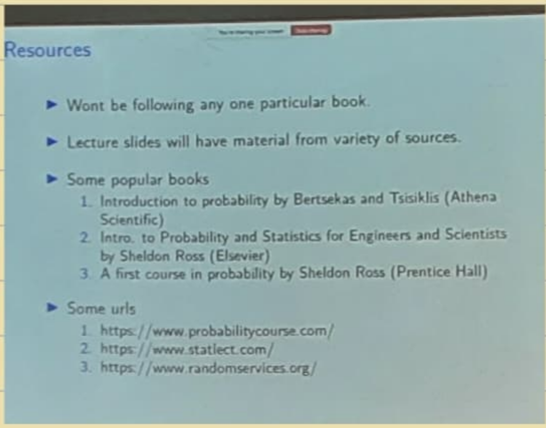
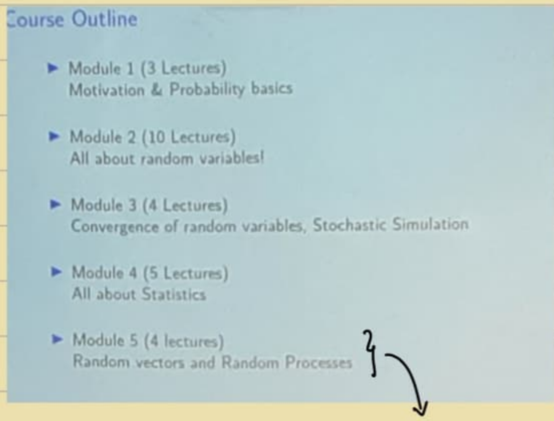
2/8

PnS





Finance

[3Blue1Brown]

→ **Random Experiments** :

Experiments involving randomness

- **Sample Space** (Ω) :

  Set of all possible outcomes of the random experiment.

  Can be finite or infinite.

e.g. $\Omega_c = \{H, T\}$

$$\Omega_{2c} = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}$$

- **Sample point / Possible outcome** : $\omega \in \Omega$
- **Event** : $A \subseteq \Omega$

  $C_1 = \{T\}$ i.e. $C_1 \longrightarrow$ Event

- Probability of event A : $\mathbb{P}(A)$

  i.e. $\mathbb{P}(C_1) = 1/2$

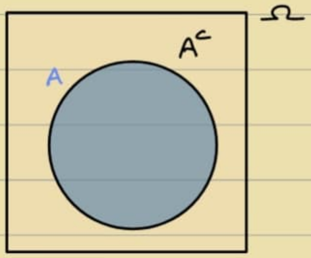
→ **Probability Theory** :

Probability is a measure.

$\hookrightarrow$ Set function

Set Theory :

• → H.W



ϕ : Empty set , $\phi \leq A, \forall A$

Union : $A \cup B$

Intersection : $A \cap B$

Difference : $A \setminus B \equiv A - B \equiv A \cap B^c$

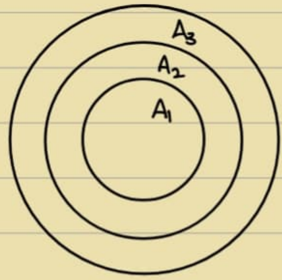
Symmetric difference : $A \Delta B \equiv (A \setminus B) \cup (B \setminus A)$

Disjoint events ≡ Mutually exclusive

Cardinality of A : $|A|$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(for n sets)

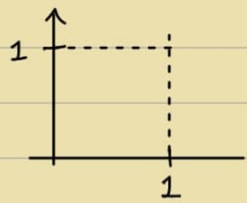


→ Increasing Sequence

$A_1 \subseteq A_2 \subseteq A_3$

Cartesian Product : $\underbrace{A \times B}$ → $(a, b)\ \forall (a \in A\ \&\ b \in B)$

$$[0,1] \times [0,1] = \{(a,b) : a \in \underbrace{[0,1]}_{\Omega_1}\ \&\ b \in \underbrace{[0,1]}_{\Omega_2}\}$$



• Powerset of A : $P(A)$

$\Omega = \{1, 2, 3, 4, 5, 6\}$

1	2	3	4	5	6	
×	×	×	×	×	×	000000
×	×	×	×	×	✓	000001
×	×	×	×	✓	×	000010

} Binary Representation

i.e. $2^6 \longrightarrow |P(\Omega)|$ i.e. 2^Ω

$$\underbrace{P([0,1])} = ? = \{(a,b) : a \leq b ; a, b \in [0,1]\} \longleftarrow \text{COMES LATER}$$

(Sigma Algebra)

→ Cardinality is uncountable infinite.

$B(\mathbb{R})$??

→ Functions :

Domain D
Range R $\Big\} f : D \longrightarrow R$

(Injection
Surjection
Bijection)

e.g. $f : \mathbb{R} \to \mathbb{R}\ ,\ f(x) = x$

- Set Functions :

  Functions which act on sets

  D : Collection of sets

→ Probability : $\mathbb{P}$ ↳ Set function

Axioms :

(1) $\mathbb{P}(\emptyset) = 0$

$\mathbb{P}(\Omega) = 1$

(2) For a set $A \subseteq \Omega$, we have $0 \leq P(A) \leq 1$

(3) For a disjoint collection of events $A_1, A_2, A_3, \ldots\ldots$, where $A_i \subseteq \Omega$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

[Union ⟶ Countable sets]

Domain of $\mathbb{P}$ in general ⟶ Powerset of A , i.e. $P(A)$

$\mathbb{P}(\Omega) = 1$

$P(\Omega) = \{A : A \subseteq \Omega\}$

6/8 Counter - example :

$\Omega = \mathbb{R} \Longrightarrow \mathbb{P}(\mathbb{R}) = 1$

Domain : $P(\mathbb{R}) \longrightarrow$ Complex

$\mathbb{P} : P(\mathbb{R}) \longrightarrow [0,1]$

$\mathbb{P}$ has the property that sets of equal 'length' have equal probability.

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n+1)\ ,\quad \text{where } [n, n+1) \in P(\mathbb{R})$$

$$\mathbb{P}(\mathbb{R}) = 1 = \sum \mathbb{P}[n, n+1)$$

∴ Restrict domain to measurable sets

Cantor's Theory

- Towards Sigma Algebra:

$\mathcal{F} = \{\phi, \Omega, \mathbb{R}_-, \mathbb{R}_+\}$

Event space or Sigma-Algebra $\mathcal{F}$ associated with a set Ω is a collection of subsets of Ω that satisfy

(1) $\phi \in \mathcal{F}$

$\Omega \in \mathcal{F}$

(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(3) $A_1, A_2, A_3, \ldots\ldots, A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

σ - Algebra is said to be closed under formation of complements & countable unions.

$$\begin{array}{l|l} A_1 \in \mathcal{F} & A_1^c \in \mathcal{F} \\ A_2 \in \mathcal{F} & A_2^c \in \mathcal{F} \end{array}$$

$(A_1^c \cup A_2^c)^c \Rightarrow$ Closed under the formation of countable intersections.

Note: When Ω is countable and finite $\Rightarrow P(\Omega)$ is the domain

$\Omega \to$ Sample space

$P(\Omega) \to$ Sigma Algebra

$(\Omega, \mathcal{F}, \mathbb{P})$: Probability Space

$\mathbb{P} \to$ Probability measure on $(\Omega, \mathcal{F})$

$\mathbb{P} : \mathcal{F} \to [0,1]$

S.T. Axioms hold true

$|\Omega| < \infty$, $\mathcal{F} = 2^n$

- Probability space for $U[0,1]$:

$\Omega = [0,1]$

Suppose, $\mathcal{F} = \{\phi, [0,1], [0,.5), [.5,1]\}$

$P([.25,.75]) = .5$

but can we show?

$\mathcal{F}^+ = \{\phi, [0,1], [0,.5), [.5,1], [.25,.75]\}$

$\to$ Not a sigma algebra

$\mathcal{F}^{++} \to$ Not a sigma-algebra

Borel-Sigma Algebra $\mathcal{B}[0,1]$

$\mathcal{B}[0,1] \longrightarrow$ σ - Algebra generated by closed sets of form $[a, b]$

$a \leq b$ & $a, b \in [0,1]$

Can also be of form $(a, b]$ or $[a, b)$

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \tfrac{1}{n}, \; b - \tfrac{1}{n}\right]$$

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, \; b + \tfrac{1}{n}\right)$$

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a + \tfrac{1}{n}, \; b\right)$$

- $\underline{\underline{\mathcal{B}(\mathbb{R})}}$:

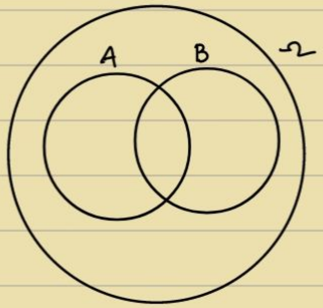
  $\Omega = \mathbb{R}$

  form $(a, b) \mid a \leq b \; ; \; a, b \in \mathbb{R}$

  $\mathcal{B}(\mathbb{R})$ contains intervals of the form

  $[a, b]$, $[a, b)$, $[a, \infty)$, (a, ∞) , $(-\infty, b]$, $(-\infty, b]$, $\{a\}$

- **Consequences of Probability Axioms** :
  - $P(A^c) = 1 - P(A)$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$A = (A \cap B) \cup (A \setminus B)$

$P(A) = P(A \cap B) + P(A \setminus B)$

H.W

- $A \subseteq B$, Then prove that $P(A) \leq P(B)$

  ($A \subseteq B \Rightarrow$ Event A implies event B)

- $P(A \cup B \cup C) = ?$

  Hint : 3^{rd} Axiom

H.W : Inclusion – exclusion principle

- Impossible event v/s Zero prob. event :

In $U[0,1]$, $P(\omega = 0.5) = ?$

$$P(\omega = 0.5) = 0$$

$$P(\omega \in [a,b]) = b - a$$

Then, $P([.5,.5]) = P(\{.5\}) = 0$

$P(\phi) = 0 \Rightarrow \phi$ is impossible

e.g $C = [0,.5] \cap [.8,1]$ $\hookrightarrow$ Null Set

$$\Omega = \bigcup_{\omega \in \Omega} \{\omega\}$$

$$P(\Omega) = P\left(\bigcup_{\omega \in \Omega} \{\omega\}\right) = \sum_{\omega \in \Omega} P(\{\omega\}) = 0 \quad \times$$

9/8

$a_1, a_2, \ldots\ldots, a_n$

$n > N_\varepsilon$

$|a_n - L| \leq \varepsilon$

Continuous : $x \to c \Rightarrow f(x) \to f(c)$

Continuous set function S, $A_m \longrightarrow A$.

$$S(A_m) \longrightarrow S(A)$$

Lemma :

$$A_n \uparrow A \text{ (or) } A_n \downarrow A \Longrightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$$

$$P(A) = \sum_{i=1}^{\infty} P(F_i) \qquad F_n = A_n - A_{n-1}$$

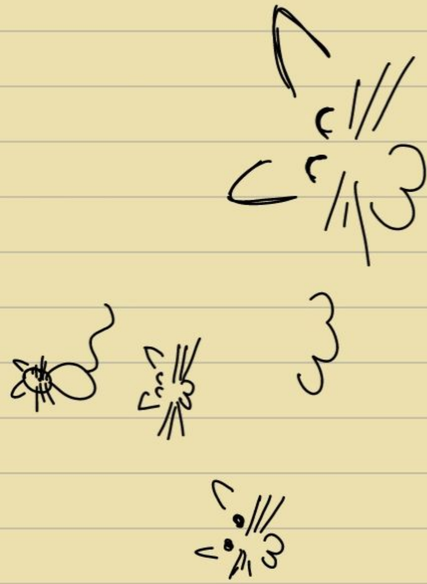
$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i)$$

$$= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} F_i\right)$$

$$= \lim_{n\to\infty} P(A_n)$$

- <u>Conditional Probability</u> :

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad , \quad P(A) > 1$$



<u>Property</u> :

$$P(A/B) \cdot P(B) = P(B/A) \cdot P(A)$$

<u>H.W</u> : $P(A/(B \cap C))$

$$P(A \cap B / C) = P(A/BC) \cdot P(B/C)$$

<u>H.W</u> : $P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/AB)$

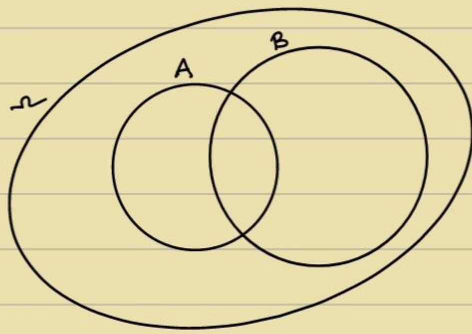
<u>H.W</u> : $P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = P(A_1) \cdot P(A_2/A_1) \cdot \ldots\ldots$

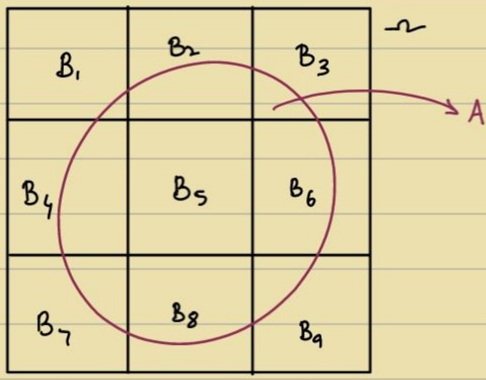
Ex. $A = (A \cap B) \cup (A \cap B^c)$

$P(A) = P(A \cap B) + P(A \cap B^c)$

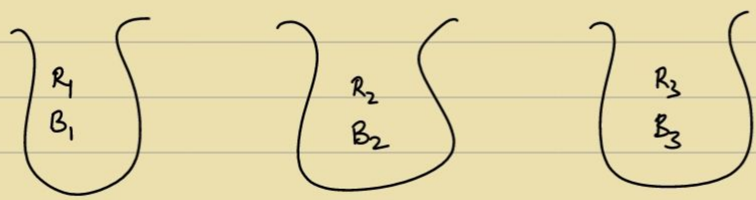
$\quad = P(A/B) \cdot P(B) + P(A/B^c) \cdot P(B^c)$

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A/B_i) \cdot P(B_i)$$





Q)



$R_1 + B_1 = M \qquad R_2 + B_2 = M \qquad R_3 + B_3 = M$

$$\frac{1}{3} \times \frac{R_1}{M} + \frac{1}{3} \times \frac{R_2}{M} + \frac{1}{3} \times \frac{R_3}{M}$$

13/8

- **Independence:**

Consider tossing a coin & rolling a dice simultaneously

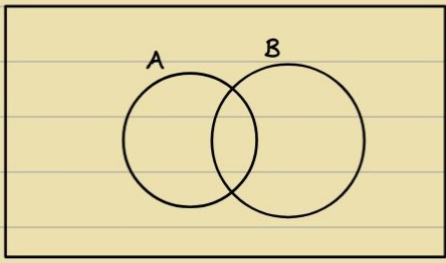
$\Omega = \{\{H,1\}, \{H,2\}, \dots\dots, \{T,1\}, \dots\}$

$\mathbb{P}(\{H,6\}) = 1/12$

$$\mathbb{P}(\{T, odd\}) = \mathbb{P}(\{\bigcup_{i=1,3,5} \{T,i\}\})$$
$$= 1/12 + 1/12 + 1/12$$
$$= 3/12$$
$$= 1/4$$

$\mathbb{P}(A \cap B) = P(A) \cdot P(B)$

$\mathbb{P}(A/B) = P(A)$



To prove:

$\mathbb{P}(A^c \cap B^c) = P(A^c) \cdot P(B^c)$

$$P(A^c) \cdot P(B^c) = (1 - P(A))(1 - P(B))$$
$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$
$$= 1 - P(A) - P(B) + P(A \cap B)$$
$$= P(A^c \cap B^c)$$

H.W: Are A and B^c independent?

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \prod_{i=1}^{n} [1 - P(A_i)] \quad \text{if } A_1, A_2, \dots A_n \text{ are independent}$$

- **Mutual Independence:**

$A_i, \ i \in I \leftarrow$ collection of events

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j) \quad \text{for any subset } J \text{ of } I$$

- **Pairwise Independence**

MI $\Rightarrow$ PI but vice-versa $\underbrace{\text{not always}}$

All pairs are independent

H.W: find examples

Q) $\Omega = \{1, 2, 3, \ldots\ldots 10\}$

Event A : number < 7 $\Rightarrow P(A) = 3/5$

Event B : number < 8 $\Rightarrow P(B) = 7/10$

Event C : number is even $\Rightarrow P(C) = 1/2$

$A \subseteq B \Rightarrow P(A \cap B) = P(A) \Rightarrow$ A and B are not independent

$P(A \cap C) = 3/10 \Rightarrow$ A and C are independent

$\therefore$ No mutual independence

A and B are positively correlated iff $P(A/B) > P(A)$

A and B are negatively correlated iff $P(A/B) < P(A)$

A and B have same correlation as A^c and B^c

A and B have opp. correlation as A and B^c

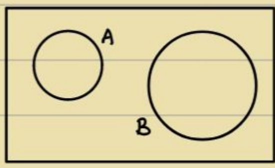
- Mutually exclusive :

One can happen $\Rightarrow$ Other event cannot occur

$P(A \cap B) = \phi$

$P(A/B) = 0$

$P(A/B^c) = \dfrac{P(A)}{P(B^c)}$



If $A \subseteq B$, then they are neither mutually exclusive nor independent.

