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Signed consensus problems on networks of agents with fixed and switching topologies

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ABSTRACT

This paper deals with signed consensus problems for networks of agents in the presence of both fixed and switching topologies. By specifying signs for the agents, they are divided into two groups and are enabled to reach agreement on a consensus value which is the same in modulus for both groups but not in sign. Using nearest-neighbour interaction rules, we propose the distributed protocols and address their exponential convergence problem. It is shown that the quasi-strong connectivity of fixed networks or joint quasi-strong connectivity of switching networks can provide a necessary and sufficient guarantee for all agents to achieve signed consensus exponentially fast. In particular, the signed consensus results can include as special cases those of bipartite consensus in signed networks with fixed and switching topologies. Numerical simulations are also provided to illustrate the exponential convergence performance of the proposed signed consensus protocols.

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Signed consensus; fixed networks; switching networks; distributed protocols; exponential convergence

1. Introduction

Consensus has been one of the hottest research topics in distributed coordination control of multi-agent systems (see e.g. surveys Cao et al., 2013; Olfati-Saber et al., 2007; Ren et al., 2007). In general, there are two classes of consensus objectives. One is to decrease all state agreement errors among agents to zero (Li et al., 2010; Meng & Jia, 2016a; Ren & Beard, 2005; Roy, 2015; Sun & Wang, 2009; Xiao & Wang, 2006; Xiong et al., 2011) and the other is to drive all agents to reach agreement on a common state (Cao et al., 2008; Jadbabaieet et al., 2003; Meng & Jia, 2016b; Moreau, 2005; Olshevsky & Tsitsiklis, 2009). In order to achieve consensus, graphic conditions have been developed for both cases regardless of fixed or switching networks (Cao et al., 2008; Jadbabaieet et al., 2003; Li et al., 2010; Meng & Jia, 2016a, 2016b; Moreau, 2005; Olshevsky & Tsitsiklis, 2009; Ren & Beard, 2005; Sun & Wang, 2009; Xiao & Wang, 2006; Xiong et al., 2011). Note that the first class consensus problem can be transformed into the scope of stability problems based on a tree-type transformation (Sun & Wang, 2009). Hence, we can benefit from the standard linear system theory to enhance the developed asymptotic consensus results to be exponentially fast especially for fixed networks. However, such benefit does not work for the second consensus problem any longer. Exponential consensus achieved at a constant state (Schenato & Fiorentin, 2011) has been studied for switching networks, for which the joint strong connectivity is required as a sufficient guarantee to implement the Lyapunov analysis. Unfortunately, it is still unclear under which network condition the complete exponential consensus holds with a necessary and sufficient guarantee.

In this paper, we are interested in discrete-time multi-agent networks and address the exponential convergence problem for all agents to reach consensus on a common state. If multi-agent networks have fixed topology, then the condition of quasi-strong connectivity, or in other words spanning tree, is necessary and sufficient to achieve their exponential consensus. Moreover, the complete exponential consensus result works for multi-agent networks with switching topologies, to which the joint quasi-strong connectivity provides a necessary and sufficient guarantee. We combine the tree-type transformation (Sun & Wang, 2009) and stability analysis of linear systems (Rugh, 1996) to establish our exponential consensus analysis. Our analysis approach admits networks to have root nodes without neighbours and also those nodes which are not neighbours of any other nodes. By contrast, the used Lyapunov analysis for exponential consensus in (Schenato & Fiorentin, 2011) is not applicable for such special networks.

By consensus under consideration in this paper, we mean a class of 'signed consensus', i.e. the agreement state

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is the same in modulus for all agents but not in sign. We can divide the agents into two groups by specifying the signs. Obviously, the classical consensus in Cao et al. (2008), Jadbabaieet et al. (2003), Li et al. (2010), Moreau (2005), Olshevsky and Tsitsiklis (2009), Ren and Beard (2005), Sun and Wang (2009), Schenato and Fiorentin (2011), Xiao and Wang (2006), and Xiong et al. (2011) can be regarded as a special case of the signed consensus by specifying the same sign for all agents. In fact, the idea of signed consensus is similar to that of bipartite consensus for signed networks in which the interaction between agents can be not only cooperative but also antagonistic (Altafini, 2013; Meng et al., 2015; Meng et al., 2016; Valcher & Misra, 2014). However, they are different and a major difference is that the sign of each agent's consensus state is specified according to the task requirement in signed consensus but is closely related to the network topology structure in bipartite consensus. It is worth noting that the existing analysis of consensus (Altafini, 2013; Meng et al., 2015; Meng et al., 2016; Valcher & Misra, 2014) and its extended formation (Hu et al., 2013) and flocking (Fan et al., 2014) tasks most focus on fixed networks, and only some remarks are given when considering switching networks. In contrast to this, it is shown in this paper that our proposed signed consensus results can be applied to deal with exponential convergence of bipartite consensus for signed networks regardless of whether their topologies are fixed or switching, and necessary and sufficient consensus conditions can be developed simultaneously. The signed consensus considered here may be included as a special case of the scaled consensus handled in Meng and Jia (2016a), Meng and Jia (2016b), and Roy (2015). But, since signed consensus may be tied together with the class of bipartite consensus (Altafini, 2013; Meng et al., 2015; Meng et al., 2016; Valcher & Misra, 2014), our obtained results may provide bipartite consensus with new developments especially for switching networks. Numerical simulations are given to illustrate that the proposed consensus protocols are effective in guaranteeing all agents to achieve signed consensus exponentially fast in the presence of both fixed and switching topologies.

The rest of this paper is organised as follows. The signed consensus problem statement and some preliminaries on stability of linear systems are presented in Section 2. We deal with the exponential convergence problem of signed consensus for fixed networks in Section 3 and for switching networks in Section 4, respectively. Simulation tests are performed to illustrate the signed consensus performance of the designed protocols in Section 5. We finally conclude this paper in Section 6.

Notation 1: Throughout this paper, $\mathcal{I}_n = \{1, 2, \dots, n\}$, $1_n = [1, 1, \dots, 1]^{\mathrm{T}} \in \mathbb{R}^n$, I and 0 are the identity and null matrices with required dimensions, respectively, diag$\{\cdot\}$ denotes a diagonal matrix whose off-diagonal elements are all zero, $\|A\|$ is the maximum row sum norm of a matrix A, and $A \geq 0$ denotes a non-negative matrix if all its elements are non-negative. In addition, we say that a matrix $A \in \mathbb{R}^{n\times n}$ is stochastic if $A \geq 0$ and satisfies $A1_n = 1_n$ and $\lim_{k\to\infty} z(k) = z_*$ exponentially fast if there exist a finite constant $\alpha > 0$ and a constant $0 \leq \beta < 1$ such that $\|z(k) - z_*\| \leq \alpha\beta^k$, $\forall k \geq 0$.

2. Problem and preliminary

2.1 Signed consensus

Consider agents with the following discrete-time dynamics:

$$x_i(k+1) = x_i(k) + u_i(k), \quad \forall i \in \mathcal{I}_n, \tag{1}$$

where $x_i(k), u_i(k) \in \mathbb{R}^m$ are the state and protocol or control input of the ith agent, respectively. Without any loss of generality, we only consider the scalar case with $m = 1$ whose analyses and results can be extended to the vector case with $m > 1$ by introducing the Kronecker product. The extensions, however, will be not detailed for simplicity.

In this paper, we adopt the following definition for signed consensus on networks of agents as well as the exponential convergence of it.

Definition 2.1: It is said that the system (1) under the action of a certain protocol achieves signed consensus (exponentially fast) if given any initial state condition $x_i(0)$ and any specified sign $\sigma_i \in \{\pm 1\}$, $\lim_{k\to\infty}\sigma_i x_i(k) = c$, $\forall i \in \mathcal{I}_n$ (exponentially fast) for some constant c.

Note that the final consensus states of agents depend upon the constant c which may not be exactly known. It is obvious that the signed consensus problem cannot be transformed into a stability problem. By Definition 2.1, it actually implies that signed consensus drives all agents to reach agreement upon a stationary state which is the same for them in modulus but not in sign. This objective is the same as that of bipartite consensus on networks of agents that possess not only cooperative but also antagonistic interactions (see e.g. Altafini, 2013; Valcher & Misra, 2014, for details). However, we call it signed consensus because the signs of all agents' consensus states are specified rather than decided by the topology structure of networks in bipartite consensus. Let $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]^{\mathrm{T}}$, and then Definition 2.1 implies $\lim_{k\to\infty} x(k)$

$= D1_n c$, where $D = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ specifies the signs of the agents' consensus states rather than heavily depends on the topology structure of the agents' networks in bipartite consensus. It will be seen that the developed signed consensus results can be extended to include the bipartite consensus results as special cases. In addition, we can divide the agents into two specified groups through choosing appropriate sign values of the parameters σ_i, $\forall i \in \mathcal{I}_n$.

