CS 302.1 - Automata Theory

Lecture 05

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Grammars

(Grammar) Formally, a *Grammar* G is a 4-tuple (V, Σ, P, S) such that

- V is the set of **Variables**
- Σ is the set of **Terminals** (disjoint from V)
- P is the set of production **Rules** [$(V \cup \Sigma)^* V (V \cup \Sigma)^* \to (V \cup \Sigma)^*$]
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

Eg: Consider the grammar G

$X \to 1X$
$X \to 0Y$
$Y \to 0X$
$Y \to 1Y$
$Y \to \epsilon$

X is the start variable of the Grammar. Variables: $\{X, Y\}$, Terminals: {0, 1}

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Grammars can be used to derive strings.

The sequence of **substitutions** (using the rules of G) required to obtain a certain string is called a **derivation**.

- Begin the **derivation** from the **Start variable.**
- Replace any variable according to a rule. Repeat until only terminals remain.
- The generated string is **derived by the grammar.**

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$X \to 0Y$
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$Y \to \epsilon$

X: Start Variable
$\{X, Y\}$: Variables
$\{0,1\}$: Terminals

The following is a derivation

$X \to 1X \to 11X \to 110Y \to 1101Y \to \mathbf{1101}$

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- To show that a string $\boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{G})$, we show that there exists a **derivation ending up in $\boldsymbol{w}$**. The fact that w can be derived using the rules of G, is expressed as $\boldsymbol{S} \overset{*}{\Rightarrow} \boldsymbol{w}$.

- The **language of the grammar, $\boldsymbol{L}(\boldsymbol{G})$** is $\{w \in \Sigma^* | S \overset{*}{\Rightarrow} w\}$

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The string $\mathbf{1101} \in \boldsymbol{L}(\boldsymbol{G})$ because there exists the following derivation

$$X \rightarrow 1X \rightarrow 11X \rightarrow 110Y \rightarrow 1101Y \rightarrow 1101$$

Grammars for Regular Languages

Regular grammar: If the *rules* of the underlying grammar G are of the form

$$Var \to Ter\ \boldsymbol{Var}$$
$$Var \to Ter$$
$$Var \to \boldsymbol{\epsilon}$$

then the language of the grammar is **regular.** Also known as **Right-linear grammar** (a single variable to the right of terminals in the RHS).

Right linear Grammar to DFA

Eg: Consider the grammar G

$X \to 1X$
$X \to 0Y$
$Y \to 1Y$
$Y \to 0X$
$Y \to \epsilon$ (indicates that Y is the final state)

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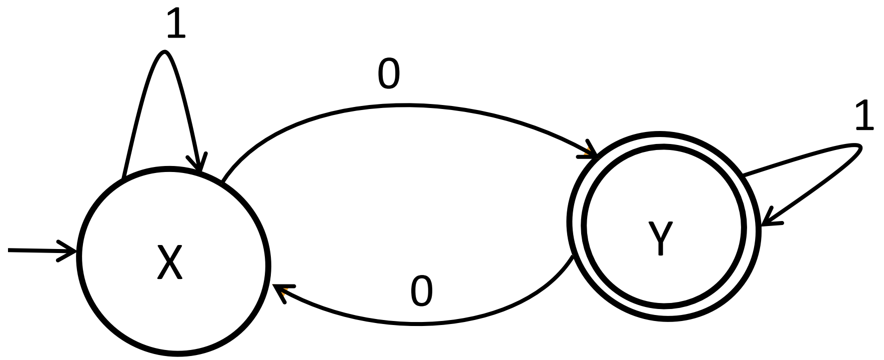

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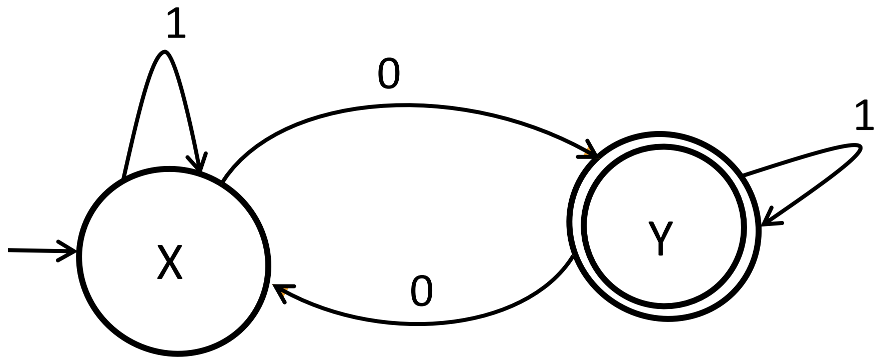

A **run** in a DFA model is analogous to a **derivation** in a linear grammar.

For the string $\mathbf{1101}$:

Derivation: $X \to 1X \to 11X \to 110Y \to 1101Y \to 1101$. So $1101 \in L(G)$

Run: $X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{0} Y \xrightarrow{1} Y$ (Accepting Run and so $1101 \in L(M)$).

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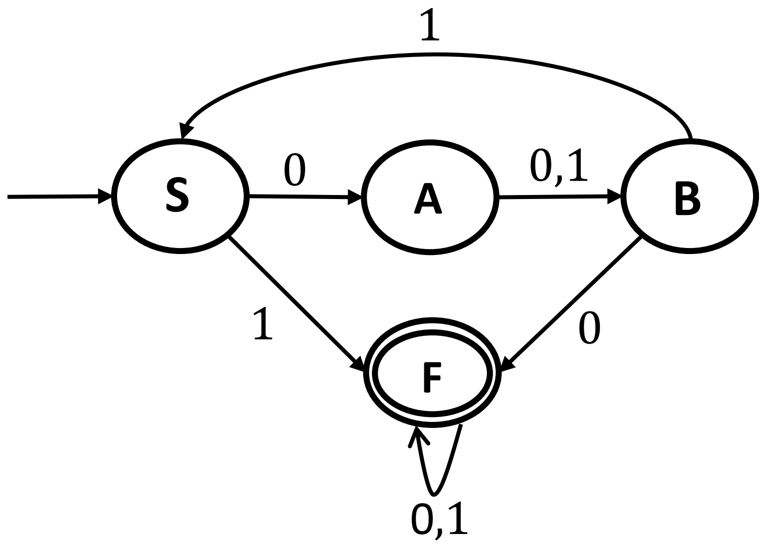
DFA to Right linear Grammar

Consider the following DFA M



The right-linear grammar G for M

$S \rightarrow 0A|1F$
$A \rightarrow 0B|1B$
$B \rightarrow 0F|1S$
$F \rightarrow 0F|1F|\epsilon$

Grammars for Regular Languages

Right-linear grammar $\equiv$ DFA $\equiv$ NFA $\equiv$ Regular Expressions

Left linear grammar: If the *rules* of the underlying grammar G are of the form

$$Var \rightarrow \boldsymbol{Var}\, Ter$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then such a grammar is called **Left-linear** (a single variable to the left of terminals in the RHS).

Right linear grammars are equivalent to Left-linear grammar (We won't be proving it here)

Grammars for Regular Languages

Right-linear grammar ≡ DFA ≡ NFA ≡ Regular Expressions

Left linear grammar: If the *rules* of the underlying grammar G are of the form

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Right linear grammars are equivalent to Left-linear grammar (We won't be proving it here)

Right-linear grammars and **Left-linear grammars** generate **Regular Languages.**

Note that mixing left-linear grammars and right-linear grammars in the same set of rules **won't generate regular languages**. (e.g: $S \rightarrow aX, X \rightarrow Sb, S \rightarrow \epsilon$)

Left-linear grammar ≡ Right-linear grammar ≡ DFA ≡ NFA ≡ Regular Expressions

Context free Grammars

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Context-Free Grammars: If the *rules* of the underlying grammar G are of the form

$$\boldsymbol{V} \rightarrow (\boldsymbol{V} \cup \boldsymbol{T})^*$$

then such a grammar is called **Context-Free.**

Any language generated by a context-free grammar is called a ***context-free language.***

Immediately we find that the *rules* are less restrictive than left-linear grammars and right-linear grammars. Context free grammars allow

$$Var \rightarrow Anything$$

$$Var \rightarrow String\ of\ Variables |\ String\ of\ Terminals | Strings\ of\ Variables\ and\ Terminals | \epsilon$$

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- So Left linear grammars and Right linear grammars are also context-free grammars.
- **Regular languages** $\subset$ **Context Free Languages**.

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$$S \rightarrow 0S1$$
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Consider the Grammar G with the following rules:

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What is the language generated by this grammar?

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Strings that can be derived from G:

$$S \rightarrow \epsilon$$

$$\{\epsilon\}$$

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Strings that can be derived from G:

$$S \rightarrow 0S1 \rightarrow 01$$

$$\{\epsilon, 01\}$$

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Strings that can be derived from G:

$$S \rightarrow 0S1 \rightarrow 00S11 \rightarrow 0011$$

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$$S \rightarrow 0S1 \rightarrow 00S11 \rightarrow 000S111 \rightarrow 000111$$

$$\{\epsilon, 01, 0011, 000111\}$$

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Strings that can be derived from G:

$$\{\epsilon, 01, 0011, 000111, 0^4 1^4, \cdots\}$$

What is the language generated by this grammar?

$$L(G) = \{\omega | \omega = 0^n 1^n, n \geq 0\}$$

So although $L(G)$ is not regular, it is context-free.

Context free Grammars

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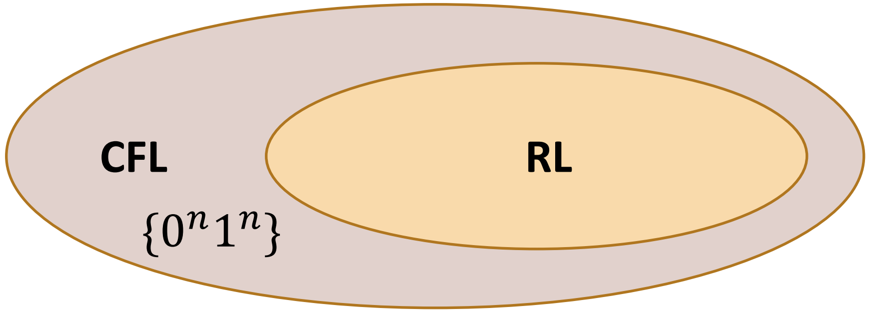

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Regular languages $\subset$ **Context Free Languages**.

