Bipartite Consensus for Multi-agent Systems on Directed Signed Networks

Jiangping Hu   Wei Xing Zheng

Abstract— Collective dynamics is a complex emergence phenomenon yielded by local interactions within multi-agent systems. When agents cooperate or compete in the community, a collective behavior, such as consensus, polarization or diversity, may emerge. In this paper, we investigate a bipartite consensus process, in which all the agents converge to a final state characterized by identical modulus but opposite sign. Firstly, the interaction network of the agents is represented by a directed signed graph. A neighbor-based interaction rule is proposed for each agent with a single integrator dynamics. Then, we classify the signed network into heterogeneous networks and homogeneous networks according to the sign of edges. Under a weak connectivity assumption that the signed network has a spanning tree, some sufficient conditions are derived for bipartite consensus of multi-agent systems with the help of a structural balance theory. At the same time, signless Laplacian matrix and signed Laplacian matrix are introduced to analyze the bipartite consensus of multi-agent systems on homogeneous networks and heterogenous networks, respectively. Finally, simulation results are provided to demonstrate the bipartite consensus formation.

I. INTRODUCTION

In recent years, there has been a surge of attention given to the study of multi-agent systems where agents interact in the neighborhood according to simple local rules, resulting in a possibly global emergence behavior. A typical collective behavior is characterized by the emergence of a global consensus, in which all agents reach the same state in the long run. Consensus occurs in many biological, sociological and physical processes as well as in engineering and computer science, and it has been investigated formally since 1960s (see [1] and references therein). Some pioneering physical models were proposed to study various consensus behaviors [2], [3], [4], [5], [6]. During the last few years consensus behaviors under the framework of multi-agent systems and complex networks have been an active area of research. In the study of consensus, a graph is normally used to model multi-agent systems, with nodes representing agents and (positive) edges describing their pairwise collaboration. It has been shown that global consensus can be reached if and only if

directed graphs associated with multi-agent systems have a spanning tree [9], [10], [11], [12], [13].

In many real world scenarios, another type of "consensus" phenomenon has been observed for a long time, where all agents reach a final state with identical magnitude but opposite sign. Hereafter, we call such kind of collective behavior as *bipartite consensus* or *anti-synchronization*. For example, a polarization often happens in a two-coalition community such that opposite opinions are held by two fractions [14], [15], [16]. Anti-synchronization phenomena have also been observed in synchronization between two pendulum clocks, salt-water oscillators experiments and chaotic systems [17], [18]. In order to study bipartite consensus, the interaction networks among agents are generally modeled by signed graphs with positive/negative edges and the evolution of the collective dynamics is analyzed by using the notion of structural balance [19], [20]. Structural balance is an important property in social network theory, which partitions signed graphs into two subgraphs such that each subgraph contains only positive edges while all edges joining different subgraphs are negative [21], [22], [23]. A triad dynamics was built to realize transformation from unbalance to balance for signed networks in [19]. Closely related with this paper is the bipartite consensus formation on signed networks, which was initially discussed in [16]. The collective dynamics was described by a so-called monotone system [24]. A sufficient and necessary condition for bipartite consensus is that signed graphs associated with multi-agent systems are strongly connected and structurally balanced.

Yet, studying the relationships between structural balance and bipartite consensus has received little attention in the context of multi-agent systems. Though some preliminary results have been obtained for bipartite consensus on undirected or directed networks in [16], the requirement of strong connectedness on signed networks seems fairly stringent, and simultaneously, sufficient conditions for bipartite consensus may be extended to include a complete category of signed networks. In fact, under a milder condition that interaction graphs have a spanning tree, in addition to bipartite consensus, more complex collective behaviors may emerge. In this paper, a detailed analysis is made of the bipartite consensus formation for multi-agent systems with Laplacian-like dynamics on directed signed networks. Efforts are devoted to determining the equilibrium and deriving the spectral properties of the signed Laplacian matrix of collective dynamics if signed networks associated with multi-agent systems have a spanning tree. According to the sign of edges, signed networks are called homogeneous if all edges

have the same sign, and heterogeneous otherwise. Since the signed Laplacian matrices of heterogeneous networks and homogeneous networks have respective expressions, it is necessary to investigate how the collective dynamics evolve on these two kinds of signed networks, respectively. Ultimately, some conditions related to structural balanced are presented for bipartite consensus.