We consider interactions between agents that are represented in the framework of directed graphs. Let a directed graph $\mathcal{G}$ be a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that consists of a node set $\mathcal{V} = \{v_i : i \in \mathcal{I}_n\}$, an edge set $\mathcal{E} \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$, and a non-negative adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ for the weights of $\mathcal{G}$, where $a_{ij} > 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. We consider no self-loops in $\mathcal{G}$, i.e. $(v_i, v_i) \notin \mathcal{E}$ and $a_{ii} = 0, \forall i \in \mathcal{I}_n$. Given an edge $(v_j, v_i) \in \mathcal{E}$, v_j is a neighbour of v_i and the index set for all neighbours of v_i is denoted as $\mathcal{N}_i = \{j : (v_j, v_i) \in \mathcal{E}\}$. A path of $\mathcal{G}$ is a sequence of edges in $\mathcal{E}$ of the form: $(v_{i_l}, v_{i_{l+1}}) \in \mathcal{E}$ for $l = 1, 2, \ldots, j-1$, where $v_{i_1}, v_{i_2}, \ldots, v_{i_j}$ are distinct nodes. We say that $\mathcal{G}$ is quasi-strongly connected (or in other words, has a spanning tree) if there exists some node that can be connected to all the other nodes through paths and is strongly connected if there exist paths that can connect any given pair of nodes. For the Laplacian of $\mathcal{G}$ associated with $\mathcal{A}$, it is denoted by $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$ whose elements are defined by (Cao et al., 2013; Olfati-Saber et al., 2007; Ren et al., 2007)

$$l_{ij} = \begin{cases} \sum_{\kappa\in\mathcal{N}_i} a_{i\kappa}, & j = i \\ -a_{ij}, & j \neq i. \end{cases}$$

2.2 Stability preliminaries

Although the considered signed consensus problem is beyond the scope of stability problems, we will incorporate some classical stability criteria into our consensus problem solving. Consider the following system:

$$z(k+1) = A(k)z(k), \quad z(k_0) = z_0, \quad \forall k \geq k_0, \quad (2)$$

where $z(k) \in \mathbb{R}^n$ is the state, z_0 is the initial state at the initial time step $k_0 \geq 0$, and $A(k) \in \mathbb{R}^{n\times n}$ is the time-varying system matrix. The following concepts related to Equation (2) are introduced (see Rugh, 1996 , for more details).

Definition 2.2: We say that the system (2) is

- *uniformly stable* if there exists a finite constant $\alpha > 0$ such that $\|z(k)\| \leq \alpha\|z_0\|$, $\forall k \geq k_0$ for any k_0 and z_0;
- *uniformly asymptotically stable* if (1) it is uniformly stable and (2) given any constant $\varepsilon > 0$, there exists a positive integer k' such that $\|z(k)\| \leq \varepsilon\|z_0\|$, $\forall k \geq k_0 + k'$ for any k_0 and z_0;
- *uniformly exponentially stable* if there exist a finite constant $\alpha > 0$ and a constant $0 \leq \beta < 1$ such that $\|z(k)\| \leq \alpha\beta^{k-k_0}\|z_0\|$, $\forall k \geq k_0$ for any k_0 and z_0.

In Definition 2.2, the uniform property means that the existing α, β, and k' are independent of the initial time k_0 (Rugh, 1996). When it comes to linear time-invariant systems, i.e. $A(k) \equiv A$, $\forall k$, the uniform (asymptotic, exponential) stability of Equation (2) collapses into (asymptotic, exponential) stability and we can directly set the initial time as zero, i.e. $k_0 = 0$.

To end this preliminary, we give the following lemma for the equivalence between uniform asymptotic stability and uniform exponential stability of the system (2) (see Rugh, 1996, Theorem 22.14, for the details).

Lemma 2.1: *For the system* (2)*, it is uniformly asymptotically stable if and only if it is uniformly exponentially stable.*

3. Fixed networks

We first consider fixed networks of agents and assume that each agent is regarded as a node in the directed graph $\mathcal{G}$ with fixed topology. In this case, we propose a distributed protocol based on the nearest-neighbour interaction rule as

$$u_i(k) = \sigma_i\gamma \sum_{j\in\mathcal{N}_i} a_{ij}\left[\sigma_j x_j(k) - \sigma_i x_i(k)\right], \quad \forall i \in \mathcal{I}_n, \quad (3)$$

where $\sigma_i \in \{\pm 1\}$ is the sign specified in Definition 2.1 for the ith agent and $\gamma > 0$ is a unified scalar gain for all agents. Under the action of this protocol, the dynamic process of the multi-agent system (1) is represented by

$$x(k+1) = (I - \gamma D\mathcal{L}D)x(k). \quad (4)$$

It is obvious that the system (4) is not exponentially or asymptotically stable because the Laplacian $\mathcal{L}$ has at least one zero eigenvalue (Cao et al., 2013; Olfati-Saber et al., 2007; Ren et al., 2007). By contrast, this system can admit signed consensus with an exponential convergence speed given quasi-strong connectivity.

Theorem 3.1: *Let the gain γ be chosen such that*

$$\gamma \max_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{N}_i} a_{ij} < 1. \quad (5)$$

The system (4) achieves signed consensus exponentially fast if and only if $\mathcal{G}$ is quasi-strongly connected.

Proof: **(Sufficiency):** Due to $\sigma_i \in \{\pm 1\}$, we have $D^2 = I$ and hence $I - \gamma D\mathcal{L}D = D(I - \gamma\mathcal{L})D$. We can easily validate that the gain selection condition (5) renders $I - \gamma\mathcal{L}$ a stochastic matrix with positive diagonal elements. With this fact, it follows that if $\mathcal{G}$ is quasi-strongly connected (or has a spanning tree), then $\lim_{k\to\infty}(I - \gamma\mathcal{L})^k = 1_n\nu^{\mathrm{T}}$ where $\nu \geq 0$ satisfies $(I - \gamma\mathcal{L})^{\mathrm{T}}\nu = \nu$ and $1_n^{\mathrm{T}}\nu = 1$ (see e.g. Ren & Beard, 2005). We can thus obtain for the system (4) that

$$\begin{aligned}\lim_{k\to\infty} x(k) &= \lim_{k\to\infty}(I - \gamma D\mathcal{L}D)^k x(0)\\ &= \lim_{k\to\infty} D(I - \gamma\mathcal{L})^k Dx(0)\\ &= D1_n c\end{aligned}$$

where $c = \nu^{\mathrm{T}}Dx(0)$. It means that the signed consensus can be achieved for the system (4). To show exponential convergence of this consensus result, we denote a non-singular matrix $Q \in \mathbb{R}^{n\times n}$ and perform a linear non-singular transformation $y(k) = QDx(k)$ on the system (4) to obtain

$$\begin{aligned}y(k+1) &= QD(I - \gamma D\mathcal{L}D)DQ^{-1}y(k)\\ &= Q(I - \gamma\mathcal{L})Q^{-1}y(k)\\ &= (I - \gamma Q\mathcal{L}Q^{-1})y(k).\end{aligned} \tag{6}$$

Let $y(k) = [y_1(k), y_2(k), \ldots, y_n(k)]^{\mathrm{T}}$ and $y'(k) = [y_2(k), y_3(k), \ldots, y_n(k)]^{\mathrm{T}}$. If we consider

$$Q = \begin{bmatrix}\xi\\ \Xi\end{bmatrix}, \quad Q^{-1} = \begin{bmatrix}1_n & \Theta\end{bmatrix}$$

where $\xi = [1, 0, \ldots, 0] \in \mathbb{R}^{1\times n}$, $\Xi = [-1_{n-1}, I] \in \mathbb{R}^{(n-1)\times n}$ and $\Theta = [0, I]^{\mathrm{T}} \in \mathbb{R}^{n\times(n-1)}$, then we can get two subsystems from Equation (6) as

$$y_1(k+1) = y_1(k) - \gamma\xi\mathcal{L}\Theta y'(k) \tag{7}$$

and

$$y'(k+1) = (I - \gamma\Xi\mathcal{L}\Theta)y'(k), \tag{8}$$

for which we employ $\mathcal{L}1_n = 0$ to arrive at

$$Q(I - \gamma\mathcal{L})Q^{-1} = \begin{bmatrix}1 & -\gamma\xi\mathcal{L}\Theta\\ 0 & I - \gamma\Xi\mathcal{L}\Theta\end{bmatrix}. \tag{9}$$

