CS 302.1 - Automata Theory

Lecture 11

Shantanav Chakraborty

Center for Quantum Science and Technology (CQST)

Center for Security, Theory and Algorithms (CSTAR)

IIIT Hyderabad

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

Quick Recap

The standard TM model is quite robust. It can simulate other seemingly “powerful” variants such as

- Lazy TM
- Multi-tape TM
- Two-way infinite tape TM
- Enumerator
- Non-deterministic TM

The set of problems that are decided by a standard TM = the set of problems decided by any of these variants

Turing Machines: some definitions

Total Turing Machines: A TM M is total if for all input strings $w \in \Sigma^*$, $\boldsymbol{M(w)}$ **accepts or rejects but never runs infinitely.**

On every input, M halts

An **Algorithm** is nothing but a Total Turing Machine.

Recursive Language/Turing Decidable/Decidable: A language L is called Recursive or Turing decidable or Decidable if there exists a Total Turing Machine M for L, i.e.

$$\left.\begin{array}{l}\forall \omega \in L, M(\omega) \text{ accepts} \\ \forall \omega \notin L, M(\omega) \text{ rejects}\end{array}\right\} \text{Halts on all inputs}$$

Total TM $M =$ On input w,
If $M(w)$ reaches an accept state, ACCEPT
If $M(w)$ reaches a reject state, REJECT

Recursively Enumerable Language/Turing Recognizable (RE): A language L is called Recursively Enumerable (RE) or Turing Recognizable if

$\forall \omega \in L, M(\omega)$ **accepts**

$\forall \omega \notin L, M(\omega)$ **doesn't accept** (rejects or runs infinitely)

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$M =$ On input w,
If $M(w)$ reaches an accept state, ACCEPT
If $M(w)$ reaches a reject state, REJECT
If $M(w)$ loops, ……….

L is in RE if $\boldsymbol{L}$ **is recognized by some Turing Machine** $\boldsymbol{M}$, i.e. $\boldsymbol{L(M) = L}$. It halts for ALL the YES instances.

All Recursive Languages are Recursively Enumerable but not vice versa.

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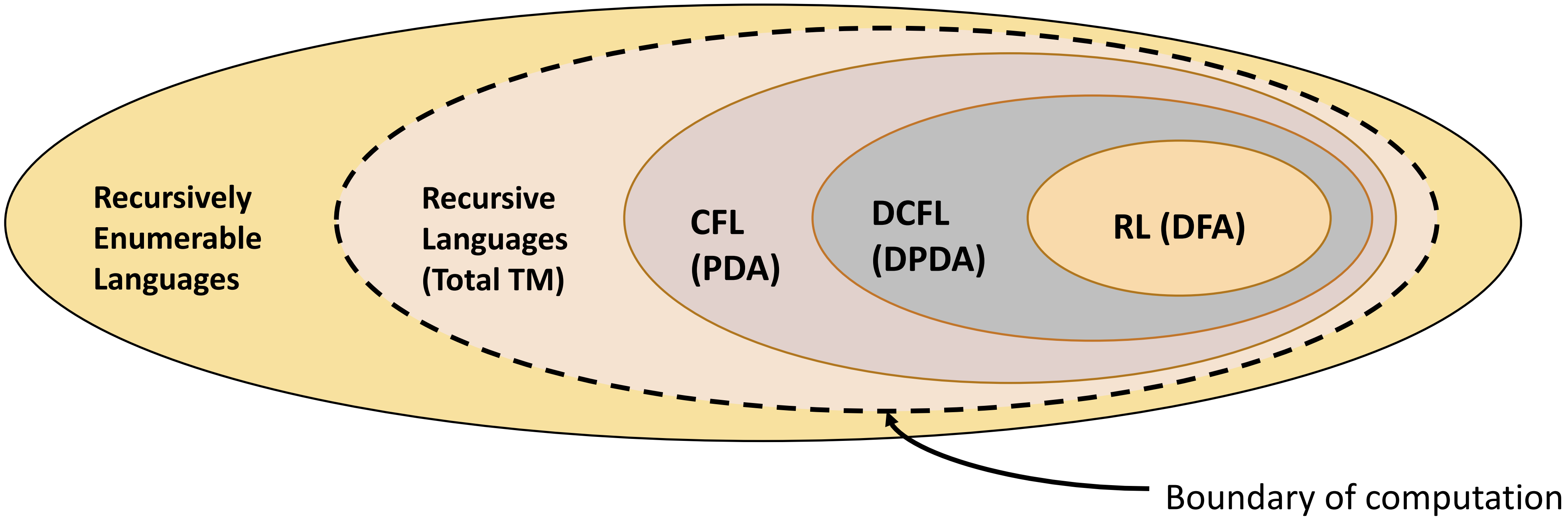
Co-Recursively Enumerable Language/co-Turing Recognizable (Co-RE/$\overline{\boldsymbol{RE}}$/nRE): A language L is **Co-Recursively Enumerable (co-RE/$\overline{\boldsymbol{RE}}$) or Co-Turing Recognizable** if

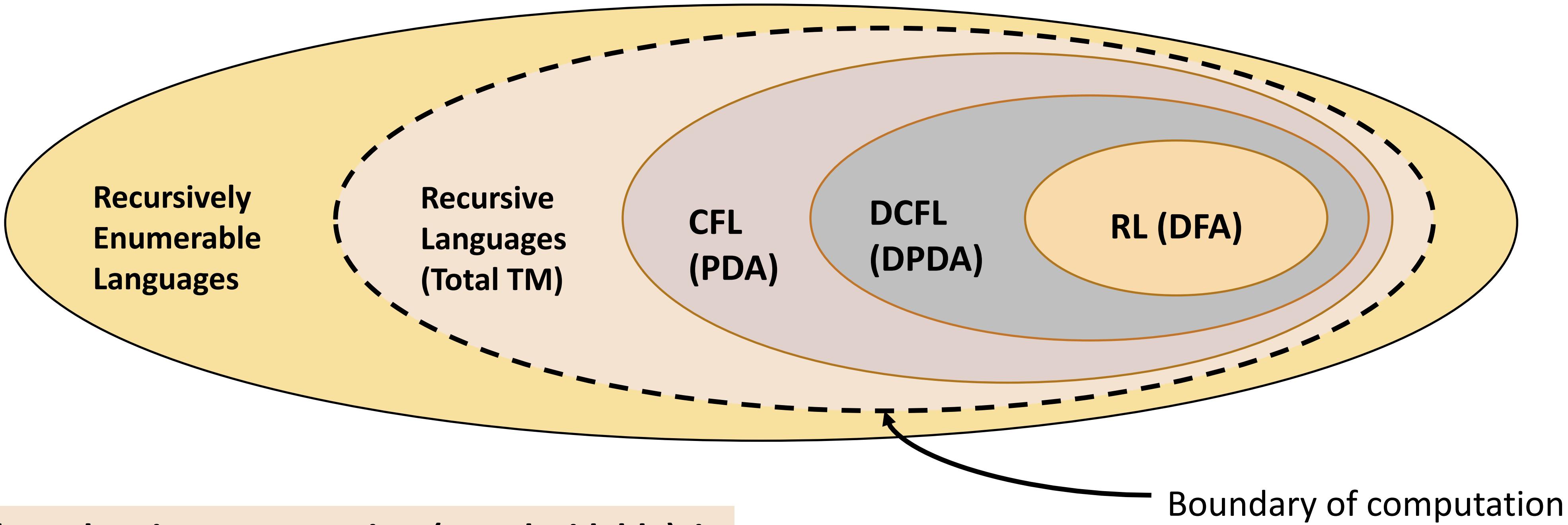
$\forall \omega \in L, M(\omega)$ **doesn't reject** (accepts or loops)

$\forall \omega \notin L, M(\omega)$ **rejects**

Hierarchy of Languages



Hierarchy of Languages



Any **problem that is not Recursive (not decidable) is called Undecidable**. There exists some input w for which the Turing Machine loops forever and hence, cannot **decide** whether or not w belongs to the Language.