Note: Zero probability events are always independent

Let $E \longrightarrow$ Zero probability event, $P(E) = 0$

For any set, $P(E \cap F) = 0$

$(E \cap F) \subseteq E$

$P(E \cap F) \leq P(E)$

- Conditional Independence :

$P(A/B) = \dfrac{P(AB)}{B}$ | $P(A/BC) = \dfrac{P(AB/C)}{P(B/C)}$

$P((AB)/C) = P(A/C) \cdot P(B/C) \quad , \quad P(C) > 0$

$\therefore$ A and B are conditionally independent

$\Rightarrow P(A/BC) = P(A/C)$

Ex. 2 coins – 1 fair & 1 fake (Both heads)

Experiment $\longrightarrow$ Choose a coin uniformly & toss twice

Event A : First coin toss results in H

$$\tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot 1 = \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$$

Event B : Second coin toss results in H

$$= \tfrac{3}{4}$$

Event C : Coin 1 is chosen

$P(A \cap B)$

$\hookrightarrow$ Both coin tosses give heads

$$= P(A \cap B / C_F) \cdot P(C_F) + P(A \cap B / C_B) \cdot P(C_B)$$
$$= \tfrac{1}{8} + \tfrac{1}{2}$$

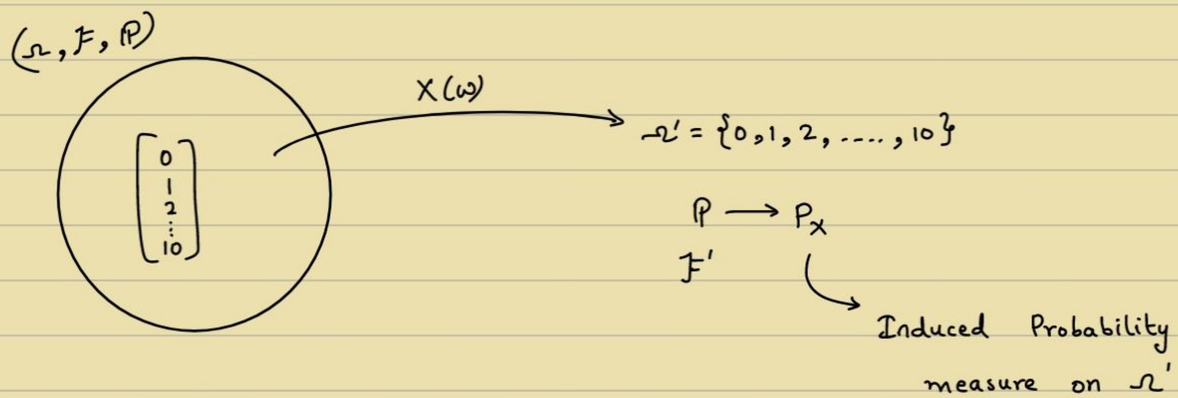
$P(A/C) = \tfrac{1}{2}$

$P(B/C) = \tfrac{1}{2}$

$P((A \cap B)/C) = \tfrac{1}{4}$

$\rightarrow$ Random variable :

$(\Omega, \mathcal{F}, \mathbb{P})$ to a simpler $(\Omega', \mathcal{F}', P_X)$



X is a function, $X : \Omega \rightarrow \Omega'$ that transforms the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega', \mathcal{F}', P_X)$ and is $(\mathcal{F}, \mathcal{F}')$ - measurable

$$X(\omega) \in \Omega' \quad \forall\, \omega \in \Omega$$

$$\forall\, B \in \mathcal{F}' \Rightarrow X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\} \quad , \text{ i.e. } X^{-1}(B) \in \mathcal{F}$$

16/8 & 20/8

Absent - Not well

23/8

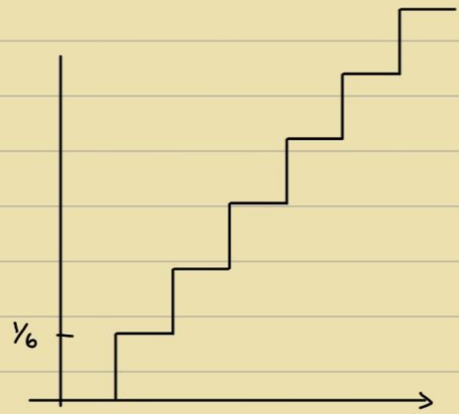
$$F_X(x) = \sum_{x \le x_1} P(x)$$

$\hookrightarrow P(X \le x_1)$

$$\mathbb{P}(\omega \subset \Omega,\ X(\omega) \le x_1)$$

- $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$

  CDF : $F_X(x) = \mathbb{P}\{\omega \in \Omega : X(\omega) \le x\}$



$\Omega = \mathbb{R}$

CDF : Continuous $\longrightarrow$ Continuous variable

CDF : Discontinuous $\longrightarrow$ Discrete variable

PDF : $f_X(x) \quad \forall x \in \mathbb{R}$

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \le x + \Delta)}{\Delta}$$

$$= \lim_{\Delta \to 0^+} \frac{P(X \le x + \Delta) - P(X < x)}{(x + \Delta) - x}$$

$$= \lim_{\Delta \to 0^+} \frac{F_X(x + \Delta) - F_X(x)}{\Delta}$$

$P(A \cap B^c) = P(A) - P(B)$

$A : X \le x_1 + \Delta$

$B : X \le x_1$

$\hookrightarrow B^c : X > x_1$

$$\left(F_X(x_1) = P(X \le x_1)\right)$$

$$= \frac{d}{dx} F_X(x)$$

$$F_X(x) = \int_{u=-\infty}^{x} f_X(u)\, du$$

$$P_X(\mathbb{R}) = \int_{u=-\infty}^{\infty} f_X(u)\, du = 1$$

$$P_X([a,b]) = F_X(b) - F_X(a)$$

$$= \int_a^b f_X(u)\, du$$

For any $B \subseteq \mathbb{R}$, $P_X(B) = \int_{u \in B} f_X(u)\, du$

$$P_X(\{a\}) = 0$$

$$P_X([a,b]) = P_X((a,b]) = P_X([a,b)) = P_X((a,b))$$

- $E[X] = \int_{-\infty}^{\infty} u\, f_X(u)\, du$

$$E[X^n] = \int_{-\infty}^{\infty} u^n \cdot f_X(u)\, du$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(u) \cdot f_X(u)\, du$$

$Var[X] = E[g(X)]$ $\qquad g(X) = (x - E[X])^2$

For $Y = aX + b$,

$E[Y] = a \cdot E[X] + b$

$\underbrace{F_Y(y)} = F_X\left(\frac{y-b}{a}\right)$ | $F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right)$ For $a < 0$

$$\left.\begin{aligned} &P(Y \le y) \\ &P(aX + b \le y) \\ &P(aX \le y - b) \\ &P\left(X \le \tfrac{y-b}{a}\right) \\ &F_X\left(\tfrac{y-b}{a}\right) \end{aligned}\right\} \ \underline{\underline{a > 0}}$$

$$\frac{d}{du} F_X(u) \cdot \frac{du}{dy} \qquad \left[u = \frac{y-b}{a}\right]$$

$$\frac{1}{a} P_X\left(\frac{y-b}{a}\right)$$

- **Uniform Random variable:**

$$f_X(x) = \frac{1}{b-a} \quad \forall x \in [a,b]$$

$$\text{CDF} : F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & x \in [a,b] \\ 1, & \text{otherwise} \end{cases}$$

$$E[X] = \frac{a+b}{2}$$

$$Var[X] = \frac{(b-a)^2}{12}$$

- **Exponential Random variable:**

Non-negative R.V with parameter λ

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0 \qquad E[X] = \frac{1}{\lambda}$$

$$F_X(x) = 1 - e^{-\lambda x} \text{ for } x \geq 0 \qquad Var(X) = 1/\lambda^2$$

Memoryless property

$$P(X > a+h \mid X > a) = \frac{e^{-\lambda(a+h)}}{e^{-\lambda(a)}} = e^{-\lambda h} = P(X > h)$$

Prep for Quiz - 1

Sample Space : Ω

Possible outcome : $\omega \in \Omega$

Event : $A \subseteq \Omega$

Probability : $\mathbb{P}(A)$

Powerset : $P(A)$

(1) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$

(2) $A \subseteq \Omega$, $0 \leq \mathbb{P}(A) \leq 1$

(3) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$, $A_i \rightarrow$ Disjoint events

→ Sigma Algebra ?

$$\mathcal{F} = \{\phi, \Omega, \mathbb{R}_-, \mathbb{R}_+\}$$

(i) $\phi \in \mathcal{F}$ & $\Omega \in \mathcal{F}$

(ii) $B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$

(iii) $A_1 \in \mathcal{F}$ & $A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2 \in \mathcal{F}$ & $A_1 \cap A_2 \in \mathcal{F}$

Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$

Measurable space : $(\Omega, \mathcal{F})$

→ Borel - Sigma Algebra ?

$\mathcal{B}[0,1]$

→ LCD:

$\lim_{n\to\infty} a_n = L$ | $x \rightarrow c \Rightarrow f(x) \longrightarrow f(c)$

Q) Defective $\rightarrow$ 98% Acc.

Not defective $\rightarrow$ 99% Acc.

0.1% $\rightarrow$ Total Defective

$\mathbb{P}$(Robot says defective | Defective) = 0.98

$\mathbb{P}$(Robot says not def. | Not Def.) = 0.99

$\Rightarrow$ $\mathbb{P}$(Robot says def. | Not def.) = 0.01

$\mathbb{P}$(Defective) = 0.01

$\Rightarrow$ $\mathbb{P}$(Not def.) = 0.99

$P(A|B)$ — B: occured

$$\mathbb{P}(\text{def.} \mid \text{Robot says def.}) = \frac{\mathbb{P}(\text{robot says def.} \mid \text{def}) \cdot \mathbb{P}(\text{def.})}{\mathbb{P}(\text{Robot says def.})}$$

$$= \frac{0.98 \times 0.01}{\frac{98}{100} \times \frac{0.1}{100} + \frac{99.9}{100} \times \frac{1}{100}}$$

$$= 8.9\%$$

Independence : $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

+ve correlation : $\mathbb{P}(A|B) > \mathbb{P}(A)$

−ve correlation : $\mathbb{P}(A|B) < \mathbb{P}(A)$

$$\mathbb{P}(A/B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A/BC) = \frac{\mathbb{P}(AB/C)}{\mathbb{P}(B/C)}$$

$$\mathbb{P}(AB/C) = \mathbb{P}(A/BC) \cdot \mathbb{P}(B/C)$$

$\rightarrow$ <u>Binomial</u> :

$$P(i) = P(X=i) = \binom{n}{i} p^i \cdot (1-p)^{n-i}$$

$$E[X] = \sum x P(x)$$

$$= \sum_{i=0}^{n} i \cdot P(i)$$

$$= \sum_{i=0}^{n} (i) \cdot \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i \cdot \frac{n!}{(n-i)!\, i!} \cdot p^i \cdot \frac{(1-p)^n}{(1-p)^i}$$

$$= n \cdot \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} \cdot p^i \cdot (1-p)^{n-i}$$

$$= np \cdot \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} \cdot (1-p)^{(n-1)-(i-1)}$$

$$= np \cdot \sum_{i=0}^{m} \binom{m}{j} p^j \cdot (1-p)^{m-j} = np$$

→ Geometric:

$$P(X=i) = (1-p)^{i-1}p$$

$$E[X] = \sum_{i=0}^{n} x \cdot P(x)$$

$$= \sum_{i=0}^{n} i \cdot (1-p)^{i-1} \cdot p$$

$$= \sum_{i=0}^{n} i \cdot (1-p)^{i} \cdot \frac{p}{1-p}$$

$$= \frac{p}{1-p} \cdot \sum_{i=1}^{n} i(1-p)^{i}$$

$$= \frac{p}{1-p} \cdot \frac{1-p}{p^2} \cdot (1-(1-p)^n)$$

$$= \frac{1}{p}$$

$$S = 1(1-p) + 2(1-p)^2 + 3(1-p)^3 + \dots$$

$$(1-p)S = \qquad 1(1-p)^2 + 2(1-p)^3 + \dots$$

$$S \cdot p = (1-p) + (1-p)^2 + (1-p)^3 + \dots \infty$$

$$= (1-p) \cdot \frac{1-(1-p)^n}{p}$$

$$\Rightarrow S = \frac{1-p}{p^2} \cdot (1-(1-p)^n)$$

→ Uniform:

$$P(X=x) = \frac{1}{b-a} \quad \forall x \in [a,b]$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot P(x)\, dx$$

$$= \int_{a}^{b} x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \cdot \left(\frac{x^2}{2}\right]_a^b$$

$$= \frac{a+b}{2}$$

→ Exponential:

$$P(X=x) = \lambda e^{-\lambda x} \quad \forall x \geq 0$$

$$E[X] = \int_{0}^{\infty} \lambda e^{-\lambda x} = e^{-\lambda x}\Big]_0^{\infty} \cdot \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}(0+1) = \frac{1}{\lambda}$$

Q) 10 min break

$\mathbb{P}(\text{rain}) = 3/10$

$\mathbb{P}(\text{crowd}) = 1/2$

$\mathbb{P}(\text{late} \mid \text{rain}) = 1/2$

$\mathbb{P}(\text{Late} \mid \text{crowded}) = 3/10$

(a) $\mathbb{P}(\text{late}) = \mathbb{P}(\text{late} \mid \text{rain}) \cdot \mathbb{P}(\text{rain}) + \mathbb{P}(\text{late} \mid \text{crowded}) \cdot \mathbb{P}(\text{crowded})$

$= 1/2 \cdot 3/10 + 3/10 \cdot 1/2$

$= 3/10$

(b) $\mathbb{P}(\text{rain} \mid \text{late}) = \dfrac{1/2 \cdot 3/10}{3/10} = 1/2$

$\mathbb{P}(\text{crowded} \mid \text{late}) = \dfrac{1/2 \cdot 3/10}{3/10} = 1/2$

(c) $\mathbb{P}(\text{rain} \cap \text{crowded}) = \mathbb{P}(\text{rain}) \cdot \mathbb{P}(\text{crowded})$

$= 3/10 \cdot 1/2$

$= 3/20$

Q) $E_{11}\ E_{12} \quad E_{21}\ E_{22}$

$(E_{11}E_{21} + E_{11}E_{22} + E_{12}E_{21} + E_{12}E_{22} + \ldots)$

$P(\text{crash}) = \overline{E_{11}}\,\overline{E_{12}} + \overline{E_{21}}\,\overline{E_{22}} - \overline{E_{11}}\,\overline{E_{12}}\,\overline{E_{21}}\,\overline{E_{22}}$

$= 2/10 \cdot 2/10 + 2/10 \cdot 2/10 - 2/10 \cdot 2/10 \cdot 2/10 \cdot 2/10$

$= 8/100 - 16/10000 = 784/10000$

$\overline{P}(\text{crash}) = 1 - 784/10000 = \dfrac{9216}{10000}$

$= \underline{\underline{0.9216}}$

Q) $P(X=k) = \dfrac{e^{-\lambda}\cdot\lambda^k}{k!}$ $\quad(\lambda>0)$, $k=0,1,2,\dots$

$$E[X] = \sum_{x=0}^{\infty} x\,P(x)$$

$$= \sum_{i=0}^{\infty} i\cdot\frac{e^{-\lambda}\cdot\lambda^i}{i!}$$

$$= e^{-\lambda}\cdot\sum_{i=0}^{\infty} i\cdot\frac{\lambda^i}{i!}$$

$$= e^{-\lambda}\cdot\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!}\cdot\lambda$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

$$= \lambda e^{-\lambda}\cdot\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$$

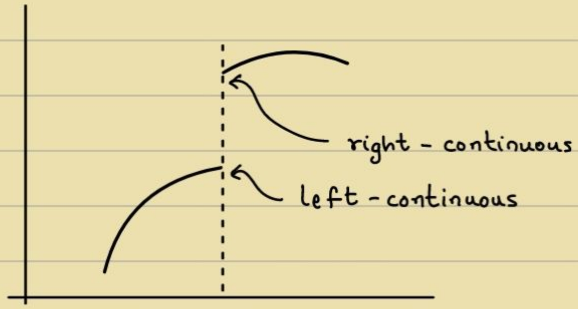
$$e^\lambda = 1+\lambda+\frac{\lambda^2}{2!}+\dots$$

$$= \lambda\cdot e^{-\lambda}\cdot e^{\lambda} = \underline{\lambda}$$

$$Var[X] = \lambda(1-\lambda/n)$$

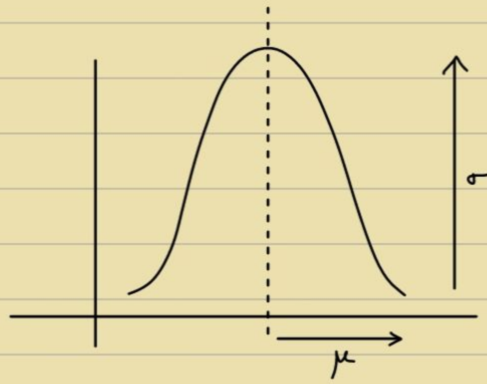
$$\to E[X^2]-(E[X])^2$$

3/9



→ Gaussian Random variable $\left(N(\underbrace{\mu}_{\text{mean}}, \underbrace{\sigma^2}_{\text{variance}})\right)$:

- Continuous RV



PDF : $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-\mu)^2}{2\sigma^2}} \quad \forall x \in \mathbb{R}$

H.W : $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$E[X] = \mu$

$Var[X] = \sigma^2$

- Standard Normal R.V : $N(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$F_X(x) : \Phi(x) \longleftarrow$ Table

- Normality preserved under Linear Transformations :

$X \sim N(\mu, \sigma^2)$

$Y = AX + B \longleftarrow N(A\mu + B, A^2\sigma^2)$

$E[Y] = A\mu + B$

$Var[Y] = A^2\sigma^2$

- Central Limit Theorem :

$\frac{1}{n}\sum_{i=1}^{n} X_i \approx N(\mu, \sigma^2/n)$ [X : Any RV with mean μ & variance σ^2]

- Multinomial Gaussian vector ← Countably finite

Gaussian Process ← Countably infinite

→ Used in Bayesian Optimization (Black-box optimization)

→ Brownian Motion

Note : Beta, Gamma, Erlang, Logistic

- $Y = aX + b$

$$F_Y(y) = P(Y \le y) = P(aX + b \le Y)$$
$$= F_X\left(\frac{y-b}{a}\right) \text{ if } a > 0$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \quad \text{when } a > 0$$

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \quad \text{if } a < 0$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{-1}{a} f_X\left(\frac{y-b}{a}\right) \quad \text{when } a < 0$$

$$\therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Generalising : [Intuition]

$$\left. Y = aX + b \Rightarrow X = \frac{Y-b}{a} \right\} \Rightarrow f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$
$$\Downarrow$$
$$\left. Y = g(X) \Longrightarrow X = h(Y) \right\} \Rightarrow f_Y(y) = \left|\frac{d\,h(y)}{dy}\right| f_X(h(y))$$

Monotone function, continuous and differentiable

$\therefore h(x) = g^{-1}(x)$

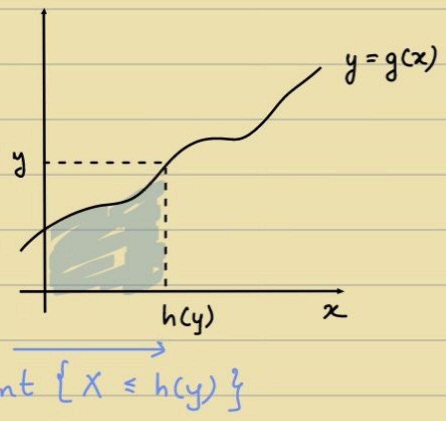
<u>Proof</u> :

