Probability and Statistics

Homework 5 Solutions

Q1: (a) $\mathbb{P}(Y \geq 1) = 0.4 \cdot \mathbb{P}(2 \geq 1) + 0.6 \cdot \mathbb{P}(Z \geq 1) = 0.4 \cdot 1 + 0.6 \cdot e^{-1} = 0.4 + 0.6e^{-1} = 0.62$, where $Z \sim \text{Exp}(1)$.

(b) CDF of Y:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 0.6(1 - e^{-y}) & \text{if } 0 \leq y < 2 \\ 0.6(1 - e^{-y}) + 0.4 = 1 - 0.6e^{-y} & \text{if } y \geq 2 \end{cases}$$

Q2: (a) For f_{XY} to be a valid PDF: $\iint f_{XY}(x, y)\, dx\, dy = 1$

$$\int_0^\infty \int_0^1 \left(\frac{1}{2} e^{-2x} + \frac{cy^2}{(1+x)^2} \right) dy\, dx = 1$$

$$\int_0^\infty \left[\frac{1}{2} e^{-2x} + \frac{c}{3(1+x)^2} \right] dx = 1$$

$$\frac{1}{2} \cdot \frac{1}{2} + \frac{c}{3} \cdot 1 = \frac{1}{4} + \frac{c}{3} = 1$$

Therefore: $c = \frac{9}{4}$

(b) $\mathbb{P}(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$:

$$\int_0^1 \int_0^{1/2} \left(\frac{1}{2} e^{-2x} + \frac{9y^2}{4(1+x)^2} \right) dy\, dx$$

$$= \int_0^1 \left[\frac{1}{4} e^{-2x} + \frac{9}{4(1+x)^2} \cdot \frac{1}{24} \right] dx$$

$$= \int_0^1 \left[\frac{1}{4} e^{-2x} + \frac{3}{32(1+x)^2} \right] dx$$

$$= \frac{1}{8}(1 - e^{-2}) + \frac{3}{32} \cdot \frac{1}{2} = \frac{1}{8}(1 - e^{-2}) + \frac{3}{64}$$

(c) $\mathbb{P}(0 \leq X \leq 1) = \int_0^1 \int_0^1 f_{XY}(x, y)\, dy\, dx$:

$$\int_0^1 \left[\frac{1}{2} e^{-2x} + \frac{9}{4(1+x)^2} \cdot \frac{1}{3} \right] dx = \int_0^1 \left[\frac{1}{2} e^{-2x} + \frac{3}{4(1+x)^2} \right] dx$$

$$= \frac{1}{4}(1 - e^{-2}) + \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{4}(1 - e^{-2}) + \frac{3}{8}$$

Q3: The random variables X_1, X_2, X_3 are independent and identically distributed (i.i.d.) Bernoulli(p) variables. This means $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$ for $i = 1, 2, 3$.

Finding $\mathbb{E}[Y]$

First, we find the expected value of each Y_i.

For $Y_1 = \max(X_1, X_2)$, Y_1 is 1 if either $X_1 = 1$ or $X_2 = 1$.

$\mathbb{P}(Y_1 = 1) = 1 - \mathbb{P}(\max(X_1, X_2) = 0) = 1 - \mathbb{P}(X_1 = 0, X_2 = 0)$.

Since X_1 and X_2 are independent, $\mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0) = (1-p)(1-p) = (1-p)^2$.

Thus, $\mathbb{P}(Y_1 = 1) = 1 - (1-p)^2 = 1 - (1 - 2p + p^2) = 2p - p^2$. The expected value of a Bernoulli random variable is its success probability, so $\mathbb{E}[Y_1] = 2p - p^2$.

By symmetry, $\mathbb{E}[Y_2] = \mathbb{E}[Y_3] = 2p - p^2$.

Using the linearity of expectation, we can find $\mathbb{E}[Y]$:

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[Y_1 + Y_2 + Y_3] \\
&= \mathbb{E}[Y_1] + \mathbb{E}[Y_2] + \mathbb{E}[Y_3] \\
&= (2p - p^2) + (2p - p^2) + (2p - p^2) \\
&= 3(2p - p^2) = 6p - 3p^2.
\end{aligned}$$

So, $\mathbb{E}[Y] = 6p - 3p^2$.