The remainder of this paper is organized as follows. In Section II, bipartite consensus problem is formulated and networks with antagonistic interaction are modeled. A complete analysis of the bipartite consensus formation on homogeneous networks and heterogeneous networks is presented for multi-agent systems in Section III. Simultaneously, some sufficient conditions are obtained to describe cases in which bipartite consensus can be achieved. In Section IV, some simulations are provided to demonstrate the collective dynamics of multi-agent systems on different signed networks. Finally, we draw a conclusion in Section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Some preliminaries

It is helpful to use directed signed graphs to represent networks with antagonistic interactions for multi-agent systems. A directed signed graph is a directed graph

$$\mathcal{G}^s = \{\mathcal{V}, \mathcal{E}, A\},$$

where $\mathcal{V} = \{1, \cdots, n\}$ is a set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, and A is an adjacency matrix describing the edge information of a positive or negative sign. The nonzero element a_{ij} of A is attached to the edge $(j, i) \in \mathcal{E}$, which is directed from node j to node i. In this way, the node j is called the parent node and i is the child node. The edge set is described by

$$\mathcal{E} = \mathcal{E}^+ \bigcup \mathcal{E}^-,$$

where

$$\mathcal{E}^+ = \{(j,i) | a_{ij} > 0\} \quad \text{and} \quad \mathcal{E}^- = \{(j,i) | a_{ij} < 0\}$$

are the sets of positive and negative edges, respectively. If $\mathcal{E}^- = \varnothing$, then the signed graph is simply a directed graph. $\mathcal{G}^s$ is simple if it has no loops or multiple edges. A directed path is a sequence of edges of the form

$$(i_1, i_2), (i_2, i_3), \cdots, (i_{l-1}, i_l)$$

with distinct nodes and with length $l - 1$. A semipath is defined as a sequence of nodes $i_1, \cdots, i_l$ such that either $(i_\kappa, i_{\kappa-1})$ or $(i_{\kappa-1}, i_\kappa)$ belongs to the set $\mathcal{E}$. A directed (semi)cycle is a directed (semi)path beginning and ending with the same nodes. A directed graph $\mathcal{G}^s$ is said to be strongly (weakly) connected if there is a directed path (semipath) between any pair of distinct nodes. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A spanning tree of a directed graph is a directed tree containing directed paths from the root into all other vertices of the graph. Obviously, the connectivity that $\mathcal{G}^s$ has a spanning tree is much weaker than the strong connectivity.

The sign of a semicycle is the product of the signs of the edges in it. A directed (semi)cycle is positive if it contains an even number of negative edges; otherwise, negative if it contains an odd number of negative edges. A directed signed graph $\mathcal{G}^s$ is said structurally balanced if all of its semicycles are positive [22], [23], and $\mathcal{G}^s$ is said structurally unbalanced if one of its semicycles is negative. Additionally, if $\mathcal{G}^s$ has no semicycles, it is said vacuously balanced [21]. A signed graph is said to be homogeneous if all edges have the same sign, and heterogeneous otherwise. Homogeneous signed graphs have two classes: all-positive graphs (i.e., all of the edges are positive) and all-negative graphs (i.e., all of the edges are negative).

A directed signed graph $\mathcal{G}^s$ is said bipartite if its nodes can be partitioned into two disjoint sets such that every edge connects two nodes in different sets. Equivalently, $\mathcal{G}^s$ is bipartite if and only if it does not contain odd-length semicycles. It is noted that a relation of inclusion does not exist between the structural balance and the bipartiteness. For example, if a structurally balanced graph is heterogeneous, then it will be non-bipartite; if a all-negative bipartite graph has no semicycles, then it is vacuously balanced.

