RECAP

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- Axioms of probability
- Sigma-algebra's as domain $\mathcal{F}$
- Borel-Sigma algebras
- Conditional probability
- Law of total probability & Bayes rule

Independence

- Independence of events
- Mutually exclusive events
- Conditional independence
- Zero probability events and independence
- Counting (HW)
- Motivate Random variables

Independence

- Consider the experiment of tossing a coin and a dice simultaneously.

- Identify its underlying probability space.

- What is $\mathbb{P}(\{H, 6\})$?

- What is $\mathbb{P}(\{T, \text{ odd }\}) = \mathbb{P}(\{\cup_{i=1,3,5}\{T, i\}\})$?

- In both cases above we have $\mathbb{P}(A \cap B) = P(A)P(B)$.

- This implies that $\mathbb{P}(A/B) = \mathbb{P}(A)$.

> - Two events A, B are independent iff $P(A/B) = P(A)$ and $P(B/A) = P(B)$.
> - Two events A, B are independent iff $P(A \cap B) = P(A)P(B)$.

Independence

- Two events A, B are independent if and only if $P(A \cap B) = P(A)P(B)$.

- If A and B are independent, then are A^c and B^c independent?
- What about A and B^c? Are they independent?
- If $A_1, A_2, \ldots, A_n$ are independent, then prove that

$$P(\cup_{i=1}^{n} A_i) = 1 - \prod_{i=1}^{n} [1 - P(A_i)]$$

Mutual and Pairwise Independence

- A collection of events $\{A_i, i \in I\}$ are said to be **mutually independent** if the $P(\cap_{j \in J} A_j) = \prod_{j \in J} P(A_j)$ for any subset J of I.
- A collection of events $\{A_i, i \in I\}$ are said to be **pairwise independent** if any pair of events from the collection are independent.
- Mutual independence implies pairwise independence but not the other way around.
- HW: Find an example where pairwise independence does not imply mutual independence.

Independence - Example

- Pick a number randomly from the set $\{1, \ldots, 10\}$.
- Event A says that the number is less than 7.
- Event B says that the number is less than 8.
- Event C says that the number is even.
- Are the events mutually independent?
- Which pair of event is independent?

Correlation between events

- Two events A, B are independent iff $P(A \cap B) = P(A)P(B)$.
- Two events A and B are positively correlated iff $P(A/B) > P(A)$.
- Two events A and B are negatively correlated iff $P(A/B) < P(A)$.
- A and B have the same correlation as A^c and B^c.
- A and B have the opposite correlation as A and B^c.

Mutually exclusive and Independence

- Two events A and B are mutually exclusive if occurrence of one implies that the other event cannot occur. Are they independent?

- If A and B are mutually exclusive, then they are not independent (and vice versa). This can be seen as follows.

A and B are Mutually Exclusive

- $P(A \cap B) = 0$
- $P(A/B) = 0$
- $P(A/B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A)}{P(B^c)}$

A and B are Independent

- $P(A \cap B) = P(A)P(B)$
- $P(A/B) = P(A)$
- $P(A/B^c) = P(A)$

- If $A \subseteq B$, then two events are neither mutually exclusive nor independent.

Zero probability events and Independence

Zero probability events are always independent!

- Let E be a zero probability event, i.e. $P(E) = 0$.
- Then for any set F, we want to show that $P(E \cap F) = 0$.
- Note that $E \cap F \subseteq E$.
- This implies that $P(E \cap F) \leq P(E)$.

Conditional independence

- Recall : $P(A/B) = \frac{P(AB)}{P(B)}$.

- Also recall : $P(A/BC) = \frac{P(AB/C)}{P(B/C)}$

- This implies $P(AB/C) = P(A/BC)P(B/C)$.

