Optimization Techniques (OT)

Optimization Techniques

These are methods used to find the best solution among a set of possible solutions to a problem. The goal of optimization is to maximize (or) minimize a specific objective function, subjected to certain constrains.

Types of optimization Techniques.

* Linear programming
* Integer programming
* non-linear programming
* Dynamic programming
* simulated annealing.

Applications of optimization Techniques.

* optimize resource allocation in industries such as manufacturing, logistics and finas.
* optimize supply chain management and transportation.
* optimize investment portfolio to maximize returns and minimize risk.
* optimize scheduling in industries such as healthcare, transportation and education.
* optimize

Advantages (or) Benefits.

* Improved Effeciency.
* Increased productivity.
* cost savings.

* Improved design making
* compitative advantage
Draw backs
* commutational complexity
* model limitations
* over optimisation.
* Interpretability
* data requirement
* sensitivity to parameters
* limited applicability.

General statement of an optimisation problems.

optimisation problem states that

* objective function
  maximize or minimize a function $f(x)$ (or) "z" $f(x) = \ldots\ldots$
* Decision variables:
  $x = x_1, x_2, x_3, \ldots\ldots x_n$
* constrains:
  $g_1(x) \leq 0, g_2(x) \leq 0, \ldots\ldots g_n(x) \leq 0$ (or) $h_1(x) \leq 0, h_2(x) \leq 0, \ldots\ldots h_n(x) \leq 0$.
* Boundaries
  $x_1 \in (a_1, b_1), x_2 \in (a_2, b_2) \ldots\ldots x_n \in (a_n, b_n)$
* Goal: find the values of "x" that optimize (maximize or minimize) the objective function $f(x)$ while satisfying all constrains and boundaries.

Notations

$f(x) \longrightarrow$ objective function

$x \longrightarrow$ vector of decision variables

$g(x)$ & $h(x) \longrightarrow$ constraint functions

a & $b \longrightarrow$ boundaries on decision variables

Design vector

A Design vector is a mathematical representation of a decision (or) a solution to an optimisation problem. It is a typically vector of decision variables that define the characteristics of the design.

A Design vector can be represented mathematica-lly as $x = (x_1, x_2, x_3 \ldots, x_n)$.

where x is the design vector and x_i are the individual decision variables.

Design constrains.

Design constrains are restrictions that are imposed on a design (or) a system. These constrains are defined can be physical, technical, economical, Environmental and social in nature.

Constraint surfaces.

A constraint surface is a geometric representation of the constraints in a design (or) optimation problem. It is a surface that defines the boundaries within which a design must operend.

Types of constraint surfaces (cs)

* linear cs
* non linear cs
* convex cs
* Non convex cs

* linear cs can be represented mathematically as $ax+by+cz \leq d$

* Non-linear cs can be represented mathematically as $f(x,y,z) \leq 0$

Objective function (OF)

A objective Function is a mathematical function that describes the goal or objective of in optimisation problem.

It is a function that is to be minimised (or) maximized subjected to certain constrains.

Objective function surface (OFS)

An OFS is a Graphical representation of the OF in an op (optimisation problems). It is a surface that shows the value of objective function for different combinations of decision variables.

classification of op

* Linear Vs Non-linear Optimisation
* convex Vs Non-Convex Optimization
* Single Objectives Vs multi objective Optimization
* Constrained Vs Non Constrained optimization.
* Deterministic Vs stochastic optimization
* Dynamic Vs static optimization.
* Discreate Vs Continuous optimization.

Problems:

Solving a problems of single variable optimization problems without constrains and with neccessary and sufficient conditions

1. ~~De~~ neccessary condition : $f'(x)=0$

sufficient conditions.

* $f''(x)<0$ then $f(x)$ is maximum
* $f''(x)>0$, then $f(x)$ is minimum
* $f''(x)=0$, then we have to go next derivation
* $f'''(x)\neq 0$, it is a inflection point.

1/ Determine the maximum & minimum values of function $f(x)=12x^5-45x^4+40x^3+5$

Sol^n: The above problem is a single variable optimization problem. we have two conditions, we are used to solve a single variable optimization problem for maximum & minimum values.

$f(x)=12x^5-45x^4+40x^3+5$

condition 1: neccessary condition

$f'(x)=0$ (or) $\frac{d}{dx}f(x)=0$

$\frac{d}{dx}(12x^5-45x^4+40x^3+5)=0$

$60x^4-180x^3+120x^2=0$

$60x^2(x^2-3x+2)=0$

$x=0$ $\quad x^2-2x-x+2=0$

$x(x-2)-1(x-2)=0$

$\boxed{x=1,2}$

Condition ②: sufficient condition.

$f'(x)=60x^4-180x^3+120x^2$

$f''(x)=240x^3-540x^2+240x$

at $x=1$, $f''(x)=240(1)^3-540(1)^2+240(1)$

$=240+240-540$

$=-60$

$f''(x)<0$, $f(x)$ is maximum.

$f_{max}=12x^5-45x^4+40x^3+5$

$=12(1)^5-45(1)+40(1)+5$

$f_{max}=12$.

at $x=2$ $f''(x)=240(2^3)-540(2^2)+240(2)$

$=240(8)-540(4)+240(2)$

$=240$

$f''(x)>0$, $f(x)$ is a minimum

$f_{min}=12(2)^5-45(2)^4+40(2)^3+5$

$=12\times32-45\times16+40\times8+5$

$f_{min}=-11$

at $x=0$, $f''(x)=240(0)-540(0)+240(0)$

$=0$

$f'''(x)=720x^2-1080x+240$

$=720(0)^2-1080(0)+240$

$=\cancel{80}\ 240$

$f'''(x)\neq0$ then it is a inflection point.

