Probability and Statistics: MA6.101

Tutorial 2

Topics Covered: Sigma Algebra, Probability spaces, Conditional Probability, and Total Probability.

Q1: A 6-sided die is rolled n times. What is the probability all faces have appeared?
Soln. First, we calculate the total number of sequences of length n with digits from 1 to 6:

$$\text{Total number of sequences} = 6^n$$

Let A_i be the set of sequences of length n that do not contain the digit i. Using the Principle of Inclusion and Exclusion (PIE), we find the number of sequences that do not contain any one of the digits:

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = \sum_{i=1}^{6} |A_i| - \sum_{1 \leq i < j \leq 6} |A_i \cap A_j| \cdots + (-1)^{6-1} |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6|$$

We calculate the intersections as follows: $|A_i| = 5^n$, $|A_i \cap A_j| = 4^n$, $|A_i \cap A_j \cap A_k| = 3^n$, ..., $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| = 0$

This simplifies to:

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = \binom{6}{1}5^n - \binom{6}{2}4^n + \binom{6}{3}3^n - \binom{6}{4}2^n + \binom{6}{5}1^n$$

The set of sequences that contain all digits from 1 to 6 is the complement of the above set. So, Number of sequences containing all digits:

$$6^n - \left(\binom{6}{1}5^n - \binom{6}{2}4^n + \binom{6}{3}3^n - \binom{6}{4}2^n + \binom{6}{5}1^n\right)$$

Finally, the probability that a sequence of length n contains all digits from 1 to 6 is:

$$\text{Probability} = \frac{6^n - \left(\binom{6}{1}5^n - \binom{6}{2}4^n + \binom{6}{3}3^n - \binom{6}{4}2^n + \binom{6}{5}1^n\right)}{6^n}$$

Q2: Let $S = \mathbb{N} = \{1, 2, 3, \dots\}$. Define

$$\mathcal{F} = \{A \subseteq \mathbb{N} \mid A \text{ is finite or } A^c \text{ is finite}\}.$$

Show that $\mathcal{F}$ is **not** a sigma-algebra.

Soln. A sigma-algebra must be closed under countable unions. Let us consider the set of even numbers:

$$E = \{2, 4, 6, 8, \dots\}.$$

Observe:

- E is infinite.
- Its complement $E^c = \{1, 3, 5, \dots\}$ is also infinite.

Therefore $E \notin \mathcal{F}$ (since neither E nor E^c is finite).

But E can be written as a countable union of finite sets:

$$E = \{2\} \cup \{4\} \cup \{6\} \cup \cdots$$

Each singleton $\{2n\}$ is finite, hence in $\mathcal{F}$.

If $\mathcal{F}$ were a sigma-algebra, it would have to contain E, because sigma-algebras are closed under countable unions. Since $E \notin \mathcal{F}$, $\mathcal{F}$ fails this property.

Conclusion: $\mathcal{F}$ is *not* a sigma-algebra because it is not closed under countable unions.

Q3: A certain disease affects about 1 out of 10,000 people. There is a test to check whether the person has the disease. The test is quite accurate. In particular, we know that:

(a) the probability that the test result is positive (suggesting the person has the disease), given that the person does not have the disease, is only 2 percent;

(b) the probability that the test result is negative (suggesting the person does not have the disease), given that the person has the disease, is only 1 percent.

A random person gets tested for the disease and the result comes back positive. What is the probability that the person has the disease?

Soln. Let D be the event that the disease occurs, and the event of not having the disease be D^c. Given $\mathbb{P}(D) = 0.0001$. Let the event of test result returning positive be T, and the event of test result returning negative be T^c. We are given the following probabilities:

(a) $\mathbb{P}(T|D^c) = 0.02$

(b) $\mathbb{P}(T^c|D) = 0.01$

We need to find $\mathbb{P}(D|T)$. From Bayes rule, we know that

$$\mathbb{P}(D|T) = \frac{\mathbb{P}(T|D) \cdot \mathbb{P}(D)}{\mathbb{P}(T)}$$

And from law of total probability, we can decompose $\mathbb{P}(T)$ into $\mathbb{P}(T|D) \cdot \mathbb{P}(D) + \mathbb{P}(T|D^c) \cdot \mathbb{P}(D^c)$

Hence, we finally get the following formula:

$$\mathbb{P}(D|T) = \frac{\mathbb{P}(T|D) \cdot \mathbb{P}(D)}{\mathbb{P}(T|D) \cdot \mathbb{P}(D) + \mathbb{P}(T|D^c) \cdot \mathbb{P}(D^c)}$$

Using the law $\mathbb{P}(A|B) + \mathbb{P}(A^c|B) = 1$, we get $\mathbb{P}(T|D) = 1 - \mathbb{P}(T^c|D) = 1 - 0.01 = 0.99$.

