

Cluster Consensus in Discrete-Time Networks of Multiagents With Inter-Cluster Nonidentical Inputs

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Abstract—In this paper, cluster consensus of multiagent systems is studied via inter-cluster nonidentical inputs. Here, we consider general graph topologies, which might be time-varying. The cluster consensus is defined by two aspects: intracluster cluster synchronization, the state at which differences between each pair of agents in the same cluster converge to zero, and inter-cluster separation, the state at which agents in different clusters are separated. For intra-cluster synchronization, the concepts and theories of consensus, including the spanning trees, scramblingness, infinite stochastic matrix product, and Hajnal inequality, are extended. As a result, it is proved that if the graph has cluster spanning trees and all vertices self-linked, then the static linear system can realize intra-cluster synchronization. For the time-varying coupling cases, it is proved that if there exists $T > 0$ such that the union graph across any T -length time interval has cluster spanning trees and all graphs has all vertices self-linked, then the time-varying linear system can also realize intra-cluster synchronization. Under the assumption of common inter-cluster influence, a sort of inter-cluster nonidentical inputs are utilized to realize inter-cluster separation, such that each agent in the same cluster receives the same inputs and agents in different clusters have different inputs. In addition, the boundedness of the infinite sum of the inputs can guarantee the boundedness of the trajectory. As an application, we employ a modified non-Bayesian social learning model to illustrate the effectiveness of our results.

Index Terms—Cluster consensus, cooperative control, linear system, multiagent system, non-Bayesian social learning.

I. INTRODUCTION

IN RECENT years, multiagent systems have obtained a broader range of applications [1]–[3]. In particular, the consensus problems of multiagent systems have attracted

increasing interest from many fields, such as physics, control engineering, and biology [4]–[6]. In a network of agents, consensus means that all agents will converge to some common state. A consensus algorithm is an interaction rule as to how agents update their states. Recently, the consensus algorithm has also been used in social learning models. Social learning focuses on the opinion dynamics in the society, which has attached a growing interests. In social learning models, individuals engage in communication with their neighbors in order to learn from their experiences. For more details, we refer readers to see [7]–[9]. A large amount of papers concerning consensus algorithms has been published [10]–[14], most of which focused on the average principle, i.e., the current state of each agent is an average of the previous states of its own and its neighbors, which is implemented by communication between agents and can be described by the following difference equations for the discrete-time cases:

$$x_i(t+1) = \sum_{j=1}^n A_{ij} x_j(t) \quad i = 1, \dots, n \quad (1)$$

where $x_i(t)$ denotes the state of agent i and $A = [A_{ij}]_{i,j=1}^n$ is a stochastic matrix. For a survey, we refer readers to [15] and the references therein.

To realize consensus, the stability of the underlying dynamical system is curial. Since the network can be regarded as a graph, the issues can be depicted by the graph theory. In the most existing literature, the concept of spanning tree is widely use to describe the communicability between agents in networks that can guarantee the consensus of (1) [16]–[18].

It is widely known that the movement or/and defaults of the agents may lead the graph topology to change through time. So, it is essential to study the stability of the consensus algorithm in a time-varying environment, which can be described by the following time-varying linear system:

$$x_i(t+1) = \sum_{j=1}^n A_{ij}(t) x_j(t) \quad i = 1, \dots, n \quad (2)$$

where each $A(t) = [A_{ij}(t)]_{i,j=1}^n$ is a stochastic matrix. Much of the literature has investigated the stability analysis of (2). Most of their results can be derived from the theories of infinite nonnegative matrix product and ergodicity of inhomogeneous Markov chain. Among them, the followings should be highlighted. In [19] and [20], Hajnal's inequality was introduced and its generalized form was proposed in [21], to describe the compression of the differences among rows in a stochastic matrix when multiplied by another stochastic

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matrix that is scrambling. In [22], it was proved that a scrambling stochastic matrix could be obtained if a certain number of stochastic matrices that have spanning trees for their corresponding graphs were multiplied. So, in most of the papers, involving stability analysis of (2), the sufficient conditions were expressed in terms of spanning trees in the union graph across time intervals of a given length [11], [18]. In addition, communication delays have been widely investigated [17], [23], [24] and nonlinear consensus algorithms have been proposed [25].

All aforementioned papers concern the complete consensus that the states of all agents converge to a common consistent state. However, this paper considered a more general phenomenon, cluster consensus. This is defined as that the agents in the network are divided into several groups (namely clusters), the agents in the same cluster synchronize, but the agents in different cluster have different state trajectories. Cluster consensus (synchronization) is considered to be more momentous in brain science [26], engineering control [27], ecological science [28], communication, engineering [29], social science [30], and distributed computation [31].

In this paper, we define the cluster consensus as follows. First, we divide the set of agents, denoted by \mathcal{V} , into disjoint clusters, $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$, with the properties:

- 1) $\mathcal{C}_p \cap \mathcal{C}_q = \emptyset$ for each $p \neq q$;
- 2) $\bigcup_{p=1}^K \mathcal{C}_p = \mathcal{V}$.

Second, letting $x(t) = [x_1(t), \dots, x_n(t)]^\top \in \mathbb{R}^n$ denote the state trajectory of all agents, of which $x_i(t)$ represents the state of $i \in \mathcal{V}$, we define cluster consensus via the following aspects:

- 1) $x(t)$ is bounded;
- 2) we say that $x(t)$ intra-cluster synchronizes if $\lim_{t \rightarrow \infty} |x_i(t) - x_{i'}(t)| = 0$ for all $i, i' \in \mathcal{C}_p$ and $p = 1, \dots, K$;
- 3) we say that $x(t)$ inter-cluster separates if $\limsup_{t \rightarrow \infty} |x_i(t) - x_j(t)| > 0$ holds for each pair of $i \in \mathcal{C}_k$ and $j \in \mathcal{C}_l$ with $k \neq l$.

We say that a system reaches cluster consensus if each solution $x(t)$ is bounded and satisfies the intra-cluster synchronization and inter-cluster separation, i.e., the items 1–3 are satisfied.

For this purpose, we introduce the following linear discrete-time system with external inter-cluster nonidentical inputs:

$$x_i(t+1) = \sum_{j=1}^n A_{ij}x_j(t) + I_i(t) \quad i \in \mathcal{C}_p, \quad p = 1, \dots, K \quad (3)$$

where $A = [A_{ij}]_{i,j=1}^n$ is a $n \times n$ stochastic matrix, $I_i(t)$ are external scalar inputs and they are different with respect to clusters, which is used to realize inter-cluster separation. Also, we consider time-varying couplings that lead the following time-varying linear system with inputs:

$$x_i(t+1) = \sum_{j=1}^n A_{ij}(t)x_j(t) + I_i(t) \quad i \in \mathcal{C}_p, \quad p = 1, \dots, K. \quad (4)$$

Related Works: Until now, much of the literature has been concerned with the global consensus. For instance, in [15] and [18], the global consensus was studied, especially for multiagent system with time-varying topologies. There are essential differences between global consensus and the cluster consensus considered this paper, which means synchronization among the same cluster but the agents in different cluster have different state trajectories. In [32]–[35], the cluster (group) consensus in networks with multiagents was addressed and [32] showed that (2) can reach cluster consensus if the graph topology is fixed and strongly connected and the number of clusters is equal to the period of agents. For continuous-time network with fixed topology, [33] proved that under certain protocol, the multiagent network can achieve group consensus by discussing the eigenvalues and eigenvectors of the Laplacian matrix. Yu and Wang [34] investigated group consensus in continuous-time network with switching topologies. However, all of these papers had a strong restriction in graph topologies and one important insight of cluster consensus: inter-cluster separation has not been deeply investigated yet. Closely relating to this paper, the authors' previous work [35] studied cluster synchronization of coupled nonlinear dynamical system and proposed several ideas, such as intra-cluster synchronization and configuration of graph topologies that cause cluster synchronization, which are shared in this paper.

Our Contributions: In this paper, we derive sufficient conditions for cluster consensus in the sense of both (3) and (4). Unlike the Lyapunov approach used in [35], in this paper, we used the algebraic theory of product of infinite matrices and graph theory to derive the main results. The enhancements in this paper, in comparison with the literature involved with (global) consensus like [15], [18] as well as the literature involved with cluster synchronization, like [35], are as follows.

- 1) We extended the concept of consensus to the cluster consensus as we mentioned above and the core concept of the algebraic graph theory, spanning tree, meaning all nodes in the graph has a common root (a node can access all other nodes in the graph), to the cluster spanning tree, as defined in Definition 1.
- 2) The main approach Hajnal inequality is extended to a cluster Hajnal inequality as Lemma 4. Accordingly, the concept of scramblingness is extended to cluster scramblingness as described in Definition 2.
- 3) We make efforts to prove inter-cluster separation, that the agents in different cluster do not converge to the same states, which is out of the scope of the existing literature, like either [15], [18] or [35].

This paper is organized as follows. In Section II, we present some graph definitions and give some notations required in this paper. In Section III, we first investigate the cluster consensus problem in discrete-time system with fixed topologies and present the cluster consensus criterion. Then we promote the criterion to the discrete-time system with switching topologies in Section IV. An application is given in Section V to verify the theoretical results. We conclude this paper in Section VI.

II. PRELIMINARIES

In this section, we first recall some necessary notations and definitions that are related to graph and matrix theories and then generalize them into the cluster sense. We also present several lemmas which will be used later. For more details about the definitions, notations and propositions about the graph and matrix, we refer readers to textbooks [36], [37].

For a matrix A , denote A_{ij} the elements of A on the i th row and j th column. If the matrix A is denoted as the result of an expression, then we denote it by $[A]_{ij}$. A^\top denotes the transpose of A . For a set S with finite elements, $\#S$ denotes the number of elements in S . E denotes the identity matrix with a proper dimension. $\mathbf{1}$ denotes the column vector with all components equal to 1 with a proper dimension. $\rho(A)$ denotes the set of eigenvalues of a square matrix A . $\|z\|$ denotes a vector norm of a vector z and $\|A\|$ denotes the matrix norm of A induced by the vector norm $\|\cdot\|$.

An $n \times n$ matrix A is called a stochastic matrix if $A_{ij} \geq 0$ for all i, j , and $\sum_{j=1}^n A_{ij} = 1$ for $i = 1, \dots, n$. A stochastic matrix A is called scrambling if for any i and j , there exists k such that both A_{ik} and A_{jk} are positive.