Finding $\mathrm{Var}(Y)$

To find $\mathrm{Var}(Y)$, we use the formula:

$$\mathrm{Var}(Y) = \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \mathrm{Var}(Y_3) + 2\,\mathrm{Cov}(Y_1, Y_2) + 2\,\mathrm{Cov}(Y_1, Y_3) + 2\,\mathrm{Cov}(Y_2, Y_3).$$

By symmetry, the variances are equal, and the covariances are equal.

$$\mathrm{Var}(Y) = 3\,\mathrm{Var}(Y_1) + 6\,\mathrm{Cov}(Y_1, Y_2).$$

The variance of a Bernoulli random variable with parameter q is $q(1-q)$. For Y_1, the parameter is $q = 2p - p^2$.

$$\begin{aligned}
\mathrm{Var}(Y_1) &= (2p - p^2)(1 - (2p - p^2)) \\
&= (2p - p^2)(1 - 2p + p^2) \\
&= (2p - p^2)(1 - p)^2 \\
&= p(2 - p)(1 - p)^2.
\end{aligned}$$

For the covariance,

$\mathrm{Cov}(Y_1, Y_2) = \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_1]\mathbb{E}[Y_2]$.

$\mathbb{E}[Y_1 Y_2] = \mathbb{P}(Y_1 = 1, Y_2 = 1)$.

$\mathbb{P}(Y_1 = 1, Y_2 = 1) = \mathbb{P}(\max(X_1, X_2) = 1 \text{ and } \max(X_1, X_3) = 1)$.

Using the inclusion-exclusion principle for events:

$$\begin{aligned}
\mathbb{P}(Y_1 = 1, Y_2 = 1) &= \mathbb{P}(X_1 = 1) + \mathbb{P}(X_2 = 1, X_3 = 1) - \mathbb{P}(X_1 = 1, X_2 = 1, X_3 = 1) \\
&= p + p^2 - p^3.
\end{aligned}$$

So, $\mathbb{E}[Y_1 Y_2] = p + p^2 - p^3$. Now, calculate the covariance:

$$\begin{aligned}\operatorname{Cov}(Y_1, Y_2) &= (p + p^2 - p^3) - (2p - p^2)(2p - p^2) \\ &= p + p^2 - p^3 - (4p^2 - 4p^3 + p^4) \\ &= p - 3p^2 + 3p^3 - p^4 \\ &= p(1 - 3p + 3p^2 - p^3) \\ &= p(1-p)^3.\end{aligned}$$

Finally, substitute the values back into the variance formula:

$$\begin{aligned}\operatorname{Var}(Y) &= 3\operatorname{Var}(Y_1) + 6\operatorname{Cov}(Y_1, Y_2) \\ &= 3p(2-p)(1-p)^2 + 6p(1-p)^3 \\ &= 3p(1-p)^2[(2-p) + 2(1-p)] \\ &= 3p(1-p)^2[2 - p + 2 - 2p] \\ &= 3p(1-p)^2(4-3p).\end{aligned}$$

Bonus: Check mutual and pairwise independence for Y_1, Y_2, Y_3.

Q4: Let C_1 = coin chosen first, C_2 = coin chosen second.

With probability $\frac{1}{2}$: C_1 = regular, C_2 = biased With probability $\frac{1}{2}$: C_1 = biased, C_2 = regular

Joint PMF calculation:

$\mathbb{P}(X = 1, Y = 1) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$\mathbb{P}(X = 1, Y = 0) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$

$\mathbb{P}(X = 0, Y = 1) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$

$\mathbb{P}(X = 0, Y = 0) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$

Therefore: $p_{XY}(x, y) = \begin{cases} \frac{1}{3} & \text{if } (x, y) = (1, 1) \\ \frac{1}{4} & \text{if } (x, y) = (1, 0) \text{ or } (0, 1) \\ \frac{1}{6} & \text{if } (x, y) = (0, 0) \\ 0 & \text{otherwise} \end{cases}$

Independence: $p_X(1) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, $p_Y(1) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, Using the marginals obtained from above

Since $\mathbb{P}(X = 1, Y = 1) = \frac{1}{3} \neq \frac{7}{12} \cdot \frac{7}{12} = p_X(1) \cdot p_Y(1)$, X and Y are **not independent**.