Consider the Grammar G with the following rules:

$$S \to 0S1|SS|\epsilon$$

Strings that can be derived by G:

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$$\{\epsilon\}$$

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Strings that can be derived by G:

$$S \rightarrow 0\boldsymbol{S}1 \rightarrow 0\boldsymbol{0S1}1 \ldots$$

$$\{\epsilon, 01, 0011, \ldots 0^n 1^n\}$$

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Strings that can be derived by G:

$$S \to \boldsymbol{0S1} \to 0\boldsymbol{SS}1 \to 0\boldsymbol{0S1}S1 \to 001S1 \to 001\boldsymbol{0S1}1 \to 001011$$

$$\{\epsilon, 01, 0011, \ldots 0^n1^n, 001011, \ldots.\}$$

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Show that the string $\boldsymbol{010101} \in \boldsymbol{L}(\boldsymbol{G})$.

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What is $\boldsymbol{L(G)}$?

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$$\{\epsilon, ()\}$$

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$$\{\epsilon, (\quad), ((\quad)), \dots.,\}$$

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$$\left\{\epsilon, (\quad), ((\quad)), \ldots, \left(\left(\left((\ldots.)\right)\right)\right), ((\quad)(\quad)), ()()(), \ldots.\right\}$$

So, $L(G)$ is the language of all strings of properly nested parentheses.

$$L(G) = \{\omega | \omega \text{ is a correctly nested parenthesis}\}$$

Context free Grammars

Constructing CFG corresponding to a Language.

There is no fixed recipe for doing this. Requires some level of creativity.

Some tips might come in handy:

- Check if the CFL is a union of simpler languages. If $L(G) = L(G_1) \cup L(G_2)$ and G_1 and G_2 are known. If S_1 is the start variable for G_1 and S_2 is the start variable for G_2 then the the rules of G:

$$\begin{aligned} S &\rightarrow S_1 | S_2 \\ S_1 &\rightarrow \cdots\cdots \\ S_2 &\rightarrow \cdots\cdots \end{aligned}$$

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$$\begin{aligned} &S \to S_1 | S_2 \\ &S_1 \to \cdots\cdots \\ &S_2 \to \cdots\cdots \end{aligned}$$

- Grammars with rules such as $S \to aSb$ help generate strings where the corresponding machine would need unbounded memory to *remember* the number of *a's* needed to verify that there are an equal number of *b's*. This was not possible with regular expressions/linear grammars.

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- Check if the CFL is a union of simpler languages.
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Example: Construct the grammar G such that $\boldsymbol{L(G) = \{\omega | \omega}$ **has equal number of** $\boldsymbol{0}$**'s and** $\boldsymbol{1}$**'s**$\}$

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- The first thing to notice is that $L_1 = \{0^n 1^n, n \geq 0\} \subset L(G)$. We know the grammar for this language.
- Any string $\omega \in L_1$ has a series of 0’s followed by an equal number of 1’s.
- Again, consider L_2 to comprise all strings that start with a series of 1’s followed by an equal number of 0’s, i.e.

$$L_2 = \{1^n 0^n, n \geq 0\}$$

- The grammar for L_2 is similar to that of L_1: replace the 0’s with 1’s and vice versa. Importantly, $L_2 = \{1^n 0^n, n \geq 0\} \subset L(G)$ also.

- Also, $L_1 \cup L_2 \subset L(G)$

Context free Grammars

Constructing CFG corresponding to a Language.

- Check if the CFL is a union of simpler languages.
- Grammars with rules such as $S \rightarrow aSb$ help generate where the portions of *a* and *b* are equal.

Example: Construct the grammar G such that $\boldsymbol{L(G)} = \{\boldsymbol{\omega}|\boldsymbol{\omega}$ **has equal number of 0's and 1's**$\}$

- So $\mathrm{L}'(\mathrm{G}') = \{0^n 1^n | n \geq 0\} \cup \{1^n 0^n | n \geq 0\} \subset L(G)$
- Grammar for L_1: $S \rightarrow 0S1|\epsilon$
- Grammar for L_2: $S \rightarrow 1S0|\epsilon$
- Grammar for $L_1 \cup L_2$:

$$S \rightarrow S_1|S_2$$
$$S_1 \rightarrow 0S_1 1|\epsilon$$
$$S_2 \rightarrow 1S_2 0|\epsilon$$

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- Grammar for $L_1 \cup L_2$:

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$$S_1 \rightarrow 0S_1 1 | \epsilon$$
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- **Is that all? Is $\boldsymbol{L_1 \cup L_2 = L(G)}$**? $L_1 \cup L_2$ contains all strings that have equal number 0's followed by equal number of 1's or vice versa.

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- What about strings such as $s_1 = 0101 \cdots$ and $s_2 = 1010 \cdots$? For this we need to be able to go from

$$0\boldsymbol{S_1}1 \rightarrow 0\boldsymbol{S_2}1 \rightarrow 0\boldsymbol{1S_2 0}1 \rightarrow \cdots$$

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- Grammar for $L_1 \cup L_2$:

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- What about strings such as $s_1 = 0101 \cdots$ and $s_2 = 1010 \cdots$? Add transitions $S_1 \rightarrow S_2$ and $S_2 \rightarrow S_1$.

$$S \rightarrow S_1 | S_2$$
$$S_1 \rightarrow 0S_1 1 | \epsilon$$
$$S_2 \rightarrow 1S_2 0 | \epsilon$$
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Context free Grammars

Constructing CFG corresponding to a Language.

Example: Construct the grammar G such that $\boldsymbol{L(G)} = \{\boldsymbol{\omega} | \boldsymbol{\omega}$ **has equal number of 0's and 1's**$\}$

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$$S_1 \to 0S_1 1 | \epsilon$$
$$S_2 \to 1S_2 0 | \epsilon$$
$$S_1 \to S_2$$
$$S_2 \to S_1$$

- Can't we simplify this? We can replace S_1 and S_2 with a single Start variable as follows: $\boldsymbol{S \to 0S1 | 1S0 | \epsilon}$
- What kind of strings does the grammar generate? Well if we use Rule $\boldsymbol{S \to 0S1}$, m times, we get to rules such as $0^m S 1^m$.
- Now applying the rule $S \to 1S0$, k times, we get $\boldsymbol{0^m 1^k S 0^k 1^m}$.
- So the strings we obtain are of the form:

$$\{0^{m_1} 1^{n_1} 0^{m_2} 1^{n_2} \cdots 0^{n_2} 1^{m_2} 0^{n_1} 1^{m_1}\} \cup \{1^{m_1} 0^{n_1} 1^{m_2} 0^{n_2} \cdots 1^{n_2} 0^{m_2} 1^{n_1} 0^{m_1}\} \in L(G)$$

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$$S \rightarrow S_1|S_2$$
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- Simplified grammar:

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- Is that all? What about strings such as $\{\mathbf{0110}, \mathbf{00111100}\}$?

- More generally, what about strings that are a concatenation of L_1 and L_2?

Context free Grammars

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Example: Construct the grammar G such that $\boldsymbol{L(G)} = \{\boldsymbol{\omega} | \boldsymbol{\omega}$ **has equal number of 0's and 1's**$\}$

$$
\begin{gathered}
S \rightarrow S_1 | S_2 \\
S_1 \rightarrow 0S_1 1 | \epsilon \\
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S_1 \rightarrow S_2 \\
S_2 \rightarrow S_1
\end{gathered}
$$

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- Is that all? What about strings such as $\{\mathbf{01}\color{red}{\mathbf{10}}, \mathbf{0011}\color{red}{\mathbf{1100}}\}$?

- More generally, what about strings that are a concatenation of L_1 and L_2?

- Adding transitions like $S \rightarrow S_1 S_2$ incorporates this.

Context free Grammars

Constructing CFG corresponding to a Language.

Example: Construct the grammar G such that $\boldsymbol{L(G)} = \{\boldsymbol{\omega} | \boldsymbol{\omega}$ **has equal number of 0's and 1's**$\}$

$$S \rightarrow S_1 | S_2 | S_1 S_2 | S_2 S_1$$
$$S_1 \rightarrow 0 S_1 1 | \epsilon$$
$$S_2 \rightarrow 1 S_2 0 | \epsilon$$
$$S_1 \rightarrow S_2$$
$$S_2 \rightarrow S_1$$

- Simplify this further.

G: $\boldsymbol{S \rightarrow SS | 0S1 | 1S0 | \epsilon}$

Parse trees for CFG

Consider the Grammar G with the following rules:

$$S \to 0S1|SS|\epsilon$$

One derivation:

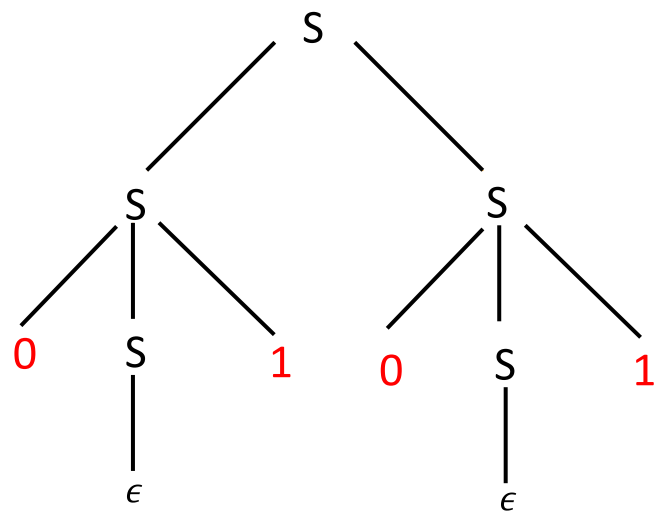
$$S \to \boldsymbol{SS} \to \mathbf{0}\boldsymbol{S}\mathbf{1}S \to 0S1\mathbf{0}\boldsymbol{S}\mathbf{1} \to \mathbf{0101}$$

Parse trees: These are ordered trees that provide alternative representations of the derivation of a grammar.

Parsing is a useful technique for compilers (Analysis of syntax eg: take sequence of tokens as input & output parse trees which provides structural representation of the input while checking for the correct syntax).

