Discrete-Time Opinion Dynamics

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***Abstract*—We discuss the Hegselmann-Krause model for opinion dynamics in discrete-time for a symmetric confidence bound. We construct an adjoint dynamics and use it to define a Lyapunov comparison function that decreases along the trajectory of the opinion dynamics. Using the Lyapunov comparison function, we develop a novel upper bound on the convergence time of the dynamics.**

I. INTRODUCTION

Recently, mathematical modeling of social networks has gained a lot of attention and several models for opinion dynamics have been proposed and studied (see for example [5], [7]). These models have also been used in distributed engineered systems to capture the dynamic interactions among the agents in the system such as, for example, in robotic networks [3]. One of the most popular models for opinion dynamics is the Hegselmann-Krause proposed by R. Hegselmann and U. Krause in [5].

The stability and convergence of the Hegselmann-Krause dynamics and its generalizations have been shown in [6] and [8]. Also, several generalizations of this dynamics for continuous time and time-varying networks have been proposed [4], [1], [2]. Among many issues that have remained opened, even for the original Hegselmann-Krause dynamics, is the provision of upper and lower bounds on the convergence time of the dynamics, which is the time required for the dynamics to reach its steady state. While it is known [1] that the convergence time is finite, the bounds on the convergence time have not been properly developed. In [3], it is shown that the termination time of the dynamics is at least in the order of m and at most in the order of m^5, where m is the number of agents in the model. In this work, we provide an upper bound of order m^2 in terms of the initial profile.

The structure of the paper is as follows: in Section II we discuss the Hegselmann-Krause dynamics as appeared in [5], and we set up the notation that we use. Then, in Section III, we introduce the concept of *adjoint dynamics* which plays a central role in our analysis. Based on the properties of the dynamics and the use of Lyapunov comparison function, we develop an upper bound for the convergence time in Section IV. We conclude our discussion in Section V.

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II. HEGSELMANN-KRAUSE OPINION DYNAMICS

Here, we discuss the Hegselmann-Krause opinion dynamics model [5], where m agents interact in time. The interactions occur at time instances indexed by non-negative integers $t = 0, 1, 2, \ldots$. At each time t, agent i has an opinion that is represented by a scalar $x_i(t) \in \mathbb{R}$. The collection of profiles $\{x_i(t), i \in [m]\}$, where $[m] = \{1, \ldots, m\}$, is termed as the *opinion profile* at time t.

Initially, each agent $i \in [m]$ has an opinion $x_i(0) \in \mathbb{R}$. At each time t, the agents interact with their neighbors and update their opinions, where the neighbors are determined based on the difference of the opinions and a prescribed maximum difference $\epsilon > 0$. The scalar ϵ is termed *confidence*, which limits the agents' interactions in time. Specifically, the set of neighbors of agent i at time t, denoted by $N_i(t)$, is given as follows:

$$N_i(t) = \{j \in [m] \mid |x_i(t) - x_j(t)| \le \epsilon\}.$$

The opinion profile evolves in time according to the following dynamics:

$$x_i(t+1) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t), \tag{1}$$

where $|N_i(t)|$ denotes the cardinality of the set $N_i(t)$. Note that the opinion dynamics is completely determined by the confidence value ϵ and the initial profile $\{x_i(0) \mid i \in [m]\}$.

First, to compactly represent the dynamics, we define the matrix $A(t)$ with entries as follows

$$A_{ij}(t) = \begin{cases} \frac{1}{|N_i(t)|} & \text{if } j \in N_i(t), \\ 0 & \text{otherwise} \end{cases}.$$

The dynamics (1) can now be written as follows:

$$x(t+1) = A(t)x(t), \tag{2}$$

where $x(t)$ is the column vector with entries $x_i(t)$, $i \in [m]$.

It is well known that Hegselmann-Krause opinion dynamics is stable and converges to a limit point in a finite time [1], [5], see also [6], [7], [3]. A lower and upper bound have been discussed in [3], where it has been demonstrated that a lower bound is at least of the order of the number m of agents, and the upper bound is of the order at most m^5. In this paper, we are interested in improving the upper bound.

