Synchronization in Random Weighted Directed Networks

Maurizio Porfiri, Daniel J. Stilwell, and Erik M. Bollt

Abstract—We assess synchronization of oscillators that are coupled via a time-varying stochastic network, modeled as a weighted directed random graph that switches at a given rate between a set of possible graphs. The existence of any graph edge is probabilistic and independent from the existence of any other edge. We further allow each edge to be weighted differently. Even if the network is always instantaneously not connected, we show that sufficient information is propagated through the network to allow almost sure local synchronization as long as the expected value of the network is connected, and that the switching rate is sufficiently fast.

Index Terms—*Ad hoc* networking, directed graphs, fast switching, random graphs, synchronization, weighted graphs.

I. INTRODUCTION

SINCE Huygen's early observations of weakly coupled clock pendula [23], synchronization has been found in a wide variety of phenomena, especially in recent years, ranging from biological systems that include fire flies in the forest [8], [37], animal gates [11], descriptions of the heart [18], [22], [53], and improved understanding of brain seizures [39], to chemistry [27], nonlinear optics [41], and meteorology [13]. Many excellent reviews are now available, including [10], [17], [36], [38], [45], and [52].

Despite the very large literature to be found on synchronization, the great majority of research activities have been focused on static networks whose connectivity and coupling strengths are constant in time. For example, static networks are assumed for the analysis of [1], [4], [31], [43], and [44]. However, there are applications where the coupling strengths and even the network topology can evolve in time. Recent work such as [5], [6], [24], [32]–[34], [49], [51], [57], and [58] are among the few to consider time-dependent couplings. There is even less previous work concerning problems of stochastic time-varying couplings as we address in this paper, see, e.g., [7], [12], and [40].

M. Porfiri is with the Department of Mechanical and Aerospace Engineering, Polytechnic Institute of New York University, Brooklyn, NY 11201 USA (e-mail: mporfiri@poly.edu).

D. J. Stilwell is with the Bradley Department of Electrical and Computer Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061 USA (e-mail: stilwell@vt.edu).

E. M. Bollt is with the Department of Mathematics and Computer Science, Clarkson University, Postdam, NY 13699 USA (e-mail: bolltem@clarkson.edu).

For the case of static networks, extensive research activity has led to many well-established and broad criteria for oscillators synchronization, see, e.g., [9], [30], [43], and [44]. In [50], it is shown that if the time-varying network topology has a uniform time-average and switches at a sufficiently fast rate, then local synchronization is asymptotically achieved if the static time-average of the network topology asymptotically supports synchronization. In this case, the concept of fast switching enables the analysis of a time-varying network to be based on a static time-average of the network. In particular, the result applies to periodically switching network topologies.

The principal contribution of this paper is to extend the deterministic result of [50] to time-varying networks whose topology changes randomly over time. We formulate the synchronization problem over a random communication network, where each communication link has a different probability to exist and communication channels are arbitrary weighted. We show that if the average network asymptotically supports synchronization, then local synchronization is achieved asymptotically almost surely if the network topology switches at a sufficiently fast rate. This result is obtained by using a mixture of findings from stochastic stability theory [28], linear system theory [48], and fast switching theory [26]. Our results are limited to local synchronization of chaotic oscillators, even though global synchronization results have been developed for statically coupled oscillators [30].

Synchronization of linear dynamics in random communication networks have been studied for example in [21], [46], and [56]. In [21], consensus problem in undirected Erdös-Rényi network is studied. In [46], the findings of [21], are generalized to weighted directed networks. In [56], synchronization of linear oscillators in weighted directed networks is analyzed.

Our present paper is motivated by a variety of emerging applications, including cooperative control [16], [47], mathematical epidemiology [49], mobile *ad hoc* networks [35], and opinion dynamics [20]. In all of these cases, information is shared along agents via a time-varying network topology that may change randomly. In the context of mathematical epidemiology, our results relate to models of how disease propagates through large network of moving agents. Such a network would be time-varying since infections depend on proximity among infected and susceptible moving agents.

The paper is organized as follows. In Section II, we review the concept of random graph and we formally define the synchronization problem. In Section III, we derive some preliminary results on stochastic linear systems needed for proving our main claim which appears in Section IV. In Section V, our main result is illustrated by a numerical simulation for a network of Rössler oscillators. Section VI is left for conclusions.

Our notation throughout is standard. $\|\cdot\|$ refers to the Euclidean norm in $\mathbb{R}^N$ or corresponding induced norm. The vector in $\mathbb{R}^N$ that consists of all unit entries is denoted $e = [1, \ldots, 1]^T$, and $\otimes$ is the standard Kronecker product. $\mathbb{Z}^+$ refers to the set of nonnegative integers, also I_N is the $N \times N$ identity matrix.

II. PROBLEM STATEMENT

A. Random Graph

For a random directed graph G with N vertices, the existence of an edge from vertex j to vertex $i \neq j$ in the set $V = \{1, \ldots, N\}$ is determined randomly and independently of other edges with probability $p_{ij} \in [0, 1]$. We assume that the graph does not have self-loops. That is, no single edge starts and ends at the same vertex. In addition, we introduce the weight matrix $W = [w_{ij}]$, where $w_{ii} = 0$ for $i \in V$. The adjacency matrix $A = [a_{ij}]$ of a directed weighted random graph is a random matrix with all zeros on the main diagonal and off-diagonal elements defined as

$$a_{ij} = \begin{cases} w_{ij}, & \text{with probability } p_{ij} \\ 0, & \text{with probability } 1 - p_{ij} \end{cases}$$

for $i \neq j$. For the case that all p_{ij} are equal to p for every $i, j \in V$, the random graph is the well known Erdös-Rényi graph [15], for which the question of percolation transitions is well understood [2]. That is, the question of when a giant component develops above a critical value of p, allowing a path between most vertices. The present approach should allow for specific popular network configurations, including small world graphs [54] and scale-free graphs [3]. The in-degree matrix $D = \text{diag}(d)$ is a random matrix whose nonzero elements are $d_i = \sum_{j=1}^n a_{ij}$. The Laplacian matrix $L = D - A$ is defined as the difference between the adjacency matrix A and the in-degree matrix D. The finite sample space of the random graph is indicated by $\mathcal{G}$, and the elementary events (possible graphs) are indicated by $G^{(j)}$, $j = 1, \ldots, |\mathcal{G}|$ where $|\cdot|$ denotes cardinality. The graph Laplacian corresponding to $G^{(j)}$ is $L^{(j)}$.