Features:

- The root node is the **Start variable**
- Branch out to nodes of the next level by following any of the rules of the grammar
- Stop when all the leaf nodes of the tree are terminals
- Read the terminals in the leaves from left to right.
- If w is the string obtained, then $\boldsymbol{S} \overset{*}{\Rightarrow} \boldsymbol{w}$ and $\boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{G})$



Parse trees for CFG

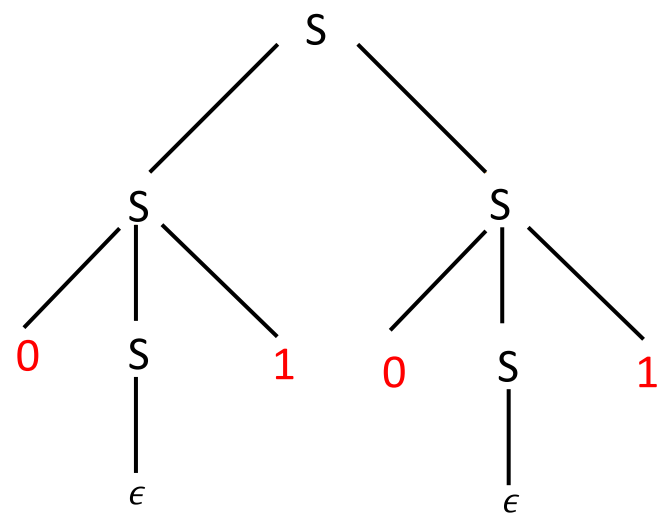
Consider the Grammar G with the following rules:

$$S \to 0S1|SS|\epsilon$$

Consider the following derivations for 0101:

1. $S \to \boldsymbol{SS} \to \boldsymbol{0S1}S \to 0S1\boldsymbol{0S1} \to \boldsymbol{0101}$
2. $S \to \boldsymbol{SS} \to \boldsymbol{0S1}S \to 01S \to 01\boldsymbol{0S1} \to \boldsymbol{0101}$
3. $S \to \boldsymbol{SS} \to S\boldsymbol{0S1} \to S01 \to \boldsymbol{0S}101 \to \boldsymbol{0101}$

- The parse trees for all these derivations are the same.



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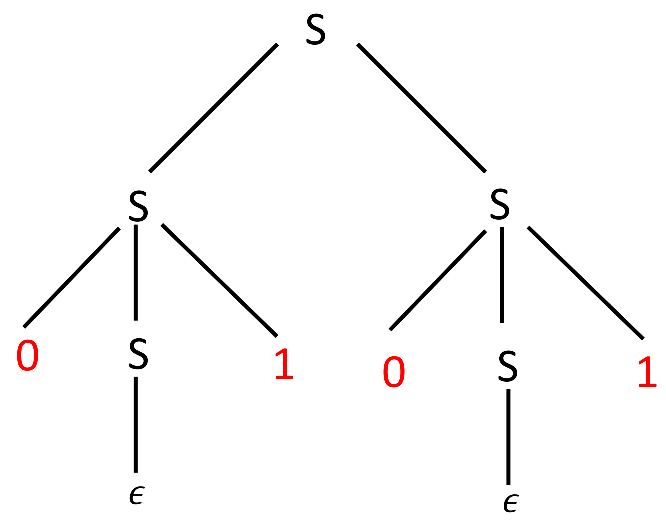
- The parse trees for all these derivations are the same.
- If a string is derived by replacing only the leftmost variable at every step, then the derivation is a **leftmost derivation.** (e.g. derivation 2.)
- ……………….rightmost variable = **rightmost derivation** (e.g. derivation 3.)
- Derivations may not always be **leftmost** or **rightmost** (e.g. derivation 1.)



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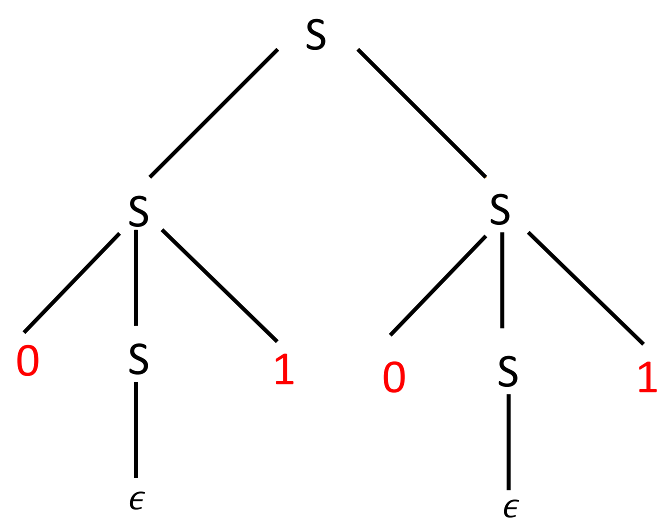

Ambiguous grammars: A CFG G is said to be **ambiguous** if there exists $\boldsymbol{\omega} \in \boldsymbol{L(G)}$, such that there are **two or more leftmost derivations for $\boldsymbol{\omega}$** (or equivalently two or more rightmost derivations) or equivalently **two or more parse trees for $\boldsymbol{\omega}$.**

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Consider the Grammar G with the following rules: $\boldsymbol{S} \rightarrow \mathbf{0}\boldsymbol{S}\mathbf{1}|\boldsymbol{SS}|\boldsymbol{\epsilon}$

Show that Grammar $\boldsymbol{G}$ is ambiguous, i.e. $\exists\, \omega \in L(G)$, such that there are two or more parse trees for ω.

Consider the string $\boldsymbol{\omega} = \mathbf{010101}$:

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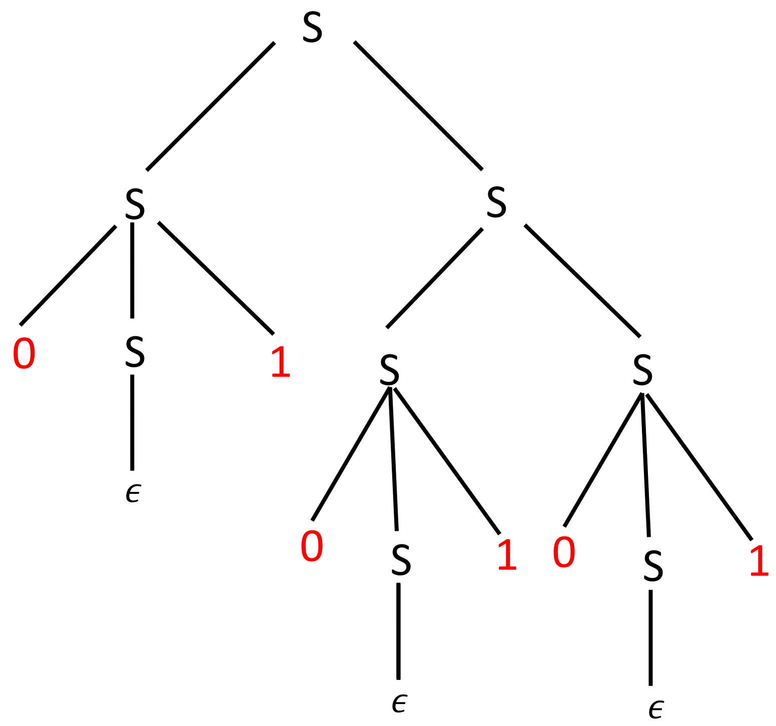

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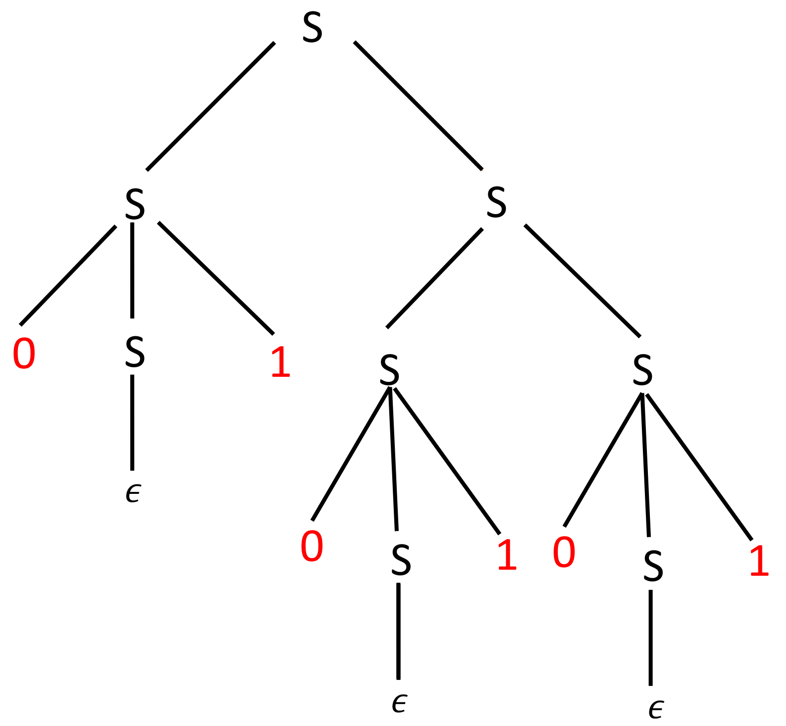

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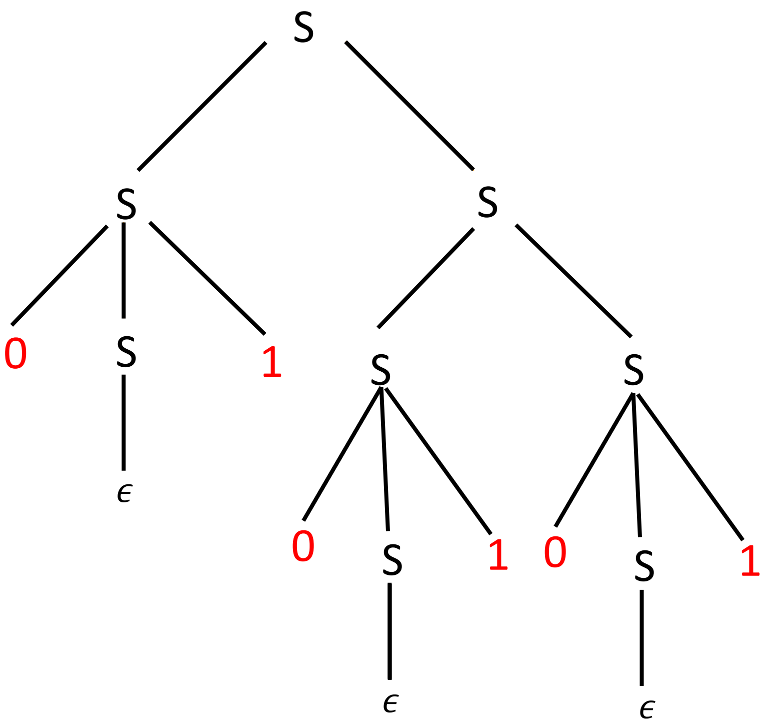

Leftmost Derivation: $S \rightarrow \boldsymbol{SS} \rightarrow \boldsymbol{0S1}S$

