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Modulus Consensus over Time-varying Digraphs

Ji Liu Dan Wang Wei Chen Tamer Başar

Abstract— **This paper considers a discrete-time modulus consensus model in which the interaction among a group of agents is described by a time-dependent, complex-valued, weighted digraph. It is shown that for any sequence of repeatedly jointly strongly connected digraphs, without any assumption on the structure of the complex-valued weights, the system asymptotically reaches modulus consensus. Sufficient conditions for exponential convergence to each possible type of limit states are provided. Specifically, it is shown that (1) if the sequence of complex-valued weighted digraphs is repeatedly jointly balanced with respect to the same type, the corresponding type of modulus consensus will be reached exponentially fast for almost all initial conditions; (2) if the sequence of complex-valued weighted digraphs is repeatedly jointly unbalanced, the system will converge to zero exponentially fast for all initial conditions.**

I. Introduction

Opinion dynamics have been widely studied in social sciences for decades. Probably the most well-known model of opinion dynamics is the classical DeGroot Model [1] in which each individual updates her opinion by taking a convex combination of the opinions of her neighbors at each discrete time step. The model is essentially the same as the discrete-time linear consensus process, on which has a large amount of literature [2]–[13]. Variants of linear consensus include quantized consensus [14], constrained consensus [15], and max consensus [16].

Ever since the DeGroot Model, various models have been proposed and developed for opinion dynamics to understand and explain the formation and evolution of opinions in a large-scale social network. Notable examples include the Friedkin-Johnsen model [17], [18], the Hegselmann-Krause model [19]–[21], the DeGroot-Friedkin model [22]–[24], and the Altafini model [25].

The discrete-time Altafini model over a fixed signed graph can be regarded as a generalization of the DeGroot model (and the linear consensus model), which deals with a network consisting of $n > 1$ agents. Each agent is able to receive information only from its neighbors. The neighbor relationships among the agents in the Altafini model are described by a signed digraph (or directed graph) in which vertices correspond to agents, directions of arcs (or directed edges) indicate directions of information flow, and signs represent the social relationships between neighboring agents. Thus, the neighbor graph is a weighted digraph whose weights equal to either 1 or -1, which respectively represent friendly and antagonistic relationships in social networks.

The discrete-time Altafini model has been studied in [26]–[31], and the continuous-time counterpart has been considered in [25], [27], [29], [32]–[34]. A generalization of the continuous-time Altafini model via point groups has been recently proposed in [35].

For the discrete-time Altafini model over time-varying signed digraphs, it has been shown in [28] that for any "repeated jointly strongly connected" sequence of signed digraphs, the absolute values of all the agents' states will asymptotically reach a consensus, which is called *consensus in absolute value* here, having standard consensus and "bipartite consensus" [32] as special cases. The result is independent of the structure of signs in the digraphs. In [30], necessary and sufficient conditions for exponentially fast consensus in absolute value with respect to each possible type of limit states are provided in terms of structural balance (or structural unbalance), a concept from social sciences [36].

In a recent paper [37], a continuous-time consensus process in complex-weighted networks is introduced. The process can be regarded as a generalization of the continuous-time Altafini model. It has been shown in [37] that when the complex-weighted neighbor graph is fixed and strongly connected, either all the agents' states converge to zero, or the magnitudes of all the agents' states reach a consensus, which is called a *modulus consensus*. The paper also considers some special time-varying graphs and a special discrete-time counterpart. Such a complex-valued consensus process has an interesting application in circular formation problems; see [37] for details.

Complex-valued weights allow the encoding of an additional dimension of information compared to the case of real-valued weights. Applications of complex-weighted graphs and the associated complex-valued adjacency matrices can be found in many different areas. A natural example is the (nodal) admittance matrix of an RLC circuit which is of complex value by nature [38]. Another motivating example comes from formation control, for instance, making satellites form a specified spatial pattern [39]. Of particular interest is circular formation [40] which is closely related to the work reported in this paper. One can also find applications of complex-weighted graphs in localization problems [41].

