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Stability of Discrete-Time Altafini's Model: A Graphical Approach*

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***Abstract*— This paper considers the discrete-time version of Altafini's model for opinion dynamics in which the interaction among a group of agents is described by a time-varying signed digraph. Prompted by an idea from [1], stability of the system is studied using a graphical approach. Necessary and sufficient conditions for exponential stability with respect to each possible type of limit states are provided. Specifically, under appropriate assumptions, it is shown that (1) a certain type of two-clustering will be reached exponentially fast for almost all initial conditions if, and only if, the sequence of signed digraphs is repeatedly jointly structurally balanced corresponding to that type of two-clustering; (2) the system will converge to zero exponentially fast for all initial conditions if, and only if, the sequence of signed digraphs is repeatedly jointly structurally unbalanced.**

I. INTRODUCTION

Over the past decade, there has been considerable interest in developing algorithms intended to cause a group of multiple agents to reach a consensus in a distributed manner [2]–[10]. Consensus processes have a long history in social science and are closely related to opinion dynamics [11]. Probably the most well-known opinion dynamics is the DeGroot model which is a linear discrete-time consensus process [12]. Recently, quite a few models have been proposed for opinion dynamics, including the Friedkin-Johnsen model [13], [14], the Krause model [15], [16], and the DeGroot-Friedkin model [17], [18]. A particularly interesting opinion dynamics model, which was first proposed by Altafini [19] and can be viewed as a more general linear consensus model, has received increasing attention lately [1], [20]–[24].

The Altafini model deals with a network of $n > 1$ agents and the constraint that each agent is able to receive information only from its "neighbors". Unlike the existing models for opinion dynamics or consensus, the neighbor relationships among the agents are described by a time-dependent, *signed*, digraph (or directed graph) in which vertices correspond to agents, arcs (or directed edges) indicate the directions of information flow, and, in particular, the signs represent the social relationships between neighboring agents in that a positive sign means friendship (or cooperation) and a negative sign indicates antagonism (or competition). Each agent i has control over a time-dependent state variable $x_i(t)$ taking values in $\mathbb{R}$, which denotes its opinion on some issue. Each agent updates its opinion based on its own current opinion, the current opinions of its current neighbors, and its relationships (friendship or antagonism) with respect to its current neighbors. Specifically, for those neighbors with friendship, the agent will trust their opinions; for those neighbors with antagonism, the agent will not trust their opinions and, instead, the agent will take the opposite of their opinions in updating, which is the key difference between the Altafini model and other opinion dynamics models.

The continuous-time Altafini model has been studied in [19], [21], [22], and papers [1], [23], [24] have studied the discrete-time counterpart. This paper will focus on the latter.

The most general result in the literature regarding the discrete-time version of the Altafini model is that for any "repeated jointly strongly connected" sequence of signed digraphs, the absolute values of all the agents' opinions will asymptotically reach a consensus, which is called *modulus consensus*, having standard consensus and bipartite consensus (or biclustering) as special cases [24]. The result is independent of the structure of signs in the digraphs which can be described by the term structural balance (or structural unbalance) from social science [25] in that different types of structural balance correspond to different clusterings of opinions in the network (Section III-A). Notwithstanding this, the following questions remain. What are necessary and sufficient conditions on the sequence of signed digraphs that will lead to a specific clustering? When will the convergence be exponentially fast? This paper aims to answer these questions and will appeal to a graphical approach prompted by an idea introduced in [1], as further discussed below.

In the recent work by Hendrickx [1], a very nice lifting approach was proposed to establish the equivalence between the Altafini model and an enlarged class of classical opinion dynamics (or consensus) models with a special structure; with this equivalence relationship, the convergence results of the Altafini model were extended under a "type-symmetry" assumption (i.e., the interaction and signs between each pair of neighboring agents are symmetric). In the discussion section of [1], it was mentioned that this lifting approach might "prove harder to treat systems where the presence of interaction is symmetric but their signs are not". This is precisely what we consider in this paper and with more generality, we make use of the lifting approach and consider the most general case where both interaction and signs between each pair of neighboring agents can be asymmetric.

The main contributions of this paper are first, development of a graphical approach to the analysis of the discrete-time version of Altafini's model, and second, derivation of necessary and sufficient conditions for exponential stability (i.e., exponentially fast convergence) with respect to different limit states.

*Proofs of some assertions in this paper are omitted due to space limitations and will be included in an expanded version of the paper.

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A. Preliminaries

For any positive integer n, we use $[n]$ to denote the index set $\{1, 2, \ldots, n\}$. We view vectors as column vectors and write x' to denote the transpose of a vector x. For a vector x, we use x_i to denote the ith entry of x. For any real number y, we use $|y|$ to denote its absolute value. For any matrix $M \in \mathbb{R}^{n\times n}$, we use m_{ij} to denote its ijth entry and write $|M|$ to denote the matrix in $\mathbb{R}^{n\times n}$ whose ijth entry is $|m_{ij}|$. A nonnegative $n \times n$ matrix is called a stochastic matrix if its row sums are all equal to 1.

