ĐSTT BVN T11 (Ma trận chuyển cơ sở)

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Đề. Cho 2 hệ vector sau là cơ sở of $\mathbb{R}^3$

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad u_3 = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$$

và

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

Hãy tìm ma trận chuyển cơ sở từ cơ sở $\mathcal{B}$ $(u_1, u_2, u_3) \to$ cơ sở $\mathcal{C}$ (v_1, v_2, v_3) và ngược lại.

Bgiải

⊕ Cminh (u_1, u_2, u_3) là 1 cơ sở of $\mathbb{R}^3$

Ta xét

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (*)$$

$$\Leftrightarrow x_1 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$\Rightarrow$ Giải $(*)$ $\Leftrightarrow$ giải ma trận

$$\left[\begin{array}{ccc|c} -1 & 1 & -2 & a_1 \\ 1 & 0 & 5 & a_2 \\ -1 & 2 & 0 & a_3 \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & a_2 \\ -1 & 1 & -2 & a_1 \\ -1 & 2 & 0 & a_3 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & a_2 \\ 0 & 1 & 3 & a_1 + a_2 \\ 0 & 2 & 5 & a_2 + a_3 \end{array}\right]$$

$$\underset{-R_3 + 2R_2 \to R_3}{\Leftrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 5 & a_2 \\ 0 & 1 & 3 & a_1 + a_2 \\ 0 & 0 & 1 & 2a_1 + a_2 - a_3 \end{array}\right] \underset{\substack{R_2 - 3R_3 \to R_2 \\ R_1 - 5R_3 \to R_1}}{\Leftrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10a_1 - 4a_2 + 5a_3 \\ 0 & 1 & 0 & -5a_1 - 2a_2 + a_3 \\ 0 & 0 & 1 & 2a_1 + a_2 - a_3 \end{array}\right]$$

Vì hệ có n. duy nhất $\begin{bmatrix} -10a_1 - 4a_2 + 5a_3 \\ -5a_1 - 2a_2 + a_3 \\ 2a_1 + a_2 - a_3 \end{bmatrix}$ nên (u_1, u_2, u_3) là cơ sở of $\mathbb{R}^3$.

⊕ Cminh (v_1, v_2, v_3) là 1 cơ sở of $\mathbb{R}^3$.

Xét

$$x_1v_1 + x_2v_2 + x_3v_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{vs } a_1, a_2, a_3 \text{ bkỳ} \in \mathbb{R}.$$

⇒ Ta xét ma trận

$$\left[\begin{array}{ccc|c} -2 & 2 & 4 & a_1 \\ 1 & 0 & 1 & a_2 \\ 3 & 1 & -1 & a_3 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Longleftrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_2 \\ -2 & 2 & 4 & a_1 \\ 3 & 1 & -1 & a_3 \end{array}\right] \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_2 \\ 0 & 2 & 6 & a_1+2a_2 \\ 0 & 1 & -4 & -3a_2+a_3 \end{array}\right]$$

$$\Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_1 \\ 0 & 1 & -4 & -3a_2+a_3 \\ 0 & 2 & 6 & a_1+2a_2 \end{array}\right] \overset{R_3-2R_2 \to R_3}{\Longleftrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_1 \\ 0 & 1 & -4 & -3a_2+a_3 \\ 0 & 0 & 14 & a_1+8a_2-2a_3 \end{array}\right]$$

$$\Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_1 \\ 0 & 1 & -4 & -3a_2+a_3 \\ 0 & 0 & 1 & \frac{1}{14}a_1+\frac{4}{7}a_2-\frac{1}{7}a_3 \end{array}\right] \Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{13}{14}a_1-\frac{4}{7}a_2+\frac{1}{7}a_3 \\ 0 & 1 & 0 & \frac{2}{14}a_1-\frac{5}{7}a_2+\frac{3}{7}a_3 \\ 0 & 0 & 1 & \frac{1}{14}a_1+\frac{4}{7}a_2-\frac{1}{7}a_3 \end{array}\right]$$

Vì hệ có n. duy nhất nên (v_1, v_2, v_3) là 1 cơ sở of $\mathbb{R}^3$.

