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Achieving consensus on networks with antagonistic interactions

C. Altafini

Abstract—Consensus protocols achieve an agreement among agents thanks to the collaborative efforts of all agents, expressed by a (connected) communication graph with nonnegative weights. The question we ask in this paper is the following: **is it possible to achieve a form of agreement in presence of antagonistic interactions, modeled as negative weights on the communication graph?** The answer to this question is affirmative: on signed networks all agents can converge to a consensus value which is the same for all except for the sign. Necessary and sufficient conditions are obtained to describe the cases when this is possible.

I. INTRODUCTION

In the literature on the distributed consensus problem, [17], [16], [18] an agreement among the agents is achieved through the cooperation of all agents. The collaborations between agents are modeled as links in an communication graph and nonnegative weights are associated to these interactions.

In this paper we are interested in networks of agents which can collaborate but also compete. Networks with antagonistic interactions are common for example in social network theory [20], [7]. They are represented as signed graphs, i.e., graphs in which the edges can assumed also negative weights. A positive/negative weight can be associated to a friend/foe (allied/adversary) relationship between the two agents linked by the edge, or, depending on the context, to a trust/distrust, like/dislike, etc. interaction. Our aim in this paper is to introduce a suitable notion of consensus in presence of antagonistic links and to investigate how and to what extent agents on signed graphs can achieve consensus through distributed protocols. In particular, we will see that under suitable conditions the agents can achieve a form of “agreed upon dissensus” (hereafter called *bipartite consensus*), in which all agents converge to a value which is the same for all in modulus but not in sign. This polarization of the community into two factions having opposite “opinions” is common in many antagonistic systems characterized by bimodal coalitions, like two-party political systems, duopolistic markets, rival business cartels, competing international alliances, etc. Engineering applications include competing robotic teams and trust networks [8], [14]. We will show that if we use distributed Laplacian-like schemes as in the current literature on consensus problems, then bipartite consensus can be achieved when and only when the signed graph of the network is structurally balanced. In social network theory, structural balance is a well-known property [9], [5], and corresponds to the possibility of bipartitioning the signed

graph into two adversary subcommunities such that all edges within each subcommunity are friendly (positive weights) while all edges joining agents of different communities are adversary (negative weights). Graphs of nonnegative weights are a special case of structural balance, in which one of the two subcommunities is empty.

We will show that Laplacian schemes are convergent also on signed graphs that are not structurally balanced. However, in this case the consensus value is always trivial (the origin), regardless of the initial condition and of the antagonistic content of the network. In fact, the Laplacian one obtains in the structurally unbalanced case is globally asymptotically stable (rather than critically stable), meaning that the (bipartite for us) agreement subspace is empty.

An equivalent characterization of structurally balanced signed graphs is that all cycles (or semicycles for directed graphs) of the graph are positive (i.e., have an even number of negative edges). Quite remarkably, all structurally balanced networks are equivalent, under a suitable change of orthant order, to nonnegative networks. Adopting the terminology used in Statistical Physics for this type of equivalence transformations (well-known in the Ising spin glass literature [2], see [13] for more details), we shall call the changes of orthant order *gauge transformations*. Gauge equivalence (or switching equivalence in the theory of signed graphs [21], or signature similarity as it is called in the field of signed pattern matrices [4]) is a finite-cardinality subclass of the similarity equivalence of matrices, which leaves the modulus of the entries of a matrix unchanged and only modifies its sign pattern. Given a structurally balanced network with its set of edge weights (possibly nonnegative), there exists a family of structurally balanced signed networks characterized by the same weights (but with different signs). All these “realizations” of the signed networks are related by gauge transformations and all are isospectral, meaning that the corresponding Laplacians enjoy the same convergence properties, although the bipartition characterizing the consensus vector differs from realization to realization. In particular, in each such family of gauge equivalent structurally balanced networks there is always one particular network with all nonnegative weights. This corresponds to a particular partial ordering of the axes of \mathbb{R}^n . Using the notion of gauge equivalence, it is therefore possible to adapt the distributed protocols developed for cooperative systems [16], [18], [17] to the more general case of structurally balanced communication graphs.

