CS 302.1 - Automata Theory

Lecture 08

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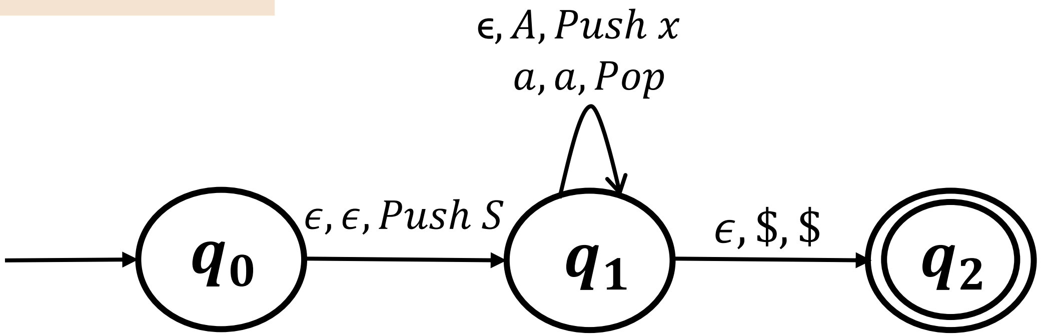
Quick Recap

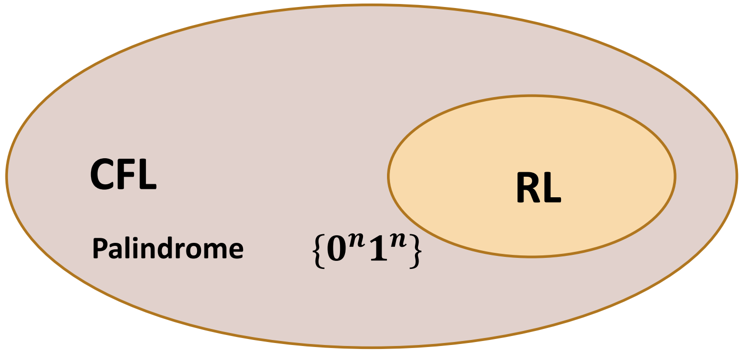
Formally, a PDA M is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q is a finite set called the ***states.***
- Σ is the set of input ***alphabets.***
- Γ is the set of ***Stack alphabets***
- $\delta: Q \times \Sigma_\epsilon \times \Gamma_\epsilon \mapsto \mathcal{P}(Q \times \Gamma_\epsilon)$ is the ***transition function*** [$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$]
- $q_0 \in Q$ is the ***start state***.
- $F \subseteq Q$ is the set of ***accepting states***.

Pushdown Automata and CFLs are equivalent

CFLs ⇒ Pushdown Automata





$$(RL \equiv Regular\ Grammar \equiv Regular\ Expressions \equiv NFA \equiv DFA) \subseteq (CFL \equiv CFG \equiv PDA)$$

Pumping Lemma for CFLs

Recall that so far, we have seen the following:

- L is a context-free language.
- L is generated by a Context Free Grammar (CFG) from which any $w \in L$ can be **derived**.
- The derivation of any CFG can be represented by **parse trees**.
- Any CFG can be expressed in Chomsky Normal Form (CNF): the number of steps required to derive any $w \in L$: $2|w| - 1$
- There exists a Pushdown Automata P such that $\mathcal{L}(P) = L$.

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- Not all languages are context free.
- Just like in the case of Regular languages, the pumping lemma helps us identify non-CFLs.
- **All CFLs satisfy the conditions of the pumping lemma**: If any language L fails to do so, it is not Context-Free.
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- The principle of the Pumping Lemma for CFLs is similar to that of Regular Languages
- In **order to recognize very long strings** in a given CFL L, the **model of computation (CFGs/parse-trees) must repeat some steps of the computation**
- These steps can be **repeated any number of times (pumped) to produce longer and longer strings all of which belong to L**.
- Conversely if **this does not hold, L is not CFL.**

Pumping Lemma for CFLs

Example:

$$A \rightarrow BC|0$$
$$B \rightarrow BA|1|CC$$
$$C \rightarrow AB|0$$

No of variables $|V| = 3$.
Consider a derivation of $w = 11100001$

Pumping Lemma for CFLs

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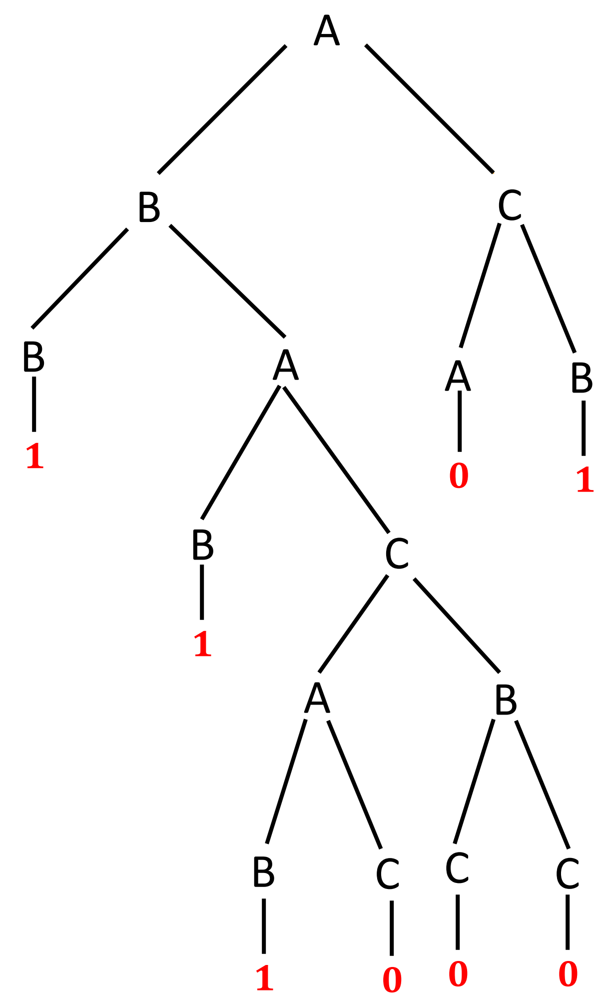
Consider the longest path in the parse tree.



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$$\boldsymbol{A \to BC|0}$$
$$\boldsymbol{B \to BA|1|CC}$$
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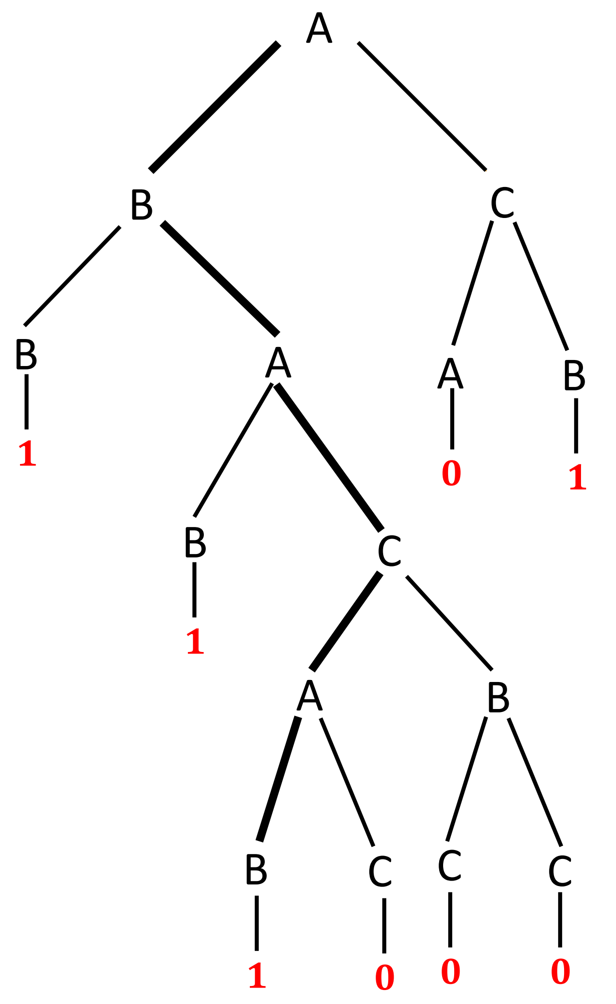
- No of variables $|V| = 3$
- Consider a derivation of $w = 11100001$
- Consider the longest path in the parse tree.
- Longest path length = 5, which is larger than $|V|$.
- There exists at least one variable that is repeated.
- For example: A – mark it.



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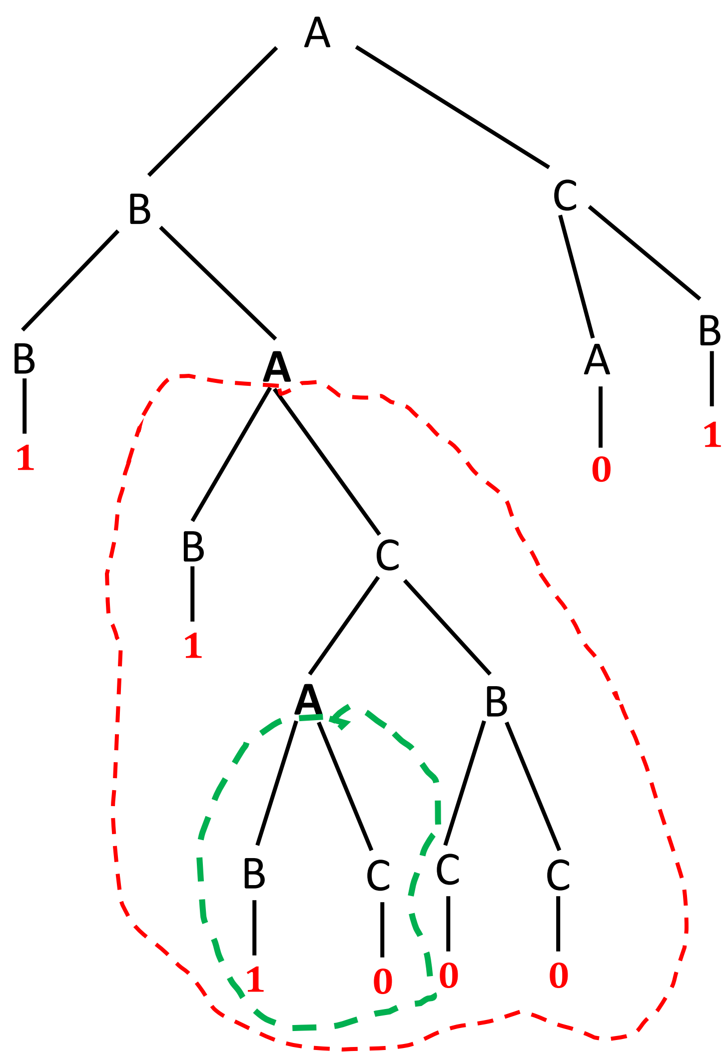
- Replace the bottom most appearance of the variable A with the subtree in the middle rooted at A to obtain a new tree.



