② Xét mặt bậc hai.

Xét mặt bậc hai sau

$$x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + x_3 = 1 \quad (*)$$

Bg giải.

⊕ Tách dc dạng toàn phương

$H = x_2.x_2 + x_3.x_3 + 2x_1x_2 + 2x_1.x_3$

=> Xét dc ma trận đ x ứng M

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

⊕

$$P_M(\lambda) = \begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -\lambda.(1-\lambda)^2 + 0 + 0 - 0 - (1-\lambda) - (1-\lambda)$$

$$= -\lambda.(1-2\lambda+\lambda^2) + 2.(\lambda-1)$$

$$= -\lambda^3 + 2\lambda^2 - \lambda + 2\lambda - 2$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$$= (\lambda-1).(\lambda-2).(\lambda+1)$$

Vậy $\lambda = 1$ $\quad \lambda = -1$ $\quad \lambda = 2$

⊕ Xét kgn us $\lambda = 1$.

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-R_1 \to R_1]{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cancel{R_1+R_2 \to R_1}$$

$$V_1 = \left\{ \begin{bmatrix} x_2 + x_3 \\ -x_3 \\ x_3 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} . x_3 \right\} = \mathcal{L}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Chọn $u_1 = (0,-1,1)$ với $V_1 = u_1 \Rightarrow e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}.(0,-1,1)$ là cstchgof V_1

⊕ Xét kgn vs $\lambda = -1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_1 - R_2 \to R_1]{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$V_{-1} = \left\{ \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} \right\} = \left\{ x_3 . \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} = \mathcal{L}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow e_2 = \frac{1}{\sqrt{6}} . (-2, 1, 1) \text{ là cstg of } V_{-1}$$

⊕ Xét kgn vs $\lambda = 2$.

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 + 2R_1 \to R_3]{-(R_2 - R_1) \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} . x_3 \right\} = \mathcal{L}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow e_3 = \frac{1}{\sqrt{3}} . (1, 1, 1) \text{ là cstg of } V_2$$

⊕ Chọn $Q = \begin{bmatrix} 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ thì $Q^t A Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

⊕ Đổi biến $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = Q^t . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{x_2}{\sqrt{2}} + \frac{x_3}{\sqrt{2}} \\ -\frac{2x_1}{\sqrt{6}} + \frac{x_2}{\sqrt{6}} + \frac{x_3}{\sqrt{6}} \\ \frac{x_1}{\sqrt{3}} + \frac{x_2}{\sqrt{3}} + \frac{x_3}{\sqrt{3}} \end{bmatrix}$ thì $H = y_1^2 - y_2^2 + 2y_3^2$

⊕ Mà $x_3 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{3}}$ nên ta có

$$(*) \Leftrightarrow y_1^2 - y_2^2 + 2y_3^2 + \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{3}} = 1$$

$$\Leftrightarrow \left(y_1^2 + 2.y_1.\frac{1}{2\sqrt{2}} + \frac{1}{8}\right) - \left(y_2^2 - 2.y_2.\frac{1}{2\sqrt{6}} + \frac{1}{24}\right) + 2.\left(y_3^2 + 2.y_3.\frac{1}{4\sqrt{3}} + \frac{1}{48}\right) = 1 + \frac{1}{8} - \frac{1}{24} + \frac{2}{48}$$

$$\Leftrightarrow \left(y_1 + \frac{1}{2\sqrt{2}}\right)^2 - \left(y_2 - \frac{1}{2\sqrt{6}}\right)^2 + 2.\left(y_3 + \frac{1}{4\sqrt{3}}\right)^2 = \frac{9}{8}$$

$$\Leftrightarrow \frac{8}{9}\left(y_1 + \frac{1}{2\sqrt{2}}\right)^2 - \frac{8}{9}\left(y_2 - \frac{1}{2\sqrt{6}}\right)^2 + \frac{16}{9}\left(y_3 + \frac{1}{4\sqrt{3}}\right)^2 = 1 \Rightarrow \text{Hình yên ngựa ??}$$