unit - 5 * Dynamic Programming *

Dynamic programming is a problem solving approch that involves breaking down complex problems into smaller sub problems, solving each sub problem only once and storing the solutions to sub problems to avoid redunded competions.

Types of Dynamic programming.

There are majarly classied into two types.

* Memorisation method (or) Calculs method (or) top down approch.
* Tabular Method (or) bottom up approch.

Stage : no. of variables.

State : no. of Constrains.

we Imagine $b_1, b_2, \ldots b_n$ are resources.

Calculs Method.

Maximize $z = 2x_1 + 5x_2$ subjected to constrains

$2x_1 + x_2 \le 430$, $2x_2 \le 460$. $x_1, x_2 \ge 0$

obj function max $z = 2x_1 + 5x_2$

constrains $2x_1 + x_2 \le 430$ — 1

$2x_2 \le 460$ — 2

Stage = 2 (no. of variables)

state = 2 (no of constrains)

Assume two resources = b_1, b_2 (based on state)

$f_i(b_1, b_2)$, $f_j(b_j)$ where i = stage j = state stage-1

$i=1$

$f_1(b_1, b_2) = \max_{0 \le x_1 \le b} 2x_1$ [from objective function]

To calculate $b = \min\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right]$

$c_1, c_2 \rightarrow$ coeffient of constraints

$$b = \min\left[\frac{430}{2}, \frac{460}{0}\right]$$

$$b = \min[215, \alpha]$$

$$\boxed{b = 215}$$

$$f_1(b_1, b_2) = \max_{0 \le x_1 \le 215} 2x_1$$

From Eq^n 1 converting into Equality

$$\boxed{x_1 = \frac{430 - x_2}{2}}$$

from Eq^n 2

$$\boxed{x_1 = 0}$$

so, minimise $\left(\frac{430 - x_2}{2}, 0\right)$

ignoring "0" because of above condition.

$$x_1 = \min\left[\frac{430 - x_2}{2}\right]$$

$$x_1 = \frac{430 - x_2}{2}$$

$$\therefore f_1(430, 460) = \max 2\left[\frac{430 - x_2}{2}\right]$$

stage 2

$i=2$

$$f_2(b_1,b_2) = \max_{0 \le x_2 \le b} 2x_1 + 5x_2$$

$$= \max 5x_2 + 2\min\left[\frac{430 - x_2}{2}\right]$$

To calculate $b = \min\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right]$

$$b = \min\left[\frac{430}{1}, \frac{460}{2}\right]$$

$$b = \min[430, 230]$$

$$\boxed{b = 230}$$

$$f_2(b_1,b_2) = \max_{0 \le x_2 \le 230} 5x_2 + 2\min\left[\frac{430 - x_2}{2}\right]$$

To calculate $x_1 = \min\left[\frac{430 - x_2}{2}\right]$ let us consider

$x_2 = 0$, then $x_1 = 215$

$x_2 = 230$ then $x_1 = 100$

at minimum $x_1 = 100$, $x_2 = 230$

obj function max $z = 2x_1 + 5x_2$

$= 2(100) + 5(230)$

$= 200 + 1150$

$= 1350$

maximize $z = 8x_1 + 7x_2$ constrains $2x_1 + x_2 \le 8$

$5x_1 + 2x_2 \le 15$ $x_1, x_2 \ge 0$

$\Rightarrow 7(8)$

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obj function max : $z = 8x_1 + 7x_2$

constrains $2x_1 + x_2 \le 8$

$5x_1 + 2x_2 \le 15$

stage = 2

state = 2

stage 1

$i = 1$

$f_1(b_1, b_2) = \max 8x_1$

$0 \le x \le b$

To calculate $b = \min\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right]$

$b = \min\left[\frac{8}{2}, \frac{15}{5}\right] \Rightarrow b = \min[4, 3]$

$b = 3$

$f_1(b_1, b_2) = \max 8x_1$

From eq^n 1 $\quad x_1 = \frac{8 - x_2}{2}$

From eq^n 2 $\quad x_1 = \frac{15 - 2x_2}{5}$

so, $x_1 = \min\left[\frac{8 - x_2}{2}, \frac{15 - 2x_2}{5}\right]$

$\therefore f_1(b_1, b_2) = \max 8 \min\left[\frac{8 - x_2}{2}, \frac{15 - 2x_2}{5}\right]$

