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Consensus and polarization in Altafini's model with bidirectional time-varying network topologies

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***Abstract*— The mechanism of reaching consensus in multi-agent systems has been exhaustively studied in recent years, motivated by numerous applications in engineering and science. Most consensus algorithms examined in the literature are based on the assumption about mutual trust and cooperation between the agents, implemented in the form of attractive couplings between the agents that render the values of the agents' states closer. However, in opinion dynamics of social groups, competition or antagonism between some pairs of agents is ubiquitous, which is usually characterized by the repulsive coupling, and may lead to clustering and polarization of opinions. A simple yet insightful model of opinion dynamics with antagonistic interactions was proposed recently by C. Altafini, which examined conventional first-order consensus algorithms with static signed interaction graphs, where the positive weight of an arc implies cooperation between the two agents and the negative one corresponds to antagonism. This protocol establishes *modulus consensus*, where the opinions become the same in modulus but may differ in sign. In the present paper, we extend the modulus consensus model to the case where the network topology is time-varying and undirected. We give necessary and sufficient conditions under which the consensus protocol with the time-varying signed Laplacian establishes agreement of opinions in moduli, whose signs may be opposite, so that the agents' opinions either reach consensus or polarize.**

I. Introduction

The phenomenon of multi-agent *consensus* caused by local interactions between the agents has recently attracted enormous attention of the research community. This interest is motivated by numerous natural phenomena and engineering designs based on synchronism between heterogeneous components (*agents*) of complex systems. Examples include, but are not limited to, flocking, swarming, and other forms of coordinated behavior of complex biological and technical systems. We refer the reader to [16], [23], [24] for excellent surveys of recent research on distributed consensus protocols and their applications, as well as historical milestones.

Originated from iterative procedures of decision-making [6] and other distributed algorithms [4], conventional consensus algorithms typically assume that during the interactions agents seek to bring their values closer. Mathematically, this manifests itself in the averaging property: each agent's state evolves in the interior of the convex hull spanned by its neighbors states, so that the convex hull of all the agents states is shrinking over time. The convergence properties of averaging consensus algorithms were thoroughly studied in the recent literature with special attention to the effect of time-variant interaction topology. The main mathematical techniques include the use of relevant algebraic results [16], [19], [23] on convergence of infinite matrix products and Lyapunov-like methods [3], [5], [13], [17], taking the diameter of the convex hull just described as a Lyapunov function. In the case of bidirectional and cut-balanced graphs the convergence of averaging consensus protocols was exhaustively examined in the very recent papers [10], [14], [26] where necessary and sufficient conditions for consensus were obtained; meanwhile, in the case of directed topology a gap between necessary and sufficient conditions still exists [5], [14], [17]. Many high-order consensus algorithms are either extensions of their first-order counterparts [23], [24] or squarely based on them [25].

Despite many natural and engineered teams of agents are known to achieve common goals due to cooperation, real-world networks often involve repulsive couplings which represent competition or antagonism between some pairs of agents. In social networks, representing real communities of individuals, such relations taking the form of friendship and hostility, respect and contempt, confidence and distrust, are ubiquitous and seriously influence the dynamics of opinions, leading in general to polarized or more general clustering behavior [28]. Unlike cooperative consensus algorithms, the protocols with both attractive and repulsive couplings are still awaiting for mathematically rigorous analysis. A simple yet instructive model of opinion dynamics where the agents may be both "friendly" or "hostile" were examined by C. Altafini [1], [2]. Altafini showed that the first-order consensus protocol with a signed interaction graph, where the positive weight of an arc implies cooperation between the two agents and the negative one corresponds to their antagonism, establishes "modulus consensus", where the opinions agree in modulus but may differ in sign. If the consensus value of modulus is non-zero, the modulus consensus includes possibility of "bi-partite consensus", where the agents divide into two groups with polarized opinions. The latter situation is possible only if the network is structurally balanced or equivalently, the community splits into two hostile camps (e.g. adherence of two political parties), where the relations inside each faction are cooperative.

