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Opinion dynamics using Altafini's model with a time-varying directed graph

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Abstract— **Distributed control policies (or protocols) for multi-agent consensus have been extensively studied in recent years, motivated by numerous applications in engineering and science. Most of these algorithms assume the agents to be mutually cooperative and "trustful" and correspondingly attractive couplings between the agents bring the values of the agents' states closer. Opinion dynamics of real social groups, however, require beyond conventional models of multi-agent consensus due to ubiquitous competition and distrust between some pairs of agents, which are usually characterized by the repulsive coupling. Antagonistic interactions prevent the averaging tendency of the opinions, which cooperative consensus protocols promote, and may lead to their polarization and clustering. A simple yet insightful model of opinion dynamics with both attractive and repulsive couplings was proposed recently by C. Altafini, who examined first-order consensus algorithms over static signed interaction graphs, where arcs of positive weights connect cooperating agents, and of negative weights correspond to antagonistic pairs. This protocol establishes modulus consensus, where the opinions become the same in modulus but may differ in sign. In the present paper, we extend the modulus consensus model to the case where the network topology is arbitrary time-varying, directed, signed graph. We show that under mild condition of uniform strong connectivity of the network, the protocol establishes agreement of opinions in moduli, whose signs may be opposite, so that the agents' opinions either reach consensus or polarize. This result is further extended to nonlinear consensus protocols. We show also that, unlike cooperative consensus algorithms, uniform strong connectivity cannot be relaxed to uniform quasi-strong connectivity (UQSC).**

I. Introduction

The striking phenomenon of global consensus in a multi-agent network, which is caused by only local interactions between the agents, has attracted long-standing interest from the research community. The interest is motivated by numerous natural phenomena and engineering designs, that are based, implicitly or explicitly, on reaching synchronism between the agents (components of a complex system). Examples include, but are not limited to, intelligence of large-scale biological populations and multi-agent robotics. We refer the reader to [14], [21], [22] for excellent surveys of recent research on distributed consensus protocols and their applications, as well as historical milestones.

Originated from iterative procedures of decision-making [6], early consensus algorithms were based on the principle of contraction, or *averaging*: every agent's state constantly evolves to the relative interior of the convex hull spanned by its own and neighbors' states. The convex hull spanned by all the agents states, driven by such a protocol, is shrinking over time. Based on the Lyapunov-like properties of this convex hull and its diameter [3], [11], [16] and relevant results on convergence of infinite products of stochastic matrices [5], [14], [21], convergence properties of averaging consensus protocols were examined intensively for multi-agent systems with special attention on the effect of time-variant interaction topology. Necessary and sufficient conditions for consensus over undirected [4], [12], [16] and cut-balanced graphs [10] boil down to infinite joint connectivity of the network. For general directed graph the condition of *uniform quasi-strong connectivity* (UQSC) [11], in spite of being only sufficient, is considered to be "the weakest assumption on the graph connectivity such that consensus is guaranteed for arbitrary initial conditions" [17]. This common belief has recently been confirmed by a fundamental result from [24], stating that the UQSC is necessary and sufficient for consensus robustness against bounded disturbances. It is also necessary and sufficient for consensus in a stronger "uniform" sense [11]. Many high-order protocols extend their first-order counterparts [21], [22] or are squarely based on them [23].

Despite many natural and engineered teams of agents are known to achieve common goals due to cooperation, real-world networks often involve repulsive couplings which represent competition or antagonism between some pairs of agents. In multi-agent swarms, repulsive couplings may prevent collisions between close agents [26]. In real social groups, such relations between the individuals taking the form of competition or hostility, are also ubiquitous and seriously influence the dynamics of opinions, leading in general to polarized or clustering behavior. Unlike cooperative consensus algorithms, the protocols with both attractive and repulsive couplings still demand more indepth mathematically rigorous analysis. A simple yet instructive model of opinion dynamics where the agents may be both "friendly" or "hostile" were examined by C. Altafini [1], [2]. Altafini considered a first-order consensus protocol over a *strongly connected* signed interaction graph, where the positive weight of an arc implies cooperation between the two agents and the negative one corresponds to their antagonism. This protocol establishes "modulus consensus", where the opinions agree in modulus but may differ in signs. If the consensus value of modulus is non-zero, the modulus consensus is referred to as the *bipartite consensus*, where the agents divide into two groups with polarized opinions (one of these groups, however, may be empty, so that all opinions

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agree). The latter situation is possible only if the network is structurally balanced, that is, the community splits into two hostile camps (e.g. adherence of two political parties), the relations inside each faction being cooperative.

The result from [1], [2] is mainly concerned with networks with static topologies and is based on techniques of gauge transformations. In the very recent paper [20] we extended these results to the case of time-varying undirected graphs and obtained complete necessary and sufficient conditions for modulus consensus, including both consensus and polarization (bipartite consensus). Similarly to classical consensus protocols, joint integral connectivity of the network is sufficient for modulus consensus and is "almost necessary", except for the degenerate situation where the opinions converge to zero independently of initial values. In the present paper, we consider Altafini's model on opinion dynamics over general directed time-varying graphs. Though one could expect the UQSC to be sufficient for the modulus consensus, under signed graph it *does not* imply consensus even for static graphs. We show, however, that if the topology is uniformly *strongly* connected, then modulus consensus is achieved. We also examine nonlinear consensus protocols analogous to those in [1], [2] under switching graphs.