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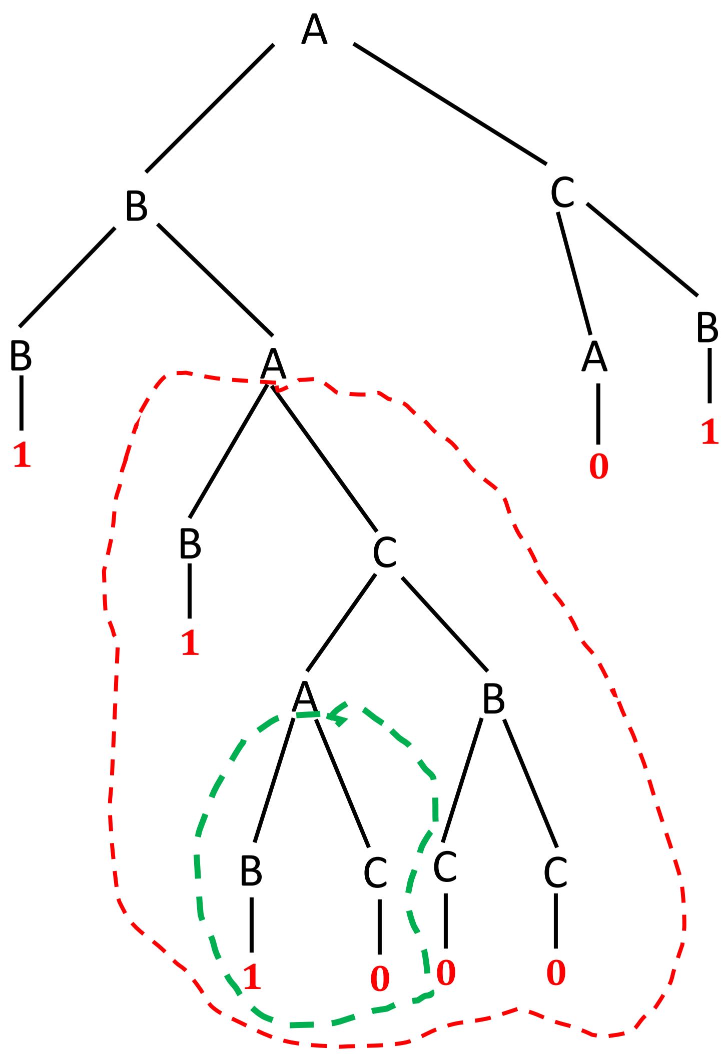

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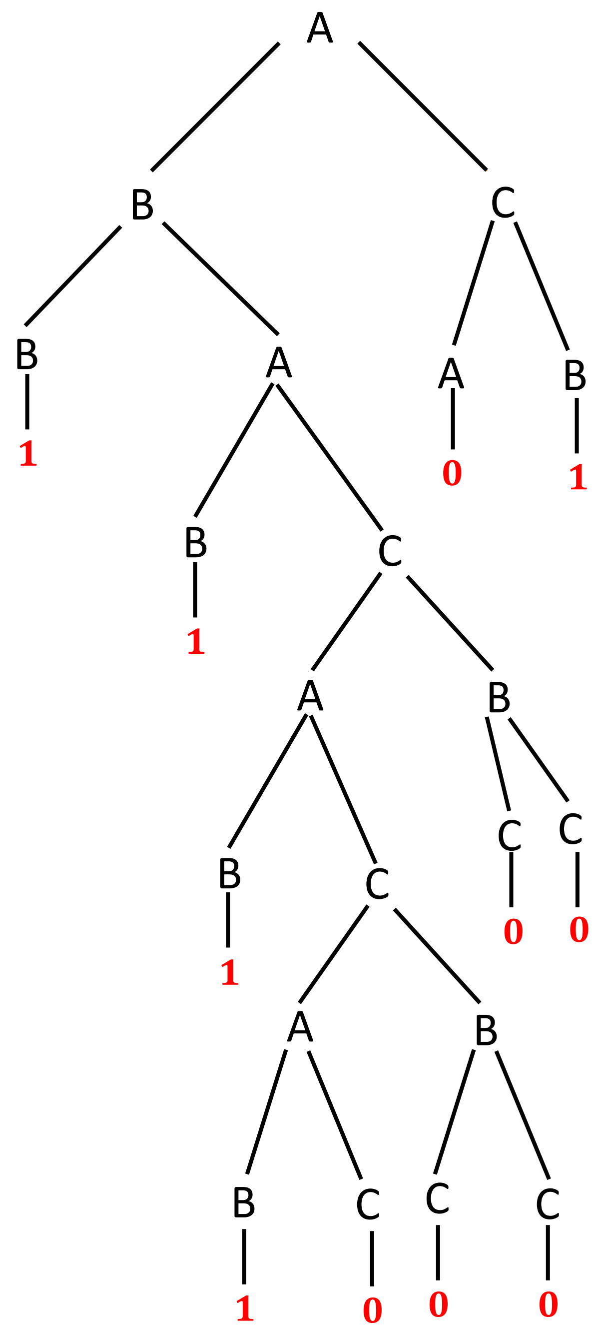
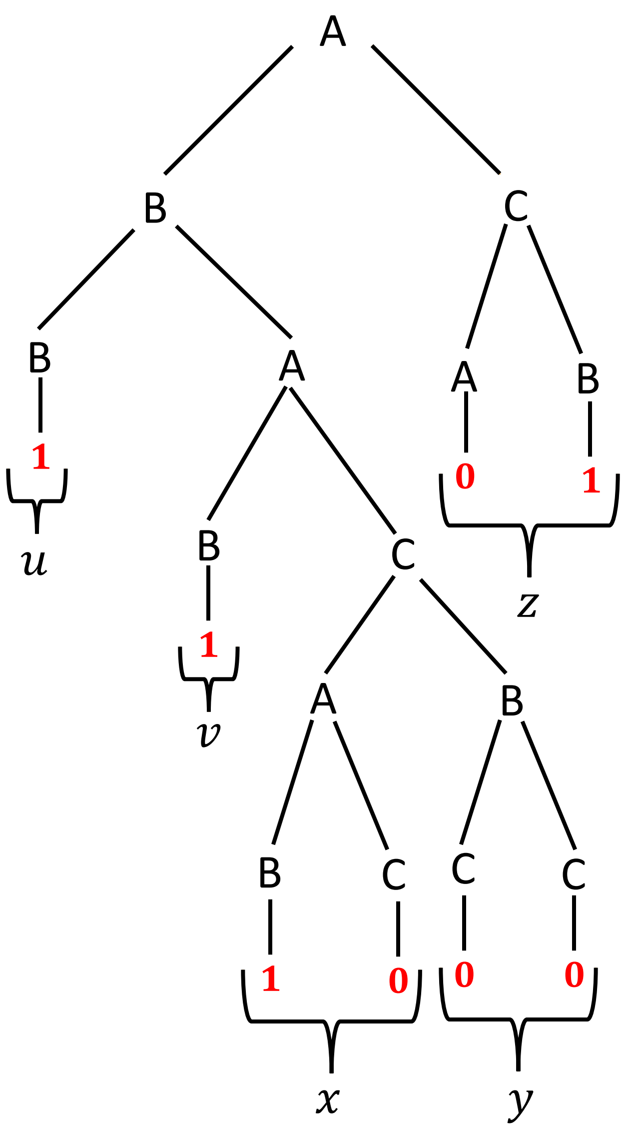




Pumping Lemma for CFLs

For the tree in the left, the input string w can be split into five parts: $\boldsymbol{w = uvxyz}$

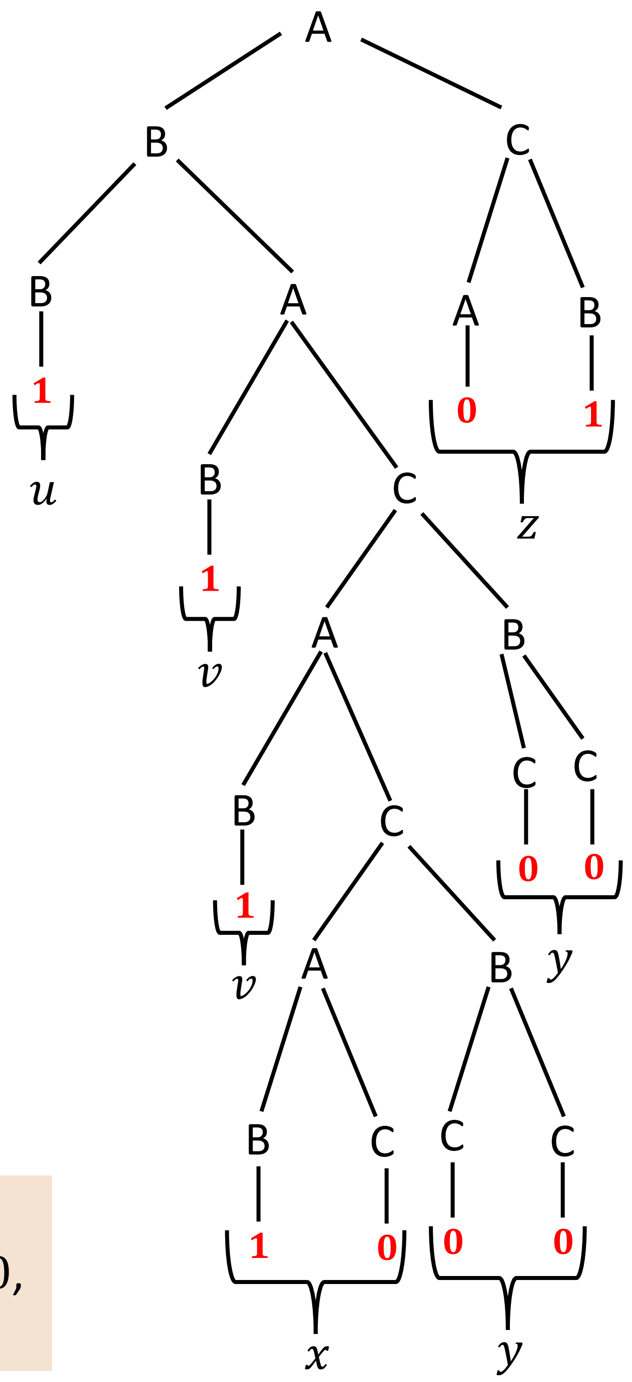
	u	**v**	**x**	**y**	**z**
L Tree	1	1	10	00	01

	u	**vv**	**x**	**yy**	**z**
R Tree	1	11	10	0000	01

By the substitution mentioned in the previous slide, we can keep pumping in v and y to get new strings of the form $\boldsymbol{w = uv^ixy^iz}$ $(i \geq 0)$, and any such $w \in L$ as it is a valid derivation.

Other conditions:
$|vy| \geq 1$, v, y cannot be both ϵ
$|vxy| \leq p$

In fact **if $\boldsymbol{L}$ is a CFL**, $\exists p$ such that $\forall w \in L$ of length $|\mathrm{w}| \geq p$, we can split $w = uvxyz$, such that $\forall i \geq 0$,
$\boldsymbol{\mathrm{w} = uv^ixy^iz \in L}$



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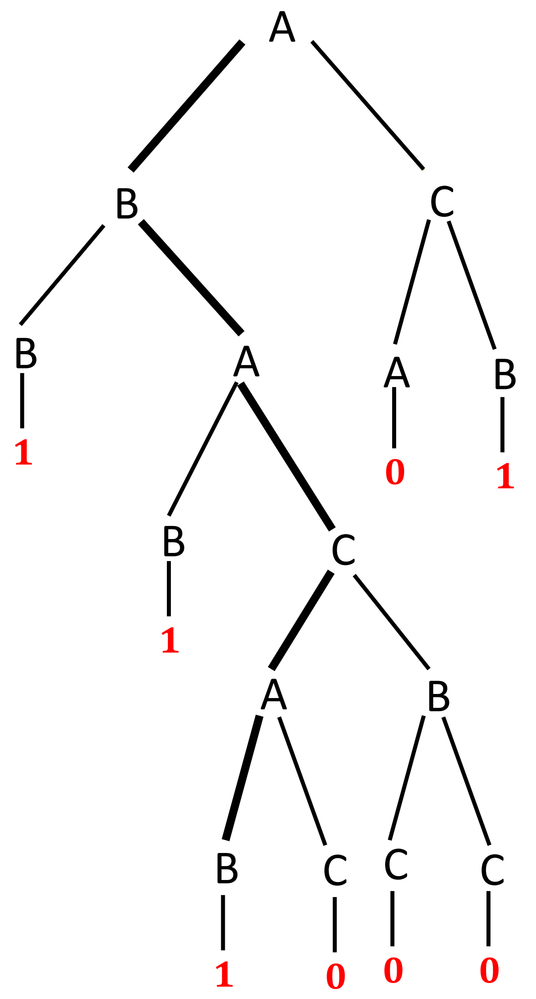
Properties of parse trees:

Let L be a CFL and G be such that $L = \mathcal{L}(G)$ and $w \in L$. Consider a parse tree T_w^G of G that yields w. Then:

- A path from the root to a leaf is sequence of variables and ends in a terminal or ϵ.



