with $D_c > 0$ and $K_c > 1$ (to satisfy the dc loop gain condition), the closed-loop system is asymptotically stable.

The closed-loop transfer function can be calculated to be

$$\mathfrak{G}(s) = \frac{\sinh s}{s\cosh s - \frac{\sinh s}{s^2+D_c s+K_c}}.$$

Consider the points $s_n := \pi/2i + n\pi i$ with $n \in \mathbb{Z}$ (which are the open-loop poles). We have $\cosh s_n = 0$ and $\sinh s_n = \pm 1$ so that $\mathfrak{G}(s_n) = -(s_n^2 + D_c s_n + K_c)$. It follows that $|\mathfrak{G}(s_n)| \to \infty$ as $n \to \infty$ so that $\mathfrak{G} \notin L^\infty(i\mathbb{R})$ and hence the closed-loop system is not input-output stable. The eigenvalues λ_n of the closed-loop state operator coincide with the poles of the closed-loop transfer function, i.e., they are the solutions of the equation

$$s\cosh s - \frac{\sinh s}{s^2 + D_c s + K_c} = 0.$$

This equation can be rewritten as

$$-\mathrm{e}^{2s} = \frac{s^3 + D_c s^2 + K_c s + 1}{s^3 + D_c s^2 + K_c s - 1}.$$

By keeping only the asymptotically largest term on the right hand-side, we obtain the asymptotic equation $-\mathrm{e}^{2s} = 1$, which has as solutions the eigenvalues of the open-loop system $s_n := \pi/2i + n\pi i$. It can be easily shown (e.g., using the method described in [15] or in a more elementary way by an application of Rouché's theorem) that $\lambda_n - s_n \to 0$, so that $\mathrm{Re}\,\lambda_n \to 0$ as $n \to \infty$. Hence the closed-loop system is not exponentially stable.

V. Conclusion

We have shown that—at least for second order systems with force control and position measurement and for finite-dimensional controllers of the same type- negative imaginary stability theory carries over to the infinite-dimensional case if stability is understood as asymptotic stability, but not if it is understood as exponential stability or as input-output stability.

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References

[1] A. Lanzon and I. R. Petersen, "A modified positive-real type stability condition," in *Proc. ECC*, 2007, pp. 3912–3918.

[2] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 1042–1046, Apr. 2008.

[3] I. R. Petersen and A. Lanzon, "Feedback control of negative imaginary systems," *IEEE Control Syst. Mag*, vol. 30, no. 5, pp. 54–72, 2010.

[4] B. Jacob, K. Morris, and C. Trunk, "Minimum-phase infinite-dimensional second-order systems," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1654–1665, Sep. 2007.

[5] W. T. Thomson, *Theory of Vibrations With Applications*, 4th ed. Bath, U.K.: Nelson Thornes, 2003.

[6] J. Fanson and T. Caughey, "Positive position feedback control for large space structures," *AIAA J*, vol. 28, no. 4, pp. 717–724, 1990.

[7] J. Xiong, A. Lanzon, and I. R. Petersen, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2342–2347, Oct. 2010.

[8] J. Xiong, I. R. Petersen, and A. Lanzon, "On lossless negative imaginary systems," in *Proc. Asian Control Conf.*, 2009, pp. 824–829.

[9] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems, ser. Monographs in Mathematics*. Basel, Switzerland: Birkhäuser Verlag, 2001, vol. 96.

[10] Z.-H Luo, B.-Z Guo, and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems With Applications, ser. Communications and Control Engineering Series*. London, U.K.: Springer-Verlag, 1999.

[11] O. Staffans, *Well-Posed Linear Systems, ser. Encyclopedia of Mathematics and its Applications*. Cambridge, U.K.: Cambridge Univ. Press, 2005, vol. 103.

[12] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups, ser. Birkhäuser Advanced Texts: Basler Lehrbücher. (Birkhäuser Advanced Texts: Basel Textbooks)*. Basel, Switzerland: Birkhäuser Verlag, 2009.

[13] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory, ser. Texts in Applied Mathematics*. New York: Springer-Verlag, 1995, vol. 21.

[14] R. Curtain and K. Morris, "Transfer functions of distributed parameter systems: A tutorial," *Automatica J. IFAC*, vol. 45, no. 5, pp. 1101–1116, 2009.

[15] P. Grabowski, "Well-posedness and stability analysis of hybrid feedback systems using shkalikov's theory," *Opuscula Math*, vol. 26, no. 1, pp. 45–97, 2006.

Consensus in Multi-Agent Systems With Coupling Delays and Switching Topology

Ulrich Münz, Antonis Papachristodoulou, and Frank Allgöwer

***Abstract*—The robustness of consensus in single integrator multi-agent systems (MAS) to coupling delays and switching topologies is investigated. It is shown that consensus is reached for arbitrarily large constant, time-varying, or distributed delays if consensus is reached without delays. This delay robustness holds under the weakest possible connectivity assumptions on the underlying graph, i.e., as long as the graph is uniformly quasi-strongly connected and switches with a dwell-time. The proof is based on a contraction argument and allows to remove technical assumptions that were used in previous publications. Moreover, the result also applies to non-scalar single integrators, an important extension toward consensus and rendezvous in higher dimensions.**

***Index Terms*—Consensus, delays, multi-agent system, rendezvous, switching topology.**

I. Introduction

Multi-agent systems (MAS) and consensus problems have a broad range of applications, e.g. multi-vehicle coordination, power network synchronization, and Internet congestion control, [1]–[4] to name a few. The most common models for MAS use single integrator agent dynamics and describe their interconnection using graphs. Most publications consider ideal interconnection networks without coupling delays, [1], [5]. Yet, in more realistic examples, the exchange of information

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on the network introduces delays between the agents, e.g., transmission delays in power networks or communication delays in multi-vehicle coordination or Internet congestion control [6]. As these delays are usually disregarded in the analysis, the question arises whether these algorithms also achieve consensus in the presence of delays, i.e., if consensus is robust to delays.

In this work, we prove that consensus in single integrator MAS is robust to arbitrarily large delays, i.e., consensus is reached under the same conditions as in the undelayed case. For this, we consider the most general case with non-scalar agent dynamics, directed, switching network topologies, heterogeneous delays, and nonlinear coupling between the agents. Moreover, our result applies to constant, time-varying, and distributed delays.

Previously, the robustness of consensus in discrete-time single-integrator MAS to arbitrarily large delays has been shown in [7], [8]. In order to derive this result, the state space was extended by delayed states. Then, results on ergodic matrices were used in order to prove consensus. Very similar results were obtained for continuous-time MAS with discrete-time communication, see [9]. These results rely on the finite dimensional nature of the MAS and are not applicable to infinite dimensional systems like continuous-time MAS with *continuous-time* communication, that we consider here. Continous-time communication with delays allows to model, for example, continuously time-varying reaction delays in car-following models or distributed transmission delays in power networks. Hence, continuous-time communication delays are an important network property that cannot be dealt with using discrete-time analysis. On the other hand, continuous-time communication delays also cover discrete-time delays as special case.

In continuous-time, delay robustness in MAS has been investigated in the frequency domain, e.g., using small gain arguments [10], [11] or the generalized Nyquist criterion in [12]–[15]. However, these results are restricted to linear coupling functions and fixed topologies. The results presented here hold also for nonlinear coupling functions and switching topologies.

In the time-domain, delay robustness of MAS has been studied using small-μ analysis [16], integral quadratic contraints [17], [18], or techniques for partial differential equations [19]. Yet, all these results consider only MAS with fixed topologies in contrast to the results for switching topologies in this technical note.

Most publications on consensus in MAS with delays rely on sums of Lyapunov-Krasovskii functionals or Lyapunov-Razumikhin functions. Sums of Lyapunov-Krasovskii functionals consist of arbitrarily large but finite sums of positive definite terms, each of which is related to one agent or one interconnection link. This approach has been used to investigate single integrator MAS [20]–[23], MAS consisting of passive agents or nonlinear agents with relative degree one [24]–[26], and MAS with relative degree two [27], [28]. The main disadvantage of this method is that the underlying graph has to be undirected or at least weight-balanced, i.e., the sum of the weights of all incoming and outgoing links of each node are the same. More general results for MAS with single integrator agent dynamics are obtained using Lyapunov-Razumikhin functions, e.g. [29], because they only require directed, uniformly quasi-strongly connected graphs [30]. In contrast to our previous publications [29], [31]–[34], we use here a completely different proof technique that allows to remove some technical assumptions required for the proofs based on Lyapunov-Razumikhin functions, see [29]. The main difference is that here we use a contraction argument extending previous results for undelayed MAS in [30] whereas we used a Barbalat-like argument in [29] based on ideas from [35]. Moreover, the new technique holds also for the more general case of MAS consisting of non-scalar integrators whereas the proof techniques in the previously mentioned publications work only with scalar MAS. Last but not least, we show that these results hold also for piecewise-continuous time-varying delays. A more detailed literature overview on MAS with delays can be found in [36].