A directed graph \mathcal{G} consists of a vertex set $\mathcal{V} = \{1, \dots, n\}$ and a directed edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, i.e., an edge is an ordered pair of vertices in \mathcal{V} . \mathcal{N}_i denotes the neighborhood of the vertex v_i , i.e., $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. A (directed) path of length l from vertex v_i to v_j , denoted by $(v_{r_1}, v_{r_2}, \dots, v_{r_{l+1}})$, is a sequence of $l+1$ distinct vertices $v_{r_1}, \dots, v_{r_{l+1}}$ with $v_{r_1} = v_i$ and $v_{r_{l+1}} = v_j$ such that $(v_{r_k}, v_{r_{k+1}}) \in \mathcal{E}(\mathcal{G})$. The graph \mathcal{G} contains a spanning (directed) tree if there exists a vertex v_i such that for all the other vertices v_j there is a directed path from v_i to v_j , and v_i is called the root vertex. Corresponding to the matrix scramblingness, we say that \mathcal{G} is scrambling if for any pair of vertices v_i and v_j there exists a common vertex v_k such that $(v_k, v_i) \in \mathcal{E}$ and $(v_k, v_j) \in \mathcal{E}$. We say that \mathcal{G} has self-links if $(v_i, v_i) \in \mathcal{E}$ for all $v_i \in \mathcal{V}$.

An ergodicity coefficient, $\mu(\cdot)$, was proposed to measure the scramblingness of a stochastic matrix. In [19] and [20], the Hajnal diameter $\Delta(\cdot)$ was introduced to measure the difference of the rows in a stochastic matrix, and established his celebrated Hajnal's inequality $\Delta(AB) \leq (1 - \mu(A))\Delta(B)$, which indicated that the Hajnal diameter of stochastic matrix product AB strictly decreases w.r.t. B , if A is scrambling, i.e., $\mu(A) < 1$. The definitions of $\mu(\cdot)$ and $\Delta(\cdot)$ can be found in [19] and [21].

An $n \times n$ nonnegative matrix A can be associated with a directed graph $\mathcal{G}(A) = \{\mathcal{V}, \mathcal{E}\}$ in such a way that $(v_j, v_i) \in \mathcal{E}$ if and only if $A_{ij} > 0$. With this correspondence, we also say A contains a spanning tree if $\mathcal{G}(A)$ contains a spanning tree. On the other hand, for a given graph \mathcal{G}_1 , we denote $\mathcal{A}(\mathcal{G}_1) = \{A | \mathcal{G}(A) = \mathcal{G}_1\}$ the subset of stochastic matrices A such that $\mathcal{G}(A) = \mathcal{G}_1$.

For an infinite stochastic matrix sequence $\{A(t)\}_{t=1}^\infty$ with the same dimension, we use the following simplified symbol for a successive matrix product from t to s with $s > t$:

$$A_t^s \triangleq A(s)A(s-1) \dots A(t).$$

For a constant matrix A , we denote its t -th power by A^t .

Reference [22] proved that if each stochastic matrix $A(t)$ has spanning trees and self-links, then A_t^s is scrambling if $s - t > n - 1$, where n is the dimension of the matrix $A(t)$ [38].

In this paper, we consider cluster dynamics of networks. First of all, for a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we define a clustering, \mathcal{C} , as a disjoint division of the vertex set, namely, a sequence of subsets of \mathcal{V} , $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$, that satisfies: 1) $\bigcup_{p=1}^K \mathcal{C}_p = \mathcal{V}$, and 2) $\mathcal{C}_k \cap \mathcal{C}_l = \emptyset$, $k \neq l$. Thus, we are able to extend the concepts of graph and matrix mentioned above to those in the cluster case.

Definition 1: For a given clustering $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$, we say that the graph \mathcal{G} has cluster-spanning-trees with respect to (w.r.t.) \mathcal{C} if for each cluster \mathcal{C}_p , $p = 1, \dots, K$, there exists a vertex $v_p \in \mathcal{V}$ such that there exist paths in \mathcal{G} from v_p to all vertices in \mathcal{C}_p . We denoted this vertex v_p as the root of the cluster \mathcal{C}_p .

It should be emphasized that the root vertex of \mathcal{C}_p and the vertices of the paths from the root to the vertices in \mathcal{C}_p do not necessarily belong to \mathcal{C}_p . It can be seen that the root vertex of a cluster does not necessarily same with the roots of other clusters. Therefore, the definition of the cluster-spanning-tree can be regarded as a generalization of that of spanning tree we mentioned above.

Definition 2: For a given clustering $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$, we say that G is cluster-scrambling (w.r.t. \mathcal{C}) if for any pair of vertices $v_{p_1}, v_{p_2} \in \mathcal{C}_p$, there exists a vertex $v_k \in \mathcal{V}$, such that both (v_k, v_{p_1}) and (v_k, v_{p_2}) belong to \mathcal{E} .

Similarly, one can see that Definition 2 is a generalization of that of scramblingness we mentioned above. For a pair of vertices that are located in different clusters, they are not necessary to have a common linked vertex.

To measure the spanning-scramblingness, as a generalization from those in Hajnal [19] and [20], we define the cluster ergodicity coefficient (w.r.t. the clustering \mathcal{C}) of a stochastic matrix A as

$$\mu_{\mathcal{C}}(A) = \min_{p=1, \dots, K} \min_{i, j \in \mathcal{C}_p} \sum_{k=1}^n \min(A_{ik}, A_{jk}).$$

It can be seen that $\mu_{\mathcal{C}}(A) \in [0, 1]$ and A is cluster-scrambling (w.r.t. \mathcal{C}) if and only if $\mu_{\mathcal{C}}(A) > 0$.

According to the definition of cluster consensus, we extend the definition of Hajnal diameter [19]–[21] to the cluster case.

Definition 3: For a matrix A , which has row vectors A_1, A_2, \dots, A_n and a given clustering \mathcal{C} , we define the cluster Hajnal diameter as

$$\Delta_{\mathcal{C}}(A) = \max_{p=1, \dots, K} \max_{i, j \in \mathcal{C}_p} \|A_i - A_j\|$$

for some norm $\|\cdot\|$.

It can be seen that $\Delta_{\mathcal{C}}(x) \rightarrow 0$ is equivalent to the intra-cluster synchronization.

Similar to the results and the proof of [38, Th. 5.1], we can prove that the product of $n - 1$ n -dimensional stochastic matrices, all with cluster-spanning-trees, is cluster-scrambling.

Lemma 1: Suppose that each $A(t)$, $t = 1, \dots, n - 1$ is an n -dimensional stochastic matrix and has cluster-spanning-trees (w.r.t. \mathcal{C}) and self-links. Then the product A_1^{n-1} is cluster-scrambling (w.r.t. \mathcal{C}), i.e., $\mu_{\mathcal{C}}(A) > 0$.

See the proof in Appendixes.

In [15], it has been proved that if a stochastic matrix A has spanning trees and all nodes self-linked, then the power matrix A^n converges to $\mathbf{1}\alpha$ for some row vector $\alpha \in \mathbb{R}^n$. Here, we conclude that the convergence can hold even without the spanning tree condition as a direct consequence from [37].

Lemma 2: If a stochastic matrix A has positive diagonal elements, then A^n is convergent exponentially.

III. CLUSTER CONSENSUS ANALYSIS OF DISCRETE-TIME NETWORK WITH STATIC COUPLING MATRIX

A. Invariance of the Cluster Consensus Subspace

To seek sufficient conditions for cluster consensus, we first consider the situation that if the initial data $x(0) = [x_1(0), \dots, x_n(0)]^\top$ has already had the cluster synchronizing structure, namely, $x_i(0) = x_j(0)$ for all $i, j \in \mathcal{C}_p$ with $p = 1, \dots, K$, then the cluster synchronization should be kept, i.e., $x_i(t) = x_j(t)$ for all $i, j \in \mathcal{C}_p$ with $p = 1, \dots, K$ and $t \geq 0$. In other words, the following subspace in \mathbb{R}^n w.r.t. the clustering \mathcal{C} :

$$\mathbb{S}_{\mathcal{C}} = \left\{ x = [x_1, \dots, x_n]^\top \in \mathbb{R}^n : x_i = x_j \text{ for all } i, j \in \mathcal{C}_p \text{ with } p = 1, \dots, K \right\}$$

named cluster-consensus subspace, is invariant through (3).

It should be emphasized that $I_i(t)$ are different with respect to clusters, which is used to realize inter-cluster separation.

Definition 4: We say that the input $I(t)$ is intra-cluster identical if $I_i(t) = I_j(t)$ for all $i, j \in \mathcal{C}_p$ and all $p = 1, \dots, K$ and the stochastic matrix A has inter-cluster common influence if for each pair of p and p' , $\sum_{j \in \mathcal{C}_{p'}} a_{ij}$ is identical w.r.t. all $i \in \mathcal{C}_p$, in other words, $\sum_{j \in \mathcal{C}_{p'}} a_{ij}$ only depends on the cluster indices p and p' but is independent of the vertex $i \in \mathcal{C}_p$.

One can see that if two stochastic matrices A and B have an inter-cluster common influence w.r.t. the same clustering \mathcal{C} , so does the product AB . In the following, similar to what we did in [35], we have the following.

Lemma 3: If the input is intra-cluster identical and the matrix A has inter-cluster common influence, then the cluster-consensus subspace is invariant through (3).

Proof: From the condition, we define

$$\beta_{p,p'} \triangleq \sum_{j \in \mathcal{C}_{p'}} a_{ij}$$

for any $i \in \mathcal{C}_p$ and $I_p(t) \triangleq I_i(t)$ for any $i \in \mathcal{C}_p$.

Assuming that $x(t) \in \mathbb{S}_{\mathcal{C}}$, we are to prove $x(t+1) \in \mathbb{S}_{\mathcal{C}}$, too. For this purpose, let $x_p(t)$ be the identical state of the cluster p at time t . Thus, for each \mathcal{C}_p and an arbitrary vertex $i \in \mathcal{C}_p$

$$\begin{aligned} x_i(t+1) &= \sum_{p'=1}^K \sum_{j \in \mathcal{C}_{p'}} a_{ij} x_j(t) + I_i(t) \\ &= \sum_{p=1}^K \beta_{p,p'} x_p(t) + I_p(t) \end{aligned}$$

which is identical w.r.t. all $i \in \mathcal{C}_p$. By induction, this completes the proof.

B. Intra-Cluster Synchronization

We assume a special sort of intra-cluster identical input as follows:

$$I_i(t) = \alpha_p u(t) \quad (5)$$

where $u(t)$ is a scalar function and $\alpha_1, \dots, \alpha_p$ are different constants.

Similar to the Hajnal inequality given in [19]–[21], we can prove the following.