We **cannot write Algorithms to decide the membership** of undecidable problems

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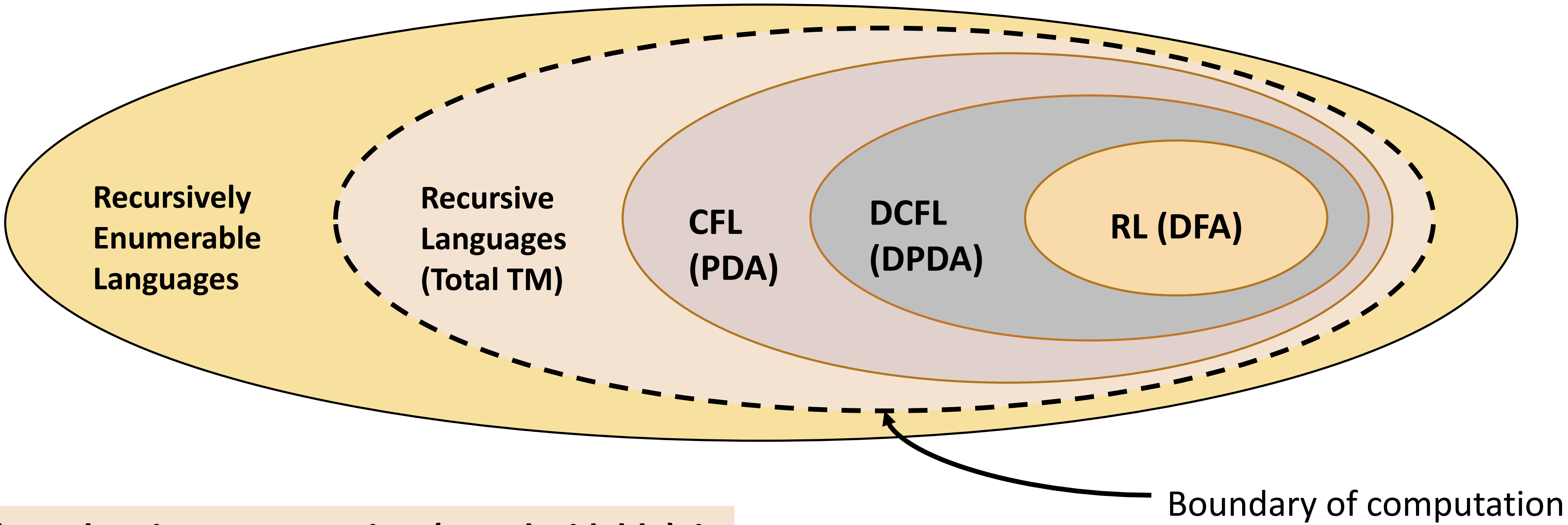

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There are **problems in $\boldsymbol{RE}$ which are not Recursive**. For such problems there exists some $\omega \notin L$, the TM **never halts but rather loops forever**. So, such problems are **undecidable**.

However, they can recognize any $\omega \in L$, so these undecidable problems are also called partially decidable.

Turing Machines: some definitions

Undecidable language: A language L is undecidable if it is not decidable/recursive. Any TM for L **will loop infinitely for some input $\boldsymbol{\omega}$**. You cannot write an algorithm to decide the membership of L.

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Undecidable languages can be of two kinds:

- **Partially decidable Language:** A language L is partially decidable if L is **Recursively Enumerable as well as Undecidable (not recursive)** (TM accepts all the YES instances and loops infinitely for at least one NO instance), i.e.

  $\forall \omega \in L, M(\omega)$ **accepts**
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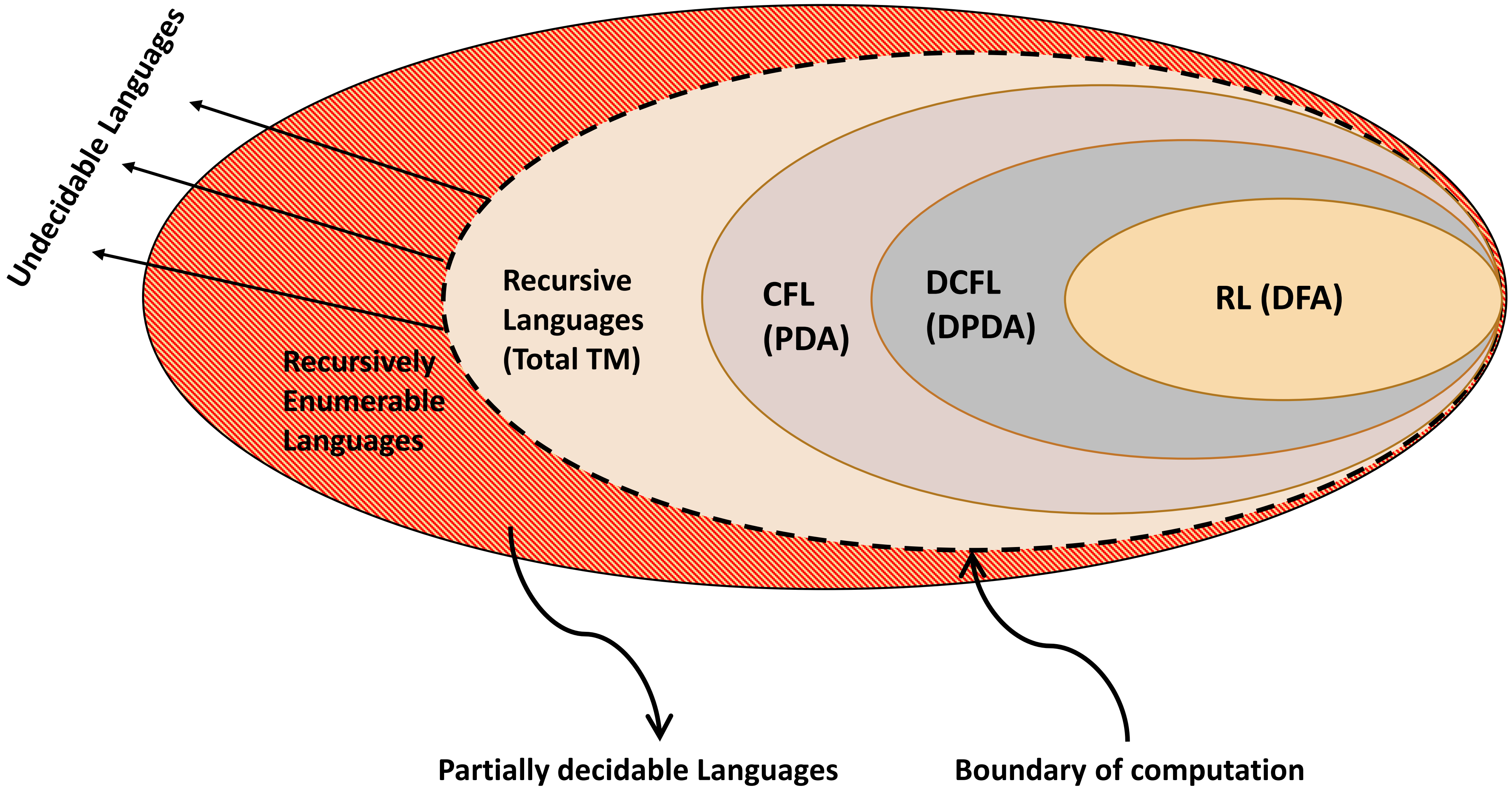
- **Completely undecidable language:** A language L is completely undecidable if $\boldsymbol{L}$ **is undecidable but not partially decidable** (TM loops infinitely for at least one YES instance), i.e.

  $\forall\omega \in L, M(\omega)$ **doesn't reject** and $\exists$ at least one instance where the program will loop forever
  $\forall\omega \notin L, M(\omega)$ **rejects/loops forever**

Hierarchy of Languages



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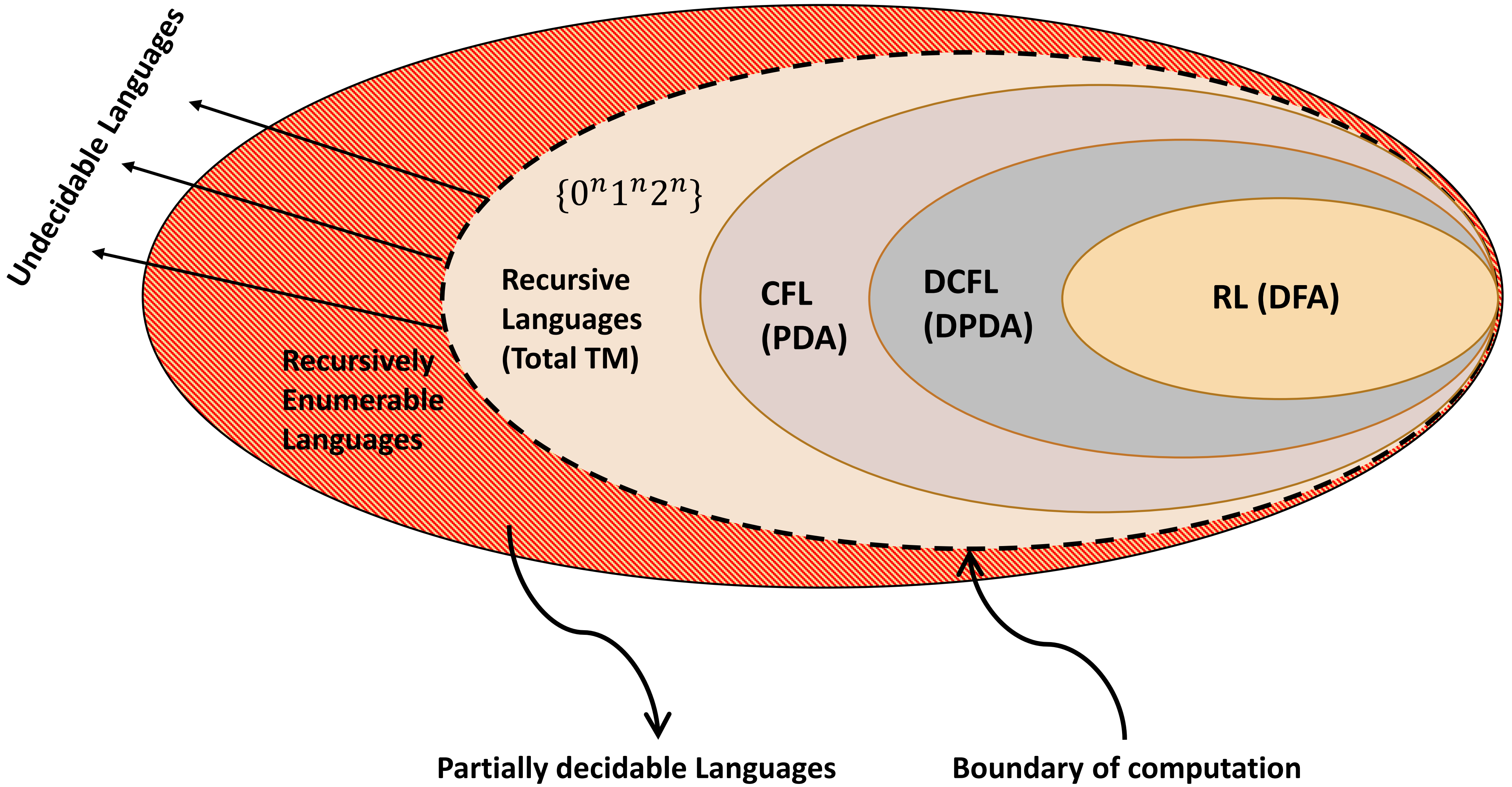