Given the directed signed graph $\mathcal{G}^s$, denote

$$C_r = [c_{r,ij}] \in \mathbb{R}^{n \times n} \quad \text{and} \quad C_c = [c_{c,ij}] \in \mathbb{R}^{n \times n}$$

the row connectivity matrix and the column connectivity matrix, respectively, where

$$c_{r,ij} = \begin{cases} \sum\limits_{j \in \mathcal{N}_i} |a_{ij}|, i = j, \\ 0, i \neq j, \end{cases}$$

$$c_{c,ij} = \begin{cases} \sum\limits_{j \in \mathcal{N}_i} |a_{ji}|, i = j, \\ 0, i \neq j, \end{cases}$$

with

$$\mathcal{N}_i = \{j \mid (i, j) \in \mathcal{E}\}$$

being the neighbor set of node i regardless of the sign of the edge (i, j). When

$$C_r = C_c,$$

the directed graph $\mathcal{G}^s$ is said weight balanced. In the sequel, assume that the weights a_{ij} are constrained to be $0, 1$ or -1. Thus, the in-degree of a weight balanced graph equals the out-degree for all nodes. The signed Laplacian matrix of a directed signed graph $\mathcal{G}^s$ is defined as

$$L^s = C_r - A, \tag{1}$$

which is generally not a symmetric matrix.

When a signed graph $\mathcal{G}^s$ is homogeneous and all-positive, the signed Laplacian matrix is identical with the Laplacian matrix of a directed graph. However, if $\mathcal{G}^s$ is homogeneous and all-negative, we use $\mathcal{G}^-$ to denote such a signed graph and define a signless Laplacian matrix as

$$L^- = C_r + A^-, \tag{2}$$

where

$$A^- = -A$$

is a nonnegative matrix. Then L^- is always a nonnegative matrix, which has been studied in the spectral graph theory [25], [26].

B. Bipartite consensus problem

Consider that a group of agents, labeled by $1, \cdots, n$, interact cooperatively or competitively on a network described by a signed graph $\mathcal{G}^s$. The collective dynamics for the multi-agent system is expressed by a first-order integrator

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}\big[x_j - \operatorname{sgn}(a_{ij})x_i\big], \ i = 1, \cdots, n, \quad (3)$$

or a compact form

$$\dot{x}(t) = -L^s x(t), \quad (4)$$

where $x_i(t) \in \mathbb{R}$ is the state of agent i,

$$x(t) = \operatorname{col}(x_1(t), \cdots, x_n(t)) \in \mathbb{R}^n$$

is the collective state of the multi-agent system, $\operatorname{col}(\cdot)$ denotes a concatenation, a_{ij} is the weight between agents i and j, $\operatorname{sgn}(\cdot)$ is the sign function,

$$\mathcal{N}_i = \{j \mid a_{ij} \neq 0\}$$

is the neighbor set of agent i, and

$$L^s = C_r - A$$

is the signed Laplacian matrix associated with a signed graph $\mathcal{G}^s$. Suppose that the initial time is $t = 0$ and the initial state is denoted by $x(0)$.

In the bipartite consensus problem under investigation, we restrict our attention to analyzing how the structurally balanced condition plays a key role in the consensus formation. More specifically, structural conditions are explored for a complete category of signed networks such that all agents reach bipartite consensus, i.e.,

$$\lim_{t \to \infty} |x_i(t)| = \mathbf{c} > 0,$$

for all $i = 1, \cdots, n$. Additionally, if the final state is

$$\mathbf{c} = \frac{1}{n}|\omega^T x(0)|$$

for some constant weight vector ω, then we say that all the agents reach an average bipartite consensus.

In the next section, the collective dynamics of multi-agent system (4) will be analyzed for homogeneous and heterogenous signed networks, respectively.

III. COLLECTIVE DYNAMICS ON SIGNED NETWORKS

A. Homogeneous signed networks

1) All-positive networks: When the interaction network $\mathcal{G}^s$ is an all-positive signed graph, then the n agents interact each other cooperatively. In order to investigate the state evolution of system (4), the following lemma plays an important role.

Lemma 1: The signed Laplacian matrix L^s has nonnegative real-part eigenvalues and 0 is its simple eigenvalues if

and only if the all-positive signed graph $\mathcal{G}^s$ has a spanning tree.

When the signed graph $\mathcal{G}^s$ is all-positive, the signed Laplacian matrix L^s turns to a conventional Laplacian matrix, whose spectral property was shown in [9].

Furthermore, a main result on consensus formation was also presented in [9], [10].

Theorem 1: All the agents reach consensus, i.e.,

$$\lim_{t \to \infty} [x_i(t) - x_j(t)] = 0, \quad \forall 1 \le i, j \le n,$$

if and only if the all-positive signed graph $\mathcal{G}^s$ has a spanning tree.

It is noted that bipartite consensus can never be reached for all the agents when the interaction network is all-positive.