Q5: First, note that since $R_{XY} = \{(x, y) | 0 \leq x, y \leq 1\}$, we find that

$$F_{XY}(x, y) = 0, \text{ for } x < 0 \text{ or } y < 0,$$

and

$$F_{XY}(x, y) = 1, \text{ for } x \geq 1 \text{ and } y \geq 1.$$

To find the joint CDF for $x > 0$ and $y > 0$, we need to integrate the joint PDF:

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u, v) du dv$$

$$= \int_0^{\min(y,1)} \int_0^{\min(x,1)} \left(u + \frac{3}{2}v^2\right) dudv.$$

For $0 \le x, y \le 1$, we obtain

$$\begin{aligned}
F_{XY}(x,y) &= \int_0^y \int_0^x \left(u + \frac{3}{2}v^2\right) dudv \\
&= \int_0^y \left[\frac{1}{2}u^2 + \frac{3}{2}v^2 u\right]_{u=0}^{x} dv \\
&= \int_0^y \left(\frac{1}{2}x^2 + \frac{3}{2}xv^2\right) dv \\
&= \left[\frac{1}{2}x^2 v + \frac{1}{2}xv^3\right]_{v=0}^{y} \\
&= \frac{1}{2}x^2 y + \frac{1}{2}xy^3.
\end{aligned}$$

For $0 \le x \le 1$ and $y \ge 1$, we use the fact that F_{XY} is continuous to obtain

$$F_{XY}(x,y) = F_{XY}(x,1) = \frac{1}{2}x^2(1) + \frac{1}{2}x(1)^3 = \frac{1}{2}x^2 + \frac{1}{2}x.$$

Similarly, for $0 \le y \le 1$ and $x \ge 1$, we obtain

$$F_{XY}(x,y) = F_{XY}(1,y) = \frac{1}{2}(1)^2 y + \frac{1}{2}(1)y^3 = \frac{1}{2}y + \frac{1}{2}y^3.$$

The joint cumulative distribution function (CDF), $F_{XY}(x,y)$, for the given problem is a piecewise function defined across different regions of the xy-plane based on the integration of the joint probability density function (PDF).

$$F_{XY}(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ \frac{1}{2}x^2 y + \frac{1}{2}xy^3 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ \frac{1}{2}x^2 + \frac{1}{2}x & \text{if } 0 \le x \le 1 \text{ and } y > 1 \\ \frac{1}{2}y + \frac{1}{2}y^3 & \text{if } x > 1 \text{ and } 0 \le y \le 1 \\ 1 & \text{if } x > 1 \text{ and } y > 1 \end{cases}$$

Marginalization is the process of finding the probability distribution of a single random variable from a joint distribution. For continuous random variables, this involves integrating the joint PDF over the domain of the other variable.

Marginal PDF of X, $f_X(x)$ To find the marginal PDF of X, we integrate the joint PDF, $f_{XY}(x,y)$, with respect to y over its entire range. The support of Y is $0 \le y \le 1$.

For $0 \leq x \leq 1$:

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \\
&= \int_0^1 \left(x + \frac{3}{2}y^2\right)\,dy \\
&= \left[xy + \frac{3}{2}\frac{y^3}{3}\right]_0^1 \\
&= \left[xy + \frac{1}{2}y^3\right]_0^1 \\
&= \left(x(1) + \frac{1}{2}(1)^3\right) - \left(x(0) + \frac{1}{2}(0)^3\right) \\
&= x + \frac{1}{2}
\end{aligned}$$

So, the marginal PDF of X is:

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal PDF of Y, $f_Y(y)$ Similarly, to find the marginal PDF of Y, we integrate the joint PDF, $f_{XY}(x,y)$, with respect to x over its entire range. The support of X is $0 \leq x \leq 1$.

For $0 \leq y \leq 1$:

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \\
&= \int_0^1 \left(x + \frac{3}{2}y^2\right)\,dx \\
&= \left[\frac{x^2}{2} + \frac{3}{2}y^2x\right]_0^1 \\
&= \left(\frac{1^2}{2} + \frac{3}{2}(1)y^2\right) - \left(\frac{0^2}{2} + \frac{3}{2}(0)y^2\right) \\
&= \frac{1}{2} + \frac{3}{2}y^2
\end{aligned}$$

So, the marginal PDF of Y is:

$$f_Y(y) = \begin{cases} \frac{1}{2} + \frac{3}{2}y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Q6: (a)

$$\iint_{x,y} f_{X,Y}(x,y)dxdy = 1$$

$$\implies \int_y \int_x f_{X,Y}(x,y)dxdy = 1$$

Since x is upper bounded by y, we take the limit of x from 0 to y. And since we are initially calculating marginal pdf of $f_Y(y)$, the outside integral will be from 0 to ∞ as y can take all these values.

