Probability and Statistics: MA6.101
Homework 4

Topics Covered: Continuous Random Variable, Functions of Random Variable

Q1: Functions of Continuous Random Variables

Let $X_1 \sim \text{Uniform}(0,1)$ and $X_2 \sim \text{Exp}(1)$, independent. Find $f_Y(y)$ for $Y = \max(X_1, X_2)$.

A: For $y \in [0,1]$:

$$F_Y(y) = P(\max(X_1, X_2) \le y) = P(X_1 \le y) \cdot P(X_2 \le y) = y(1 - e^{-y}).$$

Differentiating:

$$f_Y(y) = 1 - e^{-y} - ye^{-y}, \quad 0 \le y \le 1.$$

For $y > 1$:

$$F_Y(y) = 1 \cdot (1 - e^{-y}) = 1 - e^{-y},$$

hence

$$f_Y(y) = e^{-y}, \quad y > 1.$$

Q2: Geometric $\to$ Exponential Limit

Let Y be a Geometric(p) random variable with

$$\mathbb{P}(Y = k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$

Assume $p = \lambda h$ with $\lambda > 0$ and $h > 0$, and define the scaled variable $X = Yh$. Prove that for any fixed $x > 0$,

$$\lim_{h \downarrow 0} F_X(x) \;=\; 1 - e^{-\lambda x},$$

i.e. X converges in distribution to $\text{Exp}(\lambda)$ as $h \to 0$.

A: The CDF of variable Y is

$$F_Y(y) = P(Y \le y) = 1 - (1-p)^y = 1 - (1 - \lambda h)^y$$

Since $X = Yh$,

$$F_X(x) = P(Yh \le x) = P\left(Y \le \frac{x}{h}\right) = F_Y\left(\frac{x}{h}\right) = 1 - (1 - \lambda h)^{\frac{x}{h}}$$

We can now evaluate the limit.

$$\lim_{h \to 0} F_X(x) = \lim_{h \to 0} 1 - (1 - \lambda h)^{\frac{x}{h}}$$

Since $\lim_{m \to \infty}(1 - \frac{\lambda}{m})^{xm} = e^{-\lambda x}$ where $m = \frac{1}{h}$

$$\lim_{h \to 0} F_X(x) = 1 - e^{-\lambda x}$$

We have shown that as $p \to 0$, a geometric random variable becomes exponential.

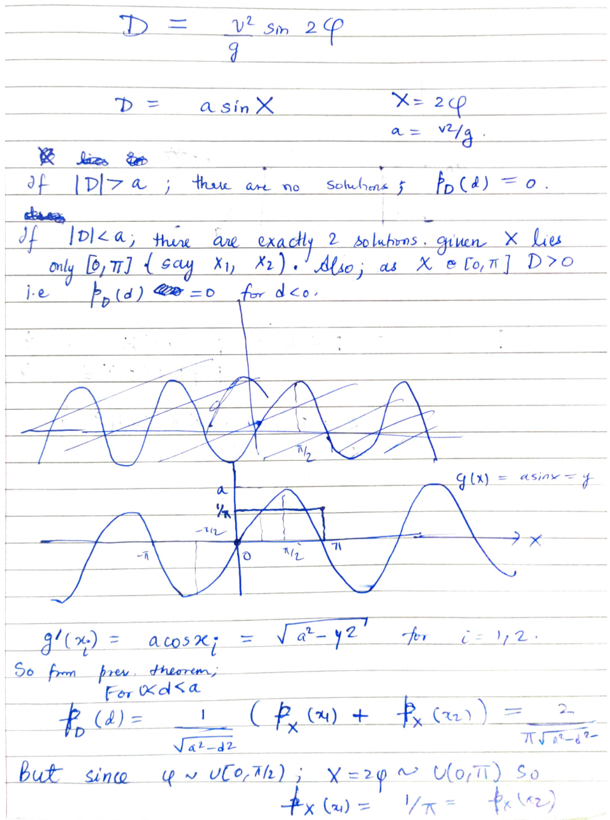


Figure 1: Q3 Ans Part 1

Q3: A particle leaves the origin under the influence of gravity, and its initial velocity v forms an angle φ with the horizontal axis. The path of the particle reaches the ground at a distance

$$d = \frac{v^2}{g}\sin(2\varphi)$$

from the origin, where g is the acceleration due to gravity.

Assuming that φ is a random variable uniformly distributed between 0 and $\pi/2$, determine:

(i) the density of d,

(ii) the probability that $d \leq d_0$.