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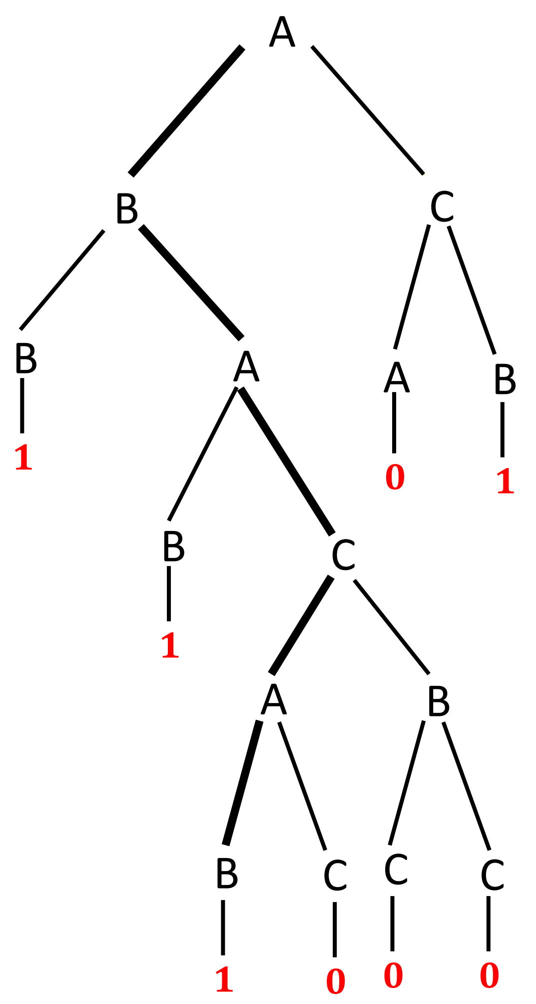
- Let d be the maximum number of variables/terminals in the RHS of any rule of G.
- For example: If G is in CNF, $d = 2$.



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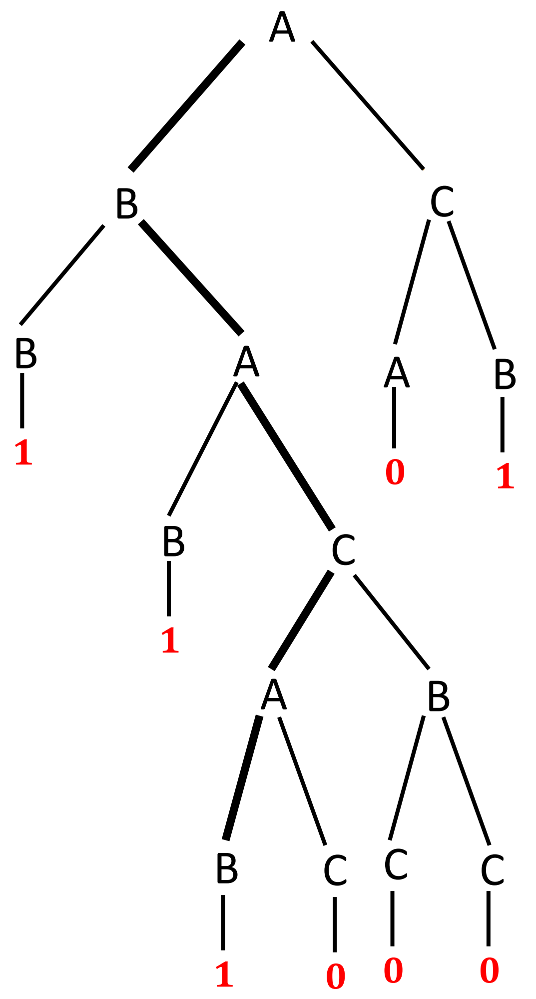
- This results in a **binary parse tree**. Henceforth $\boldsymbol{d = 2}$.
- Any T_w^G has at **level $\boldsymbol{l}$**, at most $\boldsymbol{d^l}$ **nodes**. Thus, any T_w^G of height h has at most d^h terminals.



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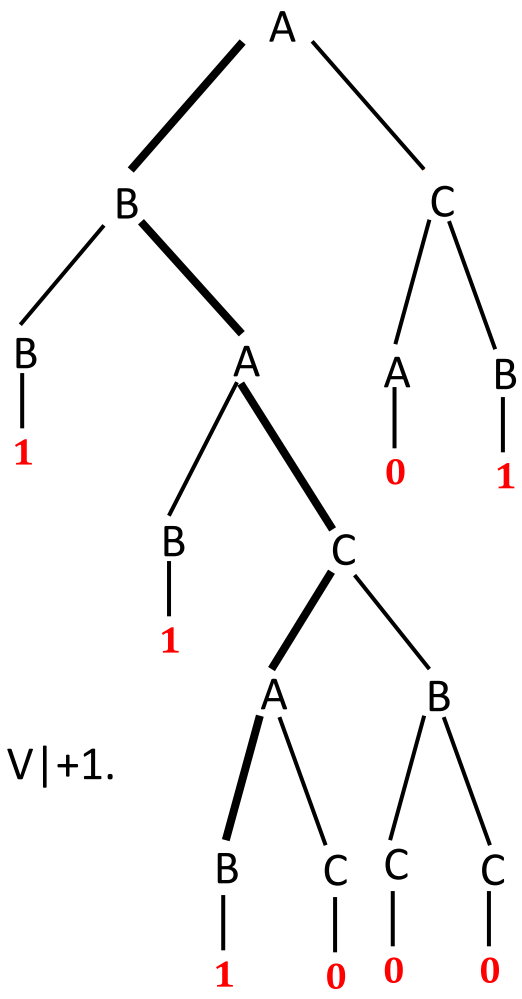

- Let |V| be the total number of variables in the Grammar G.
- If $w \in L$ such that $|w| = p = d^{|V|+1} = 2^{|V|+1}$, the underlying parse tree would have a height ≥ |V|+1.

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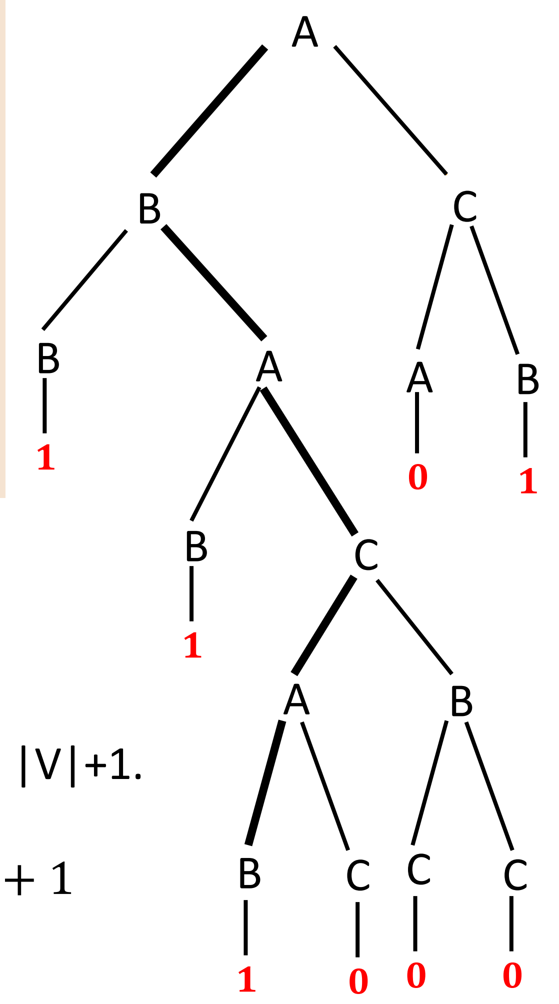

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- The longest path from the root (S) to the lowest level, is at least $|V| + 1$ (containing at least $|V| + 1$ variables).

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- The longest path from the Start Variable S to a terminal is $\geq |V| + 1$.
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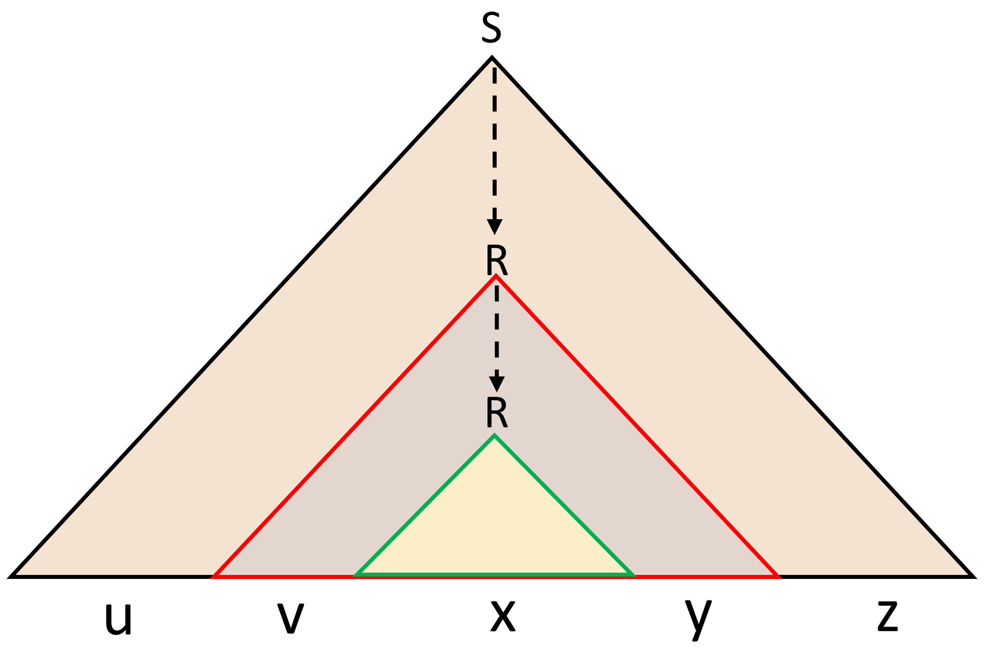

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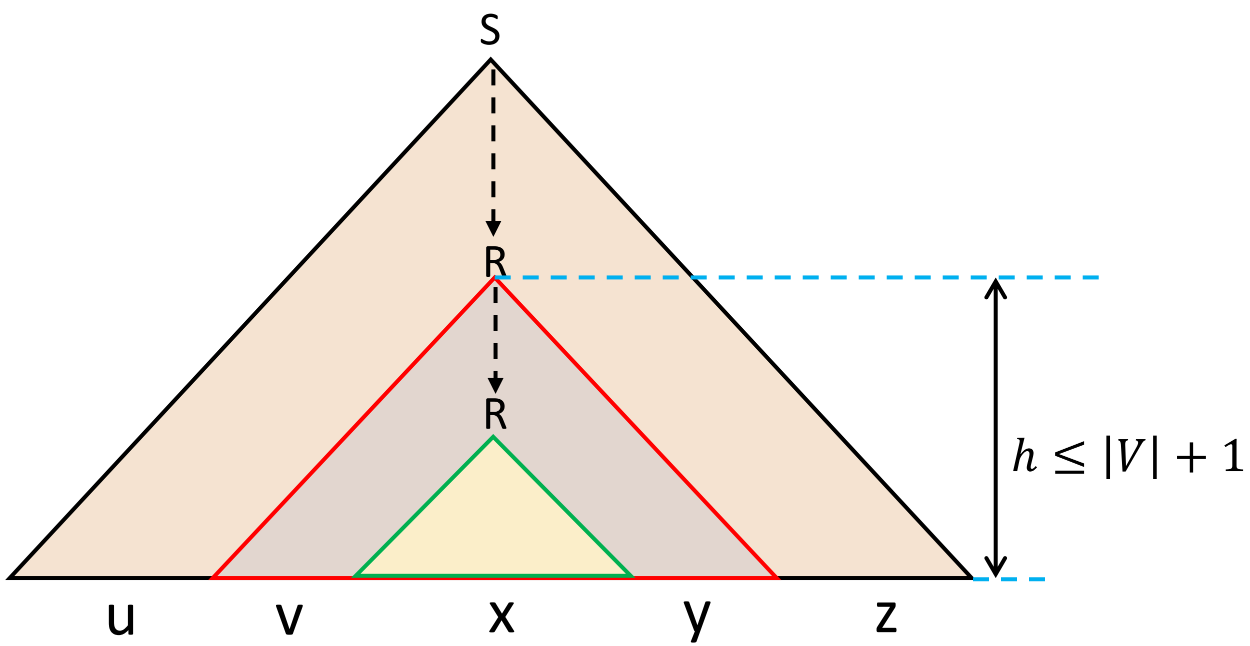