$$\int_0^{\infty}\int_0^{y} c\cdot x(y-x)e^{-y}dxdy = 1$$

$$\begin{aligned}
&\implies \int_0^{\infty} c\cdot e^{-y}dy\int_0^{y} x(y-x)dx = 1\\
&\implies \int_0^{\infty} c\cdot e^{-y}\left[\frac{yx^2}{2}-\frac{x^3}{3}\right]_0^y dy = 1\\
&\implies \int_0^{\infty} c\cdot e^{-y}\cdot\frac{y^3}{6}dy = 1\\
&\implies \frac{c}{6}\int_0^{\infty} y^3e^{-y}dy = 1\\
&\implies \frac{c}{6}\left[-(y^3+3y^2+6y+6)e^{-y}\right]_0^{\infty} = 1\\
&\implies \frac{c}{6}\cdot 6 = 1 \implies \boxed{c=1}
\end{aligned}$$

(b) For conditional PDF

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{X,Y}(x,y)}{\int_x f_{X,Y}(x,y)\cdot dx}$$

Similarly

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f(x,y)}{\int_y f(x,y)\cdot dy}$$

Let us first calculate all the marginal pdfs.

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x,y)\cdot dy$$

$$\implies f_X(x) = \int_0^{x} f_{X,Y}(x,y)\cdot dy + \int_x^{\infty} f_{X,Y}(x,y)\cdot dy$$

Since for the interval $0\le y\le x$ does not have any density

$$\begin{aligned}
\implies f_X(x) &= 0 + \int_x^{\infty} f_{X,Y}(x,y)\cdot dy\\
\implies f_X(x) &= \int_x^{\infty} x\cdot(y-x)e^{-y}\cdot dy\\
&= x\cdot\int_x^{\infty}(y-x)e^{-y}\cdot dy\\
&= x^2e^{-x}+xe^{-x}-x^2e^{-x}\\
\therefore f_X(x) &= x\cdot e^{-x}
\end{aligned}$$

Now for $f_Y(y)$

$$
\begin{aligned}
f_Y(y) &= \int_0^y f_{X,Y}(x,y)\cdot dx \\
\implies f_Y(y) &= \int_0^y x\cdot(y-x)e^{-y}\cdot dx \\
&= e^{-y}\cdot \int_0^y x\cdot(y-x)\cdot dy \\
&= e^{-y}\cdot\left(\frac{y\cdot y^2}{2}-\frac{y^3}{3}\right) \\
&= e^{-y}\cdot\left(\frac{y^3}{6}\right) \\
\therefore f_Y(y) &= e^{-y}\cdot\frac{y^3}{6}
\end{aligned}
$$

Substituting the values in the formulas specified above, we will get these expressions.

Q7: To find the Cumulative Distribution Function (CDF) of Y, we need to find $F_Y(y) = \mathbb{P}(Y \leq y)$ for all possible values of y.

First, let's determine the **range of Y**.

- When $0 \leq X \leq \frac{1}{2}$, we have $Y = X$, so Y can take any value in the interval $\left[0, \frac{1}{2}\right]$.
- When $X > \frac{1}{2}$, Y is fixed at the value $\frac{1}{2}$.

Combining these, the complete set of possible values for Y is the interval $\left[0, \frac{1}{2}\right]$. Now, we'll calculate $F_Y(y)$ by considering different cases for y.

Case 1: $y < 0$ Since the minimum value of Y is 0, it is impossible for Y to be less than 0.

$$F_Y(y) = \mathbb{P}(Y \leq y) = 0$$

Case 2: $0 \leq y < \frac{1}{2}$ For this range, the event $Y \leq y$ happens only when $X \leq y$. This is because for Y to be in this range, it must be that $Y = X$. The other possibility, $Y = \frac{1}{2}$, is greater than y. So, we can find the probability by integrating the PDF of X:

$$
\begin{aligned}
F_Y(y) &= \mathbb{P}(Y \leq y) \\
&= \mathbb{P}(X \leq y) \\
&= \int_0^y f_X(t)\,dt \\
&= \int_0^y 2t\,dt \\
&= [t^2]_0^y \\
&= y^2
\end{aligned}
$$