Few words on our notation are in place. We view vectors as columns, and use x' to denote the transpose of a vector x and A' for the transpose of a matrix A. We write e_i to denote the unit vector with ith coordinate equal to 1, and e to denote a vector with all entries equal to 1. For a vector x, we use x_i or $[x]_i$ to denote its ith coordinate value. A vector v is stochastic if $v_i \geq 0$ and $\sum_i v_i = 1$. Given a vector $v \in \mathbb{R}^m$, $\text{diag}(v)$ denotes the $m \times m$ diagonal matrix with v_i, $i \in [m]$, on its diagonal. For a matrix A, we write A_{ij} or $[A]_{ij}$ to denote its ijth entry. A matrix A is stochastic if $Ae = e$ and $A_{ij} \geq 0$ for all i, j. We use I_d to denote the $d \times d$ identity matrix. We often abbreviate double summation $\sum_{i=1}^m \sum_{j=1}^m$ with $\sum_{i,j=1}^m$. We write $|S|$ to denote the cardinality of a set S with finitely many elements.

III. Adjoint Dynamics

We next discuss the adjoint dynamics for the Hegselmann-Krause dynamics. In [9], it has been shown that for the Hegselmann-Krause dynamics there exists a sequence of stochastic vectors $\{\pi(t)\}$ such that

$$\pi'(t) = \pi'(t+1)A(t) \qquad \text{for all } t \geq 0. \tag{3}$$

We can view the above dynamics as the adjoint for the original dynamics (1). Note that this dynamics evolves backward in time. Using the relations (2) and (3) for the dynamics $\{x(t)\}$ and $\{\pi(t\}$, one can show that

$$\pi'(t+1)x(t+1) = \pi'(t)x(t) \qquad \text{for all } t \geq 0. \tag{4}$$

It can be seen that the adjoint process for the Hegselmann-Krause dynamics is not unique in general. We will construct one such process for later use.

We let T be the termination time of the dynamics $\{x(t)\}$, which is the instance when all the agent reach their corresponding steady states. Formally, the termination time T of the dynamics (2) is the smallest $t \geq 0$ such that $x_i(k+1) = x_i(k)$ for all $k \geq t$ and for all $i \in [m]$, i.e.,

$$T = \min_{t \geq 0}\{t \mid x_i(k+1) = x_i(k) \text{ for all } i \in [m] \text{ and all } k \geq t\}.$$

As shown in [1], the termination time T is finite.

At time T, the agents have reached their steady states and form a partition of the agent set $[m]$, which consists of disjoint agent's groups. Let us denote this partition by $\{G_s, s = 1, \ldots, q\}$, where q is the number of agent groups formed. For each group G_s, the agents $i \in G_s$ hold the same opinion value, i.e., for all $s = 1, \ldots, q$,

$$x_i(T) = x_j(T) \qquad \text{for all } i, j \in G_s.$$

We can define termination time for each of the groups as follows: for each $s = 1, \ldots, q$,

$$T_s = \min_{t \geq 0}\{t \mid x_i(k+1) = x_i(k) \text{ for all } i \in G_s \text{ and all } k \geq t\}.$$

Then, the termination time of the whole dynamics satisfies:

$$T = \max_{1 \leq s \leq q} T_s.$$

Let G_{s^*} be a group with the termination time equal to T, i.e., $T_{s^*} = T$. Consider the adjoint process $\{\pi(t)\}$ defined by

$$\begin{aligned} &\pi'(t) = \pi'(t+1)A(t) \qquad \text{for } t = 0, \ldots, T-1, \\ &\pi(t) = \pi(t+1) \qquad \text{for } t \geq T. \end{aligned} \tag{5}$$

We note that the vectors $\pi(t)$ are stochastic since $A(t)$ is a stochastic matrix for all t and it can be verified that such a process is in fact an adjoint process (3) and it reaches its steady state at time T.

We next establish an important property of the preceding adjoint dynamics that we use later on to derive an upper bound for the termination time T.

Lemma 1: Let $\{\pi(t)\}$ be the adjoint dynamics as given in (5). Suppose that $t < T - 1$. Then, for any $i \in [m]$ with $\pi_i(t+1) > 0$, there exists $j \in N_i(t)$ such that $\pi_j(t+1) \geq \frac{1}{2}\pi_i(t+1)$ and $N_j(t) \neq N_i(t)$.