Also, $\mathbb{P}(D^c) = 1 - \mathbb{P}(D) = 1 - 0.0001 = 0.9999$.

Substituting the values in the formula:

$$\mathbb{P}(D|T) = \frac{0.99 \times 0.0001}{0.99 \times 0.0001 + 0.02 \times 0.9999} = \frac{0.000099}{0.000099 + 0.019998} = \frac{0.000099}{0.020097} \approx 0.0049$$

Q4: Let $\mathcal{F}$ be a σ-algebra of subsets of Ω. Show that $\mathcal{F}$ is closed under countable intersections $\bigcap_n A_n$, under set differences $(A \setminus B)$, under symmetric differences $(A \Delta B)$.

Soln.

(a) $\Omega \in \mathcal{F}$.

(b) If $A \in \mathcal{F}$, then its complement A^c is also in $\mathcal{F}$.

(c) If $\{A_n\}_{n=1}^{\infty}$ is a countable sequence of sets in $\mathcal{F}$, then their union $\bigcup_{n=1}^{\infty} A_n$ is also in $\mathcal{F}$.

We will use these axioms to prove the required closure properties.

(a) **Closure under Countable Intersections**
Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets in $\mathcal{F}$. To show that $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$, we use De Morgan's laws.

- By axiom (b), since each $A_n \in \mathcal{F}$, its complement A_n^c must also be in $\mathcal{F}$.
- This gives us a countable collection of sets $\{A_n^c\}_{n=1}^{\infty}$, all of which are in $\mathcal{F}$.
- By axiom (c), the countable union of these complements is in $\mathcal{F}$:

$$\bigcup_{n=1}^{\infty} A_n^c \in \mathcal{F}$$

- Finally, by axiom (b) again, the complement of this union must also be in $\mathcal{F}$. Using De Morgan's law, we get:

$$\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c = \bigcap_{n=1}^{\infty} (A_n^c)^c = \bigcap_{n=1}^{\infty} A_n$$

- Therefore, $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

(b) **Closure under Set Differences**
Let $A, B \in \mathcal{F}$. We want to show that $A \setminus B \in \mathcal{F}$. The set difference can be expressed as an intersection:

$$A \setminus B = A \cap B^c$$

- Since $B \in \mathcal{F}$, axiom (b) guarantees that its complement B^c is also in $\mathcal{F}$.
- We now have two sets, A and B^c, which are both in $\mathcal{F}$.
- As shown in part (a), $\mathcal{F}$ is closed under countable intersections. A finite intersection is a special case of a countable intersection.
- Therefore, the intersection $A \cap B^c$ is in $\mathcal{F}$, which means $A \setminus B \in \mathcal{F}$.

(c) **Closure under Symmetric Differences**
Let $A, B \in \mathcal{F}$. We want to show that $A \Delta B \in \mathcal{F}$. The symmetric difference can be written as:

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

- From part (b), we know that if $A, B \in \mathcal{F}$, then the set differences $(A \setminus B)$ and $(B \setminus A)$ are both in $\mathcal{F}$.

- The symmetric difference is the union of these two sets.
- A finite union is a special case of a countable union. By axiom (3), $\mathcal{F}$ is closed under countable unions.
- Therefore, the union $(A \setminus B) \cup (B \setminus A)$ is in $\mathcal{F}$, which means $A \Delta B \in \mathcal{F}$.

Q5: You are standing at a fairground game where you toss rings until you win a prize. The number of tosses T you make until your first win satisfies:

$$\mathbb{P}(T \geq t) = \frac{1}{1+\frac{t}{4}}, \quad t \geq 0.$$

For example,

$$\mathbb{P}(T \geq 4) = \frac{1}{1+\frac{4}{4}} = \frac{1}{2}.$$

You have already made 4 tosses without winning. What is the probability that you win **on the 5th toss**?

Soln. Let A be the event that you win on the 5th toss (i.e., $T = 5$). Also, let B be the event that you do not win in the first four tosses (i.e., $T \geq 5$). We are interested in $P(A \mid B)$. We have

$$P(B) = P(T \geq 5) = \frac{4}{4+5} = \frac{4}{9}.$$

We also have

$$P(A) = P(T = 5) = P(T \geq 5) - P(T \geq 6) = \frac{4}{9} - \frac{4}{10} = \frac{2}{45}.$$

Finally, since $A \subset B$, we have $A \cap B = A$. Therefore,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{2}{45}}{\frac{4}{9}} = \frac{1}{10} = 0.1.$$

Q6: Two players take turns rolling two fair six-sided dice. Player A goes first, followed by player B. If player A rolls a sum of 6, they win. If player B rolls a sum of 7, they win. If neither rolls their desired value, the game continues until someone wins. What is the probability that player A wins?