In this paper, we study a discrete-time modulus consensus model over a complex-valued digraph. The model can be viewed as a generalization of the discrete-time Altafini

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model and an analog of the continuous-time model in [37]. The main contribution of this paper is two-fold. First, we consider the most general time-varying neighbor graphs, and show that an asymptotic modulus consensus will be reached for any repeatedly jointly strongly connected sequence of digraphs, independent of the values of complex-valued weights. Second, we derive sufficient conditions for exponential convergence with respect to each possible type of limit states. Specifically, we show that (1) if the sequence of complex-valued weighted digraphs is "repeatedly jointly balanced" with respect to the same type, the corresponding type of modulus consensus will be reached exponentially fast for almost all initial conditions; (2) if the sequence of complex-valued weighted digraphs is "repeatedly jointly unbalanced", the system will converge to zero exponentially fast for all initial conditions.

The remainder of this paper is organized as follows. Some notations and preliminaries are introduced in Section I-A. In Section II, the discrete-time modulus consensus process over a time-varying digraph is introduced. The main results of the paper are presented in Section III, whose proofs are given in Section IV. The paper ends with some concluding remarks in Section V.

A. Preliminaries

For any positive integer n, we use $[n]$ to denote the index set $\{1, 2, \dots, n\}$. We use $\mathbb{R}$ and $\mathbb{C}$ to denote the set of real and complex numbers, respectively. We view vectors as column vectors and write x' to denote the transpose of a vector x. For a vector x, we use x_i to denote its ith entry. For any complex number y, we use $|y|$ to denote its magnitude. For any matrix $M \in \mathbb{R}^{n\times n}$, we use m_{ij} to denote its ijth entry and write $|M|$ to denote the matrix in $\mathbb{R}^{n\times n}$ whose ijth entry is $|m_{ij}|$. A nonnegative $n \times n$ matrix is called a stochastic matrix if its row sums are all equal to 1. We use $\|\cdot\|$ to denote the induced infinity norm. For any vector v in $\mathbb{C}^n$, we use D_v to denote the $n \times n$ diagonal matrix whose ith diagonal entry is v_i, $i \in [n]$. For a square matrix S, we use $\rho(S)$ to denote the spectral radius of S. For two matrices A and B with the same size, we write $A \leq B$ to denote $a_{ij} \leq b_{ij}$.

The graph of an $n \times n$ matrix M with complex-valued entries is an n-vertex directed graph defined so that (i, j) is an arc from vertex i to vertex j in the graph whenever the jith entry of M is nonzero.

II. The Model

Consider a network of $n > 1$ agents labeled 1 through n.[1] Each agent i can receive information only from certain other agents called neighbors of agent i. Neighbor relationships among the n agents are described by a directed graph $\mathbb{N}(t)$, called neighbor graph, which may change over time. Agent j is a neighbor of agent i at time t whenever (j, i) is an arc in $\mathbb{N}(t)$. Thus, the directions of arcs indicate the directions of information flow. For convenience, we assume that each agent is always a neighbor of itself. Thus, $\mathbb{N}(t)$ has self-arcs at all n vertices for all time t. Each agent i has control over a complex-valued quantity x_i called agent i's agreement variable. Each agent i updates its variable at discrete times $t \in \{1, 2, \dots\}$ by setting

$$x_i(t+1) = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)x_j(t), \qquad i \in [n], \tag{1}$$

where $\mathcal{N}_i(t)$ denotes the set of neighbors of agent i at time t, and $a_{ij}(t)$ are complex-valued weights.

