Probability and Statistics

Solutions to Tutorial 5

Q1: (a) Marginal distribution of X:

$$p_X(x) = \sum_{y=1}^{\infty} \frac{1}{2^{x+y}} = \frac{1}{2^x} \sum_{y=1}^{\infty} \frac{1}{2^y} = \frac{1}{2^x} \cdot \frac{1/2}{1-1/2} = \frac{1}{2^x}$$

Similarly, $p_Y(y) = \frac{1}{2^y}$ for $y = 1, 2, 3, \ldots$

Since $p_{XY}(x,y) = \frac{1}{2^x} \cdot \frac{1}{2^y} = p_X(x) \cdot p_Y(y)$, X and Y are independent.

(b) $\mathbb{P}(X^2 + Y \leq 10) = \sum_{\substack{x,y \geq 1 \\ x^2+y \leq 10}} \frac{1}{2^{x+y}}$

For $x = 1$: $y \leq 9$, so $\sum_{y=1}^{9} \frac{1}{2^{1+y}} = \frac{1}{2} \sum_{y=1}^{9} \frac{1}{2^y} = \frac{1}{2} \cdot \frac{1/2-(1/2)^{10}}{1-1/2} = 1 - \frac{1}{2^9}$

For $x = 2$: $y \leq 6$, so $\sum_{y=1}^{6} \frac{1}{2^{2+y}} = \frac{1}{4} \cdot \frac{1/2-(1/2)^7}{1/2} = \frac{1}{4} - \frac{1}{2^8}$

For $x = 3$: $y = 1$, so $\frac{1}{2^4} = \frac{1}{16}$

Therefore: $\mathbb{P}(X^2 + Y \leq 10) = 1 - \frac{1}{2^9} + \frac{1}{4} - \frac{1}{2^8} + \frac{1}{16} = \frac{3}{4} + \frac{1}{16} - \frac{3}{2^9} = \frac{3}{4} + \frac{1}{16} - \frac{3}{512} = 0.813$

Q2: We are asked to compute the covariance $\mathrm{Cov}(Z, W)$, where:

$$Z = 1 + X + XY^2, \quad W = 1 + X$$

Using the properties of covariance:

$$\begin{aligned}
\mathrm{Cov}(Z, W) &= \mathrm{Cov}(1 + X + XY^2, 1 + X) \\
&= \mathrm{Cov}(X + XY^2, X) \\
&= \mathrm{Cov}(X, X) + \mathrm{Cov}(XY^2, X) \\
&= \mathrm{Var}(X) + \mathrm{E}[X^2Y^2] - \mathrm{E}[XY^2]\mathrm{E}[X]
\end{aligned}$$

Since X and Y are independent, we have:

$$\mathrm{Var}(X) = 1, \quad \mathrm{E}[X^2] = 1, \quad \mathrm{E}[Y^2] = 1, \quad \mathrm{E}[X] = 0$$

Thus:

$$\mathrm{Cov}(Z, W) = 1 + \mathrm{E}[X^2]\mathrm{E}[Y^2] - 0 = 1 + 1 - 0 = 2$$

Therefore, the covariance is:

$$\mathrm{Cov}(Z, W) = 2$$

Q3: (1) Compute the marginals:

$$f_X(x) = \int_0^{\infty} 6e^{-(2x+3y)}\, dy = 6e^{-2x} \int_0^{\infty} e^{-3y}\, dy = 6e^{-2x} \cdot \frac{1}{3} = 2e^{-2x}, \quad x \geq 0,$$

$$f_Y(y) = \int_0^{\infty} 6e^{-(2x+3y)}\, dx = 6e^{-3y} \int_0^{\infty} e^{-2x}\, dx = 6e^{-3y} \cdot \frac{1}{2} = 3e^{-3y}, \quad y \geq 0.$$

Since

$$f_X(x)f_Y(y) = (2e^{-2x})(3e^{-3y}) = 6e^{-(2x+3y)} = f_{X,Y}(x,y) \quad \text{for } x, y \geq 0,$$

we conclude that X and Y are **independent**.

