Selecting Optimal Solution from Pareto Non-inferior Solutions

Introduction

After carefully observing the distribution of the Pareto front, we find that the distribution is monotonically increasing or decreasing. This means that different variability exists in the Pareto front and that new inherent disciplines can be found.

The methods for solving Multi-objective Optimization Problems (MOP) can be divided into three categories:

- **Functional Relation Method:** Seek a functional relation between objectives to convert to single objective problem.
- **Non-dominant Relation Method:** Obtain Pareto solution by non-dominant relations.
- **Evaluation Factors Method:** Add evaluation system according to preferences.

Some common methods include:

- TOPSIS: Technique for Order Preference by Similarity to Ideal Solution.
- Fuzzy Logic
- Unsupervised ML: Automatically extract mathematical relations

All the methods have one thing in common, that is introduction of additional conditions on the basis of optimization objectives. The key to solving MOPs lie in directing the gains and losses properly.

To select the optimal solution, the rules are as follows:

- Maximize Income Ratio (Performance/Price)
- Assign priority order for objectives

The method is based on performance-price ratio and is only verified for a bi-objective problem.

Concepts

Multi-objective Optimization: h variables, r multi-objective functions and n constraints.

$$\min y = F(x) = (f_1(x), f_2(x), \cdots, f_r(x))$$

subject to

$$e(x) = (e_1(x), e_2(x), \cdots, e_n(x)) \leq 0, x \in \Omega$$

where

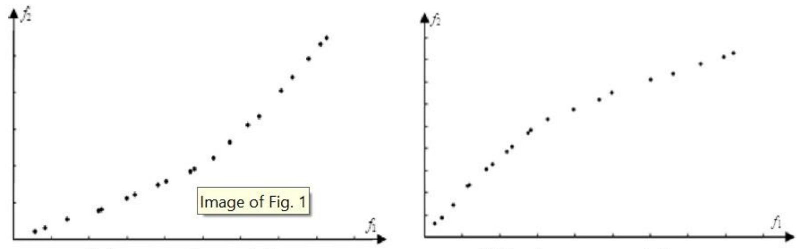
$$x = \{x_1, x_2, \cdots, x_h\}$$

$$y = \{y_1, y_2, \cdots, y_r\}$$

X_f is the feasible region and x is a solution in the region. Thus, there exists no x' such that F(x') is superior F(x) that is F(x') > F(x). Only then, x is a non-inferior solution in X_f.

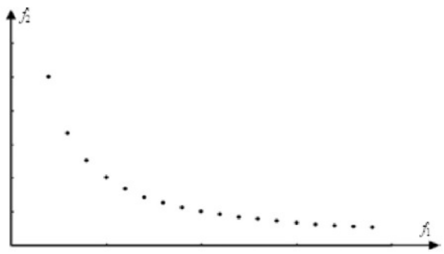
Features of Pareto Front: Say there are M number of solution in the non-inferior Pareto front. They are sorted in ascending order by the value of objective f_1 and are labelled from 1 to M.

- Max-Min/Min-Max Model: If f_1^m is increasing, f_2^m also increases.



(a) max-min model (b) min-max model

- Min-Min/Max-Max Model: If f_1^m is increasing, f_2^m decreases.



We can infer from the following models that the line connecting any two points has a positive slope for min-max model and a negative slope for max-max model.

Average Variability: Average values of slope of lines connecting two adjacent points except for endpoints.

$$k_1^{(m)} = \frac{1}{2}\left(\frac{f_2^{(m)} - f_2^{(m-1)}}{f_1^{(m)} - f_1^{(m-1)}} + \frac{f_2^{(m+1)} - f_2^{(m)}}{f_1^{(m+1)} - f_1^{(m)}}\right), m = 2, 3, \cdots M-1,$$

$$k_2^{(m)} = \frac{1}{2}\left(\frac{f_1^{(m)} - f_1^{(m-1)}}{f_2^{(m)} - f_2^{(m-1)}} + \frac{f_1^{(m+1)} - f_1^{(m)}}{f_2^{(m+1)} - f_2^{(m)}}\right), m = 2, 3, \cdots M-1,$$

For end points

$$k_1^{(1)} = \frac{f_2^{(2)} - f_2^{(1)}}{f_1^{(2)} - f_1^{(1)}}, \qquad k_1^{(M)} = \frac{f_2^{(M)} - f_2^{(M-1)}}{f_1^{(M)} - f_1^{(M-1)}},$$

$$k_2^{(1)} = \frac{f_1^{(2)} - f_1^{(1)}}{f_2^{(2)} - f_2^{(1)}}, \qquad k_2^{(M)} = \frac{f_1^{(M)} - f_1^{(M-1)}}{f_2^{(M)} - f_2^{(M-1)}}.$$

Sensitivity Ratio: Ratio of average variabilities to their respective objective function values.

$$\delta_1^m = \frac{k_1^{(m)}}{f_1^{(m)}}, m = 1, 2, \cdots M, f_1^{(m)} \neq 0,$$

$$\delta_2^m = \frac{k_2^{(m)}}{f_2^{(m)}}, m = 1, 2, \cdots M, f_2^{(m)} \neq 0.$$

Non-dimensionalization of Sensitivity Ratio:

$$\varepsilon_1^m = \frac{\delta_1^{m)}}{\sum_{i=1}^M \delta_1^{(i)}}, m = 1, 2, \cdots M, \qquad \varepsilon_2^m = \frac{\delta_2^{m)}}{\sum_{i=1}^M \delta_2^{(i)}}, m = 1, 2, \cdots M.$$

Dominance Relationship: Now that we have the sensitivity ratios, we can create a pareto subset based on it, named X*. The element x^i in X* is such that there exists no x^j in X* where $E_1^j > E_1^i$ and $E_2^j > E_2^i$.