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The result from [1], [2] is mainly concerned with networks with static topologies and is based on techniques of gauge transformations. Some extensions, removing the technical assumptions of digon-symmetry and strong connectivity in Altafini's model, are available in the very recent papers [11], [12]. In the present paper, we extend the Altafini's result to the case of networks with *time-varying* signed graphs. We confine ourselves to the case of bidirectional interactions, for which it is possible to give complete necessary and sufficient conditions for modulus consensus, classified into "trivial" consensus at zero opinion, "non-trivial" consensus, or bipartite consensus. Those necessary and sufficient conditions boil down to the essential connectivity (or joint integral connectivity) of the network, similar to cooperative consensus protocols [10], [14]. However, unlike the classical situation, trivial consensus at zero is possible without network connectivity at all. We deal with continuous-time protocols, while some results on the discrete-time case were previously obtained in [15]. Removing the restrictions of static topologies not only allows one to analyze dynamics of real social networks, where the agents may change their relations from friendship to hostility and vice versa, but also enables one to extend the result to non-linear protocols. We show, in particular, that nonlinear algorithms from [1], [2] may be examined in the common framework with linear ones, getting rid of monotonicity restrictions [1], [2], [27].

The paper is organized as follows. Section II introduces some preliminary concepts from graph theory. Section III gives technical preliminaries and the setup of the problem in question. Section IV presents the main results, and their application is discussed in Section V. Appendix gives a proof of a technical lemma.

II. PRELIMINARIES

Throughout the paper $m:n$, where $m \leq n$ are integers, stands for the sequence $\{m, m+1, \dots, n\}$. The sign of a number $x \in \mathbb{R}$ is denoted by $sgn\, x \in \{-1, 0, 1\}$. The abbreviation "a.a." stands for "almost all" (except for a zero measure set). Given a matrix $L = (L_{ij})$, let $|L|_\infty := \max_i \sum_j |L_{ij}|$, e.g. for column vector $x \in \mathbb{R}^N$ one has $|x|_\infty = \max_i |x_i|$. It is easy to show that $|L|_\infty = \sup_{|x|_\infty \leq 1} |Lx|_\infty$, where vector x is appropriately dimensioned.

A signed (weighted) graph is a triple $G = (V, E, A)$, where $V = \{v_1, \dots, v_N\}$ stands for the set of *nodes*, $E \subset V \times V$ is a set of *arcs* and $A \in \mathbb{R}^{N \times N}$ is a signed *adjacency matrix* such that $a_{jk} \neq 0$ if and only if $(v_k, v_j) \in E$. We always assume the graph has no self-loops: $a_{jj} = 0$. We say the graph is *bidirectional* if $a_{jk} \neq 0 \Leftrightarrow a_{kj} \neq 0$ and *undirected* if $A = A^T$. The graph is *digon sign-symmetric* [2] if opposite arcs cannot have different signs, that is, $a_{jk}a_{kj} \geq 0$. Throughout this paper, we confine ourselves to bidirectional digon-symmetric graphs only. A signed graph is *structurally balanced* [2], [7] if its nodes can be divided into two sets $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that $a_{jk} \geq 0$ if $(j,k) \in V_i$ and $a_{jk} \leq 0$ for $j \in V_1, k \in V_2$.

A *path* connecting nodes v and v' is a sequence of nodes $v_{i_0} := v, v_{i_1}, \dots, v_{i_{n-1}}, v_{i_n} := v'$ $(n \geq 1)$ such that $(v_{i_{k-1}}, v_{i_k}) \in E$ for $k \in 1:n$. A path where $v_{i_0} = v_{i_n}$ is referred to as a *cycle*. The cycle is *positive* if $a_{i_0 i_1} a_{i_1 i_2} \dots a_{i_{n-1} i_n} > 0$ and *negative* otherwise. The digon-symmetric graph is structurally balanced if and only if all its cycles are positive [2], [7]. A bidirectional graph is said to be *connected* if a path between any nodes exists. Disconnected bidirectional graph consists of several disjoint *connected components*, each of them being a connected graph (possible, with the only node), with no arcs connecting them.

Any matrix $A \in \mathbb{R}^{N \times N}$ may be assigned to a signed graph $G[A] = (1:N, E[A], A)$ where $E[A] := \{(j,k) : a_{kj} \neq 0\}$. Following [2], its *Laplacian* matrix $L = L[A]$ is defined by

$$L = (L_{jk})_{j,k=1}^N, \quad L_{jk} := \begin{cases} -a_{jk}, \ j \neq k \\ \sum_{m=1}^N |a_{jm}|, j = k. \end{cases} \tag{1}$$

In the case where $a_{jk} \geq 0 \, \forall j,k$ and $|a_{jk}| = a_{jk}$, the matrix (1) is the conventional Laplacian of a weighted graph [20].