$Y = g(X) \Rightarrow X = h(y)$

Monotone, continuous and differentiable function

$$F_Y(y) = P(g(X) \le y)$$
$$= P(X \le h(y))$$

$$\left[\begin{array}{l} g(X) \le y \; : \; A = \{\omega \in \Omega \mid g(X(\omega)) \le y\} \\ \Rightarrow h(g(X)) \le h(y) \\ \Rightarrow X \le h(y) \; : \; A = \{\omega \in \Omega \mid X(\omega) \le h(y)\} \end{array}\right]$$



As $g(x) \uparrow \Rightarrow h(x) \downarrow$

case - I :

$g(x) \leftarrow$ Non-decreasing

$$F_Y(y) = P(g(X) \le y) = P(X \le h(y)) = F_X(h(y))$$

$$f_Y(y) = \frac{d}{dy} F_X(h(y)) = \underbrace{f_X(h(y))}_{\text{non-dec}} \cdot \underbrace{\frac{dh}{dy}(y)}_{\ge 0}$$

case - II :

$g(x) \leftarrow$ Non-increasing

$$F_Y(y) = P(g(X) \le y) = P(X \ge h(y)) = 1 - F_X(h(y))$$

$$f_Y(y) = \frac{-d}{dy} F_X(h(y)) = - \underbrace{f_X(h(y))}_{\text{non-inc.}} \cdot \underbrace{\frac{dh}{dy}(y)}_{\le 0}$$

$$\Downarrow$$

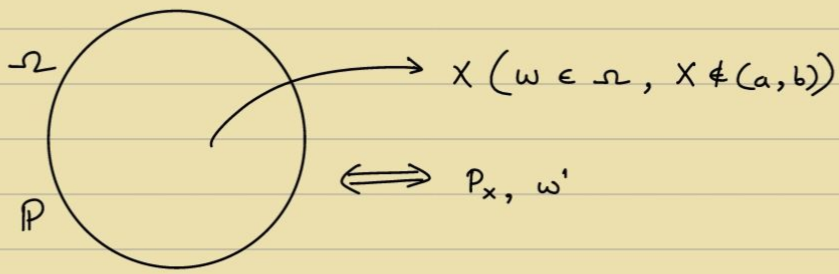
$$f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right|$$

Q) $Y = X^2$, $f_Y(y) = ?$ in terms of $f_X(x)$

6/9

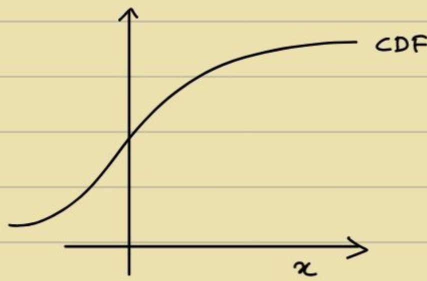
$\mathbb{P}$: Probability measure

P_X : Induced Probability measure



$$F_X(a) = P_X(-\infty, a] = P(X \le a)$$

Point mass $\rightarrow$ Discontinuity



- CDF:

(i) $F_X(\infty) = 1$

$F_X(-\infty) = 0$, when $P(-\infty < X < \infty) = 1$

(ii) $F_X : \mathbb{R} \to [0,1] \leftarrow$ Non-decreasing & right continuous

(iii) @ Points of discontinuity :

$$\left.\begin{aligned} F_X(x^+) &:= \lim_{\varepsilon \to 0} F_X(x+\varepsilon) \\ F_X(x^-) &:= \lim_{\varepsilon \to 0} F_X(x-\varepsilon) \end{aligned}\right\} \varepsilon \text{ decreases to } 0 \text{, i.e. } \varepsilon \downarrow 0$$

[$\because$ ε is positive]

$$F_X(x^+) \ne F_X(x^-)$$

Right continuous : $F_X(x) = F_X(x^+)$

if $F_X(x)$ is continuous $\Rightarrow F_X(x^+) = F_X(x^-)$

<u>Th</u> :

$F_X : \mathbb{R} \to [0, 1]$ is non-decreasing and right continuous

(i) non-decreasing :

$\forall x_1, x_2 : x_1 \leq x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$ [x_1 & x_2 are arbitrary]

[From $a \leq b \Rightarrow P(A) < P(B)$]

Define $A := \{\omega \in \Omega : X(\omega) \leq x_1\}$

Define $B := \{\omega \in \Omega : X(\omega) \leq x_2\}$

$\therefore A \subseteq B$

$\therefore \mathbb{P}(A) \leq \mathbb{P}(B)$

$F_X(x_1) = P_X((-\infty, x_1]) = \mathbb{P}(A) < \mathbb{P}(B) = F_X(x_2)$

(ii) right-continuous :

Consider seq. of no. $\{x_n\}$ decreasing to x

$$F_X(x^+) = \lim_{x_n \to x} F_X(x_n)$$

Define $A_n := \{\omega : X(\omega) \leq x_n\}$

Define $A := \{\omega : X(\omega) \leq x\}$

[When $A_n \uparrow A$ or $A_n \downarrow A \Rightarrow \lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(A)$]

$$\left[\left. \begin{array}{c} (-\infty, x_n] = A_n \\ \downarrow \\ (-\infty, x] = A \end{array} \right\} \Rightarrow P_X \right]$$

$F_X(x) = P_X(-\infty, x]$

$\therefore$ RHS $= F_X(x)$

LHS $= F_X(x_n) = F_X(x)$

If $x_n \uparrow x \Rightarrow (-\infty, x_n]$

$$\underbrace{\lim_{x_n \uparrow x} (-\infty, x_n]}_{\cup_n (-\infty, x_n]} = (-\infty, x) \quad [\because \text{Going to } x]$$

& $P_X(-\infty, x) \neq F_X(x)$ [By definition]

$\because$ There could be point mass at x.

→ **Mixed RV** :

Neither discrete nor continuous , Partly both

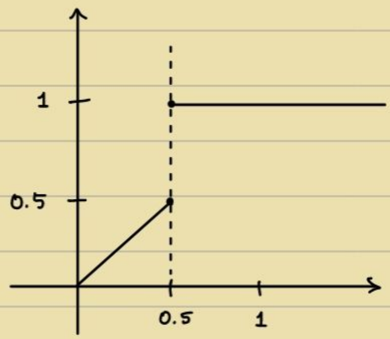
CDF ⟶ Piece - wise continuous

[Ref : 4.3.1
probabilitycourse.com]

e.g. $X = U[0,1]$ and $Y = X$ if $X \leq 0.5$

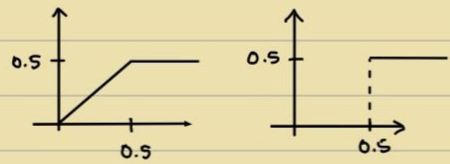
$Y = 0.5$ if $X \geq 0.5$

CDF :



$\therefore F_Y(y) = C(y) + D(y)$

$C(y)$ ↓ Continuous, $D(y)$ ↳ Discontinuous

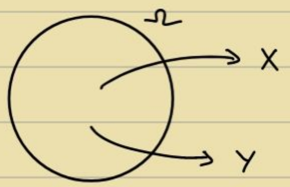


$$E[Y] = \int_{-\infty}^{\infty} x \cdot C(x)\, dx + \sum_{y_k} y_k\, P(Y = y_k)$$

where $\{y_1, y_2, \dots y_n\}$ are jump points of $D(y)$

where $P(Y = y_k) > 0$

- **Multiple R.V** :



$\Omega = \{0,1\} \times \{1,2,3,4,5,6\}$, $\mathcal{F} = 2^{\Omega}$, $P(\omega) = \frac{1}{12}$

$X_\omega \in \{0,1\}$ ← Coin

$Y_\omega \in \{1,2,3,4,5,6\}$ ← Dice

for $\omega = (1,5) \Rightarrow X(\omega) = 1$ and $Y(\omega) = 5$

Joint PMF : $P_{X,Y}(x,y)$

Joint CDF : $F_{X,Y}(x,y)$

$$P_{X,Y}(x,y) := \mathbb{P}\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = \omega\}$$

$$F_{X,Y}(x,y) := \mathbb{P}\{\omega \in \Omega : X(\omega) \leq x \text{ and } Y(\omega) \leq \omega\}$$

$$P((X,Y) \in A) = \mathbb{P}\{\omega \in \Omega : (X(\omega), Y(\omega)) \in A\}$$

$$= \sum_{(x,y) \in A} P_{X,Y}(x,y)$$

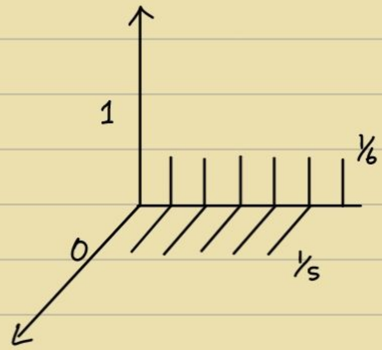
e.g. $P((X,Y) \in A)$; Given, A : coin → head & dice → 6

if H → {1, 2, 3, 4, 5, 6}

T → {1, 2, 3, 4, 5}

$$\{\overset{1/12}{(1,1)}, (1,2), \ldots \overset{1/12}{(1,6)},$$
$$\underset{1/10}{(0,1)}, (0,2), \ldots \underset{1/10}{(0,5)}\}$$

$$\begin{bmatrix} 1/12 = 1/2 \times 1/6 \\ 1/10 = 1/2 \times 1/5 \end{bmatrix}$$



IMP

- <u>Marginals</u> :

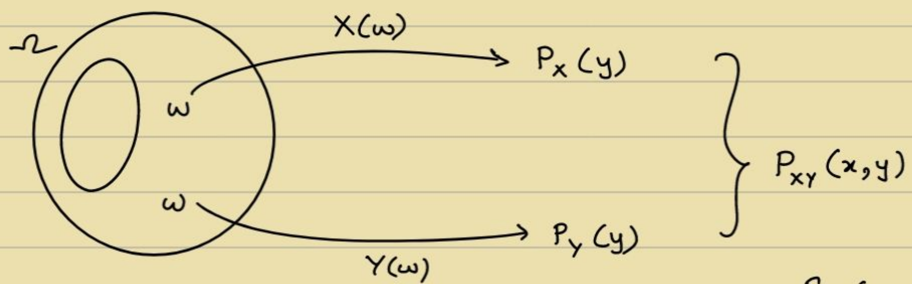
$$P(Y=3) = \frac{1}{10} + \frac{1}{12}$$
$$P(Y=6) = \frac{1}{12}$$
$$P(X=1) = \frac{1}{2}$$

$$P_{X,Y}(1,i) = 1/12$$
$$\sum_i P_{X,Y}(1,i) = P\{X \in \Omega, X(\omega) = 1\} = 1/2 = P_X(x)$$
$$P_{X,Y}(1,i) + P_{X,Y}(0,i) = \frac{1}{6} = P_Y(i)$$

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

<u>10/9</u>



$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

$$\sum_{\substack{x \le a \\ y \le b}} P_{XY}(x,y) = F_{XY}(a,b)$$
$$= P(X \le a, Y \le b)$$
$$= P(\omega \in \Omega, X(\omega) \le a, Y(\omega) \le b)$$

$$\sum_y P_{XY}(x,y) = P_X(x) = P(\omega \in \Omega, X(\omega) = x)$$
$$P_{XY}(x,y) = P(\omega \in \Omega, X(\omega) = x \;\&\; Y(\omega) = y)$$

$$\sum_y P_{XY}(x,y) = \mathbb{P}\left(\bigcup_y (\omega \in \Omega,\ X(\omega) = x \ \&\ Y(\omega) = y)\right)$$
$$= \sum_y \mathbb{P}\left((\omega \in \Omega,\ X(\omega) = x \ \&\ Y(\omega) = y)\right)$$

Note : $F_X(x) = \sum_y F_{XY}(x,y)$

- Independent :

$$P_{XY}(x,y) = P_X(x) \cdot P_Y(y)$$
$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y)$$

$$P_{XY}(x,y) = \mathbb{P}(\omega \in \Omega,\ X(\omega) = x \ \&\ Y(\omega) = y)$$

$$A = \{\omega \in \Omega,\ X(\omega) = x\}$$
$$B = \{\omega \in \Omega,\ Y(\omega) = y\}$$
$$\Rightarrow A \cap B = \{\omega \in \Omega,\ X(\omega) = x \ \&\ Y(\omega) = y\}$$

$$P_X(x) = \mathbb{P}(A)$$
$$P_Y(y) = \mathbb{P}(B)$$

$$F_{XY}(x,y) = P(X \le x,\ Y \le y)$$
$$= \mathbb{P}(\omega \in \Omega,\ X(\omega) \le x \ \&\ Y(\omega) \le y)$$

- Expectation : $E[XY]$

$$E[X] = \sum_x x \cdot P_X(x) \qquad\qquad E[Y] = \sum_y y \cdot P_Y(y)$$
$$= \sum_x \sum_y x \cdot P_{XY}(x,y) \qquad\qquad = \sum_y \sum_x y \cdot P_{XY}(x,y)$$

$$E[XY] = \sum_x \sum_y xy \cdot P_{XY}(x,y)$$

$E[XY] = E[X] \cdot E[Y]$ for independent events, X and Y given
(Converse may not be true)

e.g. When X and Y are dependent.
Roll a dice &

$$X = \begin{cases} 1 & \text{if outcome is odd} \\ 0, & \text{otherwise} \end{cases} \qquad \Bigg| \qquad Y = \begin{cases} 1 & \text{if outcome is even} \\ 0, & \text{otherwise} \end{cases}$$

$P_{XY}(0,0) = 0 \qquad P_{XY}(0,1) = \frac{1}{2}$

$P_{XY}(1,0) = \frac{1}{2} \qquad P_{XY}(1,1) = 0$

$$P_Y(0) = P_X(0) = \sum_y P_{XY}(0,y) = \frac{1}{2}$$
$$P_Y(0) = P_X(1) = \sum_y P_{XY}(1,y) = \frac{1}{2}$$

$$F_{XY}(x,y) \neq F_X(x) \cdot F_Y(y)$$

$$E[XY] \neq E[X] \cdot E[Y]$$

- Consistency Properties :

(i) $\sum_{x,y} P_{XY}(x,y) = 1$

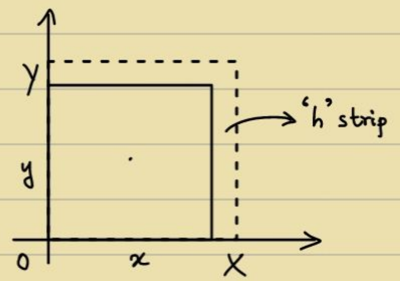
(ii) $F_{XY}(\infty, \infty) = 1$

(iii) $F_{XY}(-\infty, -\infty) = 0$

(iv) $F_{XY}(-\infty, \infty) = F_{XY}(\infty, -\infty) = 0$

(v) $\left.\begin{array}{l} F_{XY}(x, \infty) = F_X(x) \\ F_{XY}(\infty, y) = F_Y(y) \end{array}\right\} \longleftarrow$ Marginal

$F_{XY}(x,y)$:



$F_{XY}(x,y) = xy \quad \& \quad f_{XY}(x,y) = 1$

$F_{XY}(x+h, y) - F_{XY}(x,y)$

$$\frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \longleftarrow \left\{ \begin{array}{l} \dfrac{\partial F_{XY}(x,y)}{\partial x} = \lim_{h \to 0} \dfrac{F_{XY}(x+h,y) - F_{XY}(x,y)}{h} \\ \dfrac{\partial F_{XY}(x,y)}{\partial y} = \lim_{h \to 0} \dfrac{F_{XY}(x,y+h) - F_{XY}(x,y)}{h} \end{array} \right.$$

$\downarrow$

Joint PDF

$f_{X,Y}(x,y)$

$$F_{XY}(x,y) := \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(a,b)\, da \cdot db$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

Apply for ≥ 2 R.Vs

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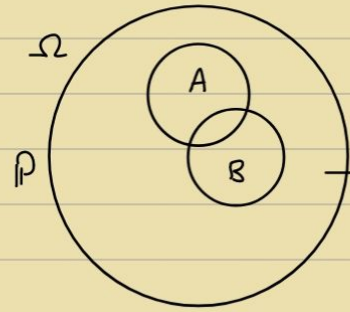
- 

$P(A|B)$

$\xrightarrow{X(\omega)}$ $\Omega' = \mathbb{R}$

$P_X(\cdot)$ $\qquad F_X(x_1) = \sum_{x \le x_1} P_X(x)$

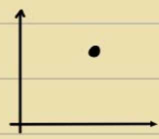
$P_X(x) = \mathbb{P}\{\omega \in \Omega,\ X(\omega) = x\}$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$P_{XY}(x,y) \leftarrow$ Joint PDF

$$F_{XY}(x_1,y_1) = \sum_{x \le x_1} \sum_{y \le y_1} P_{XY}(x,y)$$

$$f_{XY}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{XY}(x,y)$$

$\underbrace{f_{XY}(x,y)}\, dx.dy \longrightarrow$ 

$\hookrightarrow$ Probability per unit length / area / vol

i.e. Probability Density Function

- $X \longrightarrow p_X(x)$

$$\underbrace{E[g(X)]}_{z} = \sum_x g(x)\, P_X(x)$$

$$P_Z(z) = \sum_{\substack{x \\ z = g(x)}} P_X(x)$$

If $z = g(x,y)$

$P_Z(z) = ?$

$$P_Z(z) = \sum_{\substack{xy \\ g(x,y) = z}} P_{XY}(x,y)$$

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

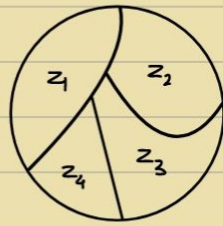
Proof : $\sum_{xy} g(x,y) \cdot P_{XY}(x,y)$

$$\left\{ \sum_{xy} (ax + by + c)\, P_{XY}(x,y) \right\}$$

$$\sum_x \sum_y P_{XY}(x,y) = 1$$

$$E[z] = \sum_{z} \sum_{\substack{x,y \\ g(x,y)=z}} z \cdot p_{xy}(x,y) \qquad \because \; p_z(z) = \sum_{\substack{x,y \\ g(x,y)=z}} p_{xy}(x,y)$$

$$E[g(x,y)] = \sum_{x,y} g(x,y)\, p_{xy}(x,y) \qquad E[z] = \sum_{z} z\, p_z(z)$$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, dx \cdot dy$$

- X and Y $\rightarrow$ continuous independent R.V

$W = \max(X, Y)$

$Z = \min(X, Y)$

$$\left\{ \begin{array}{l} \text{i.e. } p_{XY}(x,y) = p_X(x) \cdot p_Y(y) \\ F_{XY}(x,y) = F_X(x) \cdot F_Y(y) \\ E[XY] = E[X] \cdot E[Y] \end{array} \right\}$$

