This Lecture ..

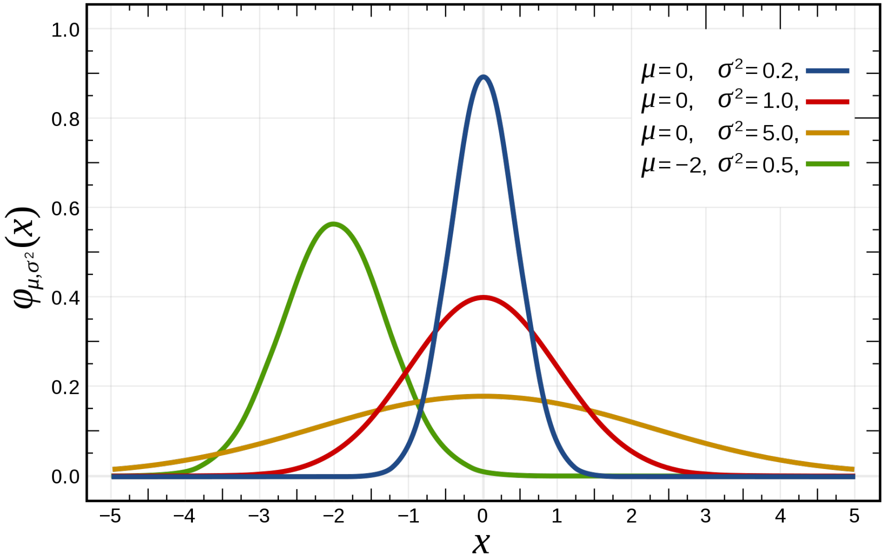
- Gaussian Random variables
- Monotone functions of continuous random variables
- Right-continuity of CDF
- Mixed random variables

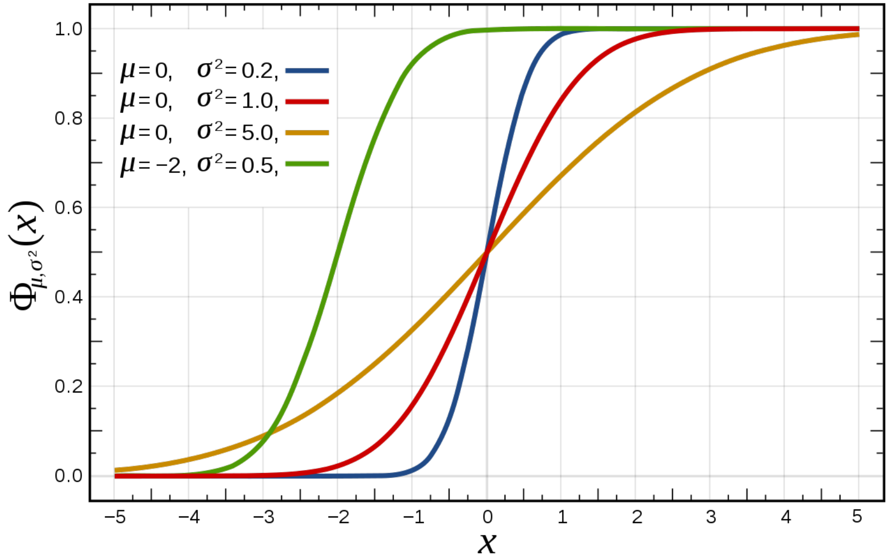
Gaussian random variable ($\mathcal{N}(\mu, \sigma^2)$)

- This is a real valued r.v. with two parameters, μ and σ.

- Its pdf $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for all $x \in \mathbb{R}$.

- Verify: $\int_{-\infty}^{\infty} f_X(x)dx = 1$, $E[X] = \mu$ and $Var(X) = \sigma^2$.





Standard Normal random variable ($\mathcal{N}(0,1)$)

- When $\mu = 0$ and $\sigma = 1$, it is called as a standard normal.
- In this case $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.
- What is $\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$? How do you even solve this? $(= \sqrt{2\pi})$
- The CDF of standard normal, denoted by $\Phi(x)$ is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

- $\Phi =$These values are recorded in a table. (Fig. 3.10 in Bertsekas)
- `https://en.wikipedia.org/wiki/Gaussian_function`

Normality preserved under Linear Transformations

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b$ is also a normal variable with $E[Y] = a\mu + b$ and variance $a^2\sigma^2$. (To be proved later)

Significance of Gaussian r.v.