III. PROBLEM SETUP

Consider a group of $N \geq 2$ agents indexed 1 through N, the opinion of the i-th agent is denoted by $x_i \in \mathbb{R}$ and we define $x := (x_1, \dots, x_N)^T \in \mathbb{R}^N$. The agents update their opinions in accordance with a distributed protocol as follows:

$$\dot{x}(t) = -L[A(t)]x(t), t \geq 0, \tag{2}$$

which may be written componentwise as

$$\dot{x}_j(t) = \sum_{k=1}^N |a_{jk}(t)|(x_k(t)\, sgn\, a_{jk}(t) - x_j(t)) \, \forall j. \tag{3}$$

Here $A(t) = (a_{jk}(t))$ is a locally bounded matrix-valued function which describes the interaction topology of the network and $a_{jj}(t) \equiv 0$. At time $t \geq 0$, the opinion of the j-th agent is influenced by those group-mates for which $a_{jk} \neq 0$. Unlike the classical first-order consensus protocol [16], [20] this influence may be both cooperative (when $a_{jk} > 0$) or competitive (when $a_{jk} < 0$). The coupling term $|a_{jk}|(x_k\, sgn\, a_{jk} - x_j)$ in (3) drives infinitesimally the opinion of the j-th agent, respectively, either towards the opinion of the k-th one (attractive coupling) or against it (repulsive coupling).

In the paper [2] the protocol (2) was thoroughly examined in the case of a constant signed interaction graph ($A(t) \equiv A$), assumed to be digon sign-symmetric and strongly connected. Unlike purely cooperative consensus algorithms ($a_{jk}(t) \geq 0$), (2) does not guarantee the convex hull of opinions to be nested. It was shown that $(-L[A])$ is a Hurwitz matrix and thus opinions converge to 0 independent of initial values, unless the graph is structurally balanced. This property implies that a community is divided into two hostile camps (such as votaries of two political parties), where each agent cooperates with its camp-mates, competing with "opponents" from the other camp. The case of unsigned weighted graph is a special case of a structurally balanced sign graph where one of the antagonistic camps is empty. For structurally balanced graphs, two situations of the network behavior are possible depending on the graph and initial condition: the opinions may reach consensus or agree in modulus, differing in sign (bipartite consensus [2]). Summarizing, the protocol (2) with $A(t) = const$ provides the *modulus consensus* [15]:

Definition 1: The protocol (2) establishes *modulus consensus*, if for any $x(0)$ a number $x_* \geq 0$ exists such that

$$\lim_{t\to+\infty} |x_i(t)| = x_*. \tag{4}$$

The notion of modulus consensus encompasses possibilities of consensus (including that with zero steady-state opinion x_*) and bipartite consensus (some of $x_i(t)$ converge to x_* and the others to $-x_*$, that is, opinions polarize).

Establishing modulus consensus was also proved in [2] for the special case of switching topology; however, those conditions are quite restrictive: the graph should be weight balanced (as follows from the proof, relying on [20, Theorem 9]) with constant signs of the weights (no pair of agents may change its behavior from cooperative to competitive and vice versa). In the present paper, we are aiming to disclose conditions which guarantee modulus consensus for time-varying topologies without those restrictions. Dealing with real-world social network, the time-invariance of such relationships between individuals as friendship and hostility is evidently a non-realistic assumption. What is more important, the opinion dynamics in social networks are usually considered to be nonlinear [8], [9], however, such models are often reducible to the linear case by introducing time-variant gains, depending on the solution. Considering general time-varying graphs allows us to examine both linear and nonlinear consensus protocols from [1], [2] in the common framework. Modulus consensus implies the vector of steady-state opinions to have the following structure.

Lemma 1: Suppose that protocol (2) establishes modulus consensus. Then there exist vectors $v, \rho \in \mathbb{R}^N$ with $\rho_1, \ldots, \rho_N = \pm 1$ such that for any solution of (2) one has

$$\lim_{t\to+\infty} x(t) = \rho v^T x(0) \Leftrightarrow \lim_{t\to+\infty} x_j(t) = \rho_j \sum_{k=1}^N v_k x_k(0).$$

The proof of Lemma 1 will be given in Appendix.