From Equation (9), we also know that each eigenvalue of $I - \gamma\Xi\mathcal{L}\Theta$ has magnitude strictly less than unity since the stochastic matrix $I - \gamma\mathcal{L}$ has exactly one eigenvalue equal to one and its other eigenvalues all have magnitude strictly less than unity. This obviously implies that the system (8) is exponentially stable (see Rugh, 1996, Theorem 22.14), i.e. $\|y'(k)\| \leq \alpha\beta^k\|y'(0)\|$ for some constant $\alpha > 0$ and some constant $0 \leq \beta < 1$. By inserting this fact into Equation (7), we can derive

$$\begin{aligned}|y_1(k+1) - y_1(k)| &= |-\gamma\xi\mathcal{L}\Theta y'(k)|\\ &\leq \|\gamma\xi\mathcal{L}\Theta\|\|y'(k)\|\\ &\leq \alpha'\beta^k,\end{aligned} \tag{10}$$

where $\alpha' = \alpha\|\gamma\xi\mathcal{L}\Theta\|\|y'(0)\|$. Given any positive integer $l > k$, we can employ Equation (10) to further arrive at

$$\begin{aligned}\left|y_1(l) - y_1(k)\right| &= \left|\sum_{j=k}^{l-1}\left[y_1(j+1) - y_1(j)\right]\right|\\ &\leq \sum_{j=k}^{l-1}\left|y_1(j+1) - y_1(j)\right|\\ &\leq \sum_{j=k}^{l-1}\alpha'\beta^j\\ &= \frac{1-\beta^{l-k}}{1-\beta}\alpha'\beta^k.\end{aligned} \tag{11}$$

Since $\lim_{k\to\infty}x(k) = D1_n c$ and $y(k) = QDx(k)$, we have $\lim_{k\to\infty}y_1(k) = \xi 1_n c = c$. Thus, we take $l \to \infty$ on both sides of Equation (11) and can get

$$\left|y_1(k) - c\right| \leq \frac{\alpha'}{1-\beta}\beta^k, \quad \forall k,$$

which implies that $\lim_{k\to\infty}y_1(k) = c$ exponentially fast. It again follows from $y(k) = QDx(k)$ that $y_1(k) = \xi Dx(k) = \sigma_1 x_1(k)$. Hence, we establish the exponential convergence of $\lim_{k\to\infty}\sigma_1 x_1(k) = c$. By following similar analysis steps, we can prove that $\lim_{k\to\infty}\sigma_i x_i(k) = c, \forall i \in \mathcal{I}_n$ exponentially fast through choosing an appropriate transformation matrix Q.

Necessity: If $\mathcal{G}$ is not quasi-strongly connected (or does not have any spanning tree), then we can follow the necessity proof of Ren and Beard (2005, Theorem 3.8) to obtain that the rank of $\lim_{k\to\infty}(I - \gamma\mathcal{L})^k$ is more than one. Thus, $\sigma_i x_i(k)$ cannot achieve consensus, which is equivalent to that the agents cannot achieve signed consensus. The proof is complete. ■

Remark 3.1: For the fixed topology case, Theorem 3.1 shows that the quasi-strong connectivity of multi-agent networks is a necessary and sufficient guarantee for the exponential convergence of their signed consensus. It can be seen that its convergence speed depends upon the non-zero eigenvalues of the Laplacian of multi-agent networks. When it particularly comes to the undirected networks, this signed consensus result collapses into the convergence speed dependence upon the algebraic connectivity of networks, which coincides with the classical consensus results (see e.g. Olfati-Saber et al., 2007). In addition, the signed consensus results are obviously degraded into a class of consensus results if the signs of all agents' consensus states are specially taken positive. This implies that the discussions of Theorem 3.1 can also provide an analysis method for exponential convergence of consensus on networks of agents.

It is interesting that Theorem 3.1 can be explored to provide exponential convergence result for bipartite consensus on networks with antagonistic interactions given quasi-strong connectivity. To make this observation clear to follow, let us denote $\mathcal{A}^s = [a^s_{ij}] \in \mathbb{R}^{n\times n}$, where $a^s_{ij} = \sigma_i\sigma_j a_{ij}$. We can easily see that $\mathcal{G}^s = (\mathcal{V}, \mathcal{E}, \mathcal{A}^s)$ is a signed directed graph (Altafini, 2013), where $a^s_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}$ and $a^s_{ij} = 0$, otherwise. The bipartite consensus of networks associated with signed directed graphs is closely tied to the notion of structural balance which is introduced in the following by considering any signed directed graph $\mathbf{G} = (\mathcal{V}, \mathcal{E}, \mathbf{A})$ with $\mathbf{A} = \left[\mathbf{a}_{ij}\right] \in \mathbb{R}^{n\times n}$ such that $\mathbf{a}_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}$ and $\mathbf{a}_{ij} = 0$ otherwise.

Definition 3.1 (Altafini, 2013): If there is a bipartition $\{\mathcal{V}^{(1)}, \mathcal{V}^{(2)}\}$ of nodes satisfying $\mathcal{V}^{(1)} \cup \mathcal{V}^{(2)} = \mathcal{V}$ and $\mathcal{V}^{(1)} \cap \mathcal{V}^{(2)} = \emptyset$ such that $\mathbf{a}_{ij} \geq 0$ for $v_i, v_j \in \mathcal{V}^{(l)}$, $\forall l \in \{1, 2\}$ and $\mathbf{a}_{ij} \leq 0$ for $v_i \in \mathcal{V}^{(l)}$ and $v_j \in \mathcal{V}^{(q)}$, $\forall l \neq q, l, q \in \{1, 2\}$, then we say that a signed directed graph $\mathbf{G}$ is structurally balanced. Otherwise, we say that $\mathbf{G}$ is structurally unbalanced.

By Definition 3.1, it follows that $\mathcal{G}^s$ is a structurally balanced signed directed graph. It is also obvious that there exists a gauge transformation performed by the matrix D such that $D\mathcal{A}^sD = \mathcal{A}$ (see Altafini, 2013). Due to $\sigma_i \in \{\pm 1\}$, it is easy to obtain

$$\operatorname{sign}\left(a^s_{ij}\right) = \sigma_i\sigma_j = \frac{\sigma_i}{\sigma_j} \quad \text{if } a_{ij} > 0,$$

which can be inserted to rewrite Equation (3) as

$$\begin{aligned} u_i(k) &= \gamma \sum_{j\in\mathcal{N}_i} \sigma_i\sigma_j a_{ij}\left[x_j(k) - \frac{\sigma_i}{\sigma_j}x_i(k)\right] \\ &= \gamma \sum_{j\in\mathcal{N}_i} a^s_{ij}\left[x_j(k) - \operatorname{sign}\left(a^s_{ij}\right)x_i(k)\right], \quad \forall i \in \mathcal{I}_n. \end{aligned} \tag{12}$$

By considering Equation (12) for Equation (1), we have

$$x(k+1) = (I - \gamma\mathcal{L}^s)x(k), \tag{13}$$

where $\mathcal{L}^s = [l^s_{ij}] \in \mathbb{R}^{n\times n}$ is the Laplacian of the signed directed graph $\mathcal{G}^s$ whose elements are given by (see e.g. Altafini, 2013)

$$l^s_{ij} = \begin{cases} \sum_{\kappa\in\mathcal{N}_i} |a^s_{i\kappa}|, & j = i \\ -a^s_{ij}, & j \neq i. \end{cases}$$

With the above development, an immediate consequence of Theorem 3.1 is the following corollary for bipartite consensus on networks with antagonistic interactions.

Corollary 3.1: *Let the gain γ satisfy*

$$\gamma \max_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{N}_i} \left|a^s_{ij}\right| < 1.$$

The system (13) achieves bipartite consensus exponentially fast if and only if the structurally balanced signed directed graph $\mathcal{G}^s$ is quasi-strongly connected.

Remark 3.2: Corollary 3.1 extends the bipartite consensus result of Altafini (2013) to discrete-time multi-agent systems. In contrast to Altafini (2013), this corollary can deal with the exponential convergence problem of bipartite consensus, which, however, only needs the quasi-strong connectivity rather than the strong connectivity of the associated fixed networks.