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II. SIGNED GRAPHS

A (weighted) signed graph \mathcal{G} is a triple $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$ where $\mathcal{V} = \{v_1, \dots, v_n\}$ is a set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, and $A \in \mathbb{R}^{n \times n}$ is the matrix of the signed weights of \mathcal{G} : $a_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}$. The adjacency matrix A alone completely specifies a signed graph. For the signed graph corresponding to A we shall use the notation $\mathcal{G}(A)$. We will not consider graphs with self-loops: $a_{ii} = 0 \forall i = 1, \dots, n$. When the graph is undirected then the order of the nodes in \mathcal{E} is irrelevant and the matrix A is symmetric. For a directed graph (digraph) we shall use the convention that on the edge $(v_j, v_i) \in \mathcal{E}$, v_j represents the tail and v_i the head of the arrow. In a digraph a pair of edges sharing the same nodes $(v_i, v_j), (v_j, v_i) \in \mathcal{E}$ is called a digon. In the digraphs of this paper we will always assume that $a_{ij}a_{ji} \geq 0$, meaning that the edge pairs of all digons cannot have opposite signs. Under this assumption (hereafter called *digon sign-symmetry*), a digraph \mathcal{G} “admits” an undirected graph $\mathcal{G}(A_u)$ defined by $A_u = (A + A^T)/2$.

A (directed) path \mathcal{P} of $\mathcal{G}(A)$ is a concatenation of (directed) edges of \mathcal{E} :

$$\mathcal{P} = \{(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots, (v_{i_{p-1}}, v_{i_p})\} \subset \mathcal{E}$$

in which all nodes v_{i_1}, \dots, v_{i_p} are distinct. The length of \mathcal{P} is $p - 1$. A (directed) cycle \mathcal{C} of $\mathcal{G}(A)$ is a (directed) path beginning and ending with the same node $v_{i_p} = v_{i_1}$. For digraphs, a semicycle is a cycle of $\mathcal{G}(A_u)$. A cycle (semicycle) is positive if it contains an even number of negative edge weights: $a_{i_1, i_2} \dots a_{i_p, i_1} > 0$. It is negative if $a_{i_1, i_2} \dots a_{i_p, i_1} < 0$. Irreducibility of A corresponds to $\mathcal{G}(A)$ which is strongly connected, i.e., $\forall v_i, v_j \in \mathcal{V} \exists \mathcal{P} \subset \mathcal{E}$ starting at v_i and ending at v_j (strong connectivity collapses into connectivity when A is symmetric).

The following is mentioned in e.g. [19].

Proposition 1 Consider a digraph $\mathcal{G}(A)$ which is strongly connected and digon sign-symmetric. $\mathcal{G}(A)$ has no negative semicycle if and only if $\mathcal{G}(A)$ has no negative directed cycle.

Given the signed digraph $\mathcal{G}(A)$, denote C_r the row connectivity matrix of A , i.e., the diagonal matrix having diagonal elements $c_{r, ii} = \sum_{j \in \text{adj}(i)} |a_{ij}|$, where $\text{adj}(\cdot)$ are the nodes adjacent to v_i in \mathcal{E} (with in-degree direction as in [16]: v_j is the tail of the arrow whose head is v_i). The column connectivity matrix C_c is defined analogously. When $A = A^T$ then $C_r = C_c = C$. More generally, a signed digraph is said *weight balanced* if $C_r = C_c$. In the consensus literature [16], [18], [17], this property is normally referred to as “balanced” *tout-court* (see [6], though).

III. CONSENSUS PROTOCOLS FOR SIGNED UNDIRECTED GRAPHS

Consider the system of integrators

$$\dot{x} = u, \quad x, u \in \mathbb{R}^n. \quad (1)$$

In the consensus problem, the task is to devise distributed feedback laws $u_i = u_i(x_i, x_j, j \in \text{adj}(i))$, i.e., feedback

laws based on the states of the node itself and of its first neighbors on the connectivity graph $G(A)$ of the network. Unlike in standard consensus problems, we do not assume that the weights of A are nonnegative.

Consider a given signed (symmetric) adjacency matrix A . The definition of a Laplacian L in the case of signed A is $L = C - A$ where in the connectivity matrix C the weights are in absolute value, see [12], [15]¹. The elements of L are therefore:

$$\ell_{ik} = \begin{cases} \sum_{j \in \text{adj}(i)} |a_{ij}| & k = i \\ -a_{ik} & k \neq i. \end{cases}$$

The corresponding Laplacian potential is

$$\begin{aligned} \Phi(x) &= x^T L x = \sum_{(v_j, v_i) \in \mathcal{E}} (|a_{ij}| x_i^2 + |a_{ij}| x_j^2 - 2a_{ij} x_i x_j) \\ &= \sum_{(v_j, v_i) \in \mathcal{E}} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j)^2 \end{aligned} \quad (2)$$

where $\text{sgn}(\cdot)$ is the sign function. The effect of a negative weight a_{ij} is to replace the usual $(x_i - x_j)^2$ term in (2) with $(x_i + x_j)^2$, which does not alter the sum of squares structure of $\Phi(x)$.