2) All-negative networks: Consider a signed network with all-negative edges, in which agents feel tension to each other. For notation simplicity, an all-negative signed network is denoted by $\mathcal{G}^-$ in this section. In order to investigate the collective dynamics of multi-agent system (4), the spectral properties of the Laplacian matrix of $\mathcal{G}^-$ need to be studied. Till today, few results have been found for spectral properties of a signless Laplacian matrix when the signed network is strongly connected or has a spanning tree. We are now in position to investigate bipartite graphs and non-bipartite graphs to gain an initial understanding of the signless Laplacian matrix L^- associated with a directed signed graph.

If the interaction network $\mathcal{G}^-$ associated with the multi-agent system (4) is bipartite, all agents are divided into two groups and the interactions among agents only exist between the two distinct groups. Since all semicycles, if any, in a directed bipartite network have even-length, a bipartite network may be structurally balanced or vacuously balanced. Conversely, if an all-negative network is structurally balanced, then it is bipartite.

Lemma 2: If a bipartite graph $\mathcal{G}^-$ has a spanning tree, then 0 is a simple eigenvalue and the rest eigenvalues have positive real-parts.

Proof: If a directed signed graph $\mathcal{G}^-$ is bipartite, no matter whether it is structurally balanced or vacuously balanced, then all the nodes in $\mathcal{G}^-$ can be partitioned into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that

$$\mathcal{V} = \mathcal{V}_1 \bigcup \mathcal{V}_2, \quad \mathcal{V}_1 \bigcap \mathcal{V}_2 = \varnothing$$

and

$$a_{ij} = 0, \ \forall i, j \in \mathcal{V}_q \ (q = 1, 2),$$
$$a_{ij} \le 0, \ \forall i \in \mathcal{V}_q, j \in \mathcal{V}_r, \ q \neq r.$$

Define a diagonal matrix

$$\Phi = \operatorname{diag}\{\phi_1, \cdots, \phi_n\}, \quad (5)$$

where $\phi_i = 1$ when node i belongs to $\mathcal{V}_1$ and $\phi_i = -1$ when node i belongs to $\mathcal{V}_2$, and $\operatorname{diag}\{\cdot\}$ denotes a diagonal matrix by placing a vector along the diagonal direction. Obviously, the inverse matrix of Φ is itself, i.e.,

$$\Phi^2 = I.$$

The adjacency matrix A of the all-negative signed graph G^- can be rearranged in the following form

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix},$$

where A_{12} and A_{21} are non-positive sub-matrices. Multiplying by Φ on the left and right sides of A yields

$$\Phi A \Phi = \begin{pmatrix} 0 & -A_{12} \\ -A_{21} & 0 \end{pmatrix},$$

which becomes a nonnegative adjacency matrix. From the definition of signless Laplacian matrix (2), it follows that

$$\Phi L^- \Phi = C_r - \Phi A \Phi,$$

which is a Laplacian matrix associated with a directed graph $\mathcal{G}$. When $\mathcal{G}$ has a spanning tree, according to Lemma 1, $\Phi L^- \Phi$ has a simple eigenvalue 0. Since L^- is similar to $\Phi L^- \Phi$, the conclusion follows. ∎

Theorem 2: If the interaction network $\mathcal{G}^s$ of a multi-agent system (3) is an all-negative bipartite graph having a spanning tree, then all the agents reach bipartite consensus. Furthermore, if $\mathcal{G}^s$ is weight balanced, then all the agents will reach an average bipartite consensus.

Proof: Define

$$L_\Phi = \Phi L^- \Phi.$$

According to Lemma 1, 0 is a simple eigenvalue of the Laplacian matrix L_Φ and

$$\mathbf{1} = \mathrm{col}(1, \cdots, 1)$$

is the corresponding right eigenvector. Let α_Φ be a left eigenvector of the eigenvalue 0 and satisfy

$$\alpha_\Phi^T \mathbf{1} = 1.$$

We can always find a nonsingular matrix U such that

$$U^{-1} L_\Phi U = J_\Phi,$$

where J_Φ is the Jordan matrix associated with L_Φ. Furthermore, one has

$$\lim_{t \to \infty} e^{-L_\Phi t} = \lim_{t \to \infty} U e^{-J_\Phi t} U^{-1} = Q_\Phi = \mathbf{1} \alpha_\Phi^T,$$