4. Switching networks

When the network topology of the multi-agent system (1) is switching, we represent it with a time-varying directed graph $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k), \mathcal{A}(k))$, where $\mathcal{E}(k) \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$ and $\mathcal{A}(k) = \left[a_{ij}(k)\right] \in \mathbb{R}^{n\times n}$ such that $a_{ij}(k) > 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}(k)$ and $a_{ij}(k) = 0$ otherwise. Accordingly, the index set for all neighbours of v_i is time varying, which is denoted by $\mathcal{N}_i(k) = \{j : (v_j, v_i) \in \mathcal{E}(k)\}$. We consider $\mathcal{G}(k)$ that switches over r topologies, i.e. $\mathcal{G}(k) \in \mathbf{G} \triangleq \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_r\}$, $\forall k \in \mathbb{Z}_+$, where $\mathcal{G}_p = (\mathcal{V}, \mathcal{E}_p, \mathcal{A}_p)$ with $\mathcal{A}_p = \left[a_{p,ij}\right]$, $\forall p \in \mathcal{I}_r$. Given any directed graphs $\mathcal{G}_o$, $\mathcal{G}_p$ and $\mathcal{G}_q$, we call $\mathcal{G}_o = \mathcal{G}_p \bigcup \mathcal{G}_q$ the union of $\mathcal{G}_p$ and $\mathcal{G}_q$ if $\mathcal{E}_o = \mathcal{E}_p \bigcup \mathcal{E}_q$.

In the presence of switching topologies, we apply a distributed protocol as

$$u_i(k) = \sigma_i\gamma(k) \sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)\left[\sigma_j x_j(k) - \sigma_i x_i(k)\right], \quad \forall i \in \mathcal{I}_n \tag{14}$$

where the specified sign $\sigma_i \in \{\pm 1\}$ for the ith agent is the same as the Definition 2.1 and the unified scalar gain $\gamma(k) > 0$ for all agents is time varying. Note that the network topology of agents switches over finite alternatives. Hence, we consider $\gamma(k)$ that takes its values from a finite set, i.e. $\gamma(k) \in \Gamma \triangleq \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$, $\forall k$, where γ_p is designed corresponding to the directed graph $\mathcal{G}_p$, $\forall p \in \mathcal{I}_r$. The application of protocol (14) renders the network process of the multi-agent system (1) time varying, which is given by

$$x(k+1) = [I - \gamma(k) D\mathcal{L}(k)D]\, x(k) \tag{15}$$

where $\mathcal{L}(k)$ is the Laplacian of $\mathcal{G}(k)$ associated with $\mathcal{A}(k)$. To develop the signed consensus of the network process (15), we consider a joint quasi-strong connectivity network condition as introduced in the following definition.

Definition 4.1: It is said that the time-varying directed graph $\mathcal{G}(k)$ is jointly quasi-strongly connected through switching over **G** if there exists an infinite sequence $\{k_i\colon k_0 = 0,\ 0 < k_{i+1} - k_i \le h,\ \forall i \in \mathbb{Z}_+\}$ for some finite positive integer h such that the collection of directed graphs $\{\mathcal{G}(k_i), \mathcal{G}(k_i+1), \ldots, \mathcal{G}(k_{i+1}-1)\}$ is jointly quasi-strongly connected for all $i \in \mathbb{Z}_+$, where it is said that such a collection is *jointly quasi-strongly connected* if the union $\bigcup_{p=k_i}^{k_{i+1}-1} \mathcal{G}(p)$ is a quasi-strongly connected directed graph.

Based on the above definition, the following theorem presents an exponential convergence result for signed consensus problems on switching networks of agents.

Theorem 4.1: *Let the time-varying gain $\gamma(k) \in \Gamma$ be chosen such that*

$$\gamma_p \max_{i\in\mathcal{I}_n} \sum_{j\in\mathcal{N}_i} a_{p,ij} < 1, \quad \forall p \in \mathcal{I}_r. \tag{16}$$

The system (15) achieves signed consensus exponentially fast if and only if $\mathcal{G}(k)$ is jointly quasi-strongly connected through switching over **G**.

Remark 4.1: Obviously, Theorem 4.1 demonstrates that the signed consensus with an exponential convergence speed can be further established for multi-agent systems in the presence of switching topologies. This not only extends the signed consensus result of Theorem 3.1 but also strengthens the classical consensus results through achieving the exponential convergence under a joint quasi-strong connectivity condition. For the selection of the time-varying gain $\gamma(k)$, we can particularly choose a constant one such that $\gamma(k) \equiv \gamma$ (i.e. $\gamma_p \equiv \gamma$) and

$$\gamma \max_{i\in\mathcal{I}_n, p\in\mathcal{I}_r} \sum_{j\in\mathcal{N}_i} a_{p,ij} < 1.$$

4.1 Preliminary lemmas

To prove Theorem 4.1, we first introduce some helpful lemmas. For notations, we denote $\Psi(k) = I - \gamma(k)D\mathcal{L}(k)D$ and $\Phi(k) = I - \gamma(k)\mathcal{L}(k)$. It follows that $\Psi(k) = D\Phi(k)D$ and $\Phi(k) \in \{I - \gamma_1\mathcal{L}_1, I - \gamma_2\mathcal{L}_2, \ldots, I - \gamma_r\mathcal{L}_r\}$, $\forall k \in \mathbb{Z}_+$, where $\mathcal{L}_p$ is the Laplacian of $\mathcal{G}_p$ associated with $\mathcal{A}_p$, $\forall p \in \mathcal{I}_r$.

Lemma 4.1: *If the time-varying gain $\gamma(k)$ satisfies Equation (16), then $\Phi(k)$ is a stochastic matrix with positive diagonal elements and further given any positive integers i and j such that $i < j$, $\prod_{k=i}^{j} \Phi(k)$ is SIA if the collection of directed graphs $\{\mathcal{G}(i), \mathcal{G}(i+1), \ldots, \mathcal{G}(j)\}$ is jointly quasi-strongly connected.*

Proof: Note that the condition (16) renders $I - \gamma_p\mathcal{L}_p$ a stochastic matrix corresponding to the directed graph $\mathcal{G}_p$, $\forall p \in \mathcal{I}_r$ and with positive diagonal elements. It follows immediately that $\Phi(k)$ is a stochastic matrix corresponding to the directed graph $\mathcal{G}(k)$, $\forall k \in \mathbb{Z}_+$ and its diagonal elements are positive. With this fact, we can adopt the same proof steps as those of Ren and Beard (2005, Lemma 3.9) to prove that $\prod_{k=i}^{j} \Phi(k)$ is SIA. ■

Let us consider the sequence $\{k_i : i \in \mathbb{Z}_+\}$ in Definition 4.1. We construct $\{\prod_{p=k_i}^{k_{i+1}-1} \Phi(p) : i \in \mathbb{Z}_+\}$ and can summarise the following property for this sequence of matrix products.

Lemma 4.2: *Let the time-varying gain $\gamma(k)$ satisfy Equation (16). If $\mathcal{G}(k)$ is jointly quasi-strongly connected through switching over* **G**, *then for each infinite sequence $\{\prod_{p=k_{i_j}}^{k_{i_j+1}-1} \Phi(p) : j = 1, 2, 3, \ldots\}$, there exists some constant vector $\nu \in \mathbb{R}^n$ such that $\lim_{l\to\infty} \prod_{j=1}^{l} \prod_{p=k_{i_j}}^{k_{i_j+1}-1} \Phi(p) = 1_n \nu^{\mathrm{T}}$.*

Proof: Based on Lemma 4.1, we can easily obtain that $\prod_{p=k_i}^{k_{i+1}-1} \Phi(p)$ under the condition (16) is a stochastic matrix with positive diagonal elements because stochastic matrices with positive diagonal elements are closed with respect to the matrix multiplication. Since $\mathcal{G}(k)$ is jointly quasi-strongly connected through switching over **G**, we obviously have that the collection of directed graphs $\{\mathcal{G}(k_{i_1}), \mathcal{G}_{k_{i_1}+1}, \ldots, \mathcal{G}_{k_{i_1+1}-1}\}$ is jointly quasi-strongly connected. Due to (see Ren & Beard, 2005, Lemma 3.1)

$$\prod_{p=k_{i_1}}^{k_{i_1+1}-1} \Phi(p) \ge \mu \sum_{p=k_{i_1}}^{k_{i_1+1}-1} \Phi(p)$$

for some $\mu > 0$, we further have that $\mathcal{G}_*(i_1)$ is quasi-strongly connected, where $\mathcal{G}_*(i)$ denotes the directed graph to which the stochastic matrix $\prod_{p=k_i}^{k_{i+1}-1}\Phi(p)$ corresponds. An obvious consequence of this is that the collection of directed graphs $\{\mathcal{G}_*(i_1), \mathcal{G}_*(i_2), \dots, \mathcal{G}_*(i_j)\}$ is jointly quasi-strongly connected. Based on this joint quasi-strong connectivity condition, we can prove that $\prod_{j=1}^{l}\prod_{p=k_{i_j}}^{k_{i_j+1}-1}\Phi(p)$ is SIA for any finite positive integer l in the same way as in the proof of Lemma 4.1. In addition, note that the sequence of matrix products $\{\prod_{p=k_i}^{k_{i+1}-1}\Phi(p) : i \in \mathbb{Z}_+\}$ has at most $r^{\,h}$ different elements. With these two facts, this proof can be established based on Ren and Beard (2005, Lemma 3.2). ■