$f(x) = 2x^3 - 3x^2 - 12x + 4$

Sol^n:

condition 1: neccessary condition.

$f'(x) = 0$ (or) $\frac{d}{dx} f(x) = 0$

$\frac{d}{dx}(2x^3 - 3x^2 - 12x + 4) = 0$

$6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x^2 - 2x + x - 2 = 0$

$(x-2)(x+1) = 0$

$\boxed{x = 2, -1}$

condition 2: sufficient condition.

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

at $x = 2$ $f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18$

$f''(x) > 0$, $f(x)$ is a minimum

$f_{min} = 2x^3 - 3x^2 - 12x + 4$

$= 2(2)^3 - 3(2)^2 - 12(2) + 4$

$= 16 - 12 - 24 + 4$

$= 20 - 36 \Rightarrow -16$

$\boxed{f_{min} = -16}$

at $x = -1$

$f''(x) = 12x - 6$

$= 12(-1) - 6$

$= -12 - 6 \Rightarrow -18$ $f''(-1) = -18$

$f''(x) < 0$, $f(x)$ is maximum

$$f_{max} = 2x^3 - 3x^2 - 12x + 4$$
$$= 2(-1)^3 - 3(-1)^2 - 12(-1) + 4$$
$$= -2 - 3 + 12 + 4$$
$$= 11$$

$$f_{max} = 11$$

$$f''(x) = 12x - 6$$

$f''(x) \neq 0$ so it is inflection point.

Determine the maximum and minimum values of the function $f(x) = 4x^3 - 18x^2 + 27x - 7$

$$f(x) = 4x^3 - 18x^2 + 27x - 7$$

condition 1: neccessary condition

$$f'(x) = 0$$
$$\frac{d}{dx}(4x^3 - 18x^2 + 27x - 7) = 0$$
$$12x^2 - 36x + 27 = 0$$
$$3(4x^2 - 12x + 9) = 0$$
$$4x^2 - 12x + 9 = 0$$
$$4x^2 - 6x - 6x + 9 = 0$$
$$2x(2x - 3) + 3(2x - 3) = 0$$
$$(2x + 3)(2x - 3) = 0$$
$$x = +3/2, 3/2$$

condition 2: sufficient condition

$f''(x) = 24x - 36$

at $x = 3/2$

$$f''(3/2) = \overset{12}{24}(3/2) - 36$$
$$= 36 - 36$$
$$= 0$$

$f''(x) = 0$, so we need to derive again.

$f'''(x) \neq 0, f'''(x) = 24$ so it is inflection point.

Determine the maximum and minimum values of the function $f(x) = 10x^6 - 48x^5 + 15x^4 + 200x^3 - 120x^2 - 480x + 100$

Multivariable Optimization.

/Find the Extreme points of $f(x, y)$ and the function is $f(z)$ and the function $z = x^2 - xy + y^2 + 3x$ by the condition of hessian matrix for maximum and minimum values.

$z = x^2 - xy + y^2 + 3x$

condition 1: neccessary condition

we use partial differentiation two find values.

$\frac{d}{dx}(z) = 0$; $\frac{d}{dy}(z) = 0$

$\frac{d}{dx}(z) = 2x - y + 3 = 0$ —— (1)

$\frac{d}{dy}(z) = -x + 2y = 0$ —— (2)

$x = 2y$ —— (3)

Substituting ③ in ①

$$2(2y) - y + 3 = 0$$

$$4y - y + 3 = 0$$

$$\boxed{y = -1}$$

Substituting x value in 3

$$x = 2(-1)$$

$$x = -2$$

$$z = x^2 - xy + y^2 + 3x$$

$$= (-2)^2 - (2)(-1) + (-1)^2 + 3(-2)$$

$$= 4 - 2 + 1 - 6$$

$$z = -3$$

Sufficient condition

$H_1 < 0,\ H_2 > 0,\ H_3 < 0 \longrightarrow$ maxima

$H_1 > 0,\ H_2 > 0,\ H_3 > 0 \longrightarrow$ minima.

$H_1 < 0,\ H_2 < 0,\ H_3 < 0 \longrightarrow$ point of inflection.

hessian matrix $= H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$

$H_1 = |f_{11}|$

$H_2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$

$H_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$

Remember that no. of variables in equation be 'n' then the matrix is $n \times n$

where $1 \rightarrow x$

$2 \rightarrow y$

$3 \rightarrow z$

:

$$f_{11} = \frac{d^2 x}{dx^2} = \frac{d}{dx}(2x-y+3) = 2$$

$$f_{11} = 2$$

$$f_{12} = \frac{d^2 x}{dx dy} = \frac{d}{dy}(2x-y+3) = -1$$

$$f_{21} = \frac{d^2 y}{dy dx} = \frac{d}{dx}(-x+2y) = -1$$

$$f_{22} = \frac{d^2 y}{dy^2} = \frac{d}{dy}(-x+2y) = 2$$

$$H_1 = 2$$

$$H_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \Rightarrow \det(H_2) = 4-1 = 3$$

Here in our equation there are only two variables, so, we are considering 2x2 matrix.

$H_1 > 0$, $H_2 > 0$ $\therefore$ $f(z)$ is minima.

Find the extreme points of the function $f(x_1, x_2) = 20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2$ the Hessian matrix of f, find the nature of these extreme points?