Soln: Dice game (A targets 6, B targets 7).
Let $p_A = \mathbb{P}(\text{sum} = 6) = 5/36$ and $p_B = \mathbb{P}(\text{sum} = 7) = 6/36 = 1/6$. Let $q_A = 1 - p_A = 31/36$ and $q_B = 1 - p_B = 5/6$. A wins on their first roll with probability p_A. If both fail in a round (A fails and then B fails), the game resets; this happens with probability $q_A q_B$. Hence

$$\mathbb{P}(\text{A wins}) = p_A \sum_{k=0}^{\infty} (q_A q_B)^k = \frac{p_A}{1 - q_A q_B} = \frac{\frac{5}{36}}{1 - \frac{31}{36} \cdot \frac{5}{6}} = \frac{30}{61} \approx 0.4918.$$

So player A wins with probability $\boxed{\frac{30}{61}}$.

Q7: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

Soln.

(a) To prove the first condition: we need to show that $\phi \in \mathcal{G}$ and $\Omega \in \mathcal{G}$. From axioms of probability, $\mathbb{P}(\phi) = 0$ and $\mathbb{P}(\Omega) = 1$. Hence by definition of $\mathcal{G}$, $\phi \in \mathcal{G}$ and $\Omega \in \mathcal{G}$.

(b) To prove the second condition:
Let $A \in \mathcal{G}$
$\implies \mathbb{P}(A) = 0$ OR $\mathbb{P}(A) = 1$.
$\implies 1 - \mathbb{P}(A) = 1$ OR $1 - \mathbb{P}(A) = 0$
$\implies \mathbb{P}(A^c) = 1$ OR $\mathbb{P}(A^c) = 0$
$\implies A^c \in \mathcal{G}$.

(c) To prove the third condition:
Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of sets in $\mathcal{G}$. We need to show that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$.
There are two possibilities to consider for each A_i:

- $P(A_i) = 0$
- $P(A_i) = 1$

First, let's consider the case where $\bigcup_{i=1}^{\infty} P(A_i) = 0$:
If $\mathbb{P}(A_i) = 0$ for all i, then the union $\bigcup_{i=1}^{\infty} A_i$ is also of measure 0. This follows from the countable subadditivity property of measures:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i) = 0.$$

Hence, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$.
Now, let's consider the case where $\bigcup_{i=1}^{\infty} P(A_i) = 1$:
If there exists at least one A_i such that $\mathbb{P}(A_i) = 1$, then the union $\bigcup_{i=1}^{\infty} A_i$ will be of measure 1. This follows because if any set in a countable collection has measure 1, the union of the entire collection must also have measure 1:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \geq P(A_i) = 1.$$

Hence, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$.

Q8: On each of its two wings a plane has 2 engines. We assume that the engines operate independently and $\mathbb{P}(\text{engine fails}) = p = 0.2$. A plane will not crash if at least one engine operates on each wing.

(a) What is the probability that it will not crash?

(b) How many engines should be installed on each wing to have the probability of not crashing at least 0.99?

(c) The plane has not crashed. What is the chance that all four engines are in a good shape?

Soln.

(a) Probability an engine works is $1 - p = 0.8$. For a wing with 2 engines, the probability both engines fail is $p^2 = (0.2)^2 = 0.04$.

Thus probability a wing *works* (at least one engine works) is

$$1 - p^2 = 1 - 0.04 = 0.96.$$

The two wings are independent, so

$$P(\text{plane not crash}) = P(\text{both wings work}) = (1 - p^2)^2 = 0.96^2 = 0.9216.$$

(b) Let n engines per wing. The probability a given wing works is $1 - p^n$. We require

$$(1 - p^n)^2 \geq 0.99 \quad \Longrightarrow \quad 1 - p^n \geq \sqrt{0.99}.$$

With $p = 0.2$ calculate small powers:

$$0.2^1 = 0.2, \quad 0.2^2 = 0.04, \quad 0.2^3 = 0.008, \quad 0.2^4 = 0.0016.$$

Compute $1 - \sqrt{0.99}$:

$$\sqrt{0.99} \approx 0.994987\ldots \quad \Rightarrow \quad 1 - \sqrt{0.99} \approx 0.0050126.$$

We need $p^n \leq 0.0050126$. Check:

$$0.2^3 = 0.008 > 0.0050126, \qquad 0.2^4 = 0.0016 < 0.0050126.$$

So the smallest integer n that works is $n = 4$.

(c) Let G = all 4 engines (both wings, 2 each) are operational. Then

$$P(G) = (1 - p)^4 = 0.8^4 = 0.4096.$$

$$P(G \mid \text{not crash}) = \frac{P(G)}{P(\text{not crash})} = \frac{0.4096}{0.9216} = \frac{4}{9}.$$