(i) $F_W(\omega_1) = P(W \le \omega_1)$

$= P(X \le \omega_1 \;\&\; Y \le \omega_1)$

$= P(X \le \omega_1) \cdot P(Y \le \omega_1)$

$= F_X(\omega_1) \cdot F_Y(\omega_1)$

$F_W(\omega_1) = F_{XY}(\omega_1, \omega_1)$, i.e. $W \le \omega_1 \Leftrightarrow X, Y \le \omega_1$

$$p_W(\omega) = \frac{d}{dw} F_W(\omega)$$

$$= F_X(\omega) \cdot f_Y(\omega) + f_X(\omega) \cdot F_Y(\omega)$$

(ii) $F_Z(z_1) = P(Z \le z_1)$

$Z \ge z_1 \Leftrightarrow X \ge z_1$ and $Y \ge z_1$

$\overline{F_Z}(z_1) = P(Z > z_1)$

$= \overline{F_{XY}}(z_1, z_1)$

$= 1 - F_{XY}(z_1, z_2)$

$= 1 - F_X(z_1) \cdot F(z_2)$

e.g. if $\underset{(\lambda_1)}{X}$ & $\underset{(\lambda_2)}{Y} \rightarrow$ exponential, Then $\underset{(\lambda_1+\lambda_2)}{Z} \rightarrow$ exponential

$$\left\{ \text{Hint: } e^{-\lambda_1 \omega} \cdot e^{-\lambda_2 \omega} = e^{-(\lambda_1+\lambda_2)\omega} \right\}$$

9) $Z = X + Y$

$$P_Z(z_1) = \sum_{\substack{x,y \\ x+y=z_1}} P_{XY}(x,y)$$
$$= \sum_{x,y} P_X(x) \cdot P_Y(y)$$
$$= \sum_x P_X(x) \cdot \sum_y P_Y(y)$$
$$= \sum_x P_X(x) \cdot P_Y(z_1 - x)$$

$$\begin{bmatrix} x + y = z \\ \Rightarrow y = z - x \end{bmatrix}$$
for satisfying (x,y)

- Covariance of X and Y:
  (Joint Variation)

$$Cov(X,Y) = E\Big[(X - E[X])(Y - E[Y])\Big]$$
$$= E[XY] - E[X] \cdot E[Y]$$

$$\left\{ \text{What is } \begin{bmatrix} cov(X,X) & cov(X,Y) \\ cov(Y,X) & cov(Y,Y) \end{bmatrix} ? \right\}$$ ← Covariance Matrix $cov[X,Y]$

Gives correlation

$$P_X(x) = \sum_y P_{XY}(x,y)$$

$$P_Y(y) = \sum_x P_{XY}(x,y)$$

$$g(x,y) = (x - E[X]) \cdot (y - E[Y])$$

$$E[g(X,Y)] = \sum_{xy} g(x,y) \cdot P_{XY}(x,y)$$

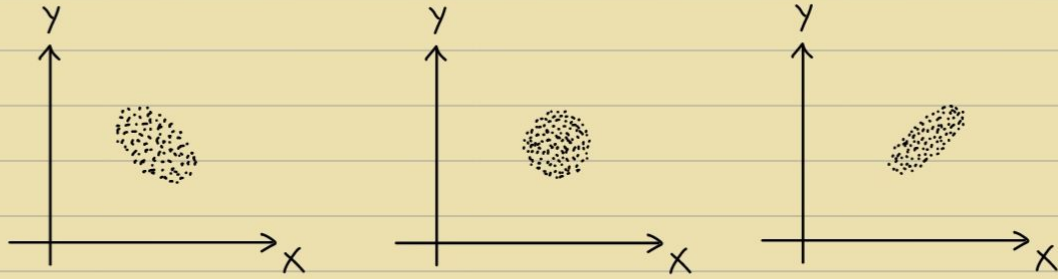
$E \to E_{P_{XY}(x,y)}$

$$E_{P_{XY}(x,y)}\Big[(X - E_{P_X(x)}[X])(Y - E_{P_Y(y)}[Y])\Big] = \sum_{xy} g(x,y) \cdot P_{XY}(x,y)$$

Note : $E[XY] = E[X] \cdot E[Y]$ only when X and Y are uncorrelated & doesn't necessarily imply independence

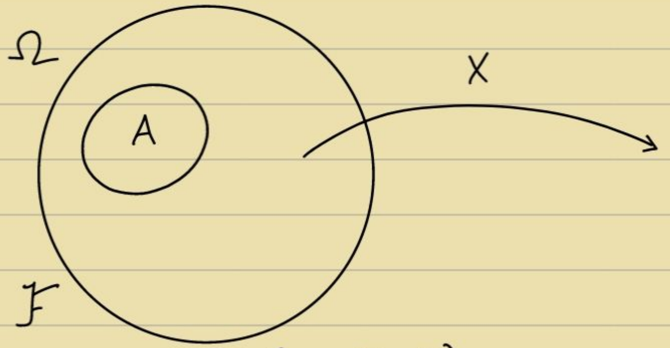
Independence $\Rightarrow$ Uncorrelated

Uncorrelated $\not\Rightarrow$ Independence (Not always)



$\mathrm{cov}(X, Y) < 0$ $\quad$ $\mathrm{cov}(X, Y) \approx 0$ $\quad$ $\mathrm{cov}(X, Y) > 0$

17/9



$(\Omega, \mathcal{F}, \mathbb{P})$

$$\begin{matrix} x_1 & x_2 & \cdots & x_{10} \\ p_1 & p_2 & \cdots & p_{10} \end{matrix}$$

$(\Omega', \mathcal{F}', P_X)$ $\quad$ $\Omega' = \mathbb{R}$

(a, b) $\quad$ $\mathcal{F} = \mathcal{B}(\mathbb{R})$

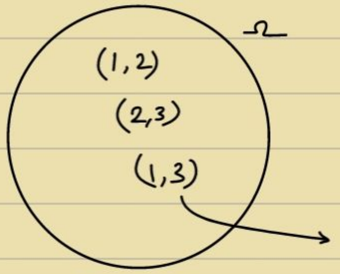
$$P_X(x) \equiv P_{X/A}(x) \rightarrow \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

$$\sum P(A_i) \cdot P_{X/A_i}(x) = P_X(x)$$

Ex. Pick 2 integers from $\{1, 2, 3\}$ without replacement

$$\Omega = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$$

$$\mathbb{P}\{\omega\} = 1/6 \quad \forall \omega \in \Omega$$



$X(\omega) = 1$
$Y(\omega) = 3$

$$P_{XY}(x,y) = 1/6$$

$$P_X(x) = \sum_{\substack{y \\ \{1,2,3\}}} P_{XY}(x,y) = 1/6 + 1/6 = 1/3$$

$$P_Y(y) = \sum_{\substack{x \\ \{1,2,3\}}} P_{XY}(x,y) = 1/6 + 1/6 = 1/3$$

$E[X] = 2$

$E[Y] = 2$

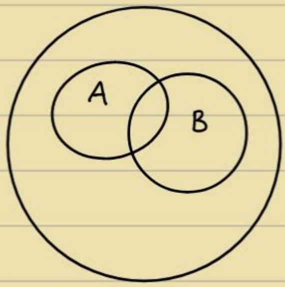
$$E[XY] = \sum g(x,y) \cdot P_{XY}(x,y)$$
$$= 2 \cdot P_{XY}(1,2) + \ldots$$
$$= 11/3$$

$$P_X(x) \cdot P_Y(y) = 1/9$$

$\longrightarrow$ R.V. X with $P_X(x)$

Event A, $A \in \mathcal{F}$

Consider event $\{\omega \in \Omega : X(\omega) = x\} \equiv \{X = x\}$



$$P(X = x \mid A) = \frac{P(\{X = x\} \cap A)}{P(A)} = P_{X|A}(x)$$

e.g. first no. $\longrightarrow$ odd, second no. $\longrightarrow$ even $\Big\}$ $A = \{(1,2), (3,2)\}$

$$P_{X|A}(1) = \frac{P(X=1 \cap A)}{P(A)} = \frac{1/6}{1/3} = \frac{1}{2}$$

Similarly, $P_{X|A}(3) = 1/2$

[Prev. $P_X(x) = 1/3$ for $x = 1, 2, 3$]

$$\sum_x P_{X|A}(x) = \sum_x \frac{P(\{X=x\} \cap A)}{P(A)}$$

$$= \frac{\sum_x P(\{X=x\} \cap A)}{P(A)}$$

$$= \frac{P(A)}{P(A)} = 1$$

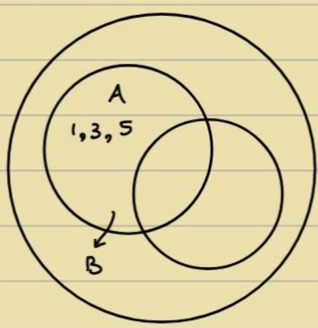
Ex. Roll a dice

A : roll is odd

$P_{X|A}(x) = ?$ $\qquad \Omega = \{1, 2, 3, 4, 5, 6\}$

$$P_{X|A}(x) = \frac{1/6}{1/2} = 1/3$$

$\{1/3, 0, 1/3, 0, 1/3, 0\}$



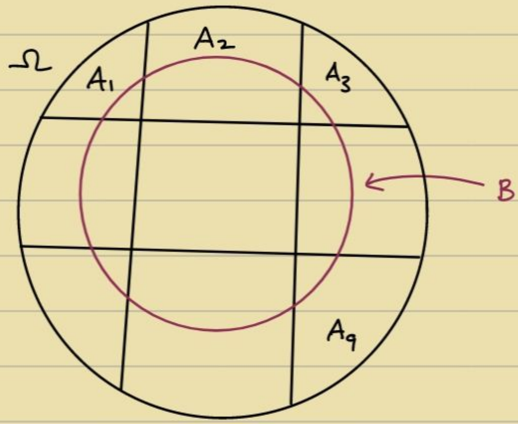
$E[X|A] = 3$

$E[X|A] = 4$ (if $A \rightarrow$ even)

$$E[X|A] = \sum_x x \, P_{X|A}(x)$$

Using LOTUS, $E[g(x)|A] = \sum_x g(x) \cdot P_{X|A}(x)$

- Law of Total Probability :



$$P(B) = \frac{\sum P(B \cap A_i)}{P(A_i)} \cdot P(A_i)$$

$$= \sum_i P(B|A_i) \cdot P(A_i)$$

$$P_X(x) = \sum_i P(A_i) \cdot P_{X|A_i}(x)$$

Proof : $B = \{\omega \in \Omega , X(\omega) = x\}$

$$P_X(x) = \sum_i P(A_i) \cdot \frac{P(\{X=x\} \cap A_i)}{P(A_i)}$$

$$= \sum_i P(\{X=x\} \cap A_i)$$

$$= P(\{X=x\})$$

$B = \bigsqcup (B \cap A_i)$

Disjoint Union

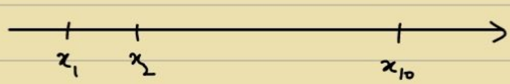
$$P(B) = \sum_i \frac{P(B \cap A_i)}{P(A_i)} \cdot P(A_i)$$

Now, $E[X] = \sum_i P(A_i) \cdot E[X|A_i]$

- e.g. $A = (X > 5) = \{6\}$
  $A \in \mathcal{F}'$

$X \in A = \{\omega \in \Omega : X(\omega) = A\}$
$P\{X \in A\} = \sum_{x \in A} P_X(x)$

$\Omega' = \{x_1, x_2, \ldots\ldots, x_{10}\}$ $\mathcal{F} = 2^{\Omega'}$
$A \in \mathcal{F}$



Let $A = \{x_1, x_9, x_{10}\}$

$$P_{X|A}(x_1) = ? = 1/3 \qquad P_{X|A}(x_1) = \frac{\mathbb{P}(\{X = x_1\} \cap A)}{\mathbb{P}(A)} = \frac{P(A \cap \overset{\{x_1\}}{B})}{P_X(A)} = \frac{P_X(x_1)}{P_X(A)}$$

$$P_{X|A}(x_2) = ? = 0$$

$$\vdots$$

$$P_{X|A}(x_{10}) = ? = 1/3$$

$$P_X(A) = \mathbb{P}(A^{-1}) = \{\omega \in \Omega,\ X(\omega) \in A\}$$

i.e. if $x \notin A \Rightarrow P_{X|A}(x) = 0$

if $x \in A \Rightarrow P_{X|A}(x) = \dfrac{P_X(x)}{\mathbb{P}(X \in A)}$

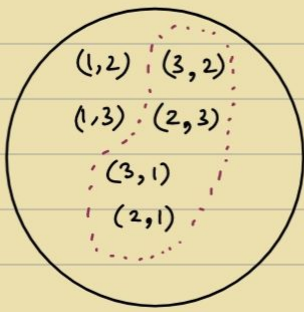
Ex. Rolling a dice, $X > 2$

i.e. $\{3, 4, 5, 6\} = A$

$$P_{X|A}(x_1) = 0$$
$$P_{X|A}(x_2) = 0$$
$$P_{X|A}(x_3) = 1/4 = \frac{1/6}{4/6}$$
$$P_{X|A}(x_4) = 1/4$$
$$P_{X|A}(x_5) = 1/4$$
$$P_{X|A}(x_6) = 1/4$$

e.g. $X \in A$ where $A = \{2, 3\}$

$P_{X|A}(x) = ?$



$P_A = 4/6$

$P_X(x) = 1/3$

- Geometric R.V :

$$P_N(k) = (1-p)^{k-1} p \quad \leftarrow \text{PMF}$$

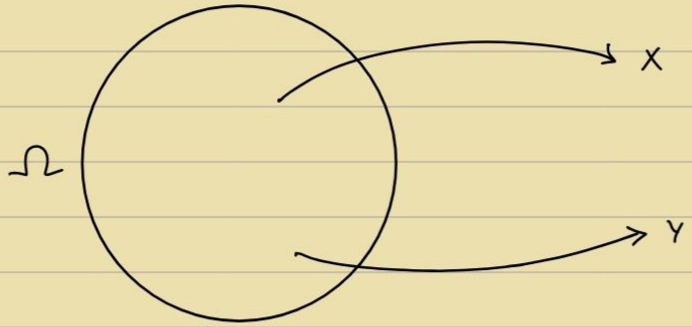
If $A := N > n$ & $B := N + j$

$P_{N|A}(k) = ?$

$$P_{N|A}(k) = 0 \quad \text{if } k < n-1$$

$$P_{N|A}(k) = \frac{\overset{P\{N=k\}}{P\{(N>n) \cap N=k\}}}{\underset{(1-p)^n}{P(N>n)}} = (1-p)^{k-1-n} \cdot p$$

20/9



$P_{XY}(x, y)$ $\quad$ @ $Y = y$

$f_{XY}(x, y)$ $\quad$ $P_{X|A}(x)$

$P_{X|Y}(x|y) \longrightarrow f_{X|Y}(x, y)$

$\hookrightarrow P(A|B) = P(X = x|\ Y = y) / P(Y = y)$

$$P_{X|Y}(x|y) = \frac{P_{XY}(x, y)}{P_Y(y)}$$

$$\Rightarrow P_{XY}(x, y) = P_{X|Y}(x|y) \cdot P_Y(y)$$

$$\left[P(A \cap B) = P(A|B) \cdot P(B)\right]$$

- <u>Independence</u> :

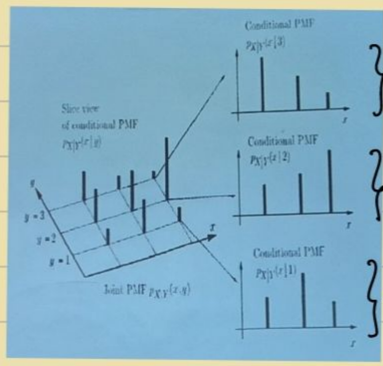
$$P_{XY}(x, y) = P_{X|Y}(x|y) \cdot P_Y(y)$$
$$\Rightarrow P_X(x) \cdot P_Y(y) = P_{X|Y}(x|y) \cdot P_Y(y)$$
$$\Rightarrow P_X(x) = P_{X|Y}(x|y)$$
$$\Rightarrow P_{X|Y}(x|y) = P_X(x)$$

$$\therefore E[XY] = E[X] \cdot E[Y]$$
$$\& \ \mathrm{cov}(X, Y) = 0$$

- $P_{XY}(x, y) = P_{X|Y}(x|y) \cdot P_Y(y)$

$$\Rightarrow P_X(x) = \sum_y P_{X|Y}(x|y) \cdot P_Y(y)$$

$\hookrightarrow$ Law of Total Probability



Normalised

Summing over x instead of y :

$$P_Y(y) = \sum_x P_{X|Y}(x|y) \cdot P_X(x)$$

$$E[X|Y=y] = \sum_x x\, P_{X|Y}(x|y)$$

$$E[X] = \sum_x x\, P_X(x)$$

(PS-34)

$$= \sum_x x \cdot \sum_y P_{X|Y}(x|y) \cdot P_Y(y)$$
$$= \sum_y \sum_x x\, P_{X|Y}(x|y) \cdot P_Y(y)$$
$$= \sum_y P_Y(y) \cdot E[X|Y=y]$$

Summary :

① $P_{X|A}(x) = \dfrac{P_X(x)}{P(A)}$ (if $x \in A$) $= \dfrac{f_X(x)}{P(A)}$ (if $x \in A$)

② $E[X|A] = \sum_x x\, P_{X|A}(x) = \int_{-\infty}^{\infty} x \cdot f_{X|A}(x)$

③ $P_X(x) = \sum_{i=1}^{n} P(A_i)\, P_{X|A_i}(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)$

④ $E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]$

⑤ $P_{XY}(x, y) = P_{X|Y}(x|y) \cdot P_Y(y)$

$f_{XY}(x, y) = f_{X|Y}(x, y) \cdot f_Y(y)$

⑥ $P_X(x) = \sum_y P_{X|Y}(x|y) \cdot P_Y(y)$

$f_X(x) = \int_y f_{X|Y}(x|y) \cdot f_Y(y)\, dy$

⑦ $E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y)$

$E[X|Y=y] = \int_x f_{X|Y}(x|y)\, dx$

⑧ $E[X] = \sum_y P_Y(y) \cdot E[X|Y=y]$

$E[X] = \int_y f_Y(y) \cdot E[X|Y=y] \cdot dy$

→ **Conditional Expectation :** $E[X|Y]$

$E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y)$

$g(y) = E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y)$

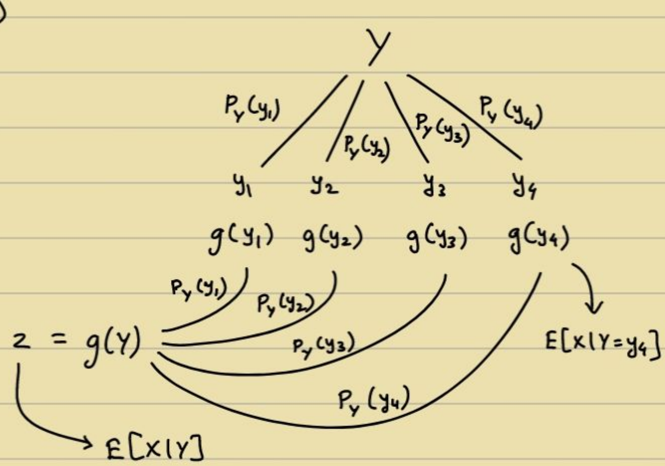
$Y = y_1, y_2, y_3, \ldots y_{10}$

$E[Z] = E[g(Y)]$

$= \sum_y P_Y(y) \cdot g(y)$

$E[E[X|Y]] = \sum_y P_Y(y) \cdot E[X|Y=y]$

$= E[X]$

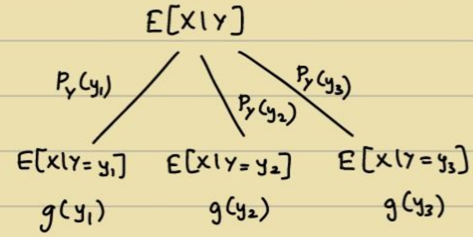


- Ex.

$$Y = \begin{cases} \lambda_1 & \text{with prob } p \\ \lambda_2 & \text{with prob } 1-p \end{cases}$$

$X \sim \text{Exponential}(Y)$

$$\left[\begin{array}{l} \because f_X(x) = \lambda e^{-\lambda x} \\ \quad X \rightsquigarrow \text{Exp}(\lambda) \\ \quad E[X] = 1/\lambda \end{array}\right]$$



$$E[E[X|Y]] = E[X]$$

$E[X] = ?$

$$E[X|Y=\lambda_1] = 1/\lambda_1$$
$$E[X|Y=\lambda_2] = 1/\lambda_2$$

$$\begin{aligned} E[X] &= \sum_{i=1}^{2} E[X|Y=y_i] \cdot P_Y(y_i) \\ &= 1/\lambda_1 \cdot (p) + 1/\lambda_2 (1-p) \\ &= p(1/\lambda_1 - 1/\lambda_2) + 1/\lambda_2 \end{aligned}$$

$$f_{X|Y}(x|\lambda_1) = \lambda_1 e^{-\lambda_1 x}$$
$$f_{X|Y}(x|\lambda_2) = \lambda_2 e^{-\lambda_2 x}$$

$$\therefore E[X] = (1/\lambda_1 - 1/\lambda_2)\,p + 1/\lambda_2$$

$$E[X] = E[\underbrace{E[X|Y]}]$$

Misuse of Notation, but intuitively.