⊕ Tìm ma trận chuyển cơ sở từ cs $\mathcal{B}(u_1, u_2, u_3)$ sang cs $\mathcal{C}(v_1, v_2, v_3)$

Xét ma trận

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -2 & -2 & 2 & 4 \\ 1 & 0 & 5 & 1 & 0 & 1 \\ -1 & 2 & 0 & 3 & 1 & -1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Longleftrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 1 \\ -1 & 1 & -2 & -2 & 2 & 4 \\ -1 & 2 & 0 & 3 & 1 & -1 \end{array}\right]$$

$$\underset{R_3+R_1 \to R_3}{\overset{R_2+R_1 \to R_2}{\Longleftrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -1 & 2 & 5 \\ 0 & 2 & 5 & 4 & 1 & 0 \end{array}\right] \overset{-R_3+2R_2 \to R_3}{\Longleftrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -1 & 2 & 5 \\ 0 & 0 & +1 & -6 & +3 & +10 \end{array}\right]$$

$$\underset{R_2-3R_3 \to R_2}{\overset{R_1-5R_3 \to R_1}{\Longleftrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 31 & -15 & -49 \\ 0 & 1 & 0 & 17 & -7 & -25 \\ 0 & 0 & 1 & -6 & 3 & 10 \end{array}\right] \quad \text{Vậy } M_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 31 & -15 & -49 \\ 17 & -7 & -25 \\ -6 & 3 & 10 \end{bmatrix}$$

Tìm ma trận chuyển cơ sở từ $\mathcal{C}(v_1, v_2, v_3)$ sang cơ sở $B(u_1, u_2, u_3)$

$$\left[\begin{array}{ccc|ccc} -2 & 2 & 4 & -1 & 1 & -2 \\ 1 & 0 & 1 & 1 & 0 & 5 \\ 3 & 1 & -1 & -1 & 2 & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Longleftrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 5 \\ -2 & 2 & 4 & -1 & 1 & -2 \\ 3 & 1 & -1 & -1 & 2 & 0 \end{array}\right]$$

$$\underset{R_3 - 3R_1 \to R_3}{\overset{R_2 + 2R_1 \to R_2}{\Longleftrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 5 \\ 0 & 2 & 6 & 3 & 1 & 8 \\ 0 & 1 & -4 & -4 & 2 & -15 \end{array}\right] \overset{R_3 \leftrightarrow R_2}{\Longleftrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 5 \\ 0 & 1 & -4 & -4 & 2 & -15 \\ 0 & 2 & 6 & 3 & 1 & 8 \end{array}\right] \overset{R_3 - 2R_2 \to R_3}{\Longleftrightarrow}$$

$$\Longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 5 \\ 0 & 1 & -4 & -4 & 2 & -15 \\ 0 & 0 & 14 & 11 & -3 & 38 \end{array}\right] \underset{\frac{R_3}{14} \to R_3}{\Longleftrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 5 \\ 0 & 1 & -4 & -4 & 2 & -15 \\ 0 & 0 & 1 & \frac{11}{14} & \frac{-3}{14} & \frac{19}{7} \end{array}\right]$$

$$\underset{R_2 + 4R_3 \to R_2}{\overset{R_1 - R_3 \to R_1}{\Longleftrightarrow}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/14 & 3/14 & 16/7 \\ 0 & 1 & 0 & -6/7 & 8/7 & -29/7 \\ 0 & 0 & 1 & 11/14 & -3/14 & 19/7 \end{array}\right]$$

$$\text{Vậy } M_{\mathcal{C}}^{B} = \begin{bmatrix} 3 & 3 & 32 \\ -6 & 8 & -29 \\ 11 & -3 & 38 \end{bmatrix}$$