Case 3: $y \geq \frac{1}{2}$ Since the maximum value of Y is $\frac{1}{2}$, any value of y that is greater than or equal to $\frac{1}{2}$ will include all possible outcomes for Y. Therefore, the probability is 1.

$$F_Y(y) = \mathbb{P}(Y \leq y) = 1$$

The Jump at $y = 1/2$ Notice that the function for Y maps an entire interval of X values (specifically, $X > \frac{1}{2}$) to the single point $Y = \frac{1}{2}$. This creates a discrete jump, or a point mass, in the distribution of Y. The probability of this specific point is:

$$\begin{aligned}
\mathbb{P}\left(Y = \frac{1}{2}\right) &= \mathbb{P}\left(X > \frac{1}{2}\right) \\
&= \int_{1/2}^{1} 2x\,dx \\
&= [x^2]_{1/2}^{1} \\
&= 1^2 - \left(\frac{1}{2}\right)^2 \\
&= 1 - \frac{1}{4} = \frac{3}{4}
\end{aligned}$$

Final CDF Combining all the cases, we get the complete CDF for Y:

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \leq y < \frac{1}{2} \\ 1 & y \geq \frac{1}{2} \end{cases}$$

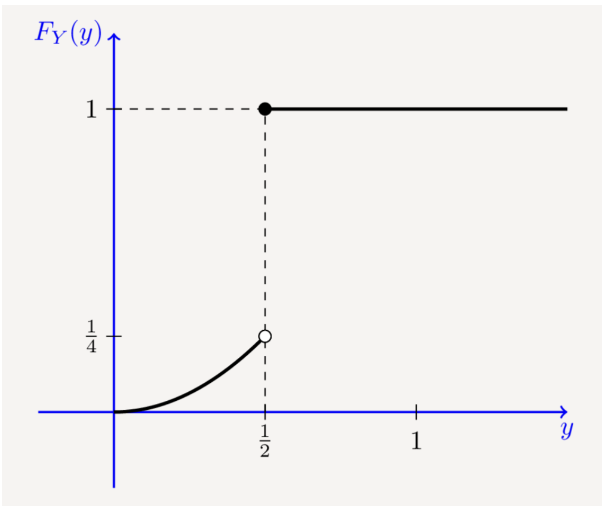

Figure 1: CDF of the Mixed random variable

Q8: **(a)** Let $g(X, Y) = X + Y$. Using LOTUS, we have:

$$\mathbb{E}[X + Y] = \sum_{(x_i, y_j) \in R_{XY}} (x_i + y_j) p_{XY}(x_i, y_j).$$

On splitting the sum:

$$\mathbb{E}[X+Y] = \sum_{(x_i,y_j)\in R_{XY}} x_i p_{XY}(x_i,y_j) + \sum_{(x_i,y_j)\in R_{XY}} y_j p_{XY}(x_i,y_j).$$

Separating the summation:

$$= \sum_{x_i\in R_X} x_i \left(\sum_{y_j\in R_Y} p_{XY}(x_i,y_j) \right) + \sum_{y_j\in R_Y} y_j \left(\sum_{x_i\in R_X} p_{XY}(x_i,y_j) \right).$$

By the property of marginal PMFs:

$$= \sum_{x_i\in R_X} x_i p_X(x_i) + \sum_{y_j\in R_Y} y_j p_Y(y_j).$$

Thus,

$$\boxed{\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].}$$

—

(b) Let $g(X,Y) = f(X) + h(Y)$. Using LOTUS, we have:

$$\mathbb{E}[f(X)+h(Y)] = \sum_{(x_i,y_j)\in R_{XY}} (f(x_i)+h(y_j))p_{XY}(x_i,y_j).$$

Splitting the sum:

$$= \sum_{(x_i,y_j)\in R_{XY}} f(x_i)p_{XY}(x_i,y_j) + \sum_{(x_i,y_j)\in R_{XY}} h(y_j)p_{XY}(x_i,yj).$$

Separating the summation:

$$= \sum_{x_i\in R_X} f(x_i) \left(\sum_{y_j\in R_Y} p_{XY}(x_i,y_j) \right) + \sum_{y_j\in R_Y} h(y_j) \left(\sum_{x_i\in R_X} p_{XY}(x_i,y_j) \right).$$

