Probability and Statistics

Homework 5

Q1: Suppose a mixed random variable Y is defined as:

- With probability 0.4, $Y = 2$.
- With probability 0.6, $Y \sim \text{Exp}(1)$.

Find the following:

- $\mathbb{P}(Y \geq 1)$
- CDF of Y

Q2: Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}e^{-2x} + \dfrac{c\,y^2}{(1+x)^2}, & 0 \leq x < \infty,\ 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find $\mathbb{P}(0 \leq X \leq 1,\ 0 \leq Y \leq \frac{1}{2})$.

(c) Find $\mathbb{P}(0 \leq X \leq 1)$.

Q3: Let X_1, X_2, and X_3 be three i.i.d Bernoulli(p) random variables and

$$\begin{aligned} Y_1 &= \max(X_1, X_2), \\ Y_2 &= \max(X_1, X_3), \\ Y_3 &= \max(X_2, X_3), \\ Y &= Y_1 + Y_2 + Y_3. \end{aligned}$$

Find EY and $\text{Var}(Y)$.

Q4: A box contains two coins: a regular coin and a biased coin with $\mathbb{P}(H) = \frac{2}{3}$. I choose a coin at random and toss it once. I define the random variable X as a Bernoulli random variable associated with this coin toss, i.e., $X = 1$ if the result of the coin toss is heads and $X = 0$ otherwise. Then I take the remaining coin in the box and toss it once. I define the random variable Y as a Bernoulli random variable associated with the second coin toss. Find the joint PMF of X and Y. Are X and Y independent?

Q5: Find the marginal and joint CDF for X and Y

$$f_{XY}(x,y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Q6: The joint density function is given as $f_{X,Y}(x,y) = cx(y-x)e^{-y}$ for $0 \leq x \leq y < \infty$.

(a) Find c.

(b) Show that:

$$f_{X|Y}(x|y) = \frac{6x(y-x)}{y^3}\,, \quad 0 \le x \le y$$
$$f_{Y|X}(y|x) = (y-x)e^{x-y}\,, \quad 0 \le x \le y < \infty$$

Q7: Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let also

$$Y = g(X) = \begin{cases} X & 0 \le X \le \frac{1}{2} \\ \frac{1}{2} & X > \frac{1}{2} \end{cases}$$

Find the CDF of Y.

Q8: For two discrete random variables X and Y, show that

(a) $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

(b) $\mathbb{E}[f(X)+h(Y)] = \mathbb{E}[f(X)] + \mathbb{E}[h(Y)]$
where $f(X)$ and $h(Y)$ are arbitrary functions of the respective random variables.

Q9: Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{xy}{6}, & 0 \le x \le 1,\ 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 \le y \le 2$, find

(a) the conditional PDF of X given $Y = y$.

(b) $\mathbb{P}\big(X < \frac{1}{2} \,\big|\, Y = y\big)$.

Q10: Roll two fair six-sided dice and let $S = X+Y$ be the sum of the faces shown, where X and Y are the outcomes on the first and second die respectively. Consider the events:

$$A_1 = \{S \le 7\}, \qquad A_2 = \{S \ge 8\}.$$

(a) Find the conditional PMFs $P_{S|A_1}(s)$ and $P_{S|A_2}(s)$ for all possible values of s.

(b) Find $\mathbb{P}(S = 6)$.

Q11: Let X and Y be two independent random variables with given joint PMF. Let g and h be two functions of X and Y, respectively. Show that the random variables $g(X)$ and $h(Y)$ are independent.

Q12: A die with r faces, numbered $1, \ldots, r$, is rolled a fixed number of times n. The probability that the ith face comes up on any one roll is denoted p_i, and the results of different rolls are assumed independent. Let X_i be the number of times that the ith face comes up.

(a) Find the joint PMF $p_{X_1,\ldots,X_r}(k_1,\ldots,k_r)$.

(b) Find the expected value and variance of X_i.

(c) Find $\mathbb{E}[X_i X_j]$ for $i \neq j$.