Lemma 4: Suppose that stochastic matrices A and B having the same dimension and inter-cluster common influence, then

$$\Delta_{\mathcal{C}}(AB) \leq (1 - \mu_{\mathcal{C}}(A)) \Delta_{\mathcal{C}}(B).$$

Proof: The idea of the proof is similar to that of the main result in [21]. Let

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}, \quad H = AB = \begin{bmatrix} H_1 \\ \vdots \\ H_n \end{bmatrix}$$

with $B_i = [B_{i1}, \dots, B_{in}]$ and $H_i = \sum_k a_{ik} B_k$, denoted by $[H_{i1}, \dots, H_{in}]$, for all $i = 1, \dots, n$.

For any pair of indices i and j belonging to the same cluster \mathcal{C}_{p_0} , we have

$$H_i = \sum_{p=1}^K \sum_{k \in \mathcal{C}_p} a_{ik} B_k, \quad H_j = \sum_{p=1}^K \sum_{k \in \mathcal{C}_p} a_{jk} B_k.$$

Let $d_k = \min\{a_{ik}, a_{jk}\}$. Define a set of index vector

$$W = \{w = [w_1, \dots, w_K] : w_p \in \mathcal{C}_p, p = 1, \dots, K\}.$$

For each $w \in W$, we define following convex combinations of B_1, \dots, B_n :

$$G_w = \sum_{p=1}^K \left[\sum_{k \in \mathcal{C}_p, k \neq w_p} d_k B_k + \left(\beta_{p,p_0} - \sum_{k \in \mathcal{C}_p, k \neq w_p} d_k \right) B_{w_p} \right].$$

It can be seen that both H_i and H_j are in the convex hull of G_w for all $w \in W$. Therefore

$$\|H_i - H_j\| \leq \max_{w, w' \in W} \|G_w - G_{w'}\|.$$

Combining with

$$\begin{aligned} \|G_w - G_{w'}\| &\leq \sum_{p=1}^K \left(\beta_{p,p_0} - \sum_{k \in \mathcal{C}_p} d_k \right) \|B_{w_p} - B_{w'_p}\| \\ &\leq (1 - \mu_{\mathcal{C}}(A)) \Delta(B) \end{aligned}$$

we have

$$\|H_i - H_j\| \leq (1 - \mu_{\mathcal{C}}(A)) \Delta_{\mathcal{C}}(B).$$

Therefore, $\Delta_{\mathcal{C}}(H) \leq (1 - \mu_{\mathcal{C}}(A)) \Delta_{\mathcal{C}}(B)$, which completes the proof due to the arbitrariness of $(i, j) \in \mathcal{C}_p$ and $p = 1, \dots, K$.

Remark 1: Lemma 4 indicates that if A has inter-cluster common influence, then the cluster-Hajnal diameter of Ax

decreases. In addition, if A is cluster-scrambling, $\Delta_C(Ax)$ is strictly less than $\Delta_C(x)$.

Based on the previous lemmas, we give the following result concerning intra-cluster synchronization of (3).

Theorem 1: Suppose that both $u(t)$ and $\sum_{k=1}^t u(k)$ are bounded, $I(t)$ is defined by (5), and A is a stochastic matrix with inter-cluster common influence, A has cluster-spanning trees and all positive diagonal elements. Then for any initial condition $x(0)$, (3) is bounded and can intra-cluster synchronize.

Proof: Let $x(k) = [x_1(k), \dots, x_n(k)]^\top$ be the solution of (3), then

$$x(t+1) = A^{t+1}x(0) + \sum_{k=0}^t A^{t-k}I(k) \quad (6)$$

where $I(t) = \varsigma u(t)$ with $\varsigma = [\varsigma_1, \dots, \varsigma_n]^\top$ and

$$\varsigma_i = \alpha_p, \quad i \in \mathcal{C}_p. \quad (7)$$

There is some $Y > 0$ such that $|u(t)| \leq Y$, $|\sum_{k=0}^t u(k)| \leq Y$ hold for all $t \geq 0$.

By Lemma 2, we have $A^t = A^\infty + \epsilon(t)$, where $\|\epsilon(t)\|_\infty \leq M\lambda^t$ for some $M > 0$ and $\lambda \in (0, 1)$. Therefore

$$\begin{aligned} \|x(t+1)\| &\leq \|A^{t+1}x(0)\|_\infty + \left\| \sum_{k=0}^t A^{t-k}\varsigma u(k) \right\| \\ &\leq \|x(0)\| + \|A^\infty\varsigma\| \left| \sum_{k=0}^t u(k) \right| + M \sum_{k=0}^t \lambda^{t-k} |u(k)| \\ &\leq \|x(0)\| + \|A^\infty\varsigma\| Y + MY \frac{1}{1-\lambda} \end{aligned}$$

which implies the solution of (3) is bounded.

By Lemma 1, we can find an integer N_1 such that for all $m \geq N_1$, A^m are cluster-scrambling. Denote $\eta = 1 - \mu(A^{N_1})$. For any t , let $t = pN_1 + l$ with some $0 \leq l < p$. We have

$$\Delta_C(A^{t+1}) \leq \eta^p \Delta_C(E_n)$$

which converges to zero as $t \rightarrow \infty$. In addition, since A^l has inter-cluster common influence and $\Delta_C(\varsigma) = 0$, then $\Delta_C(A^l\varsigma) = 0$ for all $l \geq 0$ can be concluded. Therefore, we have $\Delta_C(x(t+1)) \leq \Delta_C(A^{t+1}x(0))$ converges to zero as $t \rightarrow \infty$. This completes the proof.

C. Inter-Cluster Separation

Under the conditions of Theorem 1, the system can intra-cluster synchronize, namely, the states within the same cluster approach together. However, it is not known if the states in different clusters will approach together, too. A simple counter-example is that the matrix A with the inter-cluster common influence has (global) spanning trees with all diagonal elements positive and the input $\varsigma u(t)$ satisfies $\sum_{k=1}^\infty |u(k)|$ converges. In this case, $u(t)$ converges to zero and the influence of the input to the system disappears. One can see that $x(t)$ reaches a global consensus, i.e., $\lim_{t \rightarrow \infty} x(t) = \mathbf{1}\alpha$ for some scalar α .

In this section, we investigate this problem by assuming that $u(t)$ is periodic with a period T and $\sum_{k=1}^T I(k) = 0$, which

guarantees that the sum of $u(t)$ is bounded. Construct a new matrix: $B = [\beta_{p,q}]_{p,q=1}^K$, where

$$\beta_{p,q} = \sum_{j \in \mathcal{C}_q} a_{ij}, \quad i \in \mathcal{C}_p. \quad (8)$$

It can be seen that $\beta_{p,q}$ is independent of the selection of $i \in \mathcal{C}_p$.

Furthermore, we use the concept of “genericity” from the structural control theory [39]–[41] to investigate the inter-cluster separation. We define a set $\mathcal{T}(\mathcal{C}, \mathcal{G})$ w.r.t. a clustering \mathcal{C} and a graph \mathcal{G} , of which each element has form: $\{B, \tilde{\varsigma}, [u_1, \dots, u_{T-1}]\}$, where B is defined in (8) corresponding to the graph topology \mathcal{G} , $\tilde{\varsigma} \in \mathbb{R}^K$ is the vector to identify each cluster and defined as

$$\tilde{\varsigma}_p = \alpha_p, \quad p = 1, \dots, K \quad (9)$$

and $[u_1, u_2, \dots, u_{T-1}] \in \mathbb{R}^{T-1}$ such that

$$\begin{aligned} u(\theta + kT) &= u_\theta, \quad \theta = 1, \dots, T-1 \\ \text{and } u(kT) &= -\sum_{j=1}^{T-1} u_j \quad \forall k \geq 0. \end{aligned} \quad (10)$$

We can rewrite (3) as the following compact form:

$$x(t+1) = Ax(t) + \varsigma u(t). \quad (11)$$

Definition 5: We say that for a given set $\mathcal{T}(\mathcal{C}, \mathcal{G})$ as defined above, (11) is generically inter-cluster separative (or cluster consensus) if for almost every triple $\{B, \tilde{\varsigma}, [u_1, \dots, u_{T-1}]\} \in \mathcal{T}(\mathcal{C}, \mathcal{G})$ and almost all initial $x(0) \in \mathbb{R}^n$, (11) can inter-cluster separate (or cluster consensus).

Before presenting a sufficient condition for generical inter-cluster separation, we give the following simple lemma.

Lemma 5: Suppose that the stochastic matrix A has the inter-cluster common influence. Then, for any pair of cluster \mathcal{C}_1 and \mathcal{C}_2 , either there are no links from \mathcal{C}_2 to \mathcal{C}_1 ; or for each vertex $v \in \mathcal{C}_1$, there is at least one link from \mathcal{C}_2 to v .

Theorem 2: Suppose that:

- 1) every vertex in \mathcal{G} has a self-link;
- 2) \mathcal{G} satisfies the condition in Lemma 5 w.r.t \mathcal{C} ;
- 3) \mathcal{G} has cluster-spanning-trees.

Then (11) reaches cluster consensus generically with respect to the set $\mathcal{T}(\mathcal{C}, \mathcal{G})$. In addition, the limiting consensus trajectories are periodic, that is, there exist some scalar periodic trajectories $v_p(t)$ with the period T for each cluster \mathcal{C}_p , $p = 1, \dots, K$, such that $\lim_{t \rightarrow \infty} |x_j(t) - v_p(t)| = 0$ if $j \in \mathcal{C}_p$.

Proof: We first prove the asymptotic periodicity. Recall

$$x(t+1) = A^{t+1}x(0) + \sum_{k=0}^t A^k \varsigma u(t-k). \quad (12)$$

By Lemma 2, one can see that A^n exponentially converges to A^∞ . Thus, we can find $M > 0$ and $\lambda \in (0, 1)$ such that $\|A^t - A^\infty\| \leq M\lambda^t$. Let $Y = \max_{k=1, \dots, T} |u(k)|$. Thus, we have

$$\begin{aligned} \|x(t+lT+1) - x(t+1)\| &\leq \|(A^{t+lT+1} - A^{t+1})x(0)\| \\ &\quad + \left\| \sum_{k=0}^t A^k \varsigma [u(t+lT-k) - u(t-k)] \right\| \end{aligned}$$

$$\begin{aligned}
& + \left\| \sum_{k=t+1}^{t+lT} A^k \varsigma u(t+lT-k) \right\| \\
& = \left\| (A^{t+lT+1} - A^{t+1})x(0) \right\| + \left\| \sum_{k=t+1}^{t+lT} A^\infty \varsigma u(t-k) \right\| \\
& + \left\| \sum_{k=t+1}^{t+lT} [A^k - A^\infty] \varsigma u(t-k) \right\| \\
& \leq 2M\lambda^t \|x(0)\| + MY\|\varsigma\| \sum_{k=t}^{t+lT} \lambda^k \\
& = \left[2M\|x(0)\| + MY\|\varsigma\| \frac{1}{1-\lambda} \right] \lambda^t
\end{aligned}$$

for all l . Letting $t = mT + \theta - 1$ for any $m \in \mathbb{N}$ and $\theta = 1, \dots, T$, we have

$$\|x((m+l)T + \theta) - x(mT + \theta)\| \leq M_1 \lambda^{mT}$$

for some $M_1 > 0$. According to the Cauchy convergence principle, each $x(\theta + kT)$, $\theta = 1, \dots, T$, converges to some value as $k \rightarrow \infty$ exponentially, which implies that there exist T -periodic functions $v_p(t)$, $p = 1, \dots, K$, such that $|x_j(t) - v_p(t)| \rightarrow 0$ exponentially, if $j \in \mathcal{C}_p$.