For a digraph $\mathbb{G}$, we use (i, j) to denote a directed edge (or an arc) from vertex i to vertex j. We say that $\mathbb{G}$ has an undirected edge between vertex i and vertex j, denoted by $[i, j]$, if either (i, j) or (j, i) is a directed edge in $\mathbb{G}$. A directed walk of $\mathbb{G}$ is a sequence of vertices $v_0, v_1, \ldots, v_m$ in $\mathbb{G}$ such that (v_{i-1}, v_i) is a directed edge in $\mathbb{G}$ for all $i \in [m]$. If the vertices v_0 and v_m are the same, then the walk is called closed. If all the vertices are distinct, then the directed walk is called a directed path. If all the vertices are distinct except that vertices v_0 and v_m are the same, then the directed walk is called a directed cycle.

We write $\mathcal{G}_{sa}$ to denote the set of all digraphs with n vertices, which have self-arcs at all vertices. The graph of an $n \times n$ matrix M with nonnegative entries is an n vertex directed graph $\gamma(M)$ defined so that (i, j) is an arc from i to j in the graph only when the jith entry of M is nonzero. Such a graph will be in $\mathcal{G}_{sa}$ if and only if all diagonal entries of M are positive. For purposes of analysis, we write $\bar{\mathcal{G}}_{sa}$ to denote the set of all digraphs with $2n$ vertices which have self-arcs at all vertices.

A digraph $\mathbb{G}$ is *strongly connected* if there is a directed path between each pair of distinct vertices. A digraph $\mathbb{G}$ is *weakly connected* if there is an undirected path between each pair of distinct vertices. Note that every strongly connected graph is weakly connected. The converse statement, however, is false. We say that a finite sequence of digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_p$ with the same vertex set is *jointly strongly connected* if the union of the graphs in this sequence is strongly connected. We say that an infinite sequence of digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots$ with the same vertex set is *repeatedly jointly strongly connected* if there is a positive integer p for which each finite sequence $\mathbb{G}_{kp+1}, \mathbb{G}_{kp+2}, \ldots, \mathbb{G}_{(k+1)p}$, $k \geq 0$, is jointly strongly connected.

II. The Discrete-Time Altafini's Model

Consider a network of $n > 1$ agents labeled 1 to n. Each agent i has control over a real-valued scalar $x_i(t)$ which the agent is able to update from time to time. Each agent i can receive information only from certain other agents called agent i's *neighbors*. Neighbor relationships among the n agents are described by a signed digraph $\mathbb{G}(t)$, called *neighbor graph*, on n vertices with an arc (or directed edge) from vertex i to vertex j whenever agent i is a neighbor of agent j at time t. For simplicity, we always take each agent as a neighbor of itself. Thus, each $\mathbb{G}(t)$ has self-arcs at all n vertices. Each arc is associated with a sign, either positive "+" or negative "–", which indicates that agent i regards agent j as a cooperative neighbor if arc (j, i) is associated with a "+" sign, or a competitive neighbor if (j, i) is associated with a "–" sign. It is natural to assume that each self-arc is associated with a "+" sign since an agent cannot compete with itself.

We are interested in the following discrete-time iterative update rule. At each time t and for each $i \in [n]$, agent i updates its opinion by setting

$$x_i(t+1) = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)x_j(t) \tag{1}$$

where $\mathcal{N}_i(t)$ denotes the set of neighbors of agent i at time t, and each $a_{ij}(t)$ is a real-valued weight whose sign is consistent with the sign of the arc (j, i). The weights $a_{ij}(t)$ are assumed to satisfy the following assumption.

Assumption 1: For each $i \in [n]$, there hold $a_{ii}(t) > 0$ and

$$\sum_{j=1}^{n} |a_{ij}(t)| = 1$$

for all time t. There exists a positive number $\beta > 0$ such that $|a_{ij}(t)| \geq \beta$ when $|a_{ij}(t)| > 0$ for all $i, j \in [n]$ and t.

The update rule (1) is an analog of the continuous-time update rule in [19]. The n update equations in (1) can be combined into one linear recursion equation

$$x(t+1) = A(t)x(t) \tag{2}$$

in which $x(t)$ is a vector in $\mathbb{R}^n$ whose ith entry is $x_i(t)$ and $A(t)$ is an $n \times n$ matrix whose ijth entry is $a_{ij}(t)$. From Assumption 1, it can be seen that the infinite norm of each $A(t)$ equals 1 and $|A(t)|$ is a stochastic matrix with positive diagonal entries. In the case when all the arcs have positive signs, the system becomes the standard linear consensus process which will reach a consensus exponentially fast if and only if $\mathbb{G}(t)$ is "repeatedly jointly rooted" [5], [10]. Thus, the system (2) can be regarded as a generalized model of standard linear consensus.

