Probability and Statistics: MA6.101
Homework 4

Topics Covered: Continuous Random Variable, Functions of Random Variable

Q1: Functions of Continuous Random Variables

Let $X_1 \sim \text{Uniform}(0, 1)$ and $X_2 \sim \text{Exp}(1)$, independent. Find $f_Y(y)$ for $Y = \max(X_1, X_2)$.

Q2: Geometric $\to$ Exponential Limit

Let Y be a Geometric(p) random variable with

$$\mathbb{P}(Y = k) = (1 - p)^{k-1}p, \qquad k = 1, 2, \ldots$$

Assume $p = \lambda h$ with $\lambda > 0$ and $h > 0$, and define the scaled variable $X = Yh$. Prove that for any fixed $x > 0$,

$$\lim_{h \downarrow 0} F_X(x) \;=\; 1 - e^{-\lambda x}$$

Q3: A particle leaves the origin under the influence of gravity, and its initial velocity v forms an angle φ with the horizontal axis. The path of the particle reaches the ground at a distance

$$d = \frac{v^2}{g} \sin(2\varphi)$$

from the origin, where g is the acceleration due to gravity.

Assuming that φ is a random variable uniformly distributed between 0 and $\pi/2$, determine:

(i) the density of d,

(ii) the probability that $d \leq d_0$.

Q4: The Product of Uniforms

Let U_1 and U_2 be two independent continuous random variables, both uniformly distributed on the interval $(0, 1)$. That is, $f_{U_i}(u) = 1$ for $0 < u < 1$.

Consider the random variable $W = U_1U_2$.

(a) Find the cumulative distribution function (CDF) of W, denoted $F_W(w)$.

(b) Find the probability density function (PDF) of W, denoted $f_W(w)$.

(c) Calculate the expected value of W, $E[W]$.

Q5: The Laplace Distribution

Let X be a continuous random variable with the probability density function (PDF) given by:

$$f_X(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}} \quad \text{for } -\infty < x < \infty$$

where $\mu \in \mathbb{R}$ is the location parameter and $b > 0$ is the scale parameter. This is the PDF of a Laplace distribution with parameters μ and b.

(a) Show that $f_X(x)$ is indeed a valid PDF (i.e., it integrates to 1).

(b) Calculate the expected value of X, $E[X]$.

(c) Calculate the variance of X, $\text{Var}(X)$.

Q6: Establish the validity of the expected value rule

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx,$$

where X is a continuous random variable with PDF f_X. This rule is often called the Law of the Unconscious Statistician (LOTUS).

Q7: Suppose the number of customers arriving at a store obeys a Poisson distribution with an average of λ customers per unit time. That is, if Y is the number of customers arriving in an interval of length t, then $Y \sim \text{Poisson}(\lambda t)$. Suppose that the store opens at time $t = 0$. Let X be the arrival time of the first customer. Find the distribution of X.

Q8: Alice throws darts at a circular target of radius r and is equally likely to hit any point in the target. Let X be the distance of Alice's hit from the center.

(a) Find the PDF, the mean, and the variance of X.

(b) The target has an inner circle of radius t. If $X \leq t$, Alice gets a score of $S = 1/X$. Otherwise his score is $S = 0$. Find the CDF of S. Is S a continuous random variable?

Q9: The Rain-Cursed Fest

For each day of Felicity, the rain gods independently sample rainfall from an exponential distribution:

$$X_i \sim \text{Exp}(\lambda), \quad f_{X_i}(x) = \lambda e^{-\lambda x},\ x \geq 0,\ i = 1, 2, 3.$$

a) Find the probability that at least one of the three days receives more than the average daily rainfall $1/\lambda$.

b) Let $M = \max\{X_1, X_2, X_3\}$. (i) Find the CDF and PDF of M. (ii) Compute $\mathbb{E}[M]$.

c) Let N be the number of days on which rainfall exceeds $1/\lambda$. Find $\mathbb{E}[N]$.

Q10: A stick of length 1 is broken at a uniformly random point. Answer the following related questions.

(a) **(Distribution of the longer piece)** Let L denote the length of the longer piece. Find the CDF and PDF of L, and compute $\mathbb{E}[L]$.

(b) **(Ratio of shorter to longer piece)** Let X and Y be the lengths of the shorter and longer pieces respectively, and define

$$R = \frac{X}{Y}.$$

Find (i) the CDF and PDF of R, (ii) $\mathbb{E}[R]$, and (iii) $\mathbb{E}[1/R]$ (if they exist).

(c) **(Break the longer half again)** Break the stick at random as before. Now break the longer piece (of length L) again at a uniformly random point along that longer piece. What is the probability that the three resulting pieces can serve as side-lengths of a triangle?

Q11: The Rayleigh distribution has PDF

$$f(x) = xe^{-x^2/2}, \quad x > 0.$$

Let X have the Rayleigh distribution.

(a) Find $E(X)$.

(b) Find $E(X^2)$.

Q12: Let $Z \sim \mathcal{N}(0,1)$ and let $c \geq 0$. Find

$$\mathbb{E}\big[\max(Z - c, 0)\big]$$

in terms of the standard normal CDF Φ and PDF φ. This is the expected payoff of a European call when the stock price at maturity is assumed to be standard gaussian and strike price c.