Just like for the nonnegative weights case, one can use L for the feedback laws in (1) and study the gradient system

$$\dot{x} = -Lx, \quad (3)$$

which in components reads:

$$\dot{x}_i = - \sum_{j \in \text{adj}(i)} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j).$$

Let $\lambda_1(L) \leq \dots \leq \lambda_n(L)$ be the eigenvalues of L . From (2) it is evident that $\Phi(x) \geq 0$ and therefore that $\lambda_1(L) \geq 0$. Unlike for the case of nonnegative A , here $-L$ is however no longer a Metzler matrix² in general and its row/column sum need not be zero. The major difference with the standard theory of nonnegative adjacency matrices is that L can be positive definite.

Example 1 Consider the signed graph of Fig. 1(a) of adjacency matrix

$$A_1 = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -4 \\ -2 & -4 & 0 \end{bmatrix},$$

the corresponding Laplacian $L_1 = \text{diag}(3, 5, 6) - A_1$ has eigenvalues $\text{sp}(L_1) = \{0, 4.35, 9.65\}$, i.e. L_1 positive semidefinite ($\text{sp}(\cdot)$ = spectrum).

¹This is not the only definition of Laplacian of a signed graph available in the literature. In [3], for example, the Laplacian is defined without the absolute values in the diagonal terms. In this formulation 0 is always an eigenvalue, but negative eigenvalues may appear, rendering the Laplacian useless for convergence purposes.

²Metzler matrices, also called negated Z-matrices, are matrices with nonnegative off-diagonal entries, see [1].

Example 2 The signed graph of Fig. 1(b) instead has adjacency matrix

$$A_2 = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 4 \\ -2 & 4 & 0 \end{bmatrix}.$$

The Laplacian $L_2 = \text{diag}(3, 5, 6) - A_2$ has eigenvalues $\text{sp}(L_2) = \{1.2, 2.61, 10.18\}$, meaning that L_2 positive definite.

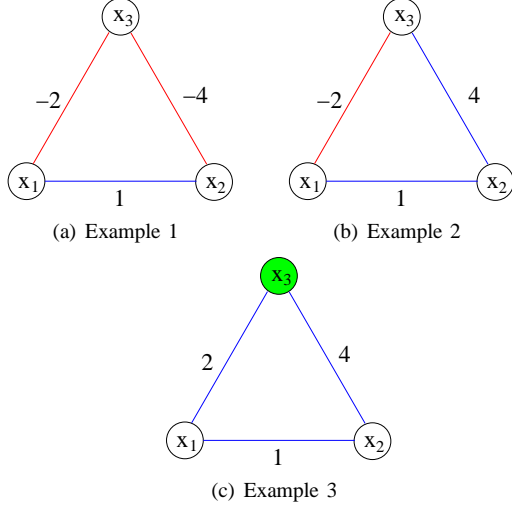


Fig. 1. Signed undirected connectivity graphs mentioned in Section III. Examples 1 and 3 are structurally balanced and differ only by the gauge transformation $D = \text{diag}(1, 1, -1)$. Example 2 is structurally unbalanced.

1) *Effect of a gauge transformation:* A partial orthant order in \mathbb{R}^n is a vector $\sigma = [\sigma_1 \dots \sigma_n]$, $\sigma_i \in \{\pm 1\}$. A *gauge transformation* is a change of orthant order in \mathbb{R}^n performed by a matrix $D = \text{diag}(\sigma)$. Denote $\mathcal{D} = \{D = \text{diag}(\sigma), \sigma = [\sigma_1 \dots \sigma_n], \sigma_i \in \{\pm 1\}\}$ the set of all gauge transformations in \mathbb{R}^n . Given the system (3), consider the change of coordinates corresponding to the gauge transformation D :

$$z = Dx, \quad D \in \mathcal{D}. \quad (4)$$

Since $D^{-1} = D$, $x = Dz$, and from (3)

$$\dot{z} = -L_D z, \quad (5)$$

where $L_D = DLD = C - DAD$ is the new Laplacian of the gauge transformed system. In components,

$$\ell_{D,ik} = \begin{cases} \sum_{j \in \text{adj}(i)} |a_{ij}| & k = i \\ -\sigma_i \sigma_k a_{ik} & k \neq i. \end{cases}$$

Proposition 2 L and L_D are isospectral: $\text{sp}(L) = \text{sp}(L_D)$. The class of gauge equivalent Laplacians $\mathcal{L}(L) = \{DLD, D \in \mathcal{D}\}$ contains at most 2^{n-1} distinct matrices.