The n update rules in (1) can be written in the form of an n-dimensional state equation. Toward this end, let $x(t)$ be a vector in $\mathbb{C}^n$ whose ith entry equals $x_i(t)$, $i \in [n]$; then,

$$x(t+1) = A(t)x(t), \tag{2}$$

where $A(t)$ is the complex-valued matrix in $\mathbb{C}^{n\times n}$ whose ijth entry equals $a_{ij}(t)$, $i, j \in [n]$. It is worth noting that the graph of $A(t)$ is the same as the neighbor graph $\mathbb{N}(t)$ for all time t.

We impose the following assumption on the weights.

Assumption 1: For all $i \in [n]$ and t, there holds $a_{ii}(t) > 0$ and $\sum_{j\in\mathcal{N}_i(t)} |a_{ij}(t)| = 1$. Further, there exists a positive number β such that $|a_{ij}(t)| \geq \beta$ for all $i \in [n]$ and $j \in \mathcal{N}_i(t)$.

This assumption implies that $|A(t)|$ is a stochastic matrix with positive diagonal entries for each time t.

We say that the system described by (1), or equivalently (2), reaches a *modulus consensus* if all $|x_i(t)|$ reach a consensus as $t \to \infty$.

Remark 1: In the case when $a_{ii}(t) > 0$ does not hold for at least one $i \in [n]$, simple examples show that the system may not reach a modulus consensus. It is worth emphasizing that positive diagonal entries are also important in establishing standard consensus for discrete-time linear consensus processes [3]. □

III. Main Results

In this section, we study the limiting behavior of the system (1).

A. Fixed Graphs

We first consider the fixed graph case in which $\mathbb{N}(t) = \mathbb{N}$ for all time t.

Proposition 1: Suppose that Assumption 1 holds and that the neighbor graph $\mathbb{N}$ is strongly connected. Then, the system (1) reaches a modulus consensus exponentially fast for all initial conditions.

To prove the proposition, we need the following concepts and lemmas.

We use $\mathbb{U}^n$ to denote the set of vectors in $\mathbb{C}^n$ such that for each $v \in \mathbb{U}^n$, the modulus of each entry of v is 1, i.e., $|v_i| = 1$ for all $i \in [n]$.

[1] The purpose of labeling of the agents is only for convenience. We do not require a global labeling of the agents in the network. We only assume that each agent can identify its neighbors.

We say that a complex-valued matrix M is *essentially nonnegative* if there exists a vector $v \in \mathbb{U}^n$ such that the matrix $D_v^{-1}MD_v$ is a nonnegative matrix, where D_v is an $n \times n$ diagonal matrix whose ith diagonal entry equals v_i, $i \in [n]$.

Lemma 1: Suppose that the neighbor graph $\mathbb{N}$ is strongly connected, and that its complex-valued weight matrix A satisfies Assumption 1. Then, $\rho(A) = 1$ and A has a simple eigenvalue at 1 if, and only if, A is essentially nonnegative.

Proof: Let $L = I - A$ where I is the identity matrix. Since A satisfies Assumption 1, L can be written as $L = D - \bar{A}$ where $\bar{A}$ is the $n \times n$ matrix whose diagonal entries all equal zero and ijth entry equals a_{ij} for all $i, j \in [n]$ and $i \neq j$, and D is the $n \times n$ diagonal matrix whose ith entry $d_i = \sum_{j=1}^{n} |a_{ij}|$. Then, it follows that

$$A = I - L = I - D + \bar{A}.$$

Since $I - D$ is a nonnegative diagonal matrix, A is essentially nonnegative if, and only if, $\bar{A}$ is essentially nonnegative. By Proposition 2.3 in [37], L has a simple eigenvalue at zero if, and only if, $\bar{A}$ is essentially nonnegative. Since $A = I - L$ and $|A|$ is an irreducible stochastic matrix, $\rho(A) = 1$ and A has a simple eigenvalue at 1 if, and only if, $\bar{A}$ is essentially nonnegative, or equivalently, A is essentially nonnegative. ■

In the case when A has an eigenvalue at 1, it must have an eigenvector in $\mathbb{U}^n$ because of the following lemma.