where Q_Φ has only one nonzero element

$$q_{\Phi,11} = 1.$$

Then we obtain that

$$\lim_{t \to \infty} x(t) = \alpha_\Phi^T \Phi x(0) \Phi \mathbf{1}.$$

Particularly, if the signed network $\mathcal{G}^s$ is also weight balanced, then the left eigenvector of L_Φ is

$$\alpha_\Phi = \frac{1}{n} \mathbf{1}$$

and all the agents will reach an average bipartite consensus, that is,

$$\lim_{t \to \infty} x(t) = \frac{1}{n} \mathbf{1}^T \Phi x(0) \Phi \mathbf{1}.$$

∎

From Theorem 2, it is found that the structural balance is not a necessary condition to ensure the bipartite consensus.

Remark 1: If the interconnection network $\mathcal{G}^s$ is all-negative but not bipartite, even though it has a spanning tree, bipartite consensus can never be reached for the multi-agent system (3). In fact, all the agents may reach consensus or have fragmental final states.

B. Heterogeneous signed networks

In a heterogeneous signed network, both positive and negative edges coexist, that is, the relationships between agents can be friendly or hostile. Such networks have gained wide applications in many social systems. For example, all agents have a uniformly distributed initial opinion on the rumor during a rumor spreading process. Everyone tells the rumor to a friend in the same form one has received, but changes the form if one passes on the rumor to a competitor. It is interesting to investigate what form the final collective opinion may take in the collective opinion dynamics. For a general collective dynamics of multi-agent systems, let us start to study the spectral properties of Laplacian matrices of different signed networks.

Lemma 3: If a heterogeneous signed network $\mathcal{G}^s$ has a spanning tree and structurally balanced, then all eigenvalues of its signed Laplacian matrix have nonnegative real-parts and 0 is a simple eigenvalue.

Proof: A similar method as in the proof of Lemma 1 can be employed. Since $\mathcal{G}^s$ is structurally balanced, it can be partitioned into two subgraphs with node sets $\mathcal{V}_i$ ($i = 1, 2$). The adjacency matrix A can be rewritten in a block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{ii} ($i = 1, 2$) are nonnegative submatrices associated with edges within the two subgraphs while A_{ij} ($i \neq j$) are non-positive submatrices associated with edges between the two subgraphs. Given a transformation matrix Φ defined in (5), it is not difficult to show that

$$\Phi L^s \Phi = C_r - \Phi A \Phi$$

is a Laplacian matrix of $\mathcal{G}$. Thus, the conclusion follows. ∎

With the help of Lemma 3, one has the following main result.

Theorem 3: If the interconnection network $\mathcal{G}^s$ associated with the multi-agent system (3) is structurally balanced and has a spanning tree, then all the agents reach bipartite consensus. Furthermore, if $\mathcal{G}^s$ is weight balanced, then an average bipartite consensus will be achieved.

Remark 2: When a signed network $\mathcal{G}^s$ is vacuously balanced, i.e., there are no semicycles in it, $\mathcal{G}^s$ can still be partitioned into two clusters such that positive edges exist only within each cluster while negative edges exist between the two clusters. The transformation matrix Φ exists for vacuously balanced networks. Therefore, the equilibrium of multi-agent system (4) is $\Phi \mathbf{1}$ and then all agents can reach bipartite consensus.

Remark 3: From [19], it is known that the collective state of the collective dynamics (4) will converge to zero if the signed network $\mathcal{G}^s$ is strongly connected and structurally unbalanced. However, if $\mathcal{G}^s$ is structurally unbalanced and has a spanning tree, then such a conclusion is no longer true. In fact, both consensus on zero and fragmentation may emerge in the collective dynamics, which will be illustrated in Section IV.

IV. SIMULATION STUDIES

In this section, we present some simulation results on heterogeneous signed networks.