The paper is organized as follows. Section III gives technical preliminaries and the setup of the problem in question. Section IV presents the main results, and their application is discussed in Section V. Appendix gives a proof of a technical lemma.

II. Preliminaries

Throughout the paper $m:n$, where $m \le n$ are integers, stands for the sequence $\{m, m+1, \dots, n\}$. The sign of a number $x \in \mathbb{R}$ is denoted by $sgn\, x \in \{-1, 0, 1\}$. The abbreviation "a.a." stands for "almost all" (except for the set of zero measure). Given a matrix $L = (L_{ij})$, let $|L|_\infty := \max_i \sum_j |L_{ij}|$, e.g. for column vector $x \in \mathbb{R}^N$, one has $|x|_\infty = \max_i |x_i|$. It is easy to show that $|L|_\infty = \sup_{|x|_\infty \le 1} |Lx|_\infty$, where x is appropriately dimensioned.

A signed (weighted) graph is a triple $G = (V, E, A)$, where $V = \{v_1, \dots, v_N\}$ stands for the set of *nodes*, $E \subset V \times V$ is a set of *arcs* and $A \in \mathbb{R}^{N\times N}$ is a signed *adjacency matrix* such that $a_{jk} \neq 0$ if and only if $(v_k, v_j) \in E$. We always assume the graph has no self-loops: $a_{jj} = 0$. We say the graph is *bidirectional* if $a_{jk} \neq 0 \Leftrightarrow a_{kj} \neq 0$. Throughout the paper, we confine ourselves to *digon sign-symmetric* graphs [2] which means that opposite arcs (if exist) cannot have different signs, e.g. $a_{jk}a_{kj} \ge 0$. Digon sign-symmetric graph is *structurally balanced* [2], [7] if the set of nodes is divided into two sets $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that $a_{jk} \ge 0$ if $(j,k) \in V_i$ and $a_{jk} \le 0$ for $j \in V_1, k \in V_2$.

A *route* connecting nodes v and v' is a sequence of nodes $v_{i_0} := v, v_{i_1}, \dots, v_{i_{n-1}}, v_{i_n} := v'$ $(n \ge 1)$ such that $(v_{i_{k-1}}, v_{i_k}) \in E$ for $k \in 1:n$. A route where $v_{i_0} = v_{i_n}$ is referred to as a *cycle*. The cycle is *positive* if $a_{i_0i_1}a_{i_1i_2}\dots a_{i_{n-1}i_n} > 0$ and *negative* otherwise. The digon-symmetric *strongly connected* graph is structurally balanced if and only if all its oriented cycles are positive [2], [7]. A node is called *root* if it can be connected with a route to any other node of the graph. A graph is *quasi-strongly connected* (QSC) if it has at least one root and *strongly connected* if any two of its nodes may be connected by a route.

Any matrix $A \in \mathbb{R}^{N\times N}$ may be assigned to a signed graph $G[A] = (1:N, E[A], A)$ where $E[A] := \{(j,k) : a_{kj} \neq 0\}$. Following [2], its *Laplacian* matrix $L = L[A]$ is defined by

$$L = (L_{jk})_{j,k=1}^N, \quad L_{jk} := \begin{cases} -a_{jk}, \, j \neq k \\ \sum_{m=1}^N |a_{jm}|, j = k. \end{cases} \qquad (1)$$

In the case where $a_{jk} \ge 0 \, \forall j,k$ and $|a_{jk}| = a_{jk}$, the matrix (1) is the conventional Laplacian of a weighted graph [18].

We say the graph $G[A]$ is strongly (respectively, quasi-strongly) ε-connected if $G[A^\varepsilon]$ is strongly (respectively, quasi-strongly) connected where $A^\varepsilon = (a_{ij}^\varepsilon)$; $a_{ij}^\varepsilon = a_{ij}$ when $|a_{ij}| \ge \varepsilon$ and $a_{ij}^\varepsilon = 0$ otherwise. Time-variant graph $G[A(t)]$, where A is locally summable, is called *uniformly* strongly connected (respectively, uniformly quasi-strongly connected, UQSC), if there exist $\varepsilon > 0, T > 0$ such that $G\big[\int_t^{t+T} A(s)\, ds\big]$ is strongly (respectively, quasi-strongly) ε-connected for any $t \ge 0$.