(2) Because of independence,

$$\mathbb{E}[Y \mid X > 2] = \mathbb{E}[Y].$$

Noting that $f_Y(y) = 3e^{-3y}$, i.e. $Y \sim \text{Exp}(3)$, we have

$$\mathbb{E}[Y] = \int_0^\infty y \cdot 3e^{-3y}\, dy.$$

Using the formula $\int uv\, dx = u \int v\, dx - \int (u' \int v\, dx)\, dx$, let $u = y$ and $v = 3e^{-3y}$. Then

$$\int_0^\infty y \cdot 3e^{-3y}\, dy = y \int_0^\infty 3e^{-3y}\, dy - \int_0^\infty \big(1 \cdot \int 3e^{-3y}\, dy\big)\, dy.$$

Now, $\int 3e^{-3y}\, dy = -e^{-3y}$. So,

$$\int_0^\infty y \cdot 3e^{-3y}\, dy = \Big[-ye^{-3y}\Big]_0^\infty + \int_0^\infty e^{-3y}\, dy = 0 + \tfrac{1}{3}.$$

Therefore,

$$\boxed{\mathbb{E}[Y \mid X > 2] = \tfrac{1}{3}.}$$

Q4: Let A denote the event "producer produces b items", so $\mathbb{P}(A) = p$ and $\mathbb{P}(A^c) = 1-p$. When A^c occurs the consumer gets 0. When A occurs the consumed amount is $C = \min(T, b)$, where $T \sim \text{Exp}(\lambda)$ and T is independent of A.

(1) Distribution of C.

Conditional distribution given A. Given A, $C = \min(T, b)$ has a mixed distribution: for $0 < c < b$ its density is

$$f_{C|A}(c) = f_T(c) = \lambda e^{-\lambda c}, \qquad 0 < c < b,$$

and there is an atom at b with

$$p_{C|A}(b) = \bar{F}_C(b) = e^{-\lambda b}.$$

Given A^c, $C = 0$ with probability 1.

Unconditional (marginal) distribution. Using the law of total probability,

- Probability mass at 0:

$$\mathbb{P}(\{C = 0\}) = \mathbb{P}(A^c) = 1 - p.$$

- For $0 < c < b$ the density of C is

$$f_C(c) = \mathbb{P}(A)\, f_{C|A}(c) = p\, \lambda e^{-\lambda c}, \qquad 0 < c < b.$$

- Probability mass at b:

$$\mathbb{P}(\{C = b\}) = \mathbb{P}(A) p_{C|A}(b) = p\, e^{-\lambda b}.$$

Thus C is a mixed random variable with a mass $1 - p$ at 0, a continuous density $p\lambda e^{-\lambda c}$ on $(0, b)$, and a mass $pe^{-\lambda b}$ at b.

(2) Expected value $\mathbb{E}[C]$. Since $C = 0$ on A^c and $C = \min(T, b)$ on A,

$$\mathbb{E}[C] = 0 \cdot \mathbb{P}(A^c) + \int_0^b t\lambda e^{-\lambda t} \cdot \mathbb{P}(A)\, dt + b \cdot \mathbb{P}(\{C = b\})$$

$$\int_0^b t\lambda e^{-\lambda t}\, dt = \Big[- te^{-\lambda t} \Big]_0^b + \int_0^b e^{-\lambda t}\, dt = -be^{-\lambda b} + \frac{1 - e^{-\lambda b}}{\lambda}.$$

So the final expectation is

$$\boxed{\mathbb{E}[C] = p\, \frac{1 - e^{-\lambda b}}{\lambda}.}$$

Q5: (a) The CDF of Z is defined as:

$$F_Z(z) = P(Z \leq z) = P\left(\frac{X}{Y} \leq z\right).$$

Since X and Y are independent and uniformly distributed over $(0, 1)$, we can write this as:

$$F_Z(z) = P(X \leq zY).$$

We now express the probability as an integral over the possible values of Y:

$$F_Z(z) = \int_0^1 P(X \leq zy \,|\, Y = y) f_Y(y)\, dy.$$

For $X \sim \text{Uniform}(0, 1)$, we have $P(X \leq zy) = \min(1, zy)$, so the CDF becomes:

$$F_Z(z) = \int_0^1 \min(1, zy)\, dy.$$

Now, let's evaluate the integral in two parts, based on the value of z.

- If $z \leq 1$, the integration of $\min(1, zy)$ is over $zy \leq 1$, which simplifies to zy for $y \in [0, 1]$.
- If $z > 1$, the minimum value becomes 1 for $y \in [0, 1]$, so the integral is over the entire interval.