The structure of the technical note is the following: The problem statement is given in Section II. The main result is provided and proven in Section III and the technical note is summarized in Section IV.

II. Problem Statement

We consider single integrator MAS consisting of N agents with dynamics

$$\dot{x}_i(t) = -\sum_{j=1}^{N} a_{ij}^{\sigma(t)} k_{ij}\left(x_i(t) - \mathbb{T}_{ij}(x_{j,t})\right) \tag{1}$$

for $i \in \mathcal{N} = \{1,\ldots,N\}$, where $x_i(t) \in \mathbb{R}^n$ is the state of agent i. The elements on the right-hand side of (1) are explained in the following paragraphs.

The agents of the MAS are interconneted on a network in order to achieve consensus, i.e.,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \text{for all } i,j \in \mathcal{N}. \tag{2}$$

We consider here a very general network description that allows for directed graphs with switching topology and transmission delays. Switching topologies are particularly interesting because they can be used to model important network properties like packet-loss or limited communication range in wireless networks. Transmission delays are important to describe packet-delays, access delays, or propagation delays in digital and analogue communication networks.

In order to describe the delays, we denote the space of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$ by $C([a,b],\mathbb{R}^n)$. For $\mathcal{T} > 0$, we define $C_n = C([-\mathcal{T},0],\mathbb{R}^n)$. A segment $x_t \in C_n$ is defined such that $x_t(\eta) = x(t+\eta)$, $\eta \in [-\mathcal{T},0]$. The *delays* are modeled by the delay operator $\mathbb{T}_{ij}(x_{j,t})$ that depends on the segment $x_{j,t} \in \mathcal{C}_n$, i.e., the states $x_j(t+\eta)$ for $\eta \in [-\mathcal{T},0]$ for some $\mathcal{T} > 0$. This delay operator represents the three most common delay models: constant delays $\mathbb{T}_{ij}(x_{j,t}) = x_j(t-\tau_{ij})$, $\tau_{ij} \in [0,\mathcal{T}]$, time-varying delays $\mathbb{T}_{ij}(x_{j,t}) = x_j(t-\tau_{ij}(t))$, $\tau_{ij} : \mathbb{R} \to [0,\mathcal{T}]$, and distributed delays $\mathbb{T}_{ij}(x_{j,t}) = \int_0^{\mathcal{T}} \phi_{ij}(\eta) x_j(t-\eta) d\eta$, with delay kernel $\phi_{ij} : \mathbb{R} \to \mathbb{R}$ that satisfies $\phi_{ij}(\eta) \geq 0$ for all $\eta \in [0,\mathcal{T}]$ and $\int_0^{\mathcal{T}} \phi_{ij}(\eta) d\eta = 1$. In all cases, we assume that the delays are uniformly bounded, i.e., the delay bound $\mathcal{T}$ is finite for all i,j. The time-varying delays can be piecewise continuous and non-chattering, i.e., there is only a finite number of discontinuities in any finite interval. Constant and time-varying delays are often used to model propagation delays or packet-delays. Distributed delays are suited for packet-delays, underlining the stochastic nature of the delays [31]. The notation using a delay operator $\mathbb{T}_{ij}$ allows to consider all three delay models at the same time.

The *switching topology* of the network is modeled using switching graphs. A switching graph $\mathcal{G}_{\sigma(t)} : \mathbb{R} \to \mathfrak{G}$ is defined on a finite set $\mathfrak{G} = \{\mathcal{G}_p\}$, $p \in \mathcal{P} = \{1,\ldots,P\}$, of P directed graphs $\mathcal{G}_p = (\mathcal{V},\mathcal{E}_p)$ with identical node set $\mathcal{V} = \{v_1,\ldots,v_N\}$, each node corresponding to one agent, but different edge set $\mathcal{E}_p \subseteq \mathcal{V}\times\mathcal{V}$ and corresponding adjacency matrix $A^p = [a_{ij}^p] \in \mathbb{R}^{N\times N}$, i.e., $a_{ij}^p > 0$ if $(v_j,v_i) \in \mathcal{E}_p$ and $a_{ij}^p = 0$ otherwise, where (v_j,v_i) represents a directed edge from vertex j to vertex i. If $(v_j,v_i) \in \mathcal{E}_p$, then j is parent of i. We define the parent set $\mathcal{N}_i^p = \{j : (v_j,v_i) \in \mathcal{E}_p\}$ of node i and the largest and smallest entry of all adjacency matrices $\overline{a} = \max_{i,j,p} a_{ij}^p > 0$ and $\underline{a} = \min_{i,j,p: a_{ij}^p \neq 0} a_{ij}^p > 0$. We assume here that the graphs $\mathcal{G}_p$ do not have self-loops, i.e., $a_{ii}^p = 0$ for all i,p. Note that all graphs $\mathcal{G}_p$ are directed, i.e., $a_{ij}^p = a_{ji}^p$ is not required. The switching between the graphs is modeled by a function $\sigma : \mathbb{R} \to \mathcal{P}$ that is piecewise constant from the right. We denote the time instances where σ switches

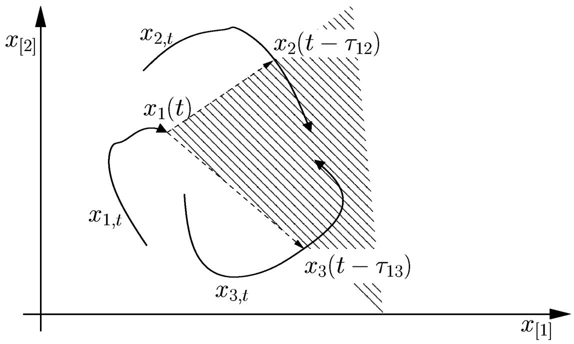

Fig. 1. Exemplary illustration of the state trajectories $x_{i,t} \in \mathcal{C}_2$, $i = 1, 2, 3$ of three agents. The axes $x_{[1]}$ and $x_{[2]}$ denote the first and second component of the state vectors $x_i(t) \in \mathbb{R}^2$. The dashed arrows indicate $x_2(t-\tau_{12}) - x_1(t)$ and $x_3(t-\tau_{13}) - x_1(t)$ located at $x_1(t)$, i.e., agent 1 can move into the shaded area in the next instant if it is connected to both agents 2 and 3.

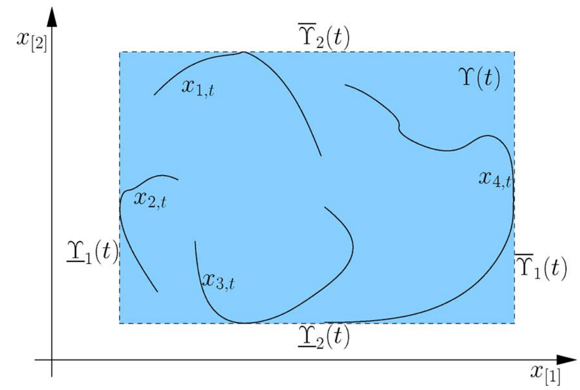

Fig. 2. Exemplary illustration of hyper-rectangle $\Upsilon(t) \subset \mathbb{R}^2$ and its surfaces $\overline{\Upsilon}_k(t)$ and $\underline{\Upsilon}_k(t)$, $k = 1, 2$. The hyper-rectangle contains the state trajectories $x_{i,t} \in \mathcal{C}_2$, $i = 1, 2, 3, 4$, of four agents. The axes $x_{[1]}$ and $x_{[2]}$ denote the first and second component of the state vectors $x_i(t) \in \mathbb{R}^2$.

$t_\varsigma > t_{\varsigma-1}$, $\varsigma = 0, 1, 2, \ldots$ with $t_0 = 0$. We assume there are infinitely many such switching times because otherwise we could just analyze the last active graph. Moreover, any two consecutive switching instants are separated by a dwell-time h_{DW}, i.e., $t_\varsigma - t_{\varsigma-1} \geq h_{DW}$. This guarantees that the switching graph is non-chattering and zeno behavior cannot occur.

