

CS 302.1 - Automata Theory

Lecture 06

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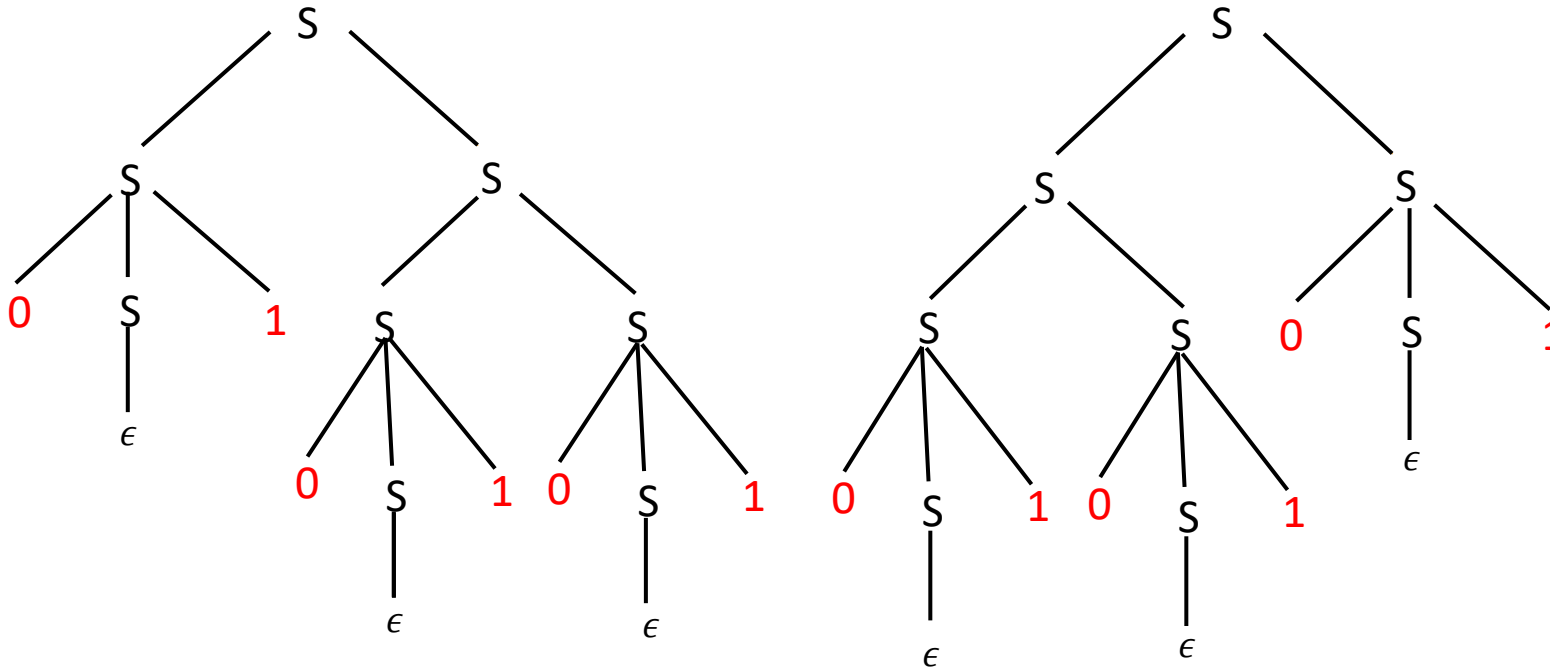
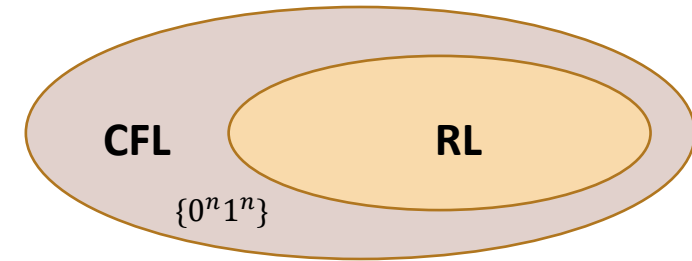
IIIT Hyderabad



Quick Recap

Context-Free Grammars: If the *rules* of the underlying grammar G are of the form
$$V \rightarrow (VUT)^*$$

then such a grammar is called **Context-Free**.



Parse trees: These are ordered trees that provide alternative representations of the derivation of a grammar.

Ambiguous grammars: There exists $\omega \in L(G)$, such that there are **two or more leftmost derivations** for ω (or equivalently two or more rightmost derivations) or equivalently **two or more parse trees** for ω . **Ambiguity** may not be desirable

Chomsky Normal Form

Often it is easier to work with CFG in a simple standardized form - the Chomsky Normal Form (CNF) is one of them.

Chomsky Normal Form

A CFG G is in CNF if every rule of G is of the form

$$Var \rightarrow Var Var$$

$$Var \rightarrow ter$$

$$Start Var \rightarrow \epsilon$$

where Var can be any variable, including the Start Variable, $Start Var$.

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Why are CNFs useful?

- Suppose you are given a CFG G and a string w as input and you have to write an algorithm that decides whether G generates w .
- Your algorithm outputs YES if G generates w and NO, otherwise.

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A CFG G is in **CNF** if every rule of G is of the form

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Why are CNFs useful?

- Suppose you are given a CFG G as and a string w as input and you have to **write an algorithm that decides whether G generates w .**
- The algorithm outputs YES if G generates w and *NO*, otherwise.
 - ❖ One idea is to go through ALL derivations one by one and output YES if any of them generates w .
 - ❖ However, infinitely many derivations may have to be tried.
 - ❖ So if G does not generate w , the algorithm will never stop.
 - ❖ So this problem appears to be **undecidable**.

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Why are CNFs useful?

Suppose you are given a CFG G as and a string w as input and you have to **write an algorithm that decides whether G generates w .**

- Converting G first to a CNF alleviates this and **makes the problem decidable**.
- It limits the number of steps in derivations required to generate any $w \in L(G)$.
- If $w \in L(G)$, then a CFG in Chomsky Normal Form has **derivations of $2n - 1$ steps** for input strings w of length n (We will prove this shortly).

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A CFG in Chomsky Normal Form has derivations of $2n - 1$ steps for generating strings $w \in L(G)$ of length n .

Why are CNFs useful?

Suppose you are given a CFG G and a string w as input and you have to **write an algorithm that decides whether G generates w .**

1. Convert G to CNF.
2. List all derivations of $2n - 1$ steps, where $|w| = n$. (There are a finite number of these)
3. If ANY of these derivations generate w , output YES, otherwise output NO.

Chomsky Normal Form

A CFG G is in **CNF** if every rule of G is of the form

$$\begin{aligned}Var &\rightarrow Var\ Var \\Var &\rightarrow ter \\Start\ Var &\rightarrow \epsilon\end{aligned}$$

where Var can be any variable, including the Start Variable, $Start\ Var$.

- 1) **A CFG in Chomsky Normal Form has derivations of $2n - 1$ steps for generating strings $w \in L(G)$ of length n .**
- 2) Any CFL can be generated by a CFG written in Chomsky Normal Form.

To prove 1) use induction!

Chomsky Normal Form

Prove that a **CFG in Chomsky Normal Form** has derivations of $2n - 1$ steps for generating strings $w \in L(G)$ of length n .

Proof: Note that any CFG in CNF can be written as:

| | |
|--------------------------|--|
| $A \rightarrow BC$ | $[B, C \text{ are not start variables}]$ |
| $A \rightarrow a$ | $[a \text{ is a terminal}]$ |
| $S \rightarrow \epsilon$ | $[S \text{ is the Start Variable}]$ |

We will prove this by **induction**.

(Basic step) Let $|w| = 1$. Then **one** application of the second rule would suffice. So any derivation of w would need $2|w| - 1 = 1$ step.

(Inductive hypothesis) Assume the statement of the theorem to be true for any string of length at most k where $k \geq 1$. Now we shall show that it holds for any $w \in L(G)$ such that $|w| = k + 1$.

Chomsky Normal Form

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Since $|w| > 1$, any derivation will start from the rule $A \rightarrow BC$. So $w = xy$, where $B \xRightarrow{*} x$, $|x| > 0$ and $C \xRightarrow{*} y$, $|y| > 0$. But since $|x|, |y| \leq k$, and we have that by the inductive hypothesis: (i) number of steps in the derivation $B \xRightarrow{*} x$ is $2|x| - 1$ and (ii) number of steps in the derivation $C \xRightarrow{*} y$ is $2|y| - 1$. So the number of steps in the derivation of w is

$$1 + (2|x| - 1) + (2|y| - 1) = 2(|x| + |y|) - 1 = 2|w| - 1 = 2(k + 1) - 1.$$

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where Var can be any variable, including the Start Variable, $Start\ Var$.

- 1) A CFG in Chomsky Normal Form has derivations of $2n - 1$ steps for generating strings $w \in L(G)$ of length n .
- 2) **Any CFL can be generated by a CFG written in Chomsky Normal Form.**

Chomsky Normal Form

Any CFL can be generated by a CFG written in Chomsky Normal Form.

Proof: The proof is constructive. Suppose we have a CFG G with a set of rules. To convert G into CNF, we do the following:

1. Add a new start variable $S' \rightarrow S$
2. Remove ϵ rules of the form $A \rightarrow \epsilon$
 - Remove nullable symbols/rules
3. Remove unit (short) rules of the form $A \rightarrow B$
 - Remove useless symbols/rules
4. Remove long rules of the form $A \rightarrow u_1 u_2 \cdots u_k$
 - Remove useless symbols/rules

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1. Add a new start variable $S' \rightarrow S$
2. **Remove ϵ rules of the form $A \rightarrow \epsilon$**

For each occurrence of A in the right side of the rule, we add a new rule with the occurrence of A deleted.

E.g.: Consider any rule $B \rightarrow uAvAw$ (u, v, w can be strings of variables and terminals)

Then new rules: $B \rightarrow uAvAw | uvAw | uAvw | uvw$

What if you had a rule such as $B \rightarrow A$? Then we would have needed to add a rule $B \rightarrow \epsilon$ (unless this rule has been already removed) as B is a **nullable variable**.

Repeat this procedure, until all ϵ -rules are removed.

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1. Add a new start variable $S' \rightarrow S$
2. Remove ϵ rules of the form $A \rightarrow \epsilon$

E.g.:
 $S \rightarrow 0|X0|ZYZ$
 $X \rightarrow Y|\epsilon$
 $Y \rightarrow 1|X$

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E.g.:
 $S \rightarrow 0|X0|ZYZ$
 $X \rightarrow Y|\epsilon$
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To remove $X \rightarrow \epsilon$, we add new rules:
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E.g.:
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 $X \rightarrow Y|\epsilon$
 $Y \rightarrow 1|X$

To remove $X \rightarrow \epsilon$, we add new rules:
 $S \rightarrow 0|X0|ZYZ$
 $X \rightarrow Y$
 $Y \rightarrow 1|X|\epsilon$

To remove $Y \rightarrow \epsilon$, we add:
 $S \rightarrow 0|X0|ZYZ|ZZ$
 $X \rightarrow Y$
 $Y \rightarrow 1|X$

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1. Add a new start variable $S' \rightarrow S$
2. Remove ϵ rules of the form $A \rightarrow \epsilon$
3. **Remove unit rules of the form $A \rightarrow B$**

We **remove the rule $A \rightarrow B$** and **whenever a rule $B \rightarrow u$ appears** (u is a string of terminals and variables), we **add a new rule $A \rightarrow u$** , unless this rule was already removed.