By the property of marginal PMFs:

$$= \sum_{x_i\in R_X} f(x_i)p_X(x_i) + \sum_{y_j\in R_Y} h(y_j)p_Y(y_j).$$

Finally, by LOTUS:

$$\boxed{\mathbb{E}[f(X)+h(Y)] = \mathbb{E}[f(X)] + \mathbb{E}[h(Y)].}$$

Q9: (a) First, compute the marginal PDF of Y:

$$f_Y(y) = \int_0^1 \left(\frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6} \right) dx = \frac{1}{12}\left(3y^2+y+1\right), \quad 0 \le y \le 2.$$

Thus, for $0 \leq y \leq 2$, the conditional density is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6}}{\frac{1}{12}(3y^2 + y + 1)} = \frac{3x^2 + 3y^2 + 2xy}{3y^2 + y + 1}, \quad 0 \leq x \leq 1.$$

So,

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{3x^2 + 3y^2 + 2xy}{3y^2 + y + 1}, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) To compute the conditional probability, we use

$$\mathbb{P}\big(X < \tfrac{1}{2} \,\big|\, Y = y\big) = \int_0^{1/2} f_{X|Y}(x|y)\, dx.$$

Substituting $f_{X|Y}(x|y)$,

$$= \int_0^{1/2} \frac{3x^2 + 3y^2 + 2xy}{3y^2 + y + 1}\, dx = \frac{1}{3y^2 + y + 1} \int_0^{1/2} \big(3x^2 + 2xy + 3y^2\big)\, dx.$$

Now, integrate each term separately:

$$\int_0^{1/2} 3x^2\, dx = \big[x^3\big]_0^{1/2} = \frac{1}{8}, \qquad \int_0^{1/2} 2xy\, dx = y\,\big[x^2\big]_0^{1/2} = \frac{y}{4},$$

$$\int_0^{1/2} 3y^2\, dx = 3y^2\,[x]_0^{1/2} = \frac{3y^2}{2}.$$

Adding them together:

$$\int_0^{1/2} \big(3x^2 + 2xy + 3y^2\big)\, dx = \frac{1}{8} + \frac{y}{4} + \frac{3y^2}{2}.$$

Therefore,

$$\mathbb{P}\big(X < \tfrac{1}{2} \,\big|\, Y = y\big) = \frac{\frac{1}{8} + \frac{y}{4} + \frac{3y^2}{2}}{3y^2 + y + 1}.$$

$$\boxed{\mathbb{P}(X < \tfrac{1}{2} \mid Y = y) = \frac{\frac{1}{8} + \frac{y}{4} + \frac{3y^2}{2}}{3y^2 + y + 1}}$$

Notice that this probability explicitly depends on y.

Q10: (a) First, we find the unconditional PMF of S. The possible values of S are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ with the following probabilities:

s	Number of ways	$\mathbb{P}(S = s)$
2	1	$1/36$
3	2	$2/36$
4	3	$3/36$
5	4	$4/36$
6	5	$5/36$
7	6	$6/36$
8	5	$5/36$
9	4	$4/36$
10	3	$3/36$
11	2	$2/36$
12	1	$1/36$

For $A_1 = \{S \leq 7\}$:

$$\mathbb{P}(A_1) = \mathbb{P}(S \leq 7) = \sum_{s=2}^{7} \mathbb{P}(S = s) = \frac{1+2+3+4+5+6}{36} = \frac{21}{36} = \frac{7}{12}$$

The conditional PMF $\mathbb{P}_{S|A_1}(s)$ is given by:

$$p_{S|A_1}(s) = \mathbb{P}(S = s|A_1) = \frac{\mathbb{P}(S = s \text{ and } A_1)}{\mathbb{P}(A_1)}$$

For $s \in \{2, 3, 4, 5, 6, 7\}$:

$$p_{S|A_1}(s) = \frac{\mathbb{P}(S = s)}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(S = s)}{\frac{7}{12}}$$

For $s \notin \{2, 3, 4, 5, 6, 7\}$:

$$\mathbb{P}_{S|A_1}(s) = 0$$

Computing each value:

$$p_{S|A_1}(2) = \frac{1/36}{7/12} = \frac{1/36 \cdot 12}{7} = \frac{1}{21} \tag{1}$$
$$p_{S|A_1}(3) = \frac{2/36}{7/12} = \frac{2/36 \cdot 12}{7} = \frac{2}{21} \tag{2}$$
$$p_{S|A_1}(4) = \frac{3/36}{7/12} = \frac{3/36 \cdot 12}{7} = \frac{3}{21} \tag{3}$$
$$p_{S|A_1}(5) = \frac{4/36}{7/12} = \frac{4/36 \cdot 12}{7} = \frac{4}{21} \tag{4}$$
$$p_{S|A_1}(6) = \frac{5/36}{7/12} = \frac{5/36 \cdot 12}{7} = \frac{5}{21} \tag{5}$$
$$p_{S|A_1}(7) = \frac{6/36}{7/12} = \frac{6/36 \cdot 12}{7} = \frac{6}{21} \tag{6}$$