- Key role in Central limit theorem.
- $\frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \dfrac{\sigma^2}{n})$ where X_i is any random variable with mean μ and variance σ^2.
- Building block for multinomial Gaussian vector and Gaussian processes (GP).
- Gaussian process are used in Bayesian Optimization (black-box optimization).
- Brownian motion is a type of GP and is used in Finance.
- GP Regression, Gaussian mixture models, used widely in ML.

List of Probability distributions ...

https://en.wikipedia.org/wiki/List_of_probability_distributions

Important ones are Beta, Gamma, Erlang, Logistic, Weibull ....

Function of continuous random variables

- Consider $Y = aX + b$ where X is a continuous random variable.
- What is $F_Y(y)$ and $f_Y(y)$?
- $F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$.
- $F_Y(y) = F_X(\frac{y-b}{a})$ if $a > 0$
- $f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X(\frac{y-b}{a})$ when $a > 0$
- $F_Y(y) = 1 - F_X(\frac{y-b}{a})$ if $a < 0$
- $f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{-1}{a} f_X(\frac{y-b}{a})$ when $a < 0$
- In general, $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$

Function of continuous random variables

Consider $Y = aX + b$ where X is a continuous random variable. Then $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$.

- What if $Y = g(X)$ where $g(\cdot)$ is continuous, differentiable and monotone. Any guess?
- Since $g(.)$ is monotone and continuous it is invertible. Let $h(.)$ denote the inverse function. Then $h(Y) = X$.

Consider $Y = g(X)$ where g is monotone, continuous, differentiable. Then $f_Y(y) = |\frac{dh}{dy}(y)| f_X(h(y))$ where h is the inverse function of g.

Function of continuous random variables

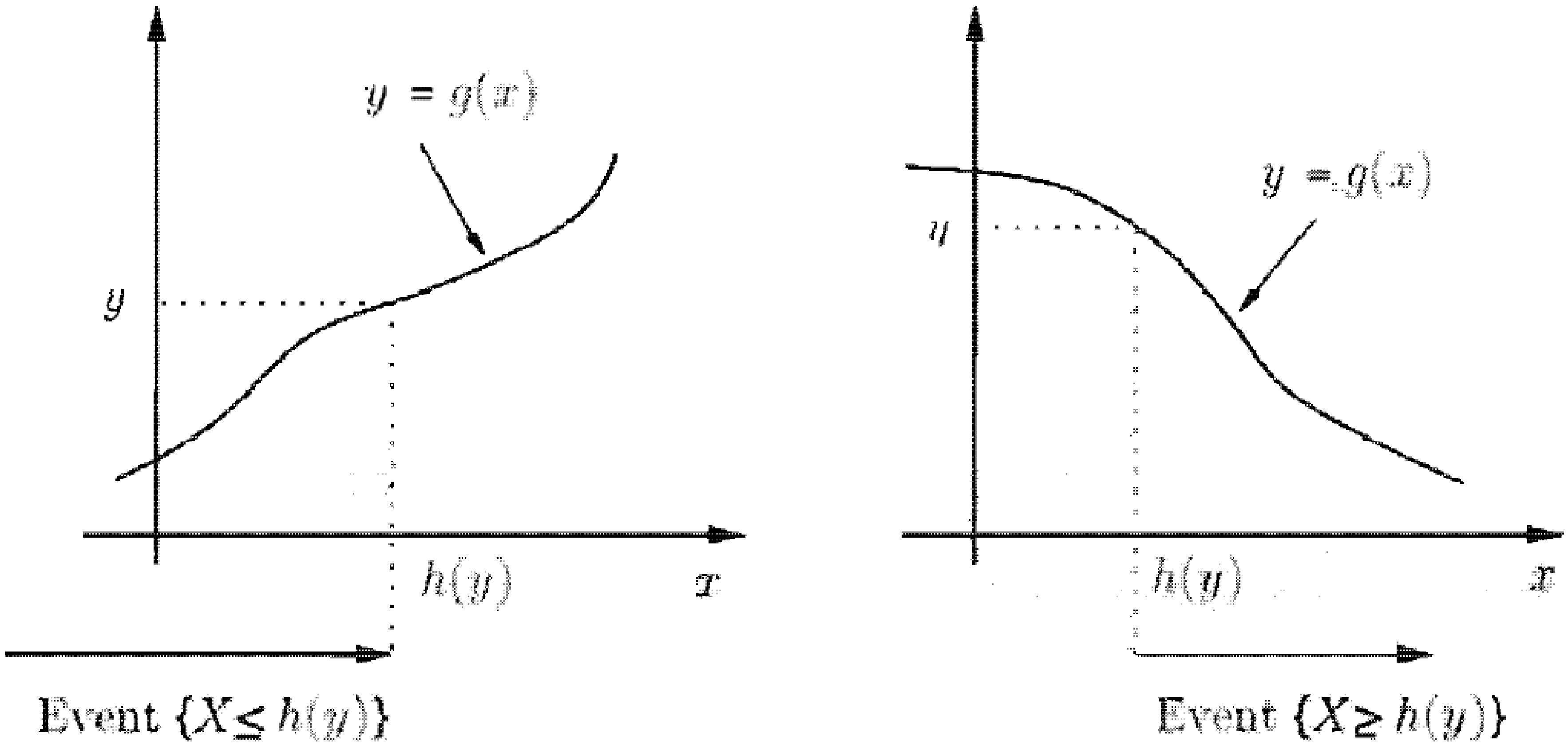
Consider $Y = g(X)$ where g is monotone, continuous, differentiable. Then $f_Y(y) = f_X(h(y))|\frac{dh}{dy}(y)|$ where h is the inverse function of g.