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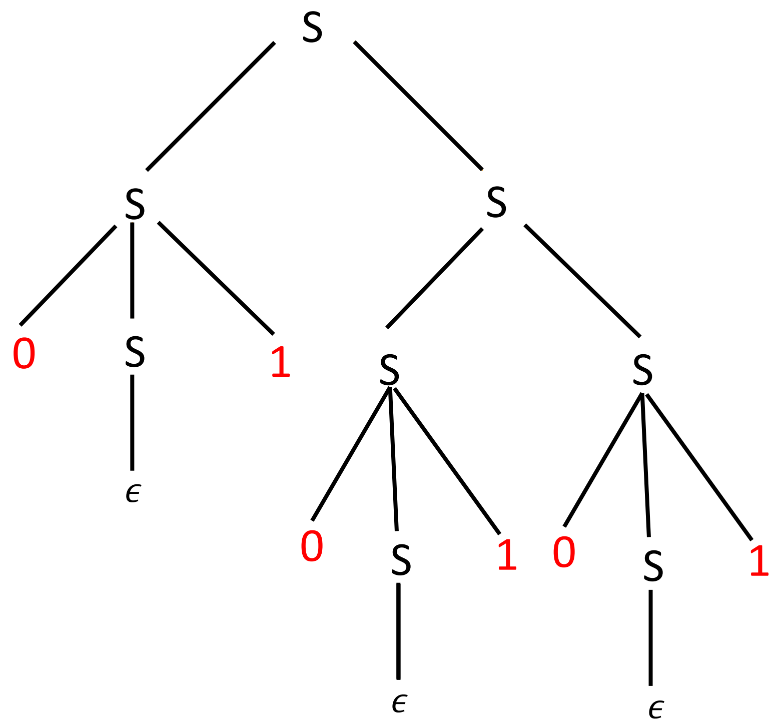

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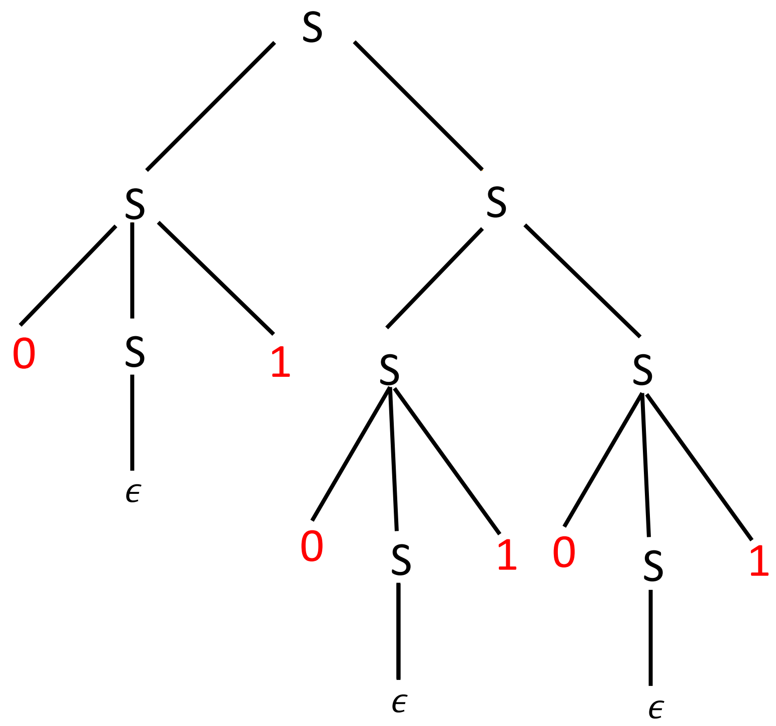

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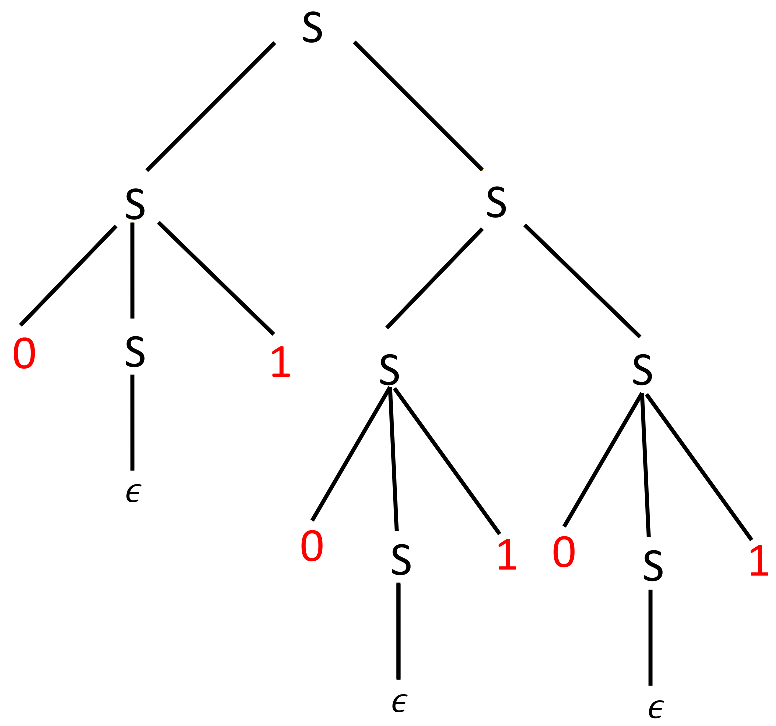

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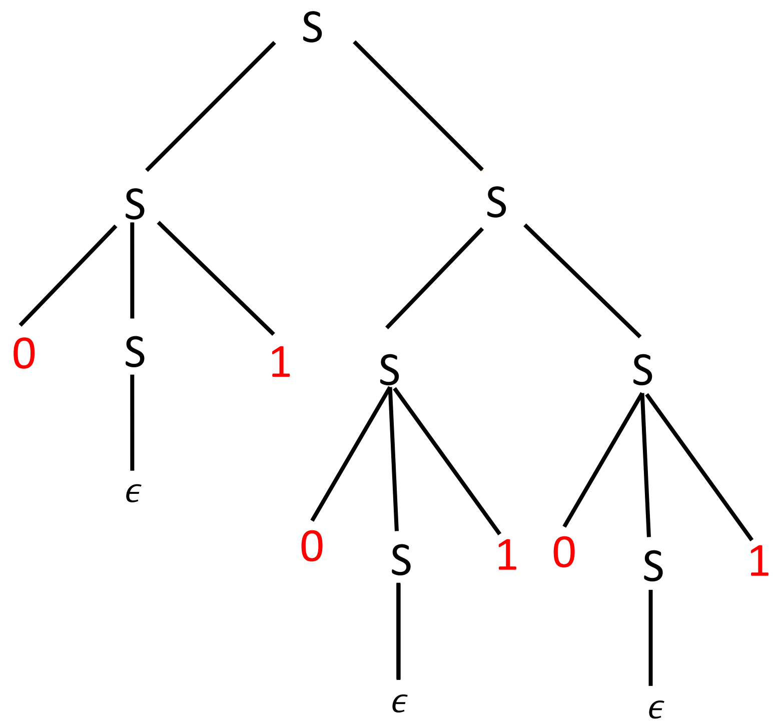

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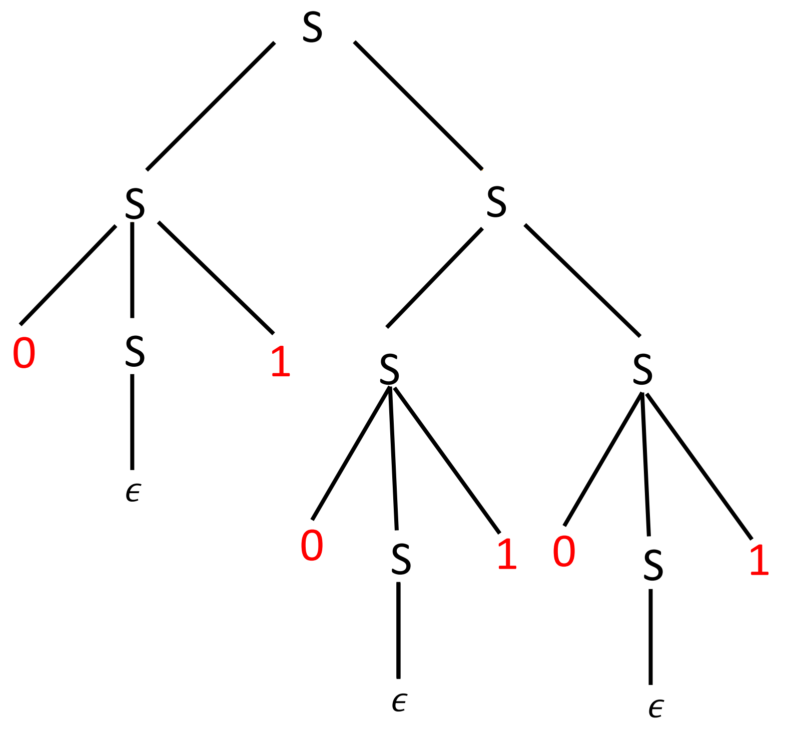

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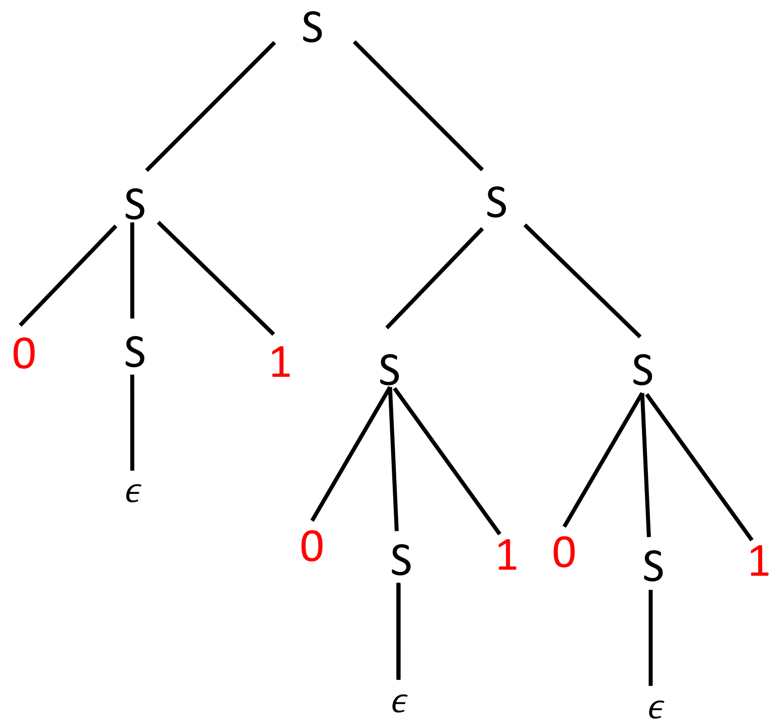