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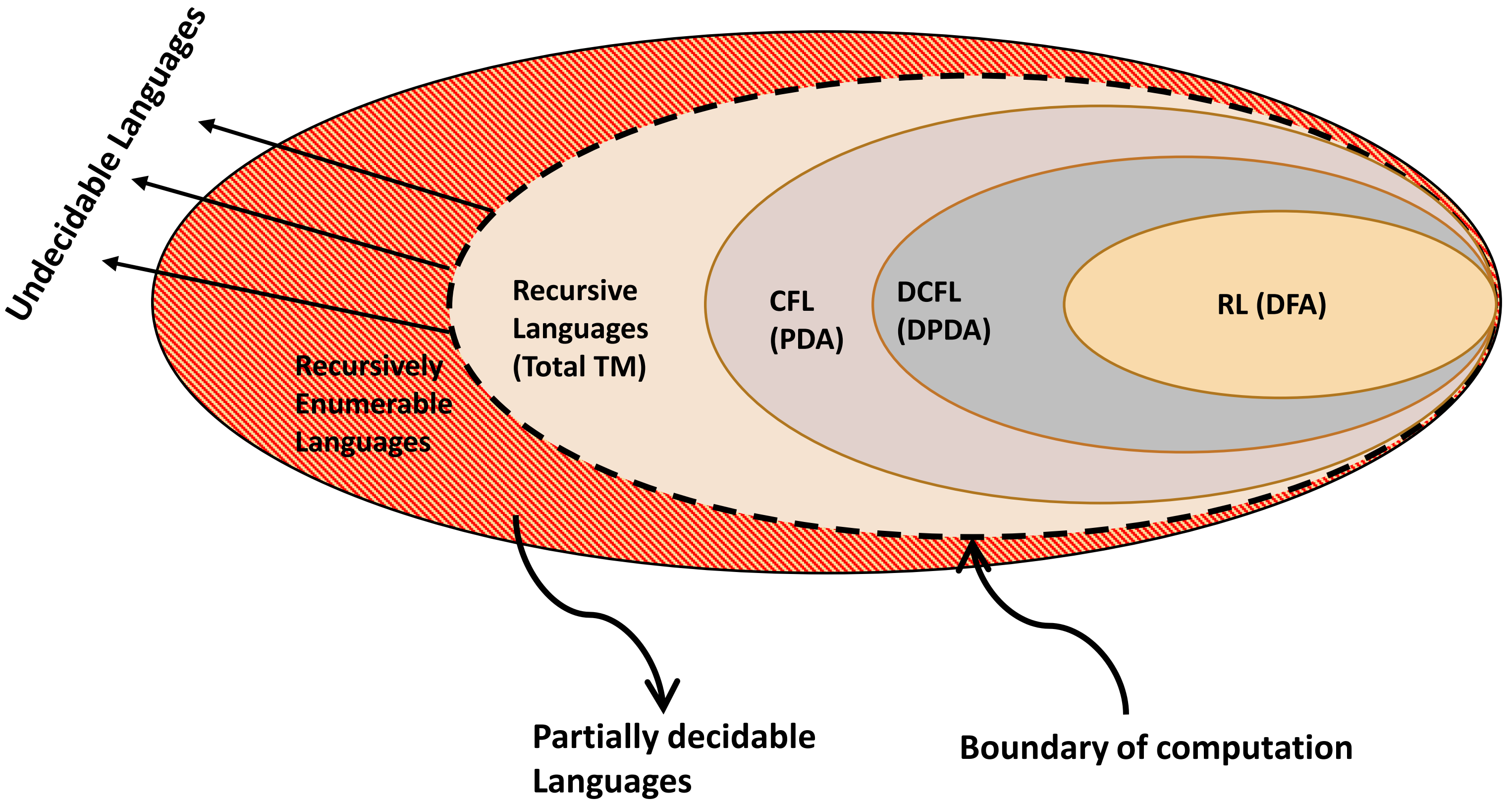
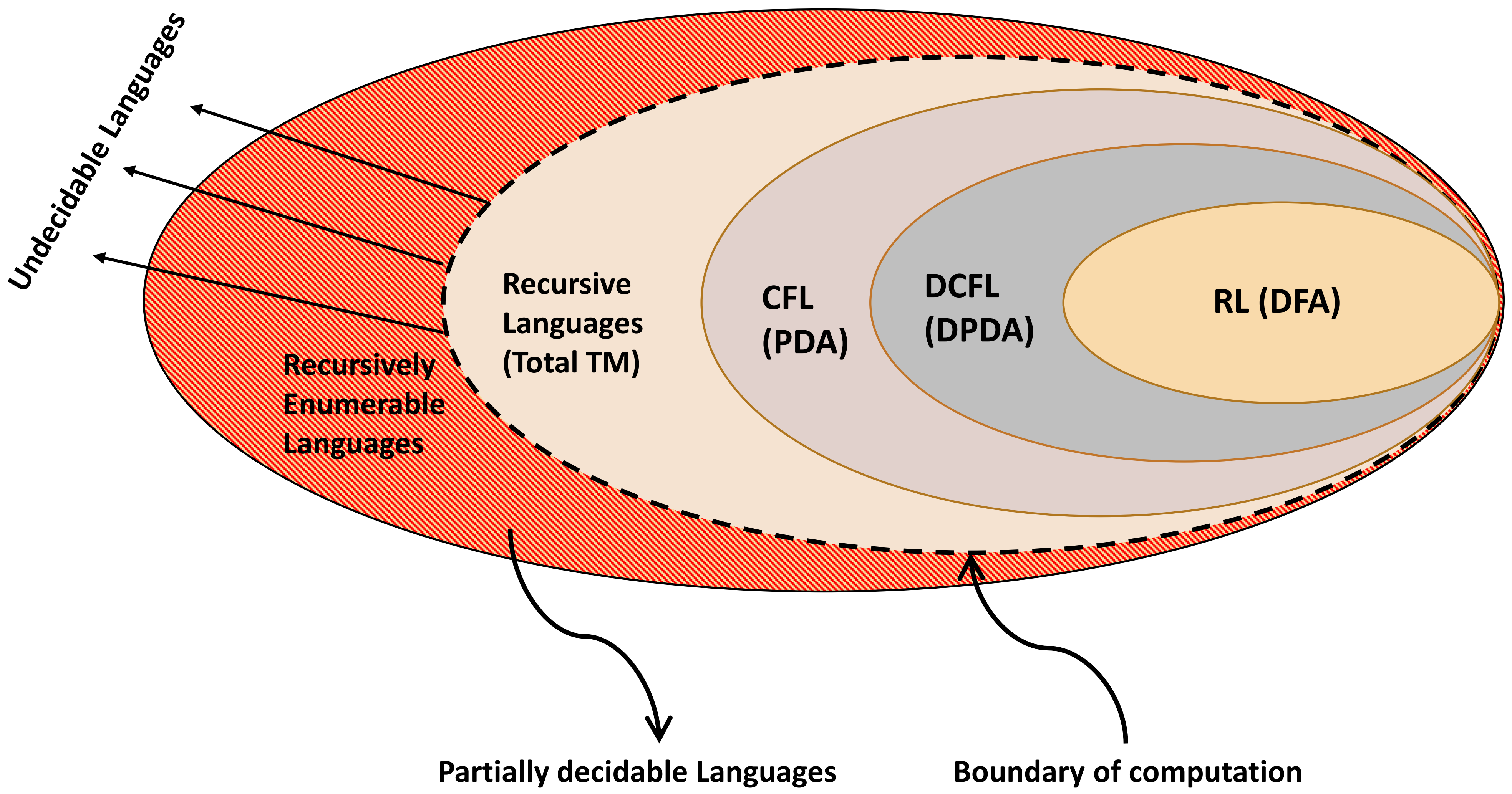


Any Language here is computable/decidable (you can write algorithms)

$$(\boldsymbol{RL \equiv DFA}) \subseteq (\boldsymbol{DCFL \equiv DPDA}) \subseteq (\boldsymbol{CFL \equiv PDA}) \subseteq (\boldsymbol{Recursive\ Lang \equiv Total\ TM}) \subseteq \boldsymbol{RE}$$

Undecidable Languages – Problems here are NOT computable

Boundary of computation



Undecidable Languages
Recursive
Languages
(Total TM)
CFL
(PDA)
DCFL
(DPDA)
RL (DFA)
Recursively
Enumerable
Languages
Partially decidable Languages
Boundary of computation

Encoding

The input to a TM are often strings/sequences of strings.