Lemma 1 shows that time-varying protocol (2) may establish modulus consensus of the following types:

1) trivial consensus: $v = 0 \Longrightarrow \lim_{t\to+\infty} x_j(t) = 0 \forall x(0)$;
2) nontrivial consensus: $v \neq 0$, $\rho_1 = \ldots = \rho_N$, so opinions agree at some value dependent on $x(0)$;
3) bipartite consensus: $v \neq 0$, ρ_i have different signs, so opinions polarize for any initial data with $v^T x(0) \neq 0$;

In the situations 2) and 3) we say the protocol leads to *non-trivial* modulus consensus. One of the principal features of the protocols with antagonistic interactions is that trivial consensus is possible in the absence of global network connectivity, e.g. the protocol (2) with constant topology is exponentially stable whenever the graph is structurally unbalanced. This property does not require connectivity and holds for instance for the union of two independent structurally unbalanced graphs. In this sense situation 1) may be considered as "degenerate": opinions converge without involving all of the agents into agreement.

Below we obtain necessary and sufficient conditions for each of the behaviors 1)-3) under additional restriction of bidirectional interaction between the agents, the interacting agents at any time being either mutual cooperators or mutual antagonists. Moreover, those relations do not degrade over time, as implied by the following assumption:

Assumption 1: For any $t \geq 0$ the graph $G[A(t)]$ is bidirectional and digon sign-symmetric, furthermore

$$|a_{jk}(t)| \leq K|a_{kj}(t)|, \quad \forall j,k \in 1:N \tag{5}$$

for some constant $K \geq 0$.

The extension of our results to the directed topology case seems to be a non-trivial problem since even for purely cooperative protocols, a visible gap between necessary and sufficient conditions for consensus still exists [14]. The sufficiency of *uniform connectivity* [25] for modulus consensus over directed graph may be shown analogous to discrete-time case in [15]; however, the classification of modulus consensus types in this case is a subject of ongoing research.

IV. Main Results

Throughout the section, we suppose Assumption 1 always holds.

As was shown in recent papers [10], [14], cooperative consensus protocol (3) ($a_{jk}(t) \geq 0$) establishes consensus if and only if the graph of *essential interactions* [14] (named in [10] as the graph of unbounded interactions) is connected (analogous criterion for consensus robustness in [26] was referred to as the "infinitely jointly connected topology"). Below we introduce an analogous concept, playing a crucial role hereinafter, for the case of signed interaction graph.

For any $\alpha \in \mathbb{R}$, we denote $\alpha^+ := (|\alpha| + \alpha)/2 \geq 0$, $\alpha^- := (|\alpha| - \alpha)/2 \geq 0$, thus $\alpha = \alpha^+ - \alpha^-$ and $|\alpha| = \alpha^+ + \alpha^-$.

Definition 2: Agents j and k are said to *essentially interact*, if $\int_0^\infty |a_{jk}(t)|dt = \infty$, *essentially cooperate*, if $\int_0^\infty a_{jk}^+(t) = \infty$ and *essentially compete* if $\int_0^\infty a_{jk}^-(t) = \infty$.

In the case of static graph $A(t) \equiv A$, the essential interaction means that $a_{jk} \neq 0$ and any interacting pair of agents either essentially cooperate or essentially compete. However, in general cases it is possible that a pair of agents is both essentially cooperative and essentially competitive.

Let $\mathcal{E}, \mathcal{E}^+ \subseteq \mathcal{E}, \mathcal{E}^- \subseteq \mathcal{E}$ be the sets of those pairs of agents (j,k) that respectively essentially interact, essentially cooperate and essentially compete. It is evident that $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$. The (undirected) *graph of essential interactions* is defined by $\mathfrak{G} := (1:N, \mathcal{E})$. If $\mathcal{E}^+ \cap \mathcal{E}^- = \emptyset$, we assign the weights $+1$ and -1 to the arcs from $\mathcal{E}^+$ and $\mathcal{E}^-$ respectively, transforming $\mathfrak{G}$ to a signed graph $\mathfrak{G}^\pm = (1:N, \mathcal{E}, (s_{jk}))$, $s_{jk} = +1$ for $(j,k) \in \mathcal{E}^+$ and $s_{jk} = -1$ for $(j,k) \in \mathcal{E}^-$.

Definition 3: The graph $G[A(\cdot)]$ is *essentially connected* if the graph $\mathfrak{G}$ is connected. The graph $G[A(\cdot)]$ is *essentially structurally balanced* if $\mathcal{E}^+ \cap \mathcal{E}^- = \emptyset$ and $\mathfrak{G}^\pm$ is structurally balanced and *essentially cooperative* if $\mathcal{E}^- = \emptyset$.

