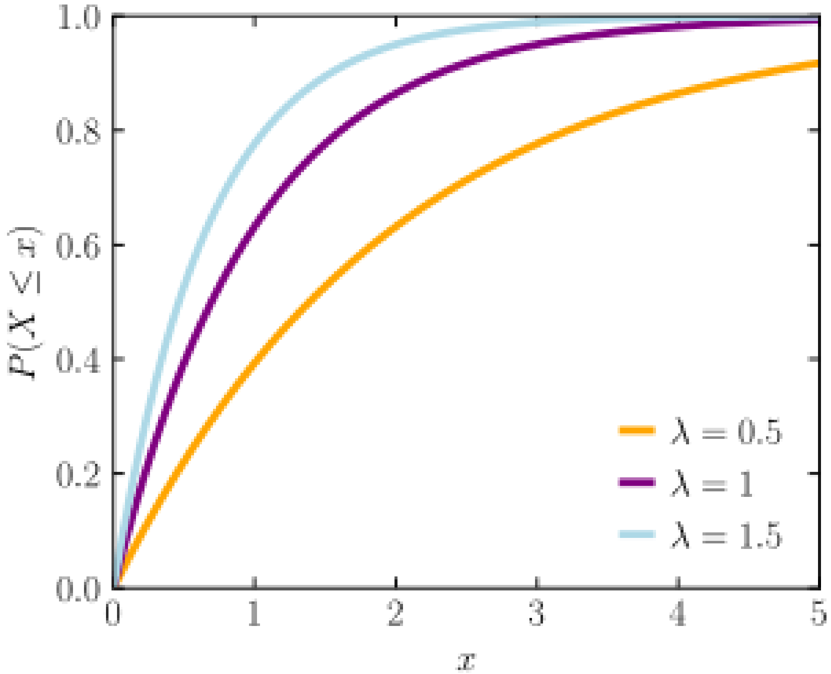




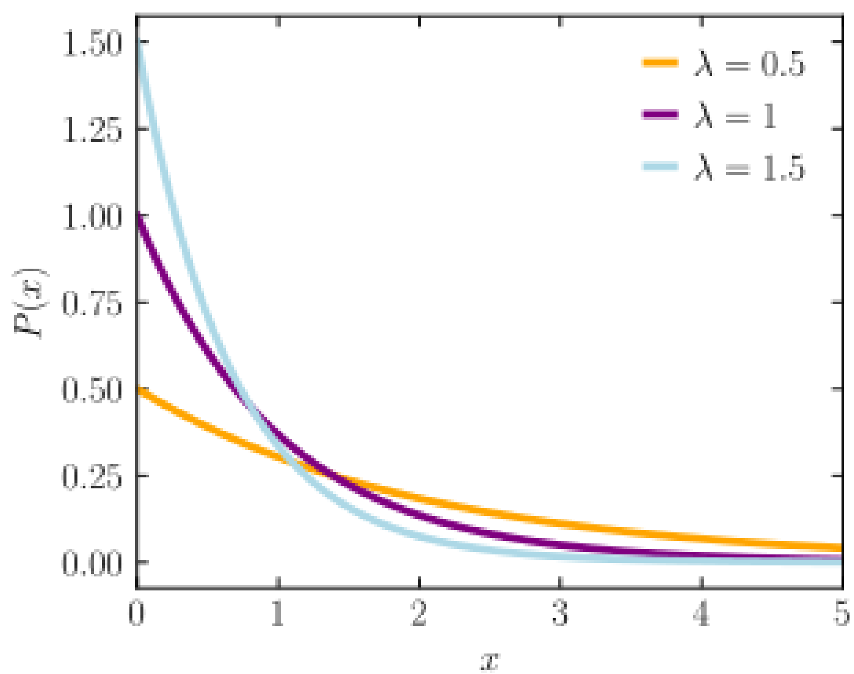
Exponential random variable ($Exp(\lambda)$)

- This is a non-negative r.v. with parameter λ.
- Its pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.
- Its CDF is given by $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.
- $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$
- $E[X^n] = \frac{n!}{\lambda^n}$

$Exp(\lambda)$



1.0
0.8
0.6
0.4
0.2
0.0
$P(X \leq x)$
0
1
2
3
4
5
x
$\lambda = 0.5$
$\lambda = 1$
$\lambda = 1.5$



1.50
1.25
1.00
0.75
0.50
0.25
0.00
$P(x)$
0
1
2
3
4
5
x
$\lambda = 0.5$
$\lambda = 1$
$\lambda = 1.5$

Significance of Exponential r.v.

- Building blocks for Continuous time Markov Chains.
- Demonstrate memory-less property.
- $P(X > a + h|X > a) = \frac{e^{-\lambda(a+h)}}{e^{-\lambda(a)}} = e^{-\lambda(h)} = P(X > h)$.
- Used extensively in Queueing theory to model inter-arrival time and service time of jobs.