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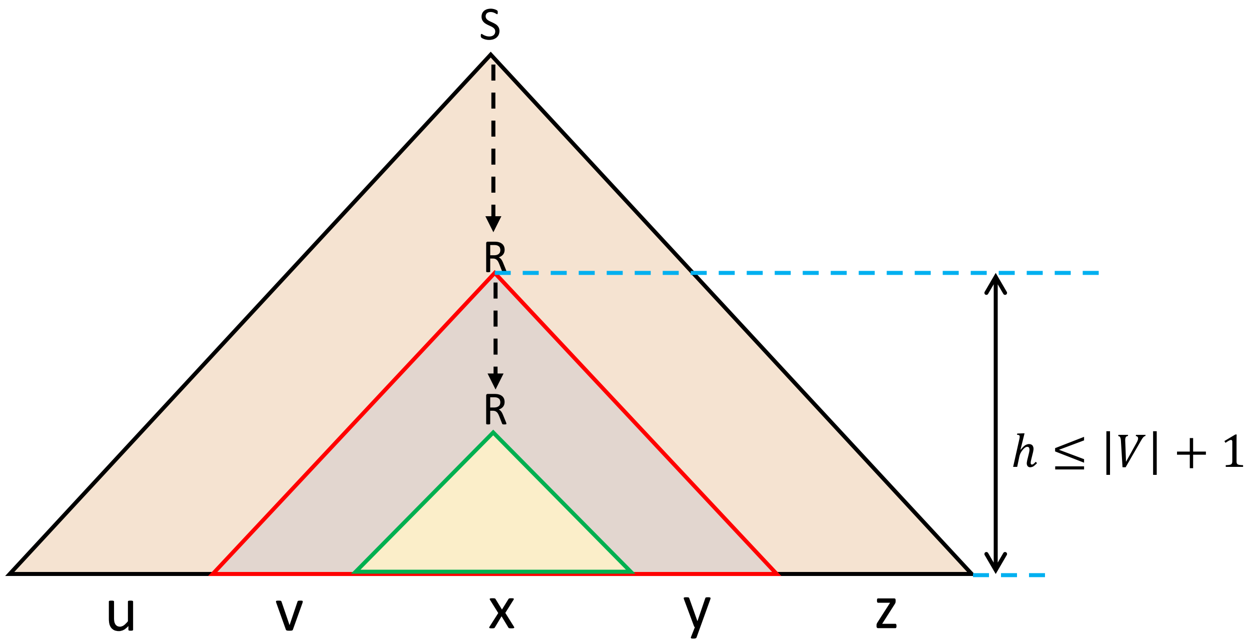
- Consider the lowest $|V| + 1$ variables in the longest path of length $\geq |V| + 1$.



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- $|vxy| \leq p$ - the uppermost R falls within the bottom $|V| + 1$ variables in the longest path and so the length of the string it can generate is $\leq d^{|V|+1} = 2^{|V|+1} = p$.

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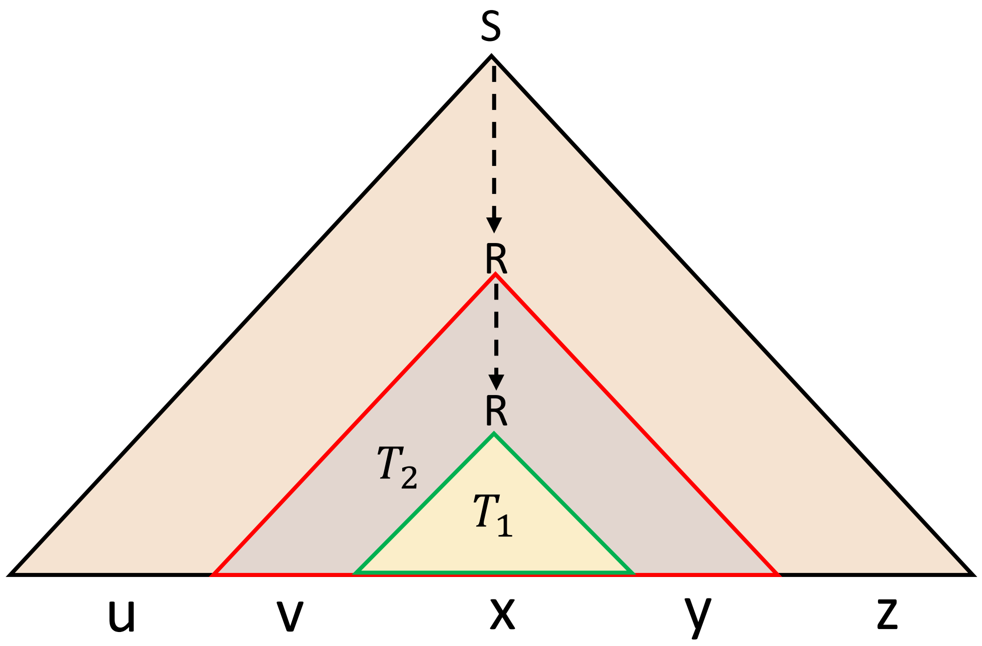

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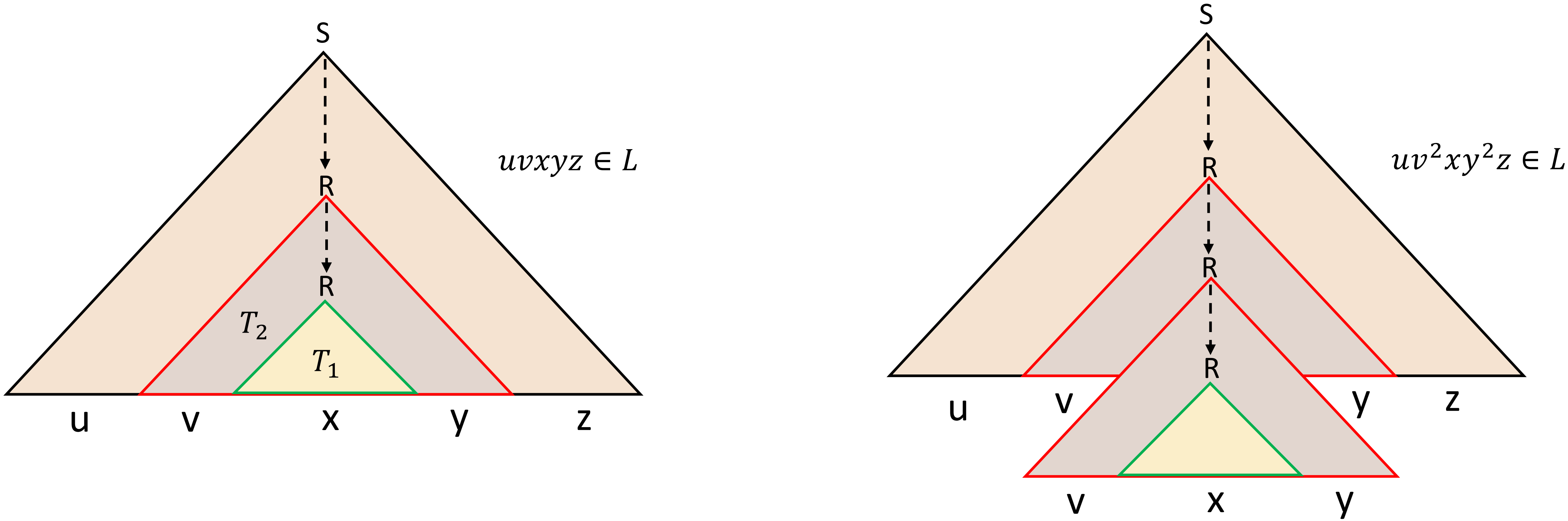

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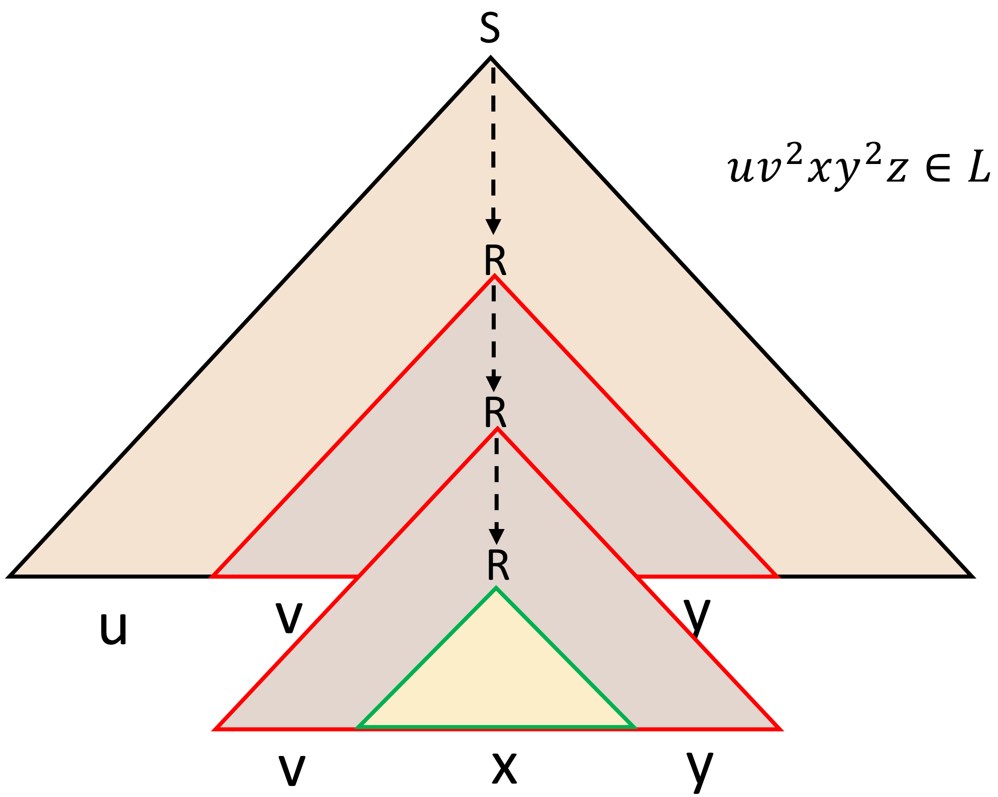