Repeat these steps until all unit rules are removed.

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E.g.:

$$S \rightarrow A|11$$

$$A \rightarrow B|1$$

$$B \rightarrow S|0$$

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E.g.:

$$\begin{aligned} S &\rightarrow A|11 \\ A &\rightarrow B|1 \\ B &\rightarrow S|0 \end{aligned}$$

Remove $S \rightarrow A$

$$\begin{aligned} S &\rightarrow 11|B|1 \\ A &\rightarrow B|1 \\ B &\rightarrow S|0 \end{aligned}$$

Remove $A \rightarrow B$

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Remove $B \rightarrow S$

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Remove $A \rightarrow S$

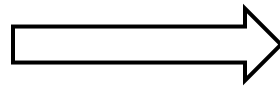
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3. Remove unit rules of the form $A \rightarrow B$
- 4. Remove long rules of the form $A \rightarrow u_1 u_2 \cdots u_k$**

Note that each u_i could be a variable or a terminal. We do the following:

- Replace $A \rightarrow u_1 u_2 \cdots u_k$, ($k \geq 3$) with the rules $A \rightarrow u_1 A_1$, $A_1 \rightarrow u_2 A_2, \dots, A_{k-2} \rightarrow u_{k-1} u_k$
- We replace any terminal u_i in the preceding rules with the new variable U_i and add the rule $U_i \rightarrow u_i$

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Remove unit rules of the form $A \rightarrow B$ (Whenever a rule $B \rightarrow u$ appears, we add a new rule $A \rightarrow u$, unless this rule was already removed. Repeat these steps until all unit rules are removed.)

Remove long rules of the form $A \rightarrow u_1 u_2 \cdots u_k$ (Replace $A \rightarrow u_1 u_2 \cdots u_k$, ($k \geq 3$) with the rules $A \rightarrow u_1 A_1$, $A_1 \rightarrow u_2 A_2, \cdots, A_{k-2} \rightarrow u_{k-1} u_k$; Replace any terminal u_i in the preceding rules with the new variable U_i and add the rule $U_i \rightarrow u_i$).

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CNF:

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Convert the CFG

$$\begin{aligned} S &\rightarrow ASA|aB \\ A &\rightarrow B|S \\ B &\rightarrow b|\epsilon \end{aligned}$$

to CNF.

1. Add a new start variable

$$\begin{aligned} S' &\rightarrow S \\ S &\rightarrow ASA|aB \\ A &\rightarrow B|S \\ B &\rightarrow b|\epsilon \end{aligned}$$

2a. Remove ϵ rules ($B \rightarrow \epsilon$)

$$\begin{aligned} S' &\rightarrow S \\ S &\rightarrow ASA|aB|a \\ A &\rightarrow B|S|\epsilon \\ B &\rightarrow b \end{aligned}$$

2b. Remove ϵ rules ($A \rightarrow \epsilon$)

$$\begin{aligned} S' &\rightarrow S \\ S &\rightarrow ASA|aB|a|AS|SA|S \\ A &\rightarrow B|S \\ B &\rightarrow b \end{aligned}$$

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Convert the CFG

$$S \rightarrow ASA|aB$$

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to CNF.

3a. Remove $S \rightarrow S$

$$S' \rightarrow S$$

$$S \rightarrow ASA|aB|a|AS|SA$$

$$A \rightarrow B|S$$

$$B \rightarrow b$$

3b. Remove $S' \rightarrow S$

$$S' \rightarrow ASA|aB|a|AS|SA$$

$$S \rightarrow ASA|aB|a|AS|SA$$

$$A \rightarrow B|S$$

$$B \rightarrow b$$

3c. Remove $A \rightarrow B$

$$S' \rightarrow ASA|aB|a|AS|SA$$

$$S \rightarrow ASA|aB|a|AS|SA$$

$$A \rightarrow S|b$$

$$B \rightarrow b$$

3d. Remove $A \rightarrow S$

$$S' \rightarrow ASA|aB|a|AS|SA$$

$$S \rightarrow ASA|aB|a|AS|SA$$

$$A \rightarrow b|ASA|aB|a|AS|SA$$

$$B \rightarrow b$$

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4a. Remove long rules

$$S' \rightarrow ASA|aB|a|AS|SA$$

$$S \rightarrow ASA|aB|a|AS|SA$$

$$A \rightarrow b|ASA|aB|a|AS|SA$$

$$B \rightarrow b$$

4b. Remove long rules

$$S' \rightarrow AU|aB|a|AS|SA$$

$$S \rightarrow AU|aB|a|AS|SA$$

$$A \rightarrow b|AU|aB|a|AS|SA$$

$$U \rightarrow SA$$

$$B \rightarrow b$$

4c. Remove long rules

$$S' \rightarrow AU|VB|a|AS|SA$$

$$S \rightarrow AU|VB|a|AS|SA$$

$$A \rightarrow b|AU|VB|a|AS|SA$$

$$U \rightarrow SA$$

$$V \rightarrow a$$

$$B \rightarrow b$$

There are other rules of the form: $\text{Var} \rightarrow ASA$

Chomsky Normal Form

CNF: $A \rightarrow BC$ [B, C are not start variables]
 $A \rightarrow a$ [a is a terminal]
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Convert the CFG

$$\begin{aligned} S &\rightarrow ASA|aB \\ A &\rightarrow B|S \\ B &\rightarrow b|\epsilon \end{aligned}$$

to CNF.

$$\begin{aligned} S' &\rightarrow AU|VB|a|AS|SA \\ S &\rightarrow AU|VB|a|AS|SA \\ A &\rightarrow b|AU|VB|a|AS|SA \\ U &\rightarrow SA \\ V &\rightarrow a \\ B &\rightarrow b \end{aligned}$$

Pushdown Automata

- For regular languages we had
 - Designed Finite automata (DFA, NFA) that recognize the strings by the language. Helped us decide whether a given string ω belongs to the language.

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 - Can we build an automata that recognizes **exactly** context free languages?
- **Finite Automaton model recognizes ALL regular languages**
- Any automata that recognizes **ALL** context free languages will need unbounded memory.

Pushdown Automata

- Finite Automaton model recognizes ALL regular languages
- Any automata that recognizes **ALL** context free languages will need unbounded memory.

Intuition to build an Automata for CFL

- It should be some **Finite State Machine** that has access to a memory device with infinite memory, i.e.

Automata for CFL = FSM + Memory device

- **FSM may choose to ignore the memory device** completely in which case it behaves like a DFA/NFA.
- FSM makes use of the Memory device to recognize “non-Regular” CFLs.

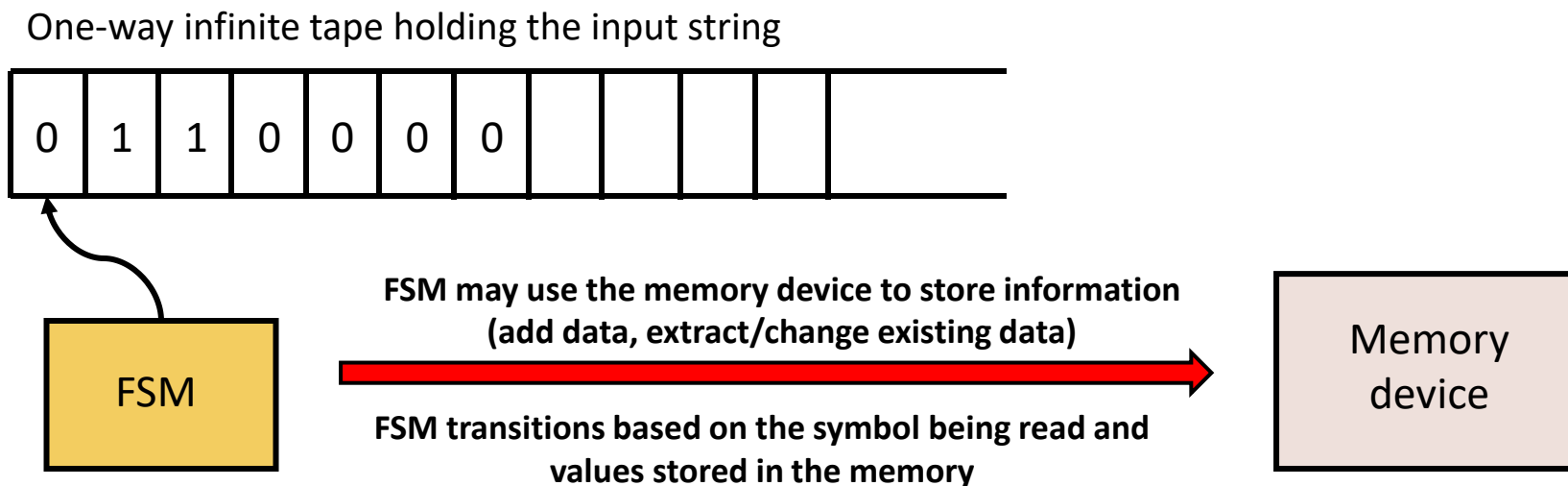
E.g.: $\{0^n 1^n, n \in \mathbb{N}\}$

Pushdown Automata

Intuition to build an Automata for CFL

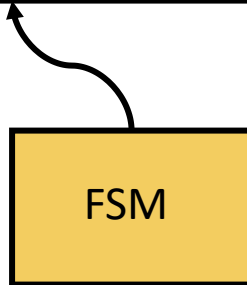
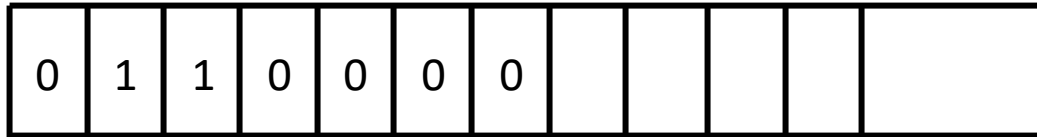
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Pushdown Automata

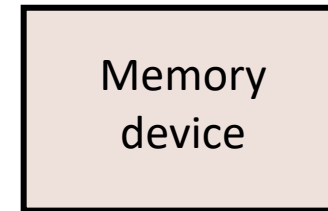
One-way infinite tape holding the input string



FSM may use the memory device to store information
(add data, extract/change existing data)



FSM transitions based on the symbol being read and
values stored in the memory



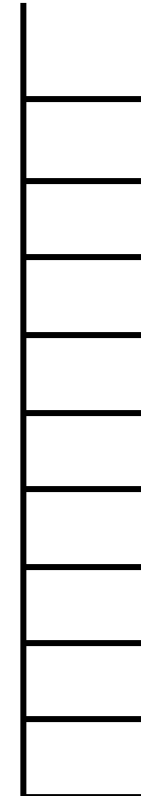
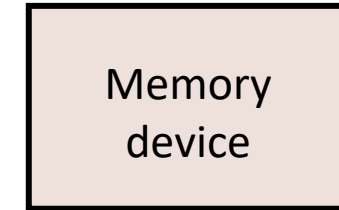
The memory device

- Simple memory device with unbounded memory.

Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, new elements can be added to the Stack (**PUSH**).
- At any stage, the element at the **top** of the STACK can be read by removing it from the stack (**POP**).



Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, elements can be pushed or popped.