Lemma 4.3: *Given any non-negative integers i and j such that $i \le j$, it follows*

$$\prod_{p=i}^{j}\Xi\Phi(p)\Theta = \Xi\left[\prod_{p=i}^{j}\Phi(p)\right]\Theta.$$

Proof: Due to $\mathcal{L}_p 1_n = 0$, we have $\Phi(p)1_n = [I - \gamma(p)\mathcal{L}(p)]1_n = 1_n$. Note that $\Xi 1_n = 0$ and $\Theta\Xi + 1_n\xi = I$. We can easily validate

$$\begin{aligned}
&[\Xi\Phi(i+1)\Theta]\,[\Xi\Phi(i)\Theta] \\
&\quad = \Xi\,[\Phi(i+1)(I - 1_n\xi)\Phi(i)]\,\Theta \\
&\quad = \Xi\,[\Phi(i+1)\Phi(i)]\,\Theta - \Xi\,[\Phi(i+1)1_n]\,\xi\Phi(i)\Theta \\
&\quad = \Xi\,[\Phi(i+1)\Phi(i)]\,\Theta - (\Xi 1_n)\,\xi\Phi(i)\Theta \\
&\quad = \Xi\,[\Phi(i+1)\Phi(i)]\,\Theta.
\end{aligned}$$

By performing the same steps repetitively, we can complete this proof. ■

4.2 Technical proof

Proof **of Theorem 4.1:(Sufficiency):** In view of the sequence $\{k_i : i \in \mathbb{Z}_+\}$ of Definition 4.1, the solution to the system (15) is given by

$$\begin{aligned}
x(k) &= \prod_{l=0}^{k-1}\Psi(l)x(0) \\
&= D\prod_{l=0}^{k-1}\Phi(l)Dx(0) \\
&= D\prod_{l=k_{i_k}}^{k-1}\Phi(l)\prod_{j=0}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p)Dx(0),
\end{aligned} \tag{17}$$

where i_k is the largest integer such that $k_{i_k} \le k - 1$. Note that $i_k \to \infty$ as $k \to \infty$. Under the proposed conditions of this theorem, $\prod_{l=k_{i_k}}^{k-1}\Phi(l)$ is stochastic by Lemma 4.1 and $\lim_{k\to\infty}\prod_{j=0}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) = 1_n\nu^{\mathrm{T}}$ for some $\nu \in \mathbb{R}^n$ by Lemma 4.2. We thus have $\|\prod_{i=k_{i_k}}^{k-1}\Phi(l)\| = 1$ and $\prod_{i=k_{i_k}}^{k-1}\Phi(l)1_n\nu^{\mathrm{T}} = 1_n\nu^{\mathrm{T}}$. Using these facts and $\|D\| = 1$, we can develop based on Equation (17) that

$$\begin{aligned}
&\left\|x(k) - D1_n\nu^{\mathrm{T}}Dx(0)\right\| \\
&= \left\|D\prod_{l=k_{i_k}}^{k-1}\Phi(l)\left[\prod_{j=0}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) - 1_n\nu^{\mathrm{T}}\right]Dx(0)\right\| \\
&\le \|D\|\left\|\prod_{l=k_{i_k}}^{k-1}\Phi(l)\right\|\left\|\prod_{j=0}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) - 1_n\nu^{\mathrm{T}}\right\|\|D\|\,\|x(0)\| \\
&= \left\|\prod_{j=0}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) - 1_n\nu^{\mathrm{T}}\right\|\|x(0)\| \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{18}$$

It is clear from Equation (18) that $\lim_{k\to\infty}x(k) = D1_n c$, where $c = \nu^{\mathrm{T}}Dx(0)$. Next, we prove its exponential convergence. Toward this end, we again consider the linear nonsingular transformation performed in the proof of Theorem 3.1. By $y(k) = QDx(k)$, we can derive from Equation (15) that

$$\begin{aligned}
y(k+1) &= QD[I - \gamma(k)D\mathcal{L}(k)D]DQ^{-1}y(k) \\
&= Q[I - \gamma(k)\mathcal{L}(k)]Q^{-1}y(k) \\
&= [I - \gamma(k)Q\mathcal{L}(k)Q^{-1}]y(k).
\end{aligned} \tag{19}$$

Since $\mathcal{L}(k)1_n = 0$, we also have the property of Equation (9) for the matrix $Q[I - \gamma(k)\mathcal{L}(k)]Q^{-1}$, i.e.

$$Q[I - \gamma(k)\mathcal{L}(k)]Q^{-1} = \begin{bmatrix} 1 & -\gamma(k)\xi\mathcal{L}(k)\Theta \\ 0 & I - \gamma(k)\Xi\mathcal{L}(k)\Theta \end{bmatrix}, \quad \forall k.$$

By inserting this into Equation (19), we separate it into two time-varying subsystems as

$$y_1(k+1) = y_1(k) - \gamma(k)\xi\mathcal{L}(k)\Theta y'(k) \tag{20}$$

and

$$y'(k+1) = [I - \gamma(k)\Xi\mathcal{L}(k)\Theta]y'(k). \tag{21}$$

Given any k_0 and $y'(k_0) = y'_0$, the solution of the system (21) is expressed by

$$\begin{aligned} y'(k) &= \prod_{l=k_0}^{k-1}[I - \gamma(l)\Xi\mathcal{L}(l)\Theta]y'(k_0) \\ &= \prod_{l=k_0}^{k-1}\Xi\Phi(l)\Theta y'(k_0) \\ &= \Xi\left[\prod_{l=k_0}^{k-1}\Phi(l)\right]\Theta y'(k_0), \quad \forall k \geq k_0, \end{aligned} \tag{22}$$

where we combine $\Xi\Theta = I$ and the fact of Lemma 4.3. It follows from Equation (22) immediately that

$$\begin{aligned} \left\|y'(k)\right\| &\leq \|\Xi\| \left\|\prod_{l=k_0}^{k-1}\Phi(l)\right\| \|\Theta\| \left\|y'(k_0)\right\| \\ &= 2\left\|y'_0\right\|, \quad \forall k \geq k_0, \end{aligned}$$

where we use $\|\Xi\| = 2$, $\|\Theta\| = 1$ and $\left\|\prod_{l=k_0}^{k-1}\Phi(l)\right\| = 1$ by considering also Lemma 4.1. Thus, the system (21) is uniformly stable based on the Definition 2.2. In addition, we can rewrite the solution (22) as

$$\begin{aligned} y'(k) = \Xi&\left[\prod_{l=k_{i_k}}^{k-1}\Phi(l)\prod_{j=\tilde{i}_{k_0}}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p)\prod_{q=k_0}^{k_{\tilde{i}_{k_0}}-1}\Phi(q)\right] \\ &\times \Theta y'(k_0), \quad \forall k \geq k_0, \end{aligned} \tag{23}$$

where $\tilde{i}_{k_0}$ is the smallest integer such that $k_{\tilde{i}_{k_0}} \geq k_0 + 1$. Since $i_k \to \infty$ as $k \to \infty$, we know that $\lim_{k\to\infty}\prod_{j=\tilde{i}_{k_0}}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) = 1_n\nu^{\mathrm{T}}$ by Lemma 4.2. Also, due to $\Xi 1_n = 0$ and $\prod_{l=k_{i_k}}^{k-1}\Phi(l)1_n = 1_n$, we can further use Equation (23) to arrive at

$$\begin{aligned} y'(k) = \Xi&\left\{\prod_{l=k_{i_k}}^{k-1}\Phi(l)\left[\prod_{j=\tilde{i}_{k_0}}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) - 1_n\nu^{\mathrm{T}}\right]\prod_{q=k_0}^{k_{\tilde{i}_{k_0}}-1}\Phi(q)\right\} \\ &\times \Theta y'(k_0), \quad \forall k \geq k_0 \end{aligned} \tag{24}$$