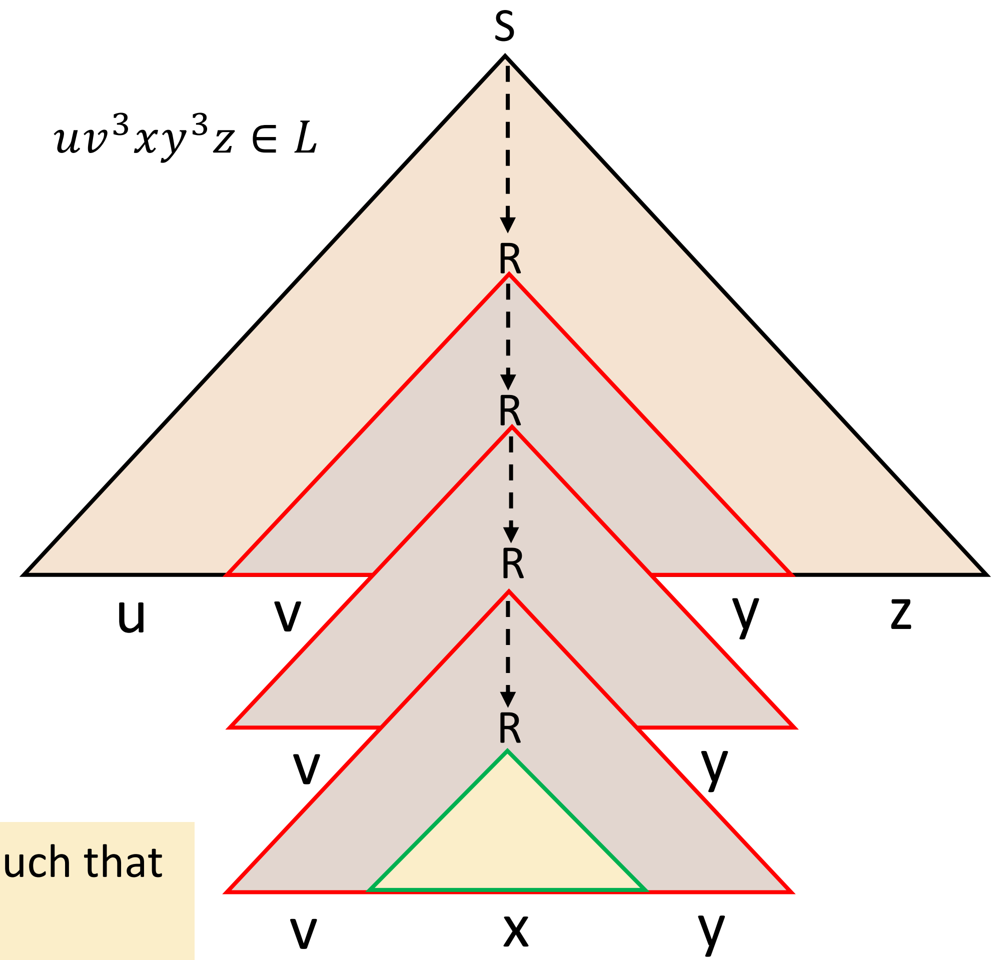

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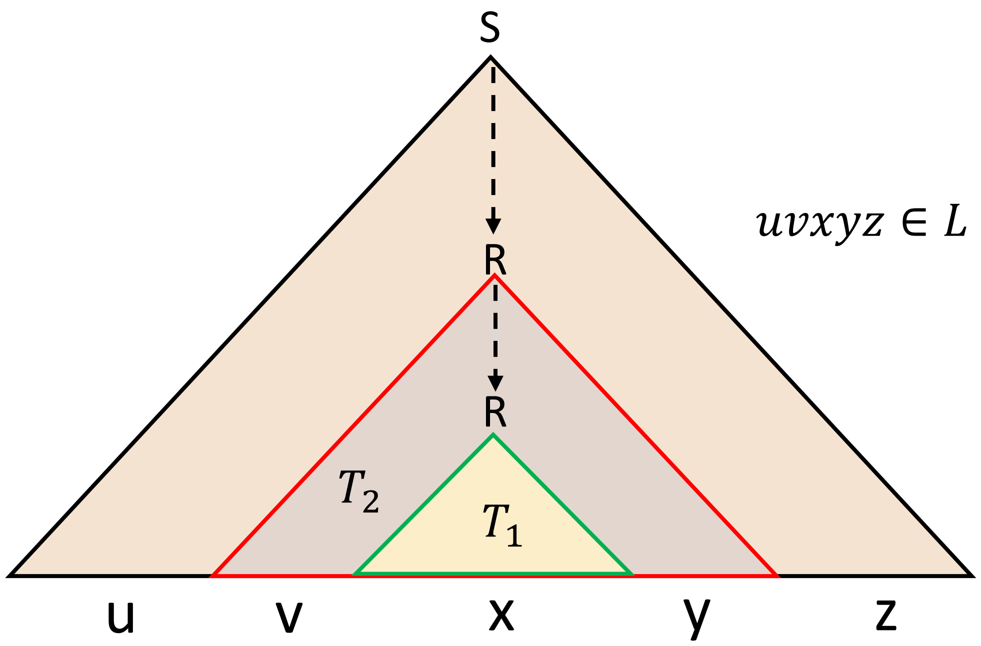

$uvxyz \in L$

$uv^0xy^0z \in L$

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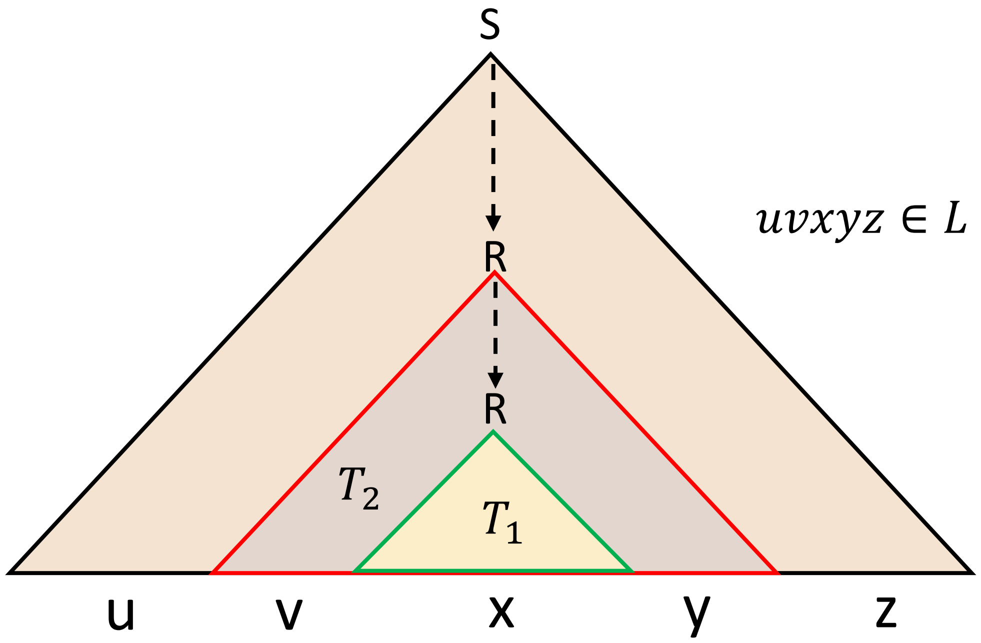

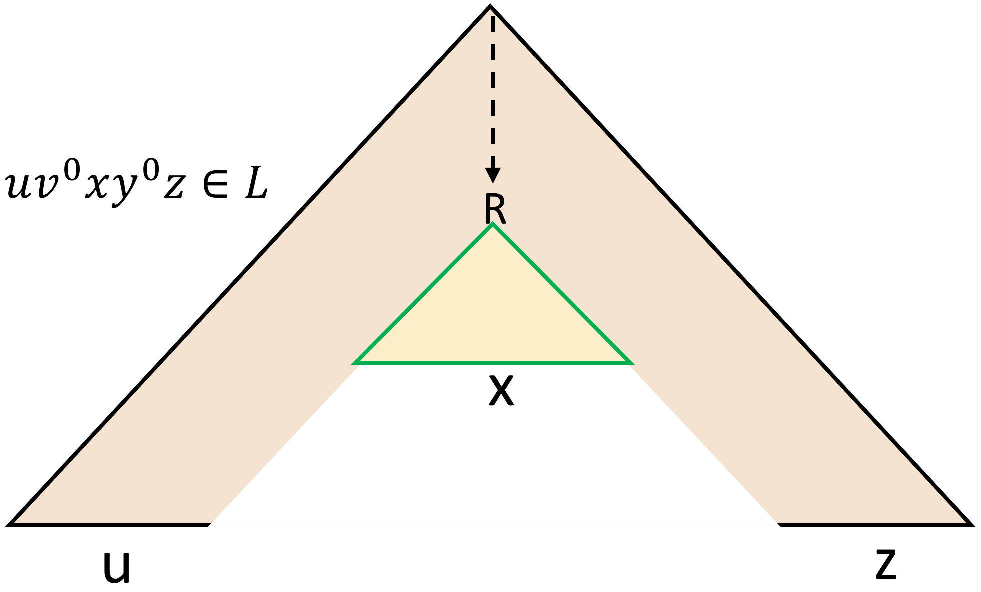

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- The longest path from the Start Variable S to a terminal is at least $|V| + 1$.
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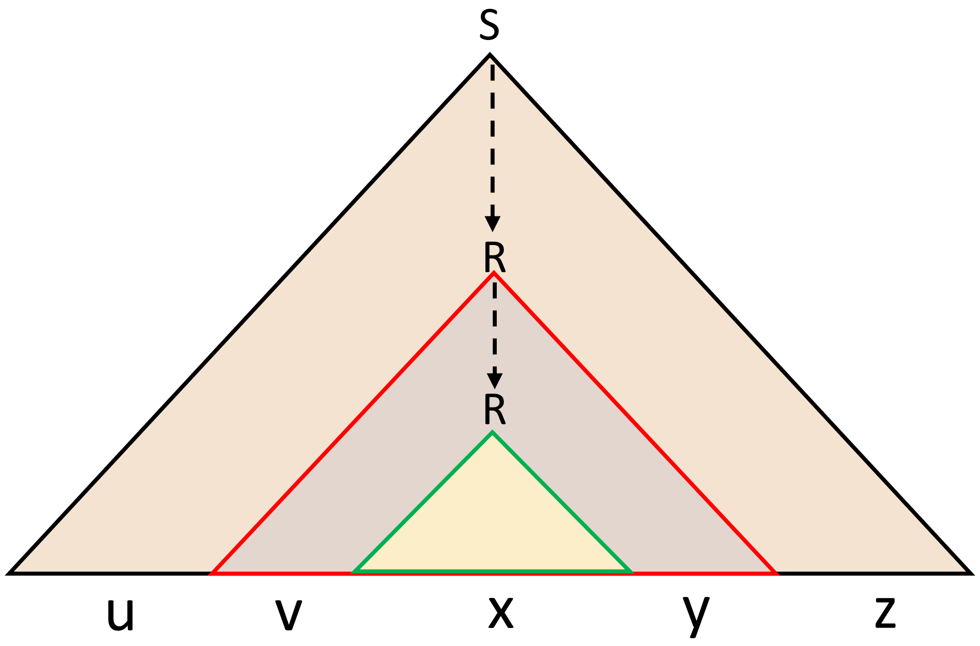

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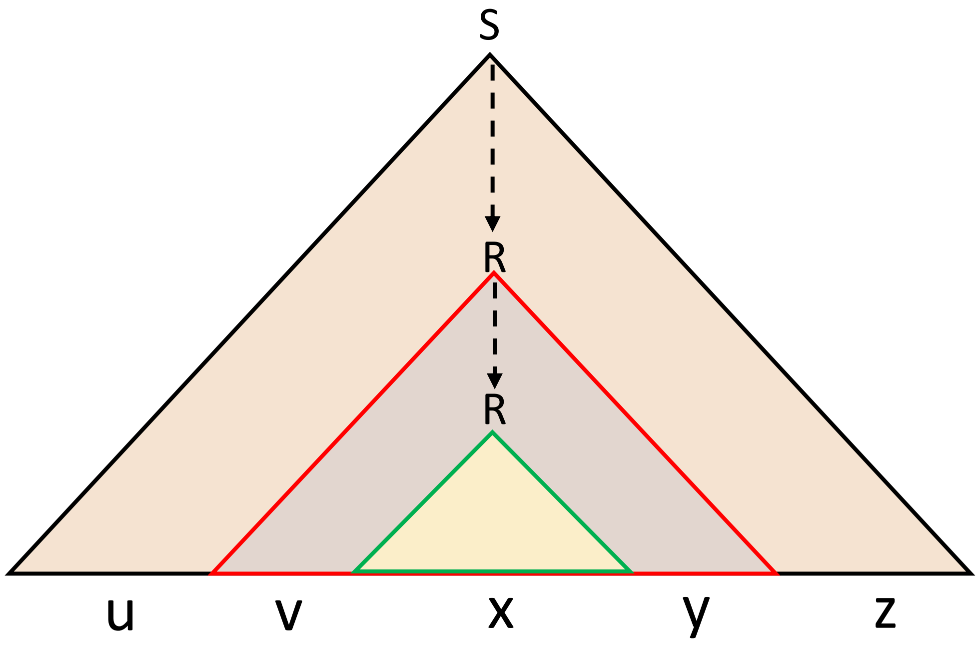

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What if G is ambiguous? More than one parse tree generates w.

Pick the one with the smallest number of nodes. So T_w^G is the smallest parse tree generating w.