We assume that the graph is uniformly quasi-strongly connected which is defined next. For this definition, we use the following notation for a union graph $\mathcal{G}_{[t_1,t_2]} = (\mathcal{V}, \bigcup_{t\in[t_1,t_2]} \mathcal{E}_{\sigma(t)})$ over the interval $[t_1, t_2]$. A union graph consists of all vertices in $\mathcal{V}$ and all edges that appear at any time $t \in [t_1, t_2]$.

Definition 1 (Uniformly Quasi-Strongly Connected [30]): A switching graph $\mathcal{G}_{\sigma(t)}$ is *uniformly quasi-strongly connected* if there exists a $\mathfrak{T} > 0$ such that, for all $t \geq 0$, the union graph $\mathcal{G}_{[t,t+\mathfrak{T}]}$ is quasi-strongly connected, i.e., $\mathcal{G}_{[t,t+\mathfrak{T}]}$ contains a spanning tree [37].

Uniform quasi-strong connectivity is a very weak assumption on the connectivity. It may hold even if none of the graphs $\mathcal{G}_p \in \mathfrak{G}$ is quasi-strongly connected, even if every graph $\mathcal{G}_p$ has just one edge. Uniform quasi-strong connectivity is in fact the weakest assumption on the graph connectivity such that consensus is guaranteed for arbitrary initial conditions. Counter examples are provided in [30], [38] which show that, in general, consensus is not achieved in single integrator MAS without delays if the graph is not uniformly quasi-strongly connected. In this work, we are interested in delay robustness which includes the undelayed case as a special case. Hence, if the graph is not uniformly quasi-stongly connected, we can construct counter examples with zero delays as in [30], [38]. In conclusion, uniform quasi-strong connectivity is also the weakest connectivity assumption for our delay robustness analysis.

The *coupling functions* $k_{ij} : \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear, Lipschitz continuous, and depend on the delayed state difference $x_i(t) - \mathbb{T}_{ij}(x_{j,t})$ between the agents i and j. In other words, each agent updates its state by comparing its own current state $x_i(t)$ to the delayed state $\mathbb{T}_{ij}(x_{j,t})$ of its parents, see Fig. 1. The coupling function satisfies the following assumption:

Assumption 1: The coupling functions $k_{ij} : \mathbb{R}^n \to \mathbb{R}^n$ are such that $k_{ij}(z) = \overline{k_{ij}}(\|z\|)(z/\|z\|)$ with nonlinear, continuous gain $\overline{k_{ij}} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that satisfies $k_{ij}(0) = \overline{k_{ij}}(0) = 0$ and

$$\overline{\mathcal{K}}\eta \geq \overline{k_{ij}}(\eta) \geq \underline{\mathcal{K}}\eta, \quad \forall \eta \in \mathbb{R}_0^+ \tag{3}$$

for arbitrary $\overline{\mathcal{K}} \geq \underline{\mathcal{K}} > 0$, i.e., $\overline{k_{ij}}(\eta)$ is in the sector $[\underline{\mathcal{K}}\eta, \overline{\mathcal{K}}\eta]$.

The sector condition (3) considerably simplifies the proof. Note however that the main result holds also for arbitrary, strictly passive $\overline{k_{ij}}$, i.e., $\overline{k_{ij}}(\|z\|) > 0$ for all $\|z\| \neq 0$, see [36].

Assumption 1 implies the following: If agent i does not have a parent at some time t, i.e., it does not receive any information, then the state of this agent remains constant $\dot{x}_i(t) = 0$. On the other hand, if agent i has at least one parent, then its state changes toward the states of its parents, or, more precisely, toward some weighted average of the delayed states of its parents, see Fig. 1. The terms $z/\|z\|$ and $\overline{k_{ij}}(\|z\|) \geq 0$ guarantee that $k_{ij}(x_i(t) - \mathbb{T}_{ij}(x_{j,t}))$ is heading in the same direction as $x_i(t) - \mathbb{T}_{ij}(x_{j,t})$. This way we can extend our previous results, e.g. [29], towards non-scalar agent dynamics. This assumption on the coupling function is quite intuitive: in order to achieve consensus, each agent changes its state such that it is getting closer to its parents. Moreover, a counter example has been given in [30] showing that Assumption 1 is indeed a necessary assumption in the undelayed case.

For completeness, we define the stack vector of states as $x(t) = [x_1^T, \ldots, x_N^T]^T$. The initial condition of the MAS (1) is $\varphi \in \mathcal{C}_{Nn} = \mathcal{C}([-\mathcal{T}, 0], \mathbb{R}^{Nn})$.

III. Consensus on Fixed and Switching Topologies

In this section, we present our main result. It states that consensus (2) is guaranteed for MAS (1) for arbitrary large but bounded delays if the coupling functions k_{ij} satisfy Assumption 1. The proof is based on a contraction argument that was used in [39] for linear MAS and in [30] for nonlinear MAS without delays. We extend this result to nonlinear MAS with heterogeneous delays. The fundamental idea of the proof is to define a time-varying convex set $\Upsilon : \mathbb{R} \to \mathbb{R}^n$, such that $\Upsilon(t)$ contains all state trajectory pieces $x_{i,t}$, i.e., all $x_i(t+\eta)$, $\eta \in [-\mathcal{T}, 0]$. Then, we show by an iterative procedure that this set contracts to a single point as $t \to \infty$.

The set $\Upsilon(t) \subset \mathbb{R}^n$ is defined as the smallest hyper-rectangle, such that the surfaces of Υ are aligned with the coordinate axes and $\Upsilon(t)$ contains all state trajectory pieces $x_{i,t}$, see Fig. 2 for an exemplary illustration. The boundary of $\Upsilon(t)$ is denoted $\partial\Upsilon(t)$. The $2n$ surfaces of $\Upsilon(t)$ are $\overline{\Upsilon}_k(t)$ and $\underline{\Upsilon}_k(t)$, $k = 1, \ldots, n$, such that $\overline{\Upsilon}_k(t) \geq x_{i[k]}(t+\eta) \geq \underline{\Upsilon}_k(t)$ for all $i \in \mathcal{N}$ and for all $\eta \in [-\mathcal{T}, 0]$, where $x_{i[k]}(t+\eta)$ denotes the kth element of $x_i(t+\eta)$. Moreover, we define the length of the hyper-rectangle in dimension k as $\widehat{\Upsilon}_k(t) = \overline{\Upsilon}_k(t) - \underline{\Upsilon}_k(t)$. Since $\Upsilon(t)$ is the smallest hyper-rectangle, there is always at least one $i \in \mathcal{N}$ such that $x_{i[k]}(t+\eta) = \overline{\Upsilon}_k(t)$ for some $\eta \in [-\mathcal{T}, 0]$ and similarly at least one $j \in \mathcal{N}$ such that $x_{j[k]}(t+\eta) = \underline{\Upsilon}_k(t)$ for some $\eta \in [-\mathcal{T}, 0]$, see Fig. 2. Clearly, the hyper-rectangle $\Upsilon(t)$ changes over time.

In the following lemma, we show that the set $\Upsilon(t)$ is non-increasing, i.e., $\Upsilon(t_2) \subseteq \Upsilon(t_1)$ for all $t_2 \geq t_1$. This implies that $\Upsilon(0)$ is positively invariant.