Lemma 2: [42] Let M be a matrix in $\mathbb{C}^{n \times n}$. Suppose that M is irreducible and that λ is an eigenvalue of M satisfying

$$|\lambda - a_{ii}| \geq \sum_{j=1, j \neq i}^{n} |a_{ij}|, \qquad i \in [n].$$

Then, the corresponding eigenvector v satisfies $|v_i| = |v_j|$ for all $i, j \in [n]$.

Lemma 3: Suppose that the neighbor graph $\mathbb{N}$ is strongly connected, and that its complex-valued weight matrix A satisfies Assumption 1. Then, $\rho(A) < 1$ if, and only if, A is not essentially nonnegative.

This lemma is a direct consequence of Lemma 1.

Proof of Proposition 1: From Lemmas 1 and 3, if A is essentially nonnegative, $x(t)$ will converge to $(u'x(1))v$ where u and v are respectively left and right eigenvectors corresponding to eigenvalue 1. Since v satisfies $|v_i| = |v_j|$ for all $i, j \in [n]$, the system (2) reaches a modulus consensus. If A is not essentially nonnegative, the system (2) will converge to zero, which is also a modulus consensus. Thus, the system (2) will always reach a modulus consensus. Since the system is a linear time-invariant system, the convergence is exponentially fast. ■

B. Time-varying Graphs

To state our main results, we need the following concepts.

A digraph $\mathbb{G}$ is *strongly connected* if there is a directed path between each pair of distinct vertices. We say that a finite sequence of digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_p$ with the same vertex set is *jointly strongly connected* if the union[2] of the digraphs in this sequence is strongly connected. We say that an infinite sequence of digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots$ with the same vertex set is *repeatedly jointly strongly connected* if there is a positive integer l for which each finite sequence $\mathbb{G}_{kl+1}, \mathbb{G}_{kl+2}, \ldots, \mathbb{G}_{(k+1)l}$, $k \geq 0$, is jointly strongly connected. It is worth emphasizing that the above connectivity concepts are also applicable to weighted graphs, without taking weights into account.

The following theorem states that the system (2) asymptotically reaches modulus consensus for all initial conditions under an appropriate connectivity assumption.

Theorem 1: Suppose that all n agents adhere to the update rule (1) and that Assumption 1 holds. Suppose that the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \ldots$ is repeatedly jointly strongly connected. Then, the system (2) asymptotically reaches a modulus consensus.

The proof of this theorem is given in Section IV.

Remark 2: It is known that for discrete-time linear consensus processes, repeatedly jointly strong connectivity guarantees exponentially fast consensus [9]. But this is not the case for modulus consensus. Examples can be generated to show that modulus consensus may be reached only asymptotically, but not exponentially fast, even with repeatedly jointly strong connectivity. □

In the sequel, we will provide sufficient conditions for each type of limit states to be reached exponentially fast. To this end, we will resort to the concept of "balanced" networks.

C. Exponential Convergence

A complex-weighted digraph $\mathbb{G}$ is called *balanced* with respect to $v \in \mathbb{U}^n$ if its weight matrix A is essentially nonnegative with respect to v, i.e., $D_v^{-1}AD_v$ is a nonnegative matrix. In the case when A satisfies Assumption 1, it can be verified that $D_v^{-1}AD_v$ is a stochastic matrix. Otherwise, $\mathbb{G}$ is called *unbalanced*.

Each $v \in \mathbb{U}^n$ corresponds to one type of modulus consensus. For each type of nonzero modulus consensus and the zero (modulus) consensus, we will provide sufficient conditions under which the modulus consensus can be reached exponentially fast. To state the results, we need the following concepts.