Therefore:

$$p_{S|A_1}(s) = \begin{cases} \dfrac{s-1}{21}, & s \in \{2, 3, 4, 5, 6, 7\} \\ 0, & \text{otherwise} \end{cases}$$

Similarly, for $A_2 = \{S \geq 8\}$, we have $\mathbb{P}(A_2) = \frac{5}{12}$ and:

$$p_{S|A_2}(s) = \begin{cases} \dfrac{13-s}{15}, & s \in \{8, 9, 10, 11, 12\} \\ 0, & \text{otherwise} \end{cases}$$

(b) To find $\mathbb{P}(S = 6)$, we use the law of total probability:

$$\mathbb{P}(S = s) = \sum_{i=1}^{2} \mathbb{P}(A_i)\mathbb{P}_{S|A_i}(s)$$

For $s = 6$:

$$\begin{aligned} \mathbb{P}(S = 6) &= \mathbb{P}(A_1)\mathbb{P}_{S|A_1}(6) + \mathbb{P}(A_2)\mathbb{P}_{S|A_2}(6) \\ &= \frac{7}{12} \cdot \frac{5}{21} + \frac{5}{12} \cdot 0 \\ &= \frac{7 \cdot 5}{12 \cdot 21} = \frac{35}{252} = \frac{5}{36} \end{aligned}$$

Therefore, $\mathbb{P}(S = 6) = \dfrac{5}{36}$.

Q11: Let $U = g(X)$ and $V = h(Y)$. Then

$$p_{U,V}(u, v) = \sum_{\{(x,y)\,:\,g(x)=u,\, h(y)=v\}} p_{X,Y}(x, y).$$

Since X and Y are independent,

$$p_{X,Y}(x, y) = p_X(x)p_Y(y),$$

so

$$p_{U,V}(u, v) = \sum_{\{x\colon g(x)=u\}} p_X(x) \sum_{\{y\colon h(y)=v\}} p_Y(y) = p_U(u)p_V(v).$$

Thus U and V are independent.

Q12: (a) The probability of a sequence of rolls where, for $i = 1, \ldots, r$, face i comes up k_i times is $p_1^{k_1} \cdots p_r^{k_r}$. Every such sequence determines a partition of the set of n rolls into r subsets with the ith subset having cardinality k_i. The number of such partitions is the multinomial coefficient

$$\binom{n}{k_1, \ldots, k_r} = \frac{n!}{k_1! \cdots k_r!}.$$

Thus, if $k_1 + \cdots + k_r = n$,

$$p_{X_1,\ldots,X_r}(k_1, \ldots, k_r) = \binom{n}{k_1, \ldots, k_r} p_1^{k_1} \cdots p_r^{k_r},$$

and otherwise $p_{X_1,\ldots,X_r}(k_1, \ldots, k_r) = 0$.

(b) The random variable X_i is binomial with parameters n and p_i. Therefore,

$$\mathbb{E}[X_i] = np_i, \qquad \mathrm{Var}(X_i) = np_i(1 - p_i).$$

(c) Suppose that $i \neq j$, and let $Y_{i,k}$ (resp. $Y_{j,k}$) be the Bernoulli random variable that takes the value 1 if face i (resp. j) comes up on the kth roll, and 0 otherwise. Note that $Y_{i,k}Y_{j,k} = 0$, and for $l \neq k$, $Y_{i,k}$ and $Y_{j,l}$ are independent, so that

$$\mathbb{E}[Y_{i,k}Y_{j,l}] = p_i p_j.$$

Therefore,

$$\mathbb{E}[X_i X_j] = \mathbb{E}[(Y_{i,1} + \cdots + Y_{i,n})(Y_{j,1} + \cdots + Y_{j,n})].$$

Expanding the product gives $n(n-1)$ cross terms, each with expectation $p_i p_j$. Hence

$$\mathbb{E}[X_i X_j] = n(n-1)p_i p_j.$$