based on which we can deduce

$$\begin{aligned} \left\|y'(k)\right\| &= \left\|\Xi\left\{\prod_{l=k_{i_k}}^{k-1}\Phi(l)\left[\prod_{j=\tilde{i}_{k_0}}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) - 1_n\nu^{\mathrm{T}}\right]\right.\right. \\ &\qquad \left.\left.\times\prod_{q=k_0}^{k_{\tilde{i}_{k_0}}-1}\Phi(q)\right\}\Theta y'(k_0)\right\| \\ &\leq \|\Xi\|\left\|\prod_{l=k_{i_k}}^{k-1}\Phi(l)\right\|\left\|\prod_{j=\tilde{i}_{k_0}}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) - 1_n\nu^{\mathrm{T}}\right\| \\ &\quad \times\left\|\prod_{q=k_0}^{k_{\tilde{i}_{k_0}}-1}\Phi(q)\right\|\|\Theta\|\left\|y'(k_0)\right\| \\ &= \underbrace{2\left\|\prod_{j=\tilde{i}_{k_0}}^{i_k-1}\prod_{p=k_j}^{k_{j+1}-1}\Phi(p) - 1_n\nu^{\mathrm{T}}\right\|}_{\to 0 \text{ as } k\to\infty}\left\|y'_0\right\| \\ &\to 0 \quad \text{as } k \to \infty, \end{aligned} \tag{25}$$

where we insert $\|\Xi\| = 2$, $\|\Theta\| = 1$, $\|\prod_{l=k_{i_k}}^{k-1}\Phi(l)\| = 1$, and $\|\prod_{q=k_0}^{k_{\tilde{i}_{k_0}}-1}\Phi(q)\| = 1$. From Equation (25), it follows that given any constant $\varepsilon > 0$, there exists a positive integer k' (independent of k_0) such that the solution of Equation (21) corresponding to k_0 and y'_0 satisfies $\|y'(k)\| \leq \epsilon\|y'_0\|$, $\forall k \geq k_0 + k'$. This together with the uniform stability of the system (21) implies that it is uniformly asymptotically stable by Definition 2.2. Thus, we can develop based on Lemma 2.1 that the system (21) is uniformly exponentially stable. Again using Definition 2.2, we know that there exist a finite constant $\alpha > 0$ and a constant $0 \leq \beta < 1$ such that $\|y'(k)\| \leq \alpha\beta^{k-k_0}\|y'_0\|$, $\forall k \geq k_0$ for any k_0 and y'_0. By inserting this into Equation (20), we can deduce

$$\begin{aligned} |y_1(k+1) - y_1(k)| &= |-\gamma(k)\xi\mathcal{L}(k)\Theta y'(k)| \\ &\leq \max_{p\in\mathcal{I}_r}\|\gamma_p\xi\mathcal{L}_p\Theta\|\|y'(k)\| \\ &\leq \alpha''\beta^{k-k_0}, \end{aligned} \tag{26}$$

where $\alpha'' = \alpha\max_{p\in\mathcal{I}_r}\|\gamma_p\xi\mathcal{L}_p\Theta\|\|y'_0\|$. Without any loss of generality, we take $k_0 = 0$ and then can follow the same lines used in the proof of Equation (11) to develop based on Equation (26) that $\lim_{k\to\infty}y_1(k) = c$ exponentially fast. That is to say, $\lim_{k\to\infty}\sigma_1 x_1(k) = c$ exponentially fast. Then for the same reason as the proof of Theorem 3.1, we can establish the exponential convergence of $\lim_{k\to\infty}\sigma_i x_i(k) = c, \forall i \in \mathcal{I}_n$.

Necessity: If $\mathcal{G}(k)$ is not jointly quasi-strongly connected through switching over $\mathbf{G}$, then we can find an integer $k_0 \in \mathbb{Z}_+$ such that for all $k' \in \mathbb{Z}_+$, the collection of directed graphs $\{\mathcal{G}(k_0), \mathcal{G}(k_0+1), \ldots, \mathcal{G}(k_0+k')\}$ is not jointly quasi-strongly connected. That is, $\bigcup_{k=k_0}^{k_0+k'}\mathcal{G}(k)$ is not quasi-strongly connected. This guarantees that the Laplacian matrix $\sum_{k=k_0}^{k_0+k'}\mathcal{L}(k)$ of $\bigcup_{k=k_0}^{k_0+k'}\mathcal{G}(k)$ associated with $\sum_{k=k_0}^{k_0+k'}\mathcal{A}(k)$ can be expressed in the form of (see Lin et al., 2005, Lemma 1; Moreau, 2005, Theorem 5; Moreau, 2005, Theorem 5; Lin et al., 2005, Lemma 1)

$$\sum_{k=k_0}^{k_0+k'} \mathcal{L}(k) = \begin{bmatrix} \mathcal{L}_{11}^{\sum}(k) & 0 & 0 \\ 0 & \mathcal{L}_{22}^{\sum}(k) & 0 \\ \mathcal{L}_{31}^{\sum}(k) & \mathcal{L}_{32}^{\sum}(k) & \mathcal{L}_{33}^{\sum}(k) \end{bmatrix}, \quad (27)$$

where $\mathcal{L}_{ij}^{\sum}(k) \in \mathbb{R}^{n_i \times n_j}$, $\forall i, j = 1, 2, 3$ and $n_1 + n_2 + n_3 = n$. Since $\mathcal{A}(k) \geq 0$, $\forall k$ and non-negative matrices of the block diagonal form are closed under matrix addition, we are not difficult to obtain from Equation (27) that $\mathcal{L}(k)$ satisfies

$$\mathcal{L}(k) = \begin{bmatrix} \mathcal{L}_{11}(k) & 0 & 0 \\ 0 & \mathcal{L}_{22}(k) & 0 \\ \mathcal{L}_{31}(k) & \mathcal{L}_{32}(k) & \mathcal{L}_{33}(k) \end{bmatrix}, \quad \forall k_0 \leq k \leq k_0 + k', \quad (28)$$

where $\mathcal{L}_{ij}(k) \in \mathbb{R}^{n_i \times n_j}$, $\forall i, j = 1, 2, 3$. As a consequence of Equation (28), we have

$$\Phi(k) = \begin{bmatrix} \Phi_{11}(k) & 0 & 0 \\ 0 & \Phi_{22}(k) & 0 \\ \Phi_{31}(k) & \Phi_{32}(k) & \Phi_{33}(k) \end{bmatrix}, \quad \forall k_0 \leq k \leq k_0 + k', \quad (29)$$

where for $\forall i, j = 1, 2, 3$, $\Phi_{ij}(k) = I - \gamma(k)\mathcal{L}_{ij}(k)$ if $j = i$ and $\Phi_{ij}(k) = -\gamma(k)\mathcal{L}_{ij}(k)$ if $j \neq i$. Given two constants $c_1 \neq c_2$, we consider the solution of the system (15) corresponding to the following initial state:

$$x(k_0) = D\begin{bmatrix} 1_{n_1}c_1 \\ 1_{n_2}c_2 \\ 1_{n_3}c_3 \end{bmatrix},$$

where we can consider $c_3 = 0$ without any loss of generality. Note that $\Phi(k)$ is a stochastic matrix by Lemma 4.1. We have $\Phi_{11}(k)1_{n_1}c_1 = 1_{n_1}c_1$ and $\Phi_{22}(k)1_{n_2}c_2 = 1_{n_2}c_2$. Hence, we can combine Equation (29) with Equation (15) to arrive at

$$\begin{aligned} x(k_0 + k' + 1) &= \prod_{l=k_0}^{k_0+k'} D\Phi(l)Dx(k_0) \\ &= D\left[\prod_{l=k_0}^{k_0+k'} \Phi(l)\right] Dx(k_0) \\ &= D\prod_{l=k_0}^{k_0+k'} \Phi(l)\begin{bmatrix} 1_{n_1}c_1 \\ 1_{n_2}c_2 \\ 1_{n_3}c_3 \end{bmatrix} \\ &= D\begin{bmatrix} 1_{n_1}c_1 \\ 1_{n_2}c_2 \\ (*) \end{bmatrix}, \end{aligned} \quad (30)$$

where (*) is a certain term that does not affect the conclusion. Since k' can be arbitrarily large, we can obtain from Equation (30) that, for any constant c,

$$\|x(k) - D1_n c\| \geq \max\{|c_1 - c|, |c_2 - c|\}, \quad \forall k \geq k_0,$$

which, due to $c_1 \neq c_2$, contradicts with $\lim_{k\to\infty} x(k) = D1_n c$. Therefore, on the contrary, we can conclude that if the system (15) achieves signed consensus, then $\mathcal{G}(k)$ is jointly quasi-strongly connected through switching over $\mathbf{G}$. ■

Remark 4.2: We benefit from the equivalence between uniform asymptotic stability and uniform exponential stability of linear time-varying systems to derive the sufficiency proof for exponential convergence of signed consensus. In particular, this benefit holds for the case when the network topology is fixed as performed in the proof of Theorem 3.1. The necessity of Theorem 4.1 is obtained with proof by contradiction, the motivation behind which incorporates the ideas from the existing consensus results for multi-agent systems in, for example, Moreau (2005) and Lin et al. (2005). By the proof of Theorem 4.1, we provide a new analysis way for the exponential convergence of multi-agent consensus problems in the presence of switching topologies.