The Laplacian matrix $L = [l_{ij}]$ is a zero row-sum matrix, and therefore the null space of L contains $e = [1, \ldots, 1]^T$. Properties of directed weighted graphs may be found in the comprehensive paper [55].

Since the graph edges are independent random variables, the expected value of the graph Laplacian, written $E[L] = [E[l_{ij}]]$, may be computed entrywise by

$$E[l_{ij}] = \begin{cases} -p_{ij} w_{ij}, & i \neq j \\ +\sum_{k=1}^N p_{ik} w_{ik}, & i = j \end{cases}. \tag{1}$$

The matrix $E[L]$ corresponds to a weighted directed graph which does not necessarily belong to $\mathcal{G}$. We refer to this graph as the *average graph*, denoted $E[G]$.

If each edge has the same probability to exist, $p_{ij} = p \neq 0$, as in the Erdös-Rényi graph, then the average graph Laplacian $E[L]$ is equal to the graph Laplacian of a fully connected graph multiplied by p, where a fully connected graph has a weighted directed edge from each node to every other node.

B. Synchronization Problem

We consider a dynamic system consisting of N identical oscillators interconnected pairwise via a stochastic, weighted, unidirectional information network

$$\dot{x}_i(t) = f(x_i(t)) - \sigma B(t) \sum_{j=1}^N l_{ij}(t) x_j$$

$$x_i(0) = x_{i0}, \ i = 1, \ldots N, \ t \geq 0 \tag{2}$$

where $x_i(t) \in \mathbb{R}^n$ is the random state of oscillator i, $x_{i0} \in \mathbb{R}^n$ is its initial condition, $f : \mathbb{R}^n \to \mathbb{R}^n$ describes the oscillators' individual dynamics, $B(t) \in \mathbb{R}^{n \times n}$ models coupling between agents, $\sigma > 0$ is a control parameter that partially assigns coupling strength between oscillators, and scalars $l_{ij}(t)$ are the element of the time-varying graph Laplacian $L(t)$. We collect all the states of the system in the nN dimensional vector $x(t) = \left[x_1(t)^T, \ldots, x_N(t)^T \right]^T \in \mathbb{R}^{nN}$. We will assume in this paper that $f(x)$ is Lipschitz, which guarantees not only existence and uniqueness of solutions for the static average network, but the condition can be shown to be sufficient to guarantee uniform asymptotic stability of the linearized systems which we will study herein [25].

The network topology $G(t)$ characterized by the corresponding graph Laplacian $L(t)$ switches at a series of time instants $\{\Delta k | k \in \mathbb{Z}^+\}$, where Δ is a fixed period. The set of (2) describes a nonlinear stochastic switched system. During each time interval $[k\Delta, (k+1)\Delta)$ with $k \in \mathbb{Z}^+$, the communication network $G(t)$ is constant and equals the random graph G_k. We assume that the G_ks are independent equally distributed random graphs as described in Section II-A.

We say that the system of oscillators is synchronized if the state vectors for all oscillators are identical. Specifically, the oscillators are synchronized if $x(t) = e \otimes s(t)$ for some $s(t) \in \mathbb{R}^n$ that is a solution of the individual oscillator

$$\dot{s}(t) = f(s(t)). \tag{3}$$

We refer to the manifold in $\mathbb{R}^{nN}$ consisting of trajectories $e \otimes s(t)$ where $s(t)$ is a solution of (3) as the *synchronization manifold*. Note also that the range of $e \otimes I_n$ contains the synchronization manifold.

C. Decomposition

Synchronization of the system of oscillators can be assessed by examining local stability with respect to the synchronization manifold. Linearizing the system of oscillators about the trajectory $e \otimes s(t) \in \mathbb{R}^{nN}$ on the synchronization manifold yields

$$\dot{z}(t) = (I_N \otimes F(t) - \sigma L(t) \otimes B(t)) z(t) \tag{4}$$

where $z(t) = x(t) - e \otimes s(t)$. To analyze asymptotic stability of the set of oscillators, we decompose the state of (4) into a component that evolves along the synchronization manifold, and a component that evolves transverse to the synchronization manifold. For analysis, it suffices to show that the component that evolves transverse to the synchronization manifold asymptotically approaches the synchronization manifold.

Decomposition of the oscillator states is based on a matrix $W \in \mathbb{R}^{N \times (N-1)}$ that satisfies $W^{\mathrm{T}} e = 0$ and $W^{\mathrm{T}} W = I_{N-1}$. Note that the vector of synchronized oscillator states $e \otimes s(t)$ is in the null space of $W^{\mathrm{T}} \otimes I_n$ and in the range of $e \otimes I_n$. The state vector $z(t)$ in (4) can be decomposed as

$$z(t) = (W \otimes I_n)\zeta(t) + e \otimes z_s(t)$$
$$\text{where } \zeta = (W \otimes I_n)^{\mathrm{T}} z$$
$$\text{and } z_s = \frac{1}{N}((e \otimes I_n)^{\mathrm{T}} z).$$

Note that $z_s(t)$ is the average of all the components in $z(t)$, and that $((W \otimes I_n)\zeta)^{\mathrm{T}}(e \otimes z_s) = 0$. Using the state transformation

$$\begin{bmatrix} z_s(t) \\ \zeta(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{N}(e \otimes I_n)^{\mathrm{T}} \\ (W \otimes I_n)^{\mathrm{T}} \end{bmatrix} z(t)$$

the linearized oscillator dynamics (4) are partitioned

$$\dot{z}_s(t) = F(t)z_s(t) - \sigma(e^{\mathrm{T}} L(t)W \otimes B(t))\zeta(t) \tag{5a}$$
$$\dot{\zeta}(t) = (I_{N-1} \otimes F(t) - \sigma W^{\mathrm{T}} L(t)W \otimes B(t))\zeta(t). \tag{5b}$$

Thus, local almost sure asymptotic synchronization of the set of oscillators (2) is achieved if $\zeta(t)$ in (5b) almost surely converges to zero, see, e.g., [29]. The definition of almost sure convergence may be found, for example, in [19, Ch. 5].