PUSH

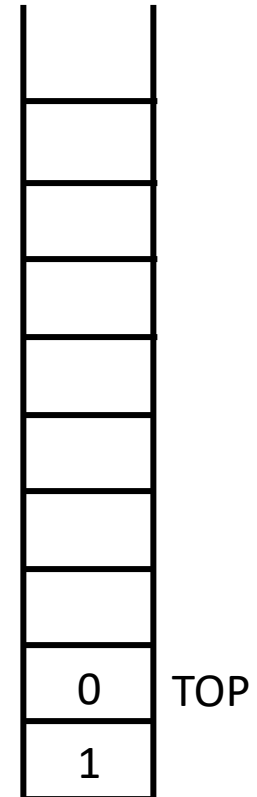
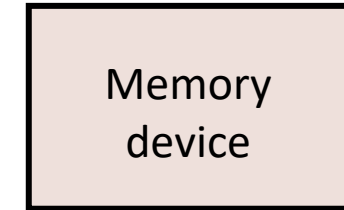
- New symbols can be **pushed** in to the STACK.

E.g: PUSH 1

- The Top of the STACK now covers the old stack top, i.e.

$TOP = TOP + 1$

- The size of the stack keeps growing.



Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, elements can be pushed or popped.

PUSH

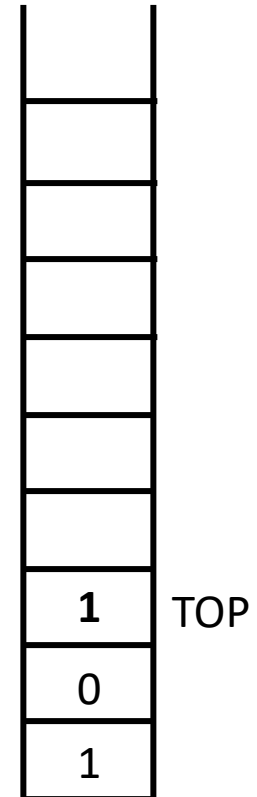
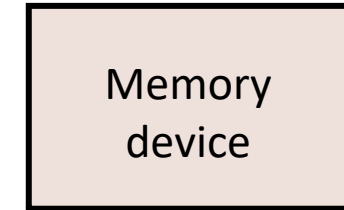
- New symbols can be **pushed** in to the STACK.

E.g: PUSH 1

- The Top of the STACK now covers the old stack top, i.e.

$TOP = TOP + 1$

- The size of the stack keeps growing.



Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, elements can be pushed or popped.

PUSH

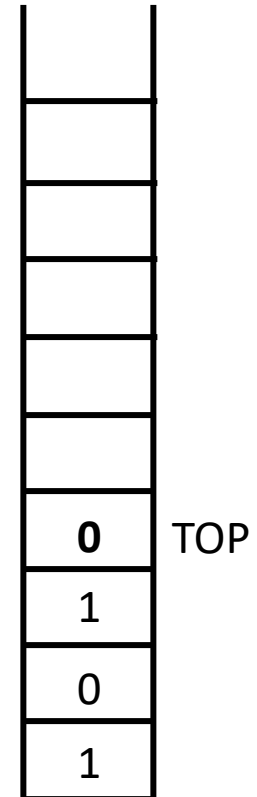
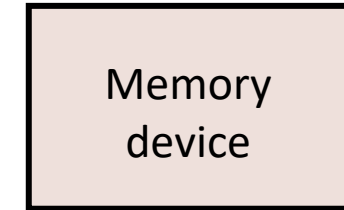
- New symbols can be **pushed** in to the STACK.

E.g: **PUSH 0**

- The Top of the STACK now covers the old stack top, i.e.

$$\text{TOP} = \text{TOP} + 1$$

- The size of the stack keeps growing.



Pushdown Automata

The memory device

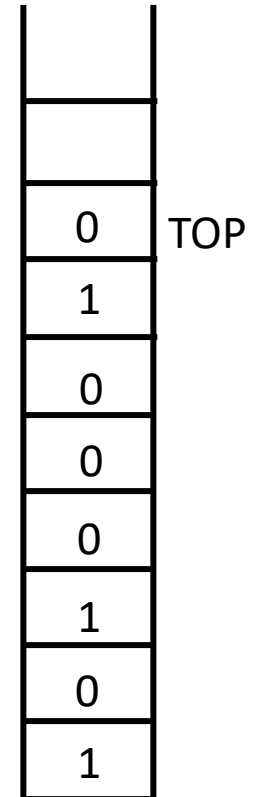
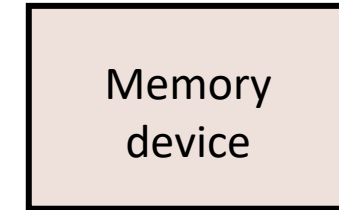
- Simple memory device with unbounded memory.
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PUSH

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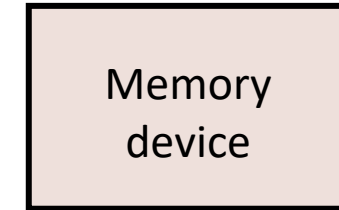
- The size of the stack keeps growing.



Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, elements can be pushed or popped.



POP

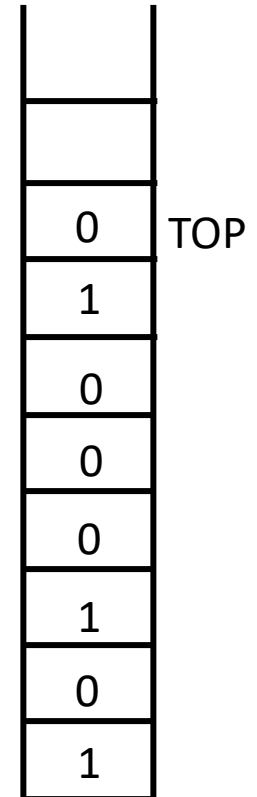
- The element from the TOP of the stack can be **popped** out

E.g.: **POP 0**

- The Top of the STACK moves to the element below.

$$\text{TOP} = \text{TOP} - 1$$

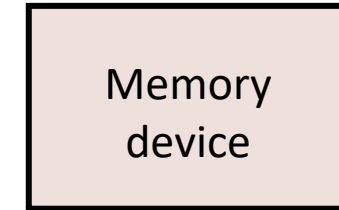
- Successive **POP** operations shrink the stack size. Elements can be popped until EMPTY.
- **Last In First Out (LIFO)**: The last element that was pushed is the first to be popped out



Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, elements can be pushed or popped.



POP

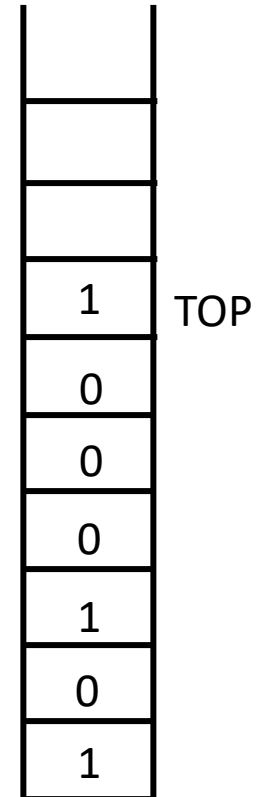
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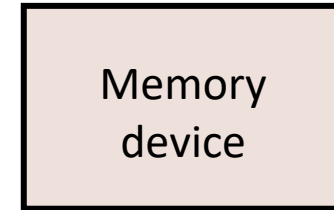
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Pushdown Automata

The memory device

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- Consider a **STACK**
- At any stage, elements can be pushed or popped.



POP

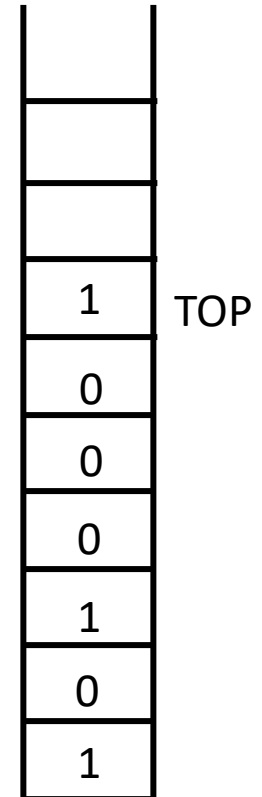
- The element from the TOP of the stack can be **popped** out

E.g.: **POP 1**

- The Top of the STACK moves to the element below.

$$\text{TOP} = \text{TOP} - 1$$

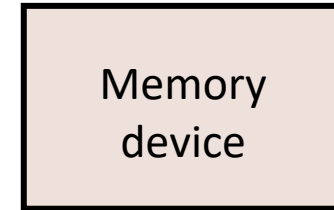
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Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, elements can be pushed or popped.



POP

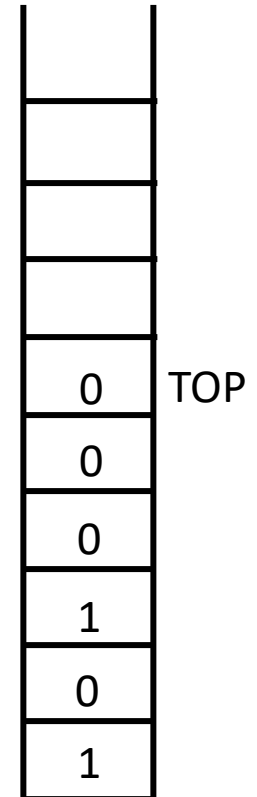
- The element from the TOP of the stack can be **popped** out

E.g.: **POP 1**

- The Top of the STACK moves to the element below.

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- Successive **POP** operations shrink the stack size. Elements can be popped until EMPTY.
- **Last In First Out (LIFO)**: The last element that was pushed is the first to be popped out



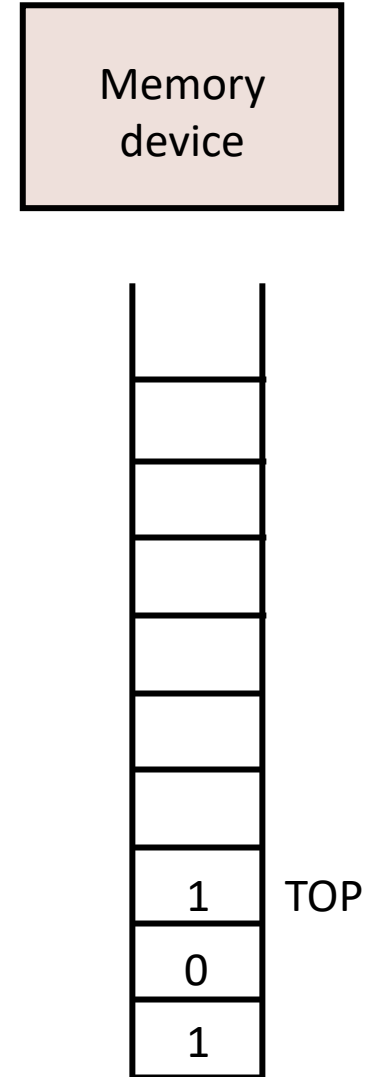
Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- Last In First Out (**LIFO**)

POP

- The element from the TOP of the stack can be **popped** out.
 - $TOP = TOP - 1$
 - Elements can be popped until STACK is EMPTY.
-
- How would you know that the STACK is EMPTY?