Proof: $D \in \mathcal{D}$ is such that $|\det D| = 1$, $D^{-1} = D = D^T$. Hence the transformation $L \rightarrow DLD$ is a similarity transformation and as such it preserves the spectrum. The set \mathcal{D} contains 2^n diagonal matrices D and each corresponding

gauge transformation changes the signs of the rows/columns corresponding to the -1 entries of D . When L connected, all L_D in $\mathcal{L}(L)$ are distinct, up to a global symmetry: $DLD = (-D)L(-D)$. \square

It follows from Proposition 2 that also $\text{sp}(A) = \text{sp}(DAD)$.

Example 3 Applying the gauge transformation $D = \text{diag}(1, 1, -1)$ to A_1 of Example 1 one gets

$$A_3 = DA_1D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix},$$

i.e., a nonnegative adjacency matrix isospectral with A_1 , see Fig. 1(c). The corresponding $L_3 = \text{diag}(3, 5, 6) - A_3$ can therefore be used in (5) to solve a standard average consensus problem. In this case the right null space $\ker(L) = \text{span}(\mathbf{1})$ is the agreement subspace and, following [17], the solution of the average consensus problem is

$$z^* = \lim_{t \rightarrow \infty} z(t) = \frac{1}{n}(\mathbf{1}^T z(0))\mathbf{1}. \quad (6)$$

2) *Structural balance and bipartite consensus:* Since Example 3 is a standard consensus problem and since $\text{sp}(L_1) = \text{sp}(L_3)$, it is intuitively clear that also for Example 1 a consensus problem can be formulated and that its solution x^* must be related to z^* of Example 3. In particular, from (4), $x_i^* = z_i^*$ $i = 1, 2$, $x_3^* = -z_3^*$, i.e., $|x| = |z|$, meaning that the components of x converge to values which agree in modulus but differ in sign. This asymptotic behavior is a form of “agreed dissensus”, which we shall denote *bipartite consensus*. More formally, we have:

Definition 1 The system (3) admits a bipartite consensus solution if $\lim_{t \rightarrow \infty} |x_i(t)| = \alpha > 0 \forall i = 1, \dots, n$.

It is not too difficult to verify that no gauge transformation $D \in \mathcal{D}$ exists able to render DA_2D nonnegative. In order to understand the difference between Example 1 (and 3) and Example 2, it is useful to introduce the notion of structurally balanced signed network and its equivalence characterizations.

Definition 2 A signed graph $\mathcal{G}(A)$ is said structurally balanced if it admits a bipartition of the nodes $\mathcal{V}_1, \mathcal{V}_2$, $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that $a_{ij} \geq 0 \forall v_i, v_j \in \mathcal{V}_q$ ($q \in \{1, 2\}$), $a_{ij} \leq 0 \forall v_i \in \mathcal{V}_q, v_j \in \mathcal{V}_r, q \neq r$ ($q, r \in \{1, 2\}$). It is said structurally unbalanced otherwise.

Lemma 1 A connected signed graph $\mathcal{G}(A)$ is structurally balanced if and only if any of the following equivalent conditions holds:

- 1) all cycles of $\mathcal{G}(A)$ are positive;
- 2) $\exists D \in \mathcal{D}$ such that DAD has all nonnegative entries;
- 3) 0 is an eigenvalue of L .