III. PRELIMINARY PROPOSITIONS

We present here two propositions on linear system stability which are used to determine sufficient conditions for almost sure asymptotic synchronization of (2). The first proposition establishes the relationship between asymptotic stability of a linear switched time-varying system and that of a derived sample-data system.

Proposition 1: Consider the stochastic linear system

$$\dot{y}(t) = (A(t) + B(t) \otimes H(t))y(t), \qquad t \geq 0 \tag{6}$$

where $y(t) \in \mathbb{R}^{m_1 m_2}$, $A(t) \in \mathbb{R}^{m_1 m_2 \times m_1 m_2}$ and $B(t) \in \mathbb{R}^{m_1 \times m_1}$ are bounded and continuous functions for all $t \geq 0$, and $H(t) \in \mathbb{R}^{m_2 \times m_2}$ is a bounded random process that for some $\Delta > 0$ is constant for all $t \in [k\Delta, (k+1)\Delta)$ and switches at time instants Δk, for all $k \in \mathbb{Z}^+$. Let $y_k \equiv y(k\Delta)$. If $y_k \overset{a.s.}{\to} 0$, then $y(t) \overset{a.s.}{\to} 0$.

Proof: For any $t \in [k\Delta, (k+1)\Delta)$, $y(t) = \Phi(t, k\Delta)y_k$ where Φ is the transition matrix of the linear system (6). Since A, B, and H are bounded, there exist positive constants α, β, and η such that for any $t \geq 0$

$$\|A(t)\| \leq \alpha, \quad \|B(t)\| \leq \beta \quad \|H(t)\| \leq \eta \tag{7}$$

where $\|\cdot\|$ is the induced Euclidean norm. From the Gronwall-Belmann inequality (see [48, Exercise 3.12])

$$\|y(t)\| \leq \|y_k\| \exp \int_{k\Delta}^{(k+1)\Delta} (\|A(\tau) + B(\tau) \otimes H(\tau)\|)dt.$$

Thus, for any $t \in [k\Delta, (k+1)\Delta)$

$$\|y(t)\| \leq \exp((\alpha + \beta\eta)\Delta)\|y_k\|$$

and claim follows. ∎

For clarity, we restate the well-known Borel-Cantelli Lemma in the form presented in [28, Lemma 1, Ch. 8].

Lemma 1: Consider the stochastic process $X_k \in \mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}$ is a nonnegative function and

$$\sum_{k=0}^{\infty} E[f(X_k)] < \infty$$

then $f(X_k) \overset{a.s.}{\to} 0$.

The following proposition generalizes the claim in [28 Theorem 1, Ch. 8] to time-varying systems. We note that arguments from stochastic stability theory were first applied to synchronization problems by [21].

Proposition 2: Consider the stochastic system

$$X_{k+1} = A_k X_k \tag{8}$$

where the A_k are mutually independent random matrices and $X_0 \in \mathbb{R}^m$. Suppose there is a sequence of symmetric positive definite matrices Q_k, such that for any k

$$E[A_k^{\mathrm{T}} Q_{k+1} A_k] - Q_k = -C_k \tag{9}$$

where E indicates expected value, and C_k is a sequence of symmetric positive definite matrices such that $C_k \geq \nu I$ with $\nu > 0$ for any k, then $X_k \overset{a.s.}{\to} 0$ for any initial condition X_0.

Proof: Our proof is similar to that of [28, Theorem 1, Ch. 8]. Define the quadratic Lyapunov function $v_k(x) = (1/2)x^{\mathrm{T}} Q_k x \; x \in \mathbb{R}^m$ and the related random process

$$V_k = v_k(X_k) = \frac{1}{2} X_k^{\mathrm{T}} Q_k X_k. \tag{10}$$

Thus, using iteratively (8), (9), and (10) and using the smoothing lemma, see [19, Lemma 1.1, Ch. 10] we obtain

$$E[V_1] = V_0 - \frac{1}{2} X_0^{\mathrm{T}} C_0 X_0$$
$$E[V_2|X_1] = v_1(X_1) - \frac{1}{2} X_1^{\mathrm{T}} C_1 X_1$$
$$E[V_k|X_{k-1}, \ldots X_1] = v_{k-1}(X_{k-1}) - \frac{1}{2} X_{k-1}^{\mathrm{T}} C_k X_{k-1}$$
$$E[V_k|X_{k-1}, \ldots X_1] = V_0 - \frac{1}{2} \sum_{h=0}^{k-1} X_h^{\mathrm{T}} C_h X_h. \tag{11}$$

Since C_k and Q_k are positive definite, (11) yields

$$V_0 \geq V_0 - E[V_k] = \frac{1}{2} \sum_{h=0}^{k-1} E[X_h^{\mathrm{T}} C_h X_h]. \tag{12}$$

Thus, the right-hand side (RHS) of (12) is bounded above. Since C_k is positive definite and bounded away from zero, the RHS of (12) may be further bounded as

$$\frac{1}{2} \sum_{h=0}^{k-1} E[X_h^{\mathrm{T}} C_h X_h] \geq \frac{\nu}{2} \sum_{h=0}^{k-1} E[X_h^{\mathrm{T}} X_h].$$

The direct application of Lemma 1 with $f(X_k) = \nu/2 X_k^{\mathrm{T}} X_k$ yields $X_k \overset{a.s.}{\to} 0$. ∎

We note that imposing $X_k \overset{a.s.}{\to} 0$ implies that (8) is almost surely asymptotically stable. Indeed, for linear systems with bounded state matrices it is simple to show that attractivity implies asymptotic stability, see [14]. For nonlinear systems attractivity does not imply asymptotic stability in general, see [25].