Q4: The Laplace Distribution

Let X be a continuous random variable with the probability density function (PDF) given by:

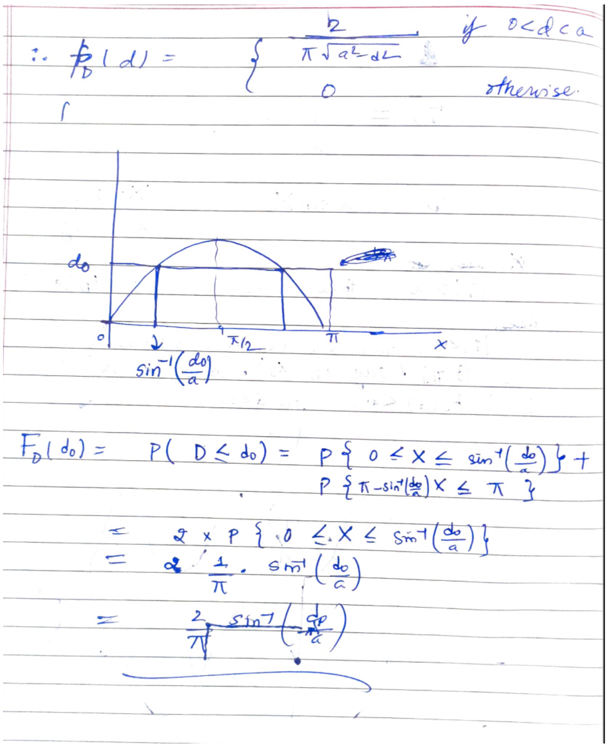
$$f_X(x) = \frac{1}{2b}e^{-\frac{|x-\mu|}{b}} \quad \text{for } -\infty < x < \infty$$

where $\mu \in \mathbb{R}$ is the location parameter and $b > 0$ is the scale parameter. This is the PDF of a Laplace distribution with parameters μ and b.

(a) Show that $f_X(x)$ is indeed a valid PDF (i.e., it integrates to 1).

(b) Calculate the expected value of X, $E[X]$.

(c) Calculate the variance of X, $\mathrm{Var}(X)$.



$$\therefore f_D(d) = \begin{cases} \dfrac{2}{\pi\sqrt{a^2-d^2}} & \text{if } 0<d<a \\ 0 & \text{otherwise} \end{cases}$$

$$F_D(d_0) = P(D \le d_0) = P\left\{0 \le X \le \sin^{-1}\left(\frac{d_0}{a}\right)\right\} + P\left\{\pi - \sin^{-1}\left(\frac{d_0}{a}\right) X \le \pi\right\}$$

$$= 2 \times P\left\{0 \le X \le \sin^{-1}\left(\frac{d_0}{a}\right)\right\}$$

$$= 2 \cdot \frac{1}{\pi} \cdot \sin^{-1}\left(\frac{d_0}{a}\right)$$

$$= \frac{2}{\pi} \sin^{-1}\left(\frac{d_0}{a}\right)$$

Figure 2: Q3 Ans Part 2

(a) To show that $f_X(x)$ is a valid PDF, we need to verify that:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

We have:

$$\int_{-\infty}^{\infty} \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}\,dx = \frac{1}{2b}\int_{-\infty}^{\infty} e^{-\frac{|x-\mu|}{b}}\,dx$$

Let's make the substitution $u = x-\mu$, so $x = u+\mu$ and $dx = du$:

$$\frac{1}{2b}\int_{-\infty}^{\infty} e^{-\frac{|u|}{b}}\,du$$

Since the integrand is an even function (symmetric about $u=0$), we can write:

$$\frac{1}{2b}\int_{-\infty}^{\infty} e^{-\frac{|u|}{b}}\,du = \frac{1}{2b}\cdot 2\int_{0}^{\infty} e^{-\frac{u}{b}}\,du = \frac{1}{b}\int_{0}^{\infty} e^{-\frac{u}{b}}\,du$$

Now, let $v = \frac{u}{b}$, so $u = bv$ and $du = b\,dv$:

$$\frac{1}{b}\int_{0}^{\infty} e^{-v}\cdot b\,dv = \int_{0}^{\infty} e^{-v}\,dv$$

Evaluating this integral:

$$\int_{0}^{\infty} e^{-v}\,dv = \left[-e^{-v}\right]_0^{\infty} = -e^{-\infty} + e^{0} = 0 + 1 = 1$$

Therefore: $\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$

(b) The expected value is:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}\, dx$$