Proof:

- 1) This is a classical result from [5]³;
- 2) From Definition 2, \mathcal{V} can be partitioned such that all and only the negative edges have a node in \mathcal{V}_1 and the other in \mathcal{V}_2 . It is enough to choose $D = \text{diag}(\sigma)$ with σ such that $\sigma_i = +1$ when $v_i \in \mathcal{V}_1$ and $\sigma_i = -1$ when $v_i \in \mathcal{V}_2$ to attain the sought gauge transformed adjacency matrix DAD with all nonnegative entries.
- 3) If A is structurally balanced then $\exists D \in \mathcal{D}$ such that DAD is nonnegative. Therefore the corresponding Laplacian $C - DAD$ has 0 as eigenvalue, and by Proposition 2 so does the Laplacian $L = C - A$. To prove the converse assume $\lambda_1(L) = 0$. Since A is symmetric, $\exists w \in \mathbb{R}^n$, $w \neq 0$, such that $Lw = w^T L = 0$, i.e., w is a left and right eigenvector of L . By contradiction, assume A has at least a negative cycle $\mathcal{C} = \{(v_{i_1}, v_{i_2}), \dots, (v_{i_p}, v_{i_1})\} \subseteq \mathcal{E}$ such that $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_p i_1} < 0$. From (2), the Laplacian potential $\Phi(x)$ can be split accordingly:

$$\begin{aligned} \Phi(x) &= \sum_{(v_i, v_j) \in \mathcal{C}} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j)^2 \\ &+ \sum_{(v_i, v_j) \in \mathcal{E} \setminus \mathcal{C}} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j)^2. \end{aligned} \quad (7)$$

Let us focus on the first summation. Without loss of generality, assume only one of the a_{ij} edges of \mathcal{C} has negative weight (since \mathcal{C} has an odd number of negative edges and each node intersects \mathcal{C} in at most two edges, it is always possible to find a $D \in \mathcal{D}$ such that only one negative edge is left in \mathcal{C} ; all our considerations are invariant to gauge transformations). Assume for example that $a_{i_1 i_2} > 0, \dots, a_{i_{p-1} i_p} > 0$ and $a_{i_p i_1} < 0$. From $w^T L w = 0$, owing to the sum of square form of $\Phi(x)$, each term in (7) must be 0 in correspondence of w . In particular, expanding the first summation in (7)

$$\begin{aligned} a_{i_1 i_2} (w_{i_1} - w_{i_2})^2 + \dots + a_{i_{p-1} i_p} (w_{i_{p-1}} - w_{i_p})^2 \\ + |a_{i_p i_1}| (w_{i_p} + w_{i_1})^2 = 0. \end{aligned} \quad (8)$$

From the first $p - 1$ terms of (8) we deduce $w_{i_1} = w_{i_2} = \dots = w_{i_p}$. But this implies that the last term in (8) cannot be zero unless $w_{i_1} = \dots = w_{i_p} = 0$. Consider now $\mathcal{V}_\mathcal{C} = \{v_{i_1}, v_{i_2}, \dots, v_{i_p}\}$ and its complement in \mathcal{V} : $\hat{\mathcal{V}}_\mathcal{C} = \mathcal{V} \setminus \mathcal{V}_\mathcal{C}$. Owing to the connectivity of $\mathcal{G}(A)$, it is always possible to find a collection of paths in $\mathcal{G}(A)$ linking all nodes of $\hat{\mathcal{V}}_\mathcal{C}$ to those of $\mathcal{V}_\mathcal{C}$. Let $\mathcal{P} = \{(v_{j_1}, v_{j_2}), \dots, (v_{j_q}, v_{i_k})\} \subset \mathcal{E}$ with $v_{j_1}, v_{j_2}, \dots, v_{j_q} \in \hat{\mathcal{V}}_\mathcal{C}$ and $v_{i_k} \in \mathcal{V}_\mathcal{C}$. When $x = w$ and $\Phi(w) = 0$, from (7) and $w_{i_k} = 0$ it follows that $w_{j_1} = \dots = w_{j_q} = 0$. Iterating the argument until all nodes of $\hat{\mathcal{V}}_\mathcal{C}$ are covered, we obtain $w = 0$, and hence we have a contradiction. \square

Remark 1 The key argument for the absence of the 0 eigenvalue in structurally unbalanced Laplacians is the impossibility of satisfying all the constraints imposed by $\Phi(x) = 0$ by choosing a combination of signs of the variables x_i . When such a combination of sign exists then we have structural balance.

This argument can be readily applied to spanning trees.

Corollary 1 *A spanning tree is always structurally balanced.*

Proof: When $\mathcal{G}(A)$ is a spanning tree no cycle is present, and, for each signature of the $n - 1$ edges a_{ij} , $\mathcal{G}(A)$ has n variables available in order to fulfill the condition $\Phi(x) = 0$ mentioned in Remark 1. \square

From conditions 2 and 3 of Lemma 1, it follows that on a structurally balanced graph L is positive semidefinite and $\ker(L) = \text{span}(D\mathbf{1})$. Lemma 1 induces also a characterization of structurally unbalanced graphs.