Given $f(x_1, x_2) = 20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2$

condition 1: necessary condition.

$$\frac{d}{dx_1} f(x_1, x_2) \Rightarrow \frac{d}{dx_1}(20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2) = 0$$

$$20 + 4x_2 - 8x_1 = 0 \quad \text{--- (1)}$$

$$\frac{d}{dx_2} f(x_1, x_2) \Rightarrow \frac{d}{dx_2}(20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2) = 0$$

$26 + 4x_1 - 6x_2 = 0$ —— (2)

multiplying equation (2) with 2.

$52 + 8x_1 - 12x_2 = 0$ —— (3)

From (1) and (3)

$$\begin{array}{r} 20 + 4x_2 - \cancel{8x_1} = 0 \\ 52 - 12x_2 + \cancel{8x_1} = 0 \\ \hline 72 - 8x_2 = 0 \end{array}$$

$$x_2 = \frac{\cancel{72}\,9}{\cancel{8}\,1}$$

$$\boxed{x_2 = 9}$$

From (1) $20 + 4(9) - 8x_1 = 0$

$$20 + 36 - 8x_1 = 0$$

$$+8x_1 = +56$$

$$x_1 = \frac{56}{8}$$

$$\boxed{x_1 = 7}$$

Substituting x_1 and x_2 values in Given equation.

$f(7,9) = 20(7) + 26(9) + 4(7)(9) - 4(7)^2 - 3(9)^2$

$= 140 + 234 + 252 - 196 - 243$

$= 187$

$f(7,9) = 187 \longrightarrow$ maxima.

condition 2: sufficient condition.

$1 \rightarrow x_1$

$2 \rightarrow x_2$

Hessian matrix $= \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = H$

$f_{11} = \frac{d^2 x_1}{dx_1^2} = \frac{d}{dx_1}(20 + 4x_2 - 8x_1) = -8$

$f_{11} = -8$

$$f_{12} = \frac{d^2x_1}{dx_1 dx_2} = \frac{d}{dx_2}(20+4x_2-8x_1)$$

$$f_{12} = 4$$

$$f_{21} = \frac{d^2x_2}{dx_2 dx_1} = \frac{d}{dx_1}(26+4x_1-6x_2)$$

$$= +4$$

$$f_{21} = 4$$

$$f_{22} = \frac{d^2x_2}{dx_2^2} = \frac{d}{dx_2}(26+4x_1-6x_2)$$

$$= -6$$

$$H_1 = -8$$

$$H_2 = \begin{bmatrix} -8 & 4 \\ 4 & -6 \end{bmatrix} \Rightarrow \det(H_2) = 48-16 = 32$$

$H_1 < 0$, $H_2 > 0$

Here H_1 and H_2 are alternate so the function $f(x_1, x_2)$ is maxima.

Solving another example on single Variable Optimization.

$$f(x) = 10x^6 - 48x^5 + 15x^4 + 200x^3 - 120x^2 - 480x + 100$$

Another Example for multivariable 2)

$z = 48y - 3x^2 - 6xy - 2y^2 + 72x$

Multivariable optimisation (Three Variables)

If $U = f(x, y, z) = -x^3 + 3xz + 2y - y^2 - 3z^2$. Find the maximum or minimum values.

Given $f(x, y, z) = -x^3 + 3xz + 2y - y^2 - 3z^2$

condition 1: neccessary condition.

$\frac{d}{dx}(u) = 0 = -3x^2 + 3z = 0$ —— 1

$\frac{d}{dy}(u) = 0 = 2 - 2y = 0 \Rightarrow y = 1$ —— 2

$\frac{d}{dz}(u) = 0 = 3x - 6z = 0$ —— 3

f_1

f_1

f_1

f_2

f_2

multiply ① with 2.

$$-6x^2+6\cancel{z}=0$$
$$3x-6\cancel{z}=0$$

$$3x-6x^2=0$$

$$3x(\cancel{3}-2x)=0$$

$$x=0,\ x=\cancel{3}/2$$

From ③ $3(0)-6(z)=0$, $3(\cancel{3}/2)-6z=0$

$z=0$ $\quad\quad$ $\cancel{\frac{9}{9}}\ \frac{3}{2}-6z=0$

$\frac{3}{2}-6z=0$ $\quad\quad$ $3\cancel{6}(3/\cancel{24})=z\ldots)$

$\frac{\cancel{3}}{2}=\cancel{6}z^2$ $\boxed{z=1/4}$

2) condition 2. Sufficient condition.

$$H=\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

$f_{11}=f_{xx}=\frac{d^2x}{dx^2}=\frac{d}{dx}(-3x^2+3z)=-6x \Rightarrow -\cancel{6}^{3}(1/\cancel{4})=-3$

$f_{12}=\cancel{\frac{d}{dy}}\ \frac{d^2x}{dxdy}=\frac{d}{dy}(-3x^2+3z)=0$

$f_{13}=\frac{d^2x}{dxd\cancel{z}}=\frac{d}{du}(-3x^2+3z)=3$

$f_{21}=\frac{d^2y}{dydx}=\frac{d}{dx}(2-2y)=0$

$f_{22}=\frac{d^2U}{dydy}=\frac{d}{dy}(2-2y)=-2$

$$f_{23} = \frac{\partial^2 u}{\partial y \partial z}(2-2y) = 0$$

$$f_{31} = \frac{\partial^2 u}{\partial z \partial x}(3x-6z) = \cancel{3x} \cdot 3$$

$$f_{32} = \frac{\partial^2 u}{\partial z \partial y}(3x-6z) = 0$$

$$f_{33} = \frac{\partial^2 u}{\partial z \partial z}(3x-6z) = -6$$

$$H = \begin{bmatrix} -3 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{bmatrix} \qquad H_1 = -3$$