Now, we will prove the consensus states in different clusters are different generically.

Since each cluster synchronizes, we can pick a single vertex state from each cluster to represent the whole state of this cluster. We can divide the space \mathbb{R}^n into the direct sum of two subspaces: $\mathbb{R}^n = V_1 \oplus V_2$, where V_1 denotes the right eigenspace of A corresponding to the eigenvalue 1 and V_2 denotes the right eigenspace of A corresponding to all other eigenvalues. Since all diagonal elements of A are positive, then the direct sum works and $AV_i \subset V_i$ holds for $i = 1, 2$. In addition, since the column vectors in A^∞ belong to \mathcal{S}_C , $V_1 \subset \mathcal{S}_C$. So, $\mathbb{R}^n = \mathcal{S}_C + V_2$.

For any initial data $x(0) \in \mathbb{R}^n$, we can find $y^0 \in \mathcal{S}_C$ with the decomposition $x(0) = y^0 + x(0) - y^0$ such that $x(0) - y^0 \in V_2$. Consider the following system restricted to \mathcal{S}_C :

$$y(t+1) = Ay(t) + I(t) \quad y(0) = y_0$$

where $y(t) \in \mathcal{S}_C$ for all t .

Let $\delta x(0) = x(0) - y^0 \in V_2$. We have

$$\delta x(t+1) = A\delta x(t) \quad \delta x(0) = x(0) - y^0$$

which implies that $\lim_{t \rightarrow \infty} \delta x(t) = 0$, that is, $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t)$. Therefore, we only need to discuss $y(t) \in \mathcal{S}_C$.

Since each component of $y(t)$ in the same cluster is identical, we can pick a single component from each cluster to lower dimensional column vector $\tilde{y} \in \mathbb{R}^K$ with $\tilde{y}_p = y_i$ for some $i \in \mathcal{C}_p$. Because of the inter-cluster common influence condition, we have

$$\tilde{y}(t+1) = B\tilde{y}(t) + \tilde{\varsigma}u(t) \quad (13)$$

where B is defined in (8) and $\tilde{\varsigma} = [\alpha_1, \dots, \alpha_K]^\top$. The inter-cluster separation is equivalent to investigate the separation among components of $\tilde{y}(t)$. One can see that for almost every B , B has K distinguishing left eigenvectors, denoted by

ϕ_1, \dots, ϕ_K , corresponding to eigenvalues v_1, \dots, v_K (possibly overlapping). So, for almost every B with K left eigenvectors, let us write down the solution (13) at time nT as follows:

$$\begin{aligned}
\tilde{y}(nT+1) &= B^{nT+1}\tilde{y}(0) + \sum_{k=0}^{nT} B^{nT-k}\tilde{\varsigma}u(k) \\
&\rightarrow Z_1\tilde{y}(0) + Z_2\tilde{\varsigma}, \text{ as } n \rightarrow \infty
\end{aligned}$$

where

$$Z_1 = \lim_{n \rightarrow \infty} B^{nT+1}, \quad Z_2 = \lim_{n \rightarrow \infty} \left[\sum_{k=0}^{nT} B^{nT-k}u(k) \right].$$

From Lemma 2, Z_1 does exist. Combined with $\sum_{k=0}^{nT-1} u(k) = 0$, we can conclude that Z_2 exists, too.

For an arbitrary fixed pair of (p, q) , with $p, q = 1, \dots, K$ and $p \neq q$, we are to show Z_2 can generically have different p -th and q -th components. In fact, for each k_1 with $|v_{k_1}| < 1$, noting that its associated left eigenvector is ϕ_{k_1} , we have

$$\begin{aligned}
\phi_{k_1} \sum_{k=0}^{nT} B^{nT-k}u(k) &= \phi_{k_1} \sum_{k=0}^{T-1} u(k) \sum_{z=0}^{n-1} v_{k_1}^{zT+k} + \phi_{k_1} v_{k_1}^{nT} u(0) \\
&\rightarrow \phi_{k_1} \frac{\sum_{k=0}^{T-1} u(k)v_{k_1}^k}{1 - v_{k_1}^T}, \text{ as } n \rightarrow \infty.
\end{aligned}$$

For each k_2 with $|v_{k_2}| = 1$, noting its associated left-eigenvector is ϕ_{k_2} , according to the fact that all diagonal elements in B are positive, from [37], we have $v_{k_2} = 1$ indeed. Then, we have

$$\phi_{k_2} \sum_{k=0}^{nT} B^{nT-k}u(k) = \phi_{k_2} \sum_{k=0}^{nT} u(k) = \phi_{k_2} u(0) = \phi_{k_2} u_T.$$

So, for almost $[u_1, \dots, u_{T-1}] \in \mathbb{R}^{T-1}$, the eigenvectors of Z_2 are the same with B and the corresponding eigenvalues are u_T and $\sum_{k=0}^{T-1} u(k)v_p^k/(1 - v_p^T)$. For almost every realization of $[u_i]_{i=1}^{T-1}$ and B , none of them is zero, which implies that Z_2 is nonsingular. That means it is impossible for each pair of its rows to be identical. So, for almost all $\tilde{\varsigma}$, the p -th and q -th component of Z_2 are not identical. Equivalently, for almost every $\tilde{\varsigma}$, $Z_2\tilde{\varsigma}$ has no pair of components identical. Therefore, we conclude that for almost every x_0 , associated with almost every $\tilde{y}(0)$, each pair of components in $Z_1\tilde{y}(0) + Z_2\tilde{\varsigma}$ are not identical.

We can arbitrarily select the cluster pair (p, q) and the exception cases of the statements above are within a set of $\mathcal{T}(\mathcal{G}, \mathcal{C})$ with Lebesgue measure zero. Since any finite union of sets with Lebesgue measure zeros still has Lebesgue measure zero, we conclude that $\lim_{n \rightarrow \infty} \tilde{y}(nT+1)$ has no identical components generically, which implies that the states of any two clusters in $\lim_{n \rightarrow \infty} y(nT+1)$ are not identical generically. This completes the proof.

Remark 2: In this paper, we make efforts to prove the inter-cluster separation rigorously; however, in [35], the inter-cluster separation was not touched (but only assumed). We argue that for general nonlinear coupled system (models in [35]), proving the inter-cluster separation is very difficult, if not impossible.

IV. CLUSTER-CONSENSUS IN DISCRETE-TIME NETWORK WITH SWITCHING TOPOLOGIES

In this section, we study the cluster-consensus in network with switching topologies described as the following time-varying linear system:

$$x_i(t+1) = \sum_{j=1}^N A_{ij}(t)x_j(t) + I_i(t) \quad \forall i \in \mathcal{C}_p, p = 1, \dots, K \quad (14)$$

where $A(t)$ is associated with a graph from the graph set $\Upsilon = \{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ w.r.t. a given clustering \mathcal{C} , each of which satisfy the property \mathcal{A} : for each pair p and q of cluster indices:

- 1) there are no links from \mathcal{C}_q to \mathcal{C}_p in each graph \mathcal{G}_l , $l = 1, \dots, m$;
- 2) or for each vertex $v \in \mathcal{C}_p$ and each graph \mathcal{G}_l , $l = 1, \dots, m$, there is at least one link from \mathcal{C}_q to it.

For the matrix sequence $A(t)$, we have the following assumptions.

- \mathcal{B}_1 : There is a positive constant $e > 0$ such that for each pair i, j and t , either $A_{ij}(t) = 0$ or $A_{ij} \geq e$ holds.
- \mathcal{B}_2 : $A_{ii}(t) \geq e$ holds for all $i = 1, \dots, n$ and $t \geq 0$.
- \mathcal{B}_3 : (Inter-cluster common influence.) There exists a $\mathbb{R}^{n,n}$ stochastic matrix $B(t) = [b_{p,q}(t)]_{p,q=1}^K$, possibly depending on time, such that

$$\sum_{j \in \mathcal{C}_q} A_{ij}(t) = b_{p,q}(t) \quad (15)$$

holds for all $i \in \mathcal{C}_p$ and each $p, q = 1, \dots, K$.

- \mathcal{B}_3^* : (Static inter-cluster common influence.) There exists a constant $\mathbb{R}^{n,n}$ stochastic matrix $B = [b_{p,q}]_{p,q=1}^K$, such that

$$\sum_{j \in \mathcal{C}_q} A_{ij}(t) = b_{p,q} \quad (16)$$

holds for all $i \in \mathcal{C}_p$ and each $p, q = 1, \dots, K$.

In other words, we define a graph set containing all possible graph induced by the matrix sequence $A(t)$. The graph set satisfies the property in Lemma 5 uniformly and each graph in the set either never occurs in the corresponding graph sequence induced by $A(t)$ or occurs frequently.

Then, we are in the position to give a sufficient condition for the cluster synchronization.

Theorem 3: Suppose that \mathcal{A} , \mathcal{B}_1 , \mathcal{B}_2 , and \mathcal{B}_3 hold. If there exists an integer $L > 0$ such that for any L -length time interval $[t, t+L)$, the union graph $\mathcal{G}[\sum_{i=t}^{t+L-1} A(i)]$ has cluster-spanning-trees, then (14) cluster synchronizes.