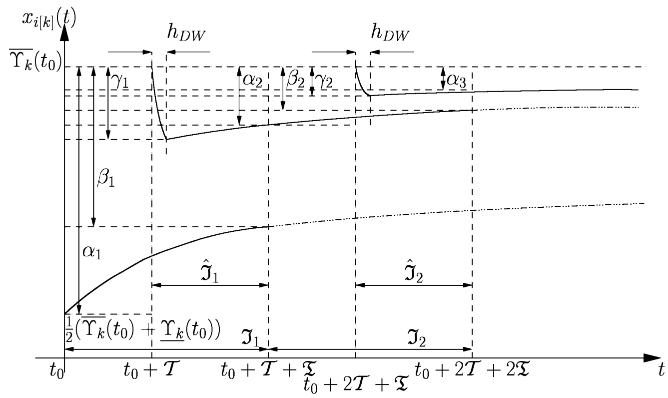

Fig. 3. Schematic illustration of contraction argument: Agents $i \in \mathcal{I}_1$ that have distance α_1 from the upper surface $\overline{\Upsilon_k}(t_0)$ at time t_0 converge at most exponentially toward $\overline{\Upsilon_k}(t_0)$ and, thus, have at least distance β_1 during the interval $\mathfrak{I}_1$, see Lemma 2. Assuming that a root of the union graph $\mathcal{G}_{\hat{\mathfrak{I}}_1}$ satisfies $i_{R_1} \in \mathcal{I}_1$, i.e., it has distance α_1 from $\overline{\Upsilon_k}(t_0)$ at time t_0, we know that during the interval $\hat{\mathfrak{I}}_1$ there is at least one agent $I \in \mathcal{N} \setminus \mathcal{I}_1$ that is attracted by on agent $J \in \mathcal{I}_1$ during the dwell time h_{DW} and, therefore, I has at least distance γ_1 from $\overline{\Upsilon_k}(t_0)$ after this dwell time and distance α_2 at time $t_0 + \mathcal{T} + \mathfrak{T}$, see Lemma 3. These steps are iterated over $2N$ consecutive intervals of length $\mathcal{T} + \mathfrak{T}$, i.e., $\mathfrak{I}_1, \ldots, \mathfrak{I}_{2N}$ with parameters $\alpha_l, \beta_l, \gamma_l$ in order to prove consensus, see Theorem 1.

Lemma 1: The set $\Upsilon(t)$ defined above and the length $\widehat{\Upsilon_k}(t)$ of $\Upsilon(t)$ in every dimension k is non-increasing for MAS (1) on networks of arbitrary size $N \in \mathbb{N}$ with arbitrary bounded heterogeneous constant, time-varying, or distributed delays if the coupling functions k_{ij} satisfy Assumption 1.

Proof: The trajectories $x_i(t)$ are piecewise continuously differentiable because the coupling functions are locally Lipschitz and both the delays and the topology changes are non-chattering. The set $\Upsilon(t)$ can only increase at time t if there is an agent i on the boundary of Υ, i.e., with $x_i(t) \in \partial\Upsilon(t)$, and if this agent is leaving $\Upsilon(t)$. We show that any $x_i(t) \in \partial\Upsilon(t)$ is not leaving $\Upsilon(t)$. Assumption 1 implies that $\dot{x}_i(t)$ is either zero or heading toward some weighted average of the past states of its parents $\mathbb{T}_{ij}(x_{j,t}), j \in \mathcal{N}_i^{\sigma(t)}$. Recall that all trajectory pieces $x_{j,t}$ are in $\Upsilon(t)$, see Fig. 2. Therefore, $\mathbb{T}_{ij}(x_{j,t}) \in \Upsilon(t)$ for constant, time-varying, and distributed delays. This also holds for distributed delays because $\int_0^{\mathcal{T}} \phi_{ij}(\eta)x_j(t-\eta)d\eta \in \mathrm{Co}\{x_j(t+\eta) : \eta \in [-\mathcal{T}, 0]\} \subseteq \Upsilon(t)$ for any non-negative delay kernel ϕ_{ij} with $\int_0^{\mathcal{T}} \phi_{ij}(\eta)d\eta = 1$, where $\mathrm{Co}\{\cdot\}$ denotes the convex hull. Thus, the derivative $\dot{x}_i(t)$ of an agent with $x_i(t) \in \partial\Upsilon(t)$ is such that the state of this agent either remains on the boundary or varies toward the interior of $\Upsilon(t)$. Summarizing, x_i is not leaving $\Upsilon(t)$ and $\Upsilon(t)$ is non-increasing. Using the same argument, we can show that $\widehat{\Upsilon_k}(t)$ is non-increasing. ■

The following two lemmas build the core of the contraction argument. The first lemma shows that agents in the interior of $\Upsilon(t)$ approach the surfaces only exponentially fast due to Assumption 1. The second lemma indicates that agents on or very close to the surfaces of $\Upsilon(t)$ are eventually torn away from these surfaces due to Assumption 1 and the uniform quasi-strong connectivity assumption. The combination of these two lemmas will be used in Theorem 1 in order to prove that $\Upsilon(t)$ contracts to a single point as t goes to infinity. Both lemmas are schematically illustrated in Fig. 3.

Lemma 2: Given an initial time $t_0 > 0$, a dimension k, and a set $\mathcal{I}_1 \subseteq \mathcal{N}$, assume that there exists an $\alpha_1 > 0$ such that

$$x_{i[k]}(t_0) \le \overline{\Upsilon_k}(t_0) - \alpha_1, \quad \text{for all } i \in \mathcal{I}_1$$

i.e., all agents $i \in \mathcal{I}_1$ have at least distance α_1 from the upper surface $\overline{\Upsilon_k}(t_0)$ in dimension k at time t_0. Then, Assumption 1 implies that these agents $i \in \mathcal{I}_1$ converge at most exponentially toward $\overline{\Upsilon_k}(t_0)$, i.e., these agents satisfy

$$x_{i[k]}(t_0 + \mathcal{T} + \mathfrak{T} + \eta) \le \overline{\Upsilon_k}(t_0) - \beta_1$$

for all $\eta \in [-(\mathcal{T} + \mathfrak{T}), 0]$, where $\beta_1 = e^{-\overline{a}\overline{\mathcal{K}}N(\mathcal{T}+\mathfrak{T})}\alpha_1$, i.e., all agents $i \in \mathcal{I}_1$ have distance β_1 from the upper surface $\overline{\Upsilon_k}(t_0)$ in dimension k during the interval $[t_0, t_0 + \mathcal{T} + \mathfrak{T}]$, see Fig. 3.

Proof: By assumption, we have $x_{i[k]}(t_0) \le \overline{\Upsilon_k}(t_0) - \alpha_1$ for all $i \in \mathcal{I}_1$. Condition (3) implies

$$\begin{aligned} \dot{x}_{i[k]}(t) &\le \overline{a}\overline{\mathcal{K}}N \max_{j \in \mathcal{N}_i^{\sigma(t)}} \left(\mathbb{T}_{ij}(x_{j,t[k]}) - x_{i[k]}(t)\right) \\ &\le \overline{a}\overline{\mathcal{K}}N \left(\overline{\Upsilon_k}(t_0) - x_{i[k]}(t)\right) \end{aligned}$$

where $\overline{a} = \max_{i,j,p} a_{ij}^p > 0$, i.e., all agents $i \in \mathcal{I}_1$ are attracted at most with convergence rate $\overline{a}\overline{\mathcal{K}}N$ toward the upper surface $\overline{\Upsilon_k}(t_0)$ because $\mathbb{T}_{ij}(x_{j,t[k]}) \le \overline{\Upsilon_k}(t_0)$ for all $t \ge t_0$ and for all $j \in \mathcal{N}$. Recall that $\overline{\Upsilon_k}$ is non-increasing. Hence, all agents $i \in \mathcal{I}_1$ satisfy $x_{i[k]}(t_0 + \mathcal{T} + \mathfrak{T} + \eta) \le \overline{\Upsilon_k}(t_0) - \beta_1$ for all $\eta \in [-(\mathcal{T} + \mathfrak{T}), 0]$, where $\beta_1 = e^{-\overline{a}\overline{\mathcal{K}}N(\mathcal{T}+\mathfrak{T})}\alpha_1$. This is illustrated in Fig. 3 for $\alpha_1 = (1/2)(\overline{\Upsilon_k}(t_0) - \underline{\Upsilon_k}(t_0))$. ■

Lemma 3: Given an initial time $t_0 > 0$, a dimension k, and a set $\mathcal{I}_1 \subset \mathcal{N}$, assume that there exists an $\beta_1 > 0$ such that

$$x_{i[k]}(t_0 + \mathcal{T} + \mathfrak{T} + \eta) \le \overline{\Upsilon_k}(t_0) - \beta_1$$

for all $i \in \mathcal{I}_1$ and all $\eta \in [-(\mathcal{T} + \mathfrak{T}), 0]$, i.e., all agents $i \in \mathcal{I}_1$ have distance β_1 from the upper surface $\overline{\Upsilon_k}(t_0)$ in dimension k during the interval $[t_0, t_0 + \mathcal{T} + \mathfrak{T}]$. Assume, moreover, that the switching graph $\mathcal{G}_{\sigma(t)}$ is uniformly quasi-strongly connected with dwell time h_{DW} and that the union graph $\mathcal{G}_{[t_0+\mathcal{T}, t_0+\mathcal{T}+\mathfrak{T}]}$ has a root $i_{R_1} \in \mathcal{I}_1$. Then, Assumption 1 implies that there is at least one agent $I \in \mathcal{N} \setminus \mathcal{I}_1$ that satisfies

$$x_{I[k]}(t_0 + \mathcal{T} + \mathfrak{T}) \le \overline{\Upsilon_k}(t_0) - \alpha_2$$

i.e., this agent has distance α_2 from the upper surface $\overline{\Upsilon_k}(t_0)$ at time $t_0 + \mathcal{T} + \mathfrak{T}$, where $\alpha_2 = e^{-\overline{a}\overline{\mathcal{K}}N\mathfrak{T}}(1 - e^{-(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N)h_{DW}})(\underline{a}\underline{\mathcal{K}}/(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N))\beta_1$.