Using the substitution $u = x - \mu$, so $x = u + \mu$ and $dx = du$:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} (u+\mu) \cdot \frac{1}{2b} e^{-\frac{|u|}{b}}\, du && (1) \\ &= \frac{1}{2b} \int_{-\infty}^{\infty} u e^{-\frac{|u|}{b}}\, du + \frac{\mu}{2b} \int_{-\infty}^{\infty} e^{-\frac{|u|}{b}}\, du && (2) \end{aligned}$$

From part (a), we know that $\int_{-\infty}^{\infty} e^{-\frac{|u|}{b}}\, du = 2b$, so:

$$\frac{\mu}{2b} \int_{-\infty}^{\infty} e^{-\frac{|u|}{b}}\, du = \frac{\mu}{2b} \cdot 2b = \mu$$

For the first integral, note that $g(u) = ue^{-\frac{|u|}{b}}$ is an odd function because:

$$g(-u) = (-u)e^{-\frac{|-u|}{b}} = -ue^{-\frac{|u|}{b}} = -g(u)$$

Therefore:

$$\int_{-\infty}^{\infty} u e^{-\frac{|u|}{b}}\, du = 0$$

Hence:

$$\boxed{E[X] = \mu}$$

(c) We use the formula $\mathrm{Var}(X) = E[X^2] - (E[X])^2$.
From part (b), we know $E[X] = \mu$, so we need to find $E[X^2]$:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}\, dx$$

Using the substitution $u = x - \mu$, so $x = u + \mu$:

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} (u+\mu)^2 \cdot \frac{1}{2b} e^{-\frac{|u|}{b}}\, du && (3) \\ &= \frac{1}{2b} \int_{-\infty}^{\infty} (u^2 + 2\mu u + \mu^2) e^{-\frac{|u|}{b}}\, du && (4) \\ &= \frac{1}{2b} \left[\int_{-\infty}^{\infty} u^2 e^{-\frac{|u|}{b}}\, du + 2\mu \int_{-\infty}^{\infty} u e^{-\frac{|u|}{b}}\, du + \mu^2 \int_{-\infty}^{\infty} e^{-\frac{|u|}{b}}\, du \right] && (5) \end{aligned}$$

We already know: - $\int_{-\infty}^{\infty} ue^{-\frac{|u|}{b}}\, du = 0$ (odd function) - $\int_{-\infty}^{\infty} e^{-\frac{|u|}{b}}\, du = 2b$
For $\int_{-\infty}^{\infty} u^2 e^{-\frac{|u|}{b}}\, du$, since $u^2 e^{-\frac{|u|}{b}}$ is an even function:

$$\int_{-\infty}^{\infty} u^2 e^{-\frac{|u|}{b}}\, du = 2 \int_{0}^{\infty} u^2 e^{-\frac{u}{b}}\, du$$

Using the substitution $v = \frac{u}{b}$, so $u = bv$ and $du = b\,dv$:

$$2\int_0^\infty (bv)^2 e^{-v} \cdot b\,dv = 2b^3 \int_0^\infty v^2 e^{-v}\,dv$$

The integral $\int_0^\infty v^2 e^{-v}\,dv = \Gamma(3) = 2! = 2$, so:

$$\int_{-\infty}^\infty u^2 e^{-\frac{|u|}{b}}\,du = 2b^3 \cdot 2 = 4b^3$$

Therefore:

$$\begin{aligned} E[X^2] &= \frac{1}{2b}\left[4b^3 + 0 + \mu^2 \cdot 2b\right] && (6)\\ &= \frac{1}{2b}\left[4b^3 + 2b\mu^2\right] && (7)\\ &= 2b^2 + \mu^2 && (8) \end{aligned}$$

Finally:

$$\text{Var}(X) = E[X^2] - (E[X])^2 = (2b^2 + \mu^2) - \mu^2 = 2b^2$$

Therefore: $\boxed{\text{Var}(X) = 2b^2}$

Q5: Establish the validity of the expected value rule

$$\mathbf{E}[g(X)] = \int_{-\infty}^\infty g(x) f_X(x)\,dx,$$

where X is a continuous random variable with PDF f_X. This rule is often called the Law of the Unconscious Statistician (LOTUS).

A: We first prove it for a non-negative function $g(x)$ and then extend it to the general case.