Consider two signed networks represented, respectively, by Figure 1(a) and Figure 1(b), where the solid and dash
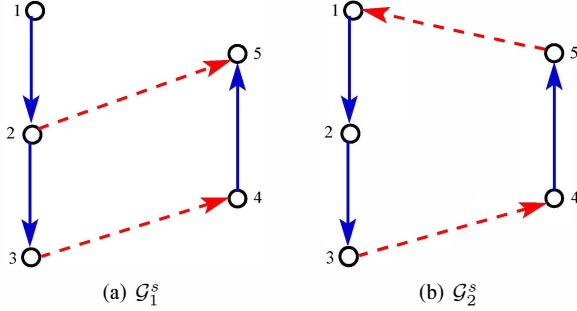

(a) $\mathcal{G}_1^s$ (b) $\mathcal{G}_2^s$

Fig. 1.   Structurally balanced signed networks

edges, respectively, denote the allied and opposed relations between agents. According to the definition on structural balance, both $\mathcal{G}_1^s$ and $\mathcal{G}_2^s$ are structurally balanced; furthermore, $\mathcal{G}_2^s$ is weight balanced as well. It is noted that both $\mathcal{G}_1^s$ and $\mathcal{G}_2^s$ have a spanning tree. Through simple calculation, the eigenvalues of the signed Laplacian matrix L_1 are $0, 1, 1, 1, 2$, and the eigenvalues of the signed Laplacian matrix L_2 are

$$0, \ 0.6910 \pm 0.9511i, \ 1.8090 \pm 0.5878i.$$

The initial state of the multi-agent system (4) is given by

$$x(0) = \mathrm{col}(1, -3, 5, -2, -1).$$

Figure 2 shows that the agents achieve a bipartite consensus if the signed network has a spanning tree and is structurally balanced. The blue and red lines, respectively, denote the state evolution of agents belonging to the two clusters

$$\mathcal{V}_1 = \{1, 2, 3\} \quad \text{and} \quad \mathcal{V}_2 = \{4, 5\}.$$

Specifically, the final state of multi-agent system (4) in Figure 2(a) is $\mathrm{col}(1, 1, 1, -1, -1)$ while the one in Figure 2(b) is $\frac{6}{5}\mathrm{col}(1, 1, 1, -1, -1)$.

When the signed network associated with the multi-agent system (3) is structurally unbalanced and has a spanning tree, different collective behaviors will appear. For example, the two signed networks $\mathcal{G}_4^s$ and $\mathcal{G}_5^s$ given in Figure 3 are both structurally unbalanced. In Figure 4(a), all the states of the agents on $\mathcal{G}_4^s$ converge to zero, which seems consistent with the conclusion mentioned above. However, the agents on $\mathcal{G}_5^s$ have three final states in Figure 4(b).
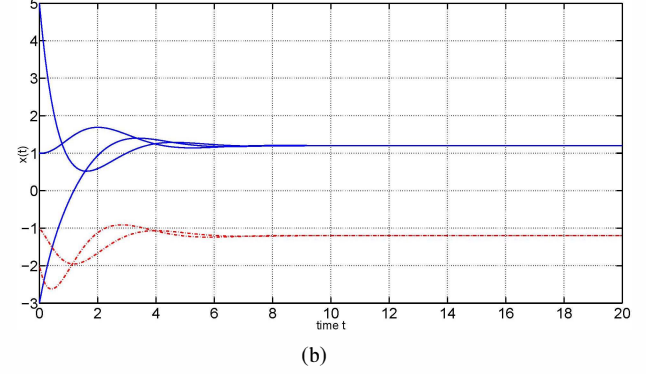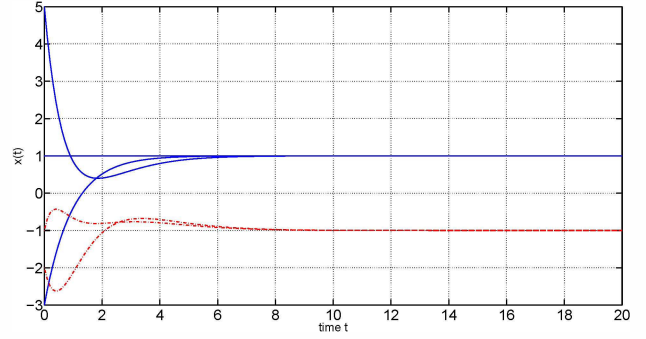

(a)

(b)