Bias Degree/Weights: Bias degrees of solutions are values in (0, 1) for different objective functions. It can be used later to select solutions on the basis of preferences by referring its bias degrees over different objectives.

$$w_1^{(m)} = \frac{\varepsilon_1^{(m)}}{\varepsilon_1^{(m)} + \varepsilon_2^{(m)}}, \; m \in M^*, \quad w_2^{(m)} = \frac{\varepsilon_2^{(m)}}{\varepsilon_1^{(m)} + \varepsilon_2^{(m)}}, \; m \in M^*,$$

$$w_1^{(m)} + w_2^{(m)} = 1, \; m \in M^*$$

Objectives might be conflicting in nature, hence increasing the weight of one, might lead to decrease in the other.

Good Solution: Based on the above analysis, a good solution is selected by using the following criteria.

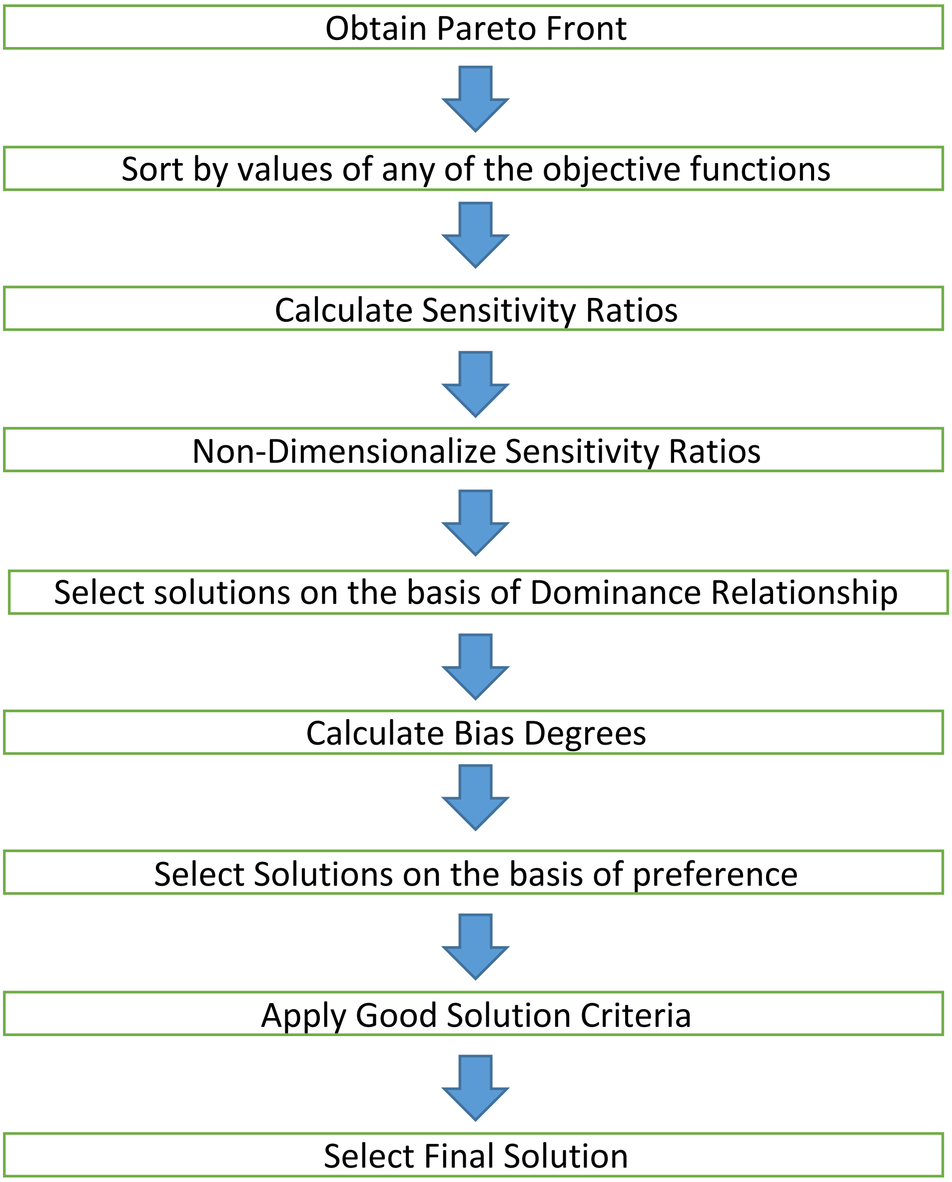
$$\Delta\varepsilon^{(m)} = \left|\varepsilon_1^{(m)} - \varepsilon_2^{(m)}\right|, \; m = 1, 2, \cdots, \mathrm{M}.$$

$(\Delta\varepsilon)_{\min}$ is the minimal $\Delta\varepsilon^{(m)}$, and

$$(\Delta\varepsilon)_{\min} = \min\left\{\Delta\varepsilon^{(1)}, \Delta\varepsilon^{(2)}, \cdots, \Delta\varepsilon^{(\mathrm{M})}\right\}.$$

The solution with minimum value of $\triangle E_{min}$ is regarded as the unbiased or good solution. In real sense, a good solution has a high performance price ratio for both the objective functions.

Flowchart:



Conclusion:

The method described above reduced the number of non inferior solutions in the set. It is a quantitative method for decision makers to obtain the preferred solution. If all objectives are considered equally, a good solution can be obtianed on the basis of performace-price ratio.

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