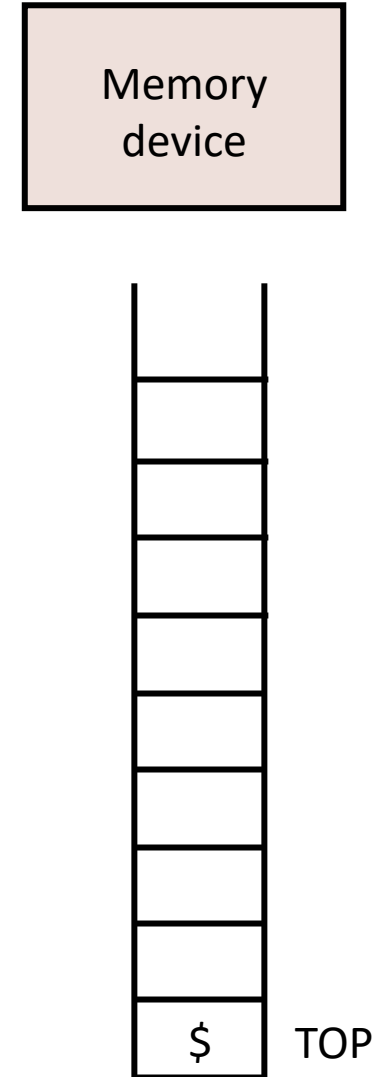
Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- Last In First Out (**LIFO**)

POP

- The element from the TOP of the stack can be **popped** out.
 - $TOP = TOP - 1$
 - Elements can be popped until STACK is EMPTY.
-
- How would you know that the STACK is EMPTY?
 - There is generally some special symbol (say \$) that demarcates the bottom of the STACK.
 - This element is Pushed at the very beginning. Whenever the popped element = \$, the STACK is EMPTY.



Pushdown Automata

Memory device of PDA: STACK

- STACK is a **LIFO** data structure of unbounded memory
- Only the TOP element can be read from the STACK.
- The bottom of the STACK contains a special symbol (\$)
- Characterized by two operations:

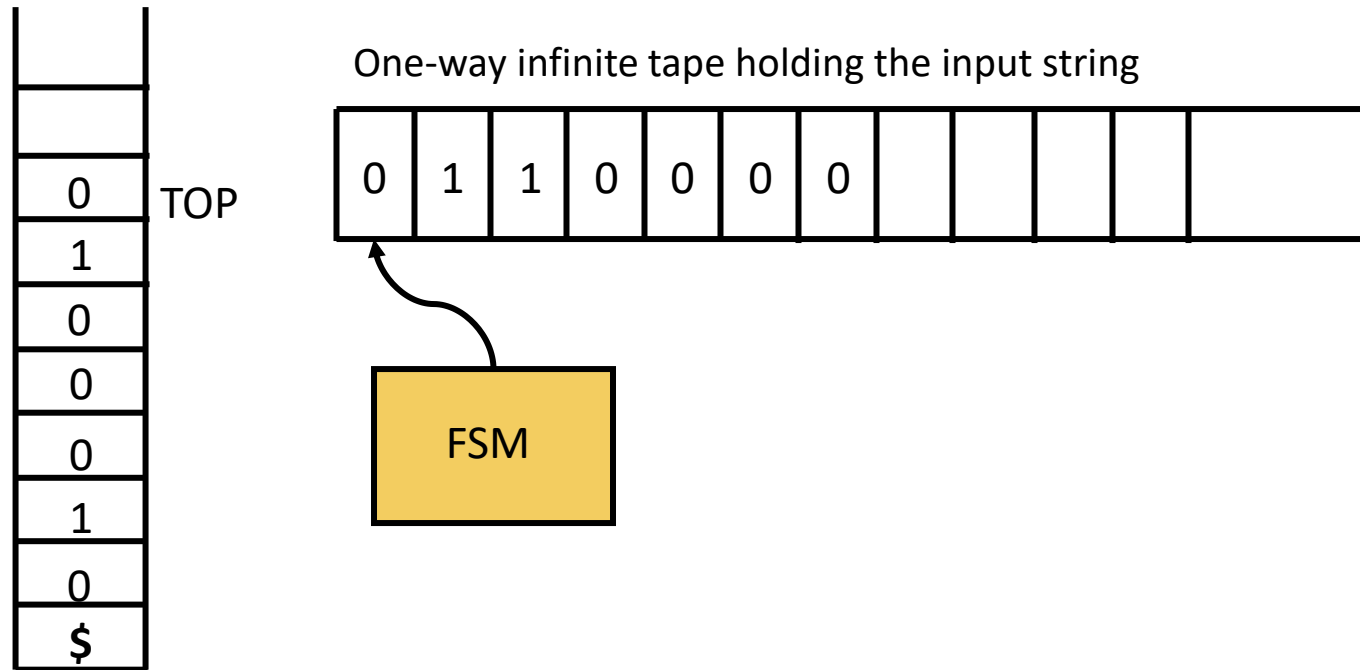
PUSH

- New symbols can be **pushed** in to the STACK.
- $TOP = TOP + 1$

POP

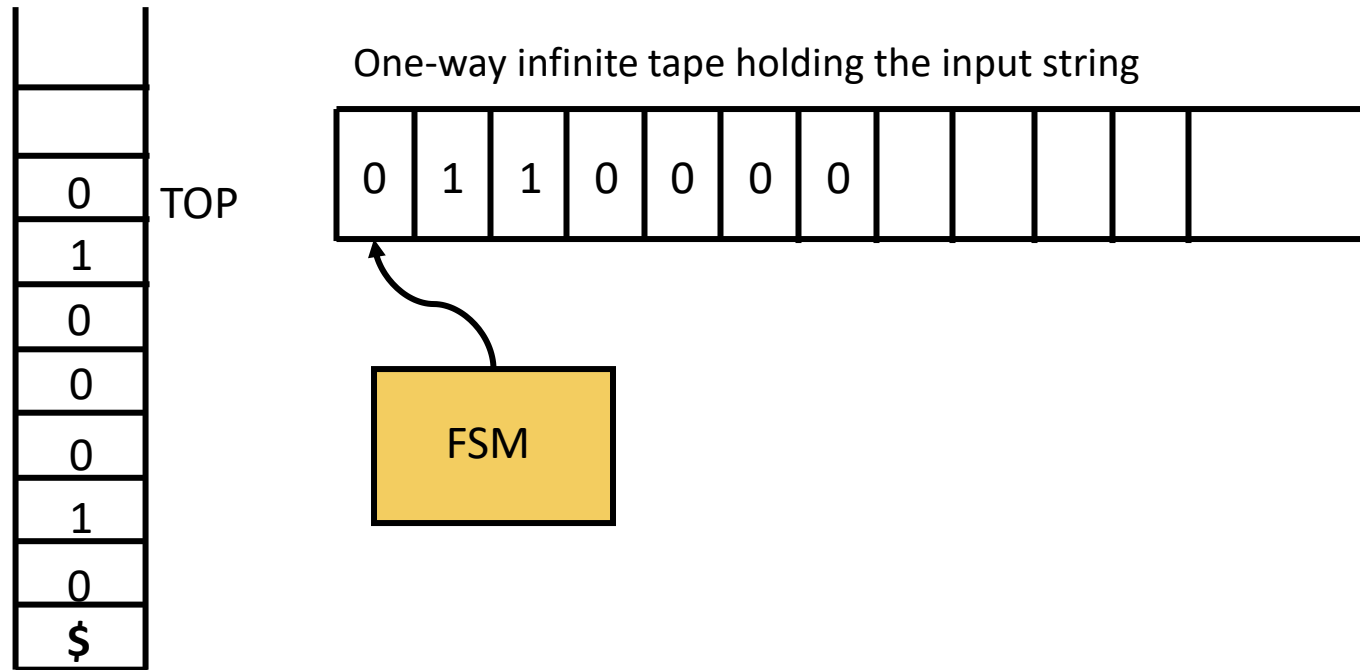
- The element from the TOP of the stack can be **popped** out.
- $TOP = TOP - 1$
- Elements can be popped until STACK is EMPTY.

Pushdown Automata



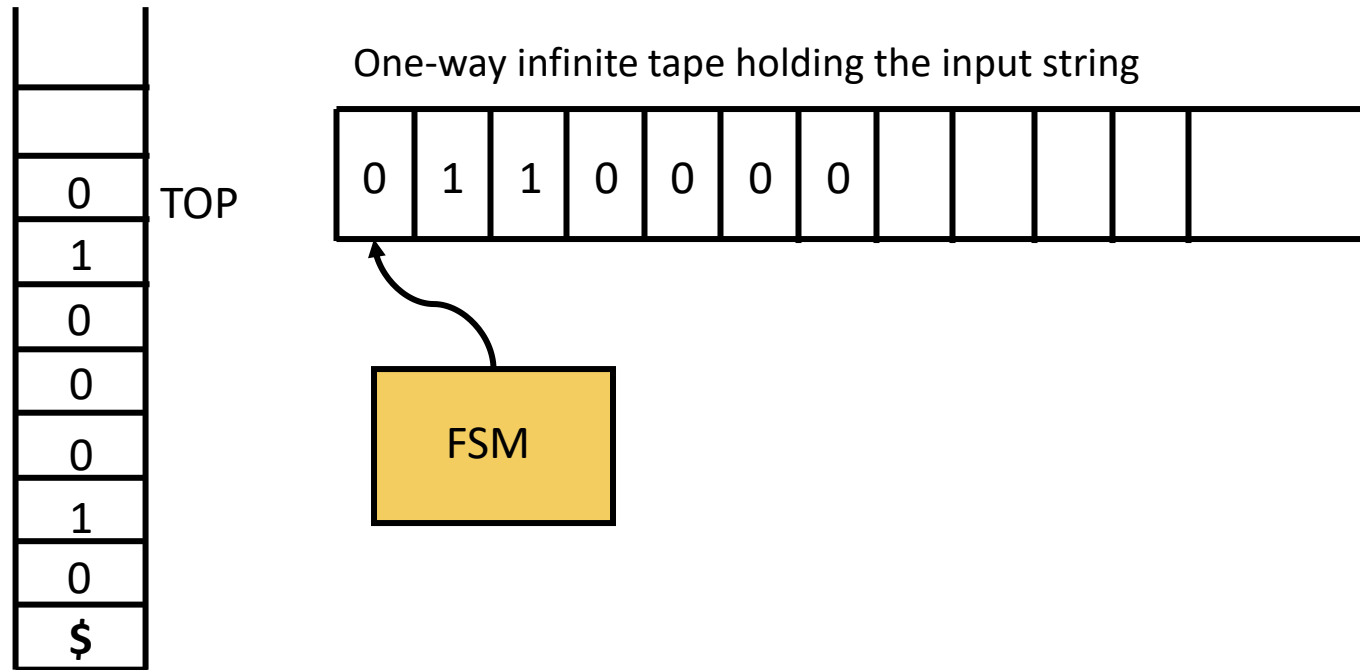
- A Pushdown Automata (PDA) is a finite automaton that has access to a stack.
- The FSM:

Pushdown Automata



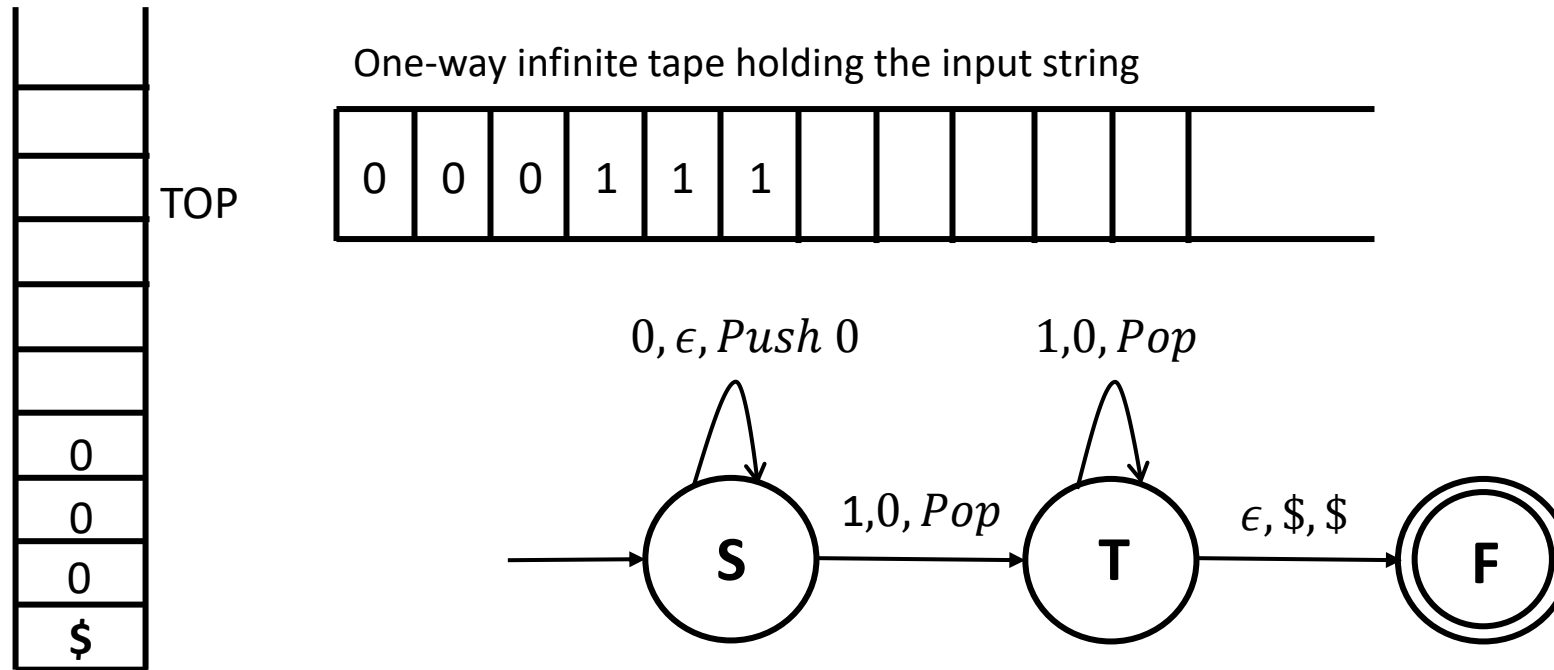
- A Pushdown Automata (PDA) is a finite automaton that has access to a stack.
- The FSM:
 - Transitions based on the Input symbol and the element at the top of the stack (e.g.: If I/P symbol = 0 & 0 is popped, transition from i to j)

Pushdown Automata



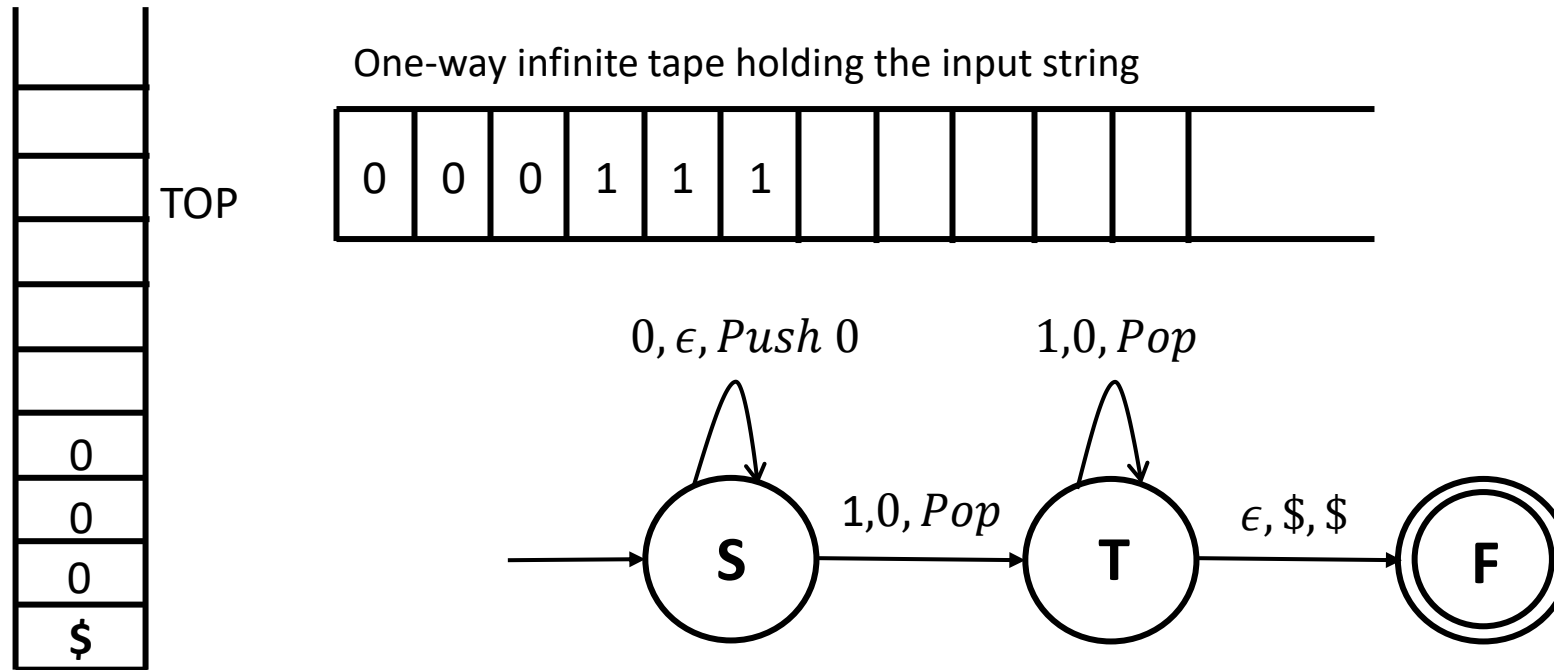
- A Pushdown Automata (PDA) is a finite automaton that has access to a stack.
- The FSM:
 - Transitions based on the Input symbol and the element at the top of the stack (e.g.: If I/P symbol = 0 & POP 0, transition from i to j)
 - Pushes new elements into the Stack (e.g.: If I/P symbol = 0, PUSH 0, transition from i to j).

Pushdown Automata



- A Pushdown Automata (PDA) is a finite automaton that has access to a stack.
- The FSM:
 - Transitions based on the Input symbol and the element at the top of the stack
 - Pops the element at the top of the Stack.
 - Pushes new elements into the Stack.

Pushdown Automata

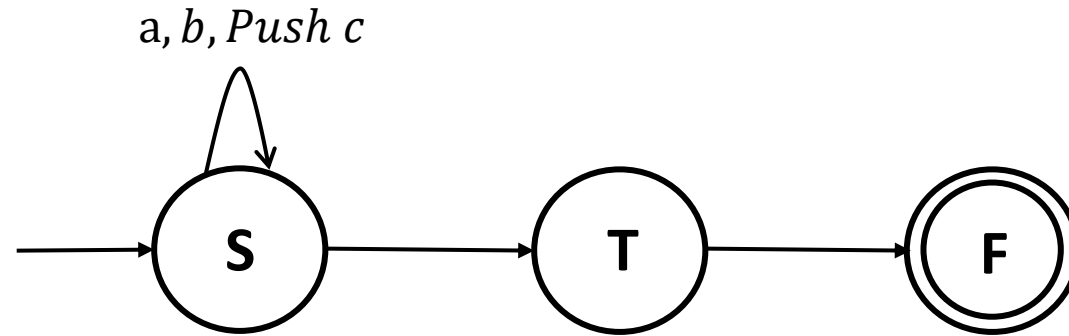


PDA's are **non-deterministic**.

- Missing transitions
- ϵ -transitions
- Multiple transitions/input symbol possible

Pushdown Automata

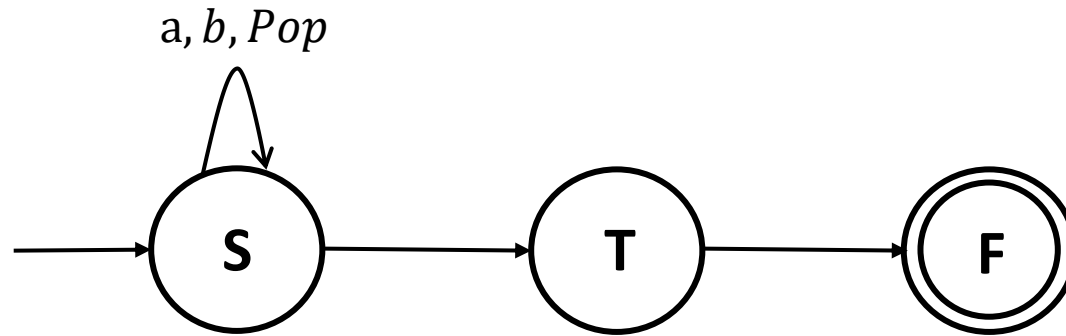
- How to represent a transition in a PDA?



If input symbol = a , Stack top = b (if b is popped), Push c onto the Stack and remain in S

Pushdown Automata

- How to represent a transition in a PDA?

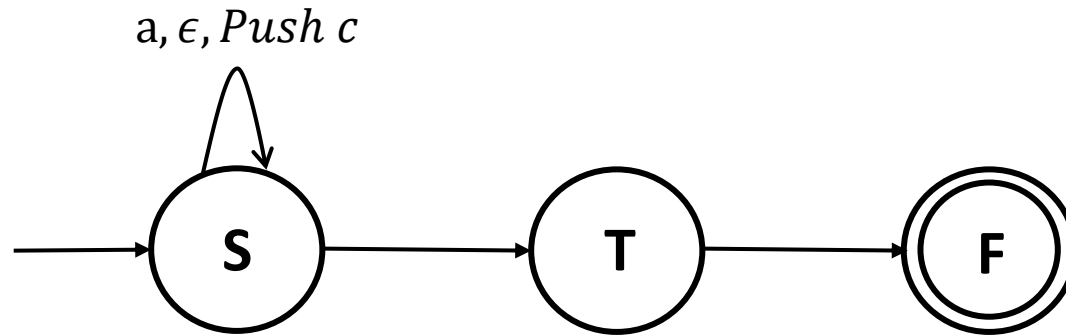


If input symbol = a , and b is popped, remain in S .