Ex. $Y = X_1 + X_2 + \dots + X_N$, $N > 0$

X_i's are independent & identically distributed with mean $E[X]$

i.i.d $\rightarrow$ PMF of all X_i are same

$E[Y] = ?$

$$\begin{aligned} E[Y|N=n] &= E[X_1 + X_2 \dots + X_n] \\ &= E[X_1] + E[X_2] + \dots + E[X_n] \\ &= n\,E[X] \end{aligned}$$

$$E[Y|N] = \sum_{i=1}^{N} E[X_i|N] = N \cdot E[X]$$

$$\begin{aligned} E[Y] &= \sum_n E[Y|N=n] \cdot P_N(n) \\ &= E[X] \cdot E[N] \end{aligned}$$

(OR) $E[E[Y|N]] = E[N \cdot E[X]]$

$\Rightarrow E[Y] = E[N] \cdot E[X]$

$\text{Var}[Y] = ?$ $\longleftarrow$ H.W

$$Var[Y] = E[Y^2] - E[Y]^2$$

As $Y = \sum X_i$

& $\because E[Y] = E[X] \cdot E[N]$

$$E[Y^2] = E\left[\left(\sum X_i\right)^2\right]$$
$$= E\left[\sum (X_i)^2 + 2 \cdot \sum_i \sum_{i+1} X_i X_j\right]$$
$$= E\left[\sum (X_i)^2\right] + 2 \cdot E\left[\sum_i \sum_{i+1} X_i X_j\right]$$
$$\therefore \; E[Y^2] = \sum E[(X_i)^2] + 2 \sum_i \sum_{i+1} E[X_i X_j]$$

$$Var[Y] = E[Y^2] - E[Y]^2$$
$$= \sum E[(X_i)^2] + 2 \sum_i \sum_{i+1} E[X_i] \cdot E[X_j] - (E[Y])^2$$
$$= \sum E[(X_i)^2] + 2 \sum_i \sum_{i+1} E[X_i] \cdot E[X_j] - \left(E\left[\sum X_i\right]\right)^2$$
$$= \sum E[(X_i)^2] + 2 \sum_i \sum_{i+1} E[X_i] \cdot E[X_j] - \left(\sum E[X_i]\right)^2$$
$$= \sum E[(X_i)^2] + 2 \sum_i \sum_{i+1} E[X_i] \cdot E[X_j] - \sum (E[X_i])^2 - 2 \sum_i \sum_{i+1} E[X_i] E[X_j]$$
$$= \sum E[(X_i)^2] - \sum (E[X_i])^2$$
$$= \sum Var[X_i]$$
$$= Var[X_1] + Var[X_2] + \dots + Var[X_N]$$

$\left\{ \begin{array}{l} \therefore \text{ For Independent R.Vs:} \\ Var[X+Y] = Var[X] + Var[Y] \end{array} \right\}$

$$Var[Y \mid N=n] = Var[X_1 + X_2 \dots + X_n]$$
$$= Var[X_1] + Var[X_2] + \dots + Var[X_n]$$
$$= n \cdot Var[X]$$
$$Var[Y \mid N] = \sum_{i=1}^{N} Var(X_i \mid N) + 2 \cdot \sum_{i<j} cov(X_i, X_j \mid N)$$

$$Var[Y] = \sum_n Var[Y \mid N=n] \cdot P_N(n)$$

$$E[Var[Y \mid N]] + Var[E[Y \mid N]]$$
$$E[Var[Y \mid N]] + Var\left[\sum_{i=1}^{N} E[X_i \mid N]\right]$$
$$= E[N] \cdot Var[X] + Var[N] \cdot (E[X])^2$$

$$E[Var[Y \mid N]] = E[N \cdot Var[X]]$$
$$= E[N] \cdot Var[X]$$
$$Var[E[Y \mid N]] = Var[N \cdot E[X]]$$
$$= Var[N] \cdot (E[X])^2$$

Mid - Sem

2022

1.

1. For a continuous non-negative random variable X prove that $E[X^2] = 2\int_x x\bar{F}_X(x)dx$ where $\bar{F}_X(x) = 1 - F_X(x)$.

$$E[X^2] = 2\int_x x\,\bar{F}_X(x)\,dx \qquad \text{Where , } \bar{F}_X(x) = 1 - F_X(x)$$

Proof :

$$E[X] = \int x \cdot f_X(x)\,dx$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$\Rightarrow 1 - F_X(x) = \overline{F_X}(x) = \int_x^{\infty} f_X(x)\,dx$$

$$2\int_x x\,\overline{F_X}(x)\,dx = 2\int_0^{\infty} x \cdot \overline{F_X}(x)\,dx$$

$$= 2\int_0^{\infty} x \cdot \int_x^{\infty} f_X(u)\cdot du \cdot dx$$

$$= \int_0^{\infty}\int_0^{u} 2x \cdot f(u) \cdot du \cdot dx$$

$$= \int_0^{\infty} u^2 f(u)\,du = E[X^2]$$

2.

2. Let X be a continuous random variable with distribution $F_X(\cdot)$ and density $f_X(x)$. Find the density and distribution for $Z = \sqrt{X}$.

$$Z = \sqrt{X}$$

$$F_X(x) = P(X \leq x)$$

$$F_Z(z) = P(Z \leq z) = P(\sqrt{X} \leq z) = P(X \leq z^2) = F_X(z^2)$$

$$f_Z(z) = 2z \cdot f_X(z^2) \qquad \left(\text{PDF} \rightarrow \frac{d}{d(\text{RV})}\text{ CDF}\right)$$

3.

3. Consider two exponential random variables X and Y with parameters λ_1 and λ_2 respectively. Consider $Z = min(X, Y)$ (min stands for minimum). Find the probability density and cumulative distribution of Z.

$$Z = \min(X, Y)$$

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$F_Z(z) = P(Z \leq z)$$

$$= 1 - P(Z > z)$$

$$= 1 - P(X > z \ \&\ Y > z)$$

$$= 1 - P(X > z) \cdot P(Y > z)$$

$$= 1 - e^{-\lambda_1 z} \cdot e^{-\lambda_2 z}$$

$$P(X \le x) = 1 - e^{-\lambda_1 x}$$

$$F_Z(z) = 1 - e^{-(\lambda_1 + \lambda_2) z}$$

$$P(X > x) = 1 - (1 - e^{-\lambda_1 x})$$

$$= e^{-\lambda_1 x}$$

$\hookrightarrow$ Exponential in $(\lambda_1 + \lambda_2)$

$$f_Z(z) = (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) z}$$

(4) 4. Let X and Y denote Gaussian random variables with mean μ_1 and μ_2 and standard deviation σ_1 and σ_2 respectively. Consider $Z = X + Y$. Using Moment generating functions, show that Z is also a Gaussian random variable, with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

$X \rightsquigarrow (\mu_1, \sigma_1)$, $Y \rightsquigarrow (\mu_2, \sigma_2)$

$Z = X + Y$

$$f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$-\frac{x^2 - 2\mu x + \mu^2}{2\sigma^2} + tx$$

$$-\frac{x^2 - 2(\mu - t\sigma^2)x + \mu^2}{2\sigma^2}$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$Z = X + Y$

$$M_Z(t) = E[e^{tz}]$$

$$= E[e^{t(x+y)}] = E[e^{tx} \cdot e^{ty}]$$

$$= E[e^{tx}] \cdot E[e^{ty}] = M_X(t) \cdot M_Y(t)$$

$$= e^{\mu_1 t + \frac{1}{2}\sigma_1 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2 t^2}$$

$$= e^{(\mu_1 + \mu_2)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2}$$

$\therefore$ Mean $\rightarrow \mu_1 + \mu_2$

Variance $\rightarrow \sigma_1^2 + \sigma_2^2$

(5) 5. Let U_1 and U_2 be two independent Uniform random variables with support $[0, 1]$. Then find the cdf or pdf of $U_1 + U_2$.

$$f_{U_1}(u) = f_{U_2}(u) = \begin{cases} 1/1-0, & \text{if } u \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$Z = U_1 + U_2 \ : \ [0, 2]$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{U_1}(u_1) \cdot f_{U_2}(z - u_1) \, du_1$$

(I) If $0 \leq z \leq 1$

$\Rightarrow 0 \leq u_1 \leq z$

$$f_Z(z) = \int_0^z f_{U_1}(u_1) \cdot f_{U_2}(z - u_1) \, du_1$$

$$= \int_0^z 1 \cdot du_1 = z$$

(II) If $1 \leq z \leq 2$

$\Rightarrow z - 1 \leq u_1 \leq 1$

$$f_Z(z) = \int_{z-1}^{1} f_{U_1}(u_1) \cdot f(z - u_1) \, du_1$$

$$= \int_{z-1}^{1} 1 \cdot du_1 = 2 - z$$

$$\therefore \ f_Z(z) = \begin{cases} z, & \text{if } 0 \leq z \leq 1 \\ 2 - z, & \text{if } 1 \leq z \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

⑥ 6. If X and Y are independent random variables, prove that $Var(X+Y) = Var(X) + Var(Y)$. (Recall that $Var(X) = E[(X - E[X])^2]$)

$$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

$$\begin{aligned} Var(X+Y) &= E[(X+Y)^2] - (E[X+Y])^2 \\ &= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\ &= E[X^2] + \cancel{E[2XY]} + E[Y^2] - (E[X])^2 - \cancel{2E[X] \cdot E[Y]} - (E[Y])^2 \\ &= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2 \\ &= Var(X) + Var(Y) \end{aligned}$$

⑦ 1. $X_1, X_2, \ldots, X_n$ are independent and identically dsitributed $Bernoulli(p)$ random variables (i.e., they take the value 1 with probability p and 0 otherwise). Consider $S_n = \sum_{i=1}^{n} X_n$. Find the PMF, MGF, mean and variance of S_n.

$$\begin{aligned} M_{S_n}(t) &= E[e^{tS_n}] \\ &= E[e^{t(X_1 + X_2 + \ldots + X_n)}] \end{aligned}$$

$$= E[e^{tX_1} \cdot e^{tX_2} \cdot \ldots \cdot e^{tX_n}]$$
$$= E[e^{tX_1}] \cdot E[e^{tX_2}] \cdot \ldots\ldots E[e^{tX_n}]$$
$$= (E[e^{tX_1}])^n = (e^0(1-p) + e^t p)^n$$
$$= (1-p+e^t p)^n$$

$\therefore$ Binomial

$$M_{S_n}(t) = E[e^{tS_n}]$$
$$= \sum_{k=0}^{n} e^{tk} \cdot f_X(x) = \sum_k e^{tk} \cdot \binom{n}{k} p^k (1-p)^{n-k}$$
$$= (1-p+e^t p)^n$$

$$E[S_n] = E[X_1 + X_2 + X_3 + \ldots + X_n]$$
$$= E[X_1] + E[X_2] + \ldots\ldots + E[X_n]$$
$$= n \cdot E[X_1] = n \cdot p = np$$

$$E[S_n] = M'_{S_n}(t)\big|_{t=0}$$
$$E[S_n^2] = M''_{S_n}(t)\big|_{t=0}$$

$$Var(S_n) = E[S_n^2] - (E[S_n])^2$$
$$= np + n^2p^2 - np^2 - n^2p^2 = np - np^2 = np(1-p)$$

8 2a (5 marks) The joint probability mass function of the discrete random variables X and Y are given by $p_{X,Y}(x,y) = \frac{1}{2^{x+y}}$, $x = 1, 2., \ldots$ and $y = 1, 2, \ldots$.

(a) Find the expression for the marginal pmf $p_X(x)$ and $p_Y(y)$ and the conditional pmf $p_{X|Y}(x|y)$.

(b) Find $E[XY]$ and determine if the RV X and Y are independent.

(a) $P_{XY}(x,y) = \dfrac{1}{2^{x+y}}$, $x = 1, 2, \ldots$ & $y = 1, 2, \ldots$

(i) Marginal PMF :

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$= \sum_{y=1}^{\infty} \frac{1}{2^{x+y}} = \sum_y \frac{1}{2^x \cdot 2^y} = \frac{1}{2^x} \cdot \sum_y \frac{1}{2^y} = \frac{1}{2^x}$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y) = \frac{1}{2^y}$$

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)} = \frac{1/2^{x+y}}{1/2^y} = \frac{1}{2^x}$$

(ii) $E[XY] = E[X] \cdot E[Y]$ $\because$ X and Y are independent

$$= \sum_x x \cdot \frac{1}{2^x} \cdot \sum_y y \cdot \frac{1}{2^y}$$
$$= 2 \cdot 2$$
$$= 4$$

2b (5 marks) The joint pdf of random variables X and Y is given by $f_{X,Y}(x,y) = \lambda\mu e^{-\lambda x-\mu y}$, $x \geq 0, y \geq 0, \lambda > 0, \mu > 0$.

(a) Find the expression for the marginal pdf's $f_X(x)$ and $f_Y(y)$ and the joint CDF $F_{X,Y}(x,y)$

(b) Are the RV X and Y independent? Give reasons.

(b) $f_{XY}(x,y) = \lambda\mu e^{-\lambda x-\mu y}$, $x \geq 0$, $y \geq 0$, $\lambda > 0$, $\mu > 0$

$$f_X(x) = \int_y f_{XY}(x,y)\,dy$$

$$= \int_{y=0}^{y=\infty} \lambda\mu e^{-\lambda x-\mu y}\,dy$$

$$= \lambda\mu \cdot \int_0^\infty e^{-\lambda x} \cdot e^{-\mu y}\,dy$$

$$= \lambda\mu e^{-\lambda x} \cdot \frac{-1}{\mu}\left(e^{-\mu y}\right]_0^\infty = -\lambda e^{-\lambda x}(0-1) = \lambda e^{-\lambda x}$$

$$f_Y(y) = \int_x f_{XY}(x,y)\,dx = \mu e^{-\mu y}$$

$$F_{XY}(x,y) = \int_0^\infty \int_0^\infty f_{XY}(x,y)\,dx\,dy$$

$$= (1-e^{-\lambda x})(1-e^{-\mu y})$$

$$= F_X(x) \cdot F_Y(y)$$

$\therefore$ X and Y are independent.

2023 :

1

1. Consider a random variable X with the following pdf. Find mean and variance for X.

$$f_X(x) = \begin{cases} 0.5\lambda e^{-\lambda x}, & x \geq 0 \\ 0.5\lambda e^{\lambda x}, & x < 0 \end{cases}$$

$$E[X] = \int_{-\infty}^{0} 0.5\lambda e^{\lambda x} \cdot x \, dx + \int_{0}^{\infty} 0.5\lambda e^{-\lambda x} x \, dx$$

$$= \frac{\lambda}{2}\left(\int_{-\infty}^{0} x e^{\lambda x} \, dx + \int_{0}^{\infty} x e^{-\lambda x} \, dx \right)$$

$$= \frac{\lambda}{2}\left(\left(\frac{1}{\lambda} x e^{\lambda x} - \frac{1}{\lambda^2} e^{\lambda x} \right]_{-\infty}^{0} + \left(\frac{-1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_{0}^{\infty} \right)$$

$$= \frac{\lambda}{2}\left(\frac{-1}{\lambda^2} - \left(+ \frac{1}{\lambda^2} \right) \right) = 0$$

D	I
$+ \; x$	$e^{-\lambda x}$
$- \; 1$	$\frac{-1}{\lambda} e^{-\lambda x}$
$+ \; 0$	$\frac{1}{\lambda^2} e^{-\lambda x}$

$\therefore \; E[x] = 0$

$$E[X^2] = \int_{-\infty}^{0} 0.5\lambda e^{\lambda x} \cdot x^2 \, dx + \int_{0}^{\infty} 0.5\lambda e^{-\lambda x} x^2 \, dx$$

$$= \frac{\lambda}{2}\left(\int_{-\infty}^{0} x^2 e^{\lambda x} \, dx + \int_{0}^{\infty} x^2 e^{-\lambda x} \, dx \right)$$

$$= \frac{2}{\lambda^2}$$

D	I
$+ \; x^2$	$e^{\lambda x}$
$- \; 2x$	$1/\lambda \, e^{\lambda x}$
$+ \; x$	$1/\lambda^2 \, e^{\lambda x}$
$- \; 1$	$1/\lambda^3 \, e^{-\lambda x}$
$+ \; 0$	$1/\lambda^4 \, e^{-\lambda x}$

$Var(X) = \frac{2}{\lambda^2} - 0 \quad \therefore \; Var(x) = 2/\lambda^2$

2

2. Let X be a Uniform $U[0, 1]$ random variable. Let $Y = e^{2X}$. FInd pdf and cdf of Y.

$F_X(x) = P(X \leq x)$

$F_Y(y) = P(Y \leq y)$

$= P(e^{2X} \leq y)$

$= P(2X \leq \ln|y|)$

$= P(X \leq \ln|y|/2)$

$= F_X\left(\frac{\ln|y|}{2}\right)$

$$f_X(x) = \begin{cases} 1/1-0 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$F_X(x) = \frac{x-0}{1-0} = x \quad , \; x \in [0,1]$

$F_Y(y) = F_X\left(\frac{\ln|y|}{2}\right) = \frac{\ln|y|}{2} \quad \text{for } 1 \leq y \leq e^2$

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \frac{\ln|y|}{2} = \frac{1}{2y} \quad \text{for } 1 \leq y \leq e^2$

6

1. Let $Y = aX^2 + b$ where X is a continuous random variable. Derive the expression for the CDF and pdf of Y in terms of the pdf of X.