$M(w_1, w_2) =$ If w_1 is a substring of w_2, ACCEPT
Otherwise, REJECT.

Not just numbers, seemingly complicated objects such as a **graph, a DFA, a CFG and even a Turing Machine** itself can be encoded as a string – and hence can be an input to a TM.

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Consider this example:

$M(\langle M_1, w \rangle) =$ Run M_1 on input w.
If $M_1(w)$ accepts, ACCEPT
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- $\langle M_1 \rangle$ is the encoding of TM M_1 as a string.
- M simulates the run of M_1 on input w.
- Observe that M can accept a description of itself as input.

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- Encoding objects such as TMs as strings will help define a Universal Turing Machine U_{TM} which is a DTM that accepts as input the encoding of a DTM M and an input string w, and simulates $M(w)$.
- To prove that problems related to regular languages, CFLs are decidable/undecidable, we need to provide encodings of DFAs/CFGs as inputs to a TM.
- How can we encode objects as strings? We will show a simple encoding of a DTM into a binary string.

Encoding a Turing Machine

- We will provide a simple mapping from a DTM to a **binary string**.
- Of course, this is not the only encoding.
- You can come up with your own encoding.

Encoding a Turing Machine

Recall that a DTM M is a 7-tuple $(Q,\ \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$.

- Let $Q = \{q_0, \cdots, q_{m-1}\}, \Sigma = \{0,1, \cdots, k-1\}, \Gamma = \{0,1, \cdots, n-1\}$. As $\Sigma \subseteq \Gamma, \mathrm{k} < \mathrm{n}$ and without loss of generality B corresponds to the last symbol $n-1$ in Γ.

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- Any state $q_i \in Q$ can be encoded as a binary string, where

$$\langle q_0 \rangle = 0, \langle q_1 \rangle = 1, \langle q_2 \rangle = 10, \cdots$$

- Any symbol in Γ (or Σ) can be encoded as

$$\langle 0 \rangle = 0, \langle 1 \rangle = 1, \langle 2 \rangle = 10, \cdots$$

- The directions $\langle L \rangle = 0$ and $\langle R \rangle = 1$. So the transition function $\delta(q_i, a) = (q_j, b, L/R)$ is just the sequence

$$\langle \langle q_i \rangle, \langle a \rangle, \langle q_j \rangle, \langle b \rangle, \langle L/R \rangle \rangle$$

- All such transitions are listed in lexicographic order into

$$\langle \delta \rangle = \langle \langle \delta_0 \rangle, \langle \delta_1 \rangle, \langle \delta_2 \rangle, \cdots \rangle$$

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Following these encodings we can simply encode the DTM M as $\boldsymbol{\langle M \rangle = (\ \langle m \rangle, \langle k \rangle, \langle n \rangle, \langle \delta \rangle, 0/1, \langle q_{accept} \rangle, \langle q_{reject} \rangle\ )}$

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We are almost there but not quite. We have to find a way to combine this tuple of binary strings into one bigger binary string. Note that $\langle \delta \rangle$ itself is a tuple of binary strings.

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We can combine multiple sequences of binary strings into one as follows. Consider the sequence

$$\big\langle \langle a_1 \rangle, \langle a_2 \rangle, \cdots, \langle a_n \rangle \big\rangle = \big\langle \langle a_1 \rangle \# \langle a_2 \rangle \# \cdots \# \langle a_n \rangle \big\rangle,$$

where a_i are binary strings of finite length.

We claim that using the following map suffices

$$0 \mapsto 00$$
$$1 \mapsto 01$$
$$\# \mapsto \mathbf{1}$$

Encoding a Turing Machine

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Why does this work?

- For a 0 in an odd position, the symbol immediately following it corresponds to the symbol that was encoded
- We can identify the delimiter as the 1 that appears in an odd position.

Encoding a Turing Machine

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To recover a_1 and a_2 from the encoding:

- For any 0 in odd positions, the symbol that follow in the even positions, belong to a_1.
- If a 1 is obtained in an odd position, it corresponds to the delimiter/partition $\Rightarrow a_1$ has been recovered, now a_2 will be obtained similarly.
- This can be generalized to multiple tuples of binary strings which is what we need to encode M.

Encoding a Turing Machine

So for any DTM M, we obtain an encoding

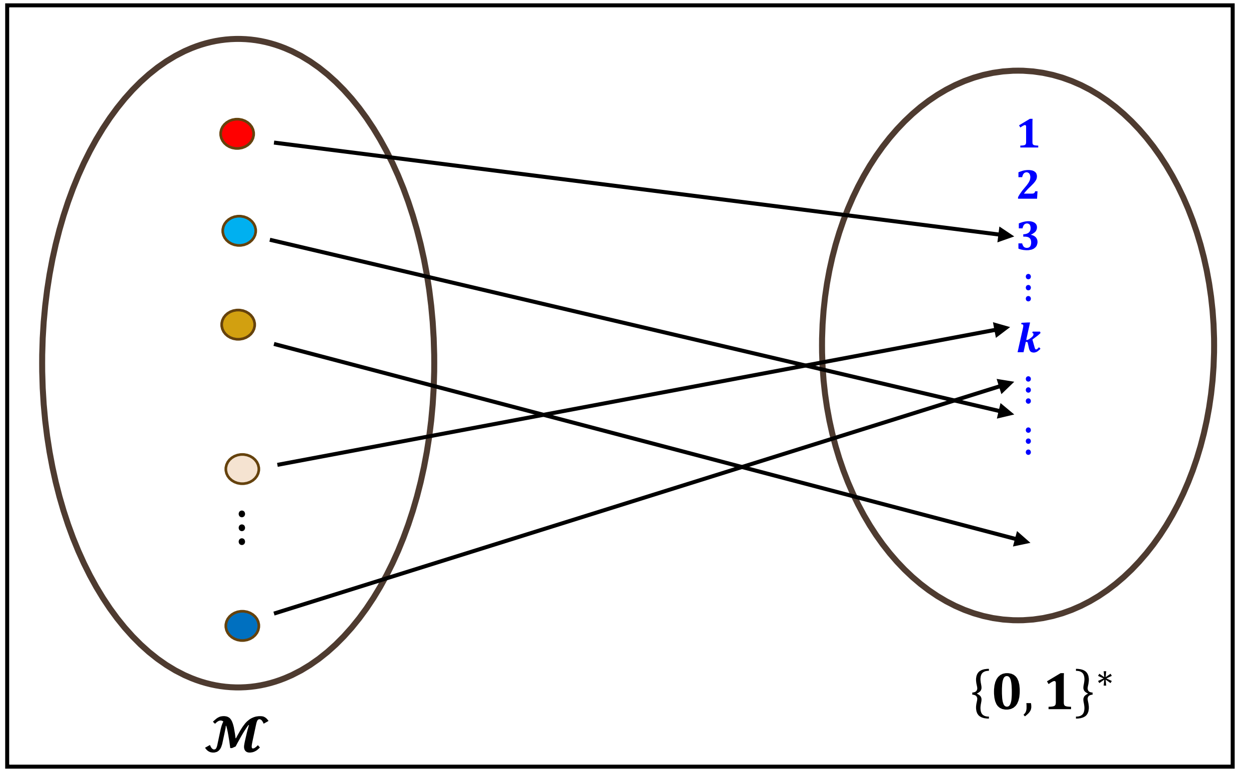
$$\langle \boldsymbol{M} \rangle = (\, \langle \boldsymbol{m} \rangle, \langle \boldsymbol{k} \rangle, \langle \boldsymbol{n} \rangle, \langle \boldsymbol{\delta} \rangle, \mathbf{0/1}, \langle \boldsymbol{q_{accept}} \rangle, \langle \boldsymbol{q_{reject}} \rangle \,)$$

such that $\langle M \rangle \in \{0,1\}^*$.

Every DTM corresponds to a binary string but the reverse is not necessarily true. Some binary strings are not valid descriptions of DTMs.