Proof: The solution of (14) is

$$x(t+1) = A(t)x(t) + \varsigma u(t) = A_0^k x(0) + \sum_{k=0}^t A_{k+1}^t u(k) \varsigma.$$

Noting that the diagonal elements of each $A(t)$ are positive, we can see that the graph $\mathcal{G}(A_t^{t+L-1})$ contains all links in the union graph $\mathcal{G}(\sum_{k=t}^{t+L-1} A(k))$ and hence has cluster-spanning-trees and positive diagonal elements for all t . By Lemma 1, we can conclude that there is an integer N such that the graph

$\mathcal{G}(A_t^{t+NL-1})$ is scrambling for all $t \geq 0$. Since the nonzero elements in each $A(t)$ are greater than some constant $e > 0$, there is some $\delta > 0$ such that

$$\inf_t \mu_{\mathcal{C}}(A_t^{t+NL-1}) \geq \delta.$$

Hence, for each t , we have

$$\Delta_{\mathcal{C}}(A_0^t x(0)) \leq (1 - \delta)^{\lfloor \frac{t}{NL} \rfloor} \Delta_{\mathcal{C}}(x(0))$$

which converges to zero as $t \rightarrow \infty$. Here, $\lfloor \cdot \rfloor$ denotes the floor function. Therefore, $\lim_{t \rightarrow \infty} \Delta_{\mathcal{C}}(A_0^t x(0)) = 0$.

Combining with the fact that $\Delta_{\mathcal{C}}(A_t^s \varsigma) = 0$ holds for all $s \geq t$ and ς , we can conclude that (14) intra-cluster synchronizes.

Remark 3: Due to the difference of the techniques used in [35] and this paper, the result of Theorem 3 is impossible to extend to general coupled nonlinear system, as the models in [35], because a Lyapunov function for time-varying coupled systems is in general unable to be found.

The inter-cluster separation can be derived by the same fashion of Theorem 2.

Theorem 4: Suppose that \mathcal{A} , \mathcal{B}_1 , \mathcal{B}_2 , and \mathcal{B}_3^* hold. If there exists an integer $L > 0$ such that for any L -length time interval $[t, t+L)$, the union graph $\mathcal{G}[\sum_{i=t}^{t+L-1} A(i)]$ has cluster spanning trees. If the input $u(t)$ and $\sum_{k=0}^t u(k)$ are both bounded, then for any initial data $x(0)$, the solution of (14) is bounded. In addition, if the input $u(t)$ is periodic with a period T and satisfies $\sum_{k=1}^{T-1} u(k) = 0$, (14) reaches cluster consensus generically and each trajectory converges to a T -periodic one.

Proof: To prove the boundedness, we are to find a solution of (14) that stays at $\mathcal{S}_{\mathcal{C}}$ and is the limiting of $x(t)$. Similar to the proof of Theorem 2, we can represent the limiting trajectory by a lower dimensional linear system (13). The \mathcal{B}_3^* implies that this linear lower dimensional system is static. So, we can prove its boundedness by the same way of the proof of Theorem 1.

Define the Lyapunov exponent of the matrix sequence $A(t)$ as follows:

$$\lambda(v) = \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \log \left(\|A_0^t v\| \right).$$

From the Pesin's theory [42], the Lyapunov exponents can only pick finite values and provide a splitting of \mathbb{R}^n . Namely, there is a subspace direct-sum division

$$\mathbb{R}^n = \oplus_{j=1}^J V_j$$

and $\lambda_1 > \dots > \lambda_J$, possibly $J < n$, such that for each $v \in V_j$, $\lambda(v) = \lambda_j$. It can be seen that $\lambda_1 = 0$ since $A(t)$, $t \geq 0$, are all stochastic matrices. Let $V = \oplus_{j>1} V_j$. We claim

Claim 1: $\mathbb{R}^n = \mathcal{S}_{\mathcal{C}} + V$.

We prove this claim in appendix. Therefore, for any $x(0) \in \mathbb{R}^n$, we can find a vector $y_0 \in \mathcal{S}_{\mathcal{C}}$ such that $x(0) - y_0 \in V$. Define a linear system

$$y(t+1) = A(t)y(t) + \varsigma u(t) \quad y(0) = y_0. \quad (17)$$

Then, letting $\delta x(t) = x(t) - y(t)$, it should satisfy

$$\delta x(t+1) = A(t)\delta x(t) \quad \delta x(0) = y(0) - x(0) \in V.$$

Since $\delta x(0) \in V$, $\lambda(\delta x(0)) < 0$. This implies $\lim_{t \rightarrow \infty} \delta x(t) = 0$. So, $\lim_{t \rightarrow \infty} [x(t) - y(t)] = 0$. We can rewrite the (17) as a lower dimensional linear system

$$\tilde{y}(t+1) = B\tilde{y}(t) + \tilde{\zeta}u(t) \quad (18)$$

which is same with (13). The \mathcal{B}_3^* guarantees that the matrix B is static. So, the proof of boundedness of $\tilde{y}(t)$ is an overlap of that of Theorem 1.

In addition, since B is static, then the inter-cluster separation can be proved as an overlap of that of Theorem 1. Therefore, we can conclude that $x(t)$ is bounded, too. This completes the proof.

Remark 4: In [32], the sufficient condition to guarantee cluster consensus is when the number of clusters is equal to the period of agents. The period of agent i is the greatest common divisor of the lengths of paths starting from agent i to itself. To apply the results in [32], the period of all agents should be no less than 2. In this paper, we assume the existence of self-links, which means the period of every agent is 1. So, the results in [32] cannot be employed in our situation.

V. NUMERICAL EXAMPLES

Cluster consensus is a new issue in the coordination control. Although much of the literature has been concerned with complete consensus, some works have been concerned with cluster consensus. Moreover, all of them cannot handle the scenario in this paper. For example, [33] and [34] investigated group consensus in continuous-time network with fixed and switching topologies, respectively. Instead, in this paper, we study the discrete-time network. Even though Chen *et al.* [32] investigated the cluster consensus in discrete-time network, it was concluded that cluster consensus can be achieved if the graph topology is fixed and strongly connected and the number of clusters equals to the period of agents. Hence, the period of agents should be larger than 1. But in this paper, the assumption that each agent has self-link means that the period of agents in our algorithm is 1. For these reasons, their results can hardly be applied to our case.

In this section, we provide an application example by a modified non-Bayesian social learning model. Social learning can be described as the process by which individuals infer information about some alternative by observing the choices of others. In [8], a new social learning model was proposed, by which an individual updates his/her belief as a convex combination of the Bayesian posterior beliefs based on its private signal and the beliefs of its neighbors at the previous time. In details, let $\Theta = \{\theta_1, \dots, \theta_m\}$ denote a finite set of possible states of the world and $\mu_{i,t}(\theta)$ denote the probability (belief in their terminology) of individual i about state $\theta \in \Theta$ at time t . Conditional on the state θ , a signal vector $\omega_t = (\omega_{1,t}, \dots, \omega_{n,t}) \in S_1 \times \dots \times S_n$ is generated by the likelihood function $l(\cdot|\theta)$, where signal $\omega_{i,t}$ is the signal privately observed by agent i at period t and S_i denotes the signal space of agent i . $l_i(\cdot|\theta)$ is the i -th marginal of $l(\cdot|\theta)$. It is assumed that every agent i knows this conditional likelihood function. The one-step-ahead forecast of agent i at time t is given by $m_{i,t}(\omega_{i,t+1}) = \sum_{\theta \in \Theta} l_i(\omega_{i,t+1}|\theta)\mu_{i,t}(\theta)$. The

k -step-ahead forecast of agent i at time t is similarly given by $m_{i,t}(\omega_{i,t+1}, \dots, \omega_{i,t+k}) = \sum_{\theta \in \Theta} (\prod_{r=1}^k l_i(\omega_{i,t+r}|\theta))\mu_{i,t}(\theta)$. Then, the belief updating rule can be written as

$$\mu_{i,t+1}(\theta) = a_{ii}\mu_{i,t}(\theta) \frac{l_i(\omega_{i,t+1}|\theta)}{m_{i,t}(\omega_{i,t+1})} + \sum_{j \in \mathcal{N}_i} a_{ij}\mu_{j,t}(\theta). \quad (19)$$

Reference [8] considered the case that each agent may face an identification problem in the sense that agent may not be able to distinguish between two states. Observational equivalence is used to reflect the identification problem. Two states are observationally equivalent from the point of view of agent i , if the conditional distributions of agent i 's signals under the two states coincide. As proved in [8], all beliefs asymptotically coincide by this algorithm. This confirms the facts that the interaction among individuals can eliminate the initial difference among them and converge to an agreement.

For any state θ , (19) can be rewritten in matrix form

$$\mu_{t+1}(\theta) = A\mu_t(\theta) + e_t(\theta) \quad (20)$$

here, $e_t(\theta) = (e_{1,t}(\theta), \dots, e_{n,t}(\theta))^T$ and $e_{i,t}(\theta) = a_{ii}\mu_{i,t}(\theta)((l_i(\omega_{i,t+1}|\theta)/m_{i,t}(\omega_{i,t+1})) - 1)$. For state $\hat{\theta}$ that is observationally equivalent to θ^* , the one-step-ahead forecasts and k -step-ahead forecasts, respectively, satisfy

$$m_{i,t}(\omega_{i,t+1}) \rightarrow l_i(\omega_{i,t+1}|\hat{\theta}) \quad t \rightarrow \infty$$

and

$$m_{i,t}(\omega_{i,t+1}, \dots, \omega_{i,t+k}) \rightarrow \prod_{r=1}^k l_i(\omega_{i,t+r}|\hat{\theta}) \quad t \rightarrow \infty \quad (21)$$

Therefore, $e_{i,t}(\hat{\theta})$ converges to zero almost surely as time goes on. Then from matrix and probability theories, the existence of $\lim_{t \rightarrow \infty} \mu_{i,t}(\hat{\theta})$ can be obtained. For state θ that is not observationally equivalent to θ^* , there exist a positive integer \hat{k}_i , a sequence of signals $(\hat{s}_{i,1}, \dots, \hat{s}_{i,\hat{k}_i})$ and constant $\delta_i \in (0, 1)$ such that $\prod_{r=1}^{\hat{k}_i} (l_i(\hat{s}_{i,r}|\theta)l_i(\hat{s}_{i,r}|\theta^*)) \leq \delta_i$, combining with the k -step-ahead forecast (21), $\mu_{i,t}(\theta) \rightarrow 0$ a.s. can be obtained.