Stage 2

$i = 2$

$F_2(b_1, b_2) = \max 8x_1 + 7x_2$

$0 \le x_2 \le b.$

$= \max 7x_2 + 8 \min\left[\frac{8 - x_2}{2}, \frac{15 - 2x_2}{5}\right]$

To Calculate $b = \min\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right] = \min\left[\frac{8}{1}, \frac{15}{2}\right]$

$\boxed{b = 8}$

$F_2(b_1, b_2) = \max 7x_2 + 8 \min\left[\frac{8 - x_2}{2}, \frac{15 - 2x_2}{5}\right] \quad 0 \le x_2 \le 8$

To calculate $x_1 = \min\left[\frac{8-x_2}{2}, \frac{15-2x_2}{5}\right]$ let consider

$x_2 = 0$, then $x_1 = \min[4, 3] \Rightarrow 3$

$x_2 = 8$, then $x_1 = \min[0, -1/5] \Rightarrow 0$

3. Use dynamic programming solve the following problem minimize $z = y_1^2 + y_2^2 + y_3^2$ is subjected to $y_1 + y_2 + y_3 = 10$ $y_1, y_2, y_3 \geq 0$

The given problem is a 3 stage problem and defined as

$$S_3 = y_1 + y_2 + y_3 = 10 \quad \text{---(1)}$$

$$S_2 = y_1 + y_2 = S_3 - y_3 \quad \text{---(2)}$$

$$S_1 = y_1 = S_2 - y_2 \quad \text{---(3)}$$

The functional recurrance relation in expressed on

Stage-1 $\quad f_1(S_1) = \min_{0 \leq y_1 \leq S_1} y_1^2$

$$f(S_1) = \min_{0 \leq y_1 \leq S_1} (S_2 - y_2)^2$$

Stage-2 $\quad f_2(S_2) = \min_{0 \leq y_2 \leq S_2} (y_1^2 + y_2^2)$

$$f_2(S_2) = \min_{0 \leq y_2 \leq S_2} [(S_2 - y_2)^2 + y_2^2]$$

$$\frac{df_2}{dy^2} = 0$$

$$\frac{df_2}{dy^2} = \min_{0 \leq y_2 \leq S_2} [(S_2 - y_2)^2 + y_2^2] = 0$$

$$\Rightarrow 2(S_2 - y_2)(-1) + 2y_2 = 0$$

$$-2S_2 + 2y_2 + 2y_2 = 0 \Rightarrow 2S_2 = 4y_2 \Rightarrow y_2 = \frac{S_2}{2}$$