$$H_2 = \begin{vmatrix} -3 & 0 \\ 0 & -2 \end{vmatrix} = \det(H_2) = 6$$

$$H_3 = \det(H) = -3(12-0) - 0(0) + 3(0+6)$$
$$= -36+18$$
$$= -18 < 0$$

Substituting x, y, z values in given equation

$$u = -\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + 2(1) - (1)^2 - 3\left(\frac{1}{4}\right)^2$$

$$= -\frac{1}{8} + \frac{3}{8} + 2 - 1 - \frac{3}{16}$$

$$\frac{1}{4} + 1 - \frac{3}{16} \Rightarrow \frac{4+1-3}{16} = \frac{\cancel{2}}{\cancel{16}\,8} = \frac{1}{8} > 0$$

$H_1 < 0$, $H_2 > 0$, $H_3 < 0$ So $f(x, y, z)$ is maxima

u is maxima at $x = \frac{1}{2}$, $y = 1$, $z = \frac{1}{4}$

Example for two variables

$z = 48y - 3x^2 - 6xy - 2y^2 + 72x$

Condition 1: neccessary condition.

$\frac{dz}{dx} = 0 \Rightarrow -6x - 6y + 72 = 0$ —— 1

$\frac{dz}{dy} = 0 \Rightarrow 48 - 6x - 4y = 0$ —— 2

From 1 & 2

$-6x - 6y + 72 = 0$

$-6x - 4y + 48 = 0$

$\quad + \quad + \quad -$

$-2y + 24 = 0$

$\boxed{y = 12}$

Substituting y in 1

$-6x - 6(12) + 72 = 0$

$-6x - 72 + 72 = 0$

$\boxed{x = 0}$

Substituting x & y in z

$48(12) - 3(0)^2 - 6(0)(12) + 2(12)^2 + 72(0)$

$48(2) + 144(2)$

$96 + 288$

$= 384$

condition 2: sufficient condition.

Hessian matrix = $\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$.

$f_{11} = f_{xx} = \frac{d^2x}{dx^2} = \frac{d}{dx}(-6x-6y+72) = -6$

$f_{12} = \frac{d^2x}{dxdy} = \frac{d}{dy}(-6x-6y+72) = -6$

$f_{21} = \frac{d^2y}{dydx} = \frac{d}{dx}(48-6x-4y) = -6$

$f_{22} = \frac{d^2y}{dy^2} = \frac{d}{dy}(48-6x-4y) = -4$

$H = \begin{bmatrix} -6 & -6 \\ -6 & -4 \end{bmatrix}$

$H_1 = -6$

$\det(H) = H_2 = \begin{vmatrix} -6 & -6 \\ -6 & -4 \end{vmatrix} = 0,$

$24 - 36 = -12$

$H_1 < 0$, $H_2 < 0$ So z is a point of inflection.

Multivariables problem of three variables (with constrains)

find the value of $f(x_1, x_2, x_3) = x_1^2 + (x_2+1)^2 + (x_3-1)^2$ is subjected to $x_1 + 5x_2 - 3x_3 = 6$.

$f(x_1, x_2, x_3) = x_1^2 + (x_2+1)^2 + (x_3-1)^2$

Constrain $= x_1 + 5x_2 - 3x_3 + 6$

$$\boxed{x_3 = \frac{x_1 + 5x_2 + 6}{3}}$$

Direct Substitution method

we have consider

$$f(x_1, x_2) = x_1^2 + (x_2+1)^2 + \left(\frac{x_1 + 5x_2 - 6}{3} - 1\right)^2$$

$$= x_1^2 + (x_2+1)^2 + \frac{1}{9}(x_1 + 5x_2 - 6 - 3)^2$$

$$= x_1^2 + (x_2+1)^2 + \frac{1}{9}(x_1 + 5x_2 - 9)^2$$

$\frac{df(x_1, x_2)}{d}$ condition 1: neccessary condition

$$\frac{df}{dx_1} = 0 \Rightarrow \frac{df(x_1, x_2)}{dx_1} = 2x_1 + \frac{2}{9}(x_1 + 5x_2 + 9)(1)$$

$$= 2x_1 + \frac{2}{9}(x_1 + 5x_2 + 9) = 0$$

$$= 18x_1 + 2x_1 + 10x_2 + 18 = 0$$

$$20x_1 + 10x_2 + 18 = 0 \quad \text{--- (1)}$$

$$\frac{df}{dx_2} = \frac{df(x_1, x_2)}{dx_2} = 2(x_2+1) + \frac{2}{9}(x_1 + 5x_2 - 9)(5)$$

$$\Rightarrow 2x_2 + 2 + \frac{10}{9}[x_1 + 5x_2 - 9] = 0$$

$$18x_2 + 18 + 10x_1 + 50x_2 - 90 = 0$$

$$68x_2 + 10x_1 - 72 = 0 \quad \text{--- (2)}$$

From (1) & (2) 3445

multiplying (2) 10 (2)

$$136x_2 + 20x_1 - 144 = 0$$
$$10x_2 + 20x_1 + 18 = 0$$

$$126x_2 - 126 = 0$$

$$x_2 = \frac{\cancel{126}}{126} = 1.\cancel{28}$$

$$\boxed{x_2 = 1}$$

$$20x_1 + 10(1.28) + 18 = 0$$
$$20x_1 + 10.8 + 18 = 0$$
$$20x_1 = -28$$
$$\boxed{\cancel{x_1 = 1.54}} \quad x_1 = \frac{28}{20} \frac{2}{5}$$
$$\boxed{x_1 = 0.4}$$