Remark 1: Suppose that there exists $M > m > 0$ such that $|a_{ij}(t)| \in \{0\} \cup [m; M]$ for any i,j. The graph $G[A(\cdot)]$ is essentially connected if and only if $G[A(t)]$ is connected for any t from a set of infinite Lebesgue measure. If entries of $A(t)$ do not change sign: $a_{ij}(t)a_{ij}(s) \geq 0 \forall t, s \geq 0$ and $G[A(t)]$ is structurally balanced for any t, the graph $G[A(\cdot)]$ is essentially structurally balanced.

The following main theorem gives necessary and sufficient conditions for reaching non-trivial modulus consensus.

Theorem 1: The protocol (2) establishes non-trivial modulus consensus if and only if $G[A(\cdot)]$ is essentially connected and essentially structurally balanced. It establishes non-trivial consensus in the case of essentially cooperative graph $G[A(\cdot)]$, and bipartite consensus otherwise. If $G[A(\cdot)]$ is essentially connected but not essentially balanced, the protocol (2) establishes consensus at zero.

In the case of purely cooperative protocol ($a_{jk}(t) \geq 0$) the topology is evidently essentially cooperative, and Theorem 1 transforms into the result obtained in [10], [14], [26]:

Corollary 1: Cooperative protocol establishes consensus if and only if the graph $G[A(\cdot)]$ is essentially connected.

Theorem 1 implies that consensus may survive the antagonism between some pairs of agents, if this competition is "dominated" by the overall collaboration.

Our next result addresses the case where $\mathfrak{G}$ is not necessarily connected and thus may be decomposed into several disjoint connected components $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \ldots \cup \mathfrak{G}_d$, $\mathfrak{G}_r = (V_r, \mathcal{E}_r)$, $d \geq 1$. In this case in any component $\mathfrak{G}_r$ modulus consensus is established, the type of which depends only on the structure of $\mathfrak{G}_r$. Let $\mathcal{E}_r^+ := \mathcal{E}_r \cap \mathcal{E}^+$ and $\mathcal{E}_r^- := \mathcal{E}_r \cap \mathcal{E}^-$. If $\mathcal{E}_r^+ \cap \mathcal{E}_r^- = \emptyset$, define a signed graph $\mathfrak{G}_r^{\pm}$ by assigning arcs from $\mathcal{E}_r^+, \mathcal{E}_r^-$ with weights $+1$ and -1 respectively.

Theorem 2: For any solution of (3) there exists limits $x_i^{\dagger} = \lim\limits_{t\to\infty} x_i(t)$ and $|x_i^{\dagger}| = |x_j^{\dagger}|$ whenever i and j are in the same connected component: $i, j \in V_r$. If $\mathcal{E}_r^+ \cap \mathcal{E}_r^- = \emptyset$ and $\mathfrak{G}_r^{\pm}$ is structurally balanced, this modulus consensus is non-trivial: $x_i^{\dagger}$ for a.a. $x(0)$. If $\mathcal{E}_r^- = \emptyset$ non-trivial consensus is established ($x_i^{\dagger} = x_j^{\dagger} \forall i, j \in V_r$), and otherwise the opinions polarize ($V_r = V_r^1 \cap V_r^2$ and $x_i^{\dagger} = -x_j^{\dagger} \, \forall i \in V_r^1, j \in V_r^2$). If $\mathcal{E}_r^+ \cap \mathcal{E}_r^- \neq \emptyset$ or the graph $\mathfrak{G}_r^{\pm}$ is structurally unbalanced, $x_i^{\dagger} = 0 \, \forall i \in V_r$ that is, trivial consensus in V_r is established.

The following criterion of trivial consensus is immediate from Theorem 2.

Corollary 2: The protocol (2) establishes trivial consensus if and only if for any connected component $\mathfrak{G}_r$ one has either $\mathcal{E}_r^+ \cap \mathcal{E}_r^- \neq \emptyset$ or $\mathfrak{G}_r^{\pm}$ being structurally unbalanced.