(If the symbol read is a and the element at the Stack TOP = b , then remain in S)

Pushdown Automata

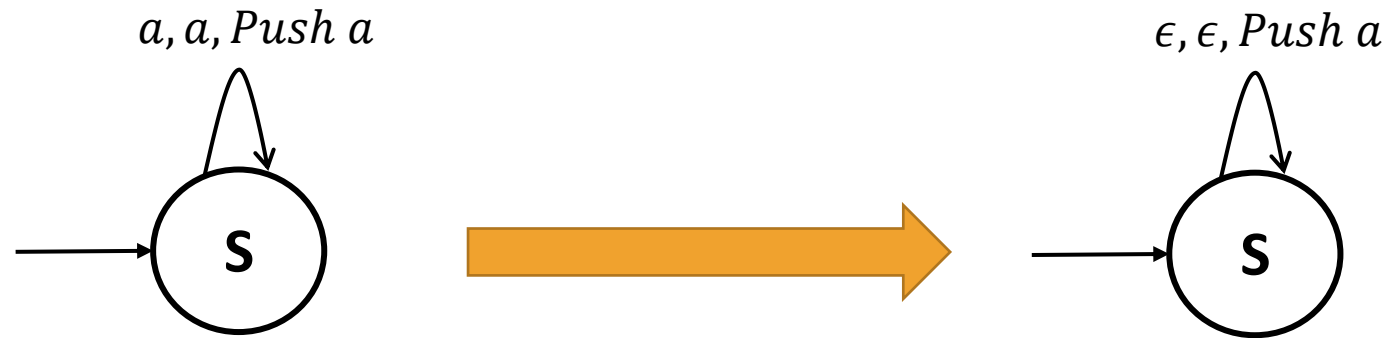
- How to represent a transition in a PDA?



If input symbol = a , then Push c

Pushdown Automata

- How to represent a transition in a PDA?

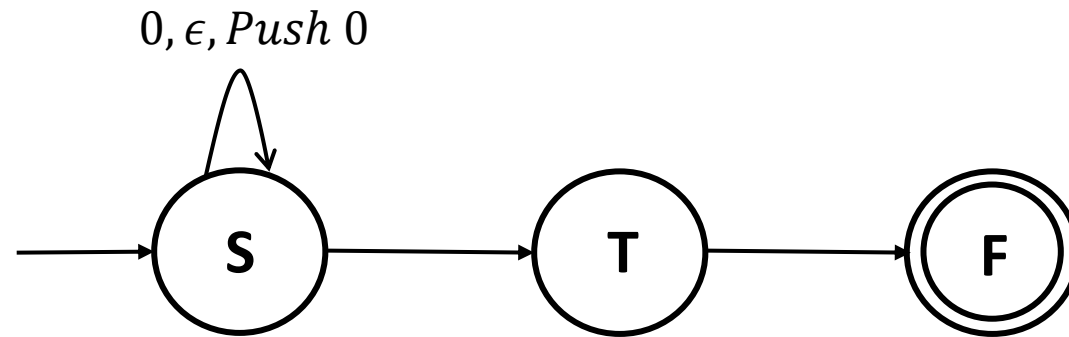


- (i) If input symbol = a , and a is popped, then Push a and remain in S .
- (ii) Push a on to the stack and remain in S .

Through Steps (i) and (ii), the PDA pushes a onto the stack if it reads a on the input tape and the element at the stack top = a .

Pushdown Automata

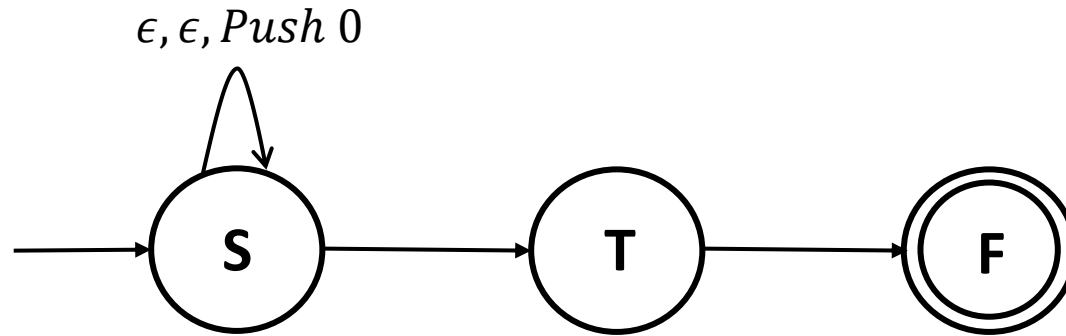
- How to represent a transition in a PDA?



If input symbol = 0, Push 0 onto the Stack irrespective of the element at the top of the stack

Pushdown Automata

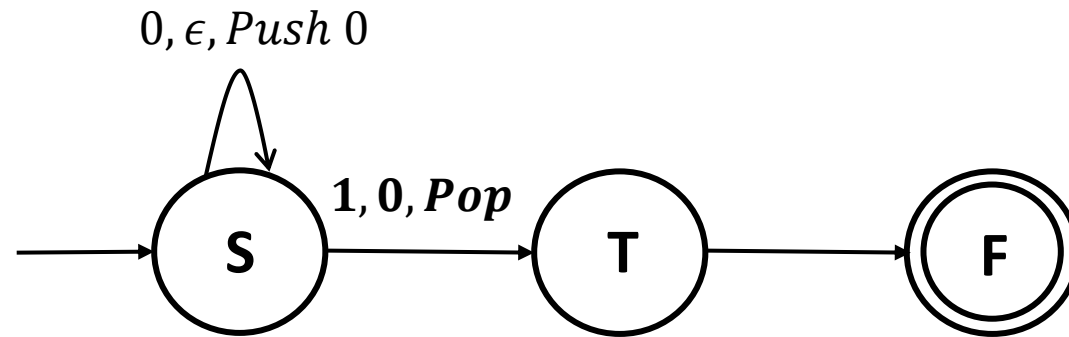
- How to represent a transition in a PDA?



Without reading the input symbol and the Stack top, Push 0 onto the Stack

Pushdown Automata

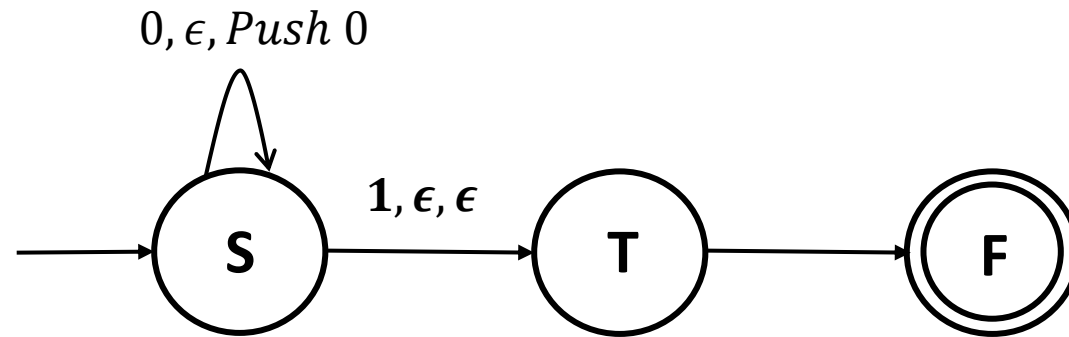
- How to represent a transition in a PDA?



If the input symbol is 1, and the element 0 is popped (**Pop 0**), then transition from S to T

Pushdown Automata

- How to represent a transition in a PDA?

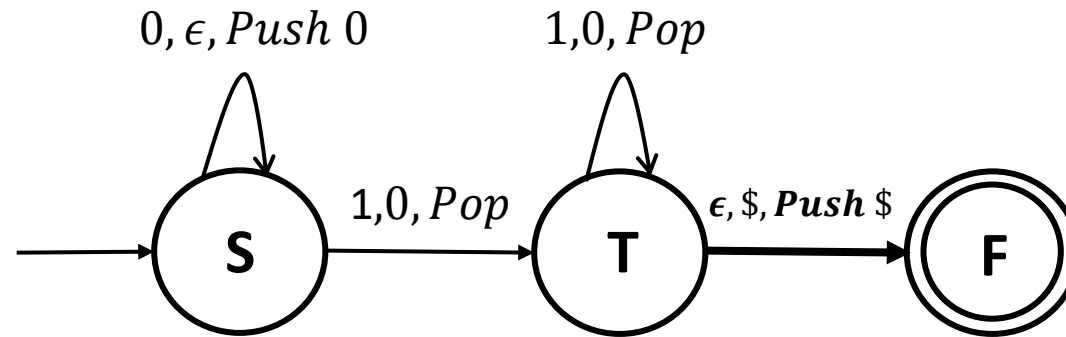


If the input symbol is 1, transition to T by ignoring the stack completely.

If this happens at every step of the execution of the PDA, then it is as powerful as an NFA.

Pushdown Automata

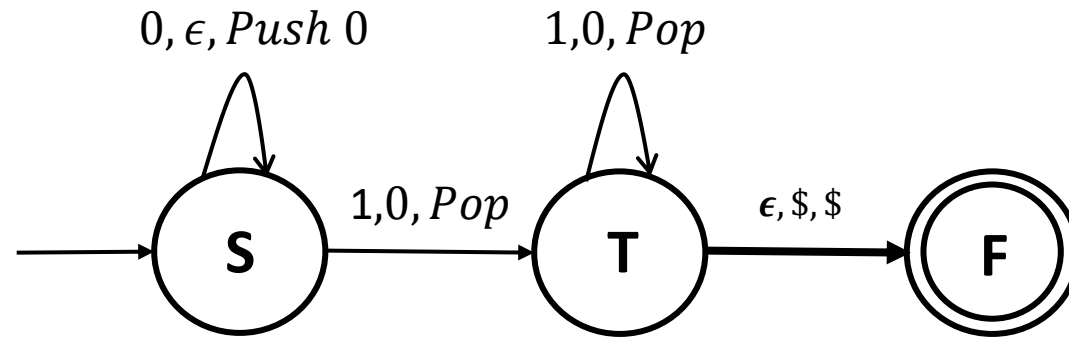
- How to represent a transition in a PDA?



Empty stack: If ϵ is popped, push the ϵ back onto the stack and transition to F from T , without reading the input

Pushdown Automata

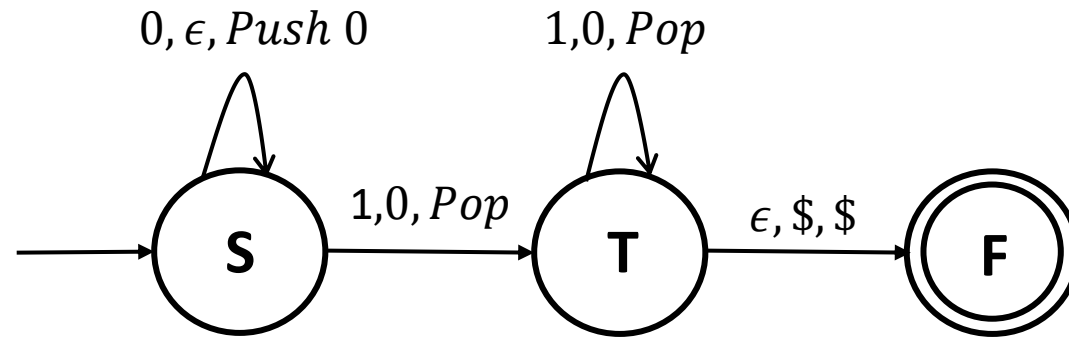
- How to represent a transition in a PDA?