Proof: Let us define the closest upper and lower neighbors of an agent $i \in [m]$, respectively, as follows

$$\bar{i}(t) = \text{argmin}_{\ell \in N_i(t)}\{x_i(t) < x_\ell(t)\},$$

$$\underline{i}(t) = \text{argmax}_{\ell \in N_i(t)}\{x_\ell(t) < x_i(t)\}.$$

Note that the closest upper (lower) neighbor of $i \in [m]$ may not exist. Define the following subsets of $N_i(t+1)$:

$$N_i^-(t+1) = \{j \in N_i(t+1) \mid x_j(t+1) \leq x_i(t+1)\},$$

$$N_i^+(t+1) = \{j \in N_i(t+1) \mid x_j(t+1) \geq x_i(t+1)\}.$$

By the definition of the adjoint dynamics in (5), we have for $t < T - 1$,

$$\begin{aligned} \pi_i(t+1) &= \sum_{j \in N_i(t+1)} \pi_j(t+2)A_{ji}(t+1) \\ &\leq \sum_{j \in N_i^-(t+1)} \pi_j(t+2)A_{ji}(t+1) \\ &\quad + \sum_{j \in N_i^+(t+1)} \pi_j(t+2)A_{ji}(t+1), \end{aligned} \tag{6}$$

where the inequality comes from $N_i^-(t+1) \cup N_i^+(t+1) = N_i(t+1)$. So, either $\frac{\pi_i(t+1)}{2} \leq \sum_{j \in N_i^-(t+1)} \pi_j(t+2)A_{ji}(t+1)$, or $\frac{\pi_i(t+1)}{2} \leq \sum_{j \in N_i^+(t+1)} \pi_j(t+2)A_{ji}(t+1)$. Without loss of generality assume that

$$\frac{\pi_i(t+1)}{2} \leq \sum_{j \in N_i^+(t+1)} \pi_j(t+2)A_{ji}(t+1).$$

Now, we consider the following two cases:
Case 1: $\bar{i}(t+1)$ exists. Then, we have $\bar{i}(t+1) \in N_i^+(t+1)$

and $N_i^+(t+1) \subseteq N_{\bar{i}(t+1)}(t+1)$. Therefore,

$$\begin{aligned}\pi_{\bar{i}(t+1)}(t+1) &= \sum_{j\in N_{\bar{i}}(t+1)} \pi_j(t+2)A_{j\bar{i}}(t+1)\\ &\geq \sum_{j\in N_i^+(t+1)} \pi_j(t+2)A_{j\bar{i}}(t+1)\\ &= \sum_{j\in N_i^+(t+1)} \pi_j(t+2)A_{ji}(t+1)\\ &\geq \frac{\pi_i(t+1)}{2},\end{aligned}$$

where in the second inequality we use the fact that the positive entries in each row of $A(t+1)$ are identical. Note that $\bar{i}(t+1) \in N_i(t)$, because otherwise

$$x_{\bar{i}(t+1)}(t) \leq x_{\bar{i}(t+1)}(t+1),$$

and also,

$$x_i(t+1) \leq x_i(t).$$

Since $\bar{i}(t+1) \in N_i^+(t+1)$, it follows $\bar{x}_i(t+1) - x_i(t+1) \leq \epsilon$. This and the preceding two relations imply $x_{\bar{i}(t+1)}(k) - x_i(t) \leq \epsilon$, i.e. $\bar{i}(t+1) \in N_i(t)$. Also, $N_{\bar{i}(t+1)}(t) \neq N_i(t)$, because otherwise $x_i(t+1) = x_{\bar{i}(t+1)}(t+1)$ which contradicts with $x_i(t+1) < x_{\bar{i}(t+1)}(t+1)$. Therefore, in this case, the assertion is true and we have $j = \bar{i}(t+1)$.