Step 1: Proof for a Non-Negative Function

Assume $g(x) \geq 0$ for all x. Let $Y = g(X)$. The expected value of any non-negative continuous random variable Y can be calculated using its tailed CDF:

$$\mathbf{E}[Y] = \int_0^\infty P(Y > y)\,dy \implies \mathbf{E}[g(X)] = \int_0^\infty P(g(X) > y)\,dy$$

The probability $P(g(X) > y)$ is the integral of the PDF $f_X(x)$ over the set $A_y = \{x : g(x) > y\}$.

$$P(g(X) > y) = \int_{\{x:g(x)>y\}} f_X(x)\,dx$$

Substituting this back gives a double integral:

$$\mathbf{E}[g(X)] = \int_0^\infty \left(\int_{\{x:g(x)>y\}} f_X(x)\,dx\right) dy$$

We can change the order of integration. The region of integration is $\{(x, y) : -\infty < x < \infty, 0 < y < g(x)\}$.

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} \left(\int_{0}^{g(x)} dy \right) f_X(x)\, dx$$

The inner integral evaluates to $\int_0^{g(x)} dy = g(x)$. This gives the desired result for non-negative functions:

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Step 2: Extension to a General Function
Any general function $g(x)$ can be written as the difference of two non-negative functions: $g(x) = g^+(x) - g^-(x)$, where $g^+(x) = \max(g(x), 0)$ and $g^-(x) = \max(-g(x), 0)$. By the linearity of expectation:

$$\mathbf{E}[g(X)] = \mathbf{E}[g^+(X) - g^-(X)] = \mathbf{E}[g^+(X)] - \mathbf{E}[g^-(X)]$$

Applying the result from Step 1 to the non-negative functions g^+ and g^-:

$$\begin{aligned}
\mathbf{E}[g(X)] &= \int_{-\infty}^{\infty} g^+(x) f_X(x)\, dx - \int_{-\infty}^{\infty} g^-(x) f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} (g^+(x) - g^-(x)) f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx
\end{aligned}$$

Q6: Suppose the number of customers arriving at a store obeys a Poisson distribution with an average of λ customers per unit time. That is, if Y is the number of customers arriving in an interval of length t, then $Y \sim \text{Poisson}(\lambda t)$. Suppose that the store opens at time $t = 0$. Let X be the arrival time of the first customer. Find the distribution of X.

A:
To show that X follows an exponential distribution, we will first find its CDF, by determining its tailed CDF, $\mathbb{P}(X > x)$.

The event that the arrival time of the first customer is after time x (i.e., $X > x$) is identical to the event that there are **zero** customers in the time interval $[0, x]$.

Let N_x be the number of customers arriving in the interval $[0, x]$. From the problem statement, we know that $N_x \sim \text{Poisson}(\lambda x)$. The probability mass function for a Poisson random variable with mean μ is given by $p_N(k) = \frac{e^{-\mu}\mu^k}{k!}$.

Therefore, the probability that the first arrival is after time x is:

$$\mathbb{P}(X > x) = p_{N_x}(0)$$

Substituting $\mu = \lambda x$ and $k = 0$ into the Poisson formula:

$$p_{N_x}(0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = \frac{e^{-\lambda x} \cdot 1}{1} = e^{-\lambda x}$$

Thus, the tailed CDF for X is $e^{-\lambda x}$ for $x \geq 0$.
The CDF is:

$$F_X(x) = \mathbb{P}(X \leq x) = 1 - \mathbb{P}(X > x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

For $x \geq 0$:

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}(1 - e^{-\lambda x}) = -(-\lambda)e^{-\lambda x} = \lambda e^{-\lambda x}$$

The complete PDF is:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Q7: Alice throws darts at a circular target of radius r and is equally likely to hit any point in the target. Let X be the distance of Alice's hit from the center.

(a) Find the PDF, the mean, and the variance of X.

(b) The target has an inner circle of radius t. If $X \leq t$, Alice gets a score of $S = 1/X$. Otherwise his score is $S = 0$. Find the CDF of S. Is S a continuous random variable?

A:

(a) First, we find the Cumulative Distribution Function (CDF), $F_X(x) = \mathbb{P}(X \leq x)$. Since the dart hit is uniform over the area, the probability is the ratio of the favorable area to the total area. The event $X \leq x$ corresponds to the dart landing in a circle of radius x.
The total area of the target is $A_{\text{total}} = \pi r^2$. The favorable area is $A_{\text{favorable}} = \pi x^2$.
For $0 \leq x \leq r$, the CDF is:

$$F_X(x) = \frac{A_{\text{favorable}}}{A_{\text{total}}} = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2}$$

The PDF, $f_X(x)$, is the derivative of the CDF:

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\left(\frac{x^2}{r^2}\right) = \frac{2x}{r^2}$$

So, the **PDF of X** is:

$$f_X(x) = \begin{cases} \frac{2x}{r^2} & \text{for } 0 \leq x \leq r \\ 0 & \text{otherwise} \end{cases}$$