Empty stack: If $\$$ is popped, push the $\$$ back onto the stack and transition to F from T , without reading the input

Pushdown Automata

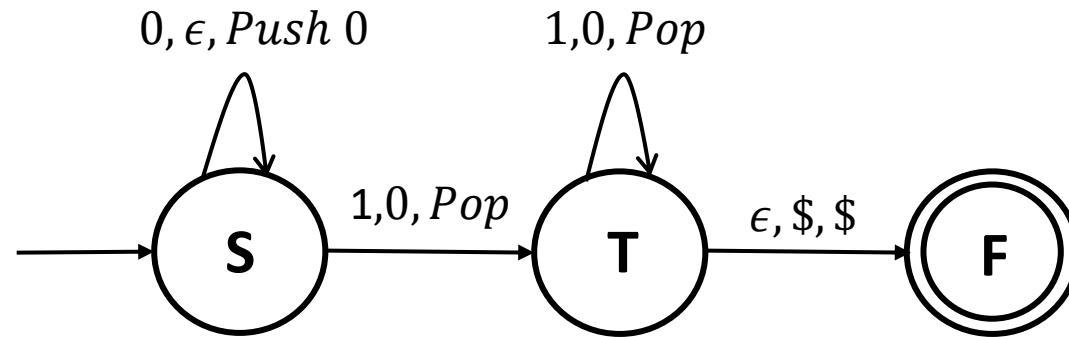
- How to represent a transition in a PDA?



What is the language recognized by this PDA?

Pushdown Automata

- How to represent a transition in a PDA?

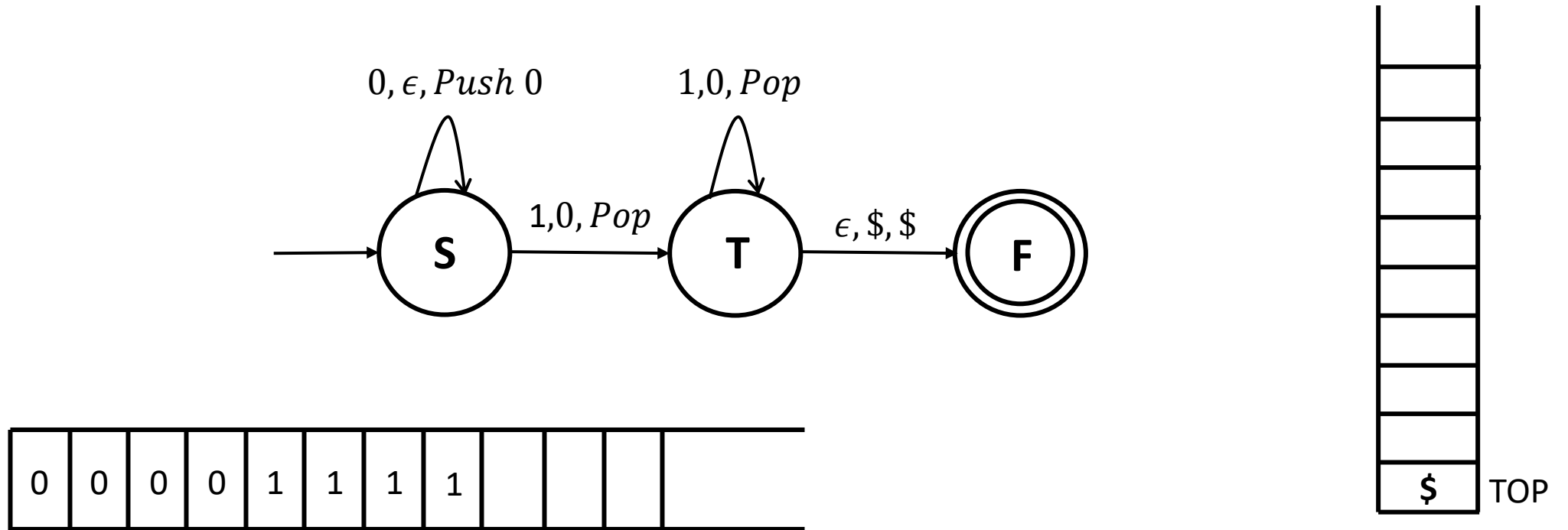


What is the language recognized by this PDA?

Verify that it is $L = \{0^n 1^n, n \geq 1\}$

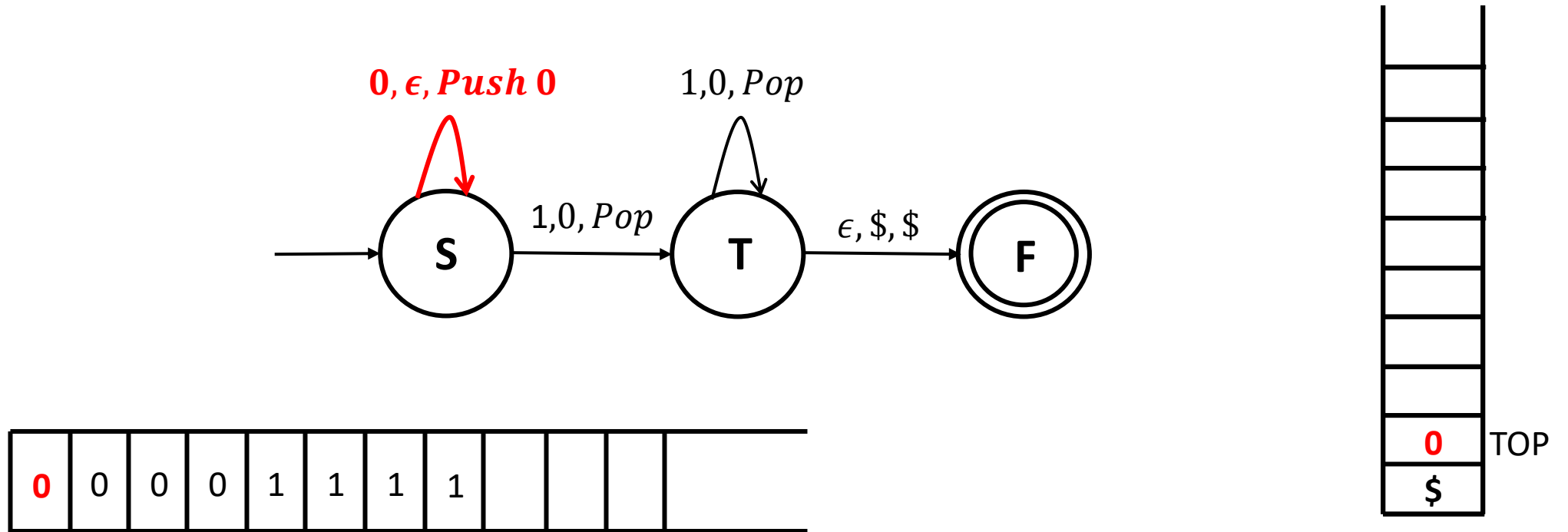
Pushdown Automata

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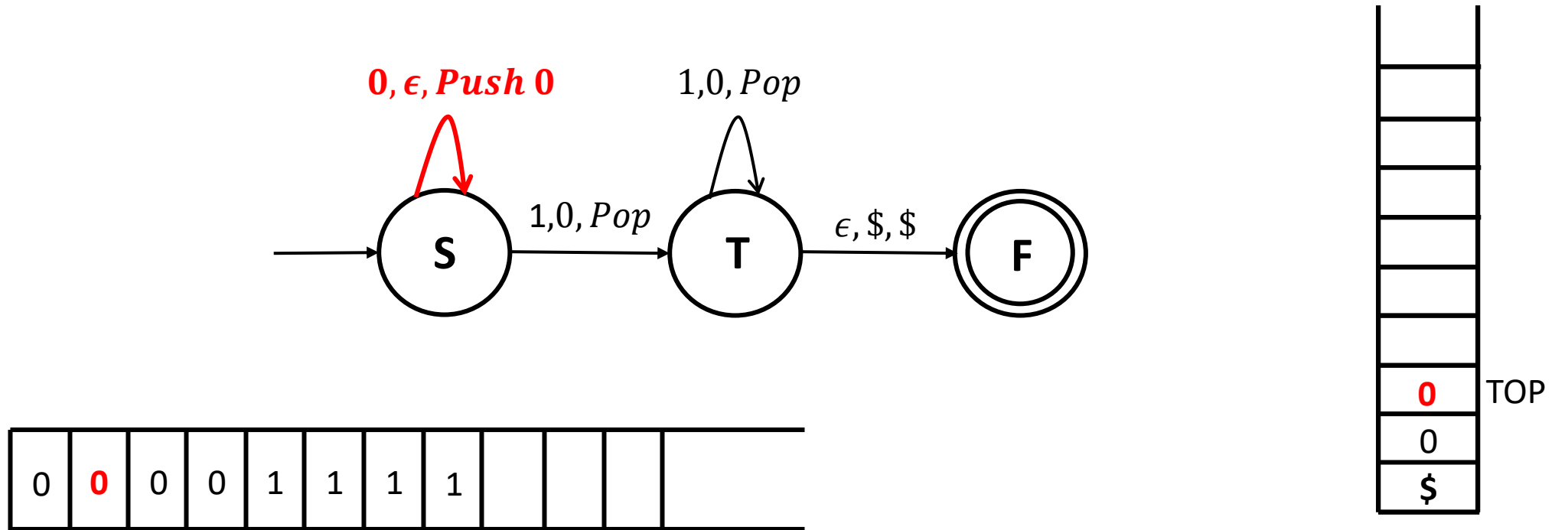
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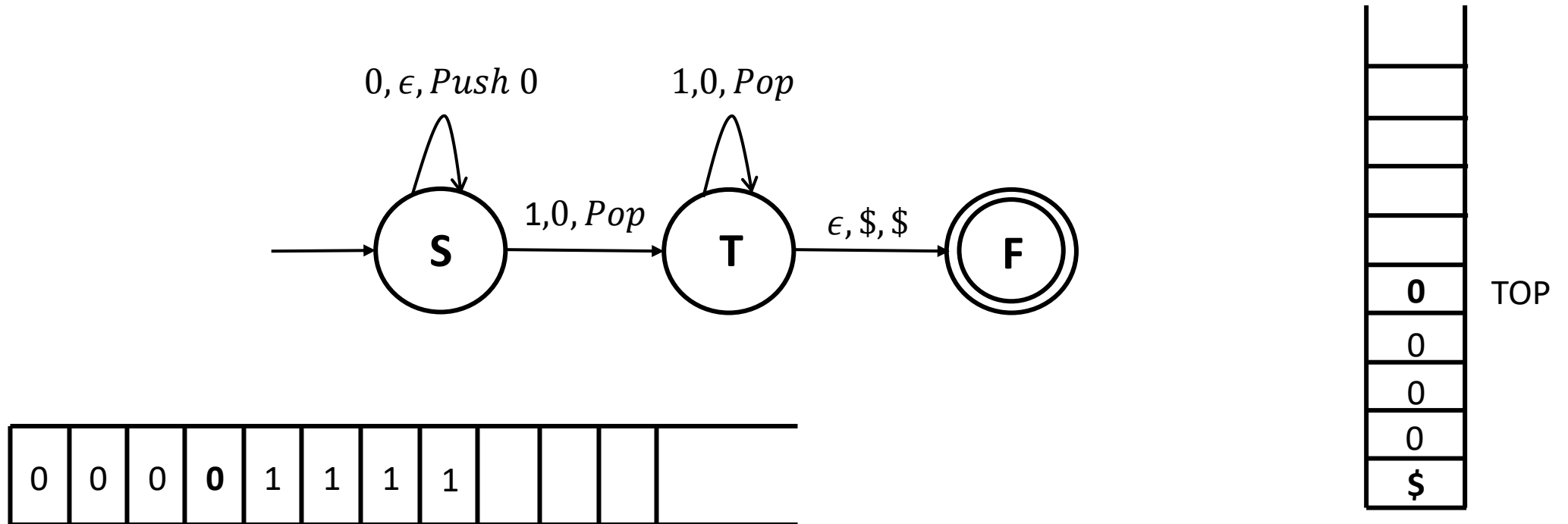
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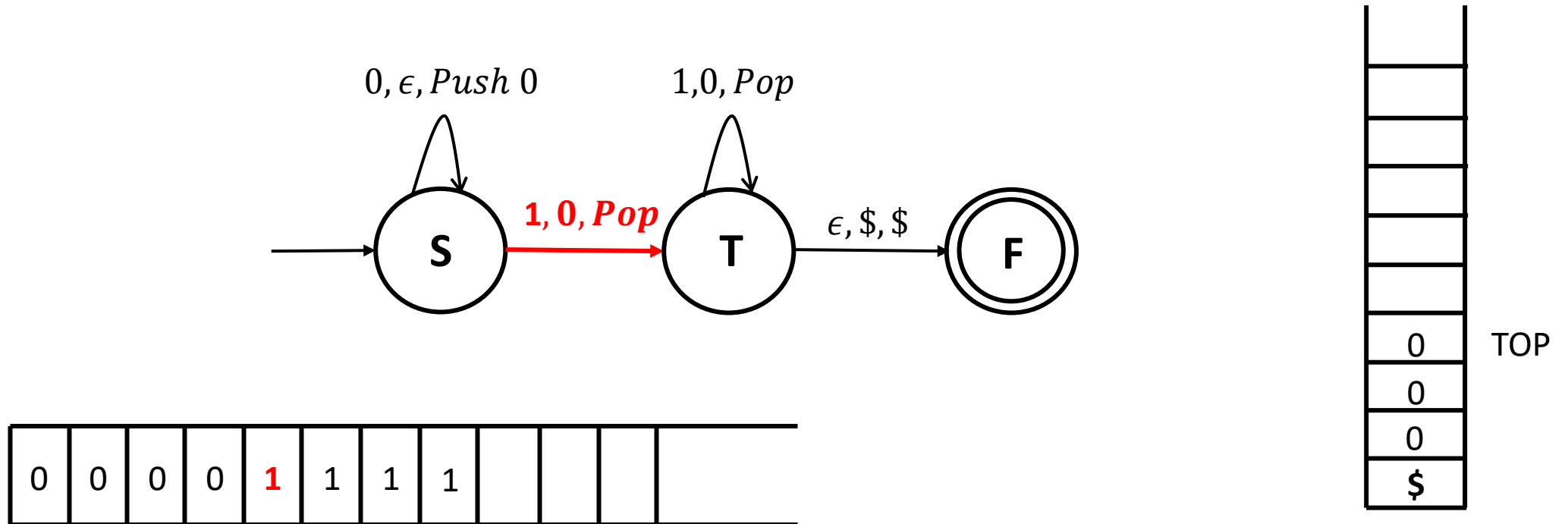
Pushdown Automata