4.3 Extensions to bipartite consensus

Next, we make discussions on possible extensions of the development of Theorem 4.1 to the exponential convergence for bipartite consensus on networks associated with signed graphs in the presence of switching topologies. Let $\mathcal{A}^s(k) = [a_{ij}^s(k)] \in \mathbb{R}^{n\times n}$, where $a_{ij}^s(k) = \sigma_i\sigma_j a_{ij}(k)$. Clearly, $\mathcal{G}^s(k) = (\mathcal{V}, \mathcal{E}(k), \mathcal{A}^s(k))$ is a time-varying signed directed graph, where $a_{ij}^s(k) \neq 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}(k)$ and $a_{ij}^s(k) = 0$ otherwise. Due to $\mathcal{G}(k) \in \mathbf{G}$, we can accordingly obtain $\mathcal{G}^s(k) \in \{\mathcal{G}_1^s, \mathcal{G}_2^s, \ldots, \mathcal{G}_r^s\} \triangleq \mathbf{G^s}$, $\forall k \in \mathbb{Z}_+$, where $\mathcal{G}_p^s = (\mathcal{V}, \mathcal{E}_p, \mathcal{A}_p^s)$ with $\mathcal{A}_p^s = [a_{p,ij}^s]$ and $a_{p,ij}^s = \sigma_i\sigma_j a_{p,ij}$, $\forall p \in \mathcal{I}_r$. It can be easily validated that

$$\operatorname{sign}\left(a_{p,ij}^s\right)\operatorname{sign}\left(a_{q,ij}^s\right) \geq 0, \quad \forall p, q \in \mathcal{I}_r, \forall i, j \in \mathcal{I}_n \quad (31)$$

due to

$$\begin{aligned}\text{sign}\left(a^s_{p,ij}\right)\text{sign}\left(a^s_{q,ij}\right) &= \text{sign}\left(a^s_{p,ij}a^s_{q,ij}\right)\\ &= \text{sign}\left(\sigma_i^2\sigma_j^2 a_{p,ij}a_{q,ij}\right)\\ &= \text{sign}\left(a_{p,ij}a_{q,ij}\right)\\ &= \text{sign}\left(a_{p,ij}\right)\text{sign}\left(a_{q,ij}\right).\end{aligned}$$

That is, we can perform the same gauge transformation with D to simultaneously render all $\mathcal{A}^s_p$, $\forall p \in \mathcal{I}_r$ (thus, all $\mathcal{A}^s(k)$, $\forall k \in \mathbb{Z}_+$) non-negative. Obviously, $\mathcal{G}^s(k)$ is a time-varying structurally balanced signed directed graph for all the time.

In addition, we can rewrite the protocol (14) as

$$u_i(k) = \gamma(k)\sum_{j\in\mathcal{N}_i(k)} a^s_{ij}(k)\left[x_j(k) - \text{sign}\left(a^s_{ij}(k)\right)x_i(k)\right], \quad \forall i \in \mathcal{I}_n.$$

If we employ this protocol under signed directed networks, we can describe the network process of agents given by Equation (1) as

$$x(k+1) = [I - \gamma(k)\mathcal{L}^s(k)]x(k), \tag{32}$$

where $\mathcal{L}^s(k)$ is the Laplacian of the time-varying signed directed graph $\mathcal{G}^s(k)$. For such a network process subject to switching topologies, we can consider Theorem 4.1 to directly establish the following corollary for the exponential convergence of its bipartite consensus.

Corollary 4.1: *Let $\mathcal{G}^s(k) \in \mathbf{G}^s$ fulfil (31) and $\gamma(k) \in \Gamma$ satisfy*

$$\gamma_p \max_{i\in\mathcal{I}_n}\sum_{j\in\mathcal{N}_i}\left|a^s_{p,ij}\right| < 1, \quad \forall p \in \mathcal{I}_r.$$

The system (32) achieves bipartite consensus exponentially fast if and only if the time-varying structurally balanced signed directed graph $\mathcal{G}^s(k)$ is jointly quasi-strongly connected through switching over $\mathbf{G}^s$.

Remark 4.3: In Corollary 4.1, we provide a necessary and sufficient guarantee to the exponential convergence of bipartite consensus for discrete-time multi-agent systems subject to antagonistic interactions and switching topologies. It considers directed networks and only needs a joint quasi-strong connectivity condition on the agents' switching network topologies, rather than imposing the requirement upon strong connectivity of all possible network topologies in Altafini (2013). However, as was argued in Altafini (2013), the assumption given by Equation (31) cannot be relaxed when seeking bipartite consensus under switching topologies.

5. Simulation results

To illustrate signed consensus performances, we give numerical simulations for fixed and switching networks, respectively. To this end, we consider a class of eight-agent systems with initial states given by $x(0) = [7, -10, -3, 2, -8, 15, 10, 12]^{\mathrm{T}}$. Without loss of generality, we will specify signs of the consensus states of agents such that $D = \text{diag}\{1, -1, -1, -1, 1, 1, -1, -1\}$. It is easy to see that the agents v_1, v_5, and v_6 will achieve consensus, and the other agents v_2, v_3, v_4, v_7, and v_8 will reach agreement. The consensus values for two groups are the same in modulus but opposite in sign. In addition, we will employ edge weights of the directed graphs of interests all equal to 1 for simplicity.

First, we perform simulations to illustrate signed consensus in fixed networks. The interactions between agents are represented in Figure 1. In this case, we apply the protocol (3) by choosing $\gamma = 0.45$ which satisfies the condition (5). Figure 2 plots states of agents with evolution along the time axis. It is shown that the agents are divided into two groups: $\{v_1, v_5, v_6\}$ and $\{v_2, v_3, v_4, v_7, v_8\}$, and each group of agents is enabled to achieve consensus. It is also shown that the final consensus values are the same in modulus for both groups but not in sign. This implies that the system (4) achieves signed consensus. According to Definition 2.1, we can further obtain $c = -5$. Figure 3

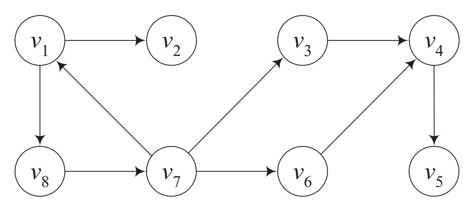

Figure 1. An example of a quasi-strongly connected directed graph for the fixed network.

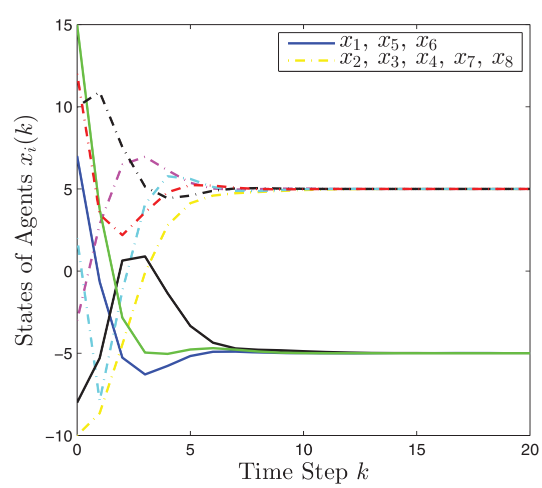

Figure 2. State evolution of the fixed network (4) with respect to time.