Here, we assume that all states $\theta_j \in \Theta$ are observationally equivalent for all individuals. Under this assumption, $(l_i(\omega_{i,t+1}|\theta_j)m_{i,t}(\omega_{i,t+1})) = 1$ always are true. This implies that the signals observed have no effect in this situation, thus we remove the conditional likelihood term in (19). In addition, we consider that the belief of each individual is affected by different religious beliefs or cultural backgrounds. This affection flags the sub-group that each individual belongs to. Consider the group with nine individuals that are divided into three groups: $\mathcal{C}_1 = \{1, 2, 3\}$, $\mathcal{C}_2 = \{4, 5, 6\}$, and $\mathcal{C}_3 = \{7, 8, 9\}$. We denote auxiliary terms, $I_i(t)$, as the external inputs to the learning model (19), in order to denote the influence of the religious beliefs and/or cultural backgrounds and they are different with respect to sub-groups (cluster). These terms are regarded as the flags that distinguish the different sub-groups (clusters). Hence, the dynamic model (19) becomes

$$\mu_{i,t+1}(\theta) = a_{ii}\mu_{i,t}(\theta) + \sum_{j \in \mathcal{N}_i} a_{ij}\mu_{j,t}(\theta) + I_i(t) \quad (22)$$

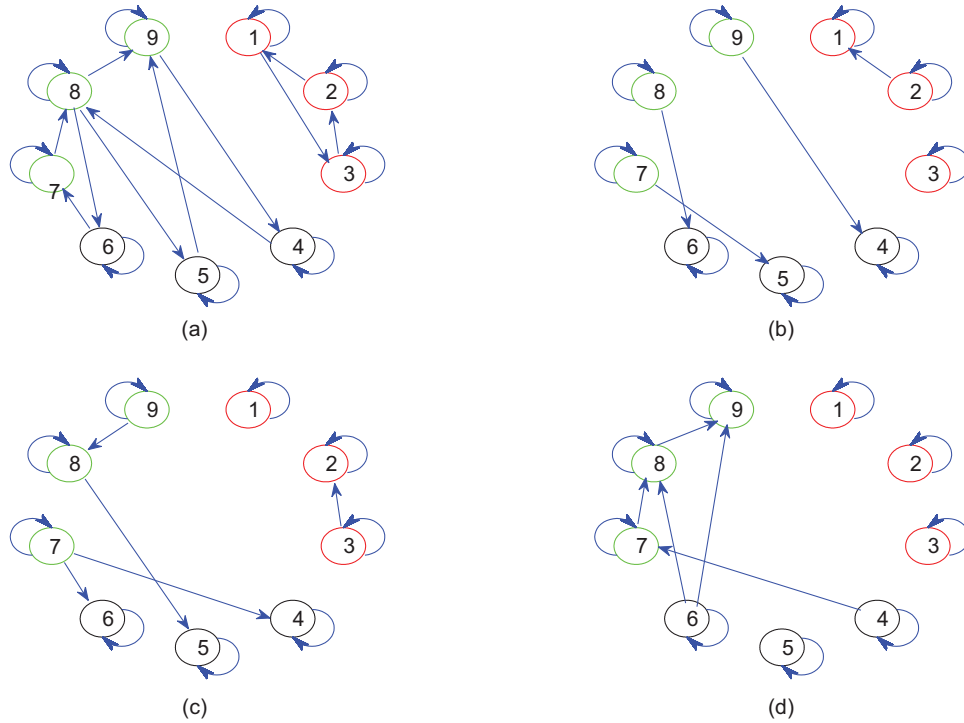


Fig. 1. Each of these graphs has self-links. (a) Example A simulates the network with fixed topology. (b)–(d) Example B simulates the network with topologies switching.

with the cultural/religious terms

$$I_i(t) = cu(t)\sigma_k(\theta) \quad i \in \mathcal{C}_k, \quad k = 1, 2, 3$$

where c denotes the influence strength. To guarantee $\mu_{i,t}(\theta) \in [0, 1]$, we assume the inter-cluster nonidentical input $u(t)$ is periodic with a period $T = 2$ and $u_k + u_{k+1} = 0$. For every i and t , to guarantee $\mu_{i,t}(\theta_1) + \mu_{i,t}(\theta_2) = 1$, we demand $\sigma_i(\theta_1) + \sigma_i(\theta_2) = 0$. It can be seen that the modified social learning model (22) is a special case of the model (3).

To illustrate the availability of our results, we consider the state space has two states: $\Theta = \{\theta_1, \theta_2\}$. The coupling matrix $A = [a_{ij}]$ satisfies the inter-cluster influence condition, and suppose $\{k | \mathcal{N}_i \cap \mathcal{C}_k \neq \emptyset\}$ is identical to all $i \in \mathcal{C}_p$, $p = 1, 2, 3$. Denote d_{iq} the number of agents in set $\mathcal{N}_i \cap \mathcal{C}_q$ and for $q \in \{k | \mathcal{N}_i \cap \mathcal{C}_k \neq \emptyset\}$, $j \in \mathcal{N}_i \cap \mathcal{C}_q$, take $a_{ij} = \beta_{pq}/d_{iq}$. For any p and any $q \in \{k | \mathcal{N}_i \cap \mathcal{C}_k \neq \emptyset\}$, $\sum_{j \in \mathcal{C}_q} (\beta_{p,q}/d_{iq}) = \sum_{j \in \mathcal{C}_q} (\beta_{p,q}/d_{i'q}) = \beta_{p,q}$ always holds for $\forall i, i' \in \mathcal{C}_p$, i.e., the coupling matrix in (22) has the common inter-cluster influence. We use $B = [\beta_{pq}]_{p,q=1}^3$ to inflect the inter-cluster influence among clusters, and choose $u(2l) = -u(2l+1) = 1$, for all $l \in \mathbb{N}$.

A. Static Topology

In this example, the graph is depicted in Fig. 1(a). We take the matrix B as

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

and can see that the graph has cluster spanning trees and the roots of groups $\mathcal{C}_{1,2,3}$ are 3, 7, and 7, respectively. Therefore,

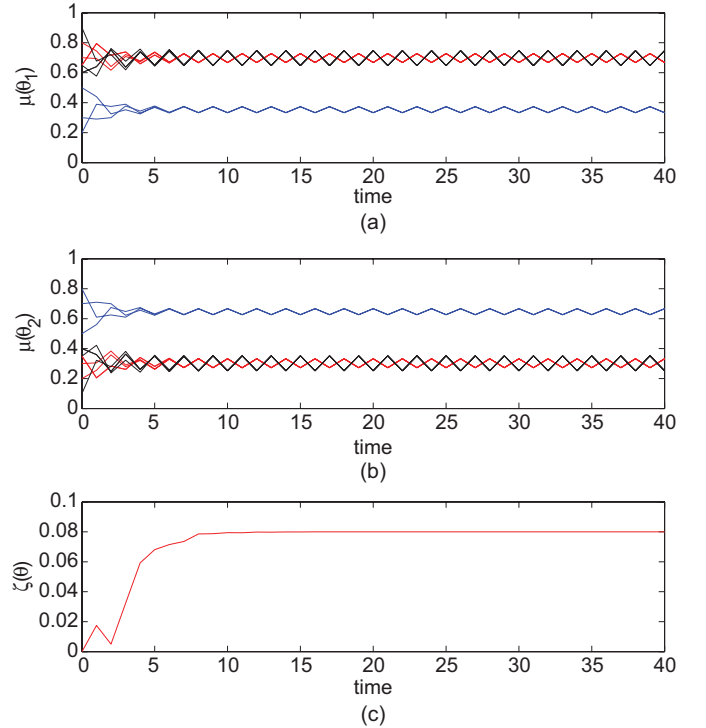


Fig. 2. Dynamical behavior of beliefs (a) $\mu(\theta_1)$, (b) $\mu(\theta_2)$, and (c) $\zeta(\theta)$ with respect to example A with randomly picked initial values. In (a) and (b), blue, red, and black curves show the dynamical behaviors of individuals in group $\mathcal{C}_1, \mathcal{C}_2$, and \mathcal{C}_3 , respectively.

all conditions in Theorem 1 hold. Then (22) reaches cluster consensus generically. The dynamical behaviors of the beliefs $\mu_{i,t}(\theta_j)$, $i = 1, \dots, 9$, $j = 1, 2$ are shown in Fig. 2(a) and (b).

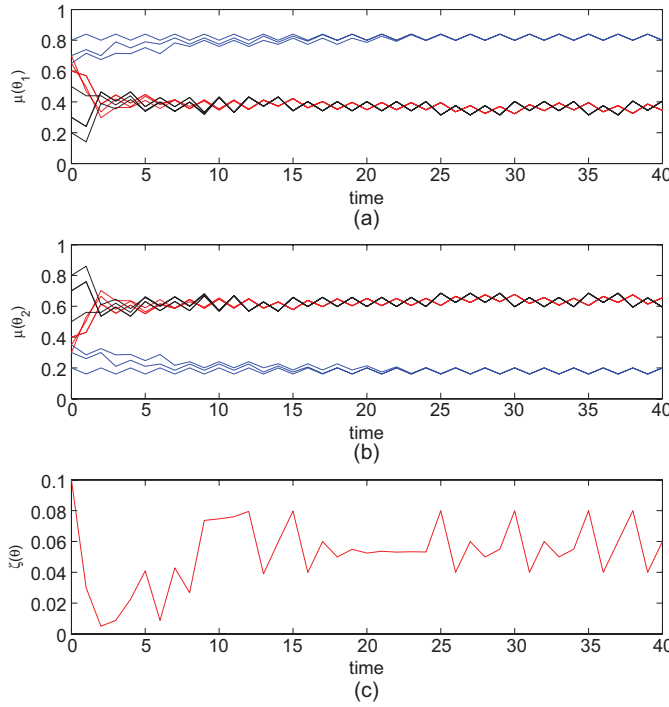


Fig. 3. Dynamical behavior of states (a) $\mu(\theta_1)$, (b) $\mu(\theta_2)$, and (c) $\zeta(\theta)$ with respect to example B with randomly picked initial values. In (a) and (b), blue, red, and black curves show the dynamical behaviors of individuals in group C_1 , C_2 , and C_3 , respectively.

It is clear that they are asymptotically convergent, which means different groups of individuals can realize intra-cluster synchronization. In Fig. 2(c), the dynamical behaviors of $\zeta(\theta_j) = |\mu_{C_2}(\theta_j) - \mu_{C_3}(\theta_j)|$, $j = 1$ are plotted and it does not converge to zero, which means that although groups C_2 and C_3 are strongly connected, the influence of different religious beliefs or cultural backgrounds still cannot be ignored.

B. Switching Topologies

In this example, the graph topology is switching among the topologies given in Fig. 1(b)–(d) periodically. Noting that none of these graphs has cluster spanning trees, i.e., the condition in Theorem 1 does not hold. However, the union graph of those in Fig. 1(b)–(d) has cluster spanning trees and the roots of groups $C_{1,2,3}$ are agents 3, 7, and 7, respectively. We pick an identical matrix B w.r.t. the clustering for the three graphs as

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Hence, all assumptions in Theorem 4 hold. Therefore, (22) with switching topologies can achieve cluster consensus. The dynamical behaviors of beliefs $\mu_{i,t}(\theta_j)$, $1 \leq i \leq 9$ are shown in Fig. 3(a) and (b), the dynamics of $\zeta(\theta_j) = |\mu_{C_2}(\theta_j) - \mu_{C_3}(\theta_j)|$, $j = 1$ is plotted in Fig. 3(c), respectively. All of them show that the cluster consensus is perfectly reached and $\mu_{i,t}(\theta_j)$, $1 \leq i \leq 9$ is convergent.