Proof:

- Since $g(.)$ is monotone and continuous it is invertible. Let $h(.)$ denote the inverse function. Then $X = h(Y)$.
- $F_Y(y) = P(g(X) \leq y)$.
- Is $P(g(X) \leq y) = P(X \leq h(y))$ always?
- Are the two events $\{g(X) \leq y\}$ and $X \leq h(y)$ same?
- If they are same, then the two probabilities are equal.

Function of continuous random variables

- Are the two events $\{g(X) \leq y\}$ and $\{X \leq h(y)\}$ same ?



- Same when g is increasing and compliments when g is decreasing.

Function of continuous random variables

- Are the two events $\{g(X) \leq y\}$ and $\{X \leq h(y)\}$ same ?
- Same when g is increasing and compliments when g is decreasing.
- CASE 1: $g(x)$ is non-decreasing
- $F_Y(y) = P(g(X) \leq y) = P(X \leq h(y)) = F_X(h(y))$.
- $f_Y(y) = \frac{d}{dy}(F_X(h(y))) = f_X(h(y))\frac{dh}{dy}(y)$ where $\frac{dh}{dy}(y) \geq 0$ as h is also non-decreasing.
- Rewritten therefore as $f_Y(y) = f_X(h(y))|\frac{dh}{dy}(y)|$

Function of continuous random variables

- Are the two events $\{g(X) \leq y\}$ and $\{X \leq h(y)\}$ same ?
- Same when g is increasing and compliments when g is decreasing.
- CASE 2: $g(x)$ is non-increasing
- $F_Y(y) = P(g(X) \leq y) = P(X > h(y)) = 1 - F_X(h(y))$.
- $f_Y(y) = -\frac{d}{dy}(F_X(h(y))) = -f_X(h(y))\frac{dh}{dy}(y)$ where $\frac{dh}{dy}(y) \leq 0$ as h is non-increasing as well.
- Rewritten therefore as $f_Y(y) = f_X(h(y))|\frac{dh}{dy}(y)|$. $\square$

HW: What about the case when g is not monotone ?

Q: Suppose $Y = X^2$, then what is $f_Y(y)$ in terms of $f_X(x)$?