We say that a finite sequence of complex-weighted digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_p$ with the same vertex set is *jointly balanced* with respect to a vector $v \in \mathbb{U}^n$ if all the corresponding weight matrices $A_1, A_2, \ldots, A_p$ are essentially nonnegative with respect to the same vector v. We say that a finite sequence of complex-weighted digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_p$ with the same vertex set is *jointly unbalanced* if there does not exist a vector $v \in \mathbb{U}^n$ such that all the corresponding weight matrices $A_1, A_2, \ldots, A_p$ are essentially nonnegative with respect to v. We say that an infinite sequence of complex-weighted digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots$ with the same vertex

[2]The union of a finite sequence of unsigned digraphs with the same vertex set is an unsigned digraph with the same vertex set and the arc set which is the union of the arc sets of all digraphs in the sequence.

set is *repeatedly jointly balanced* with respect to a vector $v \in \mathbb{U}^n$ if there exist positive integers τ_0 and l for which each finite sequence $\mathbb{G}_{\tau_0+kl}, \mathbb{G}_{\tau_0+kl+1}, \ldots, \mathbb{G}_{\tau_0+(k+1)l-1}$, $k \geq 0$, is jointly balanced with respect to v. We say that an infinite sequence of complex-weighted digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots$ with the same vertex set is *repeatedly jointly unbalanced* if there exist positive integers τ_0 and l for which each finite sequence $\mathbb{G}_{\tau_0+kl}, \mathbb{G}_{\tau_0+kl+1}, \ldots, \mathbb{G}_{\tau_0+(k+1)l-1}$, $k \geq 0$, is jointly unbalanced.

The exponential convergence results are then as follows. We begin with nonzero modulus consensus.

Theorem 2: Suppose that all n agents adhere to the update rule (1) and that Assumption 1 holds. Suppose that the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \ldots$ is repeatedly jointly strongly connected. If $\mathbb{N}(1), \mathbb{N}(2), \ldots$ is repeatedly jointly balanced with respect to a vector $v \in \mathbb{U}^n$, then system (2) reaches the corresponding nonzero modulus consensus exponentially fast for almost all initial conditions.

The proof of this theorem is given in Section IV.

Remark 3: Note that if a finite sequence of complex-valued digraphs with the same vertex set is jointly balanced with respect to a vector $v \in \mathbb{U}^n$, then each graph in the sequence is balanced with respect to the same v. By the definition of repeatedly jointly balanced graphs, there exists a finite time instant τ such that $\mathbb{G}(t)$ is balanced with respect to v for all $t > \tau$. $\square$

The next theorem addresses the zero (modulus) consensus.

Theorem 3: Suppose that all n agents adhere to the update rule (1) and that Assumption 1 holds. Suppose that the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \ldots$ is repeatedly jointly strongly connected. If $\mathbb{N}(1), \mathbb{N}(2), \ldots$ is repeatedly jointly unbalanced, then system (2) converges to zero exponentially fast for all initial conditions.

The proof of this theorem is given in Section IV.

IV. Proofs

In this section, we provide proofs for the main results presented in the previous section.

A. Proof of Theorem 1

From the update rule (1) and Assumption 1, it follows that

$$\begin{aligned} |x_i(t+1)| &\leq \sum_{j \in \mathcal{N}_i(t)} |a_{ij}(t)||x_j(t)| \\ &\leq \left(\sum_{j \in \mathcal{N}_i(t)} |a_{ij}(t)| \right) \max_{k \in [n]} |x_k(t)| \\ &= \|x(t)\|, \end{aligned}$$

which implies that $\|x(t+1)\| \leq \|x(t)\|$, and thus $\{\|x(t)\|\}$ is a bounded nonincreasing sequence. It follows that this sequence has a limit as $t \to \infty$, i.e.,

$$\lim_{t \to \infty} \|x(t)\| = c, \tag{3}$$

where c is a real-valued constant. Thus, for any $\varepsilon > 0$, there exists a time τ_1, depending on ε, such that $c - \varepsilon \leq \|x(t)\| \leq c + \varepsilon$ for all $t \geq \tau_1$.

