Pumping Lemma for CFL

If L is context-free, $\exists\, p \in \mathbb{Z}$, $p \geq 1$ (pumping length)
s.t. $\forall\, s \in L$ s.t. $|s| \geq p$, $\exists\, u, v, x, y, z \in \Sigma^*$
s.t. $s = uvxyz$
s.t. 1. $|vy| \geq 1$
2. $|vxy| \leq p$
3. $uv^n x y^n z \in L \quad \forall\, n \geq 0$.

To show L is NOT a CFL.

1. Assume L is a CFL.

2. Assume $\exists$ pumping constant p for L.
3. Pick $w \in L$ with $|w| \geq p$
4. Look at all decompositions of w into $uvxyz$ s.t. $|vy| \geq 1$, $|vxy| \leq p$ and find one i s.t. $uv^i x y^i z \notin L$.

Show that $\mathcal{L}$ is not a CFL

$\mathcal{L} = \{ ww : w \in \{0,1\}^* \}$

Suppose $\exists\ p$ for $\mathcal{L}$.

Choose $w = \underbrace{0^p 1^p}_{\mathrm{I}} \mid \underbrace{0^p 1^p}_{\mathrm{II}}$

Look at all valid decompositions into $uvxyz$

case 1 : v, y are only in I.

$\underbrace{0 \cdots 0 1 \cdots 1}_{v\ y} \mid 0 \cdots 0 1 \cdots 1$

What if we pump $i=2$?

Now new string will be like $\underbrace{0 \cdots 0 1 \cdots 1}_{\mathrm{I}'} \mid \underbrace{1\ 0 \cdots}_{\mathrm{II}''}$ new mid point

$\therefore \mathrm{I}' \neq \mathrm{II}''$ (Both start differently)

$\therefore$ The new string is not in the language.

case 2 : $0 \cdots 0 1 \cdots 1 \mid \underbrace{0 \cdots 0 1 \cdots 1}_{vy}$

new string $\underbrace{0 \cdots 0\ 1 \cdots 1 \mid 0}_{\mathrm{I}'} \mid \underbrace{0 \cdots 0\ 1 \cdots 1}_{\mathrm{II}'}$

$\therefore \mathrm{I}' \neq \mathrm{II}''$ (Both end differently)

case 3 : v, y are in the middle somewhere

$\underbrace{0 \cdots 0\ 1 \underbrace{\cdots 1}_{v}}_{\mathrm{I}} \mid \underbrace{\underbrace{0 \cdots}_{y}\ 0\ 1 \cdots 1}_{\mathrm{II}}$

Pump $i=2$. v, y both can't be empty $|vy| \geq 1$

Now I will have more 1's than II
Also II has more 0's than I $\therefore \mathrm{I} \neq \mathrm{II}$

These are the only cases as $|vxy| \leq p$
and $w = 0^p 1^p 0^p 1^p$

$\therefore \mathcal{L}$ is not a CFL. $\square$