The mean (or expected value) $\mathbb{E}[X]$ is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^r x\left(\frac{2x}{r^2}\right) dx$$

$$\mathbb{E}[X] = \frac{2}{r^2}\int_0^r x^2\, dx = \frac{2}{r^2}\left[\frac{x^3}{3}\right]_0^r = \frac{2}{r^2}\left(\frac{r^3}{3} - 0\right)$$

The **mean of X** is:

$$\mathbb{E}[X] = \frac{2r}{3}$$

The variance is $Var(X) = E[X^2] - (E[X])^2$. First, we find $E[X^2]$:

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^r x^2\left(\frac{2x}{r^2}\right) dx$$

$$\mathbb{E}[X^2] = \frac{2}{r^2}\int_0^r x^3\, dx = \frac{2}{r^2}\left[\frac{x^4}{4}\right]_0^r = \frac{2}{r^2}\left(\frac{r^4}{4} - 0\right) = \frac{r^2}{2}$$

Now, we calculate the variance:

$$Var(X) = \frac{r^2}{2} - \left(\frac{2r}{3}\right)^2 = \frac{r^2}{2} - \frac{4r^2}{9} = r^2\left(\frac{9-8}{18}\right)$$

The **variance of X** is:

$$Var(X) = \frac{r^2}{18}$$

(b) The score S is defined as: $S = \begin{cases} 1/X & \text{if } X \le t \\ 0 & \text{if } X > t \end{cases}$

We find the CDF, $F_S(s) = \mathbb{P}(S \le s)$, by considering different cases for s. The possible values for S are 0 or values in the range $[\frac{1}{t}, \infty)$.

Case 1: $s < 0$

Since the score S cannot be negative, $P(S \le s) = 0$.

Case 2: $0 \le s < 1/t$

In this range, the condition $S \le s$ is only met if $S = 0$.

$$F_S(s) = \mathbb{P}(S \le s) = \mathbb{P}(S = 0) = \mathbb{P}(X > t) = 1 - \mathbb{P}(X \le t) = 1 - F_X(t) = 1 - \frac{t^2}{r^2}$$

Case 3: $s \ge 1/t$

The condition $S \le s$ is met if $S = 0$ or if $1/t \le S \le s$.

$$F_S(s) = \mathbb{P}(S = 0) + \mathbb{P}(1/t \le S \le s)$$

The condition $1/t \le S \le s$ is equivalent to $1/t \le 1/X \le s$, which means $1/s \le X \le t$.

$$\mathbb{P}(1/s \le X \le t) = F_X(t) - F_X(1/s) = \frac{t^2}{r^2} - \frac{(1/s)^2}{r^2} = \frac{t^2}{r^2} - \frac{1}{s^2 r^2}$$

Adding the probabilities:

$$F_S(s) = \left(1 - \frac{t^2}{r^2}\right) + \left(\frac{t^2}{r^2} - \frac{1}{s^2 r^2}\right) = 1 - \frac{1}{s^2 r^2}$$

Combining these cases, the **CDF of S** is:

$$F_S(s) = \begin{cases} 0 & \text{for } s < 0 \\ 1 - \frac{t^2}{r^2} & \text{for } 0 \le s < 1/t \\ 1 - \frac{1}{s^2 r^2} & \text{for } s \ge 1/t \end{cases}$$

A random variable is continuous if its CDF is a continuous function. We check for a discontinuity at $s = 0$.

$$\lim_{s \to 0^-} F_S(s) = 0$$

$$F_S(0) = 1 - \frac{t^2}{r^2}$$

Since $\lim_{s \to 0^-} F_S(s) \neq F_S(0)$ (assuming $t < r$), the CDF has a jump discontinuity at $s = 0$.
No, **S is not a continuous random variable**. It is a **mixed random variable** because its distribution has a discrete part (a probability mass at $S = 0$) and a continuous part.

Q8: The Rain-Cursed Fest

For each day of Felicity, the rain gods independently sample rainfall from an exponential distribution:

$$X_i \sim \text{Exp}(\lambda), \quad f_{X_i}(x) = \lambda e^{-\lambda x}, \ x \ge 0, \ i = 1, 2, 3.$$

a) Find the probability that at least one of the three days receives more than the average daily rainfall $1/\lambda$.

b) Let $M = \max\{X_1, X_2, X_3\}$. (i) Find the CDF and PDF of M. (ii) Compute $\mathbb{E}[M]$.

c) Let N be the number of days on which rainfall exceeds $1/\lambda$. Find $\mathbb{E}[N]$.

