Topics

We have seen

- Conditioning
- Law of Iterated Expectations

This class ..

- Sums of random variables & Convolutions
- Bayes Rule revisited
- Variance of sums of random variables
- Moment Generating functions

Sums of independent random variable

- Consider $Z = X + Y$. What is the pdf of Z when X and Y?
- What is $p_Z(z)$ or $f_Z(z)$?
- $p_Z(z) = \sum_{\{(x,y):x+y=z\}} p_{X,Y}(x,y)$
- $f_Z(z) = \int_{\{(x,y):x+y=z\}} f_{X,Y}(x,y)dxdy$.
- If X and Y are independent $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ and $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. This gives us

Convolution formula

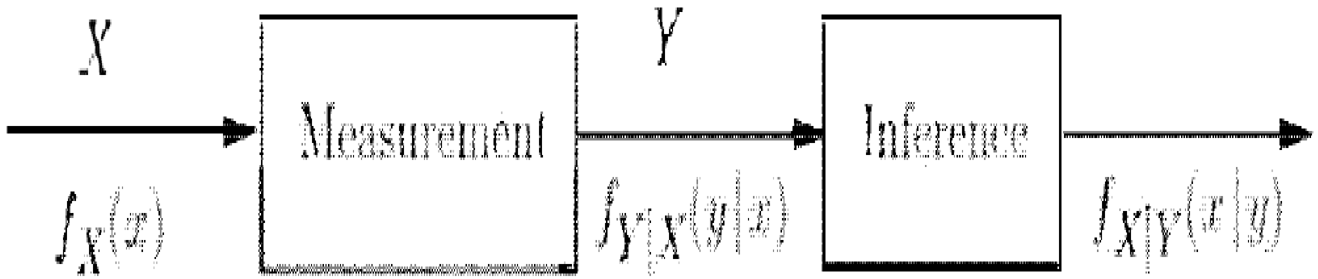
$$p_Z(z) = \sum_x p_X(x)p_Y(z-x)$$
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

HW: What if X and Y are not independent?

Examples

- EX1: Suppose X and Y are independent and $U[0, 1]$. Find the pdf and CDF of $Z = X + Y$.
- https://en.m.wikipedia.org/wiki/File:Convolution_of_box_signal_with_itself2.gif
- Ex2: Suppose X and Y are outcomes of independent roll of dice. Find the pmf of $Z = X + Y$.

Inference problem



- X is an unobservable random variable with a known distribution.
- We only observe measurements Y that takes values according to $f_{Y|X}(y|x)$.
- Objective is to draw inference about X having seen a realization of Y i.e., Obtain $f_{X|Y}(x|y)$ using only $f_X(x)$ and $f_{Y|X}(y|x)$, both of which are known.

Bayes Rule revisited

$$P(B/A) = \frac{P(A/B)P(B)}{P(A)}$$

For continuous random variables X and Y

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x)f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t)f_X(t)dt}$$

For discrete random variables X and Y

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} = \frac{p_{Y|X}(y|x)p_X(x)}{\sum_i p_{Y|X}(y|i)p_X(i)}$$

Example 3.19(Bertsekas)

Lifetime of a Phillips bulb is assumed to be an exponential random variable Y with parameter Λ. Λ itself is a uniform random variable over $[1, 1.5]$. You test a bulb and see that it has a lifetime of y units. What can you say about randomness of Λ having observed $Y = y$.?

- What is $f_\Lambda(\lambda)$?
- What is $f_{Y|\Lambda}(y|\lambda)$?
- What is $f_Y(y)$?
- $f_{\Lambda|Y}(\lambda|y) = \frac{2\lambda e^{-\lambda y}}{\int_1^{1.5} 2te^{-ty}dt}$ for $\lambda \in [1, 1.5]$.

Bayes Rule revisited

For discrete N and continuous random variable Y

$$P(N = n|Y = y) = \frac{f_{Y|N}(y|n)p_N(n)}{f_Y(y)} = \frac{f_{Y|N}(y|n)p_N(n)}{\sum_i f_{Y|N}(y|i)p_N(i)}$$

Equivalently

$$f_{Y|N}(y|n) = \frac{P(N = n|Y = y)f_Y(y)}{p_N(n)} = \frac{P(N = n|Y = y)f_Y(y)}{\int_{-\infty}^{\infty} P(N = n|Y = t)f_Y(t)dt}$$

Example 3.20 (Bertsekas)

- Suppose $X = 1$ w.p. p and $X = -1$ w.p. $1 - p$. While transmitting this signal, it is corrupted by a Gaussian noise $N \sim \mathcal{N}(0, 1)$. We observe $Y = X + N$. Suppose you observe $Y = y$, then show that

$$P(X = 1|Y = y) = \frac{pe^y}{pe^y + (1-p)e^{-y}}$$

- Intuitively, this probability goes to zero as y decreases to $-\infty$ and increases to 1 as y increases to ∞.
- $P(X = 1|Y = y) = \frac{f_{Y|X}(y|1)p_X(1)}{f_Y(y)}$
- Here $f_Y(y) = f_{Y|X}(y|1)p_X(1) + f_{Y|X}(y|-1)p_X(1)$.
- Substitute values to obtain answer.

Variance of sum of random variables

- Let $X_1, X_2, \ldots X_n$ be possibly dependent and non-identical random variables.
- Lets say you know the joint pdf/pmf for every pair of random variables from this collection.
- AIM: Calculate $Var(Z)$ where $Z = \sum_{i=1}^{n} a_i X_i$ for some scalars a_i.

Variance of sum of random variables

- Recall $Var(X) = E[X - E[X]]^2 = E[X^2] - E[X]^2$.
- Also recall $Cov(X, Y) = E[XY] - E[X]E[Y]$.
- Following properties of covariance follow (HW)

1. $Cov(X, X) = Var(X)$
2. If X, Y are independent, $Cov(X, Y) = 0$.
3. $Cov(X, Y) = Cov(Y, X)$
4. $Cov(aX, Y) = aCov(X, Y)$
5. $Cov(X + a, Y) = Cov(X, Y)$
6. $Cov(X + Z, Y) = Cov(X, Y) + Cov(Z, Y)$
7. $Cov\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j Cov(X_i, Y_j)$

Variance of sum of random variables

- AIM: Calculate $Var(Z)$ where $Z = \sum_{i=1}^{n} a_i X_i$ for some scalars a_i.
- $Var(Z) = Cov(Z, Z)$ and therefore

$$
\begin{aligned}
Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} a_j X_j\right) &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i a_j Cov(X_i, X_j) \\
&= \sum_{i=1}^{n} a_i^2 Var(X_i) \\
&+ \sum_{(i,j):i\neq j} a_i a_j Cov(X_i, X_j)
\end{aligned}
$$

$$
Var(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i^2 Var(X_i) + \sum_{(i,j):i\neq j} a_i a_j Cov(X_i, X_j)
$$

Variance of sum of random variables

$$Var(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i^2 Var(X_i) + \sum_{(i,j):i \neq j} a_i a_j Cov(X_i, X_j)$$

- Show that $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- Now if $X_i's$ are independent, what is $Var(Z)$?
- Let $\{X_i, i = 1, 2, \ldots n\}$ be i.i.d and consider $S_n = \frac{\sum_{i=1}^{n} X_i}{n}$.
- Show that $Var(S_n) = \frac{Var(X)}{n}$