Case 2: $\bar{i}(k+1)$ does not exist. In this case, for any $j \in N_i^+(t+1)$, we have $x_j(t+1) = x_i(t+1)$. Thus, $N_i^+(t+1) \subseteq N_i^-(t+1)$ and hence, $N_i^-(t+1) = N_i(t+1)$. If $\underline{i}(t+1)$ does not exist, then $N_i^-(t+1) = N_i(t+1)$, implying $x_\ell(t+1) = x_i(t+1)$ for all $\ell \in N_i(t)$. Hence, it follows that $t+1$ is the termination time, which contradicts the assumption $t < T-1$. Therefore, $\underline{i}(t+1)$ must exist. Since $N_i^-(t+1) \subseteq N_{\underline{i}(t+1)}(t+1)$, using the same line of argument as in the preceding case, we can show that

$$\pi_{\underline{i}(t+1)} \geq \frac{1}{2}\pi_i(t+1), \qquad N_{\underline{i}(t+1)}(t) \neq N_i(t),$$

implying that the assertion holds for $j = \underline{i}(t+1)$. ∎

IV. Upper Bound for Termination Time

In this section, we establish an improved upper bound for the termination time of the Hegselmann-Krause dynamics. Our analysis uses a Lyapunov comparison function that is constructed by using the adjoint dynamics in (5).

The comparison function is $V_\pi(x,t)$ defined by: for all $x \in \mathbb{R}^n$ and $t \geq 0$,

$$V_\pi(x,t) = \sum_{i=1}^m \pi_i(t)(x_i - \pi'(t)x)^2, \tag{7}$$

where $\{\pi(t)\}$ is the adjoint dynamics (3). This function has been proposed and investigated in [9]. In particular, the decrease of this comparison function along trajectories of the Hegselmann-Krause dynamics $\{x(t)\}$ and its adjoint dynamics is of particular importance in our analysis. We use the following result, as shown in [9], Theorem 4.3.

Theorem 1: For any $t \geq 0$, we have

$$V_\pi(x(t+1), t+1) = V_\pi(x(t),t) - \frac{1}{2}\sum_{i,j=1}^m H_{ij}(t)(x_i(t)-x_j(t))^2,$$

where $H(t) = A'(t)\,\mathrm{diag}(\pi(t+1))A(t)$.

As an immediate consequence of Theorem 1 and the definition of the adjoint process in (5), by summing both sides of the asserted relation for $t = 0,\ldots,\tau$, we have for all τ, $0 \leq \tau \leq T-1$,

$$\begin{aligned}V_\pi(x(\tau+1),\tau+1) &= V_\pi(x(0),0)\\ &- \frac{1}{2}\sum_{t=0}^{\tau}\sum_{i,j=1}^m H_{ij}(t)(x_i(t)-x_j(t))^2. \end{aligned}\tag{8}$$

We next derive a lower bound for the summands that appear with the negative sign in the preceding relation. We do so through the use of two auxiliary relations.

The first relation gives us a lower bound in terms of the maximal opinion spread in the neighborhood for every agent. Specifically, for every $\ell \in [m]$ and every $t \geq 0$, let

$$d_\ell(t) = \max_{i\in N_\ell(t)} x_i(t) - \min_{j\in N_\ell(t)} x_j(t).$$

Thus, $d_\ell(t)$ measures the opinion spread that agent ℓ observes at time t. We have the following result.

Lemma 2: For any $t \geq 0$, we have

$$\sum_{i,j=1}^m H_{ij}(t)(x_i(t)-x_j(t))^2 \geq \frac{1}{4m}\sum_{\ell=1}^m \pi_\ell(t+1)\, d_\ell^2(t).$$

Proof: Since $H(t) = A'(t)\,\mathrm{diag}(\pi(t+1))A(t)$, it follows that $H_{ij}(t) = \sum_{\ell=1}^m \pi_\ell(t+1)A_{\ell i}(t)A_{\ell j}(t)$. Therefore, by exchanging the order of summation we obtain

$$\sum_{i,j=1}^m H_{ij}(t) = \sum_{\ell=1}^m \pi_\ell(t+1)\sum_{i,j=1}^m A_{\ell i}(t)A_{\ell j}(t).$$

Using the definition of the matrix $A(t)$, we obtain

$$A_{\ell i}(t)A_{\ell j}(t) = \begin{cases} \frac{1}{|N_\ell(t)|^2} & \text{if } i,j \in N_\ell(t),\\ 0 & \text{otherwise,}\end{cases}$$

thus implying

$$\begin{aligned}&\sum_{i,j=1}^m H_{ij}(t)(x_i(t)-x_j(t))^2\\ &\geq \sum_{\ell=1}^m \frac{\pi_\ell(t+1)}{|N_\ell(t)|^2}\sum_{i,j\in N_\ell(t)}(x_i(t)-x_j(t))^2. \end{aligned}\tag{9}$$