Now, to better illustrate the role of the inter-cluster non-identical inputs, we give a simulation based on (22) without inputs, see Fig. 4. The dynamical behaviors of beliefs

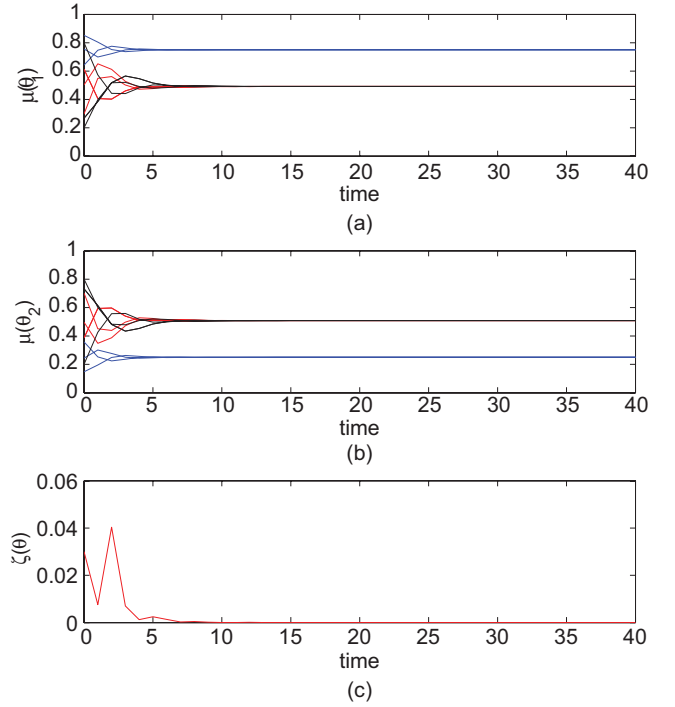


Fig. 4. Dynamical behavior of states (a) $\mu(\theta_1)$, (b) $\mu(\theta_2)$, and (c) $\zeta(\theta)$ with respect to example A with randomly picked initial values. In (a) and (b), blue, red, and black curves show the dynamical behaviors of individuals in group C_1 , C_2 , and C_3 , respectively.

$\mu_{i,t}(\theta_j)$, $i = 1, \dots, 9$, $j = 1, 2$ are shown in Fig. 4(a) and (b). In Fig. 4(c), the dynamical behavior of $\zeta(\theta_j) = |\mu_{C_2}(\theta_j) - \mu_{C_3}(\theta_j)|$ is plotted, which means the groups C_2 and C_3 cannot separate. Compare with Fig. 2(c), we can see that the inter-cluster nonidentical inputs play key roles in separating different groups.

VI. CONCLUSION

The idea for studying consensus of multiagent systems sheds light on cluster consensus analysis. In this paper, we studied cluster consensus of multiagent systems via inter-cluster nonidentical inputs. We derived the criteria for cluster consensus in both discrete-time systems with fixed or switching graph topologies. The difference between clustered states were guaranteed by the different inputs to different clusters. We posited that every cluster in the graph corresponding to the system has a spanning tree, then the multiagent system reaches cluster consensus. The analysis was presented rigorously based on algebraic graph theory and matrix theory. We used a modified non-Bayesian social learning model to illustrate our theoretical results. In this model, the beliefs of individuals were described as the probability for the states and updated by an interacted algorithms. We added an auxiliary term to flag the difference of culture and/or region of different group of individuals. The numerical results showed that the social learning algorithm can guarantee that the beliefs of individuals in the same cluster converge but the difference between any pair of groups, owing to the auxiliary external input terms, permanently exists that cannot be eliminated by the interactions.

APPENDIX
PROOF OF LEMMA 1

For each cluster \mathcal{C}_p and each pair of vertices $v_{p_1}, v_{p_2} \in \mathcal{C}_p$, let \mathcal{V}_1^t and \mathcal{V}_2^t be the neighborhoods to v_{p_1} and v_{p_2} , respectively, in the graph $\mathcal{G}(A_1^t)$. The fact that each $A(t)$ has all nodes self-linked implies that $\mathcal{V}_i^t \subset \mathcal{V}_i^{t+1}, i = 1, 2$, respectively. In the following, we are going to prove that $\mathcal{V}_1^t \cap \mathcal{V}_2^t \neq \emptyset$ holds for at least some $t \leq n$.

If $t < n$, $\mathcal{V}_1^t \cap \mathcal{V}_2^t = \emptyset$, then $\#[\mathcal{V}_1^t \cup \mathcal{V}_2^t] \geq t + 1$.

We will prove it by induction. By the assumptions, there is a cluster root in $\mathcal{G}(A(1))$ that has paths toward the vertices v_{p_1} and v_{p_2} , both \mathcal{V}_1^1 and \mathcal{V}_2^1 are not empty. If $\mathcal{V}_1^1 \cap \mathcal{V}_2^1 = \emptyset$, then $\#[\mathcal{V}_1^1 \cup \mathcal{V}_2^1] \geq 2$.

Suppose $\mathcal{V}_1^t \cap \mathcal{V}_2^t = \emptyset$ and $\#[\mathcal{V}_1^t \cup \mathcal{V}_2^t] \geq t + 1$. We will prove $\#(\mathcal{V}_1^{t+1} \cup \mathcal{V}_2^{t+1}) \geq t + 2$.

In fact, let v_1 be the root vertex in the graph $\mathcal{G}(A(t+1))$ having paths toward v_{p_1} and v_{p_2} . We select their shortest paths: $(v_{k_1}, v_{k_2}, \dots, v_{k_p})$, and $(v_{l_1}, v_{l_2}, \dots, v_{l_q})$, from v_1 to v_{p_1} and v_{p_2} , respectively, with $v_{k_1} = v_{l_1} = v_1$, $v_{k_p} = v_{p_1}$, and $v_{l_q} = v_{p_2}$. If one of the paths has one vertex not belonging to the corresponding \mathcal{V}_1^t or \mathcal{V}_2^t . Without loss of generality, we assume that $(v_{k_1}, v_{k_2}, \dots, v_{k_p})$ has vertices not belonging to \mathcal{V}_1^t and let $v_{k_{r_0}}$ be the index such that:

- 1) for each $r > r_0$, $v_{k_r} \in \mathcal{V}_1^t$;
- 2) $v_{k_{r_0}} \notin \mathcal{V}_1^t$.

This implies that

$$[A_1^{t+1}]_{v_{k_{r_0}}, v_{k_p}} \geq [A(t+1)]_{k_{r_0}, k_{r_0+1}} [A_1^t]_{v_{k_{r_0+1}}, v_{k_p}} > 0$$

holds. This implies that $v_{k_{r_0}} \in \mathcal{V}_1^{t+1}$. Hence

$$\#(\mathcal{V}_1^{t+1} \cup \mathcal{V}_2^{t+1}) \geq \#(\mathcal{V}_1^t \cup \mathcal{V}_2^t) + 1 \geq t + 2.$$

Thus, either for some $t < n$, $\mathcal{V}_1^t \cap \mathcal{V}_2^t \neq \emptyset$ or

$$\#(\mathcal{V}_1^n \cup \mathcal{V}_2^n) \geq n + 1$$

which implies $\mathcal{V}_1^n \cap \mathcal{V}_2^n \neq \emptyset$. Therefore, there exists some $t \leq n$ such that $\mathcal{V}_1^t \cap \mathcal{V}_2^t \neq \emptyset$. Proof of the lemma is completed.

Proof of Claim 1:

$$\mathbb{R}^n = \mathcal{S}_C + V.$$

For this purpose, we define a nonsingular matrix $P = [P_1, \dots, P_n] \in \mathbb{R}^{n,n}$ such that the first K column vectors compose a basis of \mathcal{S}_C . In particular, we chose each P_k , $k = 1, \dots, K$, as

$$[P_k]_i = \begin{cases} 1, & i \in \mathcal{C}_k \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$\hat{A}(t) \triangleq P^{-1} A(t) P = \begin{bmatrix} \hat{A}_{1,1} & \hat{A}_{1,2}(t) \\ 0 & \hat{A}_{2,2}(t) \end{bmatrix}$$

where the bottom-left block equals to zero since the subspace \mathcal{S}_C is invariant by $A(t)$ and the top-left block $\hat{A}_{1,1}$ is a static

matrix due to \mathcal{B}_3^* . Furthermore, since all eigenvalues of B , defined in (15), of which the modules equal to 1 should equal to 1, owing to the fact that all matrices $A(t)$ have all diagonal elements positive, we can select Q_1 with the first several columns composing of the basis of the eigenspace of the static sub-matrix $\hat{A}_{1,1}$ corresponding to eigenvalue 1 and all last $n - K$ columns were chosen to guarantee Q_1 is nonsingular. Construct a new linear transformation Q has the form as

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & I_{n-K} \end{bmatrix}.$$

Then, we further make linear transformation with Q over $\hat{A}(t)$ resulting in

$$\tilde{A}(t) \triangleq Q^{-1} \hat{A}(t) Q = \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2}(t) \\ 0 & \tilde{A}_{2,2}(t) \end{bmatrix}$$

where $\tilde{A}_{1,1}$ has the following block form:

$$\tilde{A}_{1,1} = \begin{bmatrix} \tilde{A}_{1,1}^{1,1} & 0 \\ 0 & \tilde{A}_{1,1}^{2,2} \end{bmatrix}$$

with all eigenvalues of $\tilde{A}_{1,1}^{1,1}$ equal to 1 and $\rho(\tilde{A}_{1,1}^{2,2}) < 1$. Accordingly, we write

$$\tilde{A}_{1,2}(t) = \begin{bmatrix} \tilde{A}_{1,2}^1(t) \\ \tilde{A}_{1,2}^2(t) \end{bmatrix}.$$

Thus, we define

$$\tilde{A}_0^t = \begin{bmatrix} (\tilde{A}_{1,1})^{t+1} & \tilde{A}_{1,2}^{(t)} \\ 0 & (\tilde{A}_{2,2})_0^t \end{bmatrix}$$

where $(\cdot)_0^t$ denotes the left matrix product from 0 to t , as defined before.