stage-3 $f_3(s_3) = \min(y_1^2 + y_2^2 + y_3^2)$
$0 \le y_3 \le s_3$

$\Rightarrow \min[f_2(s_2) + y_3^2]$
$0 \le y_3 \le s_3$

$\Rightarrow \min[\frac{1}{2} s_2 + y_3^2]$
$0 \le y_3 \le s_3$

$\Rightarrow \min[\frac{1}{2}(s_3 - y_3)^2 + y_3^2]$
$0 \le y_3 \le s_3$

From $y_1 + y_2 + y_3 = s_3 = 10$ then

$f_3(s_3) = \min[\frac{1}{2}(10 - y_3)^2 (-4) + y_3^2]$
$0 \le y_3 \le 10$

$\frac{df_3}{dy_3} = 0$

$f_3(s_3) = \frac{1}{2} \cdot 2(10 - y_3)(-1) + 2y_3 = 0$

$(10 - y_3)(-1) + 2y_3 = 0 \Rightarrow -10 + y_3 + 2y_3 = 10$

$3y_3 = 20 \Rightarrow y_3 = \frac{10}{3} \Rightarrow 3.33$

From (2) $s_2 = s_3 - y_3$

$s_2 = 10 - 3.34 \Rightarrow 6.67$

From (4) $y_2 = \frac{s_2}{2} \Rightarrow \frac{6.67}{2} = 3.335$

From (3) $s_1 = s_2 - y_2$

$s_1 = 6.67 - 3.335 \Rightarrow 3.335$

$s_1 = y_1 = 3.335$

Hence optimal polling are $y_1 = 3.335$, $y_2 = 3.335$, $y_3 = 3.33$

$F_3(s_3) = \frac{1}{2}[(10 - 3.33)^2 + (3.33)^2]$

$\Rightarrow \frac{1}{2}[44.48 + 11.08] \Rightarrow \cancel{27.78} \cdot 33.34$

$min(z) = (3.33)^2 + (3.33)^2 + (3.33)^2$

$= 33.26$

maximise $z = y_1 y_2 y_3$ is subjected to $y_1 + y_2 + y_3 = 10$,

$y_1, y_2, y_3 \geq 0$

$z = y_1 y_2 y_3$ — stage = 3

$y_1 + y_2 + y_3 = 10$ — state = 1

$S_3 = y_1 + y_2 + y_3 = 10$ — (1)

$S_2 = y_1 + y_2 \Rightarrow S_3 - y_3 \rightarrow$ (2)

$S_1 = y_1 = S_2 - y_2$ — (3)

stage-1 $f_1(S_1) = \max y_1$

$f_1(S_1) = S_2 - y_2$

stage-2 $f_2(S_2) = \max y_1 y_2$

$f_2(S_2) = (S_2 - y_2) \cdot y_2$

$= S_2 y_2 - y_2^2$

$\frac{df_2}{dy_2} = 0 \Rightarrow \cancel{S_2}\ S_2 - 2y_2 = 0$

$f_2(S_2) = S_2\left(\frac{S_2}{2}\right) - \left(\frac{S_2}{2}\right)^2$

$= \frac{S_2}{2}\left[S_2 - \frac{S_2}{2}\right] = \frac{S_2}{2}\left(\frac{2S_2 - S_2}{2}\right]$

$= \frac{S_2^2}{4}$

stage-3 :

$F_3(S_3) = \max y_1 y_2 y_3$

$= f_2(S_2) \cdot y_3 \Rightarrow \frac{S_2^2}{4} \cdot y_3$

From equation (2)

$f_3(S_3) = \frac{(S_3 - y_3)^2}{4} \cdot y_3 \Rightarrow \left(\frac{S_3^2 + y_3^2 - 2S_3 y_3}{4}\right) y_3$

$$= \frac{s_3^2 y_3 + y_3^3 - 2s_3 y_3^2}{4}$$

$$\frac{df_3}{dy_3} = 0$$

$$\Rightarrow \frac{1}{4}\left[s_3^2 + 3y_3 - 4 s_3 y_3\right] = 0$$

$$s_3^2 + 3y_3^2 - 4s_3 y_3 = 0 \Rightarrow s_3^2 - 4s_3 y_3 + 3y_3 = 0$$

$$s_3^2 + s_3 y_3 + 2 \qquad s_3^2 - 2s_3 y_3 + s_3 y_3 + 2y_3 = 0$$

$$s_3(s_3 - 2y_3) + y_3($$

$$s_3^2 - 4s_3 y_3 + 3y_3^2 = 0$$

$$s_3^2 - s_3 y_3 - 3s_3 y_3 + 3y_3^2 = 0$$

$$s_3(s_3 - y_3) - 3y_3(s_3 - y_3) = 0$$

$$(s_3 - 3y_3)(s_3 - y_3) = 0$$

$$s_3 = 3y_3 \quad . \quad s = y_3$$

From ① $y_3 = \frac{s_3}{3}$ from ② $s_2 = s_3 - y_3 = 0$

From ③ $s_1 = s_2 - y_2 = 0$

$$y_3 = \frac{10}{3}, \quad s_3 = \frac{10}{3}$$

This is the trial solution because we get $s_1 = 0, s_2 = 0$ i.e $y_1 = 0$ $y_2 = 0$ then objective function becomes '0'.

A Machine Manufacturing Concern as 9 medical representatives working in 3 sales areas. the profitability for each representative in 3 sales areas is as follows.

No. of representatives	profitability (1000 of rupees)		
	Area 1	Area 2	Area 3
0	32	37	44
1	47	47	56
2	62	54	62
3	72	66	72
4	81	74	84
5	92	84	97
6	100	95	104
7	107	100	112
8	102	102	112
9	92	102	112

Determine the optimum allocation of medical representative in order to maximize the profits.