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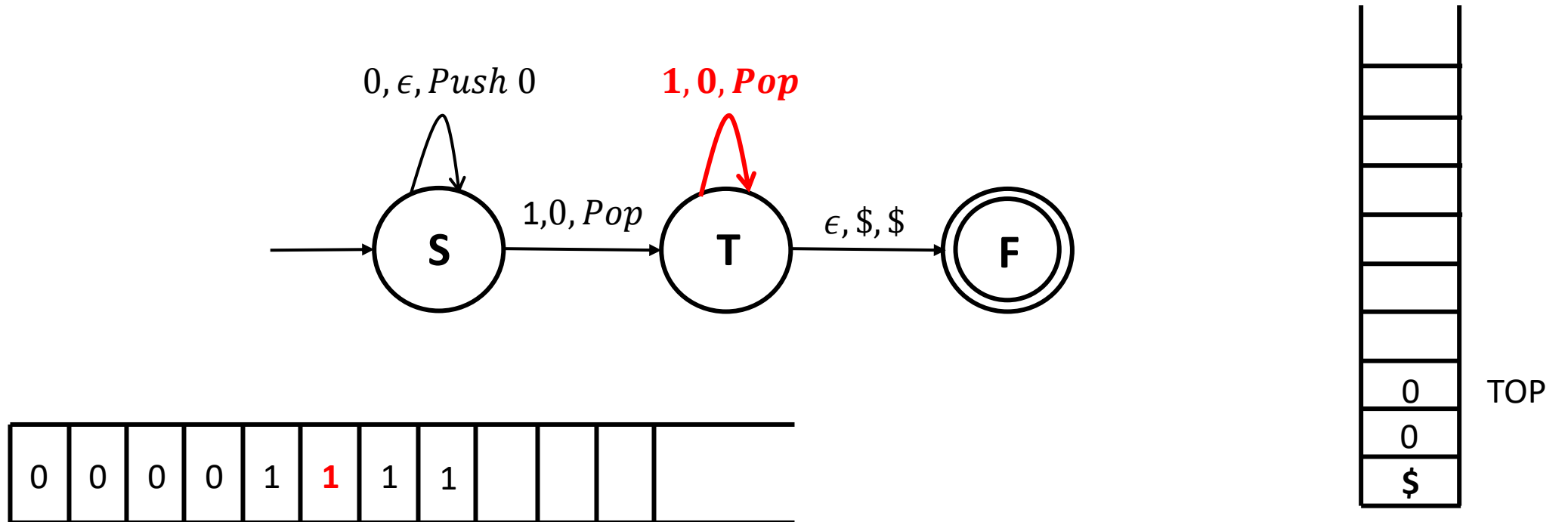
Pushdown Automata

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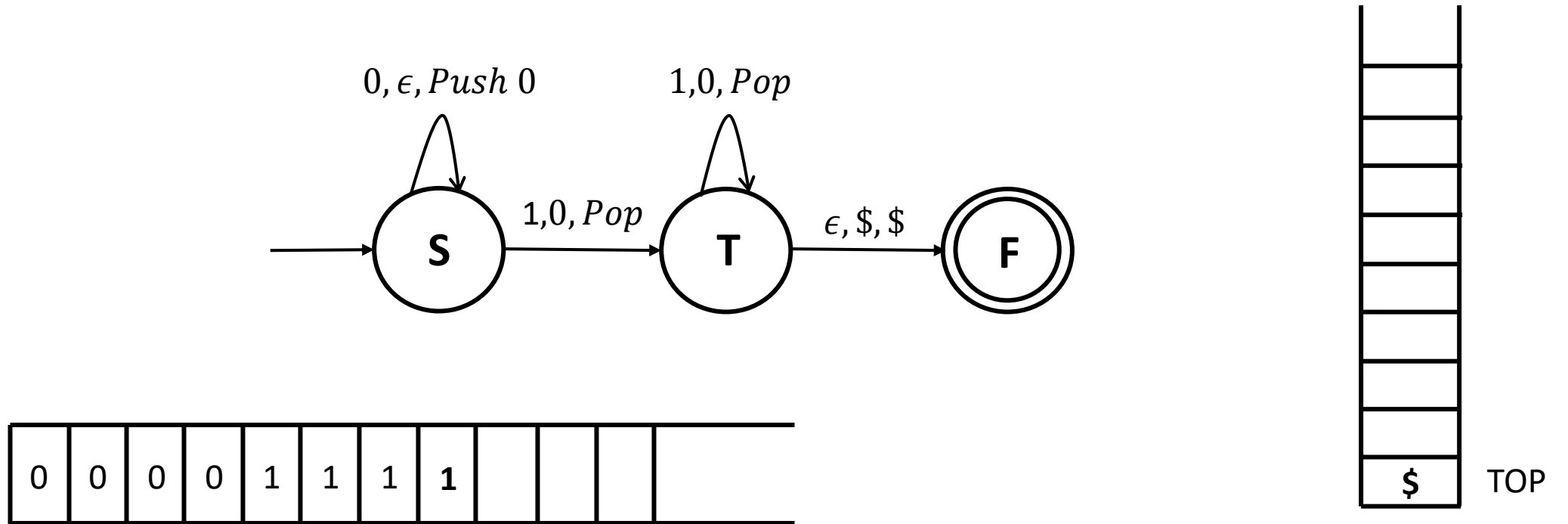
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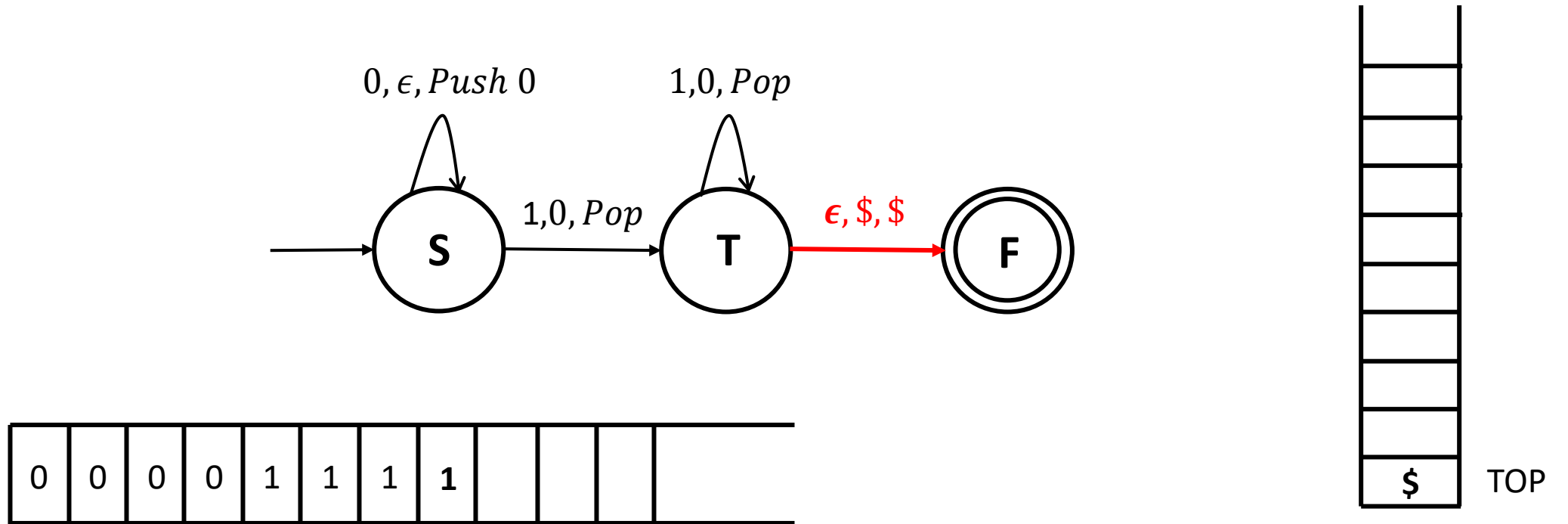
Pushdown Automata

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Pushdown Automata

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Thank You!