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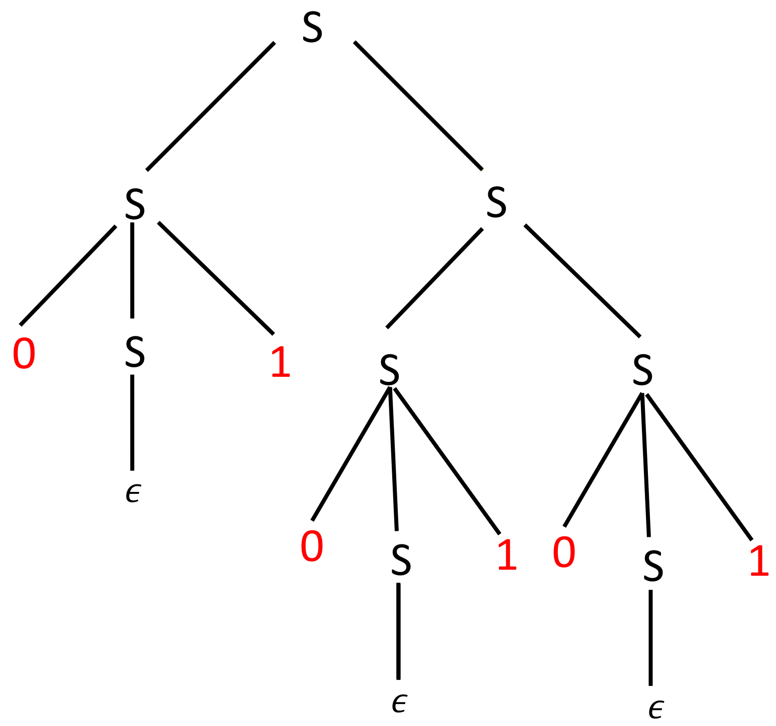

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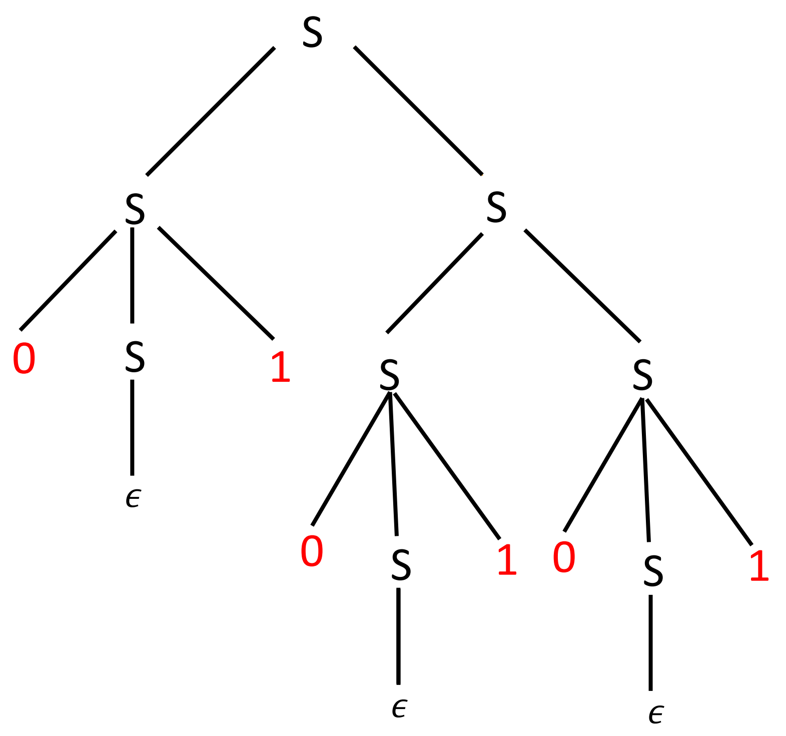

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Parse trees for CFG

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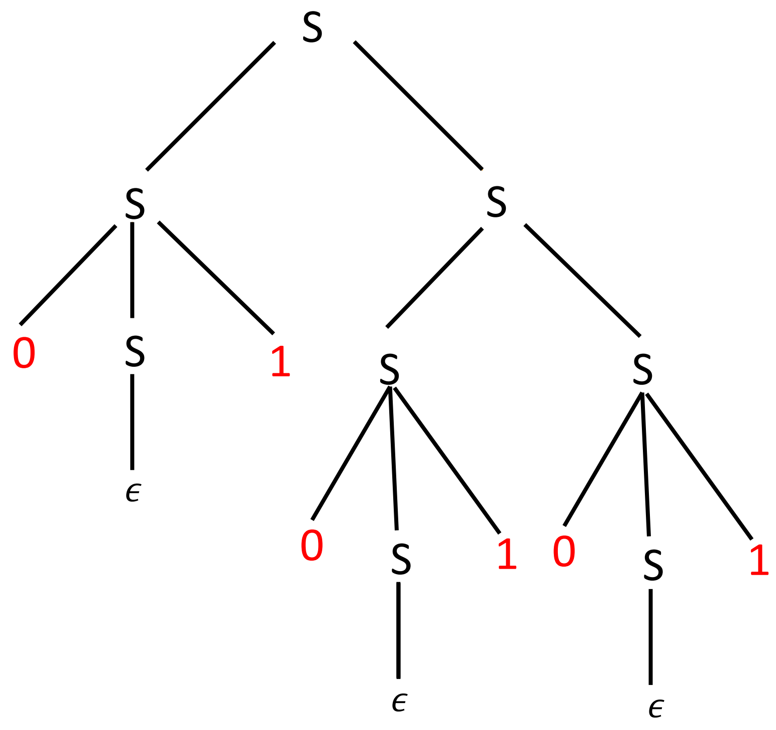
Consider the Grammar G with the following rules: $\boldsymbol{S} \rightarrow \mathbf{0}\boldsymbol{S}\mathbf{1}|\boldsymbol{SS}|\boldsymbol{\epsilon}$

Show that Grammar $\boldsymbol{G}$ is ambiguous, i.e. $\exists\, \omega \in L(G)$, such that there are two or more parse trees for ω.

Consider the string $\boldsymbol{\omega} = \mathbf{010101}$:

- Show that there exist two different parse trees for **010101**.
- Show that there exist two leftmost derivations for **010101.**



Leftmost Derivation: $S \rightarrow \boldsymbol{S}S \rightarrow 0\boldsymbol{S}1S \rightarrow 01\boldsymbol{S} \rightarrow 01\boldsymbol{S}S \rightarrow 010\boldsymbol{S}1S \rightarrow 0101\boldsymbol{S}$

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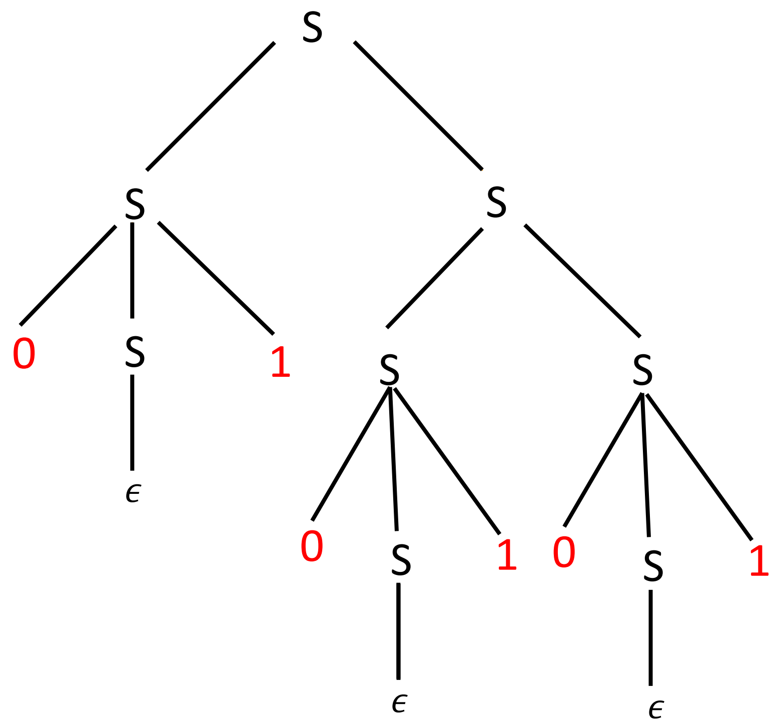

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Parse trees for CFG

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Consider the Grammar G with the following rules: $\boldsymbol{S \rightarrow 0S1|SS|\epsilon}$

Show that Grammar $\boldsymbol{G}$ is ambiguous, i.e. $\exists\, \omega \in L(G)$, such that there are two or more parse trees for ω.

Consider the string $\boldsymbol{\omega = 010101}$:

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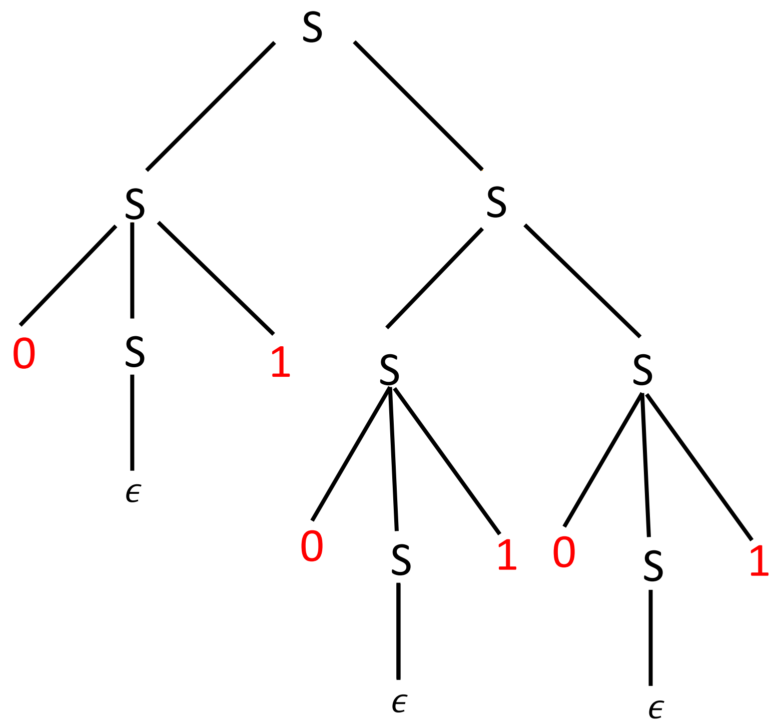
- Show that there exist two leftmost derivations for **010101.**