III. Problem Setup

Consider a group of $N \ge 2$ agents indexed 1 through N, the opinion of the i-th agent is denoted by $x_i \in \mathbb{R}$ and we define $x := (x_1, \dots, x_N)^T \in \mathbb{R}^N$. The agents update their opinions in accordance with a distributed protocol as follows:

$$\dot{x}(t) = -L[A(t)]x(t), t \ge 0, \qquad (2)$$

which may be written componentwise as

$$\dot{x}_j(t) = \sum_{k=1}^N |a_{jk}(t)|(x_k(t)\, sgn\, a_{jk}(t) - x_j(t)) \, \forall j. \qquad (3)$$

Here the matrix $A(t) = (a_{jk}(t))$ with $a_{jj}(t) \equiv 0$ is locally bounded and describes the interaction topology of the network. At time $t \ge 0$, the opinion of the j-th agent is influenced by agents for which $a_{jk}(t) \neq 0$ ("neighbors"). Unlike usual consensus protocols [18] this influence may be both cooperative (when $a_{jk} > 0$) or competitive (when $a_{jk} < 0$). The coupling term $|a_{jk}|(x_k\, sgn\, a_{jk} - x_j)$ in (3) drives infinitesimally the opinion of the j-th agent, respectively, either towards the opinion of the k-th one or against it.

In the paper [2] the protocol (2) was thoroughly examined in the case of a constant signed interaction graph ($A(t) \equiv A$), assumed to be digon sign-symmetric and *strongly connected*. Unlike purely cooperative consensus algorithms ($a_{jk}(t) \ge 0$), (2) does not guarantee shrinking of the convex hull of opinions. It was shown that $(-L[A])$ is a Hurwitz matrix and thus opinions converge to 0 independent of initial values, unless the graph is structurally balanced. This property implies that a community is divided into two hostile camps (such as votaries of two political parties), where each agent cooperates with its camp-mates, competing with "opponents" from the other camp. The case of unsigned weighted graph is a special case of a structurally balanced sign graph where one of the antagonistic camps is empty. For structurally balanced graphs, two situations of the network behavior are possible depending on the graph and initial condition: the opinions may reach consensus or agree in modulus, differing in sign (bipartite consensus [2]). Summarizing, the protocol (2) with $A(t) = const$ and $G[A]$ strongly connected provides the *modulus consensus* [13]:

Definition 1: The protocol (2) establishes *modulus consensus*, if for any $x(0)$ a number $x_* \geq 0$ exists such that

$$\lim_{t\to+\infty} |x_i(t)| = x_*. \tag{4}$$

It is easy to show that under modulus consensus the opinion of each agent has a finite limit $\lim_{t\to+\infty} x_i(t)$. The following lemma, proved in Appendix, describes the structure of these ultimate opinions.

Lemma 1: Suppose that protocol (2) establishes modulus consensus. Then there exist vectors $v, \rho \in \mathbb{R}^N$ with $\rho_1, \ldots, \rho_N = \pm 1$ such that for any solution of (2) one has

$$\lim_{t\to+\infty} x(t) = \rho v^T x(0) \Leftrightarrow \lim_{t\to+\infty} x_j(t) = \rho_j \sum_{k=1}^{N} v_k x_k(0).$$

Lemma 1 shows that time-varying protocol (2) may establish modulus consensus of the following types:

1) trivial consensus: $v = 0 \Longrightarrow \lim_{t\to+\infty} x_j(t) = 0 \forall x(0)$;
2) nontrivial consensus: $v \neq 0$, $\rho_1 = \ldots = \rho_N$, so opinions agree on some value dependent on $x(0)$;
3) bipartite consensus: $v \neq 0$, ρ_i have different signs, so opinions polarize for any initial data with $v^T x(0) \neq 0$.

The goal of the present paper is to disclose conditions which guarantee modulus consensus for general time-varying graphs. In the paper [2] this problem was considered only for the very special case where the graph is not only constantly strongly connected but also weight-balanced (although not explicitly stated, this follows from the proof relying on [18, Theorem 9]) with constant signs of the arcs. Below we relax these restrictions, requiring only the graph to be uniformly strongly connected. Dealing with real-world social networks, the time-invariance of such relationships between individuals as friendship and hostility is evidently a non-realistic assumption. What is more important, the opinion dynamics in social networks are usually considered to be nonlinear [8], [9], however, such models are often reducible to the linear case by introducing time-variant gains, depending on the solution. Considering general time-varying graphs allows us to examine both linear and nonlinear consensus protocols from [1], [2] in the common framework.

It should be noticed that the problem of determining to which behavior of 1)-3) in Lemma 1 a given graph corresponds seems to be non-trivial and is a subject of ongoing research. In the important case of *bidirectional interactions* it was solved in our recent paper [20], where necessary and sufficient criteria for each type of the modulus consensus were obtained. Another situation where this classification may be given and vectors ρ, v be explicitly found is the static topology case: $A(t) \equiv A$. The relevant result from [2] deals with strongly connected graphs only, which restriction will be discarded in Section IV-A.