Substituting x_2 in (1)

$$x_3 = \frac{1.54 + 5(1.28) + 6}{3} = \frac{1.54 + 6.8 + 6}{3} = 4.78 \text{ } \}\text{no-need.}$$

$$\boxed{x_3 = 4.78}$$

$$x_3 = \frac{0.4 + 5 + 6}{3} = \frac{\cancel{11.4}}{3} = \cancel{3.8} \quad \frac{0.4 - 1}{3} = \frac{-0.6}{3} = -0.2.$$
$$\boxed{x_3 = -0.2}$$

$$(0.4)^2 + (1+1)^2 + (4.78 - 1)^2$$
$$0.16 + 4 + 14.28$$
$$4.16 + 14.28 = 18.44$$

Sufficient condition.

$$H = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$f_{11} = \frac{d}{dx_1}(20x_1 + 10x_2 - 18) = 20$$

$$f_{12} = \frac{d}{dx_2}(1) = 10$$

$$f_{21} = \frac{d}{dx_1}(68x_2 + 10x_1 - 72) = 10$$

$$f_{22} = \frac{d}{dx_2}(2) = 68$$

$$H_1 = 20$$

$$H_2 = \begin{vmatrix} 20 & 10 \\ 10 & 68 \end{vmatrix} \Rightarrow 1200 - 100 = 1100 > 0$$

$H_1 > 0$, $H_2 > 0$, so, $f(x_1, x_2, x_3)$ is minimum.

Q/ Find the minimum value of $x^2+y^2+z^2$ is subjected to $x+y+2z=12$.

Given $f(x_1,x_2,x_3) = x^2+y^2+z^2$

Constraint: $g(x_1,x_2,x_3) = x+y+2z-12$

$$z = \frac{12-y-x}{2}$$

Direct substitution method.

we have to consider

$$f(x_1,x_2) = x^2+y^2+\left(\frac{12-y-x}{2}\right)^2$$

$$= x^2+y^2+\frac{1}{4}(12-y-x)^2$$

Condition 1: neccessary condition

$$\frac{df}{dx} = 2x+\frac{2}{4}(12-y-x)(-1)$$

$$= 2x+\frac{1}{2}(12-y-x) = 0 \Rightarrow 4x+12+y+x = 0$$

$$= 5x+y+12 = 0 \quad \text{—— (1)}$$

$$\frac{df}{dy} = 2y+\frac{2}{4}(12-y-x)(-1)$$

$$= 2y+\frac{1}{2}(-12+y+x) = 0 = 4y-12+y+x = 0$$

$$5y+x-12 = 0 \quad \text{—— (2)}$$

Solving (1) & (2) ×5

$$5x+y-12 = 0$$

$$5x+25y-60 = 0$$

$$-24y+48 = 0 \Rightarrow \boxed{y = +2}$$

Substituting y in 1

$$5x + 2 - 12 = 0$$

$$5x = 14$$

$$\boxed{x = \frac{14}{5}}$$

$$z = \frac{12 - (-2) - \left(+\frac{14}{5}\right)}{2}$$

$$= \frac{12 + 2 + \frac{14}{5}}{2} = \frac{\frac{60+10+14}{5}}{2}$$

$$= \frac{\frac{64}{5}}{2} \Rightarrow \frac{64}{10} = 5.4$$

$\boxed{z = 8.4}$ $\boxed{z = 5.4}$.

Condition 2: Sufficient Condition.

$$H = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$f_{11} = \frac{d}{dx}(5x + y - 12) = 5$$

$$f_{12} = \frac{d}{dy}(5x + y - 12) = 1$$

$$f_{21} = \frac{d}{dx}(5y + x - 12) = 1$$

$$f_{22} = \frac{d}{dx}(5y + x - 12) = 5$$

$$H = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$H_1 = 5 > 0$$

$$H_2 = 25 - 1 = 24 > 0$$

$H_1 > 0$, $H_2 > 0$, so, $f(x, y, z)$ is a minimum

Negra Legrange Method.

minimize $f(x) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$

Subjected to constraint of $x_1 + x_2 = 5$.

F value "−" vaste Maximum
"+" vaste Minimum.

$$f(\underline{x}) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$$

$$g(x) = x_1 + x_2 - 5$$

1. Neccessary condition

$$L(x_1, x_2, \lambda) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2 + \lambda(x_1 + x_2 - 5)$$

$$\frac{dL}{dx_1} = -6x_1 - 6x_2 + 7 + \lambda = 0 \quad \text{—— (1)}$$

$$\frac{dL}{dx_2} = -6x_1 - 10x_2 + 5 + \lambda = 0 \quad \text{—— (2)}$$

$$\frac{dL}{d\lambda} = x_1 + x_2 - 5 = 0 \quad \text{—— (3)}$$

From equation (1)

$$-6(x_1 + x_2) + 7 + \lambda = 0$$

$$-6(5) + 7 + \lambda = 0$$

$$\boxed{\lambda = 23}$$

Solving (1) & (2)

$$-6x_1 - 6x_2 + 30 = 0$$

$$-6x_1 - 10x_2 + 28 = 0$$

$$+ \quad + \quad -$$

$$4x_2 + 2 = 0$$

$$\boxed{x_2 = -0.5}$$

From Constraints $x_1 + x_2 = 5$

$$\boxed{x_1 = 5.5}$$

$n \times n$ $\quad$ $1 \times n$

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \quad \begin{bmatrix} g_{11} \\ g_{12} \\ g_{13} \\ \vdots \\ g_{1n} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{L} & \boxed{g_1} \\ \boxed{g_2} & \boxed{N} \end{bmatrix}$$

$$\underset{(1 \times n)}{[g_{11}\ g_{12} \cdots g_{1n}]} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$11 \rightarrow xx$

$12 \rightarrow xy$

$21 \rightarrow yx$

$22 \rightarrow yy$

In this way we can write upto $n \times n$.