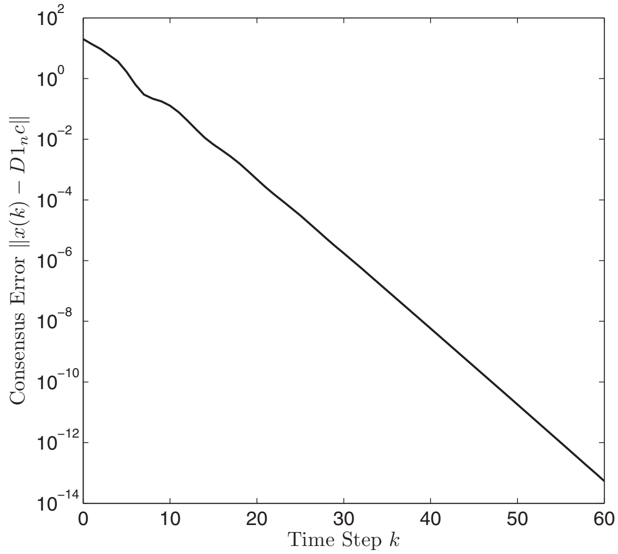

Figure 3. Consensus error evolution of the fixed network (4) with respect to time.

plots time evolution of the l_∞ norm of consensus error $x(k) - D1_n c$. It is clearly shown that the consensus error decreases to zero exponentially fast with increasing time steps.

Next, we perform simulations to illustrate signed consensus in switching networks. Let the switching graph set **G** consist of four directed graphs of Figure 4. Clearly, none of these directed graphs are quasi-strongly connected, but $\mathcal{G}_1 \bigcup \mathcal{G}_2 \bigcup \mathcal{G}_3 \bigcup \mathcal{G}_4$ is quasi-strongly connected. We thus adopt a four-state switching machine over **G** of Figure 5, which starts with $\mathcal{G}_1$ for $k = 0$ and switches from one state (e.g. $\mathcal{G}_3$) to the next (e.g. $\mathcal{G}_4$) for every two time steps. For $\mathcal{G}(k) \in \mathbf{G}$ under the action of the switching machine of

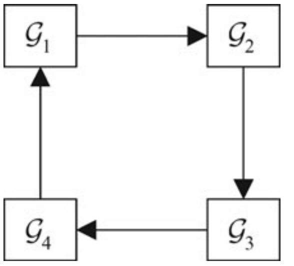

Figure 5. An example of switching machine over **G**.

Figure 5, we can easily validate that $\mathcal{G}(k)$ is jointly quasi-strongly connected through switching over **G**. To apply the protocol (14), we use $\gamma(k) \in \Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where

$$\gamma_1 = 0.35, \quad \gamma_2 = 0.9, \quad \gamma_3 = 0.7, \quad \gamma_4 = 0.8.$$

This gain selection can guarantee the condition (16). In Figure 6, we plot time evolution of the states of agents. We can observe from this figure that the system (15) achieves signed consensus. By Definition 2.1, we can further obtain $c = -4.3563$. In addition, Figure 7 plots the evolution of consensus error norm $\|x(k) - D1_n c\|$ with respect to time step. This figure further illustrates exponential convergence of signed consensus in switching networks. Due to the switching of network topologies over **G**, the curve of consensus error evolution is not smooth.

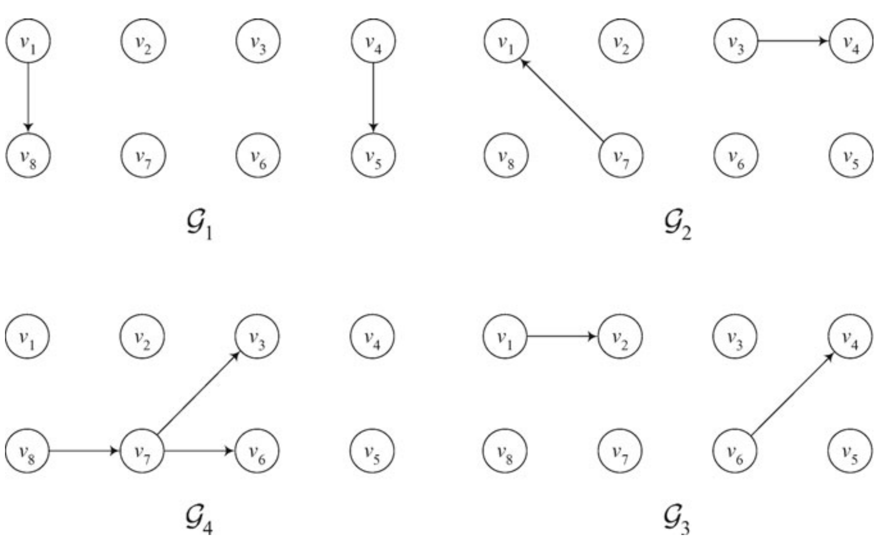

Figure 4. Four directed graphs forming the set **G** of graphs for all possible interaction topologies of agents, none of which is quasi-strongly connected.

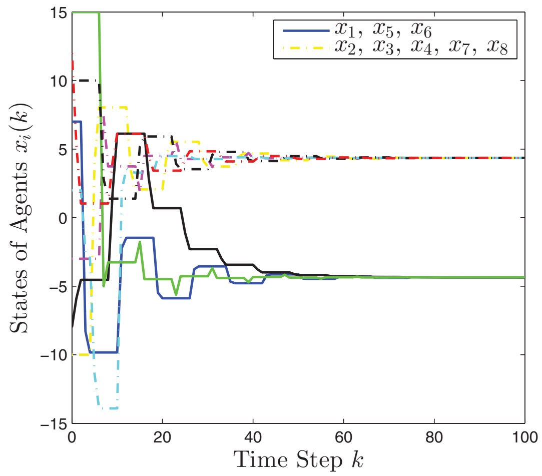

Figure 6. State evolution of the switching network (15) with respect to time.

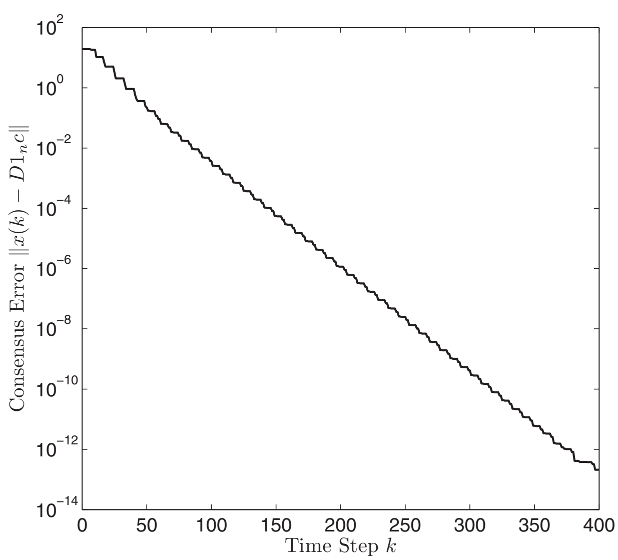

Figure 7. Consensus error evolution of the switching network (15) with respect to time.

5.1 Discussions

From Figure 1, we know that the considered fixed network is quasi-strongly connected but not strongly connected. Figures 4 and 5 illustrate that the considered switching networks are jointly quasi-strongly connected but not jointly strongly connected. Moreover, Figures 3 and 7 imply that signed consensus is achieved exponentially fast with increasing time steps in fixed and switching networks. These observations demonstrate that (joint) quasi-strong connectivity is sufficient for exponential convergence of signed consensus. Thus, the requirement of (joint) strong connectivity is relaxed regardless of fixed or switching networks (see e.g. Altafini, 2013; Schenato & Fiorentin, 2011). This coincides with the statements of Theorems 3.1 and 4.1 which supply exponential convergence of signed consensus in fixed and switching networks with the fundamental necessary and sufficient guarantee given quasi-strong connectivity and joint quasi-strong connectivity, respectively.

6. Conclusions

In this paper, we have discussed a class of signed consensus problems on multi-agent networks in the presence of both fixed and switching topologies. We have studied under what kind of topology condition the exponential convergence of signed consensus can be accomplished with a necessary and sufficient guarantee. To this end, we have further developed the graphic approach to consensus by incorporating the stability theory of linear systems. This can provide a new way to the exponential consensus analysis of multi-agent networks, which relaxes the strong topology requirement of the Lyapunov-based consensus analysis approach (see e.g. Schenato & Fiorentin, 2011). In particular, the established signed consensus results can be considered to take the discrete-time counterpart bipartite consensus results of Altafini (2013) as special cases, regardless of fixed or switching networks. Numerical simulations have been performed to illustrate the effectiveness of the proposed protocols for signed consensus.

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Disclosure statement

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