IV. SYNCHRONIZATION THROUGH FAST-SWITCHING

Our principal result is to show that the network of coupled oscillators with stochastic time-varying $L(t)$ can synchronize even if the network is insufficiently coupled to support synchronization at every instant of time. We show that synchronization is achieved as long as the average network $E[G]$, corresponding to the graph Laplacian $E[L]$ in (1), supports synchronization and the switching period Δ between new samples of the network topology is sufficiently small. In other words, analysis of synchronization for the stochastic time-varying network (2) reduces to analysis of synchronization for the static network

$$\dot{x}_i(t) = f(x_i(t)) - \sigma B(t) \sum_{j=1}^{N} E[l_{ij}]x_j, \ i = 1, \ldots N, \ t \geq 0 \tag{13}$$

along with sufficiently fast switching of the network topology. The case of static networks has been extensively addressed in the literature, and analysis tools include the well-known master-stability function [43]. We note that the analysis herein requires that the transverse dynamics is uniformly asymptotically stable, which cannot be directly inferred using the master-stability function. This popular method can be used as a starting point and as an easily checkable surrogate to indicate if the transverse dynamics is stable. Alternatively, criteria based on direct demonstration of a Lyapunov function can be used to guarantee uniform asymptotic stability as done in [30].

A similar result for consensus problems under a random directed network has been presented in [46]. In [46], it is shown that if the average network $E[G]$ supports consensus then consensus is achievable for the time-varying network as long as the graph topology changes sufficiently fast. Both this work and [46] use fast switching techniques to infer the behavior of the time-varying system from its time-average. The mathematical framework therein developed rather differs from the one presented here. In consensus problems, the dynamics of the individual oscillators are not considered. As a result, the consensus protocol is described in terms of a homogeneous Markov chain for which classical results from stochastic stability [28] are applicable. Also, the consensus problem on the average network is a linear time-invariant problem.

Theorem 1: Consider the deterministic dynamic system

$$\dot{y}(t) = (I_{N-1} \otimes F(t) - \sigma W^{\mathrm{T}} E[L] W \otimes B(t))y(t) \tag{14}$$

representing the linearized transverse dynamics of (13). Assume that $F(t)$ and $B(t)$ are bounded and continuous for all $t \geq 0$. If (14) is uniformly asymptotically stable, there is a time-scale $\Delta^* > 0$ such that for any shorter time-scale $\Delta < \Delta^*$ the stochastic system (2) asymptotically synchronizes almost surely.

Proof: We define $M(t) \equiv (I_{N-1} \otimes F(t) - \sigma W^{\mathrm{T}} L(t)W \otimes B(t))$ and rewrite the variational equation (5b) as

$$\dot{\zeta}(t) = M(t)\zeta(t). \tag{15}$$

Consequently, (14) may be compactly rewritten as

$$\dot{y}(t) = \bar{M}(t)y(t) \tag{16}$$

where $\bar{M}(t) \equiv (I_{N-1} \otimes F(t) - \sigma W^{\mathrm{T}} E[L] W \otimes B(t))$. With Proposition 1 in mind, we define the sequences $\zeta_k \equiv \zeta(k\Delta)$ and $y_k \equiv y(k\Delta)$, for $k \in \mathbb{Z}^+$. For any $k \in \mathbb{Z}^+$

$$\zeta_{k+1} = \Gamma_k \zeta_k \tag{17a}$$
$$y_{k+1} = \Theta_k y_k \tag{17b}$$

where Γ_k and Θ_k are the transition matrices of (15) and (16) over the time interval $[k\Delta, (k+1)\Delta)$, respectively. By hypothesis, y_k in (17b) asymptotically converges to zero for any initial condition y_0. Our task is to show that ζ_k in (17a) converges to zero almost surely for any initial condition if Δ is sufficiently small.

Since F and B are bounded for all $t \geq 0$, there are positive constants ϕ and β such that for any $t \geq 0$, $\|F(t)\| \leq \phi$, and $\|B(t)\| \leq \beta$. Since the event sample space is finite $\|L(t)\| \leq \lambda$, where $\lambda = \max_{j=1,\ldots|\mathcal{G}|} \|L^{(j)}\|$. By definition $\|W\| = 1$, therefore, for any $t > 0$

$$\|M(t)\| \leq \alpha, \qquad \|\bar{M}(t)\| \leq \alpha \tag{18}$$

where $\alpha = \phi + \sigma\beta\lambda$.

Since (16) is uniformly asymptotically stable and $\bar{M}(t)$ is continuous there exists symmetric continuously differentiable matrix $Q(t)$ such that (see [48, Theorem 7.2])

$$\eta I \leq Q(t) \leq \rho I. \tag{19}$$

The related Lyapunov function $v(t, y) = (1/2)y^{\mathrm{T}} Q(t)y$ satisfies $(d/dt)v(t, y(t)) \leq -\mu/2\|y(t)\|^2$, where η, ρ, and μ are positive scalars. For linear systems, uniform asymptotic stability is equivalent to uniform exponential stability, thus in $[k\Delta, (k+1)\Delta)$, (see [48, the proof of Theorem 7.4])

$$v((k+1)\Delta, y_{k+1}) - v(k\Delta, y_k) \leq \frac{\rho}{2} \left(\exp\left(-\frac{\mu\Delta}{\rho}\right) - 1 \right) \|y_k\|^2. \tag{20}$$