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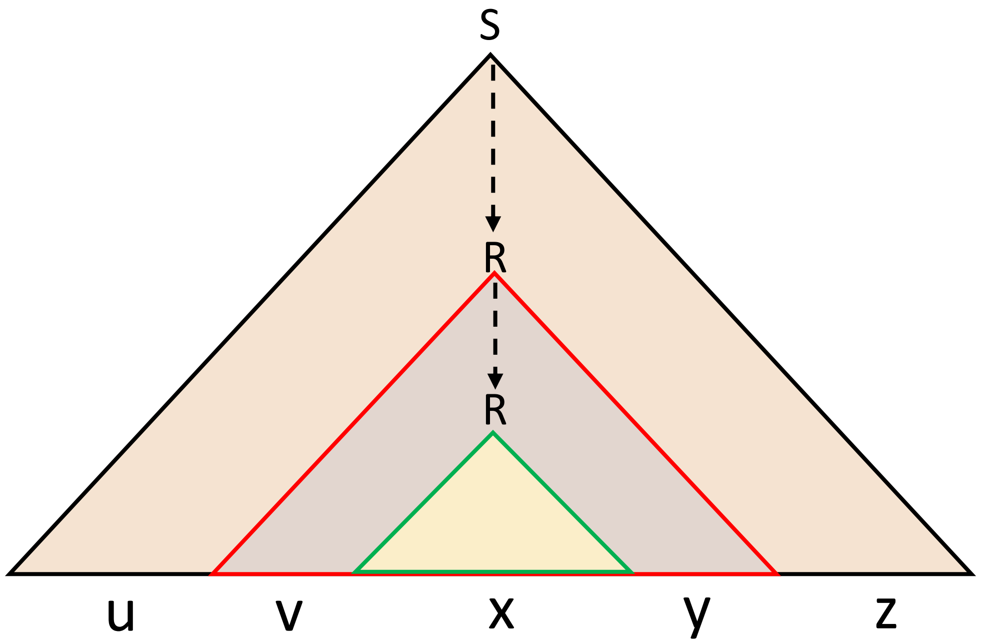
- Consider the lowest $|V| + 1$ variables in that path.
- By the pigeonhole principle, within the lowest $|V| + 1$ variables, at least one variable is repeated



Then any string w such that $|w| \geq p$ can be partitioned as $w = uvxyz$ such that

- $|vxy| \leq p$
- $uv^ixy^iz \in L, \forall i \geq 0$

T_w^G is the smallest parse tree generating w.

This leads to an additional condition!

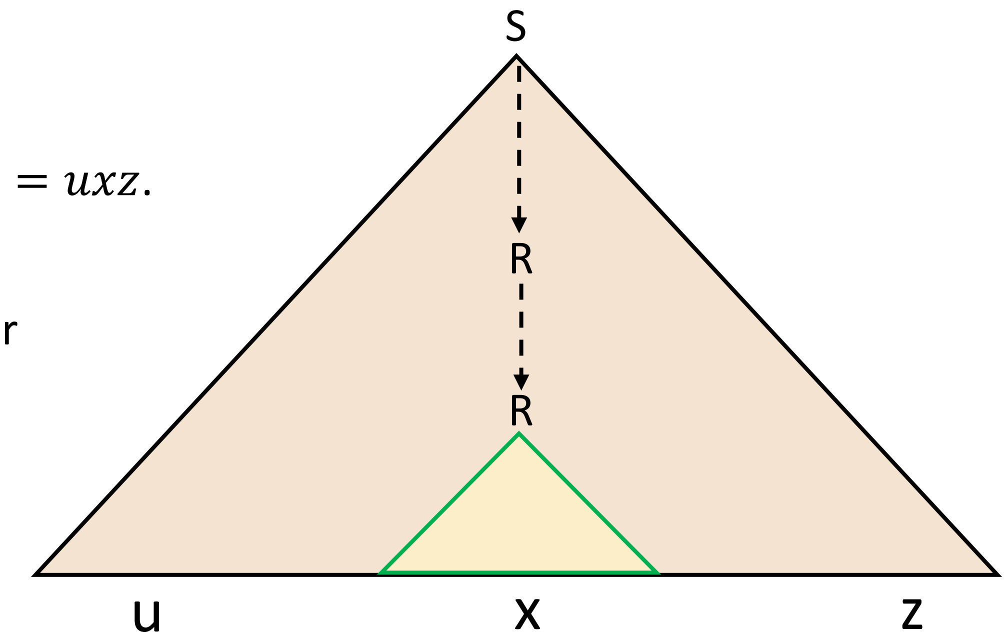
v, y cannot be both empty, i.e. $|vy| \geq 1$

Pumping Lemma for CFLs

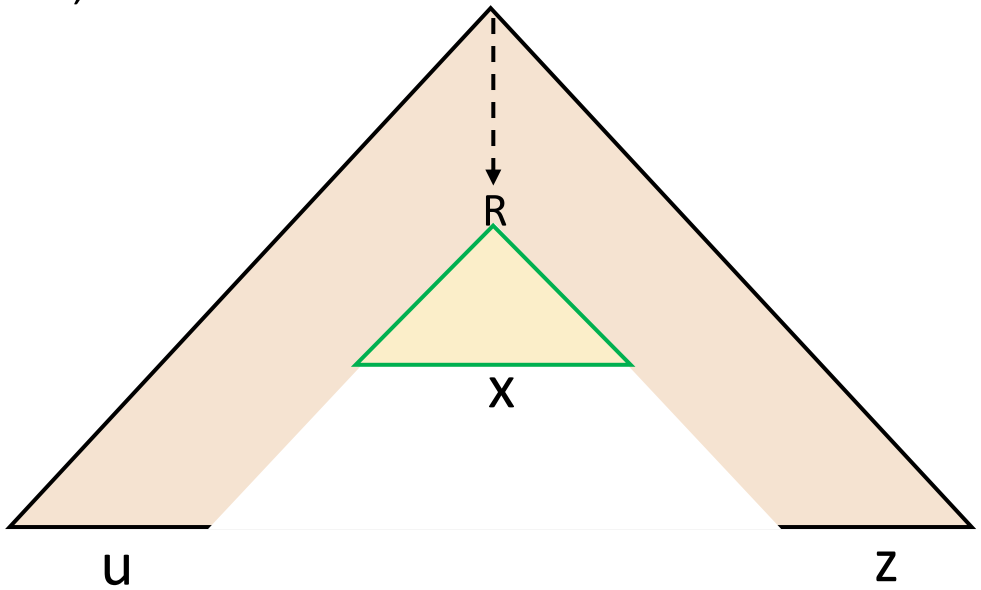
Let L be a CFL and G be such that $L = \mathcal{L}(G)$ and $w \in L$. Consider the **smallest parse tree** T_w^G of G that yields w.

v, y cannot be both empty, i.e. $|vy| \geq 1$

Proof by contradiction: Let us assume that they were both empty, i.e. $w = uxz$.
Then T_w^G would look like this.



However, if we substitute the smaller subtree rooted at R with the higher subtree, we obtain



The parse tree to the left generates w and has fewer nodes which is a **contradiction**!!

Pumping Lemma for CFLs

Putting things together:

Pumping Lemma for CFL: IF L is Context Free, **THEN** there exists $p > 0$ (pumping length), such that, for any $w \in L$ of length $|w| \geq p$, $\exists u, v, x, y, z$ such that w can be split into five parts, i.e.

$$w = uvxyz$$

satisfying the following conditions:

- $|vy| \geq 1$
- $|vxy| \leq p$
- $uv^i xy^i z \in L,\ \forall i \geq 0$

We have proved this in the previous slides.

Pumping Lemma for CFLs

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satisfying the following conditions:

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Note: $(A \Rightarrow B) \equiv (\neg B) \Rightarrow (\neg A)$

IF L is Context Free, THEN conditions of Pumping Lemma are Satisfied

$$\equiv$$

IF conditions of Pumping Lemma are NOT satisfied THEN L is NOT Context Free

In order to prove that a language is not Context Free, assume that it is Context Free and obtain a contradiction.

Non Context Free Languages

$\boldsymbol{L = \{0^n 1^n 2^n | n \geq 0\}}$ is not Context-Free.

Proof: We shall prove this by contradiction. Let L be a CFL and so it must satisfy the conditions of the Pumping Lemma. Let p be the pumping length and so $w = 0^p 1^p 2^p \in L$.

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Proof: We shall prove this by contradiction. Let L be a CFL and so it must satisfy the conditions of the Pumping Lemma. Let p be the pumping length and so $w = 0^p 1^p 2^p \in L$.

Note that $|w| = 3p$ $(\geq p)$. The pumping lemma states that w can be split into $w = uvxyz$ such that

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- $vxy = 0^k$ or 1^k or $2^k, k \leq p$: Then $w' \notin L$. (E.g: If $vxy = 0^k$, then, w' will have more $0's$ than $1's$ and $2's$.)

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- $vxy = 0^m 1^n$ or $1^m 2^n, m + n \leq p$: Again, $w' \notin L$.

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Both cases lead to a contradiction. Hence, $L \notin CFL$.

Non Context Free Languages

$L = \{0^n 1^n 2^n | n \geq 0\}$ is not Context-Free.

Other examples:

- $L = \{ww | w \in \{0, 1\}^*\}$
- $L = \{a^p | p \text{ is prime}\}$
- $L = \{0^n 1^{n^2} | n \geq 0\}$

…………………………………………………………

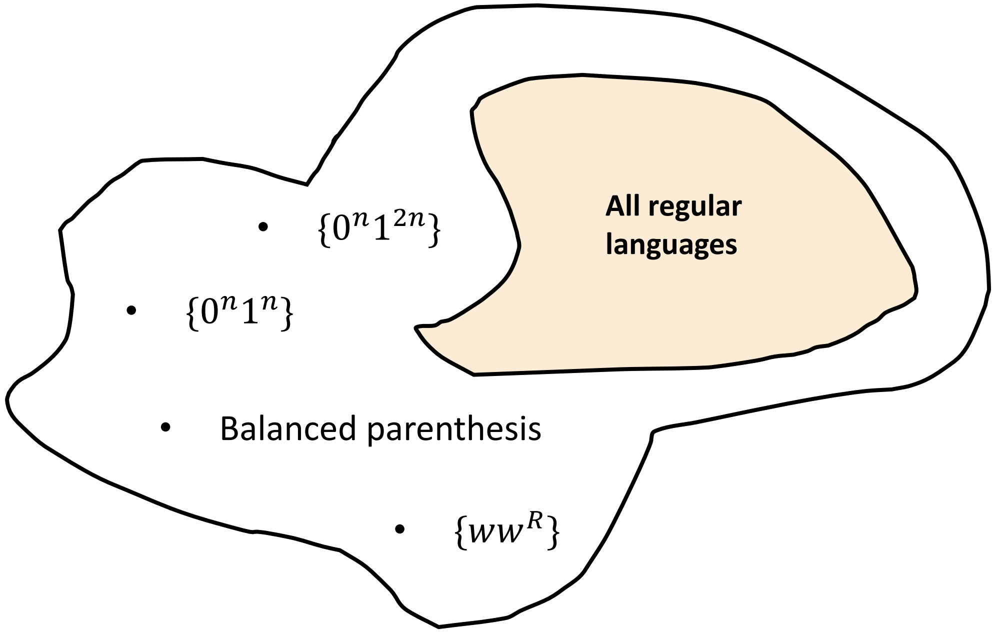
Recommend you to use Pumping Lemma and check that they are indeed not Context Free

Closure properties of CFL

Now that we know that there are languages that are not Context Free – let us investigate the closure properties of CFLs.

Recall what we mean by the statement "**CFLs are closed under some operation**"

- We pick up points within the set of all CFLs (say L_1 and L_2)
- Perform *set operations* such as Union, concatenation, Star, intersection, complement etc on them.
- Observe whether the resulting language still belongs to the set of all CFLs.
- If so, we say, CFLs are **closed** under that operation otherwise we say CFLs are not closed under that operation.