Solution:

a) The average rainfall is $\mathbb{E}[X_i] = 1/\lambda$. For a single day,

$$P(X_i > 1/\lambda) = \int_{1/\lambda}^{\infty} \lambda e^{-\lambda x}\, dx = e^{-1}.$$

For three independent days, the probability that *none* exceed $1/\lambda$ is

$$(1 - e^{-1})^3.$$

Hence the probability that at least one exceeds is

$$P(\text{at least one} > 1/\lambda) = 1 - (1 - e^{-1})^3.$$

b) (i) For any $m \geq 0$,

$$F_M(m) = (1 - e^{-\lambda m})^3,$$

$$f_M(m) = 3(1 - e^{-\lambda m})^2 \cdot \lambda e^{-\lambda m}.$$

(ii) Using the tail-integral formula,

$$\mathbb{E}[M] = \int_0^\infty P(M > m)\, dm,$$

with

$$P(M > m) = 3e^{-\lambda m} - 3e^{-2\lambda m} + e^{-3\lambda m}.$$

So

$$\mathbb{E}[M] = \frac{11}{6\lambda}.$$

c) Define indicators $I_i = \mathbf{1}\{X_i > 1/\lambda\}$. Then

$$N = I_1 + I_2 + I_3, \qquad \mathbb{E}[N] = \sum_{i=1}^{3} \mathbb{E}[I_i].$$

Each $\mathbb{E}[I_i] = P(X_i > 1/\lambda) = e^{-1}$. Thus

$$\mathbb{E}[N] = 3e^{-1} \approx 1.103.$$

Q9: A stick of length 1 is broken at a uniformly random point. Answer the following related questions.

(a) **(Distribution of the longer piece)** Let L denote the length of the longer piece. Find the CDF and PDF of L, and compute $\mathbb{E}[L]$.

(b) **(Ratio of shorter to longer piece)** Let X and Y be the lengths of the shorter and longer pieces respectively, and define

$$R = \frac{X}{Y}.$$

Find (i) the CDF and PDF of R, (ii) $\mathbb{E}[R]$, and (iii) $\mathbb{E}[1/R]$ (if they exist).

(c) **(Break the longer half again)** Break the stick at random as before. Now break the longer piece (of length L) again at a uniformly random point along that longer piece. What is the probability that the three resulting pieces can serve as side-lengths of a triangle?

Solution. Let the break location measured from the left end be $U \sim \text{Uniform}(0, 1)$. The two pieces have lengths U and $1 - U$, so

$$L = \max(U, 1 - U), \qquad L \in \left[\tfrac{1}{2}, 1\right].$$

For $t \in [\frac{1}{2}, 1]$,

$$\{L \leq t\} = \{\max(U, 1 - U) \leq t\} = \{1 - t \leq U \leq t\},$$

hence

$$F_L(t) = \mathbb{P}(L \le t) = \begin{cases} 0, & t < \frac{1}{2}, \\ \mathbb{P}(1-t \le U \le t) = 2t-1, & \frac{1}{2} \le t \le 1, \\ 1, & t > 1. \end{cases}$$

Differentiating on $(\frac{1}{2}, 1)$ gives the pdf

$$p_L(t) = 2, \qquad \tfrac{1}{2} < t < 1.$$

Therefore

$$\mathbb{E}[L] = \int_{1/2}^{1} t \cdot 2\, dt = 2\left[\frac{t^2}{2}\right]_{1/2}^{1} = 1 - \frac{1}{4} = \frac{3}{4}.$$

$$\boxed{F_L(t) = 2t-1 \ (t \in [\tfrac{1}{2}, 1]),\ p_L(t) = 2,\ \mathbb{E}[L] = \tfrac{3}{4}.}$$

(Ratio of shorter to longer piece) Let X and Y be the lengths of the shorter and longer pieces respectively, and define

$$R = \frac{X}{Y}.$$

Find (i) the CDF and PDF of R, (ii) $\mathbb{E}[R]$, and (iii) $\mathbb{E}[1/R]$ (if they exist).

Solution. As before let $U \sim \text{Uniform}(0,1)$. Then

$$X = \min(U, 1-U), \qquad Y = \max(U, 1-U).$$

For $0 < u \le \frac{1}{2}$ we have $X = u,\ Y = 1-u$ and thus

$$R = \frac{u}{1-u}, \qquad u \in (0, \tfrac{1}{2}].$$

By symmetry the same R-values arise for $u \in [\frac{1}{2}, 1)$ (with u replaced by $1-u$). Hence $R \in (0,1)$.