From (20) the following condition arises

$$\Theta_k^{\mathrm{T}} Q((k+1)\Delta)\Theta_k - Q(k\Delta) = -P_k \tag{21}$$

where P_k is a symmetric positive definite matrix satisfying

$$\|P_k\| \geq \rho \left(1 - \exp\left(-\frac{\mu\Delta}{\rho}\right) \right) \tag{22}$$

for every k. In addition, since $\bar{M}(t)$ is bounded, from the Gronwall-Belmann inequality, see [48, Exercise 3.12], we have

$$\|\Theta_k\| \leq \exp \int_{k\Delta}^{(k+1)\Delta} \|\bar{M}(t)\| dt$$

which yields

$$\|\Theta_k\| \leq \exp(\alpha\Delta). \tag{23}$$

Recalling the Peano-Baker expansion for Γ_k (see [48, Ch. 3])

$$\Gamma_k = I_{(N-1)n} + \int_{k\Delta}^{(k+1)\Delta} M(\tau_1)d\tau_1$$
$$+ \sum_{i=2}^{\infty} \int_{k\Delta}^{(k+1)\Delta} M(\tau_1) \int_{k\Delta}^{\tau_1} \cdots \int_{k\Delta}^{\tau_{i-1}} M(\tau_i)d\tau_i \cdots d\tau_1$$

we can express

$$R_k = \Gamma_k - \Theta_k \tag{24}$$

as

$$R_k = \int_{k\Delta}^{(k+1)\Delta} (M(\tau) - \bar{M}(\tau))d\tau + \varepsilon_k \tag{25}$$

where

$$\varepsilon_k = \sum_{i=2}^{\infty} \int_{k\Delta}^{(k+1)\Delta} M(\tau_1) \int_{k\Delta}^{\tau_1} \cdots \int_{k\Delta}^{\tau_{i-1}} M(\tau_i)d\tau_i \cdots d\tau_1$$
$$- \sum_{i=2}^{\infty} \int_{k\Delta}^{(k+1)\Delta} \bar{M}(\tau_1) \int_{k\Delta}^{\tau_1} \cdots \int_{k\Delta}^{\tau_{i-1}} \bar{M}(\tau_i)d\tau_i \cdots d\tau_1. \tag{26}$$

By taking the expected value of both sides of (25) we have

$$E[R_k] = E[\varepsilon_k]. \tag{27}$$

From (18), the norm of the first term on the RHS of (25) is bounded by $2\Delta\alpha$. On the other hand, from (18) and (26), the norm of ε_k may be bounded by

$$\|\varepsilon_k\| \leq \sum_{i=2}^{\infty} (2\Delta\alpha)^i = \exp(2\Delta\alpha) - 1 - 2\Delta\alpha \tag{28}$$

which implies

$$\|E[\varepsilon_k]\| \leq \exp(2\Delta\alpha) - 1 - 2\Delta\alpha. \tag{29}$$

Using (18) and (29), the norm of R_k is bounded by

$$\|R_k\| \leq \exp(2\Delta\alpha) - 1. \tag{30}$$

We emphasize that from (28) and (30)

$$\|R_k\| \leq O(\Delta), \qquad \|\varepsilon_k\| \leq O(\Delta^2). \tag{31}$$

Next, we show that for sufficiently small values of Δ, the matrix $Q_k = Q(k\Delta)$ defines a quadratic Lyapunov function for (5b) in the
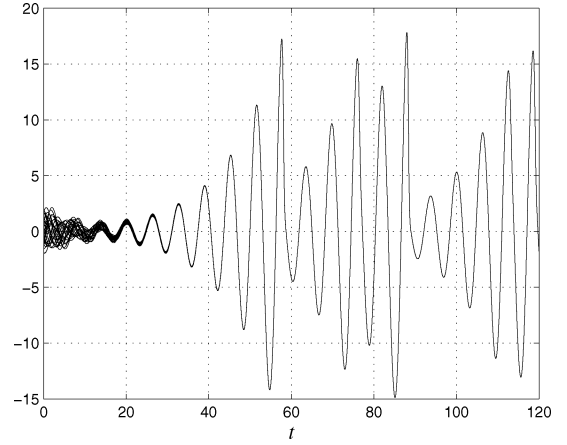

Fig. 1. Time evolution of the x coordinate of the set of coupled Rössler oscillators using the average graph.
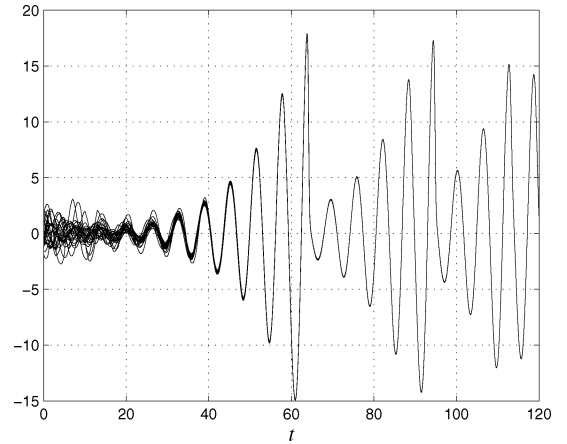

Fig. 2. Time evolution of the x coordinate of the set of coupled Rössler oscillators using the random graph.

sense of Proposition 2. Indeed by substituting (24) into (21) and by using (27), we obtain

$$E[\Gamma_k^{\mathrm{T}} Q_{k+1} \Gamma_k] - Q_k$$
$$= E[(R_k + \Theta_k)^{\mathrm{T}} Q_{k+1}(R_k + \Theta_k)] - Q_k =$$
$$- P_k + \Theta_k^{\mathrm{T}} Q_{k+1} E[\varepsilon_k] + E[\varepsilon_k]^{\mathrm{T}} Q_{k+1} \Theta_k$$
$$+ E[R_k^{\mathrm{T}} Q_{k+1} R_k]. \tag{32}$$

Combining all of the bounds in (18), (19), (22), (23), (29), and (30), (32) yields

$$E[\Gamma_k^{\mathrm{T}} Q_{k+1} \Gamma_k] - Q_k \leq -g(\Delta)I_{(N-1)n}$$

where the continuous function $g(\Delta)$ is defined by

$$g(\Delta) = \rho \left(1 - \exp\left(-\frac{\mu\Delta}{\rho}\right)\right) - 2\exp(\alpha\Delta)\rho$$
$$\times (\exp(2\Delta\alpha) - 1 - 2\Delta\alpha) - \rho(\exp(2\Delta\alpha) - 1)^2.$$