IV. Main Results

The section is organized as follows. We start with the case of static graphs (Subsection IV-A), where necessary and sufficient conditions are offered for non-trivial (Lemma 2) and trivial (Lemma 3) modulus consensus. Subsection IV-B deals with the switching topology case that is the main concern of the present paper. Lemma 4 shows that the maximal of opinion moduli is non-increasing independently of the graph properties and hence has a limit at infinity. Under additional restriction of uniform strong connectivity the protocol establishes modulus consensus; this is the main result of the paper (Theorem 1). The proofs of the mentioned results are omitted due to space limitations, available upon request and to appear in [19]. In the final Subsection IV-C we demonstrate that, unlike the cooperative consensus protocols, the conventional UQSC condition is neither necessary nor sufficient for modulus consensus over general signed graph.

A. *Modulus consensus over static signed graphs*

The first lemma gives a necessary and sufficient condition for behaviors 2) and 3) under static topology.

Lemma 2: Let the topology be static: $A(t) \equiv A$. The protocol (2) establishes non-trivial modulus consensus (with $v \neq 0$) if and only if $G[A]$ is structurally balanced and quasi-strongly connected. If this is the case, the matrix $L[A]$ has zero eigenvalue of algebraic and geometric multiplicity 1. The vectors ρ and v are respectively right and left eigenvectors at zero, that is $v^T L[A] = L[A]\rho = 0$, and $v^T \rho = 1$. For the correspondent subdivision of agents into hostile camps $V_1 \cup V_2 = 1 : N$, one may choose ρ and v so that $\rho_i = 1$ for $i \in V_1$ and $\rho_i = -1$ when $i \in V_2$. Nontrivial consensus is established only if $a_{jk} \geq 0$ for any j, k and thus either V_1 or V_2 is empty; otherwise opinions polarize.

If the graph $G[A]$ is not structurally balanced, only trivial kind of modulus consensus may be reached. An obvious obstacle to this behavior is existence of a subcommunity, which is independent of the remaining agents and is able to reach non-trivial modulus consensus. Given a graph $G = (V, E, A)$, any graph $G' = (V', E', A')$, where $V' \subseteq V$, $E' = E \cap (V' \times V')$ and A' is the corresponding submatrix of A, is said to be a *subgraph* of G. We say a subgraph G' is *isolated* if $a_{jk} = 0$ whenever $j \in V'$ and $k \notin V'$, that is, agents from V' are not aware of the opinions of their teammates from $V \setminus V'$. In accordance with Lemma 2, if the graph $G = G[A]$ has a non-empty isolated *structurally balanced* subgraph, its nodes agree on some non-zero (for a.a. initial conditions) opinion which entails that the whole community is not able to reach trivial consensus. As shown by the following lemma, the inverse proposition is also valid.

Lemma 3: The protocol (2) establishes trivial consensus, i.e. $-L[A]$ is a Hurwitz matrix, if and only if the graph $G[A]$ has no non-empty structurally balanced isolated subgraphs (in particular, $G[A]$ is not structurally balanced itself).

B. *Modulus consensus under switching signed graphs*

We start with the following useful lemma, which does not rely on any connectivity assumptions and shows, in particular, that solutions to (2) are always bounded.

Lemma 4: For any solution of system (2), the function $|x(t)|_\infty = \max_i |x_i(t)|$ is ~~monotonically non-increasing~~: $|x(t)|_\infty \leq |x(t_0)|_\infty$ whenever $t \geq t_0 \geq 0$. Equivalently, the Cauchy evolutionary matrix $\Phi(t;t_0)$ of system (2) satisfies the inequality $|\Phi(t;t_0)|_\infty \leq 1$ for $t \geq t_0$.

Lemma 4 shows, in particular, that the maximal modulus always has a limit $\lim_{t\to+\infty} |x(t)|_\infty$. However, to guarantee that all other moduli converge to the same limit, one requires additional connectivity assumption.

Assumption 1: The functions $a_{jk}(t)$ are essentially bounded: there exists $M > 0$ such that $|a_{jk}(t)| \leq M$ for a.a. $t \geq 0$. The graph $G[A(\cdot)]$ is uniformly *strongly* connected.

We are now in a position to present our main theorem which gives a sufficient condition for modulus consensus.

Theorem 1: Under Assumption 1 the protocol (2) establishes modulus consensus.

As a corollary, we obtain the result from [2] for static graphs and consensus criterion for cooperative protocols.

Corollary 1: Suppose that $A(t) \equiv A$ and $G[A]$ is strongly connected. Then protocol (3) establishes modulus consensus.

The proof is immediate since a static strongly connected graph is also uniformly strongly connected.

Corollary 2: Let Assumption 1 hold and $a_{jk}(t) \geq 0$. Then protocol (2) establishes *non-trivial* consensus.