We have established an injective map

$$\boldsymbol{G}: \boldsymbol{\mathcal{M}} \mapsto \{\mathbf{0},\mathbf{1}\}^*$$



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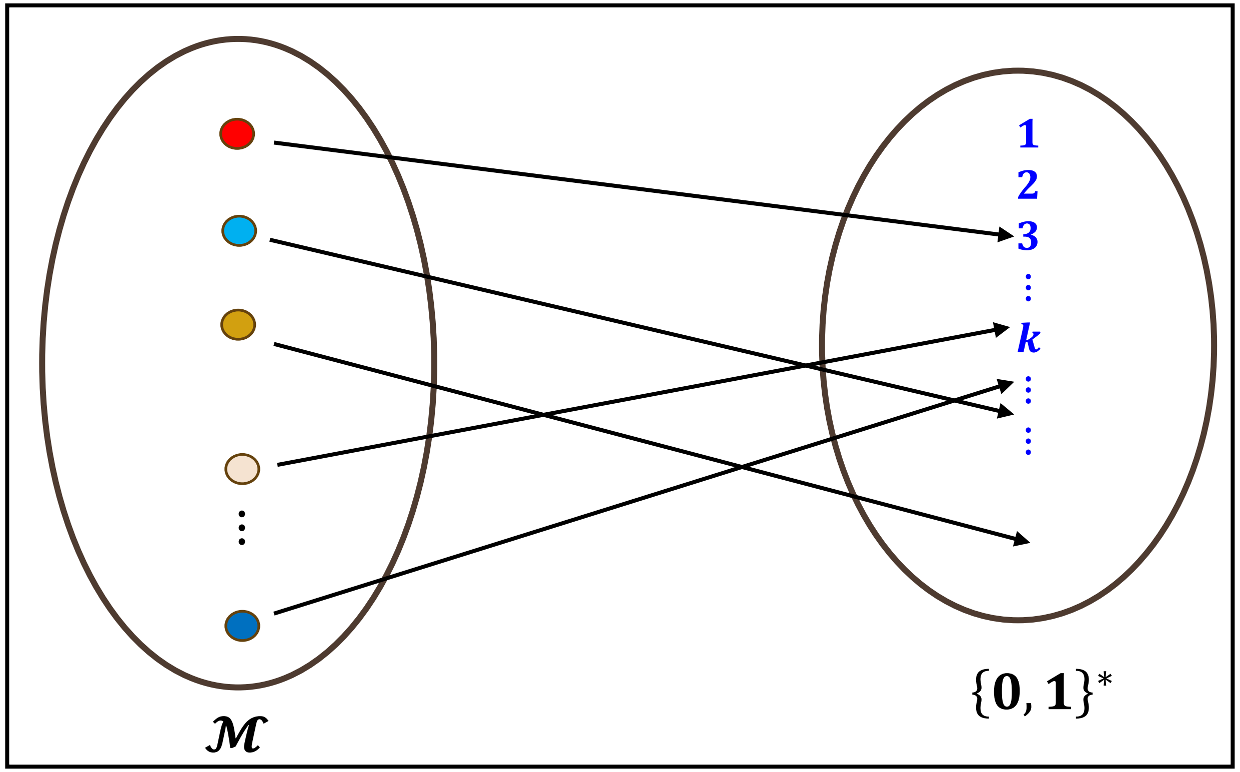

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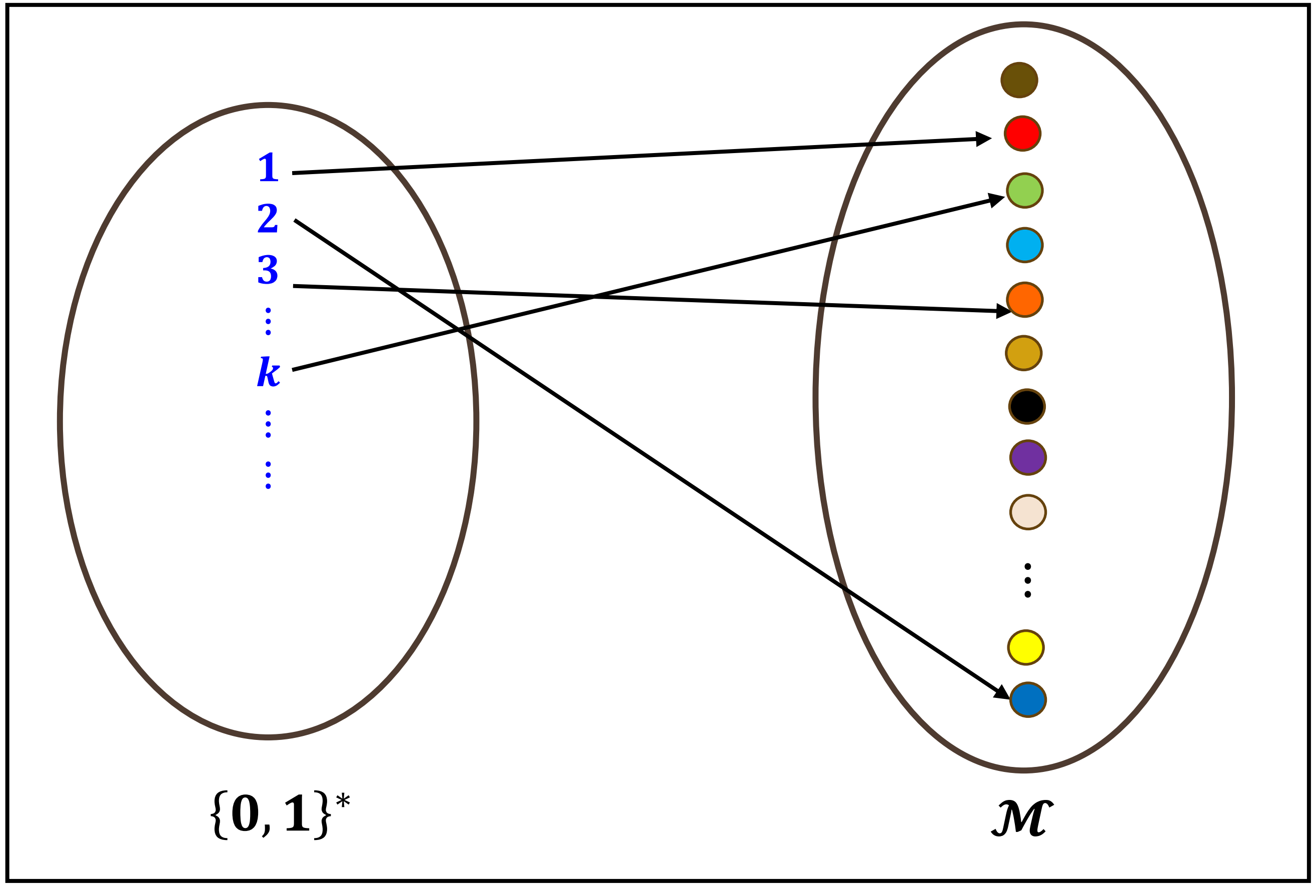
Can we make this a bijection?

- Establish an injective map $\boldsymbol{F}: \{\mathbf{0}, \mathbf{1}\}^* \mapsto \boldsymbol{\mathcal{M}}$
- **Use Bernstein-Schröder Theorem!**



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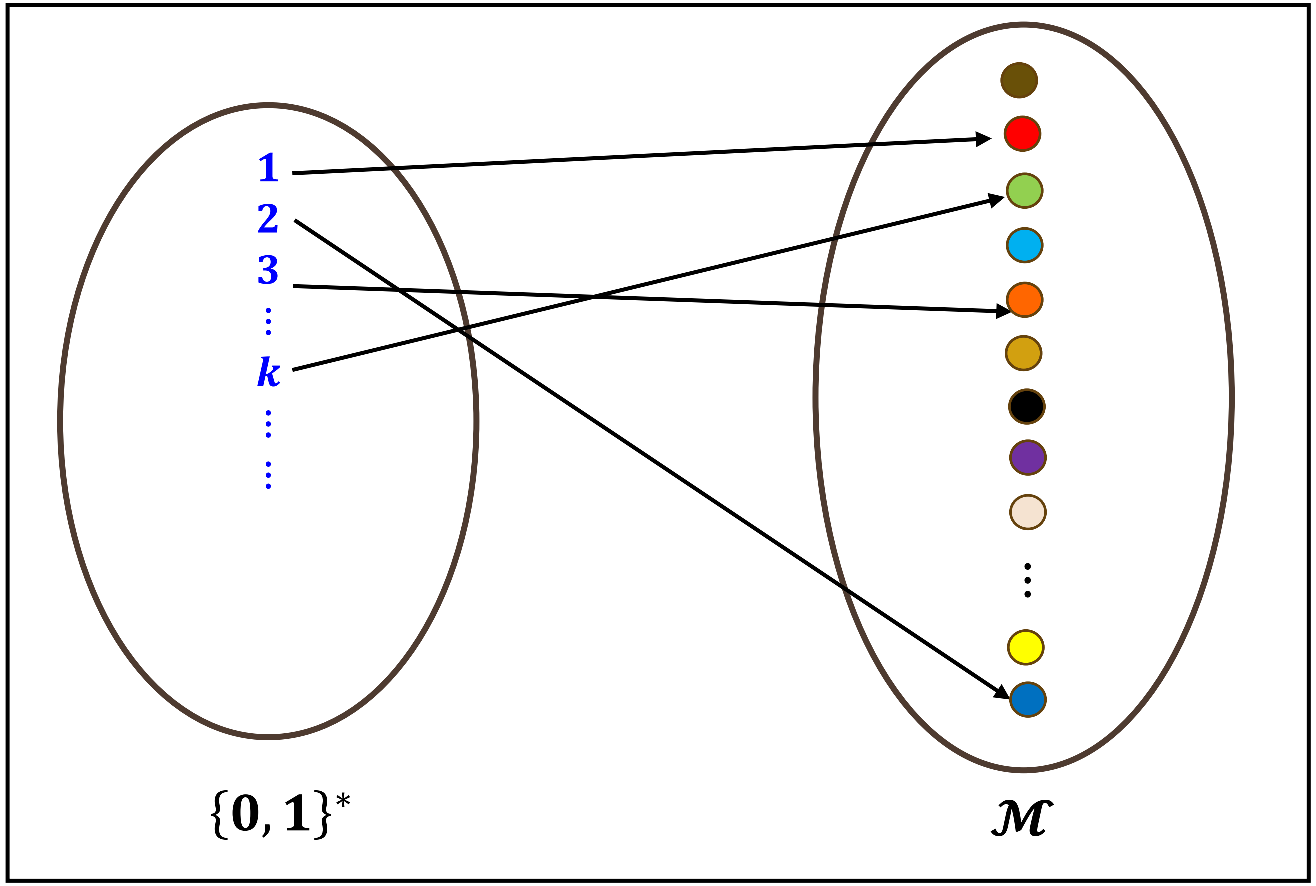

Bernstein-Schröder Theorem

For any two sets A and B, suppose there exists an injective function $f: A \mapsto B$, and an injective function $g: B \mapsto A$, then there exists a bijection $h: A \mapsto B$.