Proof: Since the root i_{R_1} of the union graph $\mathcal{G}_{[t_0+\mathcal{T}, t_0+\mathcal{T}+\mathfrak{T}]}$ is in $\mathcal{I}_1$, there is at least one agent $I \in \mathcal{N} \setminus \mathcal{I}_1$ that has a parent $J \in \mathcal{I}_1$ at some time $t \in [t_0 + \mathcal{T}, t_0 + \mathcal{T} + \mathfrak{T}]$, i.e., $a_{IJ}^{\sigma(t)} \ge \underline{a} = \min_{i,j,p: a_{ij}^p \neq 0} a_{ij}^p$, $\underline{a} > 0$. Since the graph switches with a dwell time h_{DW}, there exists a time interval $\Delta_{IJ}^1 = [\underline{t}_{IJ}^1, \overline{t}_{IJ}^1] \subseteq [t_0 + \mathcal{T}, t_0 + \mathcal{T} + \mathfrak{T}]$ that is at least as long as the dwell time h_{DW}, i.e., $\overline{t}_{IJ}^1 - \underline{t}_{IJ}^1 \ge h_{DW}$, such that $a_{IJ}^{\sigma(t)} \ge \underline{a}$ for all $t \in \Delta_{IJ}^1$. During this interval, we know that $\mathbb{T}_{IJ}(x_{J,t[k]}) \le \overline{\Upsilon_k}(t_0) - \beta_1$ because $J \in \mathcal{I}_1$. Thus, Condition (3) implies that

$$\begin{aligned} \dot{x}_{I[k]}(t) \le\; &\underline{a}k_{IJ}\left(\mathbb{T}_{IJ}(x_{J,t[k]}) - x_{I[k]}(t)\right) \\ &+ \sum_{j \in \mathcal{N}_I^{\sigma(t)} \setminus \{J\}} \overline{a}k_{Ij}\left(\mathbb{T}_{Ij}(x_{j,t[k]}) - x_{I[k]}(t)\right) \\ \le\; &\underline{a}\underline{\mathcal{K}}\left(\overline{\Upsilon_k}(t_0) - \beta_1 - x_{I[k]}(t)\right) \\ &+ \overline{a}\overline{\mathcal{K}}N\left(\overline{\Upsilon_k}(t_0) - x_{I[k]}(t)\right) \\ =\; &-(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N)x_{I[k]}(t) + (\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N)\overline{\Upsilon_k}(t_0) \\ &- \underline{a}\underline{\mathcal{K}}\beta_1 \end{aligned} \tag{4}$$

for all $t \in \Delta_{IJ}^1$. For the worst case position $x_{i[k]}(\underline{t}_{JI}^1) = \overline{\Upsilon_k}(t_0)$, we have $\overline{\Upsilon_k}(t_0) - x_{i[k]}(\overline{t}_{JI}^1) \ge \gamma_1$ where $\gamma_1 = (1 - e^{-(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N)h_{DW}})(\underline{a}\underline{\mathcal{K}}/(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N))\beta_1$, see Fig. 3

where $\underline{t}^1_{IJ} = t_0 + \mathcal{T}$ is depicted. If $x_{i[k]}(\underline{t}^1_{JI}) < \overline{\Upsilon_k}(t_0)$, then $\overline{\Upsilon_k}(t_0) - x_{i[k]}(\overline{t}^1_{JI}) \geq \gamma_1$ also holds, which can be shown using the comparison principle, e.g. [40]. After this interval Δ^1_{JI}, the topology might be such that agent I is attracted by a parent on the surface $\overline{\Upsilon_k}(t_0)$ during the interval $[\overline{t}^1_{JI}, t_0 + \mathcal{T} + \mathfrak{T}] \subset [t_0 + \mathcal{T}, t_0 + \mathcal{T} + \mathfrak{T}]$, i.e., the length of this interval is less than $\mathfrak{T}$. During this interval, I is attracted at most with convergence rate $\overline{a}\overline{\mathcal{K}}N$, see Lemma 2, and we obtain $\overline{\Upsilon_k}(t_0) - x_{i[k]}(t_0 + \mathcal{T} + \mathfrak{T}) \geq \alpha_2$ with

$$\begin{aligned}\alpha_2 &= e^{-\overline{a}\overline{\mathcal{K}}N\mathfrak{T}}\gamma_1\\ &= \left(1 - e^{-(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N)h_{DW}}\right)\frac{\underline{a}\underline{\mathcal{K}}}{\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N}e^{-\overline{a}\overline{\mathcal{K}}N\mathfrak{T}}\beta_1\end{aligned}$$

which is clearly less than β_1. ■

Now, we state the main result of this technical note:

Theorem 1: A MAS with agent dynamics (1) where the coupling functions k_{ij} satisfy Assumption 1 achieves consensus asymptotically on networks of arbitrary size $N \in \mathbb{N}$, with arbitrary uniformly quasi-strongly connected switching directed topologies, and for arbitrary bounded heterogeneous constant, time-varying, or distributed delays.

Proof: We have shown in Lemma 1 that $\Upsilon(t)$ is non-increasing. It remains to show that $\lim_{t\to\infty}\Upsilon(t) = \{x^*\}$ for some $x^* \in \mathbb{R}^n$, i.e., the hyper-rectangle contracts to a single point. This is equivalent to proving that the length of the hyper-rectangle in every direction shrinks to zero, i.e., $\lim_{t\to\infty}\widehat{\Upsilon_k}(t) = 0$ for all $k = 1,\ldots,n$. In the sequel, we consider only one dimension, i.e., one arbitrary k, and show that $\lim_{t\to\infty}\widehat{\Upsilon_k}(t) = 0$. The same argument applies for all other dimensions.

Recall that all delays are bounded by $\mathcal{T} \in \mathbb{R}$ and the graph is uniformly quasi-strongly connected, i.e., there exists a $\mathfrak{T} > 0$ such that the union graph $\mathcal{G}_{[t,t+\mathfrak{T}]}$ is quasi-strongly connected for any $t \geq 0$. Given any time $t_0 \geq 0$, we show in the sequel that

$$\widehat{\Upsilon_k}(t_0 + T_{\text{iter}}) - \widehat{\Upsilon_k}(t_0) \leq -\varepsilon\widehat{\Upsilon_k}(t_0) \tag{5}$$

where $T_{\text{iter}} = 2N(\mathcal{T} + \mathfrak{T}) + \mathcal{T}$, N is the number of agents, and $\varepsilon > 0$ is a strictly positive constant. Using (5), we show in the last step that $\lim_{t\to\infty}\widehat{\Upsilon_k}(t) = 0$.

In order to establish (5), we develop an iterative procedure that shows that all agents that are close to one of the surfaces, either $\overline{\Upsilon_k}(t_0)$ or $\underline{\Upsilon_k}(t_0)$ at time t_0, have at least distance $\varepsilon\widehat{\Upsilon_k}(t_0)$ to this surface at time $t_0 + T_{\text{iter}}$. There are two key elements of the proof corresponding to $\overline{\mathcal{K}}, \underline{\mathcal{K}}$ in (3): (i) the upper bound $\overline{\mathcal{K}}$ guarantees that agents in the interior of $\Upsilon(t)$ approach the boundary $\partial\Upsilon(t)$ at most exponentially, see also Lemma 2, and (ii) the lower bound $\underline{\mathcal{K}}$ implies that agents on the boundary eventually move to the interior of $\Upsilon(t)$ because they are at some point connected to other agents in the interior, see also Lemma 3.

