Btập 2. Chéo hóa ma trận trực giao

Hãy chéo hóa trực giao ma trận sau

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

Bgiải

+ Xét đa thức đặc trưng.

$$P_A(\lambda) = \begin{vmatrix} 3-\lambda & 3 & 3 \\ 3 & 3-\lambda & 3 \\ 3 & 3 & 3-\lambda \end{vmatrix} = (3-\lambda)^3 + 27 + 27 - 9.(3-\lambda) - 9.(3-\lambda) - 9.(3-\lambda)$$

$$= (27 - 3.9.\lambda + 3.3.\lambda^2 - \lambda^3) + 2.27 + 27.(\lambda - 3)$$

$$= -\lambda^3 + 9\lambda^2 + 27.3 - 3.27$$

$$= -\lambda^3 + 9\lambda^2$$

$$= \lambda^2.(9-\lambda)$$

Vậy $\lambda = 0$

$\lambda = 9$.

+ Với kgian riêng $\lambda = 9$

$$\begin{bmatrix} -6 & 3 & 3 \\ 3 & -6 & 3 \\ 3 & 3 & -6 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} -6 & 3 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ 3 & 3 & -6 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & -6 & 3 & 0 \\ -6 & 3 & 3 & 0 \\ 3 & 3 & -6 & 0 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 + 2R_1 \to R_2} \left[\begin{array}{ccc|c} 3 & -6 & 3 & 0 \\ 0 & -9 & 9 & 0 \\ 0 & 9 & -9 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - R_2 \to R_3]{} \left[\begin{array}{ccc|c} 3 & -6 & 3 & 0 \\ 0 & -9 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\frac{R_2}{-9} \to R_2]{\frac{R_1}{3} \to R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow V_9 = \left\{ \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} \right\} = \mathcal{L}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \Rightarrow u_1 = (1,1,1) \xrightarrow[\text{hóa}]{\text{Tgiao}} e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \cdot (1,1,1)$$

$v_1 = u_1$

Vậy e_1 là csthực chuẩn of $\mathcal{L}\left(\left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]\right)$

+ Vs kg riêng $\lambda = 0 \Rightarrow \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \xrightarrow[\frac{R_1}{3} \to R_1]{\substack{R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow V_0 = \left\{ \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_2 \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathcal{L}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Đặt $u_2 = (-1,1,0)$ $\quad u_3 = (-1,0,1)$

$v_2 = u_2 = (-1,1,0) \Rightarrow e_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}} \cdot (-1,1,0)$

$$v_3 = u_3 + pr_{v_2}(u_3) = u_3 + \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} \cdot v_2 = u_3 + \frac{1+0+0}{1+1+0} v_2$$

$$= (-1,0,1) + \frac{1}{2} \cdot (-1,1,0) = \left(-\frac{3}{2}; \frac{1}{2}; 1\right) = \frac{1}{2} \cdot (-3,1,2)$$

$$\Rightarrow e_3 = \frac{(2v_3)}{\|2v_3\|} = \frac{1}{\sqrt{14}} \cdot (-3,1,2)$$

Vậy e_2, e_3 là cơ sở trực chuẩn of V_0.

+ $\dim V_9 + \dim V_0 = 1 + 2 = 3 =$ cấp ma trận $A \Rightarrow$ chéo hóa đc.

+ Ma trận chéo hóa trực giao là

Chọn $Q = \begin{bmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -3/\sqrt{14} \\ \sqrt{3}/3 & \sqrt{2}/2 & 1/\sqrt{14} \\ \sqrt{3}/3 & 0 & 2/\sqrt{14} \end{bmatrix}$ thì ma trận đã chéo hóa trực

giao đc là $Q^T A Q = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$