In terms of cardinality, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Encoding a Turing Machine

We have established an injective map $\boldsymbol{G}: \boldsymbol{\mathcal{M}} \mapsto \{\mathbf{0}, \mathbf{1}\}^*$

Can we make this a bijection?

- For any $k \in \{0,1\}^*$, define a mapping to TM $M_k \in \mathcal{M}$ as follows:
- Irrespective of what is in its tape, M_k overwrites it with k on its tape.
  - **States of $\boldsymbol{M_k}$**: $\mathrm{q_0, q_1, \dots, q_k, q_{acc}, q_{rej}}$
  - Suppose $x_1 x_2 \cdots x_k$ are the bits of k.
  - Then, the following are the transitions of M_k, for any a $\in \Gamma$:
    - $\delta(q_0, a) = (q_1, x_1, R)$.
    - For $i \in [1, k-1]$, $\delta(q_i, a) = (q_{i+1}, x_i, R)$
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Established an injective map
$\boldsymbol{F}: \{\mathbf{0}, \mathbf{1}\}^* \mapsto \boldsymbol{\mathcal{M}}$

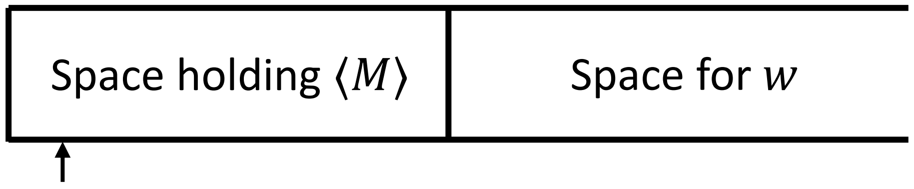
Now we have a one-one mapping (bijective relationship) between the set of finite-length binary strings and DTMs.

Universal Turing Machines

Now that we have shown how to encode objects including Turing Machines as binary strings, we can now define **Universal Turing Machines** – or Turing Machines that simulate other Turing Machines.

Universal Turing Machine: A Universal Turing Machine, denoted as $\boldsymbol{U_{TM}}$ accepts as input (i) the encoding of a Turing Machine M, (ii) an input string w and **simulates $\boldsymbol{M}$ running on $\boldsymbol{w}$**, i.e.

$$\boldsymbol{U_{TM}(\langle M, w\rangle)} = \begin{cases} \textbf{ACCEPTS, if } \boldsymbol{M(w)} \textbf{ accepts} \\ \textbf{REJECTS, if } \boldsymbol{M(w)} \textbf{ rejects} \\ \textbf{LOOPS INFINITELY, if } \boldsymbol{M(w)} \textbf{ loops infinitely} \end{cases}$$



U_{TM} checks

- the space for w to determine the symbol currently being read
- And the space containing $\langle M\rangle$ for determining the transition function to be implemented

By the **Church-Turing thesis**, a U_{TM} can perform any computation on any feasible computational device.

So, in principle using U_{TM}, Turing Machines can answer questions about Turing Machines!

Some Decidable Languages

Much like Turing Machines, DFAs, NFAs, CFGs can also be encoded as binary strings. In fact, a bijection can be established between binary strings and these objects.

This is useful as it helps answer the decidability of Languages related to them.

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Examples: The following languages are decidable

- $A_{DFA} = \{\langle DFA, w\rangle | w \in L(DFA)\}$

$M =$ On input $\langle DFA, w\rangle$:

- Simulate the run of $\langle DFA\rangle$ on w.
- If w is accepted, output $ACCEPT$
- If w is rejected, output $REJECT$

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- $A_{DFA} = \{\langle DFA, w\rangle | w \in L(DFA)\}$
- $\boldsymbol{E_{DFA} = \{\langle DFA\rangle | L(DFA) = \Phi\}}$

Some Decidable Languages

Much like Turing Machines, DFAs, NFAs, CFGs can also be encoded as binary strings. In fact, a bijection can be established between binary strings and these objects.

This is useful as it helps answer the decidability of Languages related to them.

Examples: The following languages are decidable

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$M =$ On input $\langle DFA\rangle$:

- Mark the start state of $\langle DFA\rangle$
- Repeat until no new states are marked
  - Mark any state that has an incoming transition from a marked state
- If the final state is unmarked, $ACCEPT$, else $REJECT$

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$M =$ On input $\langle CFG, w\rangle$:

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- List all derivations of $2|w| - 1$ steps
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Or, run the CYK algorithm

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Idea similar to DFAs: Check if the Start Variable leads to any terminal

Some Decidable Languages

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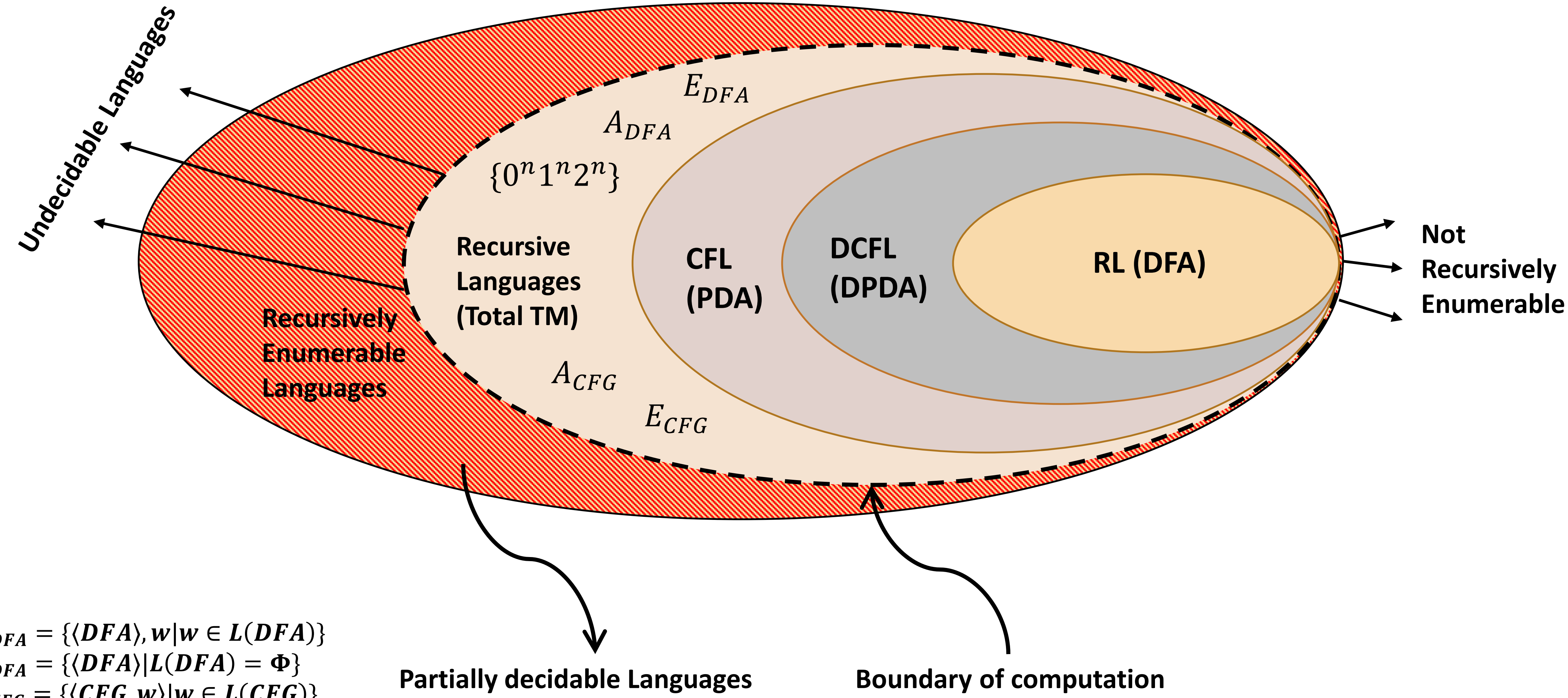
$M =$ On input $\langle CFG\rangle$:

- Mark all terminal symbols
- Repeat until no new variables are marked
  - Mark any V, s.t. $V \rightarrow X_1 X_2 \cdots X_l$.
- If S is unmarked, $ACCEPT$. Else $REJECT$

