Towards a formal definition of $\mathbb{P}$

Probability measure $\mathbb{P}$ can be defined as a set-function $\mathbb{P} : \mathcal{P}(\Omega) \to [0, 1]$ that satisfies the following 3 axioms.
Axiom 1: $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$
Axiom 2: For a set $A \subseteq \Omega$ we have $0 \leq \mathbb{P}(A) \leq 1$.
Axiom 3: For a disjoint collection of events $A_1, A_2, \ldots$ (where $A_i \subseteq \Omega$)

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

- Is there a perceivable problem with this definition?
- The following counter-example will construct a set-function $\mathbb{P}$ for which you cannot assign valid probabilities to every subsets in Ω without violating these axioms.

Counter-example

- Random exp: Pick a number uniformly from the real line.
- $\Omega = \mathbb{R}$ and hence $\mathbb{P}(\mathbb{R}) = 1$.
- Domain $\mathcal{P}(\mathbb{R})$ which is unimaginably complex!
- We have $\mathbb{P} : \mathcal{P}(\mathbb{R}) \to [0, 1]$.
- $\mathbb{P}$ has the property that sets of equal 'length' have equal probability.
- We know that $\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n+1)$ where $[n, n+1) \in \mathcal{P}(\mathbb{R})$.
- What is $\mathbb{P}[n, n+1)$?
- If we define $\mathbb{P}[n, n+1) = x$ for all $n \in \mathbb{Z}$ then $\mathbb{P}(\mathbb{R}) = \infty$!
- If we define $\mathbb{P}[n, n+1) = 0$ for all $n \in \mathbb{Z}$ then $\mathbb{P}(\mathbb{R}) = 0$!

Counter-example

- What is the takeaway from the counterexample?
- Not all set-functions (or measures) can be calibrated to measure every possible subset of your sample space.
- This is like you weighing scale at home, that is not able to weigh a piece of paper!
- What is the way out?
- Restrict your domain to only measurable sets.
- Possible domain for the counter example?
- $\mathcal{F} = \{\Phi, \Omega, \mathbb{R}_-, \mathbb{R}_+\}$. There can be many other domains one can define!

Towards sigma-algebra

- $\mathcal{F} = \{\emptyset, \Omega, \mathbb{R}_-, \mathbb{R}_+\}$.
- The domain $\mathcal{F}$ should have some nice and obvious properties.
- For example, $\emptyset$ and Ω in $\mathcal{F}$. . Also if $B \in \mathcal{F}$, then $B^c \in \mathcal{F}$.
- If A_1 and A_2 belong in $\mathcal{F}$, then $A_1 \cup A_2 \in \mathcal{F}$ and $A_1 \cap A_2 \in \mathcal{F}$.
- A domain with such nice properties is called as a *sigma-algebra*.

sigma-algebra as domain for $\mathbb{P}$

- Event space or *sigma-algebra* $\mathcal{F}$ associated with a set Ω is a collection of subsets of Ω that satisfy
  - $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$
  - $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
  - $A_1, A_2, \ldots A_n, \ldots \in \mathcal{F} \implies \cup_{n=1}^{\infty} A_n \in \mathcal{F}$

- The $\sigma-$algebra is said to be closed under formation of complements and countable unions.

- Is it also closed under the formation of countable intersections?

- When Ω is countable and finite, is $\mathcal{P}(\Omega)$ a sigma-algebra? Yes.

> When Ω is countable and finite, we will consider power-set $\mathcal{P}(\Omega)$ as the domain.

Formal definition of Probability measure $\mathbb{P}$

Definition

A probability measure $\mathbb{P}$ on the *measurable space* $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ s.t.

1. $\mathbb{P}(\emptyset) = 0, \quad \mathbb{P}(\Omega) = 1$
2. For a disjoint collection of event sets $A_1, A_2, \ldots$ from $\mathcal{F}$ we have
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$
(countable additivity)

- The trio $(\Omega, \mathcal{F}, \mathbb{P})$ is called as a probability space.
- Recall that when $|\Omega| < \infty$, we consider $\mathcal{F} = 2^{\Omega}$.

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- The trio $(\Omega, \mathcal{F}, \mathbb{P})$ is called as a probability space.
- Identify the probability space in the coin and dice experiment.

Probability space for $U[0,1]$

- $\Omega = [0,1]$.
- Suppose $\mathcal{F} = \{\Phi, [0,1], [0,.5), [.5,1]\}$. Is there a problem in using this as a sigma-algebra?
- We cannot measure probability of sets like $[.25,.75]$ although we know $P([.25,.75]) = .5$.
- So lets include $[.25,.75]$ in $\mathcal{F}$.
- Now we have $\mathcal{F}^+ = \{\Phi, [0,1], [0,.5), [.5,1], [.25,.75]\}$. Is $\mathcal{F}^+$ a sigma-algebra? No.
- Can you make it a sigma-algebra by adding missing pieces ?