2. Sufficient Condition.

$$\begin{bmatrix} L_{11}-E & L_{12} & g_{11} \\ L_{21} & L_{22}-E & g_{12} \\ g_{11} & g_{12} & 0 \end{bmatrix}$$

$-E \rightarrow$ taking polynomial so we are taking $-E$

we need to do four parts of matrix. 1. is lagrange matrix. 2 & 3 is constraint matrix & 4 is Null matrix it is used to balence the matrix. Constrain matrix we should write g_1 in [1xn] matrix in column & g_2 in row wise.

$$LH = \frac{d^2L}{dx^2} = -6$$

$$L_{12} = \frac{d^2L}{dx_1 dx_2} = -6$$

$$L_{21} = \frac{d^2L}{dx_2 dx_1} = -6$$

$$L_{22} = \frac{d^2L}{dx_2^2} = -10$$

$$H = \begin{bmatrix} -6-E & -6 & 1 \\ -6 & -10-E & 1 \\ 1 & 1 & 0 \end{bmatrix} = 0$$

$g_{11} = 1$

$g_{12} = 1$

$$\det(H) = (-6-E)[0-1] - (-6)[0-1] + 1[-6-(-10-E)] = 0$$

$$= \cancel{6} + E - \cancel{6} - 6 + 10 + E = 0$$

$$2E + 4 = 0$$

$$E = -\frac{4}{2} = -2$$

$$\boxed{E = -2}$$

Since $E = -2$, then x_1, x_2, λ correspond to maximum

Optimize the function $f(x,y) = x^2+4y^2-2x+8y$ is subjected to constrains $g(x,y) = x+2y-7$

Optimize the function $f(x,y) = 9x^2+36xy-4y^2-18x-8y$ is subjected to constrains $g(x,y) = 3x+4y=13$

Solutions:

Given $f(x,y) = x^2+4y^2-2x+8y$

$g(x,y) = x+2y-7$

A constant we need to add to given $f(x,y)$ in condition 1.

neccessary condition:

$L(x,y,\lambda) = x^2+4y^2-2x+8y+\lambda(x+2y-7)$ → legrange equation.

$$\frac{dL}{dx} = 2x-2+\lambda = 0 \quad \text{--- (1)}$$

$$\frac{dL}{dy} = 8y+8+2\lambda = 0 \quad \text{--- (2)}$$

$$\frac{dL}{d\lambda} = x+2y-7=0 \quad \text{--- (3)}$$

Solving (1)×2 & (2)

$4x-4+2\lambda=0$

$8y+8+2\lambda=0$

$4x-8y-12=0$

$x-2y-3=0$ — (4)

From (3) & (4)

$x+2y-7=0$

$x-2y-3=0$

$2x-10=0 \quad \boxed{x=5}$

From (1) → $10-2+\lambda=0 \quad \boxed{\lambda=-8}$

From (3)

$5+2y-7=0$

$2y=2 \quad \boxed{y=1}$

Sufficient condition

$$H = \begin{bmatrix} L_{11} & L_{12} & g_{11} \\ L_{21} & L_{22} & g_{12} \\ g_{11} & g_{12} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} L_{11}-E & L_{12} & g_{11} \\ L_{21} & L_{22}-E & g_{12} \\ g_{11} & g_{12} & 0 \end{bmatrix}$$

$L_{11} = \frac{d}{dx}(2x - 2 + \lambda) = 2 \qquad L_{12} = \frac{d}{dy}(2x - 2 + \lambda) = 0$

$L_{21} = \frac{d}{dx}(8y + 8 + 2\lambda) = 0 \qquad L_{22} = \frac{d}{dy}(8y + 8 + 2\lambda) = 8$

$g_{11} = \frac{d}{dx}(x + 2y - 7) = 1 \qquad g_{12} = \frac{d}{dy}(x + 2y - 7) = 2$

$$H = \begin{bmatrix} 2-E & 0 & 1 \\ 0 & 8-E & 2 \\ 1 & 2 & 0 \end{bmatrix} \qquad \det(H) = 0$$

$2-E\left[\cancel{(2-E)}(8-E\cancel{)}\right]^{-4} - 0 + 1\left[-(8-E)\right] = 0$

$-8 + 4E + 8 + E = 0$

$-16 + 5E = 0 \Rightarrow \boxed{E = \frac{16}{5}}$

Since $E = \frac{16}{5}$, then x, y, λ correspond to minimum.

2. $f(x,y) = 9x^2 + 36xy - 4y^2 + 18x - 8y$

$g(x,y) = 3x + 4y - 13$

1. neccessary condition.