Therefore, for $z \leq 1$:

$$F_Z(z) = \int_0^1 zy\, dy = \frac{z}{2}.$$

For $z > 1$:

$$F_Z(z) = \int_0^{1/z} zy\, dy + \int_{1/z}^1 1\, dy = \frac{1}{2z} + \left(1 - \frac{1}{z}\right).$$

Thus, the CDF of Z is:

$$F_Z(z) = \begin{cases} \frac{z}{2}, & \text{if } 0 \leq z \leq 1, \\ 1 - \frac{1}{2z}, & \text{if } z > 1. \end{cases}$$

(b) To find the PDF of Z, we differentiate the CDF:

$$f_Z(z) = \frac{d}{dz} F_Z(z).$$

For $z \leq 1$:

$$f_Z(z) = \frac{d}{dz}\left(\frac{z}{2}\right) = \frac{1}{2}.$$

For $z > 1$:

$$f_Z(z) = \frac{d}{dz}\left(1 - \frac{1}{2z}\right) = \frac{1}{2z^2}.$$

Thus, the PDF of Z is:

$$f_Z(z) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq z \leq 1, \\ \frac{1}{2z^2}, & \text{if } z > 1. \end{cases}$$

Q6: (a) The event $(X \leq 2, Y \leq 4)$ corresponds to the sum of the probabilities in the table where X is either 1 or 2, and Y is either 2 or 4.

$$\begin{aligned} \mathbb{P}(X \leq 2, Y \leq 4) &= \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 1, Y = 4) \\ &\quad + \mathbb{P}(X = 2, Y = 2) + \mathbb{P}(X = 2, Y = 4) \\ &= \frac{1}{12} + \frac{1}{24} + \frac{1}{6} + \frac{1}{12} \\ &= \frac{2}{24} + \frac{1}{24} + \frac{4}{24} + \frac{2}{24} = \frac{9}{24} = \frac{3}{8}. \end{aligned}$$

(b) To find the marginal PMFs, we sum the probabilities across the rows (for Y) and down the columns (for X).

Marginal PMF of X:

- $p_X(1) = \mathbb{P}(X = 1) = \frac{1}{12} + \frac{1}{24} + \frac{1}{24} = \frac{2+1+1}{24} = \frac{4}{24} = \frac{1}{6}$.
- $p_X(2) = \mathbb{P}(X = 2) = \frac{1}{6} + \frac{1}{12} + \frac{1}{8} = \frac{4+2+3}{24} = \frac{9}{24} = \frac{3}{8}$.
- $p_X(3) = \mathbb{P}(X = 3) = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} = \frac{6+3+2}{24} = \frac{11}{24}$.
- $p_X(x) = 0$ otherwise

Marginal PMF of Y:

- $p_Y(2) = \mathbb{P}(Y = 2) = \frac{1}{12} + \frac{1}{6} + \frac{1}{4} = \frac{1+2+3}{12} = \frac{6}{12} = \frac{1}{2}$.
- $p_Y(4) = \mathbb{P}(Y = 4) = \frac{1}{24} + \frac{1}{12} + \frac{1}{8} = \frac{1+2+3}{24} = \frac{6}{24} = \frac{1}{4}$.
- $p_Y(5) = \mathbb{P}(Y = 5) = \frac{1}{24} + \frac{1}{8} + \frac{1}{12} = \frac{1+3+2}{24} = \frac{6}{24} = \frac{1}{4}$.
- $p_Y(y) = 0$ otherwise

(c) Using the conditional probability formula: $\mathbb{P}(Y = y|X = x) = \frac{\mathbb{P}(X=x,Y=y)}{\mathbb{P}(X=x)}$.

$$\begin{aligned} \mathbb{P}(Y = 2|X = 1) &= \frac{\mathbb{P}(X = 1, Y = 2)}{\mathbb{P}(X = 1)} \\ &= \frac{1/12}{1/6} \\ &= \frac{1}{12} \cdot 6 = \frac{6}{12} = \frac{1}{2}. \end{aligned}$$

(d) For two random variables to be independent, their joint PMF must be equal to the product of their marginal PMFs for all pairs of values (x, y). That is, $\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$ for all x and y.