Theorems 1 and 2 in fact remain valid for some types of *directed* graphs as well, e.g. weight-balanced and *cut-balanced* [10] ones. Corresponding extensions are beyond the present work and to appear in our journal paper [22]. However, Assumption 1 cannot be fully discarded; for general dynamic graphs modulus consensus may fail even under the *uniform quasi-strong connectivity* assumption which is a conventional condition for consensus in purely cooperative networks [13], [18], see [21], [22] for details. To guarantee modulus consensus, one requires more restrictive *uniform strong connectivity* property [21], [22]. Even if modulus consensus is not established, protocol (3) guarantees the solutions to be bounded, as implied by the following result.

Lemma 2: For any solution of networked system (3) (where Assumption 1 is not necessarily valid), the function $|x(t)|_\infty = \max_i |x_i(t)|$ is monotonically non-increasing: $|x(t)|_\infty \leq |x(t_0)|_\infty$ whenever $t \geq t_0 \geq 0$. Equivalently, the Cauchy evolutionary matrix $\Phi(t; t_0)$ of system (2) satisfies the inequality $|\Phi(t; t_0)|_\infty \leq 1$ for $t \geq t_0$.

The proofs of Theorems 1, 2 and Lemma 2 have been omitted due to space limitations; they are available upon request and to appear in [22].

V. Applications

In this section we consider some applications of Theorems 1 and 2, which allow in particular to compare our results with the results from [2].

A. Static protocols

Our first example deals with static interaction graphs.

Theorem 3: Assume that $A(t) \equiv A$ and the graph $\mathfrak{G} = G[A]$ is digon-symmetric and bidirectional (in other words, $a_{ij} \neq 0 \Leftrightarrow a_{ji} \neq 0$ and $a_{ij}a_{ji} \geq 0$). The protocol (2) establishes modulus consensus in each connected component of $\mathfrak{G}_r$ of the graph $\mathfrak{G}$ which is non-trivial if and only if $\mathfrak{G}_r$ is structurally balanced. The protocol (2) establishes non-trivial modulus consensus in the whole community if and only $\mathfrak{G}$ is connected and structurally balanced, and it establishes trivial consensus if and only if each $\mathfrak{G}_r$ is structurally unbalanced.

Theorem 3, which obviously follows from Theorems 1 and 2, comprises the result of [2, Theorem 1] (addressing only the connected graph case) and gives necessary and sufficient conditions for establishing modulus consensus with a static graph. More general modulus consensus criteria under static *directed* graph may be found in [21], [22].

B. Additive Laplacian protocols

Our next examples concern with *nonlinear* consensus algorithms similar to those considered in [2]. We start with two types of nonlinear protocols that are referred in [2] as the "additive Laplacian feedback schemes". The first of them is

$$\dot{x}_i(t) = \sum_{j=1}^{N} |a_{ij}(t)| (h_{ij}(x_j(t) \, sgn \, a_{ij}(t)) - h_{ij}(x_i(t))) \, \forall i, \tag{6}$$

and the second protocol has the form

$$\dot{x}_i(t) = \sum_{j=1}^{N} |a_{ij}(t)| h_{ij}(x_j(t) \, sgn \, a_{ij}(t) - x_i(t))) \, \forall i. \tag{7}$$

We assume that $h_{ij} \in C^1(\mathbb{R})$ and is strictly increasing ($h'_{ij} > 0$), $h_{ij}(0) = 0$. Let $H_{ij}[y, z] := (h_{ij}(y) - h_{ij}(z))/(y - z)$ for $y \neq z$ and $H_{ij}[z, z] := h'_{ij}(z)$. It is easily noticed that H_{ij} is a continuous function. Under those assumptions on h_{ij}, Theorems 1 (in "sufficient" part) and 2 appear to be applicable to the protocols (6),(7), as shown by the following lemma.

Lemma 3: Let $x(t)$ be a solution to system (6), which is defined for $t \geq 0$. Define the matrix $\mathfrak{A}(t) = (\mathfrak{a}_{ij}(t))$ by $\mathfrak{a}_{ij}(t) := a_{ij}(t)H_{ij}[x_j(t)\, sgn\, a_{ij}(t), x_i(t)]$. Then

$$\dot{x}(t) = -L[\mathfrak{A}(t)]x(t), \tag{8}$$

where the matrix $\mathfrak{A}(t)$ satisfies Assumption 1 and sets $\mathcal{E}, \mathcal{E}^+, \mathcal{E}^-$ for the graphs $G[\mathfrak{A}(\cdot)]$ and $G[A(\cdot)]$ are the same. The same claims are valid for any solution of (7), taking $\mathfrak{a}_{ij}(t) := a_{ij}(t)H_{ij}[x_j(t)\, sgn\, a_{ij}(t) - x_i(t), 0]$.