Proof: By virtue of Theorem 1, modulus consensus is established. Accordingly to Lemma 1 only three types of such a consensus are possible, which are trivial consensus, polarization and consensus. The common feature of the first two types is that for a.a. $x(0)$ there exists $i \in 1:N$ such that $\lim_{t\to+\infty} x_i(t) \leq 0$. It is well known [11], [15] that the convex hull of the agents' states $\Delta(t) = [\min_i x_i(t), \max_i x_i(t)]$ is non-expanding over time and in particular, if $x_i(0) \geq 1$, then $x_i(t) \geq 1$ for any $t \geq 0$. This obviously excludes the options of trivial consensus and bipartite consensus. ∎

C. UQSC Is Not Sufficient for Modulus Consensus

In this subsection we are going to point out a crucial differences between purely cooperative protocols (2) ($a_{jk} \geq 0$) and those with antagonistic interactions. Namely, the uniform quasi-strong connectivity (UQSC) property, commonly adopted as sufficient and "almost necessary" condition for consensus with cooperative interactions, appears to be *insufficient* for modulus consensus over signed graph. As was discussed in Section II, UQSC is also not necessary.

It is widely known that the result from Corollary 2 is in fact valid under weaker assumption of uniform *quasi-strong connectivity* [11], [15], [25], whereas in a general situation the USQ property cannot be relaxed to UQSC even for the case of *static* topology $A(t) \equiv A$, as follows from Lemma 3 and is illustrated by the following simple example.

Example 1. Consider a team of $N = 3$ agents with states $x_1(t), x_2(t), x_3(t)$. Assume that $a_{12} = a_{21} = -1$, $a_{31} = a_{32} = 1$ (see Fig. 1), and hence the equations are

$$\dot{x}_1 = (-x_2 - x_1),\ \dot{x}_2 = (-x_1 - x_2),\ \dot{x}_3 = (x_1 + x_2 - x_3).$$

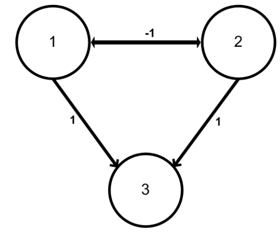

Fig. 1. Static quasi-strongly connected graph: no modulus consensus

The system has the set of equilibria $x_1 = a, x_2 = -a, x_3 = 0$, $a \in \mathbb{R}$ and hence does not reach modulus consensus.

Example 1 deals with *structurally unbalanced* static graph (agents 1 and 2 are constantly antagonistic, so the structural balance requires the agent 3 to cooperate with only one of them, competing or interacting not with the other, whereas in reality it cooperates with both of agents 1 and 2). In the same time, nodes 1 and 2 and two connecting them arcs constitute a structurally balanced isolated subgraph. Were the graph static and structurally balanced, QSC would establish modulus consensus due to Lemma 2. Moreover, this result remains valid for dynamically changing structurally balanced graph, provided that the "hostile camps" remain unchanged.

Lemma 5: Suppose that $V = 1 : N = V_1 \cup V_2$, where $a_{jk}(t) \geq 0$ for any $t \geq 0$ if $j, k \in V_1$ or $j, k \in V_2$; otherwise, $a_{jk}(t) \leq 0$ for any $t \geq 0$. If the graph $G[A(\cdot)]$ is UQSC, the protocol establishes non-trivial consensus (if $V_1 = \emptyset$ or $V_2 = \emptyset$) or bipartite consensus (when $V_1, V_2 \neq \emptyset$).

Proof: Introducing a diagonal matrix $D = diag(d_1, \ldots, d_N)$ by $d_i = 1$ for $i \in V_1$ and $d_i = -1$ for $i \in V_2$, one can easily check that the ~~gauge transformation~~ [2] $x \mapsto z := Dx$ transforms the system (2) into

$$\dot{z}(t) = -L[|A(t)|]z(t), \quad |A| = (|a_{jk}|). \tag{5}$$

Obviously $G[A]$ is UQSC if and only if $G[|A|]$ is UQSC. Therefore, protocol (5) establishes consensus, corresponding to either non-trivial consensus (if V_1 or V_2 is empty) or bipartite consensus of network (2) since $z_i = \pm x_i$. ∎

However, if the relations of friendship and hostility between the agents also evolve over time, the UQSC property is not sufficient for modulus consensus. Moreover, in the following example we construct a protocol (2) with periodic piecewise-constant matrix $A(t)$, such that the graph $G[A(t)]$ is UQSC and structurally balanced for any $t \geq 0$, but nevertheless the protocol fails to establish consensus.

Example 2. Consider the more general system

$$\begin{aligned}&\dot{x}_1(t) = (-x_2(t) - x_1(t)),\ \dot{x}_2(t) = (-x_1(t) - x_2(t)),\\&\dot{x}_3(t) = a_{31}(t)(x_1(t) - x_3(t)) + a_{32}(t)(x_2(t) - x_3(t)).\end{aligned} \tag{6}$$