Let's check the first entry: $(X = 1, Y = 2)$.

- Joint PMF: $\mathbb{P}(X = 1, Y = 2) = \frac{1}{12}$.
- Marginal PMFs: $\mathbb{P}(X = 1) = \frac{1}{6}$ and $\mathbb{P}(Y = 2) = \frac{1}{2}$.
- Product of marginals: $\mathbb{P}(X = 1) \cdot \mathbb{P}(Y = 2) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$.

In this case, $\mathbb{P}(X = 1, Y = 2) = \mathbb{P}(X = 1) \cdot \mathbb{P}(Y = 2)$.

Let's check another entry, for example $(X = 2, Y = 4)$:

- Joint PMF: $\mathbb{P}(X = 2, Y = 4) = \frac{1}{12}$.
- Marginal PMFs: $\mathbb{P}(X = 2) = \frac{3}{8}$ and $\mathbb{P}(Y = 4) = \frac{1}{4}$.
- Product of marginals: $\mathbb{P}(X = 2) \cdot \mathbb{P}(Y = 4) = \frac{3}{8} \cdot \frac{1}{4} = \frac{3}{32}$.

P Here, $\frac{1}{12} \neq \frac{3}{32}$. Since the condition $\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$ does not hold for all values, the random variables X and Y are **not independent**.

Q7: The joint PDF of X and Y is

$$f_{X,Y}(x, y) = \frac{1}{4}, \quad 0 < x < 2, 0 < y < 2.$$

We want to find

$$\mathbb{P}(\{XY < 1\}) = \iint_{xy<1} f_{X,Y}(x, y)\, dx\, dy.$$

The region $xy < 1$ in the xy-plane is bounded by the curves $y = \frac{1}{x}$, $x = 0$, $y = 0$, and the lines $x = 2$ and $y = 2$. The area of integration is thus the region where $0 < x < 2$ and $0 < y < \frac{1}{x}$. Thus,

$$\mathbb{P}(\{XY < 1\}) = \int_0^2 \int_0^{1/x} \frac{1}{4}\, dy\, dx = \int_0^2 \frac{1}{4x}\, dx = \frac{1}{4} \ln(2).$$

Q8: Conditional probability $\mathbb{P}(X > Y | X < 1/2)$ is:

$$\mathbb{P}(X > Y | X < 1/2) = \frac{\mathbb{P}(X > Y, X < 1/2)}{\mathbb{P}(X < 1/2)}$$

We find the marginal probability density $f_X(x)$ to calculate the denominator. Integrating the joint probability density function $f_{X,Y}(x, y)$ with respect to y:

$$f_X(x) = \int_0^1 (x + y)\, dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2} \quad \text{for } 0 \le x \le 1$$

Now we can calculate the probability $\mathbb{P}(X < 1/2)$:

$$\mathbb{P}(X < 1/2) = \int_0^{1/2} \left(x + \frac{1}{2}\right) dx = \left[\frac{x^2}{2} + \frac{x}{2}\right]_0^{1/2} = \frac{(1/2)^2}{2} + \frac{1/2}{2} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

We calculate the joint probability $P(X > Y, X < 1/2)$ by integrating over the region where $0 \leq y < x$ and $0 \leq x < 1/2$:

$$\mathbb{P}(X > Y, X < 1/2) = \int_0^{1/2} \int_0^x (x + y)\, dy\, dx$$

Inner integral:

$$\int_0^x (x + y)\, dy = \left[xy + \frac{y^2}{2} \right]_0^x = x^2 + \frac{x^2}{2} = \frac{3x^2}{2}$$

Outer integral:

$$\int_0^{1/2} \frac{3x^2}{2}\, dx = \left[\frac{3x^3}{6} \right]_0^{1/2} = \left[\frac{x^3}{2} \right]_0^{1/2} = \frac{(1/2)^3}{2} = \frac{1/8}{2} = \frac{1}{16}$$

Finally, we can find the conditional probability:

$$\mathbb{P}(X > Y | X < 1/2) = \frac{\mathbb{P}(X > Y, X < 1/2)}{\mathbb{P}(X < 1/2)} = \frac{1/16}{3/8} = \frac{1}{16} \times \frac{8}{3} = \frac{1}{6}$$