The arguments explained below are iterated over $2N$ consecutive intervals $\mathfrak{I}_l = [t_0 + (l-1)(\mathcal{T} + \mathfrak{T}), t_0 + l(\mathcal{T} + \mathfrak{T})]$, where the intervals and the corresponding iteration steps are indexed by $l \in \mathfrak{N} = \{1,\ldots,2N\}$. The particular structure of these intervals is explained in detail in [36]. Fig. 3 shows the first two intervals $\mathfrak{I}_1, \mathfrak{I}_2$. In each iteration interval, there is a subinterval $\hat{\mathfrak{I}}_l = [t_0 + l\mathcal{T} + (l-1)\mathfrak{T}, t_0 + l(\mathcal{T} + \mathfrak{T})] \subset \mathfrak{I}_l$ of length $\mathfrak{T}$ for every $l \in \mathfrak{N}$, see $\hat{\mathfrak{I}}_1, \hat{\mathfrak{I}}_2$ in Fig. 3. Since the graph is uniformly quasi-strongly connected, the union graph $\mathcal{G}_{\hat{\mathfrak{I}}_l}$ has at least one root. We denote the index of this root i_{R_l}, $l \in \mathfrak{N}$. If there are several roots, we choose any one of them. Note that some of these roots $i_{R_l}, l \in \mathfrak{N}$, correspond to the same node because the MAS has N agents and there are $2N$ intervals $\hat{\mathfrak{I}}_l$, i.e., there will always be $l, \tilde{l} \in \mathfrak{N}$ with $l \neq \tilde{l}$ such that $i_{R_l} = i_{R_{\tilde{l}}}$. Now, consider the kth element of the state vectors $x_{i_{R_l}[k]}(t_0)$ for $l \in \mathfrak{N}$, i.e., the state at time t_0 of those agents that will become root in one of the $2N$ intervals $\hat{\mathfrak{I}}_l$. This state $x_{i_{R_l}[k]}(t_0)$ of at least N of all $l \in \mathfrak{N}$ is either in the upper or lower half of $[\underline{\Upsilon_k}(t_0), \overline{\Upsilon_k}(t_0)]$, i.e., either $x_{i_{R_l}[k]}(t_0) \in (\overline{\Upsilon_k}(t_0) - (1/2)\widehat{\Upsilon_k}(t_0), \overline{\Upsilon_k}(t_0)]$ or $x_{i_{R_l}[k]}(t_0) \in [\underline{\Upsilon_k}(t_0), \overline{\Upsilon_k}(t_0) - (1/2)\widehat{\Upsilon_k}(t_0)]$. We assume without loss of generality that at least N of these roots are in the lower half $[\underline{\Upsilon_k}(t_0), \overline{\Upsilon_k}(t_0) - (1/2)\widehat{\Upsilon_k}(t_0)]$. We will see later on that these N roots will play a pivotal role in the proof.

The iteration works as follows: At the beginning of each interval $\mathfrak{I}_l$, there is a nonempty set $\mathcal{I}_l \subseteq \mathcal{N}$ of agents that satisfies $x_{i[k]}(t_0 + (l-1)(\mathcal{T}+\mathfrak{T})) \leq \overline{\Upsilon_k}(t_0) - \alpha_l$ for some $\alpha_l > 0$, i.e., all agents $i \in \mathcal{I}_l$ have at least distance α_l from $\overline{\Upsilon_k}(t_0)$ at the beginning of the interval, see α_1, α_2 in Fig. 3. The value α_l is initialized with $\alpha_1 = (1/2)\widehat{\Upsilon_k}(t_0)$. Thus, $\mathcal{I}_1$ contains at least N of the $2N$ roots i_{R_l} (which might correspond to one agent) because of the assumption in the previous paragraph. This guarantees that $\mathcal{I}_1$ is not empty.

Since all agents in $\mathcal{I}_l$ have at least distance α_l from $\overline{\Upsilon_k}(t_0)$ at time $t_0 + (l-1)(\mathcal{T} + \mathfrak{T})$, we can use Lemma 2 (replacing α_1 and β_1 with α_l and β_l) to determine in each interval $\mathfrak{I}_l$ a $\beta_l \in (0, \alpha_l)$ such that $x_{i[k]}(t_0 + l(\mathcal{T} + \mathfrak{T}) + \eta) \leq \overline{\Upsilon_k}(t_0) - \beta_l$ for all $\eta \in [-\mathcal{T} - \mathfrak{T}, 0]$ and all $i \in \mathcal{I}_l$, with $\beta_l = e^{-\overline{a}\overline{\mathcal{K}}N(\mathcal{T}+\mathfrak{T})}\alpha_l$. That is, all agents $i \in \mathcal{I}_l$ have distance β_l from $\overline{\Upsilon_k}(t_0)$ during the whole interval $\mathfrak{I}_l$, see β_1, β_2 in Fig. 3.

We will see later that $\alpha_{l+1} \leq \beta_l$ for all l. Therefore, we have $\mathcal{I}_l \subseteq \mathcal{I}_{l+1}$, i.e., new elements can be added to the set $\mathcal{I}_l$ in each iteration step but elements are not removed. Thus, $\mathcal{I}_l$ always contains at least N roots i_{R_l} (which might correspond to one agent). This implies that in at least N of the $2N$ intervals $\hat{\mathfrak{I}}_l$ there is a root $i_{R_l} \in \mathcal{I}_l$. We consider first those situations where $i_{R_l} \in \mathcal{I}_l$ and discuss the other ones later on. In these cases, we can apply Lemma 3 (replacing β_1, γ_1 and α_2 with β_l, γ_l and α_{l+1}) to determine an $\alpha_{l+1} \in (0, \beta_l)$ such that $x_{i[k]}(t_0 + l(\mathcal{T} + \mathfrak{T})) \leq \overline{\Upsilon_k}(t_0) - \alpha_{l+1}$ for all $i \in \mathcal{I}_l \cup \{I\}$ where agent I is a particular agent that has a parent in $\mathcal{I}_l$ at some time during the interval $\mathfrak{I}_l$. This agent indeed exists because the root $i_{R_l} \in \mathcal{I}_l$, see also proof of Lemma 3. From Lemmas 2 and 3, we have

$$\alpha_{l+1} = \left(1 - e^{-(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N)h_{DW}}\right)\frac{\underline{a}\underline{\mathcal{K}}}{\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N}e^{-\overline{a}\overline{\mathcal{K}}N(\mathcal{T}+2\mathfrak{T})}\alpha_l. \tag{6}$$

In other words, all agents $i \in \mathcal{I}_l \cup \{I\}$ have distance α_{l+1} from $\overline{\Upsilon_k}(t_0)$ at the end of the interval $\mathfrak{I}_l$.

In those iteration where $i_{R_l} \in \mathcal{I}_l$, such an I exists and the iteration step ends with $\mathcal{I}_{l+1} = \mathcal{I}_l \cup \{I\}$ and α_{l+1} given by (6). In the remaining iteration steps, i.e., if $i_{R_l} \in \mathcal{N} \setminus \mathcal{I}_l$, the iteration ends with $\mathcal{I}_{l+1} = \mathcal{I}_l$. In this case, we could use β_l as α_{l-1} for the next iteration step. However, as we do not know how often this case appears and as α_{l+1} in (6) is smaller than β_l, we continue with α_{l+1} given in (6) also in those cases where $i_{R_l} \in \mathcal{N} \setminus \mathcal{I}_l$. After $2N$ steps, we know that $\mathcal{I}_{2N+1} = \mathcal{N}$ and therefore all $i \in \mathcal{N}$ satisfy $x_{i[k]}(t_0 + 2N(\mathcal{T} + \mathfrak{T})) \leq \overline{\Upsilon_k}(t_0) - \alpha_{2N+1}$ for some $\alpha_{2N+1} > 0$. That is, all agents have distance α_{2N+1} from the upper surface $\overline{\Upsilon_k}(t_0)$ where

$$\begin{aligned}\alpha_{2N+1} &= \frac{1}{2}\widehat{\Upsilon_k}(t_0)\\ &\quad\times\left(\left(1 - e^{-(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N)h_{DW}}\right)\frac{\underline{a}\underline{\mathcal{K}}}{\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N}e^{-\overline{a}\overline{\mathcal{K}}N(\mathcal{T}+2\mathfrak{T})}\right)^{2N}.\end{aligned}$$

Then, we have to wait for an additional interval of length $\mathcal{T}$ such that $x_{i[k]}(t_0 + T_{\text{iter}} + \eta) \leq \overline{\Upsilon_k}(t_0) - \varepsilon\widehat{\Upsilon_k}(t_0)$ for all $\eta \in [-\mathcal{T}, 0]$ where $T_{\text{iter}} = 2N(\mathcal{T} + \mathfrak{T}) + \mathcal{T}$ and

$$\begin{aligned}\varepsilon = \frac{1}{2}e^{-\overline{a}\overline{\mathcal{K}}N\mathcal{T}} &\times\left(\left(1 - e^{-(\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N)h_{DW}}\right)\right.\\ &\left.\times\frac{\underline{a}\underline{\mathcal{K}}}{\underline{a}\underline{\mathcal{K}} + \overline{a}\overline{\mathcal{K}}N}e^{-\overline{a}\overline{\mathcal{K}}N(\mathcal{T}+2\mathfrak{T})}\right)^{2N} > 0\end{aligned}$$

i.e., all segments $x_{i,t_0+T_{\text{iter}}}$ are bounded away from the upper surface $\overline{\Upsilon_k}(t_0)$ at least by $\varepsilon\widehat{\Upsilon_k}(t_0)$. Hence, the hyper-rectangle Υ has shrunk in dimension k by the same magnitude, i.e., we have $\widehat{\Upsilon}_k(t_0+T_{\rm iter}) \leq \widehat{\Upsilon}_k(t_0) - \varepsilon\widehat{\Upsilon}_k(t_0)$, as already given in (5). Note that ε decreases as the number of agents N increases; yet ε is always positive for finite N.