The functions a_{31}, a_{32} are constructed as follows. Consider first system (6) with $a_{31}(t) \equiv 1, a_{32}(t) \equiv 0$ and the solution to (6) launched at the initial state $x_1(0) = 1, x_2(0) = -1, x_3(0) = -1/2$. It is evident that $x_1(t) = 1 = -x_2(t)$ for any $t \geq 0$ and $x_3(t) \uparrow 1$ as $t \to +\infty$. Therefore, there exists the first time instant $T_0 > 0$ such that $x_3(T_0) = 1/2$. Notice that in the symmetric situation where $a_{31}(t) \equiv 0, a_{32}(t) \equiv 1$ and $x(t)$ is a solution to (6) starting at $x_1(0) = 1, x_2(0) = -1, x_3(0) = 1/2$, one has $x_3(t) \downarrow -1$ and T_0 is the first instant where $x_3(T_0) = -1/2$. Taking

$$a_{31}(t) = 1 - a_{32}(t) = \begin{cases} 1, \; t \in [0; T_0) \cup [2T_0; 3T_0) \cup \dots \\ 0, \; t \in [T_0; 2T_0) \cup [3T_0; 4T_0) \cup \dots \end{cases},$$

one finally gets a $2T_0$-periodic $A(t)$ which, evidently, corresponds to uniformly quasi-strongly connected graph $G[A(\cdot)]$. Moreover, this graph is also quasi-strongly connected and structurally balanced at any time. Even so the solution to (6) starting at $x_1(0) = 1, x_2(0) = -1, x_3(0) = -1/2$ does not achieve modulus consensus. It can easily shown that $x_1(t) = -x_2(t) = 1$ for any $t \geq 0$. Since $a_{31}(t) = 1$ and $a_{32}(t) = 0$ when $t < T_0$, one has $x_3(T_0) = 1/2$ by definition of T_0. On the next interval $t \in [T_0; 2T_0)$ one has $a_{31}(t) = 0$ and $a_{32}(t) = 1$ and hence $x_3(2T_0) = -1/2$, so the solution $x(t)$ is periodic and $x_3(t) \in [-1/2; 1/2]$ whereas $|x_1(t)| = |x_2(t)| = 1$.

V. Applications: Nonlinear Protocols

In this section we consider some applications of Theorem 1 to nonlinear consensus protocols, similar to those studied in [1], [2].

A. Additive Laplacian protocols

Our first example concerns with *nonlinear* consensus algorithms that are referred in [2] as the "additive Laplacian feedback schemes". The first of them is

$$\dot{x}_i(t) = \sum_{j=1}^{N} |a_{ij}(t)| (h_{ij}(x_j(t) \, sgn \, a_{ij}(t)) - h_{ij}(x_i(t))), \quad (7)$$

and the second protocol has the form

$$\dot{x}_i(t) = \sum_{j=1}^{N} |a_{ij}(t)| h_{ij}(x_j(t) \, sgn \, a_{ij}(t) - x_i(t))) \, \forall i. \quad (8)$$

We adopt the following assumption about the nonlinearities.

Assumption 2: For any $i, j = 1, \dots, N$ the map $h_{ij} \in C^1(\mathbb{R})$ is strictly increasing (and hence $h'_{ij} > 0$) with $h_{ij}(0) = 0$.

Defining the functions $H_{ij}[y, z]$ as follows: $H_{ij}[y, z] := (h_{ij}(y) - h_{ij}(z))/(y - z)$ for $y \neq z$ and $H_{ij}[z, z] := h'_{ij}(z)$. It is easily noticed that H_{ij} is a continuous function and $h_{ij}(y) - h_{ij}(z) = H_{ij}[y, z](y - z) \, \forall y, z$. Under Assumption 2 Theorem 1 appears to be applicable to the protocols (7),(8), as shown by the following lemma.

Lemma 6: Let $x(t)$ be a solution to system (7), which is defined for $t \geq 0$. Define the matrix $\mathfrak{A}(t) = (\mathfrak{a}_{ij}(t))$ by $\mathfrak{a}_{ij}(t) := a_{ij}(t) H_{ij}[x_j(t) \, sgn \, a_{ij}(t), x_i(t)]$. Then

$$\dot{x}(t) = -L[\mathfrak{A}(t)] x(t). \quad (9)$$

If the matrix-valued function $A(\cdot)$ satisfies Assumption 1, the same is true for $\mathfrak{A}(\cdot)$. The same claims hold for the protocol (8), taking $\mathfrak{a}_{ij}(t) := a_{ij}(t) H_{ij}[x_j(t) \, sgn \, a_{ij}(t) - x_i(t), 0]$.

Proof: We consider system (7), and the protocol (8) may be studied in the same way. Equation (9) is immediate from the definitions of $\mathfrak{a}_{ij}$ and H_{ij}. As follows from Lemma 4, the solutions of (9) remain bounded since $|x(t)|_\infty \leq |x(0)|_\infty$. Since $H_{ij} > 0$ are continuous functions, there exist $M > m > 0$ such that $m \leq H_{ij}[y, z] \leq M$ whenever $|y|, |z| \leq |x(0)|_\infty$, and these inequalities hold, in particular, for $y := x_j(t) \, sgn \, a_{ij}(t)$ and $z := x_i(t)$. Therefore, $m|a_{jk}| \leq |\mathfrak{a}_{jk}| \leq M|a_{jk}|$, and hence if the matrix $A(\cdot)$ is bounded and the associated graph $G[A(\cdot)]$ is uniformly strongly connected, the same claims hold for $\mathfrak{A}(t)$. ■

Application of Theorem 1 to (9) yields the following result.