For each matrix $A(t)$ which satisfies Assumption 1, we define the graph of $A(t)$ to be a signed digraph so that (i, j) is an arc in the graph whenever $a_{ji}(t)$ is nonzero and the sign of (i, j) is the same as the sign of $a_{ji}(t)$. It is straightforward to verify that the graph of $A(t)$ is the same as the neighbor graph $\mathbb{G}(t)$. We will use this fact without special mention in the sequel.

III. Main Results

We say that the system (1) reaches a *modulus consensus* if the absolute values of all n agents $|x_i(t)|$, $i \in [n]$, converge to the same value as time $t \to \infty$. If, in addition, the limit value does not equal zero and the agents' limit values have opposite signs, the system (1) is said to reach a *bipartite consensus*. Clearly, consensus and bipartite consensus are the two special cases of modulus consensus.

The asymptotic behavior of the system (2) depends on the connectivity and sign structure of the sequence of neighbor graphs. The results in the literature appeal to the concept of "structural balance" from social science [25]. The idea of structurally balanced networks was first introduced to consensus problems in [19].

A. Structural Balance

A signed digraph $\mathbb{G}$ is called *structurally balanced* if the vertices of $\mathbb{G}$ can be partitioned into two sets such that each arc connecting two agents in the same set has a positive sign and each arc connecting two agents in different sets has a negative sign. Otherwise, the graph $\mathbb{G}$ is called *structurally unbalanced*. For our purposes, we extend the same concept of structural balance (and structural unbalance) to "signed multigraphs". A *signed multigraph* is a signed digraph which is allowed to have multiple directed edges with different sighs, i.e., for any ordered pair of two vertices i and j, there may be two directed edges from vertex i to vertex j, with positive and negative signs respectively.

An equivalent condition for checking structural balance is as follows. Let $\mathbb{G}$ be a signed digraph. For each directed (or undirected) cycle $\mathbb{C}$ in $\mathbb{G}$, we say that $\mathbb{C}$ is negative if it contains odd number of negative signs, and positive otherwise. It has been shown that a signed digraph, which can be a signed multigraph, is structurally balanced if and only if it does not have negative undirected cycles [19], [23], [25].

A simple example of a structurally unbalanced digraph is the signed digraph in which there exists one pair of agents i and j such that the arcs (i, j) and (j, i) have different signs. Another simple example is the signed multigraph in which there exist two arcs from vertex i to vertex j and they have different signs. In the both cases, the graph has an undirected cycle, consisting of the vertex sequence i, j, i, which is negative.

In the sequel, we will differentiate between different types of structurally balanced digraphs by introducing the concept of clustering. Our reason for doing this will be made clear shortly.

Let $\mathcal{I}$ be a set of vectors in $\mathbb{R}^n$ such that for each $b \in \mathcal{I}$, there hold $b_1 = 1$ and b_i equals either 1 or -1 for all $i \in [n]$ and $i \neq 1$. The set $\mathcal{I}$ is a finite set and $\mathbf{1} \in \mathcal{I}$ where $\mathbf{1}$ denotes the vector in $\mathbb{R}^n$ whose entries all equal 1. Each element b in $\mathcal{I}$ uniquely defines a *clustering* of all the agents in the network by the signs of the entries of b. Specifically, we use $\mathcal{V}_b^+$ to denote the set of indices in $[n]$ such that $b_i = 1$ for all $i \in \mathcal{V}_b^+$ and $\mathcal{V}_b^-$ to denote the set of indices in $[n]$ such that $b_i = -1$ for all $i \in \mathcal{V}_b^-$. It can be seen that $\mathcal{V}_b^+$ and $\mathcal{V}_b^-$ are disjoint and $\mathcal{V}_b^- \cup \mathcal{V}_b^+ = [n]$. Since $b_1 = 1$, it follows that $\mathcal{V}_b^+$ is nonempty. In the case when $\mathcal{V}_b^-$ is nonempty, the vector b defines a unique *biclustering* among the agents in the network. In the special case in which $\mathcal{V}_b^-$ is an empty set (i.e., $b = \mathbf{1}$), all the agents belong to the same cluster.

It is worth noting that each possible nonzero modulus consensus (i.e., all the absolute values of the opinions of the agents in the network reach a consensus at some nonzero value) can be uniquely represented by an element $b \in \mathcal{I}$ in that the agents, including agent 1, whose labels in $\mathcal{V}_b^+$ have the same sign, and the agents whose labels in $\mathcal{V}_b^-$ have the same sign. In particular, the vector $\mathbf{1}$ represents the standard consensus, and every other element in $\mathcal{I}$ represents a unique bipartite consensus. Thus, all possible nonzero limit states of modulus consensus can be partitioned into different types, each corresponding to a unique vector in $\mathcal{I}$.