Some Decidable Languages



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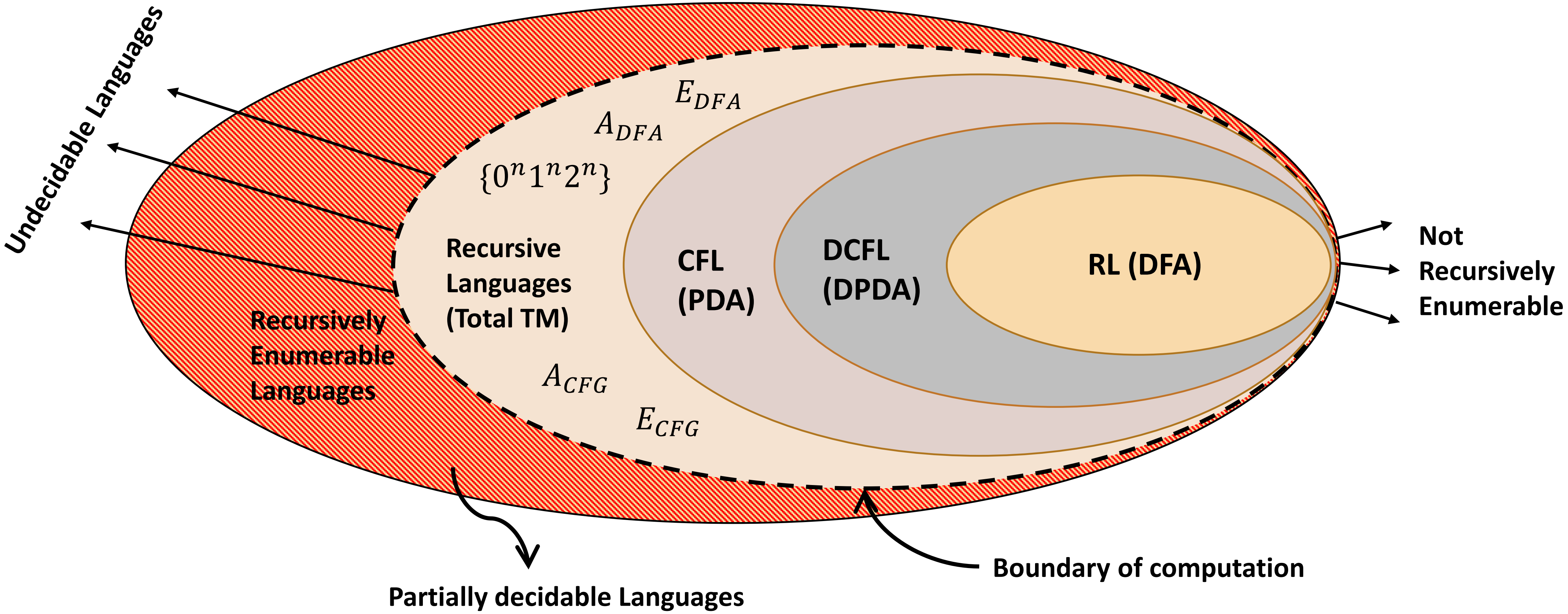

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What about undecidable languages?

An undecidable problem

$A_{TM} = \{\langle M, w\rangle | M$ accepts input $w\}$. Is A_{TM} decidable?

A_{TM}: Does there exist a Total Turing Machine A that accepts as input a Turing Machine M and an input string w and outputs ACCEPT, if $M(w)$ accepts w and REJECT, if $M(w)$ does not accept w (rejects or loops forever)?

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Every (finite length) binary string is a TM and vice versa. So, the **input may have two copies of the same string (say** $\boldsymbol{w}$**)**:

- The first copy corresponds to the encoding of some TM M_w.
- The second copy is the input string $w = \langle M_w \rangle$.

$$\boldsymbol{A}(\boldsymbol{w}, \boldsymbol{w}) = \boldsymbol{A}(\boldsymbol{M_w}, \langle \boldsymbol{M_w} \rangle)$$

In this case, A simulates the run of TM M_w on the input string w, which is the binary encoding of M_w itself

An undecidable problem

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There can be inputs such as $\boldsymbol{A}(\langle \boldsymbol{w}, \boldsymbol{w}\rangle)$

Let $\boldsymbol{w} = \langle \boldsymbol{M_w} \rangle$

$$\boldsymbol{A}(\langle \boldsymbol{w}, \boldsymbol{w}\rangle) = \begin{cases} \textbf{ACCEPTS, if } \boldsymbol{M_w}(\langle \boldsymbol{M_w}\rangle) \textbf{ accepts} \\ \textbf{REJECTS, if } \boldsymbol{M_w}(\langle \boldsymbol{M_w}\rangle) \textbf{ rejects or loops infinitely} \end{cases}$$

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- We will show that if such a Total TM A exists, we run into the following contradiction

  Using A, we can build a new Total TM for which there exists an instance for which the machine **both accepts and rejects**!

An undecidable problem

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Proof: Let us assume that a Total Turing machine A exists. Then we can construct a special Total Turing Machine D that accepts an input w and uses A as a subroutine to simulate $A(\langle w, w\rangle)$ in the following way:

An undecidable problem

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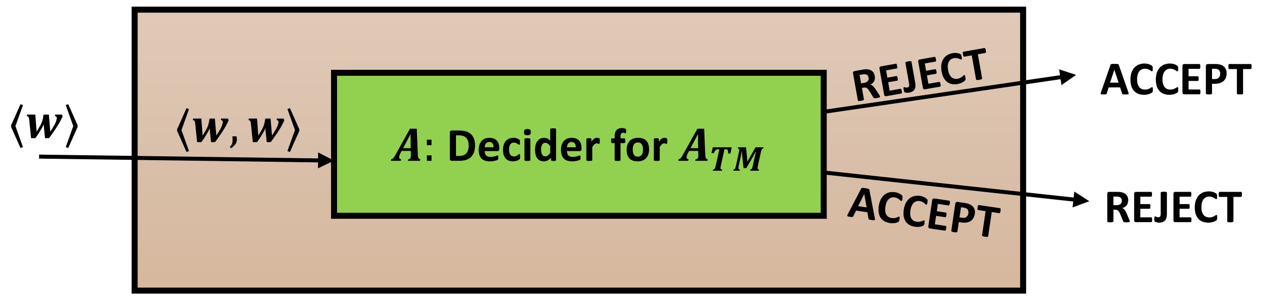

$D(w) = \{$ **Run** $A(\langle w, w\rangle)$

If $A(\langle w, w\rangle)$ **accepts,** D **outputs** $REJECT$
If $A(\langle w, w\rangle)$ **rejects,** D **outputs** $ACCEPT$

$\}$

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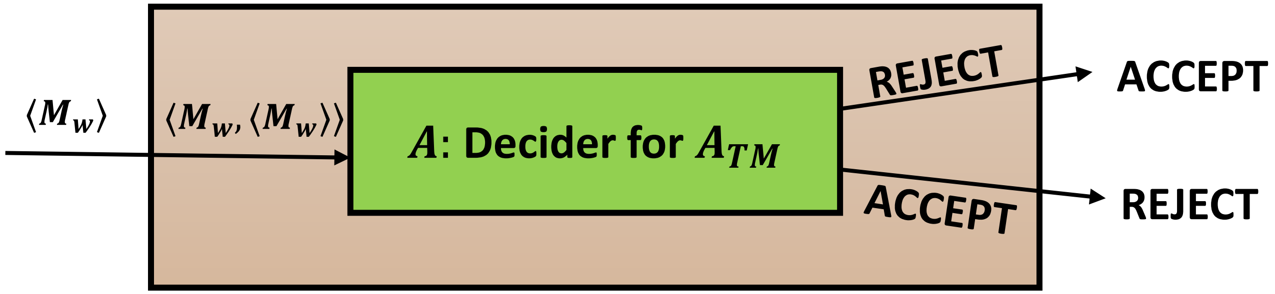

Let $w = \langle M_w\rangle$, then

$$D(\langle M_w\rangle) = \begin{cases} \textbf{ACCEPTS, if } M_w(\langle M_w\rangle) \textbf{ doesn't accept} \\ \textbf{REJECTS, if } M_w(\langle M_w\rangle) \textbf{ accepts} \end{cases}$$

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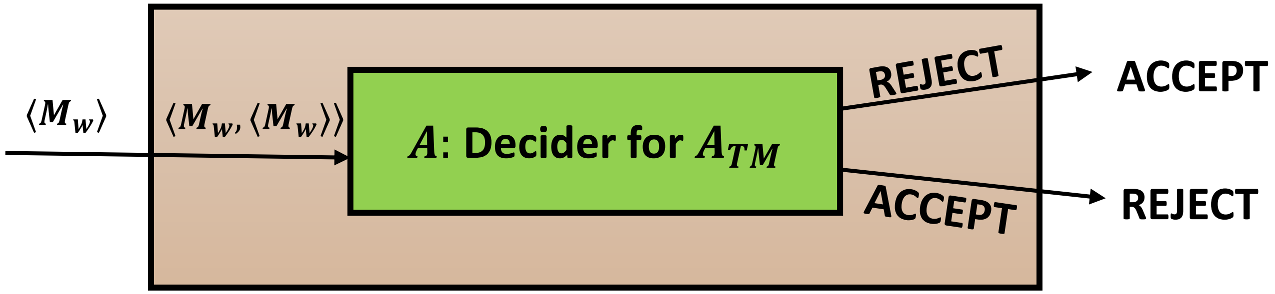

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What happens when $w = \langle D\rangle$ i.e., $M_w = D$?