In this problem the three areas representing three stages and number of representation represent state variables.

Stage-1

we start with Area 1.

No. of repres	0	1	2	3	4	5	6	7	8	9
area 1	32	47	62	72	81	92	100	107	102	92

~~Stage-2~~

stage - 2

Consider first two areas

9 representatives can be divided among two areas in 10 different ways.

Area 1	x_1	0	1	2	3	4	5	6	7	8	9
	$f_1(x_1)$	32	47	62	72	81	92	100	107	102	92
Area 2											
x_2	$f_1(x_2)$										
0	37	69	84	99	109	118	129	137	144	139	129
1	47	79	94	109	119	128	139	147	154	149	
2	54	86	101	116	126	135	146	154	161		
3	66	98	113	128	138	147	158	166			
4	74	106	121	136	146	155	166				
5	84	116	136	146	156	165					
6	95	127	142	157	167						
7	100	132	147	162							
8	102	134	149								
9	102	139									

Stage - III

9 representatives to be alloted to the three areas.

no. of rep	0	1	2	3	$\boxed{4}$	5	6	7	8	9
Total profit $f_2(x_2)+f_1(x_1)$	69	84	99	109	119	129	139	149	158	167
Rep in area 2 + Area 1 (x_2+x_1)	[0+0]	[0+1]	[0+2]	(1+2) [0+3]	[1+3]	[0+5]	(1+5)	(1+6) [3+4]	[3+5]	(6+3)
No of Rep in area 3	9	8	7	6	$\boxed{5}$	4	3	2	1	0
Profit $f_3(x_3)$	112	112	112	104	97	84	72	62	56	44
Total profit $f_3(x_3)+f_2(x_2)+f_1(x_1)$	181	196	211	213	$\boxed{216}$	213	211	209	214	211

The maximum profit for 9 representative is rupees 2160000 if 5 representative are alloted to area 3 and from the remaining 4 representative are alloted to area 2 and 3 representative alloted to area 1.

Theory part

Principal of Optimality (or) Bellman principal

- An optimal policy has the property that, what ever the initial state and initial decision are the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Work

Apply dynamic programming to maximise $z = 2x_1 + 3x_2$ is subjected to $x_1 + 2x_2 \leq 4$, $2x_1 + x_2 \leq 3$ $x_1, x_2 \geq 0$

$z = 2x_1 + 3x_2$

Constrains $x_1 + 2x_2 \leq 4$

$2x_1 + x_2 \leq 3$

Stage = 2, State = 2 b_1, b_2

Stage 1 $i = 1$

$f(b_1, b_2) = \max\limits_{0 \leq x_1 \leq b} 2x_1$

$$b = \min\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right]$$

$$b = \min\left[4/1, 3/2\right] \Rightarrow b = 3/2$$

$\therefore f(b_1, b_2) = \max 2x_1$

$0 \le x_1 \le 3/2$

Converting inequalities into equalities.

From ① $x_1 = 4 - 2x_2$ From ② $x_1 = \frac{3 - x_2}{2}$

$$x_1 = \min\left[4 - 2x_2, \frac{3 - x_2}{2}\right]$$

Stage 2 $i = 2$.

$f(b_1, b_2) = \max 2x_1 + 3x_2$

$0 \le x_2 \le b$

$$b = \min\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right] = \left[\frac{4}{2}, \frac{3}{1}\right] = 2$$

$\boxed{b = 2}$

$f(b_1, b_2) = \max 2x_1 + 3x_2$

$0 \le x_2 \le 2$.

We have $x_1 = \min\left[4 - 2x_2, \frac{3 - x_2}{2}\right]$ let us consider

$x_2 = 0$ then $x_1 = \min[4, 3/2] \Rightarrow x_1 = 3/2$

$x_2 = 2$ then $x_1 = \min[4-4, 1/2] \Rightarrow [0, 0.5] \Rightarrow 0$.

$\therefore$ obj function max: $z = 2x_1 + 3x_2$

$2(3/2) + 3(0) = 3$

$2(0) + 3(2)$

$= 6$.