Theorem 2: Under Assumption 2, the solutions to systems (7),(8) exist, are unique and infinitely prolongable for any initial condition. If Assumption 1 holds, the protocols (7),(8) establish modulus consensus.

Proof: Since the right-hand sides of (7),(8) are smooth in x, the solutions exist locally and are unique. According to Lemma 4 and 6, the solutions remain bounded and thus infinitely prolongable. Under Assumption 1, modulus consensus follows from Theorem 1, applied to (9). ■

Comparing the result of Theorem 2 with that of [2, Theorem 3,4], one notices that our assumption about the nonlinearities h_{ij} differs from [2], where they are not assumed to be smooth, but only monotonic with some integral constraint. However, unlike [2, Theorem 3,4], functions h_{ij} may be heterogeneous and not necessarily odd; the graph may be time-varying, satisfying Assumption 1.

B. Nonlinear Laplacian Flow

In this subsection we examine the nonlinear consensus protocol similar to that addressed in [2, Section IV-B]:

$$\dot{x}_i(t) = \sum_{j=1}^{N} |F_{ij}(t, x)| (x_j(t) \, sgn \, F_{ij}(t, x) - x_i(t)), \quad (10)$$

here $i \in 1 : N$ and $F_{ij} : [0; \infty) \times \mathbb{R}^N \to \mathbb{R}$ are Caratheodory maps, i.e. $F_{ij}(t, \cdot)$ are continuous for a.a. t and $F_{ij}(\cdot, x)$ are measurable for any x. We assume also that for any compact set $K \subset [0; \infty) \times \mathbb{R}^N$ one has

$$\sup\{F_{ij}(t, x) : (t, x) \in K\} < \infty \quad \forall i, j. \quad (11)$$

Theorem 3: For any initial condition $x(0)$ a solution of (10) exists for $t \geq 0$. If the matrix-valued function $A(t) := F_{ij}(t, x(t))$ satisfies Assumption 1, the protocol (10) establishes modulus consensus.

Proof: Using Lemma 4, one proves that the solution is bounded and hence its derivative also remains bounded due to (11), so any solution is infinitely prolongable. The remaining claim follows now from Theorem 1. ■

To guarantee the boundedness of the matrix $A(t)$, one in practice has to strengthen the condition (11), assuming that for any compact $C \subset \mathbb{R}^N$ one has $\sup\{|F_{ij}(t, x)| : t \geq 0, x \in C\} < \infty$. Although in general it is hard to verify the uniform strong connectivity of $G[A(\cdot)]$, where

$A(t) = F_{ij}(t, x(t))$ depends on the concrete solution, in special cases such a property may also be proved. For instance, it is implied by the *global strong ε-connectivity* [2, Section IV-B]: the graph $G(\hat{F}_{ij}(t, x))$ is strongly ε-connected for any t, x. The result of Theorem 3 extends the result from [2, Section IV-B] in several ways. First of all, it deals with time-variant gains $F_{ij}(t, x)$ and does not require them to have a constant sign. In particular, system (10) does not necessarily generate order-preserving flow [1]. Moreover, we do not assume that the graph $G[A(t)]$ is weight-balanced which can hardly be provided for nonlinear functions F_{ij}. At last, we replace global ε-connectivity where $\varepsilon > 0$ with uniform strong connectivity.

VI. Conclusion

In the present paper, we extend a model of opinion dynamics in social networks with both attractive and repulsive interactions between the agents, which was proposed in recent papers by C. Altafini, who considered the conventional first-order consensus protocols over signed graphs. Altafini showed, in particular, the possibility of opinion polarization if the interaction graph is structurally balanced. In general, the protocol establishes modulus consensus, where the agents agree in modulus but differ in signs. In the present paper, we have examined dynamics of Altafini's protocols with switching directed topologies and offer sufficient conditions for reaching modulus consensus that boil down to uniform strong connectivity of the network. We are currently working with sociologists to test the theoretical results presented in this paper using data from human social groups.

References

[1] C. Altafini. Dynamics of opinion forming in structurally balanced social networks. *PLoS ONE*, 7(6):e38135, 2012.

[2] C. Altafini. Consensus problems on networks with antagonistic interactions. *IEEE Trans. Autom. Control*, 58(4):935–946, 2013.

[3] D. Angeli and P.A. Bliman. Stability of leaderless discrete-time multi-agent systems. *Math. Control, Signals, Syst.*, 18(4):293–322, 2006.

[4] L. Cao, Y. Zheng, and Q. Zhou. A necessary and sufficient condition for consensus of continuous-time agents over undirected time-varying networks. *IEEE Trans. Autom. Control*, 56(8):1915 – 1920, 2011.

[5] M. Cao, A.S. Morse, and B.D.O. Anderson. Reaching a consensus in a dynamically changing environment: convergence rates, measurement delays, and asynchronous event. *SIAM Journal of Control and Optimization*, 47(2):601–623, 2008.