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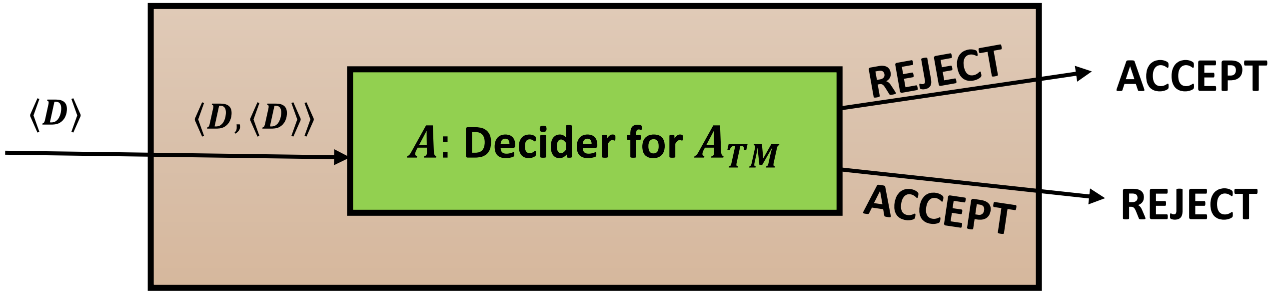

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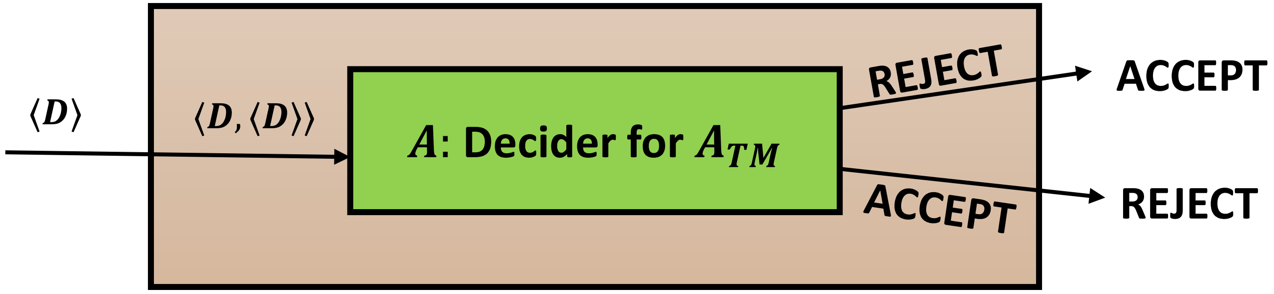

$$D(\langle D\rangle) = \begin{cases} \textbf{ACCEPTS, if } D(\langle D\rangle) \textbf{ doesn't accept} \\ \textbf{REJECTS, if } D(\langle D\rangle) \textbf{ accepts} \end{cases}$$

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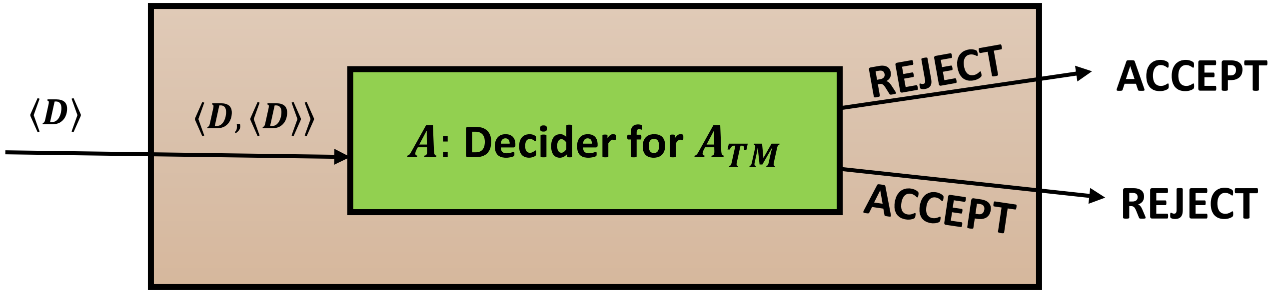

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CONTRADICTION!

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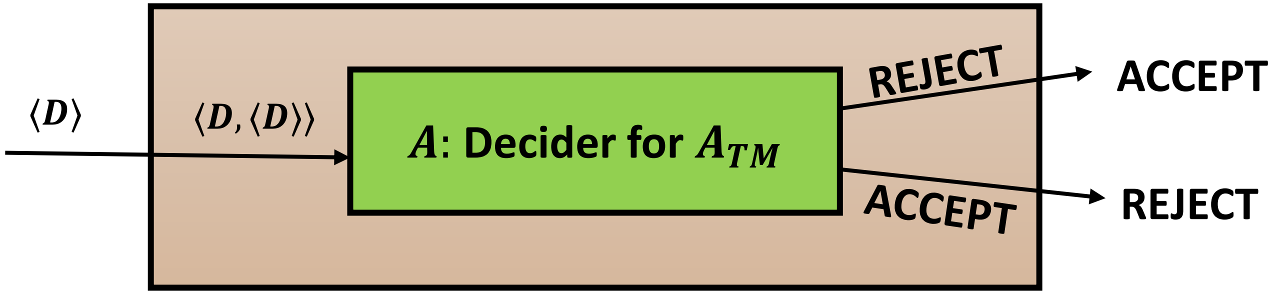

CONTRADICTION!

- D cannot be a Total TM as it cannot decide input $\langle D\rangle$.
- If a total TM A existed, we could have constructed a total TM D.
- So, a total TM A cannot exist and hence A_{TM} **is not decidable.**

An undecidable problem

$A_{TM} = \{\langle M, w\rangle | M$ **accepts input** $w\}$**. Is** A_{TM} **decidable? NO!**

$$A(\langle M, w\rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ accepts} \\ \textbf{REJECTS, if } M(w) \textbf{ rejects or loops infinitely} \end{cases}$$

Is $A_{TM} \in RE$ **?**

An undecidable problem

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Of course, $A_{TM} \in RE$ **as** A **halts whenever** M **accepts** w **and so**

$U =$ **On input** $\langle M, w\rangle$**:**

- **Simulate** M **on input** w
- **If** M **accepts** w, $ACCEPT$**; if** M **rejects** w, ***REJECT***

U recognizes A_{TM}

An undecidable problem

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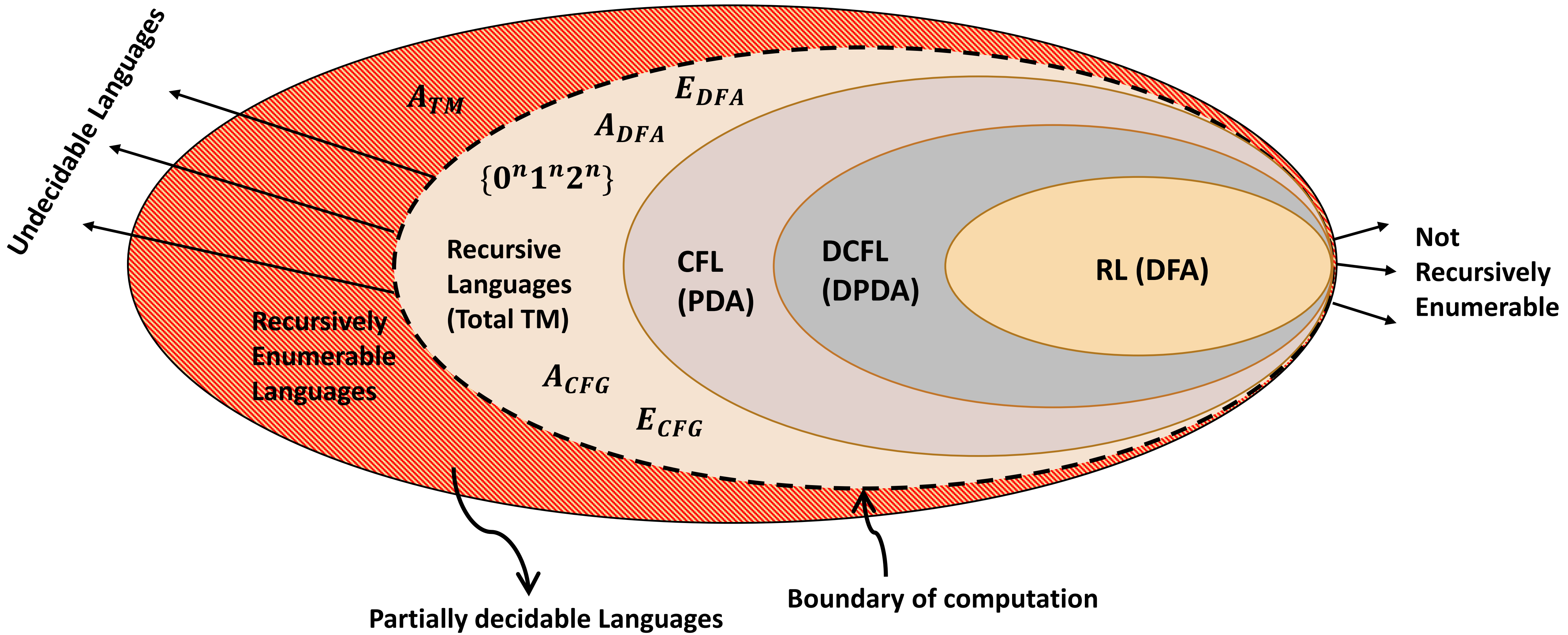
U recognizes A_{TM}

- A_{TM} **is undecidable**
- $A_{TM} \in RE$ **but not recursive**
- A_{TM} **is partially decidable**



Next Lecture:

- Halting Problem
- More on Recursive & RE languages
- Co-RE language

Thank You!