Corollary 2 *A connected signed graph $\mathcal{G}(A)$ is structurally unbalanced if and only if any of the following equivalent conditions holds:*

- 1) *one or more cycles of $\mathcal{G}(A)$ are negative;*
- 2) *$\nexists D \in \mathcal{D}$ such that DAD has all nonnegative entries;*
- 3) *$\lambda_1(L) > 0$ i.e., $\Phi(x) > 0$.*

Proof: Since structural balance and unbalance are mutually exclusive properties, the 3 conditions (and their equivalence) follow straightforwardly from Lemma 1. \square

In particular, condition 3) implies that for the structurally unbalanced case $\ker(L) = \{0\}$. This, together with Lemma 1, gives the conditions required to solve the bipartite consensus problem.

Theorem 1 *Consider a connected signed graph $\mathcal{G}(A)$. The system (3) admits a bipartite consensus solution if and only if $\mathcal{G}(A)$ is structurally balanced. If $D \in \mathcal{D}$ is the gauge transformation that renders DAD nonnegative, then the bipartite solution of (3) is $\lim_{t \rightarrow \infty} x(t) = \frac{1}{n} (\mathbf{1}^T D x(0)) D \mathbf{1}$. If instead $\mathcal{G}(A)$ is structurally unbalanced then $\lim_{t \rightarrow \infty} x(t) = 0 \forall x(0) \in \mathbb{R}^n$.*

Proof: The first part follows straightforwardly from condition 3 of Lemma 1. The second from the observation that for the gauge transformed system $z = Dx$ the problem is a usual average consensus problem on an undirected, connected graph, whose solution is (6). That such a D exists is guaranteed by condition 2 of Lemma 1. In the structurally unbalanced case, the Laplacian potential $\Phi(x)$ is positive definite, which implies the last sentence. \square

A comparison of the steady state values reached in the Examples 1-3 is shown in Fig. 2. The gauge transformation $D = \text{diag}(1, 1, -1)$ allows to pass from Example 1 (bipartite consensus) to Example 3 (standard consensus).

³This condition is often taken as definition of structural balance [5], [7]. For our purposes, the bipartition of Definition 2 is more evocative as definition.

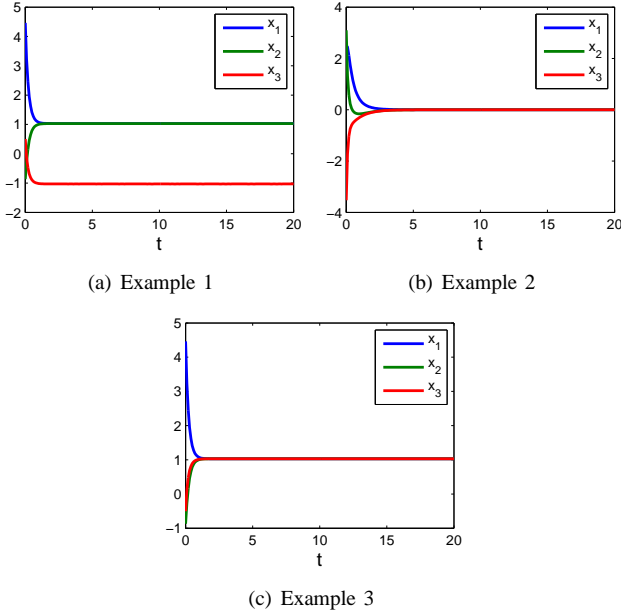


Fig. 2. Consensus time courses for the examples of Fig. 1. While in Example 1 (structurally balanced $\mathcal{G}(A)$) the 3 agents converge to bipartite consensus, in Example 2 (structurally unbalanced) all 3 agents converge to 0 (i.e., no consensus is achieved). Example 3 is the gauge transformation of Example 1 in which all 3 agents have moved to the same side of the bipartition: in this case the problem becomes a standard consensus problem on a nonnegative weighted graph.

IV. CONSENSUS PROTOCOLS FOR SIGNED DIRECTED GRAPHS

Given a digraph $\mathcal{G}(A)$, we follow the convention of the consensus literature and call (row) Laplacian of A the matrix $L = C_r - A$. When A is digon sign-symmetric then we define $A_u = (A + A^T)/2$ as symmetrized adjacency matrix of the underlying undirected graph. Notice that in general $L_u = (L + L^T)/2 = C_r - A_u$ is different from $\hat{L}_u = C_u - A_u$ where $C_u = (C_r + C_c)/2$. $L_u = \hat{L}_u$ if and only if $\mathcal{G}(A)$ is weight balanced.