$L(x,y,\lambda) = 9x^2 + 36xy - 4y^2 - 18x - 8y + \lambda(3x + 4y - 13)$

$\frac{dL}{dx} = 18x + 36y - 18 + 3\lambda = 0 \Rightarrow 6x + 13y - 6 + \lambda = 0$ — (1)

$\frac{dL}{dy} = 36x - 8y - 8 + 4\lambda = 0 \Rightarrow 9x - 2y - 2 + \lambda = 0$ — (2)

$\frac{dL}{dx} = 3x + 4y - 13 = 0$ — (3)

From (1) & (2)

$6x + 13y - 6 + \lambda = 0$

$9x - 2y - 2 + \lambda = 0$

$\quad - \quad + \quad + \quad -$

$-3x + 15y - 4 = 0$ — (4)

From (3) & (4)

$3x + 4y - 13 = 0$

$-3x + 15y - 4 = 0$

$19y - 17 = 0 \Rightarrow y = \frac{17}{19} = 0.89$

From (3)

$3x + 4\left(\frac{17}{19}\right) - 13 = 0$

$3x = 13 - 4\left(\frac{17}{19}\right)$

$3x = \frac{247 - 68}{19} = \frac{179}{19}$

$x = \frac{179}{57} = 3.1$

$\boxed{x = 3.14}$

From (1)

$6x + 13y - 6 + \lambda = 0$

$6(3.14) + 13(0.89) - 6 + \lambda$

$18.84 + 11.57 - 6 + \lambda = 0$

$30.41 - 6 + \lambda = 0$

$24.41 + \lambda = 0$

$\boxed{\lambda = -24.41}$

Sufficient Condition :

$$H = \begin{bmatrix} L_{11} - E & L_{12} & g_{11} \\ L_{21} & L_{22} - E & g_{12} \\ g_{11} & g_{12} & 0 \end{bmatrix}$$

$L_{11} = \frac{d^2L}{dx^2} = \frac{d}{dx}(18x + 36y - 18 + 3\lambda) = 18 \quad L_{12} = \frac{d}{dy}(1) = 36$

$L_{21} = \frac{d}{dy}(36x - 8y - 8 + 4\lambda) = 36 \quad L_{22} = \frac{d}{dy}(2) = -8$

$g_{11} = \frac{d}{dx}(3x + 4y - 13) = 3 \quad g_{12} = \frac{d}{dy}(3x + 4y - 13) = 4$

$$H = \begin{bmatrix} 18 - E & 36 & 3 \\ 36 & -8 - E & 4 \\ 3 & 4 & 0 \end{bmatrix} \qquad \det(H) = 0$$

$(18 - E)[-16] - 36[-12] + 3[144 - 3(-8 - E)] = 0$

$-288 + 16E + 432 + 3[144 + 24 + 3E] = 0$

$-288 + 16E + 432 + 432 + 72 + 9E = 0$

$-288 + 16E + 864 + 72 + 9E = 0$

$25E = -648 \Rightarrow E = \frac{-648}{25} = -25.92$

Since $F = -25.92$ so x, y, λ correspond to maximum.

Find the minimum value of $x^2+y^2+z^2$ subjected to $xyz = a^3$.

given let $f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = xyz - a^3$

Condition 1. Neccessary Condition.

$L(x,y,z,\lambda) = x^2+y^2+z^2+\lambda(xyz-a^3)$

$\frac{\partial L}{\partial x} = 2x + \lambda yz = 0$ —— (1)

$\frac{\partial L}{\partial y} = 2y + \lambda xz = 0$ —— (2)

$\frac{\partial L}{\partial z} = 2z + \lambda xy = 0$ —— (3)

$\frac{\partial L}{\partial \lambda} = xyz - a^3$ —— (4)

From (1) $\times x$ & (2) $\times y$

$2x^2 + \lambda \cancel{x}yz = 0$

$2y^2 + \lambda \cancel{x}yz = 0$

$2x^2 - 2y^2 = 0 \Rightarrow x^2 = y^2$

$\boxed{x = y}$ —— (5)

$2y + \lambda yz = 0$

$y(2+\lambda z) = 0$

$\boxed{y = 0}$

From (2) $\times y$ & (3) $\times z$

$2y^2 + \lambda x\cancel{z}y = 0$

$2z^2 + \lambda x\cancel{z}y = 0$

$2y^2 - 2z^2 = 0$

$\boxed{y = z}$ —— (6)

From (5) & (6)

$x = y = z$

From given Constrain

$x^3 = a^3 \Rightarrow x = a.$

$x = y = z = a.$

For $f(x,y,z) = x^2+y^2+z^2$

$= a^2+a^2+a^2$

$= 3a^2.$

So, $f(x,y,z)$ is minimum value

$f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = x+y+z = 3a.$

Condition 1: neccessary condition.

$L(x,y,z,\lambda) = x^2+y^2+z^2+\lambda(x+y+z-3a)$

$\frac{dL}{dx} = 2x+\lambda = 0$ —— (1)

$\frac{dL}{dy} = 2y+\lambda = 0$ —— (2)

$\frac{dL}{dz} = 2z+\lambda = 0$ —— (3)

From (1) & (2)

$2x+\cancel{\lambda} = 0$

$2y+\cancel{\lambda} = 0$

$2x-2y = 0$

$\boxed{x=y}$ —— (4)

From (2) & (3)

$2y+\cancel{\lambda} = 0$

$2z+\cancel{\lambda} = 0$

$2y+2z = 0$

$\boxed{y=z}$ —— (5)

From (4) & (5)

$\therefore x=y=z.$

From given Constrain.

$x+x+x = 3a$

$\cancel{3}x = \cancel{3}a \Rightarrow \boxed{x=a}$

$\boxed{y=a}$ & $\boxed{z=a}$

$f(x,y,z) = a^2+a^2+a^2$

$= 3a^2$

So, $f(x,y,z)$ is minimum.

KT method

Kuhn Tucker method.

