Limit Rules

L'Hoptial: $\lim_{x->c} \frac{f(x)}{g(x)} = \lim_{x->c} \frac{f'(x)}{g'(x)}$ if lim g(x) = lim f(x) = 0 / inf

Replacement: f(x) = g(x) for all x, except x=c. $\lim_{x->c} f(x) = \lim_{x->c} g(x)$

Squeeze: $g(x) \leq f(x) \leq h(x)$. $\lim_{x->c} g(x) = \lim_{x->c} h(x) = L \rightarrow \lim_{x->c} f(x) = L$

Exp.: $\lim_{x->c} f(x)^k \leftrightarrow \exp\left(lim(k * \ln(f(x)))\right)$

Polynomials: $\lim_{x->\infty} \frac{P(x) = Ax^\alpha + ...}{Q(x) = Bx^\beta + ...}$ 0 if $\alpha < \beta$, $\frac{A}{B}$ if $\alpha = \beta$, infinity if otherwise

Differentiation

Parametric: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{g'(t)}{f'(t)}$, and, $\frac{d^2y}{dx^2} = \frac{g''(t)f'(t) - g'(t)f''(t)}{f'(t)^3}$, x=f(t) y=g(t)

Point-slope equation: $y - y_1 = m(x - x_1)$ / $y = f(a) + f'(a)(x-a)$

Constant-power: $\frac{d}{dx} a^x = a^x \ln(a)$

Series Test	Formula/Form	Converges	Diverges	Inconclusive
N'th term	$\lim_{n->inf} a_n$	-	DNE/!=0	0
Integral	$\int_1^\infty f(x)\, dx$	Fin. num	Inf number	(f(x) is continuous positive decreasing)
Comparison	$0 \leq a_n \leq b_n$	B conv. -> a conv.	A div. -> B div.	
Ratio	$\lim_{n->\infty} \lvert\frac{a_{n+1}}{a_n}\rvert$	$0 \leq L < 1$	L > 1	L=1
Root	$\lim_{n->\infty} \sqrt[n]{\lvert a_n\rvert}$	$0 \leq L < 1$	L>1	L=1
Alternating	$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$	$\lim_{n->\infty} b_n = 0$ $0 \leq b_n \geq b_{n+1}$	-	(Note: try to show that the bn is decreasing if can)
Geometric Series	$\sum_{n=1}^{\infty} ar^{n-1}$	$\lvert r\rvert < 1$	-	-
P-series	$\sum_{n=1}^{\infty} \frac{1}{n^P}$	P>1	$P \leq 1$	-

Volume of Solids

Disk: $\pi\left(\int_a^b f(x)^2 - \int_a^b g(x)^2\right) dx$

Shell: $2 * \pi \int_a^b x |f(x) - g(x)| dx$

Get a and b by checking intercepts. Check which fn >fn2

Partial Fractions

$\frac{P(x)}{Q(x)}$ Where Px >= Qx

1. Factor Q(x): $\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)}$
2. Write out: $\frac{5x-4}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$
3. Factor in roots and solve for A/B

Power Series

$\sum_{n=0}^{\infty} c_n(x-a)^n$ Is a power series centered about a. We can check

Power series either **a)** converge at x=a only, **b)** converge for all x, **c)** There is a positive R s.t. the series converges absolutely if |x-a|<R & diverges if |x-a|>R

Radius of Convergence: $\lim_{n->\infty} \left(\lvert\frac{c_{n+1}}{c_n}\rvert\ OR\ \sqrt{\lvert c_n\rvert}\right) = L$, R = $\frac{1}{L}$.

Integration By Substitution

1. Let u = g(x)
2. Find $\frac{du}{dx}$
3. Manipulate $dx = \frac{du}{u'}$
4. Find new bounds if necessary; sub in u=g(x)
5. Sub in u, change dx to du
6. If (-1)du, flip sign of integration

You can also use du = u' * dx

Integration By Parts

1. Identify "reducible" derivative (Log, inv trig, Algebra, trig)
2. $\int uv\, dx = u\int v - \int u' * \left(\int v\, dx\right) dx$

Series Manipulation

Constants: $\sum_{n=1}^{\infty} ca_n = c * \sum_{n=1}^{\infty} a_n$ *f(x) can be a constant if outside of the series

Addition: $\sum_{n=0}^{\infty}\left(a_n + b_n\right) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

Power Series Representation

We can represent infinite power series as polynomial functions. Manipulate to the form of the 4 series below:

Geometric: The geometric series $\sum_{n=0}^{\infty} ar^n = \frac{1}{1-x}$ for |x|<1