Since $g(0) = 0$, $(d/d\Delta)g(0) = \mu > 0$, and $g(\Delta) \to -\infty$ as $\Delta \to \infty$, there exists Δ^* such that $g(\Delta) > 0$ for all $\Delta \in (0, \Delta^*)$. By applying Proposition 2 the claim follows. ∎
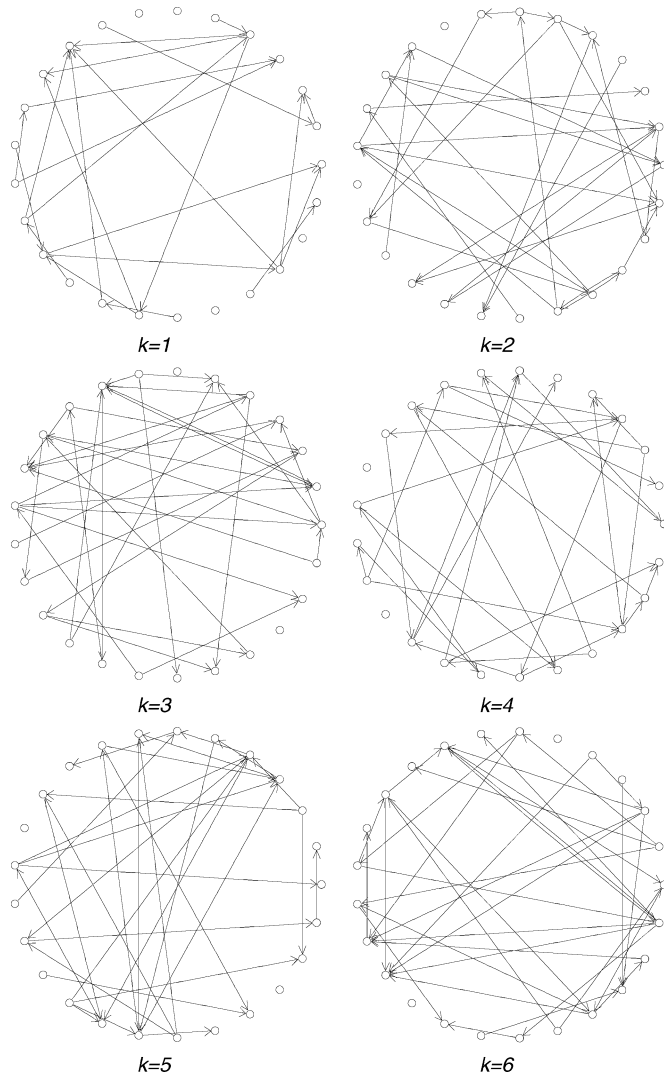
Fig. 3. Dynamics of the random network.

where $i = 1, \ldots, r$, $a = 0.165$, $b = 0.2$, $c = 10$, and $\sigma = 0.3$. Oscillators are coupled through the x_i variables via l_{ij} which changes randomly at the transition instants $k\Delta$. Each edge is assigned the same probability to exists, $p_{ij} = p = 0.05$ and the same unitary weight. One may naturally ask if such a directed random graph is close to percolation, since it is closely related to the undirected Erdös-Rényi graph [15] for which the percolation question can be directly addressed. However, the percolation problem is not relevant here, since it is not necessary for our network to be instantaneously connected. Indeed, it is possible for the network to instantaneously consist of many subcomponents when the system is changed with a fast enough Δ. We consider as example, a network of 25 oscillators, and we assume that switching occurs with a period $\Delta = 1$. Figs. 1 and 2 depict the x coordinate of the set of coupled Rössler oscillators using the average graph and the random graph respectively, versus the time t. Fig. 3 reports the dynamics of the network topology for the first six time intervals.

We note that our results are limited to local synchronization despite the fact that presented numerical results seem to indicate that the stronger property of global synchronization might in fact be exhibited by the case-study system.

VI. Conclusion

New generalizations on synchronization of mutually coupled oscillators are presented. We pose the synchronization problem in a stochastic framework where communication among nodes is modeled as a weighted directed random switching graph. We utilize tools based on fast switching and stochastic stability, and show that local synchronization is asymptotically achieved if the average communication network asymptotically supports synchronization and if the network topology is changing with a sufficiently fast rate. A numerical simulation illustrates the theoretical achievements of the present paper.

References

[1] F. M. Atay, T. Biyikoglu, and J. Jost, "Synchronization of networks with prescribed degree distributions," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, no. 1, pp. 92–98, Jan. 2006.

[2] D. ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems*. Cambridge, U.K.: Cambridge Univ. Press, 2000.

[3] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Rev. Modern Phys.*, vol. 74, no. 1, pp. 49–98, 2002.

[4] M. Barahona and L. M. Pecora, "Synchronization in small-world systems," *Phys. Rev. Lett.*, vol. 89, no. 5, 2002.

[5] V. N. Belykh, I. V. Belykh, and M. Hasler, "Connection graph stability method for synchronized coupled chaotic systems," *Physica D*, vol. 195, no. 1–2, pp. 159–187, 2004.

[6] I. V. Belykh, V. N. Belykh, and M. Hasler, "Blinking model and synchronization in small-world networks with a time-varying coupling," *Physica D*, vol. 195, no. 1–2, pp. 188–206, 2004.

[7] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, "The synchronization of chaotic systems," *Phys. Rep.*, vol. 366, no. 1–2, pp. 1–101, 2002.

[8] J. Buck and E. Buck, "Synchronous fireflies," *Scientif. Amer.*, vol. 234, pp. 74–85, 1976.

[9] M. Chen, "Some simple synchronization criteria for complex dynamical networks," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, no. 11, pp. 1185–1189, Nov. 2006.