P. minimize $f = x_1^2 + x_2^2 + 60x_1$ subjected to constrain

$g_1 = x_1 - 80 \geq 0$, $g_2 = x_1 + x_2 - 120 \geq 0$

Given $f = x_1^2 + x_2^2 + 60x_1$

$g_1 = x_1 - 80 \geq 0$ $\quad$ $g_2 = x_1 + x_2 - 120 \geq 0$

$K(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + 60x_1 + \lambda_1(x_1 - 80) + \lambda_2(x_1 + x_2 - 120$

neccessary condition $\quad$ case A

$\frac{dK}{dx_1} = 2x_1 + 60 + \lambda_1 + \lambda_2 = 0$ —(1)

$160 + 60 - 80 + \lambda = 0$ $\quad$ $\lambda_1 = -140$

$\frac{dK}{dx_2} = 2x_2 + \lambda_2 = 0 = -80$ —(2)

$\frac{dK}{d\lambda_1} = x_1 - 80 = 0$ $\quad$ 80 —(3)

$\frac{dK}{d\lambda_2} = x_1 + x_2 - 120 = 0 = 40$ —(4)

case B.

$\lambda_1(x_1 - 80) = 0$ —(5)

$\lambda_2(x_1 + x_2 - 120) = 0$ —(6)

Case C

$\lambda_1(x_1 - 80) \geq 0$ —(7)

$\lambda_2(x_1 + x_2 - 120) \geq 0$ —(8)

Case D

$\lambda_1(x_1 - 80) \leq 0$ —(9)

$\lambda_2(x_1 + x_2 - 120) \leq 0$ —(10)

From the equation (5)

$\lambda(x_1-80)=0$

$\lambda_1=0 \quad x_1=80$

Case I $\rightarrow$ only x values base chesukovali

at $\lambda_1=0$

we have to substitute λ_1 value in eq^n (1) & (2)

$2x_1+60+\lambda_2=0 \qquad 2x_2+\lambda_2=0$

$x_1=\frac{-\lambda_2-60}{2} \qquad x_2=\frac{-\lambda_2}{2}$

$x_1=-\frac{\lambda_2}{2}-30$

Sub x_1, x_2 value in eq^n (6)

$\lambda_2(x_1+x_2-120)=0$

$\lambda_2\left(-\frac{\lambda_2}{2}-30+\left(-\frac{\lambda_2}{2}-120\right)\right)=0$

$-\lambda_2-150=0 \quad \boxed{\lambda_2=-150}$

$\boxed{\lambda_2=0}$

$\lambda_2=0, \lambda_2=-150$

$\lambda_1=0, \lambda_2=0$ substitute in (1)

$2x_1+60=0$

$x_1=\frac{-60}{2}=-30 \Rightarrow \boxed{x_1=-30}$

From (2)

$2x_2-0=0$

$\boxed{x_2=0}$

check these values in (7) & (8) So, these points will doesnot satisfy these values so going next.

The values of x_1, x_2 violetes equation (7) Hence rejecting the points.

$\lambda_1 = 0, \lambda_2 = -150$

$2x_1 + 60 + 0 - 150 = 0$

$2x_1 - 90 = 0$

$x_1 = 45$

From (2)

$2x_2 - 150 = 0$

$x_2 = \frac{\cancel{150}^{75}}{\cancel{2}} = 75$

$x_2 = 75$

These values of x_1, x_2 also violetes in equation 7 & (8). Hence rejecting the points.

Case 2

Considering x_1, x_2 values.

we know that $x_1 = 80$

Substituting in (1) & (2)

$2(80) + 60 + \lambda_1 + \lambda_2 = 0$

$160 + 60 + \lambda_1 + \lambda_2 = 0$

$\lambda_2 = -220 - \lambda_1$ — (9)

From (2)

$2x_2 + \lambda_2 = 0 \Rightarrow 2x_2 - 220 - \lambda_1 = 0$

$\lambda_2 = -2x_2$ $\boxed{\lambda_1 = 2x_2 - 220}$

Substituting λ_1, λ_2 in Case c (4)

$\lambda_1 \cancel{(x_1 - 80)} = 0$

$\lambda_2 (x_1 + x_2 - 120) = 0$

$-2x_2 (x_1 + x_2 - 120) = 0$

$x_2 = 0$ $80 + x_2 - 120 = 0$

$\boxed{x_2 = 40}$

At $x_1 = 80, x_2 = 0$ substituting in (1) & (2)

$2(80) + 60 + \lambda_1 + \lambda_2 = 0$

$160 + 60 + 2(0) - 220 + \lambda_2 = 0$

~~220~~ $-2(40)$

$220 - 80 + \lambda_1 = 0 \Rightarrow \boxed{\lambda_1 = -140}$

$2(0) + \lambda_2 = 0$

$\boxed{\lambda_2 = 0}$

These points are also not satisfying case 3 so rejecting.

$x_1 = 80, x_2 = 40$ substituting (1) & (2)

$2(80) + 60 + \lambda_1 + \lambda_2$

$\boxed{\lambda_1 = -140}$

$2(40) + \lambda_2 = 0$

$\boxed{\lambda_2 = -80}$

Substituting $\lambda_1, \lambda_2, x_1, x_2$ in

Case 6.

$\lambda_1 (x_1 - 80) \geq 0$

$\lambda_2 (x_1 + x_2 - 120) \geq 0$

$-140(80 - 80) = 0$

$-80(80 + 40 - 120) = 0$

These points are satisfying the conditions.

$\therefore$ minimize values are

$x_1 = 80, x_2 = 40, \lambda_1 = -140,$

$\lambda_2 = -80$