$F_Y(y) = P(Y \leq y)$

$= P(aX^2 + b \leq y) = P(aX^2 \leq y - b)$

$$= P\left(X^2 \le \frac{y-b}{a}\right)$$

$$= P\left(|X| \le \sqrt{\frac{y-b}{a}}\right)$$

$$F_Y(y) = F_X\left(\sqrt{\frac{y-b}{a}}\right) - F_X\left(-\sqrt{\frac{y-b}{a}}\right) \qquad [\text{for } a>0]$$

$$= \int_{-\sqrt{\frac{y-b}{a}}}^{\sqrt{\frac{y-b}{a}}} f_X(x)\,dx \quad \& \quad 1 - \int_{-\sqrt{\frac{y-b}{a}}}^{\sqrt{\frac{y-b}{a}}} f_X(x)\,dx \quad (\text{if } a<0)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

7

2. Let X be a uniform random variable with support $[a, b]$, . Let Y be Poisson random variable with parameter λ. Derive the expression for the mean and variance of X and Y.

$X \rightsquigarrow U[a,b]$

$Y \rightsquigarrow P(\lambda)$

(a) $f_X(x) = \frac{1}{b-a}$ if $x \in [a,b]$, 0 otherwise

$$E[X] = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right]_a^b = \frac{1}{b-a} \cdot \frac{b^2-a^2}{2} = \frac{a+b}{2}$$

$$E[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(\frac{x^3}{3}\right]_a^b = \frac{a^2+ab+b^2}{3}$$

$$\Rightarrow Var[X] = \frac{(b-a)^2}{12}$$

(b) $f_X(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$ for $x = 1, 2, \ldots\ldots$

$$E[X] = \sum_{i=0}^{\infty} x \cdot f_X(x) = \lambda$$

$$E[X^2] = \sum_{i=0}^{\infty} x^2 \cdot f_X(x) = \lambda^2 + \lambda \Rightarrow Var[X] = \lambda^2$$

2024

① Q1: Let X be a continuous random variable with distribution $F_X(.)$ and density $f_X(x)$.Find the probability density and cummulative distribution for Y = $X^2 + 4$. [Anush]

$$F_Y(y) = P(Y \le y) = P(X^2 + 4 \le y)$$
$$= P(X^2 \le y-4)$$
$$= P(|X| \le \sqrt{y-4}) \quad \text{for } y > 4$$
$$= P(-\sqrt{y-4} \le X \le \sqrt{y-4})$$
$$= F_X(\sqrt{y-4}) - F_X(-\sqrt{y-4})$$
$$= \int_{-\sqrt{y-4}}^{\sqrt{y-4}} f_X(x)\, dx$$

$$F_Y(y) = \int_{-\sqrt{y-4}}^{\sqrt{y-4}} f_X(x)\, dx \quad \text{for } y > 4 \text{ \& } 0 \text{ otherwise}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-\sqrt{y-4}}^{\sqrt{y-4}} f_X(x)\, dx$$
$$= \frac{1}{2\sqrt{y-4}} \cdot f_X(\sqrt{y-4}) - \frac{-1}{2\sqrt{y-4}} f_X(-\sqrt{y-4})$$

$$\therefore f_Y(y) = \frac{1}{2\sqrt{y-4}} \left(f_X(\sqrt{y-4}) + f_X(-\sqrt{y-4}) \right) \quad \text{for } y > 4,\ 0 \text{ otherwise}$$

② Q2: Suppose X and Y are independent and exponential random variables with parameter a and b respectively. Then find $P(X < Y)$. [Ronak]

$$P_{XY}(x, y) = \underbrace{P_X(x)}_{ae^{-ax},\ a > 0} \cdot \underbrace{P_Y(y)}_{be^{-bx},\ b > 0}$$

$$P(X < Y)$$
$$= P(X < \forall y_i)$$
$$= \int_y P(X < y \cap Y = y)$$
$$= \int_{y=0}^{\infty} P(X < y \mid Y = y)\ f_Y(y)\, dy$$
$$= \int_{y=0}^{\infty} P(X < y)\ f_Y(y)\, dy \qquad [P(X<y) = F_X(y) \text{ \& Independence}]$$
$$= \int_0^{\infty} (1 - e^{-ay})(be^{-by})\, dy$$

$$= \int_0^\infty \left(be^{-by} - be^{-(a+b)y}\right) dy$$

$$= b \cdot \left[\int_0^\infty e^{-by}\, dy - \int_0^\infty e^{-(a+b)y}\, dy\right]$$

$$= b\left(\frac{1}{b} - \frac{1}{a+b}\right)$$

$$= \frac{a}{a+b}$$

(3) Q3: We are given random variables $X_1, X_2, \ldots, X_n$ with finite variances, and they are not necessarily independent. We need to find the variance of $S_n = \sum_{i=1}^{n} X_i$ and then check how the expression changes when the X_i's are independent. [Gopal]

$$S_n = X_1 + X_2 + \ldots\ldots + X_n$$

$$Var[S_n] = Var[X_1 + X_2 + \ldots + X_n]$$

$$= E[(X_1 + X_2 + \ldots + X_n)^2] - (E[X_1 + X_2 + \ldots + X_n])^2$$

$$E[S_n^2] - (E[S_n])^2 = E[(X_1 + X_2 + \ldots + X_n)^2] - (E[X_1] + E[X_2] + \ldots + E[X_n])^2$$

$$= E\left[\sum_1^n X_i^2 + 2\sum_1^n \sum_{i+1}^n X_i X_j\right] - \left(\sum_1^n E[X_i]\right)^2$$

$$= E\left[\sum_1^n X_i^2\right] + 2\sum_1^n \sum_{i+1}^n E[X_i X_j] - \left(\sum_1^n E[X_i]\right)^2$$

$$= \sum_1^n E[X_i^2] + 2 \cdot \sum_1^n \sum_{i+1}^n E[X_i X_j] - \sum_1^n (E[X_i])^2 - 2\sum_1^n \sum_{i+1}^n E[X_i] \cdot E[X_j]$$

$$= \sum_1^n E[X_i^2] - (E[X_i])^2 + \sum_1^n \sum_{i+1}^n E[X_i X_j] - E[X_i] \cdot E[X_j]$$

$$Var[S_n] = \sum_1^n Var[X_i] + 2\sum_1^n \sum_{i+1}^n cov(X_i, X_j)$$

If X_i's are independent $\Rightarrow Var[S_n] = \sum_{i=1}^n Var[X_i]$

(4) Q4: Let X be a random variable having Binomial distribution with parameters N and p where N is itself a random variable having Poisson distribution with mean λ. Find the probability mass function of the random variable X. Also find $E[X]$.

$X \rightsquigarrow$ Binomial (N, p)

$N \rightsquigarrow$ Poisson (λ)

$P_X(x) = ?$ & $E[X] = ?$

$$P_N(n) = \frac{e^{-\lambda} \cdot \lambda^n}{n!}, \quad \lambda > 0 \quad \text{for } n = 0, 1, 2, \ldots\ldots$$

$$E[N] = \lambda$$

$$E[X] = \sum x \cdot P_X(x)$$

$$P(X=x) = P(X=x \ \forall n)$$

$$= \sum_{n=x}^{\infty} P(X=x \cap N=n)$$

$$= \sum_{n=x}^{\infty} P(X=x \mid N=n) \cdot P(N=n)$$

$$= \sum_{n=x}^{\infty} P(X=x) \cdot P(N=n)$$

$$= \sum_{n=x}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!}$$

$$= \sum_{n=x}^{\infty} \frac{n!}{(n-x)!\,x!} \cdot p^x \cdot (1-p)^{n-x} \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!}$$

$$= \frac{p^x}{x!} \cdot e^{-\lambda} \sum_{n=x}^{\infty} \frac{(1-p)^{n-x}}{(n-x)!} \cdot \lambda^n$$

$$= \frac{p^x}{x!} \cdot e^{-\lambda} \cdot \frac{\lambda^x}{e^{-\lambda(1-p)}} \cdot \underbrace{\sum_{n=x}^{\infty} \frac{(\lambda(1-p))^{n-x}}{(n-x)!} \cdot e^{-\lambda(1-p)}}$$

$$\therefore P(X=x) = e^{-\lambda p} \cdot \frac{(\lambda p)^x}{x!} \qquad \hookrightarrow \sum_{\forall x} PMF = 1$$

$$\therefore E[X] = \lambda p$$

⑤ Q5: Let X be a standard normal variable (Gaussian with zero mean and unit variance). Let $Z = \sigma X + \mu$. Obtain the pdf and cdf of Z. [Kavin]

$$X \sim N(0,1)$$

$$Z = \sigma X + \mu$$

$$F_Z(z) = P(Z \leq z)$$

$$= P(\sigma X + \mu \leq z)$$

$$= P\left(X \leq \frac{z-\mu}{\sigma}\right)$$

$$= F_X\left(\frac{z-\mu}{\sigma}\right)$$

$$\therefore F_Z(z) = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \Phi(x)$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \Phi\left(\frac{z-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma} f_X\left(\frac{z-\mu}{\sigma}\right)$$

$$\therefore f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

⑥ Q1: Suppose U_1 and U_2 are independent uniform random variables on the segments $[-1,1]$ and $[0,1]$ respectively. Let $Z = U_1 + U_2$. Derive an expression for the pdf and cdf of Z.

$$\left.\begin{array}{l} U_1 \rightarrow [-1,1] \\ U_2 \rightarrow [0,1] \end{array}\right\} \text{Independent}$$

$Z = U_1 + U_2$

$X = U_1$ & $Y = U_2$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\cdot f_Y(z-x)\cdot dx = \int_{-\infty}^{\infty} f_Y(y)\cdot f_X(z-y)\cdot dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y)\cdot f_X(z-y)\,dy$$

$$= \int_0^1 f_Y(y)\cdot f_X(z-y)\,dy$$

$$= \int_0^1 1\cdot f_X(z-y)\,dy$$

$$f_Y(y) = \begin{cases} 1, & \text{if } x\in[0,1] \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 1/2, & \text{if } x\in[-1,1] \\ 0, & \text{otherwise} \end{cases}$$

$0 \le y \le 1$

$z \ge z-y \ge z-1$

$z \ge x \ge z-1$

If $z > 2 \Rightarrow x \notin [-1,1] \Rightarrow f_X(x) = 0$

If $z < -1 \Rightarrow x \notin [-1,1] \Rightarrow f_X(x) = 0$

case - I : $z > 2$

$$f_Z(z) = \int_0^1 f_X(z-y)\,dy = 0$$

case - II : $z < -1$

$$f_Z(z) = \int_0^1 f_X(z-y)\,dy = 0$$

case - III : $-1 \le z \le 2$

$$\left\{\begin{array}{l} -1 \le z-y \le 1 \\ -1 \le x \le 1 \;\;\&\;\; 0 \le y \le 1 \Rightarrow -1 \le x+y \le 2 \end{array}\right\}$$

ⓐ if $z \in [-1,0]$

$-2 \le z-y \le 0$

but $z-y \ge -1 \Rightarrow y \le z+1$

$$f_Z(z) = \int_0^{z+1} f_X(z-y)\,dy + \int_{z+1}^{1} f_X(z-y)\,dy$$

$$= \int_0^{z+1} f_X(z-y)\,dy = \int_0^{z+1} \frac{1}{2}\,dy = \frac{1}{2}(z+1)$$

(b) if $z \in [0, 1]$

$-1 \le z - y \le 1$

$$f_Z(z) = \int_0^1 f_X(z-y)\, dy$$

$$= \int_0^1 \frac{1}{2} dy = \frac{1}{2}$$

(c) If $z \in [1, 2]$

$0 \le z - y \le 2$

But $z - y \le 1 \Rightarrow y \ge z - 1$

$$f_Z(z) = \int_0^{z-1} f_X(z-y)\, dy + \int_{z-1}^{1} f_X(z-y)\, dy$$

$$= \int_{z-1}^{1} f_X(z-y)\, dy$$

$$= \int_{z-1}^{1} \tfrac{1}{2}\, dy = \frac{1}{2}(2-z)$$

$$\therefore f_Z(z) = \begin{cases} \frac{1}{2}(z+1), & z \in [-1, 0] \\ \frac{1}{2}, & z \in [0, 1] \\ \frac{1}{2}(2-z), & z \in [1, 2] \\ 0, & \text{otherwise} \end{cases}$$

$$F_Z(z) = \int f_Z(z)$$

(7)

Q2: Let X and Y be jointly continuous random variables with joint probability density function (PDF):

$$f_{X,Y}(x, y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x, y \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

We are required to solve the following:

(a) Find $E[X]$ and $E[Y]$.
(b) Are X and Y independent? Justify.
(c) Find $E[Y|X > 2]$.
(d) Find $P(X > Y)$.

$$f_{XY}(x, y) = \begin{cases} 6e^{-(2x+3y)}, & \text{if } x, y \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

(a)

$$f_X(x) = \int_0^\infty f_{XY}(x, y)\, dy$$

$$= \int_0^\infty 6e^{-2x} \cdot e^{-3y}\, dy$$

$$= 6e^{-2x} \int_0^\infty e^{-3y}\, dy$$

$$= 6e^{-2x} \cdot \frac{-1}{3} \cdot \left(e^{-3y}\right]_0^\infty$$

$$= -2e^{-2x} \cdot (0 - 1)$$

$$\therefore f_X(x) = 2e^{-2x}$$

$$f_Y(y) = \int_0^\infty f_{XY}(x, y)\, dx$$

$$= \int_0^\infty 6e^{-2x} \cdot e^{-3y}\, dx$$

$$= 6e^{-3y} \int_0^\infty e^{-2x}\, dx$$

$$= 6e^{-3y} \cdot \frac{-1}{2} \cdot \left(e^{-2x}\right]_0^\infty$$

$$= -3e^{-3y} \cdot (0 - 1)$$

$$\therefore f_Y(y) = 3e^{-3y}$$

(b) $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$

$\therefore$ Independent

(c) $E[Y \mid X > 2]$

$= E[Y]$

$= 1/3$

(d) $P(X > Y)$

$= P(X > \forall y)$

$= P(X > y \cap Y = \forall y)$

$= \int_y P(X > y \cap Y = y)\ dy$

$= \int_y P(X > y \mid Y = y) \cdot P(Y = y)\ dy$

$= \int_y P(X > y) \cdot P(Y = y)\ dy$

$= \int_0^\infty e^{-2y} \cdot f_Y(y)\ dy$

$= \int_0^\infty e^{-2y} \cdot 3e^{-3y}\ dy = 3/5$

$$\left(\begin{aligned} P(X > y) &= 1 - P(X \leq y) \\ &= 1 - (1 - e^{-2y}) \\ &= e^{-2y} \end{aligned} \right)$$

End 2024

1. Q1: Let $Z = X_1 + X_2 + \cdots + X_N$, where X_i are i.i.d. random variables and N is a positive discrete random variable. Prove that:

$$M_Z(t) = M_N(\log M_X(t)).$$

$Z = X_1 + X_2 + \dots + X_N$

X_i : i.i.d R.V

N : +ve discrete R.V

To prove :

$M_Z(t) = M_N(\log M_X(t))$

$$\begin{aligned}
\text{LHS}: \quad M_Z(t) &= E[e^{tZ}] \\
&= E_N\left[E[e^{tZ}|N]\right] \\
&= E_N\left[E[e^{t\sum X_i}|N]\right] \\
&= E_N\left[E[\Pi e^{tX_i}|N]\right] \\
&= E_N\left[\Pi E[e^{tX_i}|N]\right] \\
&= E_N\left[(E[e^{tX_1}])^N\right] \\
\therefore M_Z(t) &= E_N\left[(M_X(t))^N\right]
\end{aligned}$$

$$\begin{aligned}
E[Z|N=n] &= E[X_1 + X_2 \dots + X_n] \\
&= E[X_1] + E[X_2] + \dots + E[X_n] \\
&= n\,E[X]
\end{aligned}$$

$E[Z|N] = N \cdot E[X]$

$$\begin{aligned}
E[Z] &= \sum_n E[Z|N=n] \cdot P_N(n) \\
&= E[X] \cdot E[N]
\end{aligned}$$

(OR)

$$\begin{aligned}
E[Z] = E\left[E[Z|N]\right] &= E[N \cdot E[X]] \\
&= E[N] \cdot E[X]
\end{aligned}$$

$M_N(t) = E[e^{tn}]$

If $u = \ln(M_X(t))$

$\Rightarrow M_N(u) = E[(M_X(t))^n]$

$$M_Z(t) = M_N(u) = M_N(\ln(M_X(t)))$$

$$M_Z(t) = M_N(\ln(M_X(t)))$$

2. Q2: Let $Y = e^X$, where $X \sim \mathcal{N}(\mu, \sigma^2)$. Obtain the pdf of Y.

$$\begin{aligned}
F_X(x) &= P(X \le x) \\
F_Y(y) &= P(Y \le y) \\
&= P(e^X \le y) \\
&= P(X \le \ln(y)) \\
&= F_X(\ln(y)), \quad y > 0
\end{aligned}$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\begin{aligned}
\frac{d}{dy} F_Y(y) &= \frac{d}{dy}\left(F_X(\ln(y))\right) \\
f_Y(y) &= \frac{1}{y} \cdot f_X(\ln(y))
\end{aligned}$$

$$\therefore f_Y(y) = \frac{1}{y} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}, \quad y > 0$$

5

Q5: Let X and Y be independent random variables with common distribution function F.

