Probability and Statistics: MA6.101

Homework 2

Topics Covered: Sigma Algebra, Probability spaces, Conditional Probability, and Total Probability.

Q1: A four-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment? Also find the sigma algebra.

Q2: A vendor is heading to the iconic Narendra Modi Stadium Mortera for the IPL final between CSK and RCB. He carries two bags of jerseys:

- Bag 1: 100 CSK jerseys
- Bag 2: 100 RCB jerseys

On the way, an accident causes a *swap of k jerseys each way*:

- k RCB jerseys move into Bag 1
- k CSK jerseys move into Bag 2

After the swap, each bag still has exactly 100 jerseys. A spectator will first choose a bag at random with $\Pr(\text{Bag 1}) = p$ and $\Pr(\text{Bag 2}) = 1 - p$, then draw a jersey from the chosen bag. The overall probability of drawing an RCB jersey is known to be 0.4.

(a) Find p in terms of k.

$$\boxed{p = \frac{60 - k}{100 - 2k}} \qquad (k \neq 50).$$

(b) Determine which integer values for $k \in \{0, 1, \ldots, 100\}$ are valid.

$$\boxed{k \in \{0, 1, \ldots, 40\} \ \cup \ \{60, 61, \ldots, 100\}}$$

(c) Suppose if $k = 30$, check if any p can make the overall probability of RCB draw equal to 0.4. If so, find it.

$$\boxed{0.75}$$

(d) For any valid k from part (b), if a bag is chosen and the jersey drawn is CSK, what is the probability that the jersey is from Bag 1? Substitute the values from part (c) and return the final answer.

$$\boxed{\frac{7}{8} = 0.875}.$$

Q3: Alice searches for her term paper in her filing cabinet. which has several drawers. She knows that she left her term paper in drawer j with probability $p_i > 0$. The drawers are so messy that even if she correctly guesses that the term paper is in drawer i. the probability that she finds it is only d_i. Alice searches in a particular drawer say drawer i. but the search is unsuccessful. Conditioned on this event, find the probability that her paper is in drawer j in terms of p_i, p_j, d_i and d_j.

Q4: You have a 10 minute break between class and the tutorial during which you want to buy a cup of coffee from VC. It will rain during the break with probability 0.3, VC will be crowded with probability 0.5. Answer the following:

(a) Assuming that VC cannot be crowded when it is raining. If it rains you will be late to the tutorial with probability 0.5 and if it is crowded the probability is 0.3. What is the probability that you will be late to the tutorial?

$$\boxed{Ans = 0.30}$$

(b) In the scenario from (a), given that you are late, which is more probable, rain or VC being crowded?

$$\boxed{\mathbb{P}(R \mid \text{late}) = 0.5, \qquad \mathbb{P}(C \mid \text{late}) = 0.5.}$$

(c) Assuming that VC being crowded and rain are independent events. If both occur you will be late with probability 0.75, if it rains but VC is not crowded probability of being late to the tutorial is 0.5 and if it is crowded but not raining the probability is 0.3.

$$\boxed{Ans = 0.2925}$$

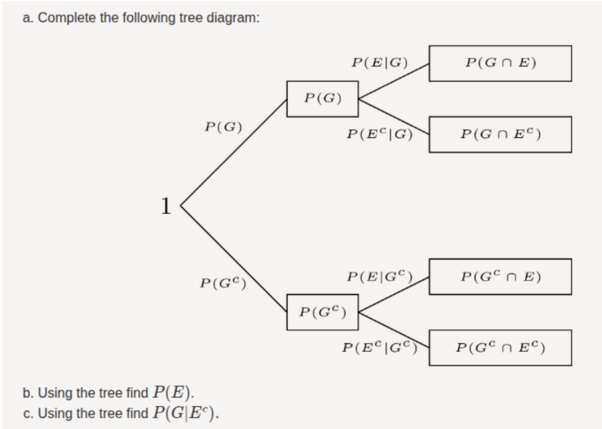
Q5: Given set A:

$$A\{(-\infty, x] : x \in \mathbb{R}\}$$

Prove that the sigma algebra generated by A is the Borel σ-algebra.

Q6: Let $\{A_1\}, \{A_2\}, ...$ be a finite or countable partition of a non-empty set Ω (i.e., A_i are pairwise disjoint and their union is Ω). What is the σ-algebra generated by the collection of subsets $\{A_n\}$?