For each $i \in [n]$, define

$$\begin{aligned} p_i &= \limsup_{t \to \infty} |x_i(t)|, \\ q_i &= \liminf_{t \to \infty} |x_i(t)|. \end{aligned}$$

Since $q_i \leq p_i \leq c$, to prove the theorem, it would be sufficient to show that $q_i = p_i = c$ for all $i \in [n]$. Suppose therefore, to the contrary, that there exists some $k \in [n]$ such that $q_k < c$. Then, there exist a time τ_2, depending on ε, and a real-valued constant $b < c$ such that $|x_k(\tau_2)| \leq b$.

Since the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \ldots$ is repeatedly jointly strongly connected, by definition (see Section III), there is a positive integer l for which each finite sequence $\mathbb{N}(kl+1), \mathbb{N}(kl+2), \ldots, \mathbb{N}((k+1)l)$, $k \geq 0$, is jointly strongly connected. Using the same argument as in the proof of Theorem 2.1 in [28], it can be shown that

$$\|x(t)\| < c - \frac{(c-b)\beta^{(n-1)l}}{2},$$

where β is given in Assumption 1, which constitutes a contradiction to (3). Thus, there holds $q_i = p_i = c$ for all $i \in [n]$, which completes the proof. ■

B. Proof of Theorem 2

Suppose that the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \ldots$ is repeatedly jointly strongly connected and balanced with respect to a vector $v \in \mathbb{U}^n$. Then, for each $A(t)$, the matrix $D_v^{-1}A(t)D_v$ is a stochastic matrix. Note that the graph of $A(t)$ is the same as the neighbor graph $\mathbb{N}(t)$ for all t. Since the sequence of the graphs of $A(1), A(2), \ldots$ is repeatedly jointly strongly connected, so is the sequence of the (unsigned) graphs of $D_v^{-1}A(1)D_v, D_v^{-1}A(2)D_v, \ldots$. With these facts and the well-known result of standard discrete-time linear consensus processes [9], it is straightforward to verify that the matrix product $A(t) \cdots A(2)A(1)$ converges to a rank one matrix of the form vc' exponentially fast, where c is a nonzero vector, and thus $x(t)$ converges to $vc'x(1)$. It follows that system (2) will reach the corresponding nonzero modulus consensus if $c'x(1)$ does not equal zero. Since the set of those vectors x which satisfy the equality $c'x = 0$ is a thin set, the nonzero modulus consensus will be reached exponentially fast for almost all initial conditions. ■

C. Proof of Theorem 3

To prove Theorem 3, we need the following lemmas.

Lemma 4: [42] Suppose that A is an $n \times n$ complex-valued matrix and B is an $n \times n$ nonnegative matrix. If $|A| \leq B$, then $\rho(A) \leq \rho(|A|) \leq \rho(B)$.

Lemma 5: [42] Suppose that A is an $n \times n$ irreducible nonegative matrix and $B \neq 0$ is an $n \times n$ nonnegative matrix. Then, $\rho(A+B) > \rho(A)$.

Lemma 6: Suppose that P and Q satisfy Assumption 1 and the graphs of P and Q are jointly strongly connected. If P and Q are jointly unbalanced, then $\rho(PQ) < 1$.

Proof: Let $R = PQ$ and $W = |P||Q|$; thus, R is a complex-valued matrix and W is a real-valued matrix. Since P and Q satisfy Assumption 1 and the graphs of P and Q are jointly strongly connected, W is an irreducible stochastic matrix. Thus, $\rho(W) = 1$ and 1 is a simple eigenvalue of W. Note that

$$|r_{ij}| = |\sum_{k=1}^{n} p_{ik}q_{kj}| \leq \sum_{k=1}^{n} |p_{ik}||q_{kj}| = w_{ij}, \tag{4}$$

it follows that $|R| \leq W$. By Lemma 4, $\rho(|R|) \leq \rho(W) = 1$. In the sequel, we consider two cases satisfying $|R| \leq W$.