We next show that for all $t \geq 0$,

$$\sum_{i,j\in N_\ell(t)}(x_i(t)-x_j(t))^2 \geq \frac{1}{4}|N_\ell(t)|d_\ell^2(t).$$

If $d_\ell(t) = 0$, then the assertion follows immediately. So, suppose that $d_\ell(t) \neq 0$. Let us define $lb = \mathrm{argmin}\{x_i(t) \mid i \in N_\ell(t)\}$ and $ub = \mathrm{argmax}\{x_j(t) \in N_\ell(t)\}$. In other words, lb and ub are the agents with the smallest and largest opinion values that agent ℓ observes in his neighborhood.

Therefore, $d_\ell(t) = x_{ub}(t) - x_{lb}(t)$ and since $d_\ell(t) \neq 0$, we have $lb \neq ub$. Noting that $\sum_{i,j\in N_\ell(t)}$ denotes the double summation $\sum_{i\in N_\ell(t)} \sum_{j\in N_\ell(t)}$, we have

$$\begin{aligned}
\sum_{i,j\in N_\ell(t)} (x_i(t) - x_j(t))^2 &\geq \sum_{j\in N_\ell(t)} (x_{ub}(t) - x_j(t))^2 \\
&+ \sum_{i\in N_\ell(t)} (x_i(t) - x_{lb}(t))^2 \\
&= \sum_{j\in N_\ell(t)} \left\{(x_{ub}(t) - x_j(t))^2 + (x_j(t) - x_{lb}(t))^2\right\} \\
&\geq \sum_{j\in N_\ell(t)} \frac{1}{4}(x_{ub}(t) - x_{lb}(t))^2 = \frac{1}{4}|N_\ell(t)| d_\ell^2(t). \qquad (10)
\end{aligned}$$

The last inequality follows from

$$(x_{ub}(t) - x_j(t))^2 + (x_j(t) - x_{lb}(t))^2 \geq \frac{1}{4}(x_{ub}(t) - x_{lb}(t))^2,$$

which holds since the function $s \to (a-s)^2 + (s-b)^2$ attains its minimum at $s = \frac{a+b}{2}$. By combining relations (9) and (10), we obtain

$$\sum_{i,j=1}^{m} H_{ij}(t)(x_i(t) - x_j(t))^2 \geq \frac{1}{4} \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|N_\ell(t)|} d_\ell^2(t).$$

The desired relation follows by using $|N_i(t)| \leq m$ for all t. ■

Now, we estimate the sum $\sum_{\ell=1}^m \pi_\ell(t+1)\, d_\ell^2(t)$.

Lemma 3: For all $0 \leq t < T-1$ we have

$$\sum_{\ell=1}^{m} \pi_\ell(t+1)\, d_\ell^2(t) \geq \frac{\epsilon^2}{4m}.$$

Proof: Let $\ell^* = \text{argmax}_{\ell\in[m]}\pi_\ell(t+1)$. Since, $\pi(t+1)$ is a stochastic vector, it follows that $\pi_{\ell^*}(t+1) \geq \frac{1}{m}$. Suppose that $d_{\ell^*}(t) > \frac{\epsilon}{2}$. In this case, it immediately follows

$$\sum_{\ell=1}^{m} \pi_\ell(t+1)\, d_\ell^2(t) \geq \pi_{\ell^*}(t+1)\, d_{\ell^*}^2(t) > \frac{1}{m}\frac{\epsilon^2}{4},$$

thus showing the desired relation.

Suppose now that $d_{\ell^*}(t) \leq \frac{\epsilon}{2}$. Since $t < T-1$, by Lemma 1, there exists $j^* \in N_{\ell^*}(t)$ such that

$$\pi_{j^*}(t+1) \geq \frac{1}{2}\pi_{\ell^*}(t+1) \geq \frac{1}{2m}.$$

Moreover, $N_{j^*}(t) \neq N_{\ell^*}(t)$. Next, we prove that $d_{j^*}(t) \geq \epsilon$. Since $d_{\ell^*}(t) \leq \frac{\epsilon}{2}$ and $j^* \in N_{\ell^*}(t)$, it follows that $|x_{\ell^*}(t) - x_{j^*}(t)| \leq \frac{\epsilon}{2}$. Also, for any $i \in N_{\ell^*}(t)$, we have:

$$|x_i(t) - x_{j^*}(t)| \leq |x_i(t) - x_{\ell^*}(t)| + |x_{\ell^*}(t) - x_{j^*}(t)| \leq \epsilon,$$

implying $i \in N_{j^*}(t)$. Thus, $N_{\ell^*}(t) \subset N_{j^*}(t)$. Let $i \in N_{j^*}(t) \setminus N_{\ell^*}(t)$. Then, $|x_i(t) - x_{\ell^*}(t)| > \epsilon$, and hence, $d_{j^*}(t) > \epsilon$. Therefore, we obtain

$$\sum_{\ell=1}^{m} \pi_\ell(t+1)\, d_\ell^2(t) \geq \pi_j^*(t+1)\, d_{j^*}^2(t) \geq \frac{1}{2m}\, \epsilon^2.$$

All in all, in any case for $t < T-1$, we have

$$\sum_{\ell=1}^{m} \pi_\ell(t+1)\, d_\ell^2(t) \geq \frac{\epsilon^2}{4m}.$$

■

We are now ready to provide our main result, which yields an upper bound for the termination time T.

Theorem 2: For the Hegselmann-Krause dynamics we have

$$T - 2 \leq \frac{32m^2}{\epsilon^2}\, V_\pi(x(0), 0).$$

Proof: From Lemma 2 and relation (8), with $t = T-2$, we obtain

$$\begin{aligned}
V_\pi(x(T-1), T-1) &= V_\pi(x(0), 0) \\
&- \frac{1}{2}\sum_{t=0}^{T-2}\sum_{i,j=1}^{m} H_{ij}(t)(x_i(t) - x_j(t))^2.
\end{aligned}$$

By using Lemma 2, we further obtain

$$V_\pi(x(T-1), T-1) = V_\pi(x(0), 0) - \frac{1}{8m}\sum_{t=0}^{T-2}\sum_{\ell=1}^{m}\pi_\ell(t+1)d_\ell^2(t).$$

Finally, we use Lemma 3 to arrive at

$$V_\pi(x(T-1), T-1) = V_\pi(x(0), 0) - \frac{\epsilon^2}{32m^2}\sum_{t=0}^{T-2} 1.$$

Therefore,

$$(T-2)\frac{\epsilon^2}{32m^2} \leq V_\pi(x(0), 0),$$

and the desired relation follows. ■

The bound of Theorem 2 shows that the convergence time is of the order of m^2, provided that the value $V_\pi(x(0), 0)$ does not depend on m. One may further analyze the bound to take $V_\pi(x(0), 0)$ into account. We do so by taking the worst case into consideration.

By further bounding $V_\pi(x(0), 0)$, in the worst case, we have

$$\begin{aligned}
V_\pi(x(0), 0) &= \sum_{i=1}^{m} \pi_i(0)\left(x_i(0) - \pi'(0)x(0)\right)^2 \\
&\leq \sum_{i=1}^{m} \pi_i(0)\left(x_i(0) - \min_{j\in[m]} x_j(0)\right)^2 \\
&\leq d^2(x(0))\sum_{i=1}^{m}\sum_{i=1}^{m}\pi_i(0),
\end{aligned}$$

where $d(x(0)) = \max_{i\in[m]} x_i(0) - \min_{j\in[m]} x_j(0)$. Since $\pi(0)$ is a stochastic vector, we obtain

$$V_\pi(x(0), 0) \leq d^2(x(0)) \leq m^2\epsilon^2.$$

Thus, as the worst case bound, we have the following result

$$T - 2 \leq 32m^4.$$

Thus, $32m^4$ is an upper estimate for the termination time of the Hegselmann-Krause opinion dynamics. The bound is of the order m^4, which is by the factor of m better than the previously known bound of [3].

V. DISCUSSION

In this paper, we have considered the Hegselmann-Krause model for opinion dynamic. By choosing an appropriate adjoint dynamics and Lyapunov comparison function, we have established an upper bound for the termination time of the dynamics. For a collection of m agents, our bound is of the order m^2 under some assumptions, while it may be of the order of m^4 in worst case. In the worst case, our bound improves the previously known bound by a factor of m. It however, remains to explore the tightness of these bounds.

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