Lemma 2 *A strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(A)$ is structurally balanced if and only if any of the following equivalent conditions holds:*

- 1) $\mathcal{G}(A_u)$ is structurally balanced;
- 2) all directed cycles of $\mathcal{G}(A)$ are positive;
- 3) $\exists D \in \mathcal{D}$ such that DAD has all nonnegative entries;
- 4) 0 is an eigenvalue of L .

Proof: From the digon sign-symmetry, $a_{ij}a_{ji} \geq 0$, which implies that for each entry of A_u $\text{sgn}(a_{u,ij}) = \text{sgn}(a_{ij})$ if $a_{ij} \neq 0$ and $\text{sgn}(a_{u,ij}) = \text{sgn}(a_{ji})$ if $a_{ji} \neq 0$ (or both, if $a_{ij}, a_{ji} \neq 0$). Digon sign-symmetry implies also that the signs of the semicycles are the signs of the cycles of $\mathcal{G}(A_u)$. From Proposition 1 this implies that $\mathcal{G}(A)$ cannot have any negative directed cycle and viceversa, since otherwise $\mathcal{G}(A_u)$ cannot be structurally balanced. The third implication follows consequently from Lemma 1. As for the fourth condition, one direction is obvious: if A is structurally

balanced then $\exists D \in \mathcal{D}$ such that DAD has all nonnegative entries, and we are in the usual consensus setting for nonnegative networks. To prove the converse, assume $\lambda_1(L) = 0$ is an eigenvalue of L . By construction, $\ell_{ii} = \sum_{j \neq i} |a_{ij}|$, which means that $\lambda_1(L) = 0$ is on the boundary of all the Geršgorin disks

$$\left\{ z \in \mathbb{C} \text{ s.t. } |z - \ell_{ii}| \leq \sum_{j \neq i} |a_{ij}| = \ell_{ii} \right\}. \quad (9)$$

Then from Lemma 6.2.3 of [11], since L is irreducible, it follows that the right eigenvector of 0, i.e., $w \neq 0$ for which $Lw = 0$, is such that $|w_i| = |w_j| \forall i, j = 1, \dots, n$. We also have $w^T L^T = 0$ and hence $w^T (L + L^T)w = 0$. Since $L_u = (C_r - C_c)/2 + \hat{L}_u$, then

$$\frac{1}{2} w^T (L + L^T) w = w^T \left(\frac{C_r - C_c}{2} \right) w + w^T \hat{L}_u w = 0. \quad (10)$$

For the first term of (10), denoting $\omega = |w_i| \forall i = 1, \dots, n$,

$$\begin{aligned} w^T (C_r - C_c) w &= \text{tr}(C_r - C_c) \omega^2 \\ &= \left(\sum_i \sum_{j \neq i} |a_{ij}| - \sum_i \sum_{j \neq i} |a_{ji}| \right) \omega^2 = 0 \quad \forall \omega. \end{aligned}$$

As for the second term of (10), it represents the Laplacian potential (computed in w) of an undirected graph, hence as in (2) it is in the form of a sum of squares. Assume now by contradiction that A has a negative semicycle. This implies that also A_u has to have a negative undirected cycle. The proof by contradiction now carries over from Lemma 1. \square

Notice that if $\mathcal{G}(A)$ is weight balanced then, since A is structurally balanced iff A_u is, the last statement follows also from the well-known inequality (see e.g. [11], p. 187)

$$\min \text{sp}(L_u) \leq \text{Re}(\text{sp}(L)) \leq \max \text{sp}(L_u).$$

In fact, $\min \text{sp}(L_u) = 0$ iff A is structurally balanced. If not, $\min \text{sp}(L_u) > 0$, hence $\min \text{Re}(\text{sp}(L)) > 0$.

Corollary 3 *A strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(A)$ is structurally unbalanced if and only if any of the following holds:*

- 1) $\mathcal{G}(A_u)$ is structurally unbalanced;
- 2) $\mathcal{G}(A)$ has at least one negative directed cycle;
- 3) $\nexists D \in \mathcal{D}$ rendering DAD nonnegative;
- 4) $\lambda_1(L) > 0$, i.e., $-L$ is Hurwitz.