We define the projection radius (w.r.t. \mathcal{C}) of $A(t)$ as follows:

$$\rho_{\mathcal{C}}(A(\cdot)) = \overline{\lim}_{t \rightarrow \infty} \left\{ \|(\tilde{A}_{2,2})_0^{t-1}\| \right\}^{1/t}$$

and the cluster Hajnal diameter (w.r.t. \mathcal{C}) of $A(t)$ as follows:

$$\Delta_{\mathcal{C}}(A(\cdot)) = \overline{\lim}_{t \rightarrow \infty} \left\{ \|\Delta_{\mathcal{C}}(A_0^{t-1})\| \right\}^{1/t}$$

for some norm $\|\cdot\|$ that is induced by vector norm. It can be seen that the projection radius and cluster Hajnal diameter are independent of the selection of the matrix norm and the matrix P . First, we shall prove that the projection radius equals to the Hajnal diameter.

Lemma 6: $\rho_{\mathcal{C}}(A(\cdot)) = \Delta_{\mathcal{C}}(A(\cdot))$.

Proof: The proof is quite similar to that in [43] and can be regarded as a generalization of [43, Lemma 2.4]. For any $d > \rho_{\mathcal{C}}(A(\cdot))$, there exists $T > 0$ such that the inequality

$$\|(\tilde{A}_{2,2})_0^{t-1}\| < d^t$$

for all $t > T$. Then

$$\begin{aligned} \left\| \tilde{A}_0^{t-1} - \begin{bmatrix} E_K \\ 0 \end{bmatrix} [(\tilde{A}_{1,1})^{t-1}, \tilde{A}_{1,2}^{(t-1)}] \right\| \\ = \left\| \begin{bmatrix} 0 & 0 \\ 0 & (\tilde{A}_{2,2})_0^{t-1} \end{bmatrix} \right\| \leq C d^t \end{aligned}$$

for some $C > 0$ and all $t > T$. Thus

$$\left\| A_0^{t-1} - P \begin{bmatrix} E_K \\ 0 \end{bmatrix} \begin{bmatrix} (\tilde{A}_{1,1})^{t-1}, \tilde{A}_{1,2}^{(t-1)} \end{bmatrix} P^{-1} \right\| \leq C_1 d^t$$

for some $C_1 > 0$ and all $t > T$. Let

$$G = P \begin{bmatrix} E_K \\ 0 \end{bmatrix} = [P_1, \dots, P_K]$$

$$H = \begin{bmatrix} (\tilde{A}_{1,1})^{t-1}, \tilde{A}_{1,2}^{(t-1)} \end{bmatrix} P^{-1}.$$

Since each $P_k \in \mathcal{S}_C$ for all $k = 1, \dots, K$, each column vector in the matrix $G \cdot H$ should belong to \mathcal{S}_C , too. So, according to the definition of Hajnal diameter, we have

$$\Delta_C(A_0^{t-1}) \leq 2C_1 d^t$$

for all $t \geq T$. This implies that $\Delta_C(A(\cdot)) \leq d$. According to the arbitrariness of d , we have $\Delta_C(A(\cdot)) \leq \rho_C(A(\cdot))$.

On the other hand, for any $d > \Delta_C(A(\cdot))$, there exists $T > 0$ such that $\Delta_C(A_0^{t-1}) < d^t$ holds for all $t > T$. Without loss of generality, we suppose that the clustering \mathcal{C} is successive, i.e., $\mathcal{C}_1 = \{1, 2, \dots, n_1\}$, $\mathcal{C}_2 = \{n_1 + 1, n_1 + 2, \dots, n_2\}$, ..., $\mathcal{C}_K = \{n_{K-1} + 1, n_{K-1} + 2, \dots, n_K\}$ with $n_K = n$. Select one single row in A_0^{t-1} from each cluster and compose them into a matrix, denoted by H . Let $G = [P_1, \dots, P_K]$. Then the rows of $G \cdot H$ corresponding to the same cluster is identical. So

$$\|A_0^{t-1} - G \cdot H\| \leq C_2 d^t$$

for some $C_2 > 0$ and $t > T$. Then

$$\|P^{-1} A_0^{t-1} P - P^{-1} G \cdot H P\| \leq C_3 d^t$$

$$\left\| \begin{bmatrix} (\hat{A}_{1,1})^{t-1} & \hat{A}_{1,2}^{(t-1)} \\ 0 & (\hat{A}_{2,2})_0^{t-1} \end{bmatrix} - \begin{bmatrix} Y & Z \\ 0 & 0 \end{bmatrix} \right\| \leq C_3 d^t$$

for some matrices Y and Z , $C_3 > 0$ and all $t > T$. This implies that $\|(\hat{A}_{2,2})_0^{t-1}\| \leq C_4 d^t$ holds for some $C_4 > 0$ and all $t > T$. It can be seen that $(\hat{A}_{2,2})_0^{t-1} = (\tilde{A}_{2,2})_0^{t-1}$. Therefore, $\rho_C(A(\cdot)) \leq d$. The arbitrariness of d can guarantee $\Delta_C(A(\cdot)) \geq \rho_C(A(\cdot))$. From both sides, we have $\Delta_C(A(\cdot)) = \rho_C(A(\cdot))$. This completes the proof of this lemma.

From Theorem 3, we can conclude $\Delta_C(A(\cdot)) < 1$. Thus, $\rho_C(A(\cdot)) < 1$. For any n -dimensional vector w_0 , we can write it as

$$w_0 = \begin{bmatrix} z_0 \\ u_0 \\ v_0 \end{bmatrix}$$

where z_0 corresponds to the sub-matrix $\tilde{A}_{1,1}^{1,1}$, u_0 corresponds to the sub-matrix $\tilde{A}_{1,1}^{2,2}$ and $v_0 \in \mathbb{R}^{n-K}$. We rewrite w_0 as a sum of $w_0^1 + w_0^2$ with

$$w_0^1 = \begin{bmatrix} z_0^1 \\ 0 \\ 0 \end{bmatrix}, \quad w_0^2 = \begin{bmatrix} z_0^2 \\ u_0 \\ v_0 \end{bmatrix}$$

where $z_0^1 + z_0^2 = z_0$ that will be determined in the following. It is clear that PQw_0^1 corresponds a vector in \mathcal{S}_C . So, if we could pick a suitable z_0^2 such that $\lim_{t \rightarrow \infty} (\tilde{A})_0^t w_0^2 = 0$, that is, PQw_0^2 corresponds a vector in V . Therefore, for any n -dimensional vector x_0 , we can find some w_0 , such that

$x_0 = PQw_0 = PQw_0^1 + PQw_0^2 \in \mathcal{S}_C + V$. This could complete the proof of the claim.

For this purpose, we consider the following linear system:

$$\tilde{w}(t+1) = \tilde{A}(t)\tilde{w}(t) \quad \tilde{w}(0) = w_0^2$$

which can be rewritten as the following component-wise form:

$$\begin{cases} \tilde{w}_1(t+1) = \tilde{A}_{1,1}^{1,1} \tilde{w}_1(t) + \tilde{A}_{1,2}^1(t) \tilde{w}_3(t) \\ \tilde{w}_2(t+1) = \tilde{A}_{1,1}^{2,2} \tilde{w}_2(t) + \tilde{A}_{1,2}^2(t) \tilde{w}_3(t) \\ \tilde{w}_3(t+1) = \tilde{A}_{2,2}(t) \tilde{w}_3(t) \end{cases}$$

$$\text{with } \tilde{w}_1(0) = z_0^2, \quad \tilde{w}_2(0) = u_0, \quad \tilde{w}_3(0) = v_0.$$

It can be seen that $\lim_{t \rightarrow \infty} \tilde{w}_3(t) = 0$ exponentially because of $\rho_C(\tilde{A}(\cdot)) < 1$ and $\lim_{t \rightarrow \infty} \tilde{w}_2(t) = 0$ exponentially because of $\rho(\tilde{A}_{1,1}^{2,2}) < 1$ and the boundedness of $\tilde{A}_{1,2}^2(t)$. Without loss of generality, since $\rho_C(\tilde{A}) < 1$ and all eigenvalues of $(\tilde{A}_{1,1}^{1,1})^{-1}$ equal to 1, for any $\epsilon_0 \in (0, |\lambda_2|/2)$, we have $\|(\tilde{A}_{2,2})_0^t\| \leq M_2 \exp[-(|\lambda_2| - \epsilon_0)t]$, $\|(\tilde{A}_{1,1}^{1,1})^{-1}\| < \exp(\epsilon_0)$ and $\|\tilde{A}_{1,2}^1(t)\| \leq M_0$ for some $M_0 > 0$, $\lambda_0 > 0$, all $t \geq 0$ and some norm $\|\cdot\|$. Note that

$$\tilde{w}_1(t) = (\tilde{A}_{1,1}^{1,1})^t z_0^2 + \sum_{k=0}^t (\tilde{A}_{1,1}^{1,1})^{t-k} \tilde{A}_{1,2}^1(k) (\tilde{A}_{2,2})_0^k v_0.$$

Since

$$\begin{aligned} \|(\tilde{A}_{1,1}^{1,1})^{-k}\| \cdot \|\tilde{A}_{1,2}^1(k)\| \cdot \|(\tilde{A}_{2,2})_0^k\| &\leq \exp(\epsilon_0 k - [|\lambda_2| - \epsilon_0]k) M_2^2 \\ &\leq \exp(-[|\lambda_2| + 2\epsilon_0]k) M_2^2 \end{aligned}$$

we let

$$R = \sum_{k=0}^{\infty} (\tilde{A}_{1,1}^{1,1})^{-k} \tilde{A}_{1,2}^1(k) (\tilde{A}_{2,2})_0^k$$

of which the limit exists in the norm sense and the operator R is well-defined. Let us consider a subspace of \mathbb{R}^n

$$\tilde{V} = \left\{ [z^\top, u^\top, v^\top]^\top \in \mathbb{R}^n : z = -Rv \right\}.$$

If we pick z_0^2 such that $w_0^2 \in \tilde{V}$, then we have

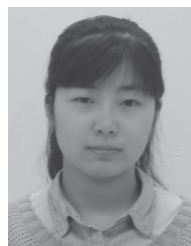
$$\begin{aligned} (\tilde{A}_{1,1}^{1,1})^{-t} \tilde{w}_1(t) &= z_0^2 + \sum_{k=0}^t (\tilde{A}_{1,1}^{1,1})^{-k} \tilde{A}_{1,2}^1(k) (\tilde{A}_{2,2})_0^k v_0 \\ &\rightarrow z_0^2 + Rv_0 = 0 \end{aligned}$$

exponentially as $t \rightarrow \infty$. So, $(\tilde{A})_0^t w_0^2$ converges to zero exponentially. This completes the proof.

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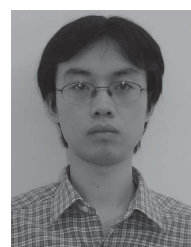
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