(i) Fix $r \in (0,1)$. Solve

$$\frac{u}{1-u} \le r \quad \Longleftrightarrow \quad u \le \frac{r}{1+r}.$$

Each side of the symmetry contributes probability $\dfrac{r}{1+r}$, so

$$F_R(r) = \mathbb{P}(R \le r) = 2 \cdot \frac{r}{1+r} = \frac{2r}{1+r}, \qquad 0 < r < 1.$$

Differentiating,

$$p_R(r) = \frac{2}{(1+r)^2}, \qquad 0 < r < 1.$$

(ii) Compute expectation:

$$\mathbb{E}[R] = \int_0^1 r\, p_R(r)\, dr = \int_0^1 r \cdot \frac{2}{(1+r)^2}\, dr.$$

Let $t = 1 + r$ ($r = t - 1$, $t \in [1, 2]$):

$$\mathbb{E}[R] = 2\int_1^2 \frac{t-1}{t^2}\, dt = 2\int_1^2 \Big(\frac{1}{t} - \frac{1}{t^2}\Big)\, dt = 2\big[\ln t + \tfrac{1}{t}\big]_1^2 = 2\ln 2 - 1.$$

(iii) For $\mathbb{E}[1/R]$,

$$\mathbb{E}[\tfrac{1}{R}] = \int_0^1 \frac{1}{r}\, p_R(r)\, dr = \int_0^1 \frac{2}{r(1+r)^2}\, dr,$$

which diverges near $r = 0$ (integrand behaves like $2/r$ - this can also be seen by writing out the partial fractions). Hence

$$\mathbb{E}[\tfrac{1}{R}] = \infty.$$

$$\boxed{F_R(r) = \frac{2r}{1+r},\ p_R(r) = \frac{2}{(1+r)^2},\ \mathbb{E}[R] = 2\ln 2 - 1,\ \mathbb{E}[1/R] = \infty.}$$

(Break the longer half again) Break the stick at random as before. Now break the longer piece (of length L) again at a uniformly random point along that longer piece. What is the probability that the three resulting pieces can serve as side-lengths of a triangle?

Solution. Denote the first break by $U \sim \text{Uniform}(0, 1)$, and let

$$S = \min(U, 1 - U), \qquad L = \max(U, 1 - U).$$

Break the length-L piece at random at relative position $V \sim \text{Uniform}(0, 1)$ (independent). The two subpieces from the long piece are VL and $(1 - V)L$. The three final pieces are

$$S, \quad VL, \quad (1 - V)L,$$

which sum to 1.

Three positive numbers that sum to 1 form a triangle iff the largest of them is $< \frac{1}{2}$. Because $S \le \frac{1}{2}$ always, the triangle condition is that both $VL < \frac{1}{2}$ and $(1 - V)L < \frac{1}{2}$. Equivalently,

$$V < \frac{1}{2L} \quad \text{and} \quad 1 - V < \frac{1}{2L},$$

so

$$1 - \frac{1}{2L} < V < \frac{1}{2L}.$$

For fixed $L = t \in (\frac{1}{2}, 1)$ this interval has length

$$\ell(t) = \frac{1}{2t} - \Big(1 - \frac{1}{2t}\Big) = \frac{1-t}{t}.$$

Therefore the conditional probability (over L) that the pieces form a triangle is $p(t) = \dfrac{1-t}{t}$.
Unconditioning using the pdf of L from part (a), $p_L(t) = 2$ for $t \in (\frac{1}{2}, 1)$, we get

$$\mathbb{P} = \int_{1/2}^{1} p(t)\, p_L(t)\, dt = 2\int_{1/2}^{1} \frac{1-t}{t}\, dt = 2\int_{1/2}^{1} \Big(\frac{1}{t} - 1\Big)\, dt.$$

Compute:

$$\mathbb{P} = 2\big[\ln t - t\big]_{1/2}^{1} = 2\big(0 - 1 - (\ln\tfrac{1}{2} - \tfrac{1}{2})\big) = 2\ln 2 - 1.$$

$\boxed{\mathbb{P} = \ 2\ln 2 - 1 \approx 0.38629436.}$

Q10: The Rayleigh distribution has PDF

$$f(x) = xe^{-x^2/2}, \quad x > 0.$$

Let X have the Rayleigh distribution.

(a) Find $E(X)$.
(b) Find $E(X^2)$.