> Two events A and B are said to be conditionally independent of event C $(P(C) > 0)$ if $P((AB)/C) = P(A/C).P(B/C)$

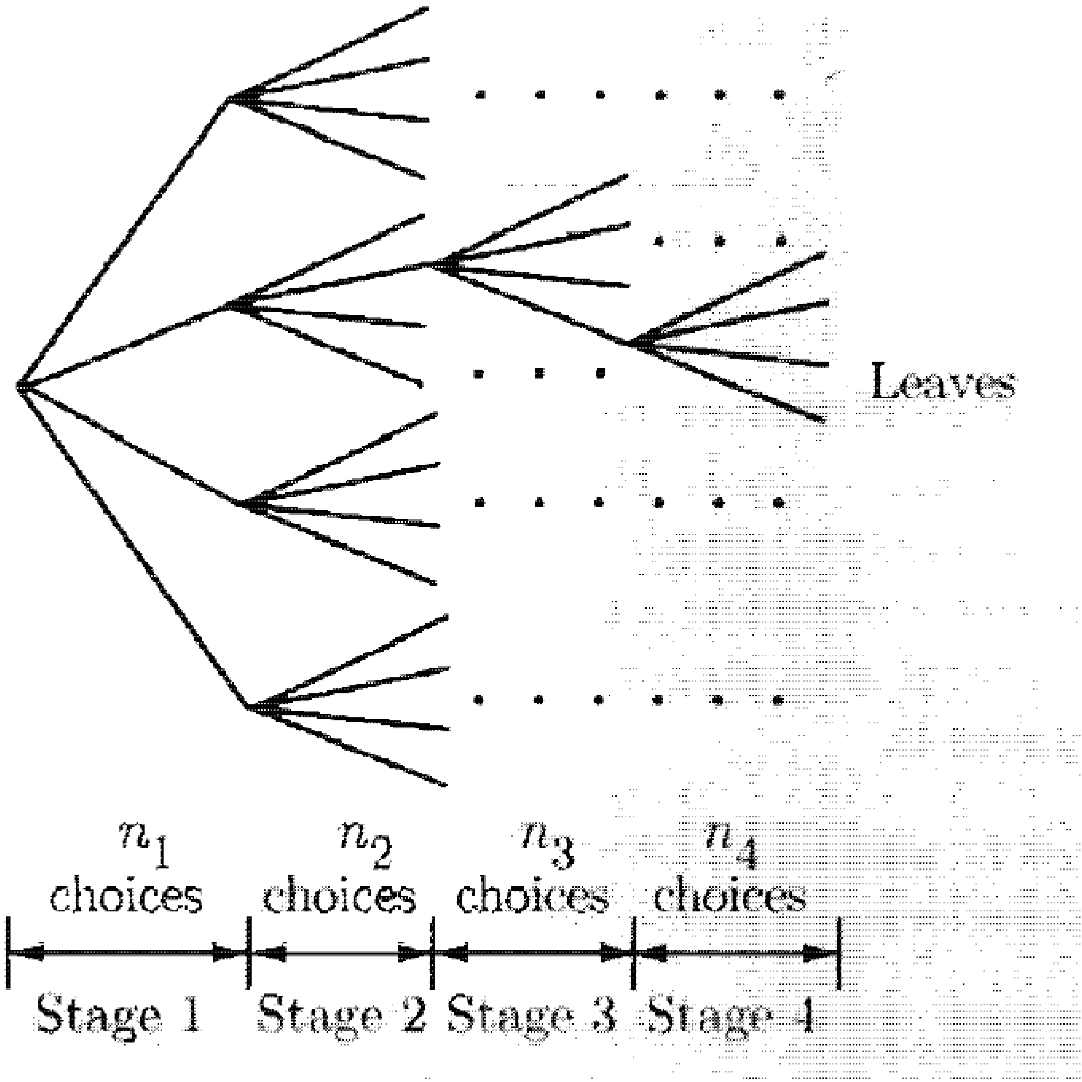
- As a consequence $P(A/BC) = P(A/C)$

HW: Verify if events A and B are conditionally independent of event C (in the experiment of picking number randomly in $\{1, .., 10\}$)

Conditional independence – example

- There are two coins, one fair and other fake (both heads). The experiment is to choose a coin uniformly and toss twice.
- Event A: First coin toss results in H. What is its probability? P(A) $= 3/4$.
- Event B: Second coin toss results in H. What is its probability? P(B) $= 3/4$.
- Event C: Coin 1 is chosen.
- What is $P(A/C)$ and $P(B/C)$? $1/2$
- What is $P((A \cap B)/C)$? $1/4$ Hence A and B are conditionally independent given C.
- Are A and B independent? HW

First principle of counting



Principle of counting

- Given n objects, in how many ways can you arrange them? n!

- Given n objects, how many distinct pairs can you form?
  ${}^{n}C_2 = \binom{n}{2} = \frac{n!}{n-2!2!}$.