Theorem 1 provides a sufficient condition for almost sure asymptotic synchronization of randomly couple oscillators, while in [56] sufficient conditions for asymptotic synchronization in probability are discussed. The condition derived in this paper is based on a mixture of fast-switching and stochastic stability results, whereas the conditions proposed in [56] are based on the direct algebraic analysis of the switching networks.

V. Illustration

To illustrate fast switching concepts applied to synchronization of a set of oscillators coupled through a randomly switching graph, we consider a set of r Rössler attractors

$$\dot{x}_i(t) = -y_i(t) - z_i(t) - \sigma \sum_{j=1}^{r} l_{ij} x_j(t)$$
$$\dot{y}_i(t) = x_i(t) + ay_i(t)$$
$$\dot{z}_i(t) = b + z_i(t)(x_i(t) - c)$$

[10] G. Chen and Y. Xinghuo, *Chaos Control Theory and Applications Series: Lecture Notes in Control and Information Sciences*. New York: Springer Series, 2003, vol. 292.

[11] J. J. Collins and I. Stewart, "Coupled nonlinear oscillators and the symetries of animal gaits," *Science*, vol. 3, no. 3, pp. 349–392, 1993.

[12] S. Dorogovtsev and J. F. F. Mendes, "Evolution of networks," *Adv. Phys.*, vol. 51, no. 4, pp. 1079–1187, 2002.

[13] G. S. Duane, P. J. Webster, and J. B. Weiss, "Go-occurrence of Northern and Southern Hemisphere blocks as partially synchronized chaos," *J. Atmospher. Sci.*, vol. 56, no. 24, pp. 4183–4205, 1999.

[14] L. H. Duc, A. Ilchmann, and S. Siegmund, "On stability of linear time-varying second-order differential equations," *Quart. Appl. Math.*, vol. 64, pp. 137–151, 2006.

[15] P. Erdös and A. Rényi, "On the evolution of random graphs," *Publ. Math. Inst. Hungar. Acad. Sci.*, vol. 5, pp. 17–61, 1960.

[16] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, pp. 1465–147, 2004.

[17] L. Glass and M. C. Mackey, *From Clocks to Chaos: The Rythms of Life*. Princeton, NJ: Princeton Univ. Press, 1988.

[18] M. R. Guevara, A. Shrier, and L. Glass, "Phase-locked rhythms in periodically stimulated heart cell aggregates," *Amer. J. Physiol.*, vol. 254, Heart Circ. Physiol. 23, pp. H1–H10, 1988.

[19] A. Gut, *Probability: A Graduate Course*. Berlin, Germany: Springer, 2005.

[20] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence: Models, analysis and simulation," *JASSS-The J. Artif. Soc. Social Simul.*, vol. 5, no. 3, 2002.

[21] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Trans. Autom. Control*, vol. 50, pp. 1867–1872, 2005.

[22] J. Honerkamp, "The heart as a system of coupled nonlinear oscillators," *J. Math. Biol.*, vol. 19, pp. 69–88, 1983.

[23] C. Hugenii, *Horoloquium Oscilatorium*, A. F. Muguet, Ed. Paris: , 1673, Engl. translation, *The Pendulum Clock*. Ames, IA: Iowa State Univ. Press, 1986.

[24] J. Ito and K. Kaneko, "Spontaneous structure formation in a network of chaotic units with variable connection strengths," *Phys. Rev. Lett.*, vol. 88, no. 2, p. 028701, 2002.

[25] H. K. Khalil, *Nonlinear Systems*, Second ed. Upper Saddle River, NJ: Prentice-Hall, 1996.

[26] R. L. Kosut, B. D. O. Anderson, and I. M. Y. Mareels, "Stability theory for adaptive systems: Method of averaging and persistency of excitation," *IEEE Trans. Autom. Control*, vol. 1, pp. 26–34, 1987.

[27] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*. Berlin, Germany: Springer, 1984.

[28] H. Kushner, *Introduction to Stochastic Control*. New York: Holt, Rinehart and Winston, 1971.

[29] H. Kushner, *Stochastic Stability and Control*. New York: Academic, 1967.

[30] J.-P. Jiang, W. K.-T. Tang, and G. Chen, "A simple global synchronization criterion for coupled chaotic systems," *Chaos Solitons Fractals*, vol. 15, pp. 925–935, 2003.

[31] Z. Li and G. Chen, "Global synchronization and asymptotic stability of complex dynamical networks," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, pp. 28–33, 2006.

[32] J. Lü, X. Yu, and G. Chen, "Chaos synchronization of general complex dynamical networks," *Phys. A*, vol. 334, no. 1–2, pp. 281–302, 2005.

[33] J. Lü and G. Chen, "A time-varyng complex dynamical network model and its controlled synchronization network," *IEEE Trans. Autom. Control*, vol. 50, pp. 841–846, 2005.

[34] J. G. Lu and D. J. Hill, "Impulsive synchronization of chaotic Lure systems by linear static measurement feedback: An LMI approach," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 54, no. 8, pp. 710–714, 2007.

[35] M. Mauve, J. Widmer, and H. Hartenstein, "A survey on position-bosed routing in mobile ad hoc networks," *IEEE Network*, vol. 15, no. 6, pp. 30–39, 2001.

[36] J. M. Gonzalez-Miranda, *Synchronization and Control of Chaos*. London, U.K.: Imperial College Press, 2004.

[37] R. Mirollo and S. Strogatz, "Synchronization of pulsed-coupled biological oscillators," *SIAM J. Appl. Math.*, vol. 50, no. 6, pp. 1645–1662, 1990.

[38] E. Mosekilde, Y. Maistrenko, and D. Postnov, *Chaotic Synchronization: Applications to Living Systems*, ser. A. Singapore: World Scientific, 2002, Nonlinear Science.

[39] T. I. Netoff and S. J. Schiff, "Decreased neuronal synchronization during experimental seizures," *J. Neurosci.*, vol. 22, no. 16, pp. 7297–7307, 2002.

[40] M. E. J. Newman, "The structure and function of complex networks," *SIAM Rev.*, vol. 45, no. 2, pp. 167–256, 2003.