Set of all CFLs

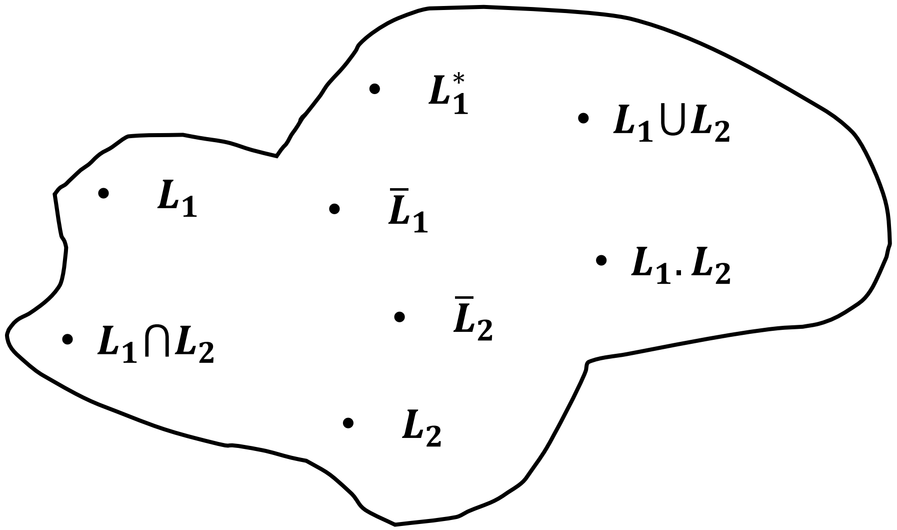
Closure properties of CFL

Some operations: Let L_1 and L_2 be languages.

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1 \text{ or } x \in L_2\}$
- **Concatenation:** $L_1 . L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
- **Intersection:** $L_1 \cap L_2 = \{x | x \in L_1 \text{ and } x \in L_2\}$
- **Star:** $L_1^* = \{x_1 x_2 \cdots x_k | \ k \geq 0 \text{ and each } x_i \in L\}$
- **Complementation:** $\bar{L} = \{x | x \notin L\}$

Recall that for Regular languages: RL are closed under

- **Union**
- **Intersection**
- **Star**
- **Complement**
- **Concatenation**



Set of all regular Languages

What about CFLs?

Closure properties of CFL

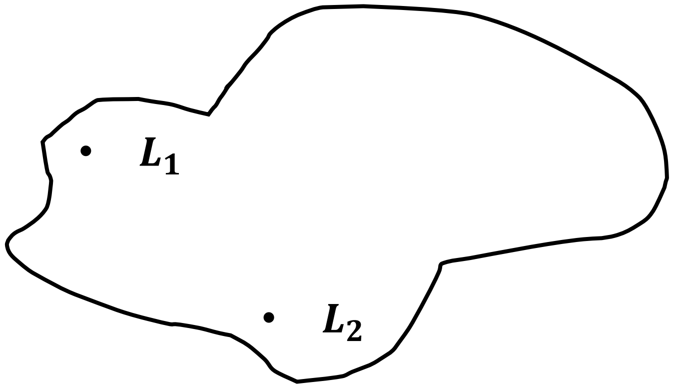
Q: Is the set of all CFLs **closed under union**?

Suppose L_1 and L_2 are CFLs. Is $L = L_1 \cup L_2$ also a CFL?

Proof: Suppose G_1 and G_2 be grammars such that $L(G_1) = L_1$ and $L(G_2) = L_2$.

Suppose:

Rules of G_1: $S_1 \rightarrow \ldots$
Rules of G_2: $S_2 \rightarrow \ldots$



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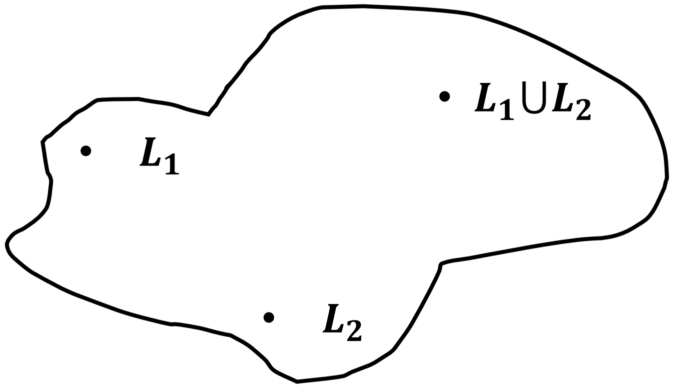


Set of all CFLs

Also suppose that the rules of G_1 and G_2 have different variables.

Then the grammar for $L_1 \cup L_2$ contains all the variables of G_1 and G_2, all the terminals of G_1 and G_2.

Add a new start symbol S and the rules of $G_1 \cup G_2$:

$$S \rightarrow S_1 | S_2$$

followed by rules of G_1 and rules of G_2. So CFLs are closed under union.

Closure properties of CFL

Q: Is the set of all CFLs **closed under Concatenation**?

Suppose L_1 and L_2 are CFLs. Is $L = L_1 . L_2$ also a CFL?

Proof: Suppose G_1 and G_2 be grammars such that $L(G_1) = L_1$ and $L(G_2) = L_2$.

Suppose:

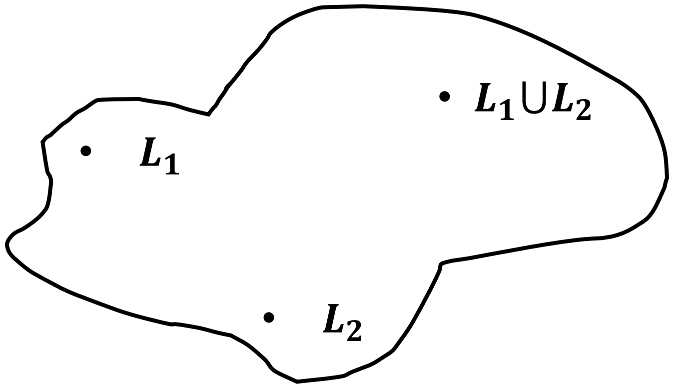
Rules of G_1: $S_1 \rightarrow \ldots$
Rules of G_2: $S_2 \rightarrow \ldots$

Also suppose that the rules of G_1 and G_2 have different variables. Then define G' such that $L(G') = L_1 . L_2$, as the grammar containing all the variables of G_1 and G_2, all the terminals of G_1 and G_2, with a new start symbol S. The new rules:

$$S \rightarrow S_1 . S_2$$

followed by rules of G_1 and rules of G_2.

So CFLs are closed under concatenation.



Set of all CFLs

Closure properties of CFL

Q: Is the set of all CFLs **closed under Concatenation**?

Suppose L_1 and L_2 are CFLs. Is $L = L_1 . L_2$ also a CFL?

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Suppose:

Rules of G_1: $S_1 \rightarrow \ldots$
Rules of G_2: $S_2 \rightarrow \ldots$

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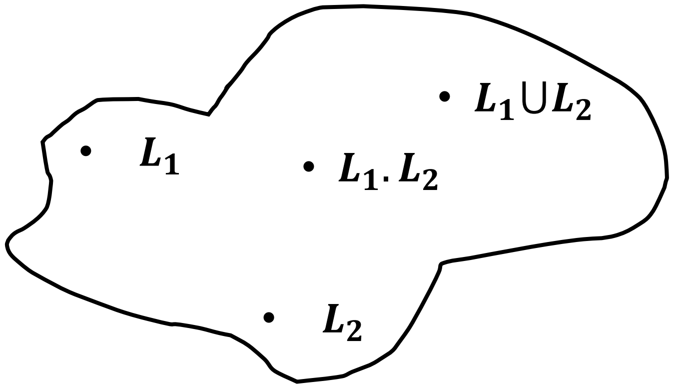


Set of all CFLs

Closure properties of CFL

Q: Is the set of all CFLs **closed under Star**?

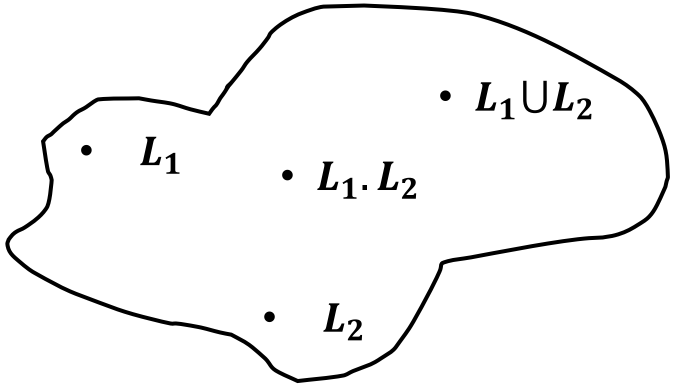
Suppose L is a CFL. Is L^* also a CFL?

Proof: Suppose G be a grammar such that $L(G) = L_1$

Suppose:

Rules of G: $S_1 \rightarrow \ldots$

Then the grammar G' such that $L(G) = L^*$ is the same as G with a new start symbol and the additional rules



Set of all CFLs

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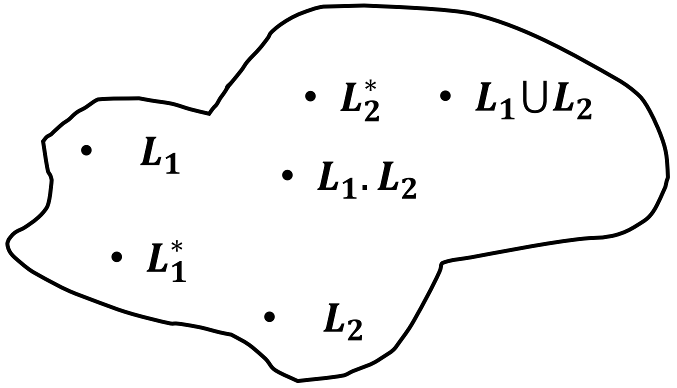
Then the grammar G' such that $L(G) = L^*$ is the same as G with a new start symbol and the additional rules

$$S \rightarrow S_1 S | \epsilon$$

So CFLs are closed under Star.



Set of all CFLs

Closure properties of CFL

Q: Is the set of all CFLs **closed under intersection**?

Suppose L_1 and L_2 are CFLs. Is $L = L_1 \cap L_2$ also a CFL?

Proof: We will prove that CFLs are NOT closed under intersection by using this simple counterexample. Let

Closure properties of CFL

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Suppose L_1 and L_2 are CFLs. Is $L = L_1 \cap L_2$ also a CFL?

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$\boldsymbol{L_1 = \{0^n 1^n 2^m | m, n \geq 0\}}$ and $\boldsymbol{L_2 = \{0^m 1^n 2^n | m, n \geq 0\}}$

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Note that $L_1, L_2 \in$ CFL – each of them are concatenation of two CFLs.

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Note that $L_1, L_2 \in$ CFL – each of them are concatenation of two CFLs.

E.g: L_1 is a concatenation of $\{0^n 1^n | n \geq 0\}$ and $\{2^m | m \geq 0\}$ and the rules of the corresponding grammar are

$$S \rightarrow AB$$
$$A \rightarrow 0A1 | \epsilon$$
$$B \rightarrow 2B | \epsilon$$

What is $L_1 \cap L_2$?

Closure properties of CFL

Q: Is the set of all CFLs **closed under intersection**?

Suppose L_1 and L_2 are CFLs. Is $L = L_1 \cap L_2$ also a CFL?

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$$S \to AB$$
$$A \to 0A1 | \epsilon$$
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$\boldsymbol{L_1 \cap L_2 = \{0^n 1^n 2^n | n \geq 0\}}$ which is not a CFL.

Hence **CFLs are NOT closed under intersection!**

Closure properties of CFL

Q: Is the set of all CFLs **closed under complementation**?

Suppose L is a CFL. Is $\overline{L}$ also a CFL?

Proof: ??????????

Closure properties of CFL

Q: Is the set of all CFLs **closed under complementation**?

Suppose L is a CFL. Is $\bar{L}$ also a CFL?

Proof: Let us assume that CFLs are closed under complementation. Then if L_1 and L_2 are context free, then $\boldsymbol{\bar{L}_1}$ and $\boldsymbol{\bar{L}_2}$ are also context free. This would imply that

$$\boldsymbol{\bar{L}_1 \cup \bar{L}_2 \in CFL}$$

Closure properties of CFL

Q: Is the set of all CFLs **closed under complementation**?