Alt. Geometric: $\frac{1}{1+x}$ is equivalent to $\sum_{n=0}^{\infty}(-1)^n x^n$, for |x|<1

Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges for any n/x

Alt. Harmonic: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges to ln(2)

Taylor & Maclaurin Series

A Taylor series centered about c: $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} * (x-c)^n$

A Maclaurin series is a special case of the taylor series centered about 0: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Discriminant

$$D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

D>0	$f_{xx}(a,b) > 0$	Local Min
D>0	$f_{xx}(a,b) < 0$	Local max
D<0	-	Saddle Point
D=0	-	Inconclusive

Critical Point: $f_x(a,b)\ AND\ f_y(a,b) = 0$

Polar Coordinates

R = {(r, θ) : 0 ≤ a ≤ r ≤ b, α ≤ θ ≤ β}

$$\iint f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r * \cos(\theta), r * \sin(\theta)) * r\ dr\, d\theta$$

Limit Tips

$\lim f(x)^{g(x)} = \exp\left(\lim\left(g(x) * \ln(f(x))\right)\right)$

Rationalize roots by multiplying by conjugate

If you see a series, might be squeeze theorem. Try taking the smallest element of the series and using that as your lower/upper bound. Same wrt to oppostie.

General Stuff

Equation of sphere: $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$, where c=(h,k,l), radius = r

Reciprocal Identities:

$$co\sec(\theta) = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

For finding bounds of double integration, try subbing z=0 so you can find relation between x and y.

PLANES AND SHIT

Distance 2 points: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Unit Vector: $\frac{u}{||u||}$ **Dot Product:** $||u|| * ||v|| * \cos(\theta)$

Cross Product: $||a|| * ||b|| * \sin(\theta), remember + - det$ (normal to a&b)

Projection: $Proj_a\, b = \frac{b \cdot a}{||a||^2} * a$ **Composition:** $Comp_a\, b = \frac{a \cdot b}{||a||}$

Angle between 2 planes/normals: $\frac{n_1 \cdot n_2}{||n_1|| * ||n_2||}$

Line Eqn: $< x,y,z > = < x_0,y_0,z_0 > + t < a,b,c >$. 2 points to find <abc>

Plane Eqn: Find 2 vectors in plane then $p \times q$ to find normal. Using any point in plane sub into $a(x - x_0) + b(y - y_0) + c(z - z_0) + d = 0$

Point to Plane: $\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$

Tangent Plane: $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

ODE

Separable: $\frac{dy}{dx} = f(x)\, g(y)$ – turn into $\int \frac{1}{g(y)} dy = \int f(x)\, dx + c$

Reductible: $y' = g\left(\frac{y}{x}\right)$ – let v = $\frac{y}{x}$, then y = vx & $y' = v + xv'$.

Manipulate and get $v' = \frac{g(v) - v}{x}$. Now apply separable ODE.

LFODE: $\frac{dy}{dx} + P(x)y = Q(x)$. Let I(X) = e^{$\int P(x)\, dx$}. Solve $y * I(x) = \int Q(x) * I(x)\, dx$

Bernoulli: $y' + p(x)y = Q(x) * y^n$. Let $u = y$^{1 – n}. Then, equation is the LFODE $u' + (1-n)P(x)u = (1-n)Q(x)$

Functions of Several Variables

Arc length curves: y=f(x) -> $\int_a^b \sqrt{1 + f'(x)^2} dx$

Arclength: <f,g> -> $\int_a^b \sqrt{f'(t) + g'(t)}$ where f/g represent x/y axes.

Chain Rule: z = f(x,y), x=g(s,t), y = h(s,t): $\frac{dz}{dt} = \frac{df}{dx} * \frac{df}{dt} + \frac{df}{dy} * \frac{dy}{dt}$ (resp. s)

Implicit D: $\frac{dz}{dx/y} = -\frac{F_{x/y}(x,y,z)}{F_z(x,y,z)}$

Directional D: $D_u f = \nabla f(x,y) * u = < f_x, f_y > \cdot u$

Level Surface to Tangent Plane: $\nabla F(x_0,y_0,z_0) * < x - x_0, y - y_0, z - z_0 > = 0$

Double Integrals

Surface Area: $\iint \sqrt{f_x^2 + f_y^2 + 1}\, dA$

Type 1: y=f(x), arrow pointing UP

Type 2: x = f(y), arrow pointing RIGHT

Concavity

Second Derivative Test: If f''(c) > 0, then f is concave up at c. If f''(c)<0, then concave down at c. If f''(c) = 0, or f''(c) does not exist, it is a point of inflection.