Proof: The first three statements follow straightforwardly from Lemma 2. As for the fourth, L is diagonally dominant, hence from the Geršgorin disk theorem the eigenvalues of L are located in the union of the disks (9). Then $\text{Re}(\text{sp}(L)) \geq 0$, and, from Lemma 2, $\text{Re}(\text{sp}(L)) > 0$ if and only if A is structurally unbalanced. \square

It also follows from Lemma 2 that for strongly connected digraphs $\text{rank}(L) = n - 1$ if and only if A is structurally balanced. In particular, any acyclic digraph is structurally balanced, hence these cases can be treated analogously to

their nonnegative weight counterparts (i.e., rooted trees admit a bipartite consensus [18]).

When $\mathcal{G}(A)$ is structurally balanced, denote ν a nonzero left eigenvector of DLD normalized such that $\nu^T \mathbf{1} = 1$, where $D \in \mathcal{D}$ s.t. DAD is nonnegative. We can now state the analogous of Thm. 1 for directed graphs, whose proof is straightforward, given Lemma 2 and Corollary 3.

Theorem 2 Consider a strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(A)$. The system (3) admits a bipartite consensus solution if and only if $\mathcal{G}(A)$ is structurally balanced. In this case $\lim_{t \rightarrow \infty} x(t) = \nu^T D x(0) D \mathbf{1}$, where $D \in \mathcal{D}$ is the gauge transformation such that DAD nonnegative. When $\mathcal{G}(A)$ is weight balanced $\lim_{t \rightarrow \infty} x(t) = \frac{1}{n} (\mathbf{1}^T D x(0)) D \mathbf{1}$. If instead $\mathcal{G}(A)$ is structurally unbalanced then $\lim_{t \rightarrow \infty} x(t) = 0 \forall x(0) \in \mathbb{R}^n$.

Proof: The first part follows from Lemma 2. The second from the observation that once we gauge transform the system via $z = Dx$ we have a standard consensus problem on a nonnegative directed graph. The final part instead follows from Corollary 3. \square

From Lemma 2, the (nontrivial) bipartite consensus solution exists if and only if all directed cycles (or semicycles) of $\mathcal{G}(A)$ have positive sign, which is true if and only if $\lambda_1(L) = 0$. All these conditions are verifiable in polynomial time, meaning that verifying structural balance is an easy computational problem even in large-scale signed graphs. See [10] for an example of algorithm computing explicitly the bipartition.

Example 4 Fig. 3 shows in (a) the bipartite consensus achieved on a strongly connected structurally balanced signed digraph $\mathcal{G}(A)$ of $n = 1000$ agents. As soon as the sign is changed even on a single edge of $\mathcal{G}(A)$, structural balance is lost, and the agreement subspace becomes empty. From Theorem 2, $\lim_{t \rightarrow \infty} x(t) = 0 \forall x(0) \in \mathbb{R}^n$, although the convergence rate can be very slow, see Fig. 3(b).

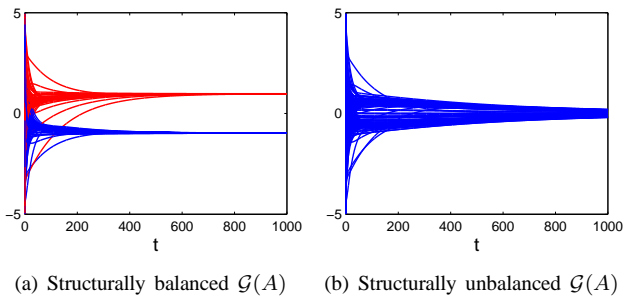


Fig. 3. Bipartite consensus on a strongly connected signed digraph with $n = 1000$. In (a) $\mathcal{G}(A)$ is structurally balanced, hence from Theorem 2 the agents split into two groups (in red and in blue). In (b) instead, a few edges have changed sign, unbalancing the graph. Bipartite consensus is now lost, and all agents converge (slowly) to 0.

V. CONCLUSION

This paper extends the notion of consensus and its distributed feedback designs to networks containing interactions which are competitive in nature, modeled as negative weights on the communication edges. In this broader scenario, the consensus reached is bipartite, i.e., the agents agree to a common (absolute) value but polarize themselves in two opposite fronts. The conditions under which a bipartite consensus is achievable are formally analogous to those that characterize monotonicity of a system. This property will be used in a forthcoming paper to obtain nonlinear distributed consensus protocols alternative to those already available in the literature.

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