Q7: Consider a communication system. At any given time, the communication channel is in good condition with probability 0.8, and is in bad condition with probability 0.2. An error occurs in a transmission with probability 0.1 if the channel is in good condition, and with probability 0.3 if the channel is in bad condition. Let G be the event that the channel is in good condition and E be the event that there is an error in transmission.


a. Complete the following tree diagram:

b. Using the tree find $P(E)$.
c. Using the tree find $P(G|E^c)$.

Part (a): Complete the tree diagram

Missing probabilities:

- $\mathbb{P}(E^c|G) = 0.9$
- $\mathbb{P}(E^c|G^c) = 0.7$

Joint probabilities:

- $\mathbb{P}(G \cap E) = 0.08$
- $\mathbb{P}(G \cap E^c) = 0.72$
- $\mathbb{P}(G^c \cap E) = 0.06$
- $\mathbb{P}(G^c \cap E^c) = 0.14$

Part (b): Find $\mathbb{P}(E)$

$$\boxed{\mathbb{P}(E) = 0.14}$$

Part (c): Find $\mathbb{P}(G|E^c)$

$$\boxed{\mathbb{P}(G|E^c) = \frac{36}{43} \approx 0.837}$$

Q8: $E_1, E_2, \ldots, E_n$ be n events, each with positive probability. Prove that

$$\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) = \mathbb{P}\{E_1\} \cdot \mathbb{P}\{E_2 \mid E_1\} \cdot \mathbb{P}\{E_3 \mid E_1 \cap E_2\} \cdots \mathbb{P}\left\{E_n \mid \bigcap_{i=1}^{n-1} E_i\right\}.$$

Q9: The alarm system at a nuclear power plant is not completely reliable. If there is something wrong with the reactor, the probability that the alarm goes off is 0.99. On the other hand, when nothing is actually wrong the alarm still goes off on 0.02 of the days. Suppose that something is wrong with the reactor only one day out of 100.

(a) What is the probability that the alarm goes off on a randomly chosen day?

$$\boxed{0.0297.}$$

(b) Given that the alarm goes off on a particular day, what is the probability that there is actually something wrong with the reactor on that day?

$$\boxed{\frac{1}{3}}.$$

Q10: A survey shows 56% of all American workers have a workplace retirement plan, 68% have health insurance, and 49% have both benefits. We select a worker at random.

(a) What is the probability he has neither health insurance nor a retirement plan?

$$\boxed{0.25}$$

(b) What is the probability he has health insurance, if he has a retirement plan?

$$\boxed{0.875.}$$

(c) Are having health insurance and a retirement plan independent events? Are these two benefits mutually exclusive?

Q11: IIITH is managing two independent government projects. The SERC lab is responsible for the first project, and the CSTAR lab is responsible for the second. Let A be the event that the SERC lab's project is successful, and let B be the event that the CSTAR lab's project is successful. The probabilities of success are $P(A) = 0.4$ and $P(B) = 0.7$.

(a) If the SERC lab's project is unsuccessful, what is the probability that the CSTAR lab's project is also unsuccessful?

$$\boxed{0.3}.$$

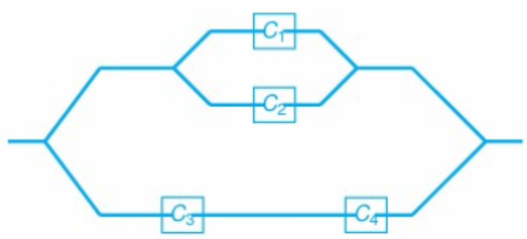
(b) What is the probability that at least one of the two projects is successful?

$$\boxed{0.82}.$$

(c) Given that at least one of the projects succeeds, what is the probability that only the SERC lab's project is successful?

$$\boxed{\frac{6}{41}}.$$

Q12: Consider the system of components connected as in the figure given below. Components C_1 and C_2 are connected in parallel, so that subsystem S_1 works if either C_1 or C_2 works; since C_3 and C_4 are connected in series, that subsystem S_2 works if both C_3 and C_4 work. If components work independently of one another, with $P(\text{component works}) = 0.9$, calculate $P(\text{system works})$.



$$\boxed{0.9981}.$$