(a) PDF of $Z_1 = \max(X, Y)$

(b) PDF of $Z_2 = \min(X, Y)$

Sol:

(a)
$$\begin{aligned} F_{Z_1}(z_1) &= P(Z \le z_1) \\ &= P(\max(X, Y) \le z_1) \\ &= P(X \le z_1 \text{ and } Y \le z_1) \\ &= P(X \le z_1) \cdot P(Y \le z_1) \\ &= F_X(z_1) \cdot F_Y(z_1) \end{aligned}$$

$$F_{Z_1}(z_1) = F(z_1)^2$$

$$\frac{d}{dz_1} F_Z(z_1) = \frac{d}{dz_1} F(z_1)^2$$

$$f_Z(z_1) = 2 \cdot F(z_1) \cdot f(z_1)$$

(b)
$$\begin{aligned} F_Z(z_2) &= P(Z \le z_2) \\ &= P(\min(X, Y) \le z_2) \\ &= 1 - P(\min(X,Y) > z_2) \\ &= 1 - P(X > z_2 \cap Y > z_2) \\ &= 1 - P(X > z_2) \cdot P(Y > z_2) \\ &= 1 - (1 - P(X \le z_2)) \cdot (1 - P(Y \le z_2)) \\ &= 1 - (1 - F_X(z_2))(1 - F_Y(z_2)) \\ &= 1 - (1 - F_X(z_2) - F_Y(z_2) + F_X(z_2) \cdot F_Y(z_2)) \\ &= F_X(z_2) + F_Y(z_2) - F_X(z_2) \cdot F_Y(z_2) \\ &= 2F(z_2) - F(z_2)^2 \end{aligned}$$

$$f_Z(z_2) = \frac{d}{dz_2} F_Z(z_2) = \frac{d}{dz_2}\left(2F(z_2) - F(z_2)^2\right)$$

$$= 2f_Z(z_2) - 2F_Z(z_2) \cdot f_Z(z_2)$$

10

$$f_X(x) = \begin{cases} \lambda e^{-\lambda(x-\mu)}, & \text{if } x \ge \mu \\ 0, & \text{otherwise} \end{cases}$$

(a) $E[e^{tx}] = ?$

$$= \int_{\mu}^{\infty} e^{tx} \cdot \lambda e^{-\lambda(x-\mu)} \, dx$$

$$= \lambda \int_{\mu}^{\infty} e^{tx} e^{-\lambda x} \cdot e^{\lambda\mu} \, dx$$

$$= \lambda \cdot e^{\lambda\mu} \int_{\mu}^{\infty} e^{(t-\lambda)x} \, dx$$

$$= \lambda \cdot e^{\lambda\mu} \cdot \frac{1}{t-\lambda} \cdot \left(e^{(t-\lambda)x} \right]_{\mu}^{\infty}$$

$$= \lambda \cdot e^{\lambda\mu} \cdot \frac{1}{\lambda - t} \left(0 - (-1) \right), \quad \text{for } t < \lambda$$

$$= \frac{\lambda e^{\lambda\mu}}{\lambda - t}, \text{ for } t < \lambda \quad \& \ 0 \text{ if } \lambda < t$$

(b) $E[X] = M_X'(0)$

$$M_X'(t) = \frac{d}{dt} \lambda e^{\mu t} (\lambda - t)^{-1} = \frac{\mu \cdot \lambda \cdot e^{\mu t}}{\lambda - t} + \frac{\lambda e^{\mu t}}{(\lambda - t)^2}$$

$$E[X] = M_X'(0) = \frac{\lambda\mu}{\lambda} + \frac{\lambda}{\lambda^2} = \mu + \frac{1}{\lambda}$$

$$E[X^2] = M_X''(0)$$

$$M_X''(t) = \frac{\mu^2 \lambda e^{\mu t}}{\lambda - 1} + 2 \cdot \frac{\mu \lambda \cdot e^{\mu t}}{(\lambda - t)^2} + \frac{2\lambda e^{\mu t}}{(\lambda - t)^3}$$

$$\Rightarrow M_X''(0) = \frac{\mu^2 \lambda}{\lambda} + 2 \cdot \frac{\mu\lambda}{\lambda^2} + \frac{2\lambda}{\lambda^2}$$

$$= \left(\mu + \frac{1}{\lambda} \right)^2 + \frac{1}{\lambda^2}$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = \left(\mu + \frac{1}{\lambda} \right)^2 + \frac{1}{\lambda^2} - \left(\mu + \frac{1}{\lambda} \right)^2$$

$$= 1/\lambda^2$$

2023 end

② $N(\mu, \sigma^2)$

$$M_X(t) = E[e^{tx}]$$
$$= \int e^{tx} \cdot P_X(x)\, dx$$
$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{tx} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 tx - (x-\mu)^2}{2\sigma^2}}\, dx$$

$$2\sigma^2 tx - x^2 + 2\mu x - \mu^2$$
$$-\left(\left(x - (t\sigma^2 + \mu)\right)^2 + t^2\sigma^4 + 2\mu t\sigma^2\right.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x - t\sigma^2 - \mu)^2}{2\sigma^2}} \cdot e^{\frac{\sigma^2 t^2}{2} + \mu t}\, dx$$
$$= \frac{e^{\sigma^2 t^2/2 + \mu t}}{\sigma\sqrt{2\pi}} \cdot \sigma\sqrt{2\pi}$$

$$u = \frac{x - t\sigma^2 - \mu}{\sigma}$$
$$du = dx$$

$$= e^{\sigma^2 t^2/2 + \mu t}$$

$$M'(t) = (\sigma^2 t + \mu) \cdot e^{\sigma^2 t^2/2 + \mu t}$$
$$M'(0) = \mu$$
$$M''(t) = (\sigma^2 t + \mu)^2 e^{\sigma^2 t^2/2 + \mu t} + e^{\sigma^2 t^2/2 + \mu t} \cdot \sigma^2$$
$$M''(0) = \mu^2 + \sigma^2$$

2022 q3

(2) **Question 2**

If A and B are exponential random variables with parameters a and b respectively, then prove $P(A < B) = E[e^{-bA}]$. Further show that this is equal to a/(a+b).

(4 marks)

$$P(A < B) = P(A < b \ \forall b)$$
$$= \int_x P(B > a \cap A = x)$$
$$= \int_x P(B > a \mid A = x) \cdot f_A(x)\, dx$$

$$\left(\begin{array}{l} A \longrightarrow ae^{-ax} = f_A(x) \\ B \longrightarrow be^{-bx} = f_B(x) \end{array} \right)$$

$$= \int_x P(B > a) \cdot f_A(x)\, dx = \int_x e^{-bx} \cdot f_A(x)\, dx$$
$$= E[e^{-bA}]$$

$$E[e^{-bA}] \rightarrow \int_x e^{-bx} \cdot ae^{-ax}\, dx = \int_0^\infty a \cdot e^{-(a+b)x}\, dx = \frac{a}{a+b}$$

(3) **Question 3**

Derive the expression for the Moment generating function of the following random variables

1. Standard Normal with mean 0 and variance 1 (3 marks)
2. Poisson random variable with parameter λ (3 marks)

$$M_X(t) = E[e^{tx}]$$

(i) $N(0,1)$

$$M_X(t) = \int_x e^{tx} \cdot P_X(x)\, dx$$
$$= \int_x e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$
$$= \int_x e^{tx} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \cdot \int_x e^{tx - \frac{x^2}{2}}\, dx$$

$$-\frac{x^2 - 2tx + t^2}{2} + \frac{t^2}{2}$$

$$-\frac{(x-t)^2}{2} + \frac{t^2}{2}$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_x e^{-\frac{(x-t)^2}{2}}\, dx$$

$$u = \frac{x-t}{\sqrt{2}}$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \cdot \sqrt{2\pi}$$

$$\therefore M_X(t) = e^{t^2/2}$$

(ii) $P_X(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$M_X(t) = E[e^{tx}]$$
$$= \sum e^{tx} \cdot P_X(x) \, dx$$
$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$
$$= \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \cdot e^{-\lambda}$$
$$= e^{-\lambda} \cdot e^{\lambda e^t}$$
$$\therefore M[e^{tx}] = e^{\lambda e^t - \lambda}$$

4

Question 4

Let X be an exponential random variable with parameter 1. Find

1. Conditional PDF and CDF given $X > 1$
2. $E[X|X > 1]$

$f_X(x) = e^{-x}$ for $x > 0$, 0 otherwise

(i) $P(X = x \mid X > 1) = f_{X|X>1}(x)$

$$X > 1 \; : \; P(A) = \int_1^{\infty} e^{-x} \, dx = \frac{1}{e}$$

$$f_{X|X>1}(x) = \frac{f_X(x)}{1/e} = \frac{e^{-x}}{e^{-1}} = e^{1-x} \quad \text{for } x > 1$$

$$F_{X|X>1}(x) = \frac{F_X(x) - F_X(1)}{P(A)} = 1 - e^{1-x}, \quad \text{for } x > 1$$

(ii) $E[X \mid X > 1] = \int_1^{\infty} x \cdot f_{X|X>1}(x) \, dx$

$$= \int_1^{\infty} x \cdot e^{1-x} \, dx$$
$$= e \int_1^{\infty} x \cdot e^{-x} = e \cdot \frac{2}{e} = 2$$

5

Question 5

Let X and Y be independent and identically distributed discrete random variables taking values on positive integers. Their pmf is $p(x) = C2^{-x}$ for $x \geq 1$. Find

1. The value of C that makes it a valid pmf
2. $P(\min\{X,Y\} \leq x)$
3. $P(X \text{ divides } Y)$

X and $Y \leftarrow$ i.i.d

$P(x) = c \cdot 2^{-x}$ for $x \geq 1$

(a) $\displaystyle\int_1^\infty c \cdot 2^{-x}\, dx \Rightarrow c \cdot \int_1^\infty 2^{-x}\, dx = 1$

$\Rightarrow c \cdot \left(\frac{-2^{-x}}{\ln(2)} \right]_1^\infty = 1$

$\Rightarrow \frac{c}{2\ln 2} = 1 \Rightarrow c = 2\ln 2$

If Continuous

$\displaystyle\sum_{x=1}^{\infty} c \cdot 2^{-x} = 1 \Rightarrow c\left(\frac{1}{2} + \frac{1}{4} + \dots\right) = 1$

$\Rightarrow \underline{\underline{c = 1}}$

Discrete

$\therefore P(x) = 2^{-x} \quad \forall x \geq 1$

(ii) $P(\min\{X, Y\} \leq x)$

$= 1 - P(\min(X, Y) > x)$

$= 1 - P(X > x \cap Y > x)$

$= 1 - P(X > x) \cdot P(Y > x)$

$= 1 - (1 - P(X \leq x)) \cdot (1 - P(Y \leq x))$

$= 1 - (1 - F_X(x))(1 - F_Y(x))$

$= 1 - (1 - F_X(x) - F_Y(x) + F_X(x) \cdot F_Y(x))$

$= F_X(x) + F_Y(x) - F_X(x) \cdot F_Y(x)$

$= 2F(x) - F(x)^2$

$F_X(x) = P(X \leq x) = \displaystyle\sum_{x=-\infty}^{x} 2^{-x} = 1 - 2^{-x}$

$\therefore F(x) = 1 - 2^{-x}$

(iii) $P(X \text{ divides } Y)$

$= \sum \sum P(X = n \cap Y = kn)$

$= \sum 2^{-n} \sum 2^{-kn}$

$= \sum 2^{-n} \cdot \frac{2^{-n}}{1 - 2^{-n}}$

$= \sum \frac{2^{-n}}{2^n - 1}$

$= \sum \frac{1}{2^n - 1} - \frac{1}{2^n}$

$= \displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n - 1}$

6

Question 1

Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2. Let $Z = X + Y$. Then find the pmf of Z.

$$P_X(x) = \frac{e^{-\lambda_1}\lambda_1^x}{x!}$$

$$P_Y(y) = \frac{e^{-\lambda_2}\cdot\lambda_2^y}{y!}$$

$$Z = X + Y$$

Method - I :

$$\begin{aligned} M_Z(t) &= E[e^{tz}] \\ &= E[e^{tx+ty}] \\ &= E[e^{tx}\cdot e^{ty}] \\ &= E[e^{tx}]\cdot E[e^{ty}] \\ &= e^{\lambda_1(t-1)}\cdot e^{\lambda_2(t-1)} \\ &= e^{(\lambda_1+\lambda_2)(t-1)} \end{aligned}$$

$$\begin{aligned} E[e^{tx}] &= \sum_x e^{tx}\cdot\frac{e^{-\lambda_1}\cdot\lambda_1^x}{x!} \\ &= e^{-\lambda_1}\cdot\sum_x\frac{(e^t\lambda_1)^x}{x!} \\ &= e^{-\lambda_1}\cdot e^{t\lambda_1} \\ &= e^{\lambda_1(t-1)} \end{aligned}$$

$Z \sim \text{Poisson}(\lambda_1+\lambda_2)$

$$\therefore P_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)}\cdot(\lambda_1+\lambda_2)^z}{z!} \qquad (\lambda_1+\lambda_2>0)$$

Method - II :

$$\begin{aligned} P(Z = X+Y = k) &= \sum_{i=0}^{k} P(X+Y=k \cap X=i) \\ &= \sum_{i=0}^{k} P(X=i \cap Y=k-i) \\ &= \sum_i P(X=i)\cdot P(Y=k-i) \\ &= \sum_i \frac{e^{-\lambda_1}\cdot\lambda_1^i}{i!}\cdot\frac{e^{-\lambda_2}\cdot\lambda_2^{(k-i)}}{(k-i)!} \\ &= \frac{(\lambda_1+\lambda_2)^k}{k!}\cdot e^{-(\lambda_1+\lambda_2)} \end{aligned}$$

$$f_X(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$\therefore p_X(z) = \sum_{-\infty}^{\infty} f_X(x)\cdot f_Y(z-x)$$

(9) **Question 4**

Let X have a Poisson distribution with parameter Λ, where Λ is an exponential random variable with parameter μ. Show that X has a geometric distribution.

$$P_X(x) = \frac{e^{-\Lambda}\cdot\Lambda^x}{x!}, \quad \Lambda > 0$$

$$f_\Lambda(\lambda) = \mu e^{-\mu\lambda}, \quad \mu > 0$$

$$\begin{aligned} \text{Now, } P(X=x) &= P(X=x \cap \Lambda = \forall\lambda) \\ &= \int_\lambda P(X=x \cap \Lambda=\lambda)\, d\lambda \\ &= \int_{\lambda=0}^{\infty} P(X=x \mid \Lambda=\lambda)\cdot f_\Lambda(\lambda)\, d\lambda \\ &= \int_{\lambda=0}^{\infty}\frac{e^{-\lambda}\cdot\lambda^x}{x!}\cdot f_\Lambda(\lambda)\, d\lambda, \quad x \geq 0 \end{aligned}$$

$$= \int_{\lambda=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \cdot \mu e^{-\mu\lambda} \, d\lambda$$

$$= \frac{\mu}{x!} \cdot \int_{0}^{\infty} e^{-\lambda} \cdot \lambda^{x} \cdot e^{-\mu\lambda} \, d\lambda$$

$$= \frac{\mu}{x!} \int_{0}^{\infty} \lambda^{x} \cdot e^{-\lambda(\mu+1)} \, d\lambda$$

$$= \frac{\mu}{x!} \cdot \frac{x!}{(1+\mu)^{x+1}}$$

$$= \frac{1}{(1+\mu)^{x}} \cdot \frac{\mu}{\mu+1}$$

$$\therefore P(X=x) = \left(\frac{1}{1+\mu}\right)^{x} \cdot \left(1 - \frac{1}{1+\mu}\right)$$

$$\therefore X \sim \mathrm{Geo}\left(\frac{1}{1+\mu}\right)$$

27/9

$\rightarrow Z = X + Y$

$X, Y, Z \rightarrow$ discrete RV

if X and Y are dependent :

$$P_Z(z) = \sum_{\substack{x,y \\ (x+y=z)}} P_{XY}(x,y)$$

$X, Y, Z \rightarrow$ continuous RV

if X and Y are dependent :

$$f_Z(z) = \int_{\substack{x,y \\ (x+y=z)}} f_{XY}(x,y)\, dx\, dy$$

If X and Y are independent,

$$P_Z(z) = \sum_{\substack{x,y \\ (x+y=z)}} P_X(x) \cdot P_Y(y)$$

$$= \sum_{x} P_X(x) \cdot P_Y(z-x)$$

When X and Y are dependent,

$$P_Z(z) = \sum_{\substack{x,y \\ (x+y=z)}} P_{XY}(x,y)$$

$$= \sum_{\substack{x,y \\ (x+y=z)}} P_X(x) \cdot P_{Y|X}(y|x)$$

$$= \sum_{x} P_X(x) \cdot P_{Y|X}(z-x|x)$$

If X and Y are independent,

$$f_Z(z) = \int_{\substack{x,y \\ (x+y=z)}} f_X(x) \cdot f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

When X and Y are dependent,

$$f_Z(z) = \int_{\substack{x,y \\ (x+y=z)}} f_{XY}(x,y)\, dx\, dy$$

$$= \int_{\substack{x,y \\ (x+y=z)}} f_X(x) \cdot f_{Y|X}(y|x)\, dx \cdot dy$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_{Y|X}(z-x|x)\, dx$$

Refer : Convolution of box signals with itself

✓ H.W : X and Y $\rightarrow$ independent & Uniform $[0,1]$

Find PDF and CDF of $Z = X + Y$

Sol:
$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

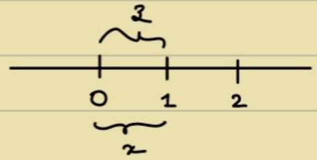
$$f_Y(y) = \begin{cases} 1, & y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$
$= 1$, if $x \in [0,1]$ and $y \in [0,1]$
$= 0$, otherwise

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx \qquad z \in [0,2]$$

$$= \int_{0}^{1} 1 \cdot f_Y(z-x)\, dx$$

If $z \in [0,1]$:



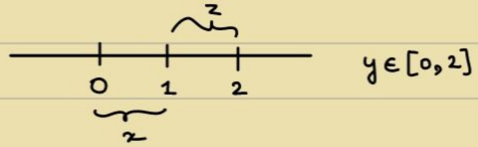
$y \in [0,1]$

$0 \le z - x \le 1$ (from diagram)

But $0 \le z - x \le 1$ (from $y = z - x$)

$$\Rightarrow \int_0^1 1 \, dx = (x]_0^1 = 1$$

If $z \in [1,2]$:



$\therefore \; 0 \leq z - x \leq 2$

But $z - x \leq 1 \Rightarrow x \geq z - 1$

$$\Rightarrow \int_0^{z-1} f_Y(z-x) \, dx + \int_{z-1}^{1} f_Y(z-x) \, dx$$

$$\Rightarrow \int_{z-1}^{1} f_Y(z-x) \, dx$$

$$\Rightarrow (x]_{z-1}^{1} \Rightarrow 1 - (z-1) = 2 - z$$

$$f_Z(z) = \begin{cases} 1, & \text{if } z \in [0,1] \\ 2-z, & \text{if } z \in [1,2] \\ 0, & \text{otherwise} \end{cases}$$

$$F_Z(z) = \int_{-\infty}^{z} f_Z(z) \, dz$$

If $z < 0$: $F_Z(z) = 0$

If $z \in [0,1]$:

$$F_Z(z) = \int_{-\infty}^{0} f_Z(z) \, dz + \int_0^z f_Z(z) \, dz$$

$$= \int_0^z 1 \, dz = z$$

If $z \in [1,2]$:

$$F_Z(z) = \int_{-\infty}^{0} f_Z(z) \, dz + \int_0^1 f_Z(z) \, dz + \int_1^z f_Z(z) \, dz$$

$$= 1 + (2z - z^2/2]_1^z$$

$$= 1 + 2z - z^2/2 - 2 + 1/2$$

$$= -z^2/2 + 2z - 1/2$$

If $z > 2$:

$$F_Z(z) = \int_{-\infty}^{0} f_Z(z) \, dz + \int_0^1 f_Z(z) \, dz + \int_1^2 f_Z(z) \, dz + \int_2^z f_Z(z) \, dz$$

$$= 1 + (2z - z^2/2]_1^2$$

$$= 1 + 4 - 2 - 2 + \tfrac{1}{2}$$
$$= \tfrac{3}{2}$$

$$\therefore F_Z(z) = \begin{cases} 0, & \text{if } z < 0 \\ z, & \text{if } z \in [0,1] \\ -\frac{z^2}{2} + 2z - \frac{1}{2}, & \text{if } z \in [1,2] \\ \frac{3}{2}, & \text{if } z > 2 \end{cases}$$

H.W: X and $Y \rightarrow$ outcomes of independent rolls of dice
$Z = X + Y$, find PDF and CDF of Z.