First, if $|R| \neq W$, then there must exist a positive scalar $0 < \alpha < 1$ such that $|R| + \alpha(W - |R|)$ is a nonnegative matrix. Since W is irreducible, so is $|R| + \alpha(W - |R|)$. By Lemmas 4 and 5, it follows that

$$\rho(R) \leq \rho(|R|) \leq \rho(|R| + \alpha(W - |R|)) < \rho(W) = 1.$$

Second, if $|R| = W$, then from (4), all $p_{ik}q_{kj}$, $k \in [n]$, have the same angle. Since $\rho(R) \leq \rho(W) = 1$, to prove the lemma, it would be sufficient to show that 1 is not an eigenvalue of R. Suppose therefore that, to the contrary, 1 is an eigenvalue of R.

Since R satisfies Assumption 1 and is irreducible, by Lemma 1, R is essentially nonnegative. Thus, there exists a vector $v \in \mathbb{U}^n$ such that $D_v^{-1}RD_v$ is nonnegative, i.e, $v_i^{-1}(\sum_{k=1}^{n} p_{ik}q_{kj})v_j$ is nonnegative for all $i, j \in [n]$. Since all $p_{ik}q_{kj}$, $k \in [n]$, have the same angle, from the fact that $p_{ii} > 0$ and $q_{jj} > 0$ for all $i, j \in [n]$, it follows that $v_i^{-1}p_{ij}v_j$ and $v_i^{-1}q_{ij}v_j$ are both nonnegative for all $i, j \in [n]$, which implies that P and Q are jointly essentially nonnegative. This contradicts the assumption of the lemma. Thus, R does not have an eigenvalue at 1, which completes the proof. ■

Using arguments similar to those as in the proof of Lemma 6, we have the following generalized result.

Proposition 2: Suppose that $P_1, P_2, \ldots, P_m$ satisfy Assumption 1 and the graphs of $P_1, P_2, \ldots, P_m$ are jointly strongly connected. If $P_1, P_2, \ldots, P_m$ are jointly unbalanced, then $\rho(P_m \cdots P_2P_1) < 1$.

We will also need the following result.

Lemma 7: Let P and Q be two matrices in $\mathbb{C}^{n\times n}$. Let W and S be two $n \times n$ positive stochastic matrices. Suppose that $\rho(P) < 1$, $\rho(Q) < 1$, $|P| \leq W$, and $|Q| \leq S$. Also, suppose that if $|P| = W$, then $p_{ii} > 0$ for all $i \in [n]$, and that if $|Q| = S$, then $q_{ii} > 0$ for all $i \in [n]$. Then, there exists a constant $\lambda \in [0, 1)$ such that for any $x \in \mathbb{C}^n$, if $z = QPx$, then $\|z\| \leq \lambda\|x\|$.

Proof: Let $y = Px$, and thus $z = Qy$. Since

$$\begin{aligned} |y_i| &= |\sum_{j=1}^{n} p_{ij}x_j| \leq \sum_{j=1}^{n} |p_{ij}||x_j| \\ &\leq \sum_{j=1}^{n} w_{ij}|x_j| \leq \sum_{j=1}^{n} w_{ij}\|x\| = \|x\|, \end{aligned} \tag{5}$$

it follows that if there is some $k \in [n]$ such that $|x_k| < \|x\|$, then $\|y\| < \|x\|$, which implies that $\|z\| < \|x\|$. Next suppose that $|x_i| = \|x\|$ for all $i \in [n]$. We consider the following two cases.

First, if $|P| \neq W$, then there must exist some $k, j \in [n]$ such that $|p_{kj}| < w_{kj}$. From (5), there holds $|y_k| < \|x\|$. If $|y_k| = \|y\|$, then $\|y\| < \|x\|$, which implies that $\|z\| < \|x\|$. If $|y_k| < \|y\|$, then using the same argument, $\|z\| < \|y\|$, which implies that $\|z\| < \|x\|$.