Suppose L is a CFL. Is $\bar{L}$ also a CFL?

Proof: Let us assume that CFLs are closed under complementation. Then if L_1 and L_2 are context free, then $\boldsymbol{\bar{L}_1}$ and $\boldsymbol{\bar{L}_2}$ are also context free. This would imply that

$$\boldsymbol{\bar{L}_1 \cup \bar{L}_2 \in CFL}$$

Finally, this would imply $\boldsymbol{\overline{\overline{L_1} \cup \overline{L_2}} \in CFL}$. However,

$$\boldsymbol{L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}}$$

But this would imply $\boldsymbol{L_1 \cap L_2 \in CFL}$, which is a contradiction.

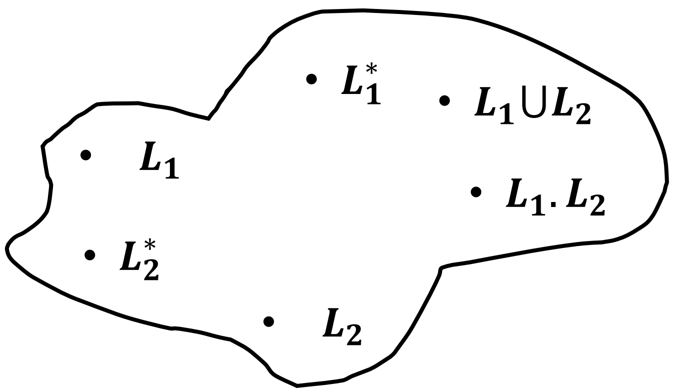
Thus CFLs are NOT closed under complementation.

Closure properties of CFL

Recall that for Regular languages:

RLs are closed under

- **Union**
- **Intersection**
- **Star**
- **Complement**
- **Concatenation**



Set of all CFLs

For CFLs:

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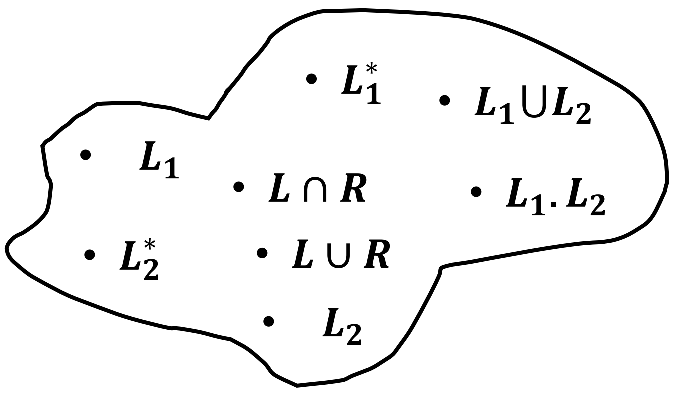


Set of all CFLs

For CFLs:

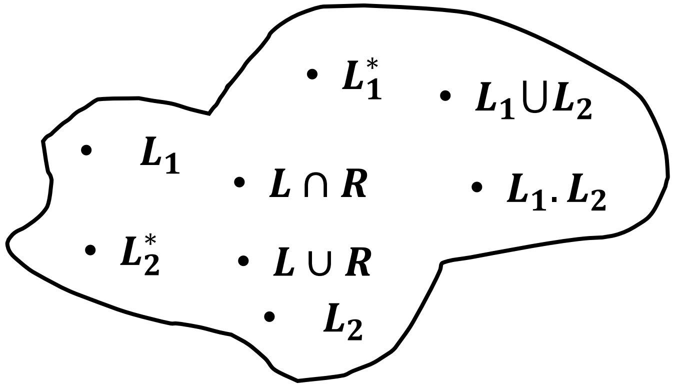
CFLs are closed under
- **Union**
- **Star**
- **Concatenation**

CFLs are NOT closed under
- **Complementation**
- **Intersection**

If L is a CFL and R is a regular language then
$L \cap R$ is a CFL.
$L \cup R$ is a CFL.

Closure properties of CFL

- CFLs are closed under **Union, Star, Concatenation**
- CFLs are NOT closed under **Complementation, Intersection**



Set of all CFLs

If L is a CFL and R is a regular language then
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Proof intuition: Construct a **Product PDA**.

If the states of the PDA P: $Q = (q_1, q_2, \cdots, q_m)$ and DFA D: $Q' = (d_1, d_2, \cdots, d_n)$, then states of **Product PDA X:**

$Q = \{(q, d), \forall q \in Q, \forall d \in Q'\}$ **Start state:** (q_1, d_1)

Closure properties of CFL

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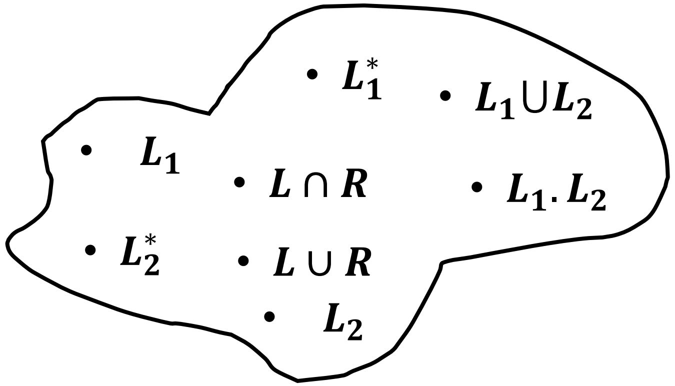

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If the states of the PDA P: $Q = (q_1, q_2, \cdots, q_m)$ and DFA D: $Q' = (d_1, d_2, \cdots, d_n)$, then states of **Product PDA X:**

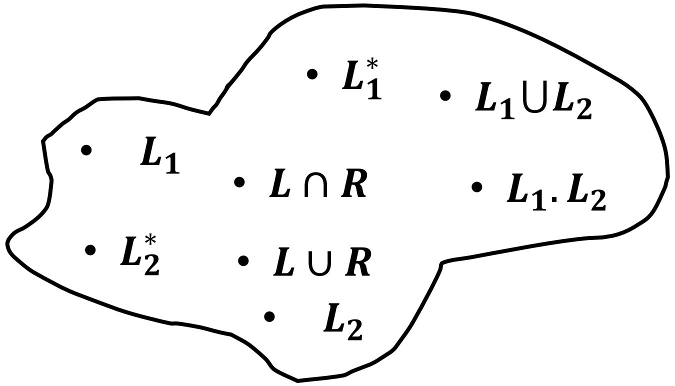
$Q = \{(q, d), \forall q \in Q, \forall d \in Q'\}$ **Start state:** (q_1, d_1)

If $\delta(q_i, a, b) = (q_j, c)$ and $\delta(d_k, a) = d_l$, then for X: $\delta\big((q_i, d_k), a, b\big) = \big((q_j, d_l), c\big)$.

So X is a PDA.

Closure properties of CFL

- CFLs are closed under **Union, Star, Concatenation**
- CFLs are NOT closed under **Complementation, Intersection**



Set of all CFLs

If $\boldsymbol{L}$ is a CFL and $\boldsymbol{R}$ is a regular language then
$\boldsymbol{L \cap R}$ is a CFL.
$\boldsymbol{L \cup R}$ is a CFL.

Proof intuition: Construct a **Product PDA**.

If the states of the PDA P: $Q = (q_1, q_2, \cdots, q_m)$ and DFA D**:** $Q' = (d_1, d_2, \cdots, d_n)$, then states of **Product PDA $\boldsymbol{X}$:**

$$Q = \{(q, d), \forall q \in Q, \forall d \in Q'\}$$

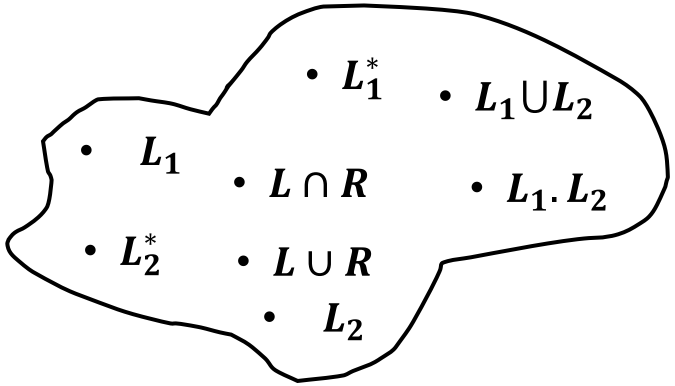
Start state: (q_1, d_1)

If $\delta(q_i, a, b) = (q_j, c)$ and $\delta(d_k, a) = d_l$, then for X: $\delta\big((q_i, d_k), a, b\big) = \big((q_j, d_l), c\big)$.

If $\boldsymbol{\delta(q_i, \epsilon, b) = (q_j, c)}$ and $\boldsymbol{\delta(d_k, \epsilon) = \Phi}$, then for $\boldsymbol{X}$: $\boldsymbol{\delta\big((q_i, d_k), \epsilon, b\big) = \big((q_j, d_k), c\big)}$.

- $\boldsymbol{L(X) = L(P) \cap L(R)}$ if the final state, say (q_r, d_s) is such that q_r and d_s **are both final states of $\boldsymbol{P}$ AND D respectively.**

Closure properties of CFL



Set of all CFLs

- CFLs are closed under **Union, Star, Concatenation**
- CFLs are NOT closed under **Complementation, Intersection**

If L is a CFL and R is a regular language then
$L \cap R$ is a CFL.
$L \cup R$ is a CFL.

Proof intuition: Construct a **Product PDA**.

If the states of the PDA P: $Q = (q_1, q_2, \cdots, q_m)$ and DFA D**:** $Q' = (d_1, d_2, \cdots, d_n)$, then states of **Product PDA X:**

$$Q = \{(q, d), \forall q \in Q, \forall d \in Q'\}$$

Start state: (q_1, d_1)

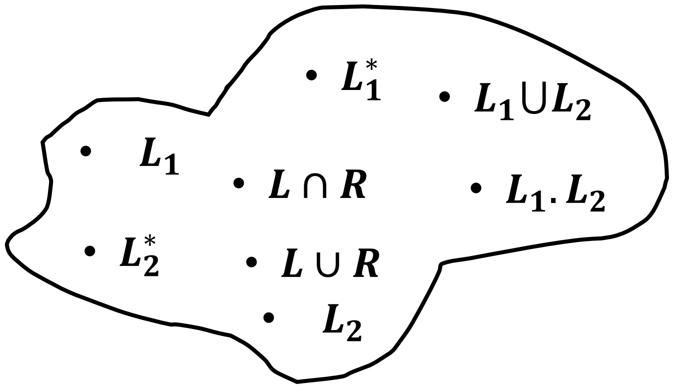
If $\delta(q_i, a, b) = (q_j, c)$ and $\delta(d_k, a) = d_l$, then for X: $\delta\big((q_i, d_k), a, b\big) = (\big(q_j, d_l\big), c)$.
If $\delta(q_i, \epsilon, b) = (q_j, c)$ and $\delta(d_k, \epsilon) = \Phi$, then for X: $\delta\big((q_i, d_k), \epsilon, b\big) = (\big(q_j, d_k\big), c)$.

- $\boldsymbol{L(X) = L(P) \cap L(R)}$ if the final state, say (q_r, d_s) is such that q_r and d_s **are both final states of P AND D respectively.**
- $\boldsymbol{L(X) = L(P) \cup L(R)}$ if the final state, say (q_r, d_s) is such that **EITHER** q_r **or** d_s **are final states of P OR D respectively.**

Closure properties of CFL

Recall that for Regular languages:

RL are closed under
- **Union**
- **Intersection**
- **Star**
- **Complement**
- **Concatenation**



Set of all CFLs

For CFLs:

CFLs are closed under
- **Union**
- **Star**
- **Concatenation**

CFLs are NOT closed under
- **Complementation**
- **Intersection**

If L is a CFL and R is a regular language then
$L \cap R$ is a CFL.
$L \cup R$ is a CFL.

Next lecture:

- **Turing Machine**

Thank You!