Probability and Statistics: MA6.101

Homework 1

Topics Covered: Introductory Calculus, Maxima & Minima, Stationary Points, Sets, Basic Sequences and Series

Permutation and Combinations

1. Eight students are running for the titles of "Felicity Coordinator" and "Corporate Head". In how many different ways can the two positions be assigned?
   A. 28
   B. 56
   C. 56
   D. 64
   E. 72
2. A trivia competition team must consist of 4 students chosen from 15. How many different teams are possible?
   A. 1345
   B. 1260
   C. 1365
   D. 1512
   E. 1780
3. Twelve distinct prizes are to be awarded to the top 3 winners (first, second, third). In how many ways can the prizes be distributed?
   A. 132
   B. 720
   C. 1320
   D. 1440
   E. 1560
4. At a food stall, a student selects one main course, one side, and one drink from 5, 3, and 4 options respectively. How many total meal combinations are possible?
   A. 45
   B. 60
   C. 80
   D. 72
   E. 120
5. A club with 20 members is electing a president, vice president, and secretary. Each must be distinct. In how many ways can the officers be elected?

A. 684

B. 68400

C. 6840

D. 6080

E. 7200

6. Six cards are to be placed in six envelopes numbered 1 to 6 such that no card is placed in the envelope with the same number, and card 1 is placed in envelope 2. How many such arrangements exist?

   A. 44

   B. 52

   C. 53

   D. 64

   E. 68

7. Evaluate the sum: $\sum_{r=1}^{15} \frac{r^2\binom{15}{r}}{\binom{15}{r-1}}$

   A. 540

   B. 600

   C. 640

   D. 680

   E. 720

8. A bag has 10 white and 3 black balls. Balls are drawn one-by-one without replacement until all 3 black balls are drawn. What is the probability that the last black ball appears on the 7th draw?

   A. $\frac{15}{286}$

   B. $\frac{1}{13}$

   C. $\frac{20}{286}$

   D. $\frac{13}{286}$

   E. $\frac{1}{26}$

Linear Algebra

Q1: a. Let $P, Q \in \mathbb{R}^{n\times n}$ be permutation matrices (i.e., each is obtained by reordering the rows of the identity matrix I_n, or in other words a, square matrix with exactly one entry of 1 in each row and each column, with all other entries being 0). Prove that

$$P - Q$$

is singular.

(Think along the lines of Q2 of Linear Algebra from Tutorial 1)

Q2: Using properties of determinants, prove that:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} b & h & e \\ a & g & d \\ c & i & f \end{vmatrix}$$

Q3: (Recommended) Consider a graph G with the corresponding adjacency matrix A. Prove that the number of walks from node i to node j of length m is given by the entry $(A^m)_{ij}$.

Introductory Calculus

Q1: Evaluate the following limit

$$\lim_{n\to\infty} \prod_{k=1}^{n} \cos\left(\tfrac{x}{2^k}\right).$$

Q2: Let

$$u(x) \ = \ \int_0^x |t| \, \mathrm{d}t.$$

(a) Find $u'(x)$ for all real x.

(b) Discuss the continuity of $u'(x)$ at $x = 0$.

Maxima / Minima and Stationary Points

Q1: **True or False:** (Fun Question)
Every continuous function on a closed and bounded interval must attain a maximum.

What if the function is defined on the half-open interval $[0, 1)$ instead?

Sets

Q1: Prove or disprove:
If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.

Miscellaneous

Q1: In a logistics warehouse, a robotic system operates over a period of 30 days to stack boxes according to the following rules:

- On the k-th day ($1 \le k \le 30$):
  - The robot stacks exactly k boxes.
  - Each box weighs $\dfrac{1}{2^k}$ kilograms.
  - Placing each box takes k minutes.
  - The robot's overall efficiency decreases exponentially: the effective time to place each box is scaled by a factor of $\dfrac{1}{2^k}$.

Let:

$$W = \sum_{k=1}^{30} k \cdot \frac{1}{2^k} \quad \text{and} \quad T = \sum_{k=1}^{30} k^2 \cdot \frac{1}{2^k}$$

Tasks

(a) Evaluate W in closed form.

(b) Simplify or evaluate T.

(c) Compute the average time per kilogram over the 30-day period:

$$\frac{T}{W}$$

(d) Assume the robot continues this stacking process indefinitely (i.e., $k \to \infty$). Compute the exact values of:

$$W_\infty = \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k}, \qquad T_\infty = \sum_{k=1}^{\infty} k^2 \cdot \frac{1}{2^k}$$

Hence, determine:

$$\lim_{n \to \infty} \frac{T_n}{W_n}$$

Q2: Let $G = a, ar, ar^2, ar^3, \ldots$ be a geometric progression with positive real terms. Suppose the sum of the first $2n$ terms is equal to S_{2n}, and the sum of the terms from the $(n+1)^{\text{th}}$ term to the $2n^{\text{th}}$ term is equal to T_n. If it is known that $\frac{T_n}{S_n} = r^n$, where S_n is the sum of the first n terms, find the common ratio r.

Q3: Extreme Points Find the local extrema of the function:

$$f(x, y) = x^3 - 3x + y^2 - 4y$$

and classify each critical point.