Fig. 2.   State evolution of agents on structurally balanced signed networks $\mathcal{G}_1^s$ and $\mathcal{G}_2^s$
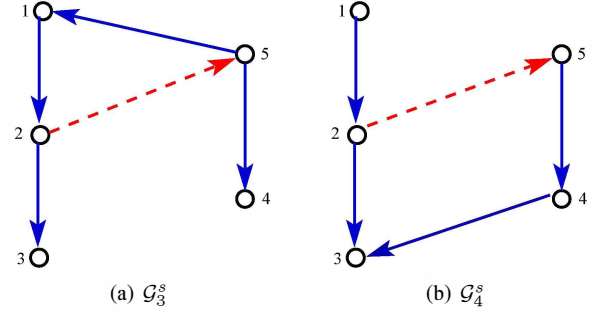

(a) $\mathcal{G}_3^s$ (b) $\mathcal{G}_4^s$

Fig. 3.   Structurally unbalanced signed networks

The differences between Figure 4(a) and Figure 4(b) lie in the different equilibrium of the multi-agent system (4) on $\mathcal{G}_3^s$ and $\mathcal{G}_4^s$. According to the interaction relations of the agents on $\mathcal{G}_3^s$, the equilibrium should satisfy

$$x_1 = x_2 = x_3 = x_4 = x_5$$

along the spanning tree with root node 5, and simultaneously,

$$x_2 = -x_5$$

on the negative edge between nodes 2 and 5, therefore, the equilibrium has to be zero. For the agents on $\mathcal{G}_4^s$, the equilibrium satisfies

$$x_1 = x_2, \ x_4 = x_5, \ x_2 = -x_5 = -x_4$$
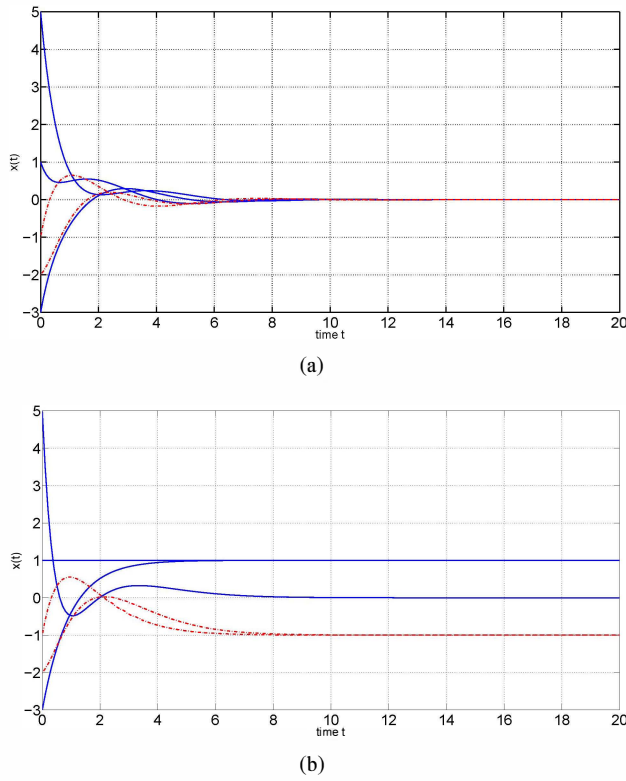
and

$$x_2 + x_4 - 2x_3 = 2x_3 = 0,$$

(a)

(b)

Fig. 4. State evolution of agents on structurally unbalanced signed networks $\mathcal{G}_3^s$ and $\mathcal{G}_4^s$

thus the equilibrium belongs to the positive invariant set

$$\{c\mathrm{col}(1,1,0,-1,-1) \mid c \in \mathbb{R}\}.$$

V. CONCLUSION

In this paper, we have given sufficient conditions under which consensus, bipartite consensus and fragment appear for multi-agent systems. We have adopted a weak connectivity assumption that the interaction networks associated with multi-agent systems have a spanning tree. In the case of heterogenous signed networks, bipartite consensus has been reached for multi-agent systems on structurally balanced or vacuously balanced networks, but the collective state approaches to zero or becomes fragmental on structurally unbalanced networks. The convergence of multi-agent systems has been analyzed by using the spectral properties of signed Laplacian matrices, which determine the equilibrium of the collective dynamics. In the case of homogeneous signed networks, there are three cases: i) consensus appears on all-positive networks, ii) bipartite consensus forms on all-negative bipartite networks, iii) both convergence to zero and fragment may occur on structurally unbalanced networks. A special signed Laplacian matrix, signless Laplacian matrix, has been introduced to explain the distinct convergence of the collective dynamics as the structurally balanced condition is changed.

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