Sol:

$$P(X = x) = \tfrac{1}{6} \quad \forall x \in [1,6],\ i \in \mathbb{N}$$
$$P(Y = y) = \tfrac{1}{6} \quad \forall y \in [1,6],\ i \in \mathbb{N}$$

$$P_Z(z) = \sum_{\substack{x,y \\ x+y=z}} P_X(x) \cdot P_Y(y)$$
$$= \sum_x P_X(x) \cdot P_X(z-x)$$
$$= P_X(1) \cdot P_Y(z-1) + P_X(2) \cdot P_Y(z-2) + \ldots + P_X(6) \cdot P_Y(z-6)$$
$$= \tfrac{1}{6}\left(P_Y(z-1) + P_Y(z-2) + \ldots + P_Y(z-6)\right)$$

$$\left.\begin{array}{l} 1 \leq x \leq 6 \\ 1 \leq y \leq 6 \end{array}\right\} \Rightarrow 2 \leq z \leq 12$$

If $z = 2$: $P_Z(z) = \frac{1}{6}(P_Y(1)) = \frac{1}{36}$
If $z = 3$: $P_Z(z) = \frac{1}{6}(P_Y(2) + P_Y(1)) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6}) = \frac{1}{18}$
If $z = 4$: $P_Z(z) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6} + \frac{1}{6}) = \frac{1}{12}$
If $z = 5$: $P_Z(z) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}) = \frac{1}{9}$
If $z = 6$: $P_Z(z) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}) = \frac{5}{36}$
If $z = 7$: $P_Z(z) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}) = \frac{1}{6}$
If $z = 8$: $P_Z(z) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}) = \frac{5}{36}$
If $z = 9$: $P_Z(z) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}) = \frac{1}{9}$
If $z = 10$: $P_Z(z) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6} + \frac{1}{6}) = \frac{1}{12}$
If $z = 11$: $P_Z(z) = \frac{1}{6}(\frac{1}{6} + \frac{1}{6}) = \frac{1}{18}$
If $z = 12$: $P_Z(z) = \frac{1}{6}(\frac{1}{6}) = \frac{1}{36}$

$$P_2(z) = \begin{cases} 1/6, & \text{if } z = 7 \\ 5/36, & \text{if } z = 6 \text{ or } 8 \\ 1/9, & \text{if } z = 5 \text{ or } 9 \\ 1/12, & \text{if } z = 4 \text{ or } 10 \\ 1/18, & \text{if } z = 3 \text{ or } 11 \\ 1/36, & \text{if } z = 2 \text{ or } z = 12 \\ 0, & \text{otherwise} \end{cases}$$

$$F_2(z) = \sum_{i=2}^{z} f_z(z)$$

If $z = 2 \Rightarrow F_2(z) = 1/36$

If $z = 3 \Rightarrow F_2(z) = 1/36 + 1/18 = 1/12$

If $z = 4 \Rightarrow F_2(z) = 1/12 + 1/12 = 1/6$

If $z = 5 \Rightarrow F_2(z) = 1/6 + 1/9 = 5/18$

If $z = 6 \Rightarrow F_2(z) = 5/18 + 5/36 = 5/12$

If $z = 7 \Rightarrow F_2(z) = 15/36 + 1/6 = 7/12$

If $z = 8 \Rightarrow F_2(z) = 7/12 + 5/36 = 13/18$

If $z = 9 \Rightarrow F_2(z) = 13/18 + 1/9 = 5/6$

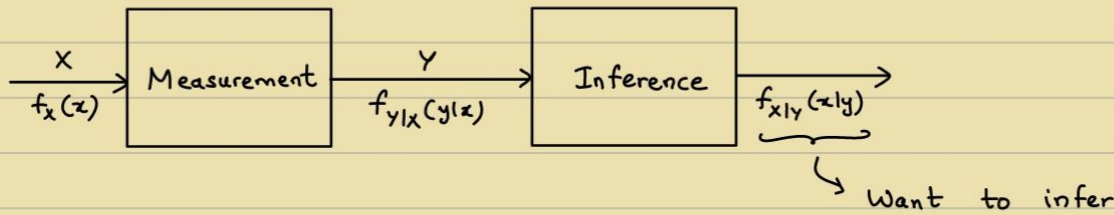
If $z = 10 \Rightarrow F_2(z) = 5/6 + 1/12 = 11/12$

If $z = 11 \Rightarrow F_2(z) = 11/12 + 1/18 = 35/36$

If $z = 12 \Rightarrow F_2(z) = 35/36 + 1/36 = 1$

$$F_2(z) = \begin{cases} 1/36, & \text{If } z = 2 \\ 1/12, & \text{If } z = 3 \\ 1/6, & \text{If } z = 4 \\ 5/18, & \text{If } z = 5 \\ 5/12, & \text{If } z = 6 \\ 7/12, & \text{If } z = 7 \\ 13/18, & \text{If } z = 8 \\ 5/6, & \text{If } z = 9 \\ 11/12, & \text{If } z = 10 \\ 35/36, & \text{If } z = 11 \\ 1, & \text{If } z = 12 \\ 0, & \text{Otherwise} \end{cases}$$

→ **Inference Problem** :



$X : f_X(x) \longrightarrow$ Not observable but of interest } ⟶ Can be calculation

$Y = y \longrightarrow$ Observable

$f_{Y|X}(y|x) \longrightarrow$ Given

$f_{X|Y}(x|y) \longrightarrow$ Need to find

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y|x)\cdot f_X(x)}{\underbrace{f_Y(y)}_{\int_x f_{Y|X}(y|t)\cdot f_X(t)\, dt}}$$

If Discrete,

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)} = \frac{P_{Y|X}(y|x)\cdot P_X(x)}{P_Y(y)} = \frac{P_{Y|X}(y|x)\cdot P_X(x)}{\sum_i P_{Y|X}(y|i)\cdot P_X(i)}$$

Ex.

$Y \rightsquigarrow Exp(\Lambda)$

$\Lambda \longrightarrow U[1, 1.5]$

$$f_\Lambda(\lambda) = \begin{cases} 2, & \lambda \in [1, 1.5] \\ 0, & \text{otherwise} \end{cases}$$

$f_Y(y) = \Lambda e^{-\Lambda y}$, $y > 0$, $\Lambda \geq 0$

$f_Y(y \mid \Lambda = \lambda) = \lambda e^{-\lambda y}$

$$\Rightarrow f_Y(y) = \int_{\forall \Lambda} f(y \mid \Lambda = \lambda)\cdot f_\Lambda(\lambda)\cdot d\lambda$$

$$= \int_\lambda f_{Y|\Lambda}(y|\lambda)\cdot f_\Lambda(\lambda)\cdot d\lambda$$

$$f_{\Lambda|Y}(\lambda|y) = \frac{f_{Y|\Lambda}(y|\lambda)\cdot f_\Lambda(\lambda)}{f_Y(y)}$$

$$= \int_1^{1.5} \lambda e^{-\lambda y}\cdot (2)\, d\lambda$$

$$= 2\left(\frac{-\lambda}{y} e^{-\lambda y} - \frac{1}{y^2} e^{-\lambda y} \right]_1^{1.5}$$

$$\left(\begin{array}{ccc} & D & I \\ + & \lambda & e^{-y\lambda} \\ - & 1 & -\frac{1}{y} e^{-\lambda y} \\ + & 0 & \frac{1}{y^2} e^{-\lambda y} \end{array}\right)$$

$$= 2\left(\frac{-\lambda}{y} e^{-\lambda y} - \frac{1}{y^2} e^{-\lambda y}\right]_1^{1.5}$$

$$= 2\left[\left(\frac{-1.5}{y} e^{-1.5y} - \frac{1}{y^2} e^{-1.5y}\right) - \left(\frac{-1}{y} e^{-y} - e^{-y}\right)\right]$$

$$\therefore f_Y(y) = 2\left(\frac{-3}{2y} e^{-3y/2} - \frac{1}{y^2} e^{-3y/2} + e^{-y}/y + e^{-y}\right)$$

<u>Note</u> :

$$P(N=n \mid Y=y) = \frac{f_{Y|N}(y|n) \cdot P_N(n)}{\underbrace{f_Y(y)}_{\sum_t f_{Y|N}(y|t) \cdot P_N(t)}} = \frac{f_{Y|N}(y|n) \cdot P_N(n)}{\sum_t f_{Y|N}(y|t) \cdot P_N(t)}$$

$$f(N=n \mid Y=y) = \frac{P(N=n|Y=y) \cdot f_Y(y)}{\underbrace{P_N(n)}_{\int_t P(N=n|Y=t) \cdot f_Y(t)\, dt}} = \frac{P(N=n|Y=y) \cdot f_Y(y)}{\int_{-\infty}^{\infty} P(N=n|Y=t) f_Y(t)\, dt}$$

<u>H.W</u>

Q) $X = 1$ with probability p
$X = -1$ with probability $1-p$ $\Big\}$ $P_X(x) = p^{\frac{x+1}{2}} \cdot (1-p)^{\frac{-x+1}{2}}$

$N \sim N(0,1)$

$Y = X + N$, Observe $Y = y$

Prove that

$$P(X=1 \mid Y=y) = \frac{pe^{y}}{pe^{y} + (1-p)e^{-y}}$$

As $y \to -\infty$, Prob. $\to 0$
$y \to \infty$, Prob. $\to 1$

<u>Sol</u> :

$$P(X=1 \mid Y=y) = \frac{f_{Y|X}(y|1) \cdot P_X(1)}{\underbrace{f_Y(y)}_{f_{Y|X}(y|1) \cdot P_X(1) + f_{Y|X}(y|-1) \cdot P_X(-1)}} = \frac{(p) \cdot f_{Y|X}(y|1)}{(p) \cdot f_{Y|X}(y|1) + (1-p) \cdot f_{Y|X}(y|-1)}$$

$Y = N + X$

$$\left[f_Y(y) = \int_x P_X(x) \cdot f_N(y-x)\, dx\right] \leftarrow \text{Intuition}$$

$$\therefore f_{Y|X}(y|X=x) = P_X(x) \cdot f_N(y-x) = p^{(1+x)/2} \cdot (1-p)^{(1-x)/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}$$

$$\therefore P(X=1 \mid Y=y) = \frac{(p) \cdot (p) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}}{(p) \cdot (p) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}} + (1-p)(1-p) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}}}$$

$$= \frac{p \cdot e^{\frac{-(y-1)^2}{2}}}{p \cdot e^{\frac{-(y-1)^2}{2}} + (1-p) \cdot e^{\frac{-(y+1)^2}{2}}} = \frac{p}{p + (1-p) \cdot e^{\frac{-(y+1)^2}{2} - \frac{-(y-1)^2}{2}}}$$

→ <u>Variance of sum of R.V</u>:

$$S_n = X_1 + X_2 + X_3 + \dots + X_n$$

$$Var[S_n] = ?$$

$$Var[S_n] = E[(X_1+X_2+\dots+X_n)^2] - (E[X_1+X_2+\dots+X_n])^2$$

$$= E[(X_1+X_2+\dots+X_n)^2] - (E[X_1]+E[X_2]+\dots+E[X_n])^2$$

$$= E\left[\sum_1^n X_i^2 + 2\sum_1^n\sum_{i+1}^n X_iX_j\right] - \left(\sum_1^n E[X_i]\right)^2$$

$$= E\left[\sum_1^n X_i^2\right] + 2\sum_1^n\sum_{i+1}^n E[X_iX_j] - \left(\sum_1^n E[X_i]\right)^2$$

$$= \sum_1^n E[X_i^2] + 2\cdot\sum_1^n\sum_{i+1}^n E[X_iX_j] - \sum_1^n (E[X_i])^2 - 2\sum_1^n\sum_{i+1}^n E[X_i]\cdot E[X_j]$$

$$= \sum_1^n E[X_i^2] - (E[X_i])^2 + \sum_1^n\sum_{i+1}^n E[X_iX_j] - E[X_i]\cdot E[X_j]$$

$$Var[S_n] = \sum_1^n Var[X_i] + 2\sum_1^n\sum_{i+1}^n cov(X_i, X_j)$$

If X_i's are independent $\Rightarrow Var[S_n] = \sum_{i=1}^n Var[X_i]$

- $cov(X, Y) = E[XY] - E[X]\cdot E[Y]$

Properties : (Prove : H.W)

(1) $cov(X, X) = Var[X] = E[E-E[X]]^2$

(2) If X and Y are independent, $cov(X, Y) = 0$

(3) $cov(X, Y) = cov(Y, X)$

(4) $cov(aX, Y) = a\cdot cov(X, Y)$

(5) $cov(X+a, Y) = a\cdot cov(X, Y)$

(6) $cov(X+Z, Y) = cov(X, Y) + cov(Z, Y)$

(7) $cov\left(\sum_{i=1}^m a_iX_i, \sum_{j=1}^n b_jY_j\right) = \sum_{i=1}^m\sum_{j=1}^n a_ib_j\, cov(X_i, Y_j)$

Ex. $Z = \sum a_i X_i$

$$var[Z] = cov(Z, Z)$$

$$= cov\left(\sum_{i=1}^n a_iX_i, \sum_{j=1}^n a_jX_j\right)$$

$$= \sum_{i=1}^m\sum_{j=1}^n a_ia_j\, cov(X_i, X_j)$$

$$= \sum_{i=1}^n a_i^2\cdot Var[X_i] + \sum_{\substack{i,j\\(i\neq j)}} a_ia_j\cdot cov(X_i, X_j)$$

9) Let $\{X_i,\ i=1,2,\dots,n\}$ be i.i.d

$$S_n = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$Var[Z] = \sum_{i=1}^{n} Var[X_i] = n \cdot Var[X]$$

$$Var[S_n] = Var\left[\frac{Z}{n}\right] = \frac{1}{n^2} Var[Z]$$
$$= \frac{1}{n^2} \cdot n \cdot Var[X]$$
$$= \frac{1}{n} \cdot Var[X]$$

$$E[S_n] = E[X]$$
$$Var[S_n] = \frac{1}{n} \cdot Var[X]$$

— Properties :

(1) $cov(X,Y) = E[XY] - E[X] \cdot E[Y]$

$cov(X,X) = E[X^2] - (E[X])^2$
$= Var[X]$

(2) Independence $\Rightarrow E[XY] = E[X] \cdot E[Y]$

$cov(X,Y) = E[XY] - E[X]\,E[Y]$
$= E[X]\,E[Y] - E[X] \cdot E[Y] = 0$

(3) $cov(X,Y) = E[XY] - E[X] \cdot E[Y]$
$= E[YX] - E[Y] \cdot E[X]$
$= cov(Y,X)$

(4) $cov(aX, Y) = E[(aX)Y] - E[aX] \cdot E[Y]$
$= E[aXY] - a \cdot E[X] \cdot E[Y]$
$= a\{E[XY] - E[X] \cdot E[Y]\}$
$= a \cdot cov(X,Y)$

(5) $cov(X+a, Y) = E[(X+a)Y] - E[X+a] \cdot E[Y]$
$= E[XY + aY] - (E[X] + a)\,E[Y]$
$= E[XY] + aE[Y] - E[X] \cdot E[Y] - aE[Y]$
$= a\{E[XY] - E[X] \cdot E[Y]\}$
$= a \cdot cov(X,Y)$

$$\begin{aligned}(6)\ \operatorname{cov}(X+Z, Y) &= E[(X+Z)Y] - E[X+Z]\cdot E[Y] \\ &= E[XY + ZY] - (E[X] + E[Z])(E[Y]) \\ &= E[XY] + E[ZY] - E[X]\cdot E[Y] - E[Z]\cdot E[Y] \\ &= (E[XY] - E[X]\cdot E[Y]) + (E[ZY] - E[Z]\cdot E[Y]) \\ &= \operatorname{cov}(X+Y) + \operatorname{cov}(Z+Y)\end{aligned}$$

Similarly, $\operatorname{cov}(X, Y+Z) = \operatorname{cov}(X, Y) + \operatorname{cov}(X, Z)$

$$(7)\ \operatorname{cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \operatorname{cov}(a_1X_1 + a_2X_2 + \dots + a_mX_m, b_1Y_1 + b_2Y_2 + \dots + b_nY_n)$$

[from property -6]

$$\begin{aligned} &= \operatorname{cov}(a_1X_1, b_1Y_1) + \operatorname{cov}(a_1X_1 + b_2Y_2) + \dots + \operatorname{cov}(a_1X_1 + b_nY_n) \\ &\quad + \operatorname{cov}(a_2X_2, b_1Y_1) + \operatorname{cov}(a_2X_2, b_2Y_2) + \dots + \operatorname{cov}(a_2X_2, b_nY_n) \\ &\quad + \dots \\ &\quad + \operatorname{cov}(a_mX_m, b_1Y_1) + \dots + \operatorname{cov}(a_mX_m, b_nY_n) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \operatorname{cov}(X_i, Y_j)\end{aligned}$$