- In general, given n objects, we can make ${}^{n}C_k = \binom{n}{k} = \frac{n!}{n-k!k!}$ distinct combination of k objects.

- Note that in each combination or group of k objects, the ordering within each group is immaterial. What if we also want to count this?

- ${}^{n}P_k = {}^{n}C_k \times k!$

Experiments with Sampling

- Sampling: Sampling from a set means choosing an element from the set.
- Sampling uniformly at random: All items in the set have equal probability of being chosen.
- Sampling can be with replacement or without replacement.
- Sampling can be ordered or unordered.
- In ordered sampling, $(a, b, c) \neq (c, b, a)$.
- This leaves us with 4 combinations.
  1. Ordered sampling with replacement
  2. Ordered sampling without replacement
  3. Unordered sampling with replacement
  4. Unordered sampling without replacement

Ordered sampling with replacement

- Suppose you want to sample k out of n objects with replacement and where the ordering of the k objects matters.
- Because we sample with replacement, repetition is allowed.
- How many ways can you choose k objects out of n this way?
- a) nk? b) $\binom{n}{k}$ c) k^n d) n^k ?
- There are k positions and n choices for every position.
- Total n^k.

Ordered sampling without replacement

- Suppose you want to sample k out of n objects now without replacement and where the ordering of the k objects matters.
- Because we sample without replacement, repetition is not allowed.
- How many ways can you choose k objects out of n this way?
- a) nk? b) $\binom{n}{k}$ c) k^n d) none ?
- There are k positions and $n - i + 1$ choices for every i^{th} position.
- Total $n \times (n-1) \times \ldots (n-k+1) = \frac{n!}{(n-k)!} = {}^nP_k$.

Unordered sampling without replacement

- Here you want to sample k out of n objects without replacement and the ordering of the k objects does not matters.
- Because we sample without replacement, repetition is not allowed.
- How many ways can you choose k objects out of n this way?
- a) nk? b) $\binom{n}{k}$ c) k^n d) none ?
- Essentially we want to count distinct k sized subsets from n objects without caring for ordering.
- ${}^nC_k$.

Unordered sampling with replacement

- Here you want to sample k out of n objects with replacement and the ordering of the k objects does not matters.
- Because we sample with replacement, repetition is allowed.
- How many ways can you choose k objects out of n this way?
- In any such sampling, any object i can appear atmost k times.
- Let x_i denote the number of times object i is chosen in k samples.
- Then any sampling satisfies $\sum_{i=1}^{n} x_i = k$
- How many solutions to the above equation tells you how many ways you can do the above sampling.
- $\binom{n+k-1}{k}$ Think(HW).

Example 1

- How many different 7-plate licenses are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?
- ANS: $26^3 10^4$.
- What if the alphabets and numbers are not to repeat?

Example 2

- How many functions defined on n points are possible if each functional value is either 0 or 1?
- ANS: 2^n

Example 3

- How many different letter arrangements can be formed using the letters PEPPER?
- If the P's and E's are distinguished as P_1, P_2, P_3 and E_1, E_2, R then 6!.
- But we don't want to distinguish the P's and E's.
- For every indistinguishable arrangement, say PPPREE, there are $3! \times 2!$ different distinguished arrangements.
- Using principles of counting, the number of indistinguishable arrangements are $\frac{6!}{3!2!1!}$

Example 4

- How many different permutations of n objects can be formed when n_1 are alike, n_2 are alike, ..., n_r are alike?
- ANS: $\frac{n!}{n_1!n_2!\ldots n_r!}$ where $\sum_{i=1}^{r} n_i = n$.
- When $r = 2$, we have $\frac{n!}{n_1!n-n_1!} = {}^nC_{n_1} = {}^nC_{n-n_1}$.
- Now suppose there are n distinct items and you want to divide them in r groups where group i has size n_i and where $\sum_{i=1}^{r} n_i = n$. How many ways can you do this in?
- ANS: $\binom{n}{n_1}\binom{n-n_1}{n_2}\ldots\binom{n-n_1-n_2\ldots-n_{r-1}}{n_r}$
- This is same as $\frac{n!}{n_1!n_2!\ldots n_r!}$

Example 5

- There are n red balls and r bins. How many ways can you put these balls in bins such that no bin is empty ?
- ANS: This is same as finding the number of solutions to $\sum_{i=1}^{r} x_i = n$ where $x_i > 0$.
- Arrange all n balls in a line.
- There are $n-1$ spaces between these n balls where you want to place $r-1$ partitions (or sticks).
- No two partitions can be in the same space else that would mean a bin is empty.
- Select $r-1$ out of $n-1$ (unordered without replacement)
- $\binom{n-1}{r-1}$.