Extrema tests

Critical Values: If c is not an endpoint, and f'(c) = 0 or f'(c) DNE

FDT abs extrema: Find critical points. Evaluate function at critical points. By EVT, highest/lowest value is abs max/min

FDT loc extrema: Find critical points. If f' changes from positive to negative at c, local maximum. (respectively for neg-pos)

SDT: Find critical points. If f''(c) < 0, local max. f''(c) > 0, local minimum. =0 inconclusive

Extrema Theorems

Extreme value theorem states that on [a,b], f has absolute extrema

Rolle's theorem states on (a,b), if f(a)=f(b) anywhere in the interval (including end point a,b), there is at least on c such that f'(c) = 0.

Mean Value Theorem states that if f is continuous on [a,b] and differentiable on (a,b), there is at least one c such that $f'(c) = \frac{f(b) - f(a)}{b-a}$. (think of a sliding door – the gradient from a to b "slides" towards the tangent of point c)

Volume of Solids

X-axis rotation – Disk method wrt to x (↑), shell wrt to y (→).

Y-axis rotation – Disk method wrt y (→), shell wrt x (↑).

Limits Tips

Limits need to converge to 1 number to exist.

If not dividing by 0 or invalid, just sub in the value

Substitute trigonometric values if can and use squeeze: $-5 \le 3\sin(g(x)) - 2\cos(h(x)) \le 5$

Differentiation RULES

Constant Rule: d/dx c = 0

Constant multiple: d/dx cu = c du/dx u

Sum Rule: d/dx (u + v) = d/dx u + d/dx v

Product Rule: d/dx(uv) = du/dx * v + dv/dx * u

Quotient Rule: $\frac{d}{dx}\frac{u}{v} = \frac{\frac{du}{dx} * v - u * \frac{dv}{dx}}{v^2}$

|f(x)| != |f'(x)|. Open abs value to –f(x) and f(x) first

Fundamental Theorem of Calculus 1

If f is a function and G is the antiderivative of F (that is, G' = f):

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

$$\int_a^b G'(x)\,dx = G(b) - G(a)$$

Fundamental Theorem of Calculus 2

If g(x) = $\int_{h(x)}^{u(x)} f(t)\,dt$, **THEN**

g'(x) = $f(u(x)) * u'(x) - f(h(x)) * h'(x)$

IF f is continuous on [a,b], f is differentiable on (a,b)

Differentiability

If f' is differentiable at a point, then:

$$\lim_{h->0} \frac{f(x)_0 + h - f(x_0)}{h}$$

$$or \lim_{x->x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exist and are equal to the derivative at the point.

For piecewise functions, if they are differentiable, then f(a)=f(b) (no sharp spikes) and their limits are equal at the "meeting point" of the piecewise fnc

Natural Logarithm Rules

$$\ln(a) = x \rightarrow a = e^x$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

$$\ln(a^n) = n * \ln(a)$$

$$\ln(1) = 0$$

$$\ln(e) = 1$$

Integration

Don't be stupid - dx/dy directly corresponds to the axis

Optimization problems

Problems where you're trying to make something <u>as large/small</u> as possible

1. *Read the problem:* Write down the givens, and what you're optimizing for
2. *Draw a picture*, label any part important
3. *Introduce Variables*: List every relation as an equation/algebra, and identify which is the unknown variable
4. *Test Critical points and unknowns:* Find critical points

E.g. An open top box is made by cutting squares from a 12-by-12 sheet and bent up. How large should the squares cut from the corners be to make the box hold as much as possible?

From problem diagram: V(x) = $x * (12 - 2x)^2$, where x is the size of the corner cut. Find critical point by taking derivative (basic), and then check which is the maximum point and sub into original equation

Trigonometry Stuff

$$\sin(0) = 0, \sin(\pi) = 0, \sin\left(\frac{\pi}{2}\right) = 1$$

$$\cos(0) = 1, \cos(\pi) = 1, \cos\left(\frac{\pi}{2}\right) = 0$$

$$\sin(\theta) = \frac{opp}{hypotenuse}$$

$$\cos(\theta) = \frac{adj}{hypotenuse}$$

$$\tan(\theta) = \frac{opp}{adj} = \frac{\sin(\theta)}{\cos(\theta)}$$

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Implicit Differentiation

Apply d/dx to every term off the equation. Treat y as a function of x, and apply chain rule:

$$\frac{d}{dx}y = \frac{dy}{dx}$$

Solve for dy/dx by collecting all the terms

Power rule: $\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$

Product Rule: $\frac{d}{dx}u(x)y(x) = u'(x)y + u(x)\frac{dy}{dx}$

Chain Rule: $\frac{d}{dx}f(y) = f'(y)\frac{dy}{dx}$

Example:

Figuring out bounds

Let's say you did some discriminant/differentiation stuff and you get an equation like:

x=y+1

How do you find out the values that hold true for this?