Leftmost Derivation: $S \rightarrow \boldsymbol{SS} \rightarrow \boldsymbol{0S1}S \rightarrow 01\boldsymbol{S} \rightarrow 01\boldsymbol{S}S \rightarrow 01\boldsymbol{0S1}S \rightarrow 0101\boldsymbol{S} \rightarrow 0101\boldsymbol{0S1} \rightarrow \boldsymbol{010101}$

Parse trees for CFG

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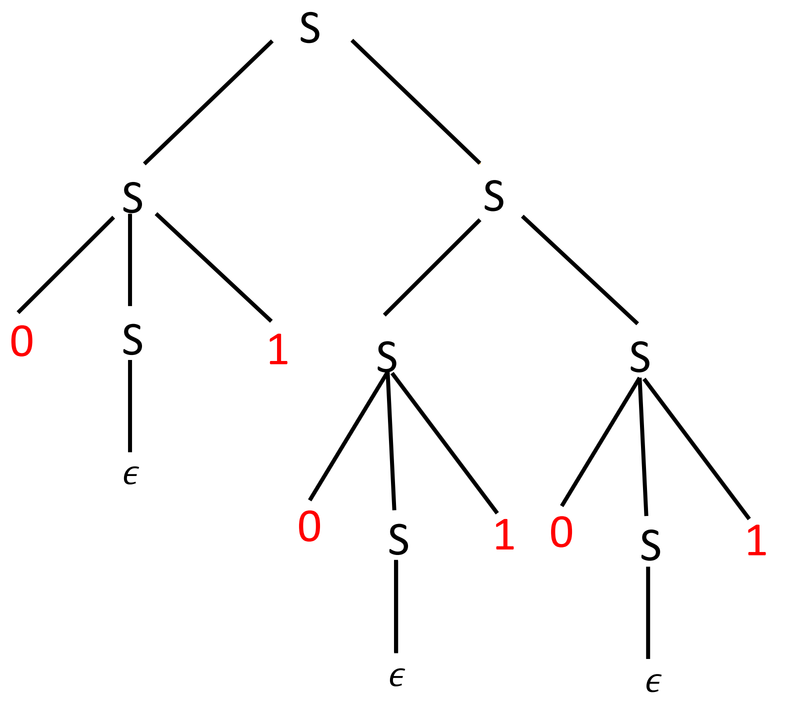
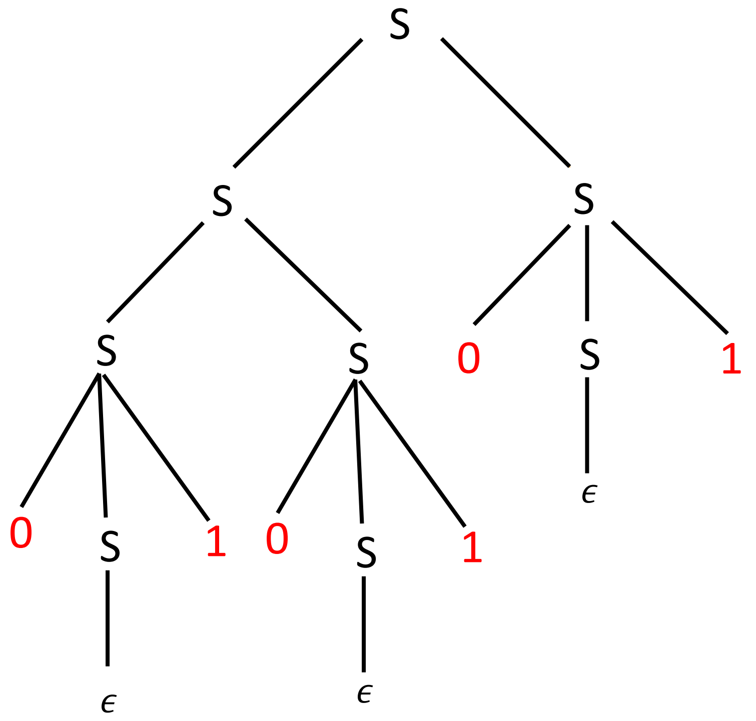


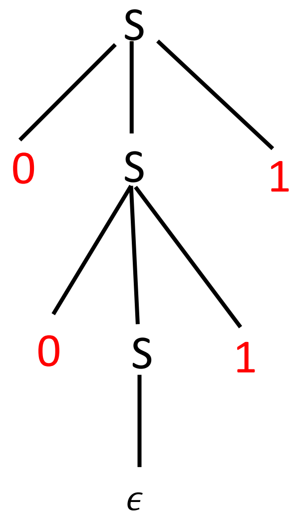


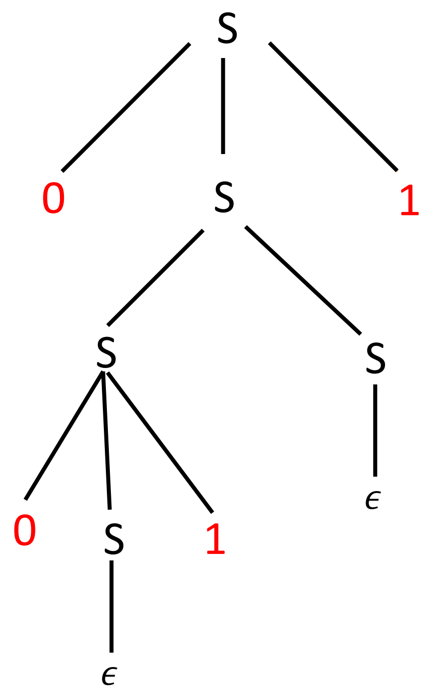
Leftmost Derivation: $S \rightarrow \boldsymbol{SS} \rightarrow \boldsymbol{SSS} \rightarrow \mathbf{0}\boldsymbol{S}\mathbf{1}SS \rightarrow 01SS \rightarrow 01\mathbf{0}\boldsymbol{S}\mathbf{1}S \rightarrow 0101\boldsymbol{S} \rightarrow 0101\mathbf{0}\boldsymbol{S}\mathbf{1} \rightarrow \mathbf{010101}$

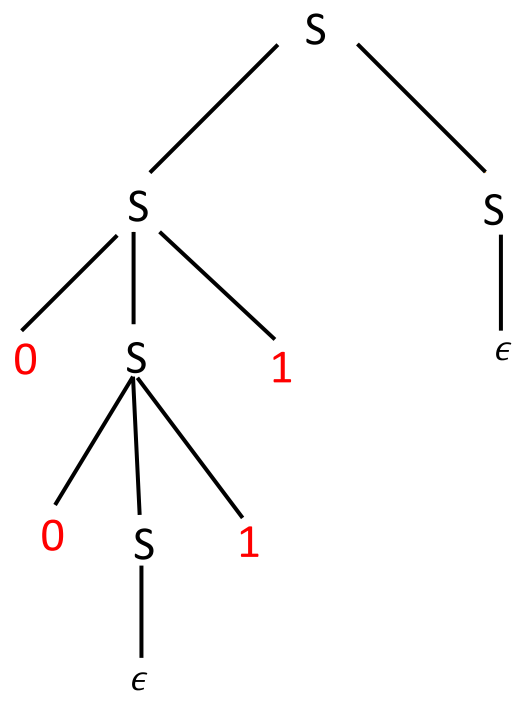
Parse trees for CFG

Show that the Grammar G with the following rules: $\boldsymbol{S \rightarrow 0S1 | SS | \epsilon}$ is ambiguous.

Consider string $\boldsymbol{\omega = 0011}$







LD: $S \rightarrow \mathbf{0S1} \rightarrow 0\mathbf{0S1}1 \rightarrow \mathbf{0011}$

LD: $S \rightarrow \mathbf{0S1} \rightarrow 0\mathbf{SS}1 \rightarrow 0\mathbf{0S1}S1 \rightarrow 001S1 \rightarrow \mathbf{0011}$

LD: $S \rightarrow \mathbf{SS} \rightarrow \mathbf{0S1}S \rightarrow 0\mathbf{0S1}1S \rightarrow 0011\mathbf{S} \rightarrow \mathbf{0011}$

Ambiguity

Unique structures are important. For example:

- The syntax of a programming language can be represented by a CFG.

- A compiler
  - translates the code written in the programming language into a form that is suitable for execution.
  - checks if the underlying programming language is syntactically correct.

- Parse trees are data structures that represent such structures.
- Parse tree for the code helps analyze the syntax. So ambiguity might lead to different interpretations and hence, different outcomes for the same code.

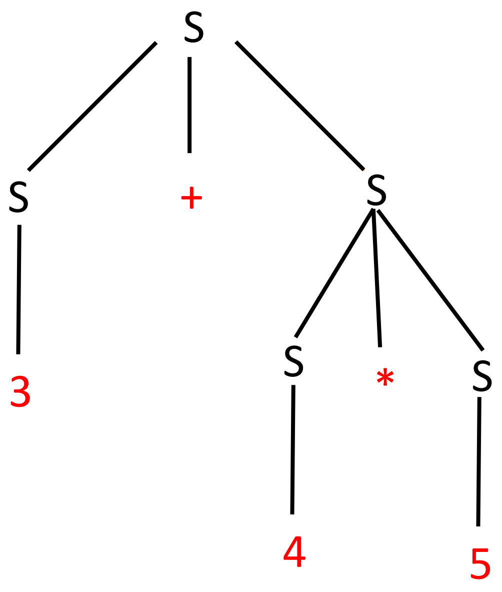
Ambiguity may not be desirable.

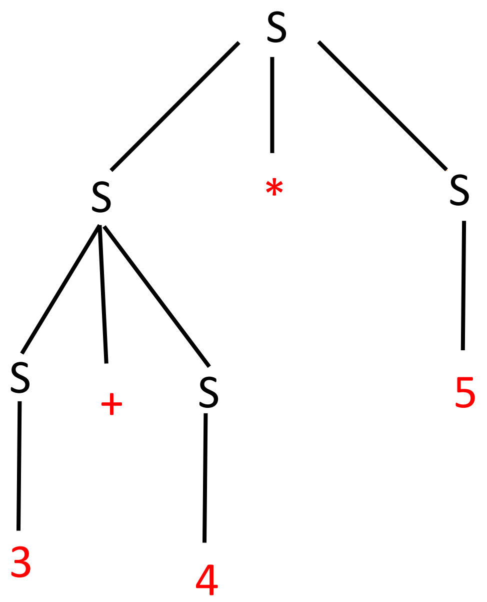
Ambiguity

Ambiguity may not be desirable.