Second, if $|P| = W$, we claim that there exists some $k \in [n]$ such that $|y_k| < \|x\|$. To establish the claim, we suppose, to the contrary, that $|y_i| = \|y\| = \|x\|$ for all $i \in [n]$. Then, from (5), all $p_{ij}x_j$, $j \in [n]$, have the same angle. Since $p_{ii} > 0$, it follows that all $p_{ij}x_j$, $j \in [n]$, have the same angle as x_i. Thus, $x_i^{-1}p_{ij}x_j$ is nonnegative for all $i, j \in [n]$, i.e., $D_x^{-1}PD_x$ is a nonnegative matrix, which implies that P is essentially nonnegative. By Lemma 1, P has an eigenvalue at 1. This contradicts the assumption that $\rho(P) < 1$. Thus, there exists some $k \in [n]$ such that $|y_k| < \|x\|$, which implies that $\|z\| < \|x\|$.

From the preceding discussion, we have thus proved that for any $x \in \mathbb{C}^n$, there holds $\|QPx\| < \|x\|$. Let

$$\lambda = \max_{x'x=1} \frac{\|QPx\|}{\|x\|},$$

which must be strictly less than 1. Therefore, $\|z\| < \lambda\|x\|$, which completes the proof. ■

Using arguments similar to those as in the proof of Lemma 7, we have the following generalized result.

Proposition 3: Let $P_1, P_2, \ldots, P_m$ be matrices in $\mathbb{C}^{n\times n}$. Let $S_1, S_2, \ldots, S_m$ be $n \times n$ positive stochastic matrices. Suppose that $\rho(P_i) < 1$ and $|P_i| \leq S_i$ for all $i \in [m]$. Also, suppose that if $|P_i| = S_i$, $i \in [m]$, then the diagonal entries of P_i are all positive. Then, there exists a constant $\bar{\lambda} \in [0, 1)$ such that for any $x \in \mathbb{C}^n$, if $z = P_m \cdots P_2P_1x$, then $\|z\| \leq \bar{\lambda}\|x\|$.

We are now in a position to prove Theorem 3.

Proof of Theorem 3: Since $\mathbb{N}(1), \mathbb{N}(2), \ldots$ is repeatedly jointly strongly connected and the corresponding matrices $|A(1)|, |A(2)|, \ldots$ are all stochastic matrices with positive diagonal entries, from Proposition 4 in [9], there exists a positive integer q such that for any $\tau \geq 1$, the product $|A(\tau + q)| \cdots |A(\tau + 2)||A(\tau + 1)|$ is a positive stochastic matrix. Let $P_\tau = A(\tau + q) \cdots A(\tau + 2)A(\tau + 1)$ and $S_\tau = |A(\tau + q)| \cdots |A(\tau + 2)||A(\tau + 1)|$. Then, the two sets of all such P_τ and S_τ matrices are both compact. Note that these P_τ and S_τ matrices satisfy the assumptions in Proposition 3, which implies that any product of infinitely many P_τ matrices converges to the zero matrix exponentially fast. Therefore,

$$\lim_{t\to\infty} A(t) \cdots A(2)A(1) = 0,$$

and the convergence is exponentially fast, which completes the proof. ■

V. Conclusions

In this paper, a discrete-time modulus consensus model has been studied. We have shown that for any sequence of repeatedly jointly strongly connected digraphs, the system asymptotically reaches modulus consensus. We have also provided sufficient conditions for exponential convergence to each possible type of limit states.

As part of future work, we plan to study the limiting behavior of the system without the strong connectivity assumption and establish necessary conditions for exponential convergence to each possible type of limit states. We will also seek applications of our results in distributed network problems.

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