Finally, we show that (5) implies that $\lim_{t\to\infty}\widehat{\Upsilon}_k(t) = 0$. For any $\upsilon \in \mathbb{N}$ and for all t_0, we have

$$\begin{aligned}\widehat{\Upsilon}_k(t_0+\upsilon T_{\rm iter}) &\leq \widehat{\Upsilon}_k(t_0) - \varepsilon\widehat{\Upsilon}_k(t_0) - \varepsilon\widehat{\Upsilon}_k(t_0+T_{\rm iter})\\ &\quad - \ldots - \varepsilon\widehat{\Upsilon}_k\left(t_0+(\upsilon-1)T_{\rm iter}\right)\\ &\leq \widehat{\Upsilon}_k(t_0) - \upsilon\varepsilon\widehat{\Upsilon}_k\left(t_0+(\upsilon-1)T_{\rm iter}\right)\\ &\leq \widehat{\Upsilon}_k(t_0) - \upsilon\varepsilon\widehat{\Upsilon}_k(t_0+\upsilon T_{\rm iter})\end{aligned}$$

because $\widehat{\Upsilon}_k$ is non-increasing. We conclude that, for given $\widehat{\Upsilon}_k(t_0)$ and any arbitrarily small $\delta > 0$, there exists a $T^* = \upsilon T_{\rm iter}$ with υ sufficiently large such that $\widehat{\Upsilon}_k(t_0+t) < \delta$ for all $t \geq T^*$. This can be shown by contradiction. Assume $\widehat{\Upsilon}_k(t_0+\upsilon T_{\rm iter}) \geq \delta$ for all υ. Then, the above inequalities provide $\delta \leq \widehat{\Upsilon}_k(t_0) - \upsilon\varepsilon\delta$ which is false for sufficiently large υ because $\varepsilon\delta > 0$. Therefore, $\lim_{t\to\infty}\widehat{\Upsilon}_k(t) = 0$ for any k. Hence, consensus is achieved. ∎

Theorem 1 extends the main results of [30] from nonlinear MAS without delays to nonlinear MAS with heterogeneous delays taking all three delay models into account. It is very interesting to see that consensus is achieved for arbitrary large but bounded delays under exactly the same assumptions as in the undelayed case. In fact, [30] provides several counter-examples illustrating that consensus is not guaranteed if these assumptions are relaxed. Clearly, these counter examples also apply for the MAS with delays discussed here because the delay robustness analysis also includes the undelayed case.

Theorem 1 can be extended to the case of dynamic graphs, where the weights of the adjacency matrix are piecewise continuous, uniformly bounded functions $a_{ij} : \mathbb{R} \to \mathbb{R}_0^+$. We only need to assume that $a_{ij}^p(t) > \underline{a}, \forall t$, if $(v_j, v_i) \in \mathcal{E}_p$ for some $\underline{a} > 0$, as in previous publications [39]. The proof for dynamic graphs follows directly from the proof presented here. Further extensions of this result are possible to non-identical nonlinear systems with relative degree one and integrating behavior [36].

Theorem 1 holds of course also for fixed topologies. This is expressed in the following corollary, that is stated here for completeness without proof.

Corollary 1: A MAS with agent dynamics (1) where the coupling functions k_{ij} satisfy Assumption 1 achieves consensus asymptotically on networks of arbitrary size $N \in \mathbb{N}$, with arbitrary quasi-strongly connected, directed topology, and for arbitrary bounded heterogeneous constant, time-varying, or distributed delays.

There are numerous applications for single integrator MAS with nonlinear coupling, like the well-known Kuramoto oscillator. The main results of this technical note can be readily applied to these models, see [31], [36] for details. These application examples have been treated exhaustively in the literature.

IV. Conclusion

In this technical note, we showed that consensus in single-integrator MAS is robust to arbitrary large delays, i.e., consensus is achieved under the same conditions as in the undelayed case. This holds even for non-identical agent dynamics, directed, switching topologies, heterogeneous delays, and nonlinear coupling functions. Even though consensus is guaranteed for arbitrary large delays, it is relatively easy to understand that the convergence rate tends to decrease as the delays increase. In other words, consensus is reached slower for larger delays. One of the main future challenges for consensus in MAS with delays is the formulation of suitable, scalable bounds on the convergence rate. A first step in this direction has been presented in [14].

References

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[2] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, “Flocking in fixed and switching networks,” *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 863–868, May 2007.

[3] F. Dörfler and F. Bullo, “Transient stability analysis in power networks and synchronization of non-uniform Kuramoto oscillators,” in *Proc. Amer. Control Conf.*, Jun./Jul. 2010, pp. 930–937.

[4] A. Papachristodoulou and A. Jadbabaie, “Delay robustness of nonlinear internet congestion control schemes,” *IEEE Trans. Autom. Control*, vol. 55, no. 6, pp. 1421–1427, Jun. 2010.

[5] W. Ren and R. W. Beard*, Distributed Consensus in Multi-Vehicle Cooperative Control*. London, U.K.: Springer, 2008.

[6] J. R. Moyne and D. M. Tilbury, “The emergence of industrial control networks for manufacturing control, diagnostics, and safety data,” *Proc. IEEE*, vol. 95, no. 1, pp. 29–47, Jan. 2007.

[7] F. Xiao and L. Wang, “Consensus protocols for discrete-time multiagent systems with time-varying delays,” *Automatica*, vol. 44, no. 10, pp. 2577–2582, 2008.

[8] M. Cao, A. S. Morse, and B. D. O. Anderson, “Reaching a consensus in a dynamically changing environment: A graphical approach & convergence ranges, measurement delays, and asynchronous events,” *SIAM J. Control Optim.*, vol. 47, no. 2, pp. 575–623, 2008.

[9] F. Xiao and L. Wang, “Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays,” *IEEE Trans. Autom. Control*, vol. 53, no. 8, pp. 1804–1816, Sep. 2008.

[10] G. Scutari, S. Barbarossa, and L. Pescosolido, “Distributed decision through self-synchronizing sensor networks in the presence of propagation delays and asymmetric channels,” *IEEE Trans. Signal Proc.*, vol. 56, no. 4, pp. 1667–1684, Apr. 2008.

[11] D. Lee and M. W. Spong, “Agreement with non-uniform information delays,” in *Proc. Amer. Contr.Conf.*, Minneapolis, MN, 2006, pp. 756–761.

[12] I. C. Lestas and G. Vinnicombe, “Heterogeneity and scalability in group agreement protocols: Beyond small gain and passivity approaches,” *Automatica*, vol. 46, no. 7, pp. 1141–1151, 2010.

[13] K. Jacobsson, L. L. H. Andrew, and A. Tang, “Stability and robustness conditions using frequency dependent half planes,” in *Proc. IEEE Conf. Decision Control*, Shanghai, China, 2009, pp. 6166–6171.

[14] U. Münz, A. Papachristodoulou, and F. Allgöwer, “Delay robustness in consensus problems,” *Automatica*, vol. 46, no. 8, pp. 1252–1265, 2010.

[15] Y.-P. Tian and C.-L. Liu, “Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations,” *Automatica*, vol. 45, no. 5, pp. 1347–1353, 2009.