Consider the grammar: $\boldsymbol{S \to S + S \mid S * S \mid 0 \mid 1 \mid 2 \mid \cdots \mid 9}$

and the derivation of the string $\mathbf{3 + 4 * 5}$





- The grammar contains no information on the precedence relations of the various arithmetic operations.
- The grammar may group $+$ before $*$

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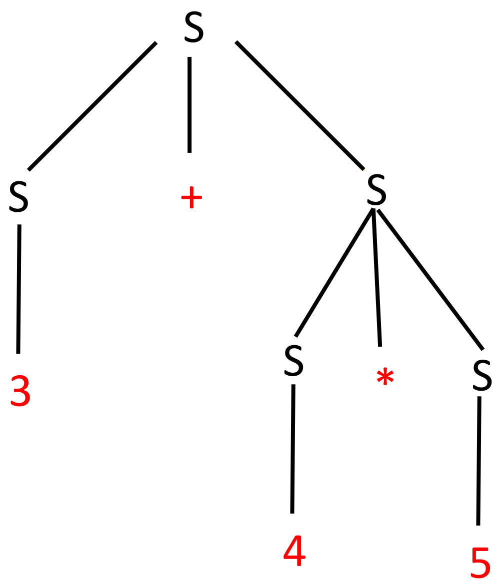

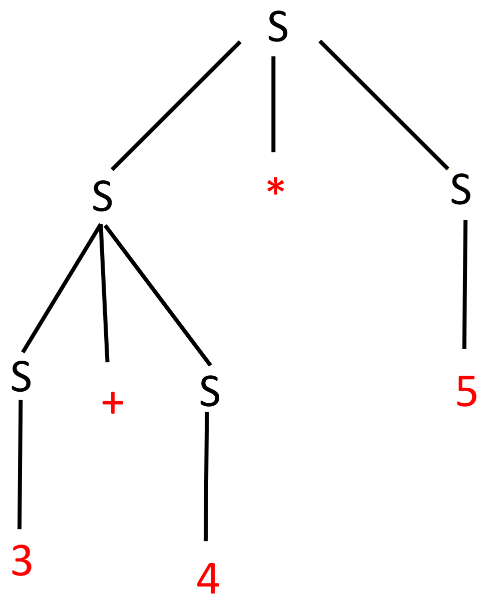

- What will be the result obtained from each of these ***parsings***?

Ambiguity

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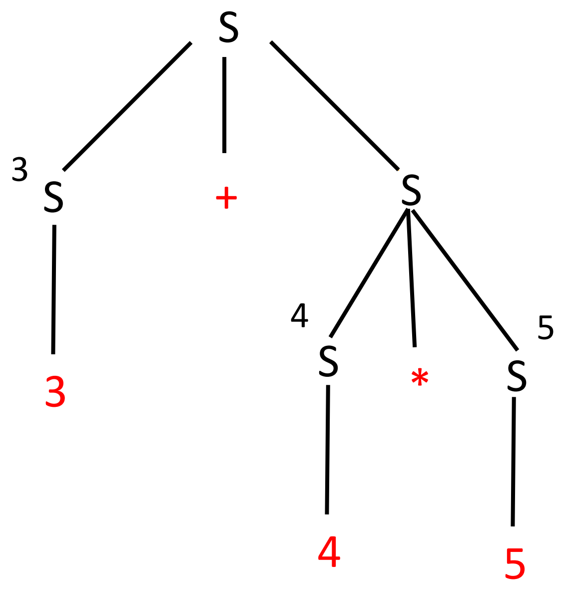

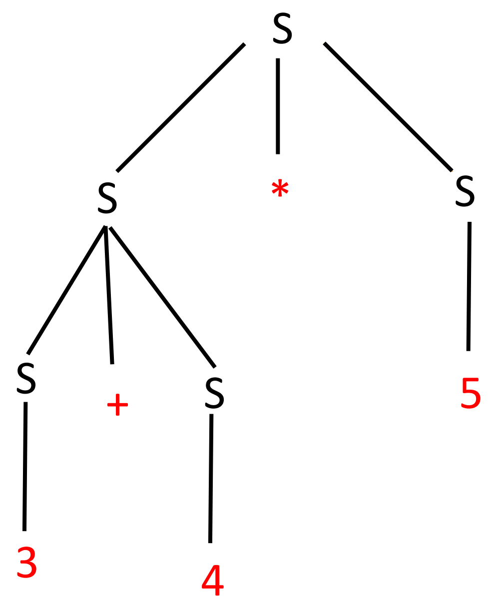

- If the compiler compiles the left parse tree

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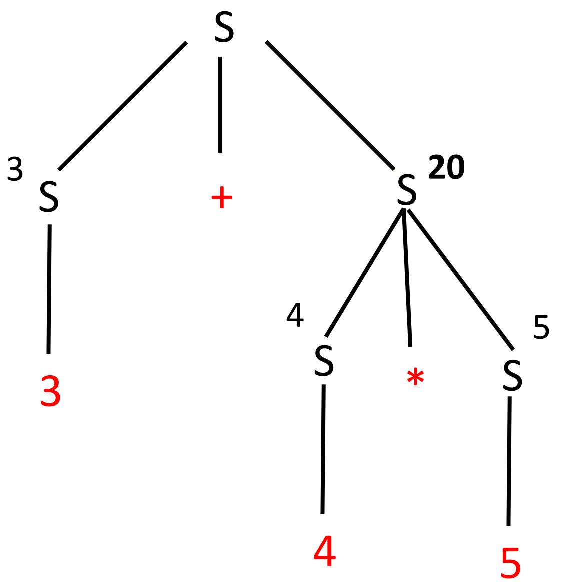

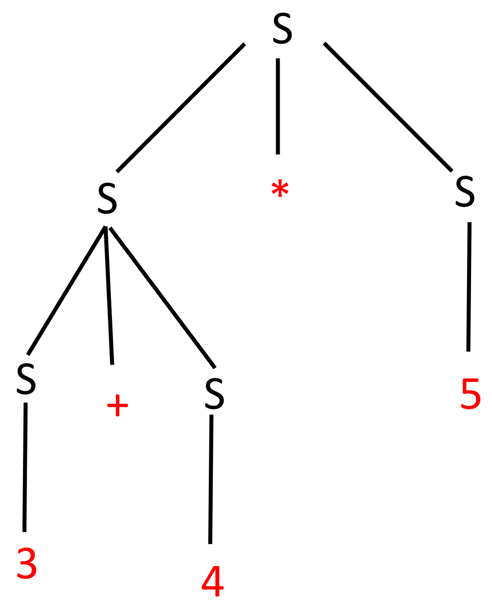

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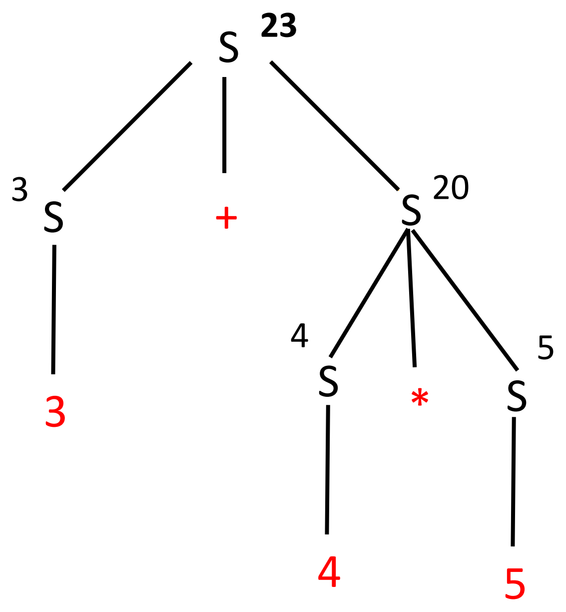

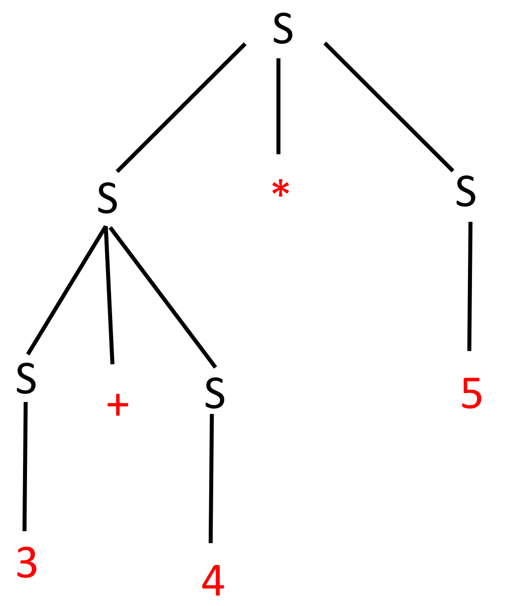

- If the compiler compiles the left parse tree. Outcome $= \mathbf{23}$

Ambiguity

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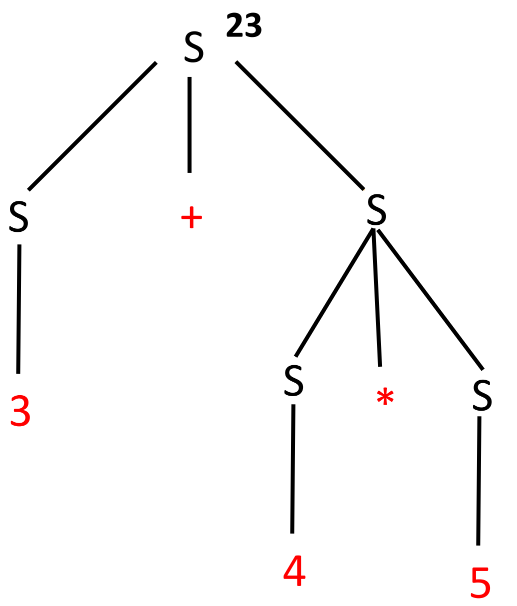

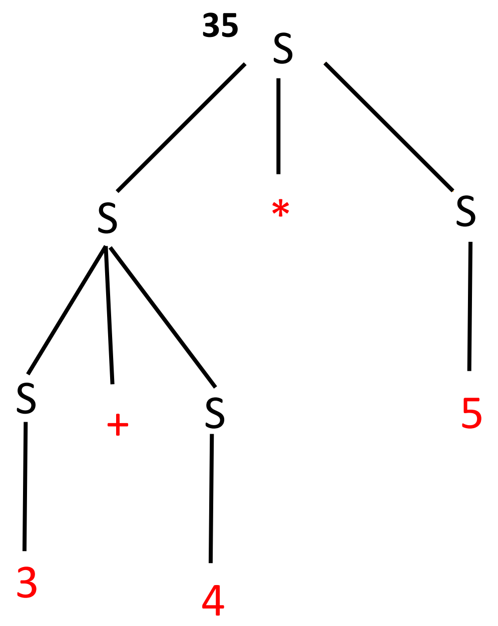

- If the compiler compiles the **right** parse tree. Outcome $= \mathbf{35}$

Ambiguity

Ambiguity may not be desirable.