Probability space for $U[0,1]$

- $\mathcal{F}^+ = \{\Phi, [0,1], [0,.5), [.5,1], [.25,.75]\}$
- Can you make it a sigma-algebra by adding missing pieces ?
- Recall that sigma-algebras are closed under complements, union and intersection.
- Intersection and union of $[.25,.75]$ with sets in $\mathcal{F}^+$ gives the collection $\{[.25,.5), [.5,.75], [.25,1], [0,0.75]\}$.
- Adding complements, the collection enlarges by $\{[.25,.5), [.5,.75], [.25,1], [0,0.75], [0,.25) \cup [.5,1], [0,.5) \cup (.75,1], [0,.25), (0.75,1]\}$.
- Lets call it $\mathcal{F}^{++} = \{\Phi, [0,1], [0,.5), [.5,1], [.25,.75], [.25,.5), [.5,.75], [.25,1], [0,0.75], [0,.25) \cup [.5,1], [0,.5) \cup (.75,1], [0,.25), (0.75,1]\}$

Probability space for $U[0,1]$

- $\mathcal{F}^{++} =$
  $\{\Phi, [0,1], [0,.5), [.5,1], [.25,.75], [.25,.5), [.5,.75], [.25,1], [0,0.75], [0,.25) \cup [.5,1], [0,.5) \cup (.75,1], [0,.25), (0.75,1]\}$
- Notice different type of sets with different brackets $[), (], ()$ that appear.
- But $\mathcal{F}^{++}$ is still not a sigma-algebra as each red set will demand a furthermore sets to be added.
- This operation we attempted is called generating a sigma-algebra!.
- Continuing on these lines, the resulting sigma algebra is called a borel-sigma algebra $\mathcal{B}[0,1]$.

Borel sigma-algebra $\mathcal{B}[0,1]$

- Borel $\sigma-$*algebra* $\mathcal{B}[0,1]$: When $\Omega = [0,1]$ the $\mathcal{B}[0,1]$ is the $\sigma-$*algebra* generated by closed sets of the form $[a,b]$ where $a \le b$ and $a, b \in [0,1]$.

- Does this set contain sets of the form (a,b) or $[a,b)$ or $(a,b]$?

- $\boxed{(a,b) = \cup_{n=1}^{\infty}[a+\frac{1}{n}, b-\frac{1}{n}].}\ \boxed{(a,b] = \cap_{n=1}^{\infty}(a, b+\frac{1}{n})}$

Borel $\sigma-$*algebra* $\mathcal{B}[0,1]$: $\mathcal{B}[0,1]$ is the $\sigma-$*algebra* generated by sets of the form $[a,b]$ or (a,b) or $(a,b]$ or even $[a,b)$ where $a \le b$ and $a, b \in [0,1]$.

Borel sigma-algebra $\mathcal{B}(\mathbb{R})$

- Borel sigma-algebra $\mathcal{B}(\mathbb{R})$:
  If $\Omega = \mathbb{R}$, then $\mathcal{B}(\mathbb{R})$ is the sigma-algebra generated by open sets of the form (a, b) where $a \leq b$ and $a, b \in \mathbb{R}$.

- $\mathcal{B}(\mathbb{R})$ contains intervals of the form

$$[a, b]$$
$$[a, b)$$
$$(a, \infty)$$
$$[a, \infty)$$
$$(-\infty, b]$$
$$(-\infty, b)$$
$$\{a\}$$

- How would you define $\mathcal{B}(\mathbb{R}^2)$?

Consequences of the Probability Axioms

- $P(A^c) = 1 - P(A)$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

- If $A \subseteq B$, prove that $P(A) \leq P(B)$. ($A \subseteq B$ has the interpretation that Event A implies event B)

- $P(\cup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} P(B_i)$ (Boole's/Bonferroni's inequality). HW

- What is $P(A \cup B \cup C)$?

- State and prove the inclusion-exclusion principle for $P(\cup_{i=1}^{n} A_i)$

Impossible event v/s Zero prob. event

- In $U[0,1]$ what is $P(\omega = 0.5)$? $= 0$.
- Intuitive reasoning for this is that a point has zero length!
- If $P(\omega \in [a,b]) = b - a$ then $P([.5,.5]) = P(\{.5\}) = 0$.
- This is a zero probability event. In fact, every outcome of this experiment is a zero probability event.
- This implies that events of zero probability can happen and they are not impossible events.
- $P(\emptyset) = 0$, then is $\emptyset$ also possible ?No!
- What is $P(\omega \in [0,.25] \cap [.75,1])$?
- $P([0,.25] \cap [.75,1]) = P(\emptyset) = 0$ This event will never happen.

Impossible event v/s Zero prob. event

- Note that in the $U[0,1]$ experiment, $\Omega = \bigcup_{\omega \in \Omega} \{\omega\}$
- $P(\Omega) = P(\bigcup_{\omega \in \Omega} \{\omega\}) = \sum_{\omega \in \Omega} P(\{\omega\}) = 0.$
- What is the problem above ?
- Ω is an uncountable set and the probability set-function only has a countable additive property.
- $\bigcup_{\omega \in \Omega} \{\omega\}$ is an uncountable disjoint union!