Example 6

- There are n red balls and r bins. How many ways can you put these balls in bins such that bins can be empty ?
- ANS: This is same as finding the number of solutions to $\sum_{i=1}^{r} x_i = n$ where $0 \leq x_i \leq n$.
- This is same as finding the number of solutions to $\sum_{i=1}^{r} y_i = n + r$ where $0 < y_i < n$. (substitute $y_i = x_i + 1$ above!)
- $\binom{n+r-1}{r-1}$.

Example 6 – Alternative solution

- There are n red balls and r bins. How many ways can you put these balls in bins such that bins can be empty ?
- ANS: This is same as finding the number of solutions to $\sum_{i=1}^{r} x_i = n$ where $0 \leq x_i \leq n$.
- Represent x_i by that many vertical lines.
- Total n vertical lines and $r - 1$ + signs.
- $n + r - 1$ objects where n are alike and $r - 1$ are alike.
- $\binom{n+r-1}{r-1}$.

Example 7

- Toss a biased coin n times with p as the probability of head. What is the probability that you have k heads ?
- ANS: $\binom{n}{k} p^k (1-p)^{n-k}$.
- When $p = 1$ and $k = n$, we will have the convention that $0^0 = 1!$. Check the following link
- `https://en.wikipedia.org/wiki/Zero_to_the_power_of_zero`
- What is the probability that you will get head for the first time at the r^{th} toss where $r \leq n$?
- ANS: $(1-p)^{r-1} p$.

Example 8

- Suppose you roll a dice n times, what is probability that half of them show 1 and remaining half show 6? (n is even)
- ANS: $\binom{n}{n/2}\left(\frac{1}{6}\right)^{n}$
- What is the probability that n_1 of them show 1 and n_2 show 6?
- $\frac{n!}{n_1!n_2!(n-n_1-n_2)!}\left(\frac{1}{6}\right)^{n_1}\left(\frac{1}{6}\right)^{n_2}\left(\frac{4}{6}\right)^{(n-n_1-n_2)}$

Example: Monty Hall

- There are 3 doors, 2 goats and 1 car.
- Please choose one door! Suppose you choose Door 1.
- As a presenter, I open a door which has a goat (say door 2) and give you an option to change your choice. Will you ?
- Sol: Let C_i denote the event that door i conceals a car and G denote the event that a goat is shown at door 2.
- $P(G) = 1$ since the presenter has shown a goat behind door 2.
- What is $P(C_3/G)$? Verify if the following is correct.
- $P(C_3/G) = \frac{P(C_3 \cap G|C_1)P(C_1) + P(C_3 \cap G|C_1^c)P(C_1^c)}{P(G)} = 2/3 > P(C_3)$

Motivation to random variables

Random variable

- Given a random experiment with associated $(\Omega, \mathcal{F}, \mathbb{P})$, it is sometimes difficult to deal directly with $\omega \in \Omega$. eg. rolling a dice ten times.
- Notice that each sample point $\omega \in \Omega$ is not a number but a sequence of numbers.
- Also, we may be interested in functions of these sample points rather than samples themselves. eg: Number of times 6 appears in the 10 rolls.
- In either case, it is often convenient to work in a new *simpler* probability space rather than the original space.
- Random variable is a device which precisely helps us make this mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to a 'simpler' $(\Omega', \mathcal{F}', P_X)$.
- P_X is called as an induced probability measure on Ω'.

Random variable as a measurable function

A random variable X is a function $X : \Omega \to \Omega'$ that transforms the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega', \mathcal{F}', P_X)$ and is '$(\mathcal{F}, \mathcal{F}')$-measurable'.

- The map $X : \Omega \to \Omega'$ implies $X(\omega) \in \Omega'$ for all $\omega \in \Omega$.
- For event $B \in \mathcal{F}'$, the pre-image $X^{-1}(B)$ is defined as $X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$

The '$(\mathcal{F}, \mathcal{F}')$-measurability' implies that for every $B \in \mathcal{F}'$, we have $X^{-1}(B) \in \mathcal{F}$.