Consider the grammar: $\boldsymbol{S \to S + S \mid S * S \mid 0 \mid 1 \mid 2 \mid \cdots \mid 9}$

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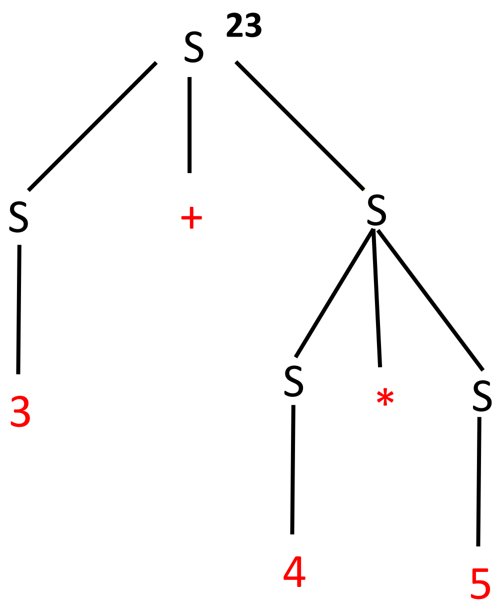
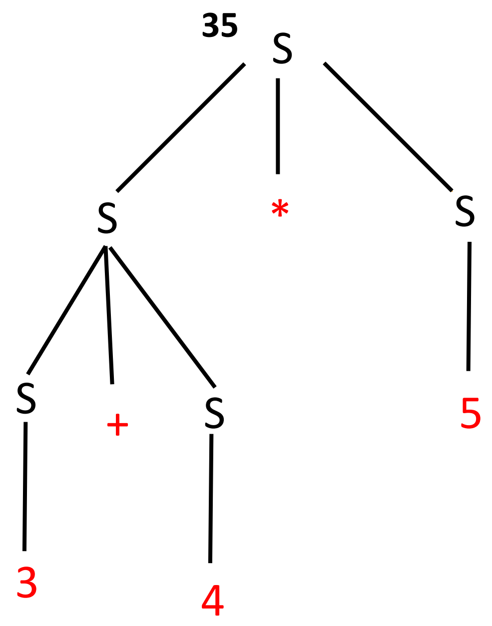




- **How can we get rid of this ambiguity?**

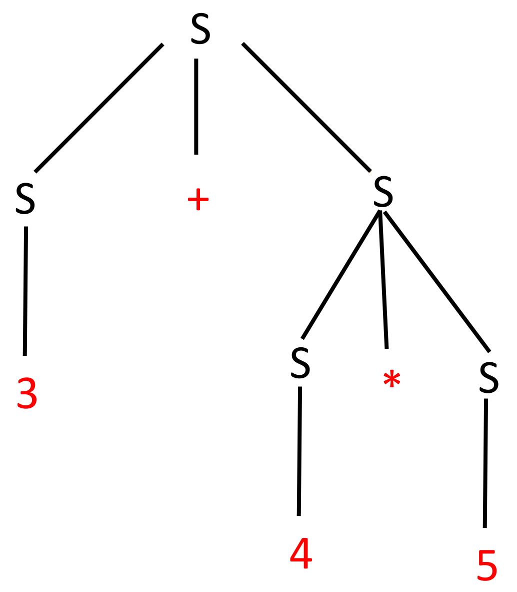
Ambiguity

Consider the grammar: $\boldsymbol{S} \rightarrow \boldsymbol{S} + \boldsymbol{S} \mid \boldsymbol{S} * \boldsymbol{S} \mid \mathbf{0} \mid \mathbf{1} \mid \mathbf{2} \mid \cdots \mid \mathbf{9}$

How can we get rid of this ambiguity? Change the production rules

1) Add parenthesis

New Grammar: $\boldsymbol{S} \rightarrow (\boldsymbol{S} + \boldsymbol{S}) \mid (\boldsymbol{S} * \boldsymbol{S}) \mid \mathbf{0} \mid \mathbf{1} \mid \mathbf{2} \mid \cdots \mid \mathbf{9}$



Old Parse tree (before adding parenthesis)

Ambiguity

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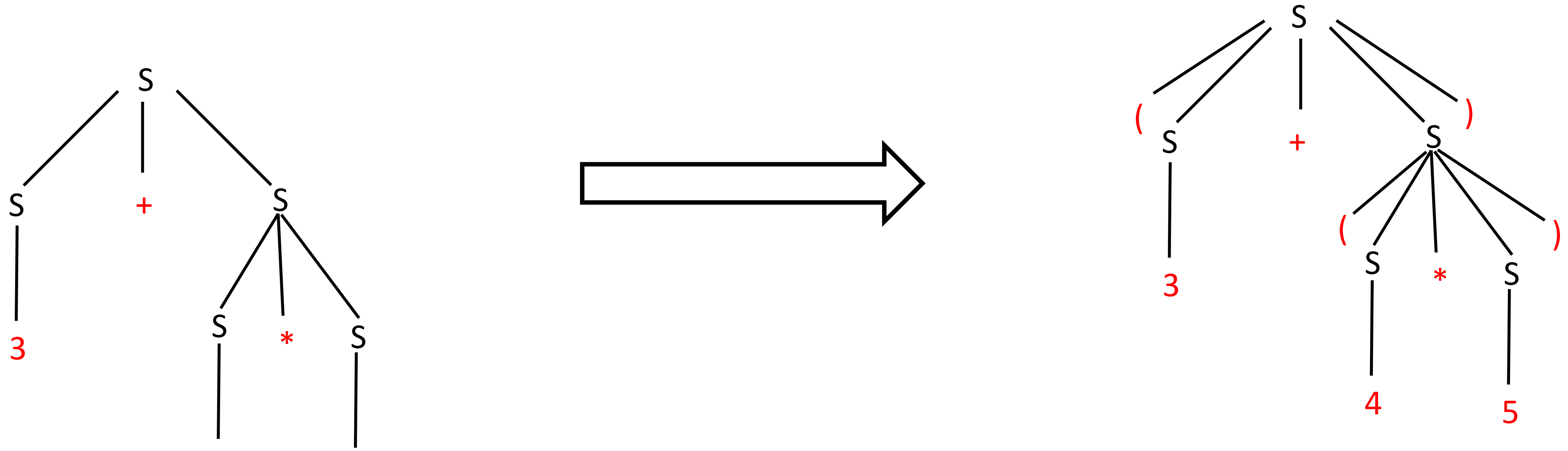

$\big(3 + (4 * 5)\big) = 23$

Ambiguity

Consider the grammar: $\boldsymbol{S \rightarrow S + S \mid S * S \mid 0 \mid 1 \mid 2 \mid \cdots \mid 9}$

How can we get rid of this ambiguity? Change the production rules

1) **Add parentheses**
2) **Add new variables**

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New Grammar:

$$\boldsymbol{E \to E + T \mid T}$$
$$\boldsymbol{T \to T * F \mid F}$$
$$\boldsymbol{F \to 0 \mid 1 \mid 2 \mid \cdots \mid 9 \mid E}$$

Ambiguity

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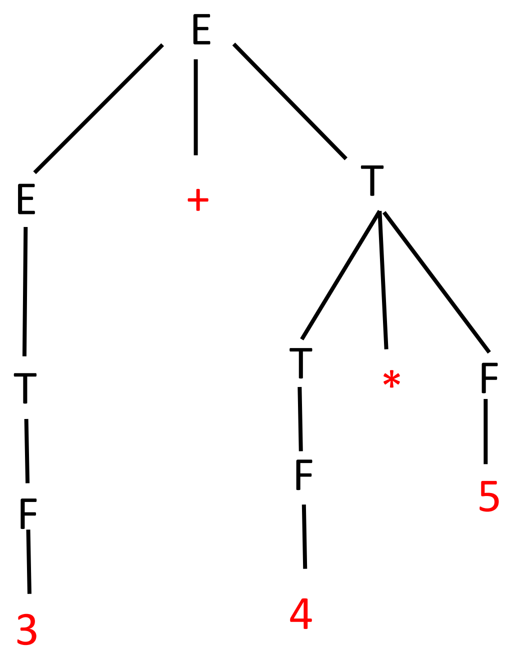
Parse tree to derive: $\mathbf{3 + (4 * 5)}$



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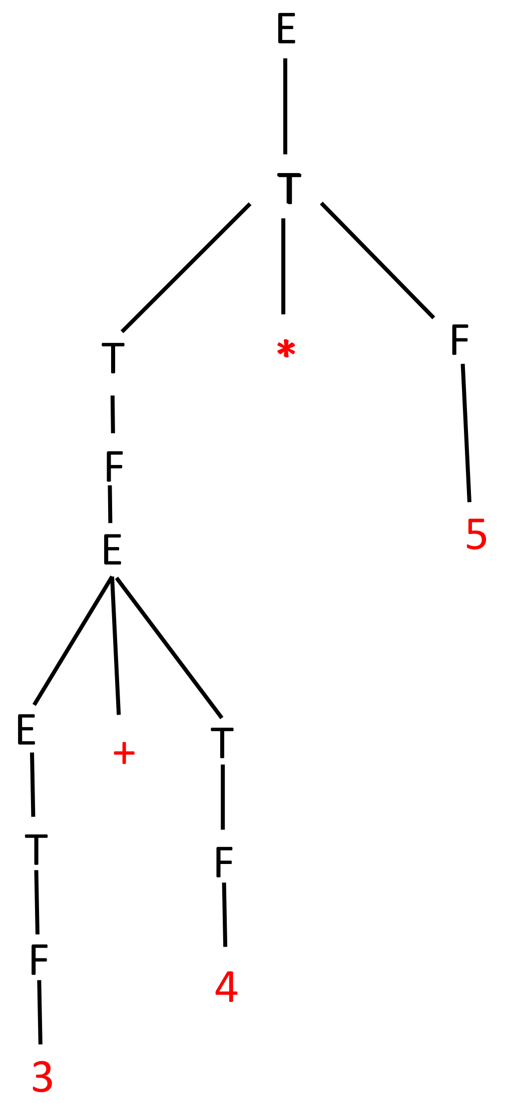
Parse tree to derive: $\boldsymbol{(3 + 4) * 5}$



Ambiguity

How can we get rid of this ambiguity? Change the production rules

1) **Add parentheses**
2) **Add new variables**

- In general, it is not possible to write an algorithm that takes as input a grammar G and outputs, YES if G is ambiguous and NO, otherwise. **(Undecidable)**

- So removing ambiguity is impossible in general.

Thank You!