(a) Find $E(X)$.
Solution: The expected value of X is given by the integral of $x{\cdot}p(x)$:

$$E(X) = \int_0^\infty xp(x)\, dx = \int_0^\infty x(xe^{-x^2/2})\, dx = \int_0^\infty x^2 e^{-x^2/2}\, dx.$$

We can relate this integral to the variance of a standard Normal random variable. Let $Z \sim \text{Normal}(0,1)$. The PDF of Z is $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.
We know that the mean of a standard Normal is $E(Z) = 0$ and its variance is $Var(Z) = 1$. The variance is also defined as $Var(Z) = E(Z^2) - (E(Z))^2$. Substituting the known values, we find $E(Z^2) = Var(Z) + (E(Z))^2 = 1 + 0^2 = 1$.
Now, let's write out the integral for $E(Z^2)$:

$$E(Z^2) = \int_{-\infty}^{\infty} z^2\phi(z)\, dz = \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = 1.$$

From this equation, we can deduce that

$$\int_{-\infty}^{\infty} z^2 e^{-z^2/2}\, dz = \sqrt{2\pi}.$$

The integrand $g(z) = z^2 e^{-z^2/2}$ is an even function (since $g(-z) = g(z)$), so its integral over the entire real line is twice the integral over the positive real line:

$$\int_{-\infty}^{\infty} z^2 e^{-z^2/2}\, dz = 2\int_0^\infty z^2 e^{-z^2/2}\, dz.$$

Combining these results, we get:

$$2\int_0^\infty z^2 e^{-z^2/2}\,dz = \sqrt{2\pi} \implies \int_0^\infty z^2 e^{-z^2/2}\,dz = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}.$$

This is precisely the integral we derived for $E(X)$. Therefore,

$$E(X) = \sqrt{\frac{\pi}{2}}.$$

(b) Find $E(X^2)$.
Solution: Using the Law of the Unconscious Statistician (LOTUS), the second moment $E(X^2)$ is given by:

$$E(X^2) = \int_0^\infty x^2 p(x)\,dx = \int_0^\infty x^2(xe^{-x^2/2})\,dx = \int_0^\infty x^3 e^{-x^2/2}\,dx.$$

we use the substitution $u = x^2/2$. This implies $x^2 = 2u$, and differentiating with respect to x gives $du = x\,dx$. The limits of integration, $x = 0$ and $x \to \infty$, correspond to $u = 0$ and $u \to \infty$.
We can rewrite the integral by grouping terms:

$$E(X^2) = \int_0^\infty x^2(e^{-x^2/2} \cdot x\,dx).$$

Substituting u and du into the integral yields:

$$E(X^2) = \int_0^\infty (2u)e^{-u}\,du = 2\int_0^\infty ue^{-u}\,du.$$

The integral is the expected value of an exponential rv with mean 1. $U \sim \text{Expo}(1)$.
Substituting this known result back into our expression for $E(X^2)$:

$$E(X^2) = 2\int_0^\infty ue^{-u}\,du = 2E(U) = 2(1) = 2.$$

Thus, $E(X^2) = 2$.

Q11: Let $Z \sim \mathcal{N}(0,1)$ and let $c \geq 0$. Find

$$\mathbb{E}\big[\max(Z - c, 0)\big]$$

in terms of the standard normal CDF Φ and PDF φ. This is the expected payoff of a European call when the stock price at maturity is assumed to be standard gaussian and strike price c.

Solution. Write $(x)_+ = \max(x, 0)$. By LOTUS (law of the unconscious statistician),

$$\mathbb{E}\big[(Z-c)_+\big] = \int_{-\infty}^\infty (z-c)_+\,\varphi(z)\,dz = \int_c^\infty (z-c)\,\varphi(z)\,dz,$$

since $(z-c)_+ = 0$ for $z \le c$. Split the integral:

$$\mathbb{E}\big[(Z-c)_+\big] = \int_c^\infty z\,\varphi(z)\,dz \;-\; c\int_c^\infty \varphi(z)\,dz.$$

Use the standard identities

$$\int_c^\infty \varphi(z)\,dz = 1 - \Phi(c), \qquad \int_c^\infty z\,\varphi(z)\,dz = \varphi(c),$$

(the latter follows from $\varphi'(z) = -z\varphi(z)$, so $\int_c^\infty z\varphi(z)\,dz = [-\,\varphi(z)]_c^\infty = \varphi(c)$). Hence

$$\mathbb{E}\big[(Z-c)_+\big] = \varphi(c) - c\big(1-\Phi(c)\big).$$

As a check, for $c = 0$ this gives $\mathbb{E}[(Z)_+] = \varphi(0) = \frac{1}{\sqrt{2\pi}}$, which matches $\frac{1}{2}\mathbb{E}|Z| = \frac{1}{\sqrt{2\pi}}$.

$$\boxed{\mathbb{E}\big[\max(Z-c,0)\big] = \varphi(c) - c\big(1-\Phi(c)\big).}$$