Proof: We consider system (6), and the protocol (7) may be studied in the same way. Equation (8) is immediate from the definitions of $\mathfrak{a}_{ij}$ and H_{ij}. As implied by Lemma 2 (independent of Assumption 1), any solution of (8) is globally bounded; moreover, $|x(t)|_\infty \leq |x(0)|_\infty$. . Since $H_{ij} > 0$ are continuous functions, there exist $M > m > 0$ such that $m \leq H_{ij}[y, z] \leq M$ whenever $|y|, |z| \leq |x(0)|_\infty$, and these inequalities hold, in particular, for $y := x_j(t)\, sgn\, a_{ij}(t)$ and $z := x_i(t)$. Therefore, $m|a_{jk}| \leq |\mathfrak{a}_{jk}| \leq M|a_{jk}|$, and hence (5) holds for $\mathfrak{A}(t)$ as well, replacing K with KM/m. Also we have $ma^+_{jk} \leq \mathfrak{a}^+_{jk} \leq Ma^+_{jk}$ and $ma^-_{jk} \leq \mathfrak{a}^-_{jk} \leq Ma^-_{jk}$. Thus agents i and j essentially interact (respectively, essentially cooperate or compete) over the graph $G[A(\cdot)]$ if and only if they essentially interact (respectively, essentially cooperate or compete) over $G[\mathfrak{A}(\cdot)]$, which proves coincidence of the sets $\mathcal{E}, \mathcal{E}^+, \mathcal{E}^-$. ■

Application of Theorems 1,2 to (8) results in the following.

Theorem 4: Suppose that Assumption 1 holds and $h_{ij} \in C^1(\mathbb{R})$ are increasing functions, $h_{ij}(0) = 0$. If the graph $G[A(\cdot)]$ is essentially connected, the protocols (6),(7) establish modulus consensus. This consensus is trivial unless the graph $G[A(\cdot)]$ is essentially structurally balanced. If it is essentially cooperative, non-trivial consensus is established. Otherwise, the protocols (6),(7) lead to bipartite consensus. If the graph $G[A(\cdot)]$ is not essentially connected, modulus consensus in each connected component $\mathfrak{G}_r$ is established which is trivial unless $\mathfrak{G}^\pm_r$ is well defined and structurally balanced.

Notice that we cannot directly apply the "necessity" part of Theorem 1, stating that essential connectivity and essential structural balance are necessary for non-trivial modulus consensus, since it addresses the protocol (2) where matrix $A(t)$ is common for all of the solutions, whereas $\mathfrak{A}(t)$ depends on concrete solution. In fact, necessity can also be proved for the protocols (6),(7) by retracing arguments from the proof of Theorem 1, which is given in our journal paper [22].

Comparing the result of Theorem 4 with results of [2, Theorem 3,4], one notices that our assumption about the nonlinearities h_{ij} differs from [2], where nonlinearities are not assumed to be smooth, but only monotonic with some integral constraint. However, unlike [2, Theorem 3,4], functions h_{ij} may be heterogeneous and not necessarily odd. Also, our result is applicable to time-variant graphs, satisfying Assumption 1.

C. Nonlinear Laplacian Flow

At last, we examine nonlinear consensus protocol similar to that addressed in [2, Section IV-B]:

$$\dot{x}_i(t) = \sum_{j=1}^{N} |F_{ij}(t, x)|(x_j(t)\, sgn\, F_{ij}(t, x) - x_i(t)), \tag{9}$$

here $i \in 1 : N$ and $F_{ij} : [0; \infty) \times \mathbb{R}^N \to \mathbb{R}$ are Caratheodory maps, i.e. $F_{ij}(t, \cdot)$ are continuous for a.a. t and $F_{ij}(\cdot, x)$ are measurable for any x. We make the following assumption on those mappings.

Assumption 2: For any compact set $C \subset [0; \infty) \times \mathbb{R}^N$ one has $\sup\{|F_{ij}(t, x)| : (t, x) \in C\} < \infty$ for any i, j and

$$K^{-1}|F_{ji}(t, x)| \leq |F_{ij}(t, x)| \leq K|F_{ji}(t, x)|\, \forall (t, x) \in C.$$

Here $K = K(C) \geq 1$ does not depend on concrete (t, x). For any t, x we have $F_{ij}(t, x)F_{ji}(t, x) \geq 0$.