Sub it into the original equation and solve!

ODE problems

Look for *rates of change*, these usually indicate ODE.

1. *Read the problem*: Find the rates given, create them if required
2. *Identify the type of ODE*
3. *Solve the ODE*
4. *Find constant*: Using initial value given in question
5. *Finish*: Calculate some value using the equation if needed

Example: In an oil refinery, a storage tank contains 10,000L of gas that initially has 50kg of additive dissolved in it. Gasoline containing 0.2kg/l of additive is pumped into the tank at a rate of 200L/min. The solution is pumped out at a rate of 220L/min. How much of the additive is in the tank 20mins after the pumping begins?

Let y be kgs of additive in tank at time t. When y=50, t=0. The number of liters of gasoline and additive in the solution is given:

$$V(t) = 10{,}000L + (200L/min - 220L/min) * t\,mins$$
$$= (10{,}000 - 20t)L$$

With this, we can model how much additive comes in and out of the tank:

Rate out = $\frac{y(t)}{V(t)} * outflow = \frac{y}{10{,}000 - 20t} * 200 = \frac{220y}{10{,}000 - 20t}\,kg/\min$

Rate in = 0.2 $kg/l * 200l = 40kg/min$

We can model a differential equation:

$$\frac{dy}{dt} = 40 - \frac{220y}{10{,}000 - 20t}\,kg/\min$$

In standard form:

$$\frac{dy}{dt} + \frac{220}{10{,}000 - 20t} * y = 40$$

This is a LFODE so we get the integrating factor:

$$I(x) = e^{\int P\,dt} = e^{\int \frac{220}{10{,}000 - 20t}}$$

$$= e^{-11 * \ln(10{,}000 - 20^t)} = (10{,}000 - 20t)^{-11}$$

Follow the steps for LFODE and multiply both sides and integrate:

$$(10{,}000 - 20t)^{-11} * \left(\frac{dy}{dt} + \frac{220}{10{,}000 - 20t} * y\right) = 40\,(10{,}000 - 20t)^{-1}$$

Skipping a few steps we get:

$$y = 0.2(10{,}000 - 20t) + C(10{,}000 - 20t)^{11}$$

From this, we solve the initial value problema and get C. Done.

Imposing restrictions

Let's say a question gives you a function of several variables f(x,y,z) = w = xyz^2.

Additionally, f(x,y,z) must fulfill the constraint equation $2x^3z^3 - xyz = 1$. You are tasked to find the exact value of dw/dx when (x,y,z)=(1,1,1). Lastly, you must regard w as a function of variables x and y only.

Firstly, you perform implicit differentiation on the constraint equation, treating z as a non-variable:

$$\frac{d}{dx}(2x^3z^3 - xyz)$$

$$= 6x^3z^3 + 6x^3z^2\frac{dz}{dx} - yz - xy\frac{dz}{dz} = 0$$

$$= \frac{dz}{dx} = -\frac{z}{x}$$

Then, we differentiate the original equation wrt to x, treating z as a function of x and y:

$$\frac{dw}{dx} = yz^2 + 2xyz\frac{dz}{dx}$$

Finally, we sub in (1,1,1) in the differentiated constraint equation to find

Interchanging bounds of integration

$$\int_0^4\int_{\sqrt{x}}^2 g(x)\,dy\,dx$$

let's say you have this integral, and want to interchange the bounds of integration. The best way is to draw a graph of sqrt x and observe how y and x change. However, you could also try this:

Write out the bounds:

$$[0 \le x \le 4]\ and\ [\sqrt{x} \le y \le 2]$$

Write out y in terms of x:

$$[x \le y^2 \le 4]$$

Combine the bounds:

$$[0 \le x \le y^2 \le 4]$$

Separate and swap:

$$[0 \le y^2 \le 4]\ and\ [0 \le x \le y^2]$$

Always double-check, though.

Maclaurin Series tiny example

Find the maclaurin series for $\frac{1}{1-3x}$

This is the geometric series, where u =3x.

Hence, $\sum_{n=0}^{\infty}(3x)^n$

Then, to find the 6th derivative of the function $\frac{1}{1-3x}$ evaluated at 0, we take: $c_n = \frac{f^{(n)}(0)}{n!}$, which when rearranged: $f^{(6)}(0) = 6! * 3^6$