The Black-Scholes Model

Theory and Practice

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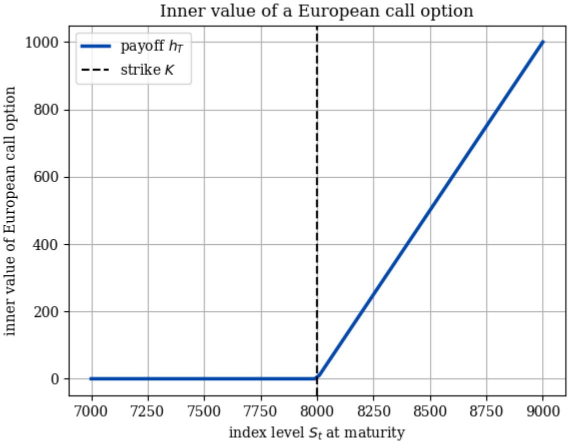
Introduction: The Problem of Option Pricing

- What is an option? (Call and Put)



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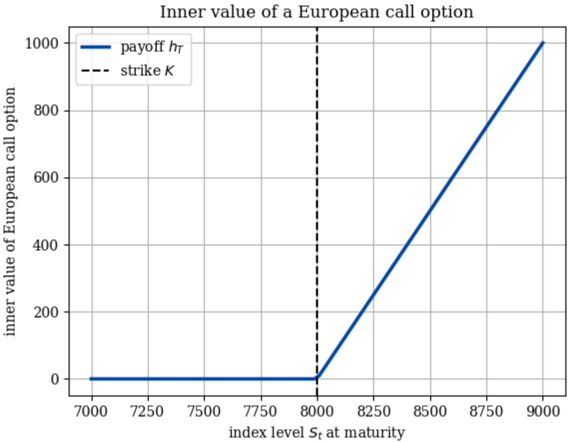
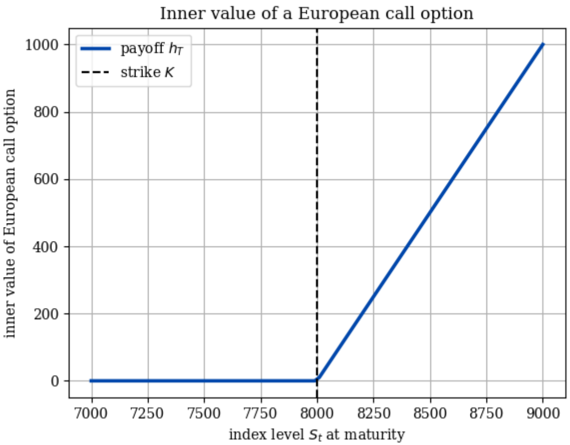


Introduction: The Problem of Option Pricing

- What is an option? (Call and Put)
- Why is option pricing important? (Risk management, speculation, hedging)
- The challenge: How to determine a "fair" price, given its dependence on the underlying asset?



The Black-Scholes Model: A Historical Perspective

- Introduced in 1973 by Fischer Black and Myron Scholes (Robert Merton later contributed).
- Revolutionized option pricing and risk management.
- Awarded the Nobel Prize in Economics in 1997 (Scholes and Merton). Black was deceased.

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- European option (can only be exercised at maturity).

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Constructing a Risk-Free Portfolio

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- Let Π be the value of the portfolio: $\Pi = -C + \Delta S$
- We need to find Δ such that it eliminates risk.

Applying Ito's Lemma

- Ito's Lemma: If $C = f(S, t)$, then:

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (dS)^2$$

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- Substituting and simplifying:

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW$$

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- Rearranging:

$$d\Pi = \left(\Delta\mu S - \frac{\partial C}{\partial t} - \mu S\frac{\partial C}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right)dt + \left(\Delta\sigma S - \sigma S\frac{\partial C}{\partial S}\right)dW$$

The No-Arbitrage Argument

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- By the no-arbitrage principle, the return on this risk-free portfolio must equal the risk-free rate: $d\Pi = r\Pi dt$

The Black-Scholes Partial Differential Equation

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- Boundary condition for a call option: $C(S, T) = \max(S - K, 0)$

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- Alternatively, risk-neutral pricing approach can be used to solve the PDE, by taking expectation of discounted pay-off of the option.

The Black-Scholes Formula

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- For a European put option:

$$P = Ke^{-rT} N(-d_2) - SN(-d_1)$$

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- Implied volatility is the volatility that yields the observed market price of the option.

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- Rho ($\rho = \frac{\partial C}{\partial r}$): Change in option price with respect to the risk-free rate.

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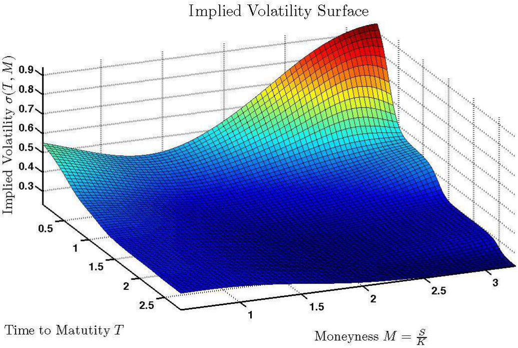
- European Option: The model is designed for European option.

Volatility Smile/Skew

- A graph of implied volatility versus strike price.
- Implied volatility is not constant across different strike prices.
- Possible reasons: demand for out-of-the-money puts for downside protection.



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- Finite difference/Monte Carlo methods for more complex options.

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- Be aware of its limitations and consider more advanced models when necessary.
- The model continues to be a benchmark against which other models are compared.

References

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