[16] W. Yang, A. Bertozzi, and X. Wang, “Stability of a second order consensus algorithm with time delay,” in *Proc. IEEE Conf. Decision Control*, Cancun, Mexico, 2008, pp. 2962–2931.

[17] C.-Y. Kao, U. Jönsson, and H. Fujioka, “Characterization of robust stability of a class of interconnected systems,” *Automatica*, vol. 45, no. 1, pp. 217–224, 2009.

[18] U. T. Jönsson and C.-Y. Kao, “A scalable robust stability criterion for systems with heterogeneous LTI components,” in *Proc. Amer. Control Conf.*, St. Louis, MO, 2009, pp. 2898–2903.

[19] P.-A. Bliman and G. Ferrari-Trecate, “Average consensus problems in networks of agents with delayed communications,” *Automatica*, vol. 44, no. 8, pp. 1985–1995, 2008.

[20] V. S. Chellaboina, W. M. Haddad, Q. Hui, and J. Ramakrishnan, “On system state equipartitioning and semistability in network dynamical systems with arbitrary time-delays,” in *Proc. IEEE Conf. Decision Control*, San Diego, CA, 2006, pp. 3461–3466.

[21] R. Ghabcheloo, A. P. Aguiar, A. Pascoal, and C. Silvestre, “Synchronization in multi-agent systems with switching topologies and nonhomogeneous communication delays,” in *Proc. IEEE Conf. Decision Control*, New Orleans, LA, 2007, pp. 2327–2332.

[22] A. Papachristodoulou and A. Jadbabaie, “Synchronization in oscillator networks: Switching topologies and non-homogeneous delays,” in *Proc. IEEE Conf. Decision Control Eur. Control Conf.*, Seville, Spain, 2005, pp. 5692–5697.

[23] U. Münz, A. Papachristodoulou, and F. Allgöwer, “Multi-agent system consensus in packet-switched networks,” in *Proc. Europ. Control Conf.*, Kos, Greece, 2007, pp. 4598–4603.

[24] N. Chopra and M. Spong, "Output synchronization of nonlinear systems with relative degree one," in *Recent Advances in Learning and Control*, V. D. Blondel, S. P. Boyd, and H. Kimura, Eds. Berlin, Germany: Springer, 2008, pp. 51–64.

[25] N. Chopra and M. Spong, "Passivity-based control of multi-agent systems," in *Advances in Robot Control*, S. Kaxamura and M. Svinin, Eds. Berlin, Germany: Springer, 2006, pp. 107–134.

[26] N. Chopra, D. M. Stipanović, and M. W. Spong, "On synchronization and collision avoidance for mechanical systems," in *Proc. Amer. Contr. Conf.*, Seattle, WA, 2008, pp. 3713–3718.

[27] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Consensus controller design for nonlinear relative degree two multi-agent systems with communication constraints," *IEEE Trans. Autom. Control*, vol. 56, no. 1, pp. 145–151, Jan. 2011.

[28] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Robust rendezvous of heterogeneous Lagrange systems on packet-switched networks," *at—Automatisierungstechnik*, vol. 58, no. 4, pp. 184–191, 2010.

[29] A. Papachristodoulou, A. Jadbabaie, and U. Münz, "Effects of delay in multi-agent consensus and oscillatior synchronization," *IEEE Trans. Autom. Control*, vol. 55, no. 6, pp. 1471–1477, Jun. 2010.

[30] Z. Lin, "Coupled Dynamic Systems: From Structure Towards Stability and Stabilizability," Ph.D. dissertation, Univ. Toronto, Toronto, ON, Canada, 2006.

[31] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Consensus reaching in multi-agent packet-switched networks," *Int. J. Control*, vol. 82, no. 5, pp. 953–969, 2009.

[32] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Output consensus controller design for nonlinear relative degree one multi-agent systems with delays," in *Proc. IFAC Workshop Time-Delay Syst.*, Sinaia, Romania, 2009, [CD ROM].

[33] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Nonlinear multi-agent system consensus with time-varying delays," in *Proc. IFAC World Congress*, Seoul, Korea, 2008, pp. 1522–1527.

[34] A. Papachristodoulou and A. Jadbabaie, "Synchronization of oscillator networks with heterogeneous delays, switching topologies and nonlinear dynamics," in *Proc. IEEE Conf. Decision Control*, San Diego, CA, 2006, pp. 4307–4312.

[35] J. P. Hespanha, D. Liberzon, D. Angeli, and E. D. Sontag, "Nonlinear norm-observability notions and stability of switched systems," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 154–168, Feb. 2005.

[36] U. Münz, *Delay Robustness in Cooperative Control*. Berlin, Germany: Logos Verlag, 2010 [Online]. Available: http://www.ist.uni-stuttgart.de/~muenz/muenz_delay_robustne_in_cooperative_control.pdf

[37] K. Thulasiraman and M. N. S. Swamy, *Graphs: Theory and Algorithms*. New York: Wiley, 1991.

[38] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.

[39] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *Proc. IEEE Conf. Decision Control*, 2004, pp. 3998–4003.

[40] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice-Hall, 2002.

A Note on Greedy Policies for Scheduling Scalar Gauss-Markov Systems

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***Abstract*—It is being stated in recent literature that greedy policies provide optimal scheduling for identical scalar Gauss-Markov systems. Here, the performance index is the sum of the covariances of the systems from time 1 to a time horizon N. The scheduling decisions from time 0 to $N-1$ constitute the policy. In this note we show that when $N = 1$, the statement is true. Our main result is to show that when $N = 2$ the greedy policy fails to be optimal for identical scalar Gauss-Markov systems, in contradiction to the statement, and illustrating the complexity of the problem.**

***Index Terms*—Gauss-Markov systems.**

I. Introduction

One primary goal of tracking systems is the maintenance of kinematic tracks on moving targets in the environment [1]. When the targets under observation are far apart, the uncertainty associated with the targets constitute the metric of success. A reasonable objective is to minimize the overall uncertainty of the collection of targets under the measurement constraints of the sensor system. When time slotted measurements are considered, measurement constraints can be represented by having only one of the targets being observed at each time slot. The problem then becomes one of scheduling these measurements among the various targets in a way that the overall uncertainty is minimized. Following [2], let the targets be represented as two identical scalar Gauss-Markov systems ([3, p.12]) of the form:

$$p_{n+1}^{(i)} = f p_n^{(i)} + w_n^{(i)} \tag{1}$$

$$z_n^{(i)} = p_n^{(i)} + v_n^{(i)} \tag{2}$$

where $p_n^{(i)}$ is the state of the ith system at time k $(i = 1, 2)$, $z_n^{(i)}$ is the corresponding measurement and $\{w_n^{(i)}\}$ and $\{v_n^{(i)}\}$ are sequences of zero mean, independent, identically distributed Gaussian random variables with $\mathbb{E}(w_a^{(i)} w_b^{(i)}) = \Sigma^2 \delta_{ab}$ and $\mathbb{E}(v_a^{(i)} v_b^{(i)}) = R^2 \delta_{ab}$, where δ_{ab} is the Kronecker delta. Here, Σ^2, $R^2 > 0$ and $f \neq 0$. Let the initial probability density of the state $p_0^{(i)}$ be Gaussian with mean $\hat{p}_0^{(i)}$, covariance $\sigma_0^{2(i)}$ and independent of $\{w_n^{(i)}\}$ and $\{v_n^{(i)}\}$. The Kalman filter provides the probability density of the state conditioned on the measurements, with covariance $\sigma^{2(i)}$ given by the recursion [3]

$$\sigma_{n+1}^{2(i)} = \left[R^2 \left(f^2 \sigma_n^{2(i)} + \Sigma^2\right)\right] \left[f^2 \sigma_n^{2(i)} + \Sigma^2 + R^2\right]^{-1}. \tag{3}$$

In the formulation in [2], one and only one system is observed at each time slot. A policy is a sequence $\{u_n\}$ of the integers 1, 2. If $u_n = i$, then the ith system is observed on the nth time slot. From the point of view of the estimators, the absence of a measurement can be represented by setting $R^2 \to \infty$ in (3). This allows one to write

$$\sigma_{n+1}^{2(i)} = \begin{cases} \left[R^2 \left(f^2 \sigma_n^{2(i)} + \Sigma^2\right)\right] & \\ \quad \times \left[f^2 \sigma_n^{2(i)} + \Sigma^2 + R^2\right]^{-1}, & \text{if } u_n = i \\ f^2 \sigma_n^{2(i)} + \Sigma^2, & \text{otherwise} \end{cases} \tag{4}$$

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