Probability and Statistics

01 Probability basics

Sample Space = collection of all outcomes of a random experiment

Example : Roll a dice

$$S = \{1, 2, 3, 4, 5, 6\}$$

Event = a subset of the sample space

Example : A = 'Even'

$$= \{2, 4, 6\}$$

$$\subset S$$

Sigma Algebra = set of all events

Example : $\mathcal{A} = \{\emptyset, \{1\}, \ldots\ldots, \{6\}, \{1, 2\}, \ldots\ldots, \{2, 4, 6\} \ldots\}$

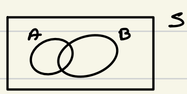
Probability Measure = $P : \mathcal{A} \to [0, 1]$

Example : Roll of a die, $A = \{\text{even}\}$

$$P(A) = \frac{1}{2}$$

... but we still need to define P properly

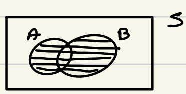
Combining Events :



$A, B \in \mathcal{A}$

Union = "...... or .... or both"

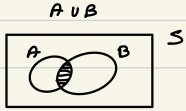
$A \cup B$



$A \cup B$

Intersection = "...... and ...."

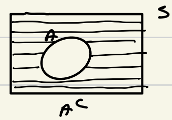
$A \cap B$



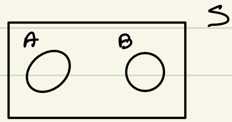
$A \cap B$

complement = "not in ...."

$A^c = \overline{A}$



Disjoint = 2 events are disjoint if they have no overlap, i.e. intersection is $\emptyset$



A and B are disjoint

Properties of probability measure

1. $A \in \mathcal{A}$: $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. $A_1\ A_2\ A_3 \ldots\ldots \in \mathcal{A}$ disjoint

$$P\left(\bigcup_i^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Consequences

1. What is $P(\emptyset)$?

$S, \emptyset, \emptyset, \ldots\ldots\ldots \in \mathcal{A}$ disjoint

$$\underbrace{P(S \cup \emptyset \cup \emptyset \ldots\ldots)}_{\substack{S \\ \shortparallel \\ P(S)}} = P(S) + P(\emptyset) + \ldots\ldots$$

$\therefore\ 0 = P(\emptyset) + P(\emptyset) + \ldots\ldots$

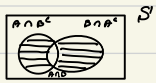
so $P(\emptyset) = 0$

2. $A, B \in \mathcal{A}$

$P(A \cup B)$

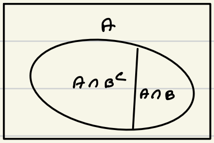
$= P[(A \cap B^c) \cup (A \cap B) \cup (B^c \cap A)]$

$= P(A \cap B^c) + P(A \cap B) + P(B \cap A^c)$



$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (B \cap A^c)$

Intermerro



$$P(A) = P[(A \cap B^c) \cup (A \cap B)]$$

$$= P(A \cap B^c) + P(A \cap B)$$

Similarly $\quad P(B) = P(B \cap A^c) + P(A \cap B)$

$$= [P(A) - P(A \cap B)] + P(A \cap B) + [P(B) - P(A \cap B)]$$

$$= P(A) + P(B) - P(A \cap B)$$

Example:

$A = \{ 1^{st}$ train is on time $\}$

$B = \{ 2^{nd}$ train is on time $\}$

$P($ on time in Lausanne $)$

$= P(A \cap B)$

$= P(A) + P(B) - P(A \cup B)$

$= 0.95 + 0.80 - 0.98$

$= 0.77$

Calculating Probabilities

Laplace's definition of probability: If our sample space has outcomes that are equally likely, then

$$P(A) = \frac{|A|}{|S|} = \frac{\#\text{ outcomes in } A}{\text{tot } \# \text{ outcomes}}$$

Example: Roll a dice, $A = \{\text{even}\}$

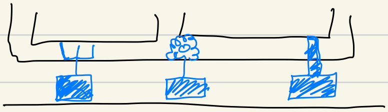
$$P(A) = \frac{|A|}{|S|} = \frac{3}{6} = 0.5$$

Counting

① Permutations :

n distinguishable objects

n available places



$3\times$	$2\times$	$1\times$
1	2	3

$3! = 3 \times 2 \times 1 = 6$

Hence $n! = n \times (n-1) \times \dots\dots \times 2 \times 1$

number of ways flowers can be placed.

Example: pack of cards (52) number of ways to order $52!$

② Combinations

3 pots	$n_1 = 3$	$3!$ ordering
2 flowers →	$n_2 = 2$	$2!$ ordering
1 cactus	$n_3 = 1$	$1!$ ordering

Same order appears : $3! \cdot 2! \cdot 1!$

There are $6!$ of them being distinguishable

numbers of ways with same arrangement $\dfrac{6!}{3! \cdot 2! \cdot 1!} = \dfrac{720}{12} = 60$

Hence Combinations : $\dfrac{n!}{n_1! \cdot n_2! \cdot n_3!}$

1. 5 Green balls

3 black balls

7 red balls

a) Total no of balls = 15

$$\frac{5}{15}\cdot\frac{4}{14} + \frac{3}{15}\cdot\frac{2}{14} + \frac{7}{15}\cdot\frac{6}{14}$$

$$\Rightarrow \frac{1}{3}\cdot\frac{2}{7} + \frac{1}{5}\cdot\frac{1}{7} + \frac{7}{15}\cdot\frac{3}{7}$$

$$\Rightarrow \frac{2}{21} + \frac{1}{35} + \frac{21}{105}$$

$$\Rightarrow \frac{10 + 3 + 21}{105}$$

$$\Rightarrow \frac{13+21}{105}$$

$$\Rightarrow \underline{\underline{\frac{34}{105}}}$$

$$\binom{5}{2} + \binom{3}{2} + \binom{7}{2}$$

$10 + 3 + 21$

$13 + 21$

34

$$\frac{34}{\binom{15}{2}} = \boxed{\frac{34}{105}}$$

$$\begin{array}{r} 15 \\ \underline{14} \\ 60 \\ \underline{150} \\ \underline{210} \\ \underline{2} \\ 105 \end{array}$$

(b) $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\binom{3}{2}}{\binom{15}{2}}$$

$$\binom{15}{2} \rightarrow 105$$

$$= \frac{3}{105}$$

$$\therefore \quad \frac{\frac{3}{105}}{\frac{34}{105}}$$

$$\Rightarrow \quad \frac{3}{\cancel{105}} \times \frac{\cancel{105}}{34} = \underline{\underline{\frac{3}{34}}}$$

8 professors

3. 6 associate

13 assistant

$$\begin{bmatrix} 8 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 1 \end{bmatrix} +$$

$$\begin{bmatrix} 8 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad 28 \cdot 6 \cdot 13 + 8 \cdot 15 \cdot 13 + 8 \cdot 6 \cdot 78$$

$$2184 + 1560 + 3744$$

$$7560$$

13.1 Distribution of sample mean.

Sample mea