[6] M.H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69:118–121, 1974.

[7] D. Easley and J. Kleinberg. *Networks, Crwods and Markets. Reasoning about a Highly Connected World*. Cambridge Univ. Press, Cambridge, 2010.

[8] A. Fläche and M.W. Macy. Small worlds and cultural polarization. *Journal of Math. Sociology*, 35(1–3):146–176, 2011.

[9] R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artifical Societies and Social Simulation (JASSS)*, 5(3):1–33, 2002.

[10] J.M. Hendricx and J. Tsitsiklis. Convergence of type-symmetric and cut-balanced consensus seeking systems. *IEEE Trans. Autom. Control*, 58(1):214–218, 2013.

[11] Z. Lin, B. Francis, and M. Maggiore. State agreement for continuous-time coupled nonlinear systems. *SIAM Journ. of Control and Optimization*, 46(1):288–307, 2007.

[12] A. Matveev, I. Novinitsyn, and A. Proskurnikov. Stability of continuous-time consensus algorithms for switching networks with bidirectional interaction. In *Proceedings of European Control Conference ECC-2013*, pages 1872–1877, 2013.

[13] Z. Meng, G. Shi, K.H. Johansson, M. Cao, and Y. Hong. Modulus consensus over networks with antagonistic interactions and switching topologies. *http://arxiv.org/abs/1402.2766*, 2014.

[14] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, Princeton and Oxford, 2010.

[15] L. Moreau. Stability of continuous-time distributed consensus algorithms. In *Proc. IEEE Conf. Decision and Control (CDC 2004)*, pages 3998 – 4003, 2004.

[16] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Trans. Autom. Control*, 50(2):169–182, 2005.

[17] U. Münz. *Delay robustness in cooperative control (PhD thesis)*. Stuttgart, 2010, 2010.

[18] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Autom. Control*, 49(9):1520–1533, 2004.

[19] A. Proskurnikov, A. Matveev, and M. Cao. Opinion dynamics in the coexistence of hostile camps: Altafini's model with a dynamic graph. *IEEE Trans. Autom. Contr. (submitted)*.

[20] A. Proskurnikov, A. Matveev, and M. Cao. Consensus and polarization both as possible outcomes in the coexistence of hostile camps. In *Proceedings of IEEE CDC 2014 (accepted)*, Los Angeles, CA, 2014.

[21] W. Ren and R. Beard. *Distributed consensus in multi-vehicle cooperative control: theory and applications*. Springer, 2008.

[22] W. Ren and Y. Cao. *Distributed Coordination of Multi-agent Networks*. Springer, 2011.

[23] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45:2557–2562, 2009.

[24] G. Shi and K.H. Johansson. Robust consensus for continuous-time multi-agent dynamics. *SIAM J. Control Optim*, 51(5):3673–3691, 2013.

[25] U.Muñz, A. Papachristodoulou, and F. Allgöwer. Consensus in multi-agent systems with coupling delays and switching topology. *IEEE Trans. Autom. Control*, 56(12):2976–2982, 2011.

[26] W. Yu, G. Chen, M. Cao, J. Lü, and H.T. Zhang. Swarming behaviors in multi-agent systems with nonlinear dynamics. *Chaos*, 23:043118, 2013.

Appendix
Proof of Lemma 1

Consider the protocol (2), establishing modulus consensus. Let $\Phi(t)$ stands for the Cauchy evolutionary operator so that $x(t) = \Phi(t)x(0)$. Note that since functions $x_i(t)$ are continuous, the existence of the limits $\lim\limits_{t\to+\infty} |x_i(t)| = x_*$ implies that the limits $\lim\limits_{t\to+\infty} x_i(t)$ also exist (and equal to $\pm x_*$). Therefore $\Phi(t) \underset{t\to\infty}{\longrightarrow} \Phi_* := [\phi_1, \ldots, \phi_N]$ as $t \to \infty$, where the columns ϕ_j have entries with equal modules. The same applies to any linear combination $\sum_{j=1}^N \alpha_j \phi_j$. If $\Phi_* = 0$, the statement of Lemma 1 is evident, taking $v = 0$. Assume that one of ϕ_j, say, ϕ_1 is nonzero, thus $\phi_1 = v_1\rho$ where $v_1 \neq 0$ and ρ is a vector with entries ± 1. Notice that for any real numbers $\alpha, \beta \neq 0$ we have $|\alpha - \beta| \neq |\alpha + \beta|$. Therefore, if $\phi_j \neq 0$ for some $j \neq 1$, all entries of $\phi_j - \phi_1$ have the same module if and only if $\phi_j = v_j\rho$, $v_j \neq 0$. If $\phi_j = 0$, we put by definition $v_j = 0$. Therefore, $\phi_j = v_j\rho$ for any j and $\lim\limits_{t\to\infty} x_j(t) = \Phi_* = \rho v^T x(0)$, where $v := (v_1, \ldots, v_N)^T$. ■