Theorem 5: Under Assumption 2, for any initial data $x(0)$ a solution of (9) exists for $t \geq 0$ and has finite limit $\lim_{t\to\infty} x(t) = x^0$, moreover, $|x(t)|_\infty \leq |x(0)|_\infty\, \forall t \geq 0$. If the graph $G[A(\cdot)]$, corresponding to the matrix $A(t) := F_{ij}(t, x(t))$, is essentially connected, the solution reaches modulus consensus: $|x^0_i| = |x^0_j|$ for any i, j. Otherwise, modulus consensus in any connected component of the correspondent graph $\mathfrak{G}$ is reached.

Proof: Thanks to Lemma 2 the function $|x(t)|_\infty$ is non-increasing along any solution, and hence $|x(t)|_\infty \leq |x(0)|$ on any interval where the solution $x(t)$ exists. Hence, any solution may be prolonged to infinity. The remaining claims follow from Theorems 1,2. ■

Although it is hard to verify the essential connectivity of $G[A(\cdot)]$, where $A(t)$ depends on the solution, in special cases such a property may be proved. For instance, it is implied by the *global ε-connectivity* [2]: the graph $G(\hat{F}_{ij}(t, x))$, where $\hat{F}_{ij}(t, x) = F_{ij}(t, x)$ if $|F_{ij}(t, x)| > \varepsilon$ and $\hat{F}_{ij}(t, x) = 0$ otherwise, is connected for any t, x. The result of Theorem 5 extends the result from [2, Section IV-B] in several ways. First of all, it deals with time-variant gains $F_{ij}(t, x)$ and does not require them to have constant sign. In particular, system (9) is not necessarily monotonous [1], [27]. Moreover, it does not require weight balance which can hardly be provided for nonlinear functions F_{ij}, requiring only Assumption 2. At last, it replaces constant connectivity (e.g. global ε-connectivity where $\varepsilon > 0$) with essential connectivity.

VI. Conclusion

In the present paper, we extend a model of opinion dynamics in social network with both attractive and repulsive interactions between the agents, which was proposed in recent papers by C. Altafini, who considered conventional first-order consensus protocol over signed graph. Altafini showed, in particular, the possibility of opinion polarization if the interaction graph is structurally balanced, that is, the community splits into two competing camps, where agents in each camp cooperate. In general, the protocol establishes modulus consensus, where the agents agree in modulus but differ in sign. In the present paper, we have examined dynamics of Altafini's protocols with switching bidirectional topologies and offered necessary and sufficient conditions for reaching modulus consensus. We are currently working with sociologists to test the theoretical results presented in this paper using data from human social groups.

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APPENDIX
PROOF OF LEMMA 1

Consider the protocol (2), establishing modulus consensus. Note that since functions $x_i(t)$ are continuous, existence of the limits $\lim\limits_{t\to+\infty} |x_i(t)| = x_*$ implies that the limits $\lim\limits_{t\to+\infty} x_i(t)$ also exist (and equal to $\pm x_*$). Therefore $\Phi(t|0) \underset{t\to\infty}{\longrightarrow} \Phi_* := [\phi_1, \ldots, \phi_N]$ as $t \to \infty$, where columns ϕ_j have entries with equal modules (here $\Phi(t|t_0)$ stands for the Cauchy evolutionary matrix of the system (3)). The same applies to any linear combination $\sum_{j=1}^N \alpha_j \phi_j$. If $\Phi_* = 0$, the statement of Lemma 1 is evident, taking $v = 0$. Assume that one of ϕ_j, say, ϕ_1 is nonzero, thus $\phi_1 = v_1\rho$ where $v_1 \neq 0$ and ρ is a vector with entries ± 1. Notice, that for any real numbers $\alpha, \beta \neq 0$ we have $|\alpha - \beta| \neq |\alpha + \beta|$. Therefore, if $\phi_j \neq 0$ for some $j \neq 1$, all entries of $\phi_j - \phi_1$ have the same module if and only if $\phi_j = v_j\rho$, $v_j \neq 0$ If $\phi_j = 0$, we put by definition $v_j = 0$. Therefore, $\phi_j = v_j\rho$ for any j and $\lim\limits_{t\to\infty} x_j(t) = \Phi_* = \rho v^T x(0)$, where $v := (v_1, \ldots, v_N)^T$. ■