[41] J. Ohtsubop, "Feedback induced instability and chaos in semiconductor lasers and their applications," *Opt. Rev.*, vol. 6, no. 1, pp. 1–15, 1999.

[42] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, no. 8, pp. 825–821, 1990.

[43] L. M. Pecora and T. L. Carroll, "Master stability function for synchronized coupled systems," *Phys. Rev. Lett.*, vol. 80, no. 10, pp. 2109–2112, 1998.

[44] L. Pecora, T. Carroll, G. Johnson, D. Mar, and K. S. Fink, "Synchronization stability in coupled oscillator arrays: Solution for arbitrary configurations," *Int. J. Bifurc. Chaos*, vol. 10, no. 2, pp. 273–290, 2000.

[45] A. Pikovsky, M. Rosemblum, and J. Kurths, *Synchronization, A Universal Concept in Nonlinear Sciences*. Cambridge, U.K.: Cambridge Univ. Press, 2001.

[46] M. Porfiri and D. J. Stilwell, "Consensus seeking over random weighted directed graphs," *IEEE Trans. Autom. Control*, vol. 52, pp. 1767–1773, 2007.

[47] M. Porfiri, D. G. Roberson, and D. J. Stilwell, "Tracking and formation control of multiple autonomous agents: A two-level consensus approach," *Automatica*, vol. 43, no. 8, pp. 1318–1328, 2007.

[48] W. J. Rugh, *Linear Systems Theory*, Second ed. Upper Saddle River, NJ: Prentice-Hall, 1996.

[49] J. D. Skufca and E. Bollt, "Communication and synchronization in disconnected networks with dynamic topology: Moving neighborhood networks," *Math. Biosci. Eng. (MBE)*, vol. 1, no. 2, pp. 347–359, 2004.

[50] D. J. Stilwell, E. M. Bollt, and D. G. Roberson, "Sufficient conditions for fast switching synchronization in time varying network topologies," *SIAM J. Appl. Dynam. Syst.*, vol. 5, no. 1, pp. 140–156, 2006.

[51] T. Stojanovski, L. Kocarev, U. Parlitz, and R. Harris, "Sporadic driving of dynamical systems," *Phys. Rev. E*, vol. 55, no. 4, pp. 4035–4048, 1997.

[52] S. Strogatz, *Sync: The Emerging Science of Spontaneous Order*. New York: Hyperion, 2003.

[53] V. Torre, "A theory of synchronization of two heart pacemaker cells," *J. Theoret. Biol.*, vol. 61, pp. 55–71, 1976.

[54] D. J. Watts and S. Strogatz, "Collective dynamics of small-world networks," *Nature*, vol. 393, pp. 409–410, 1998.

[55] C. W. Wu, "Algebraic connectivity of directed graphs," *Linear Multilinear Algebra*, vol. 53, no. 3, pp. 203–223, 2005.

[56] C. W. Wu, "Synchronization and convergence of linear dynamics in random directed networks," *IEEE Trans. Autom. Control*, vol. 51, pp. 1207–1210, 2006.

[57] D. H. Zanette and A. S. Mikhailov, "Dynamical systems with time-dependent coupling: Clustering and critical behavior," *Physica D*, vol. 194, no. 3–4, pp. 203–218, 2004.

[58] J. Zhou and T. Chen, "Synchronization in general complex delayed dynamical networks," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 53, no. 3, pp. 733–744, Mar. 2006.

Maurizio Porfiri was born in Rome, Italy, in 1976. He received the M.Sc. and Ph.D. degrees in engineering mechanics from the Virginia Polytechnic Institute and State University (Virginia Tech), Blacksburg, in 2000 and 2006; the "Laurea" in electrical engineering (with honors) and the Ph.D. degree in theoretical and applied mechanics from the University of Rome "La Sapienza" and the University of Toulon (dual degree program), in 2001 and 2005, respectively.

From 2005 to 2006, he held a Postdoctoral position with the Electrical and Computer Engineering Department, Virginia Tech. He is currently an Assistant Professor of mechanical engineering with the Polytechnic Institute of New York University, Brooklyn, NY. His research interests include dynamical systems theory, mathematical control theory, mechanics of advanced materials, MEMS, and underwater robotics.

Dr. Porfiri is a recipient of the NSF CAREER award in 2008.

Daniel J. Stilwell received the B.S. degree is computer engineering from the University of Massachusetts, Amherst, in 1991, the M.S. degree in electrical engineering from the Virginia Polytechnic Institute and State University (Virginia Tech), Blacksburg, in 1993, and the Ph.D. degree in electrical engineering from Johns Hopkins University, Baltimore, MD, in 1999.

He is currently an Associate Professor with the Bradley Department of Electrical and Computer Engineering, Virginia Tech, where he teaches courses on controls and robotics. He conducts and supervises research in the areas of distributed estimation and decentralized control, environmental sampling, and the autonomous vehicle systems.

Dr. Stilwell received the NSF CAREER award and Office of Naval Research Young Investigator Program award, both in 2003. In 2004, he received the College of Engineering Outstanding Assistant Professor award.

Erik M. Bollt received the B.A. degree in applied mathematics from the University of California at Berkeley, in 1990, the M.S. degree in applied mathematics from the University of Colorado at Boulder in 1992, and the Ph.D. degree in 1995 in applied mathematics from the University of Colorado at Boulder.

He has been a member of the Clarkson University Mathematics and Computer Science Faculty, together with the Physics Department, since 2002, where he is currently a full professor. His core research area has been in dynamical systems, including fundamental issues of topological dynamics, symbolic dynamics, measurable dynamics and transfer operators, as well as stochastic dynamical systems. Additionally, as an applied mathematician, he has worked with engineers and other scientists in problems involving control theory applications including air foil flutter suppression, information processing, image object detection, control of chaos, physiological data processing and modeling, as well as developing a deeper understanding of both dynamics of networks and dynamics on networks including applications in communications and social and epidemiological settings.