Each element b in $\mathcal{I}$, as well as the unique associated clustering, also corresponds to a class of signed digraphs with n vertices. To be more precise, each element $b \in \mathcal{I}$ such that $b \neq \mathbf{1}$ corresponds to a class of structurally balanced digraphs, in which the arcs connecting two vertices in $\mathcal{V}_b^+$ (or $\mathcal{V}_b^-$) have positive signs and the arcs connecting one vertex in $\mathcal{V}_b^+$ and one vertex in $\mathcal{V}_b^-$ have negative signs, and the element $\mathbf{1}$ corresponds to the class of signed digraphs whose signs are all positive. We call each of the above classes of signed digraphs a *structurally balanced class* and denote it by $\mathcal{C}_b$, $b \in \mathcal{I}$. The remaining signed digraphs with n vertices are all structurally unbalanced and we call this class of graphs the *structurally unbalanced class*, denoted by $\mathcal{C}_u$. In general, a signed digraph $\mathbb{G}$ may belong to different classes. But in the case when $\mathbb{G}$ is weakly connected, it belongs to a unique class.

B. Stability

For each type of nonzero modulus consensus (i.e., the corresponding type of clustering) and the zero modulus consensus (i.e, all the opinions of the agents reach a consensus at zero), we will provide necessary and sufficient conditions under which the modulus consensus can be reached exponentially fast. To state our main results, we need the following concepts.

The union of two signed digraphs $\mathbb{G}_p$ and $\mathbb{G}_q$ with the same vertex set is the signed digraph with the same vertex set, and the signed arc set being the union of the signed arcs of the two digraphs. It is clear that the union can be a signed multigraph. Since we are interested in signed digraphs with positive self-arcs at all vertices, the signed digraph generated by the union operation will also have positive self-arcs at all vertices.

We say that a finite sequence of signed digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_p$ with the same vertex set is *jointly structurally balanced* with respect to a clustering $b \in \mathcal{I}$ (or *jointly structurally unbalanced*) if the union of the graphs in this sequence is structurally balanced with respect to the clustering b (or structurally unbalanced). We say that an infinite sequence of signed digraphs $\mathbb{G}_1, \mathbb{G}_2, \ldots$ with the same vertex set is *repeatedly jointly structurally balanced* with respect to a clustering $b \in \mathcal{I}$ (or *repeatedly jointly structurally unbalanced*) if there is a positive integer l for which each finite sequence $\mathbb{G}_{kl+1}, \mathbb{G}_{kl+2}, \ldots, \mathbb{G}_{(k+1)l}$, $k \geq 0$, is jointly structurally balanced with respect to the clustering b (or jointly structurally unbalanced).

The main results of this paper are then as follows. We begin with the cases of nonzero modulus consensus.

Theorem 1: Suppose that all n agents adhere to the update rule (1) and Assumption 1 holds. Suppose that the sequence of neighbor graphs $\mathbb{G}(1), \mathbb{G}(2), \ldots$ is repeatedly jointly strongly connected. Then, for each $b \in \mathcal{I}$, the system (2) reaches the corresponding nonzero modulus consensus exponentially fast for almost all initial conditions if, and only if, the graph sequence $\mathbb{G}(1), \mathbb{G}(2), \ldots$ is repeatedly jointly structurally balanced with respect to the clustering b.[1]

Remark 1: Suppose that the sequence of neighbor graphs $\mathbb{G}(1), \mathbb{G}(2), \ldots$ is repeatedly jointly strongly connected and structurally balanced with respect to a clustering $b \in \mathcal{I}$. Let B be the $n \times n$ diagonal matrix whose ith diagonal entry equals b_i for all $i \in [n]$. Note that $B^2 = I$ and for each $A(t)$, the matrix $BA(t)B'$ is a stochastic matrix. With these facts, it is straightforward to verify that the matrix product $A(t) \cdots A(2)A(1)$ converges to a rank one matrix of the form bc', where c is a nonzero vector, and thus $x(t)$ converges to $bc'x(1)$. It follows that the system (2) will reach the corresponding nonzero modulus consensus if $c'x(1)$ does not equal zero. Since the set of those vectors x which satisfy the equality $c'x = 0$ is a thin set, the nonzero modulus consensus will be reached for almost all initial conditions.

Remark 2: If a finite sequence of signed digraphs with the same vertex set is jointly structurally balanced with respect to a clustering $b \in \mathcal{I}$, then each graph in the sequence is structurally balanced with respect to the clustering b. Thus, from the definition of repeatedly jointly structurally balanced, the statement of Theorem 1 implies that if the system reaches a nonzero modulus consensus corresponding to a clustering b, then all those signed digraphs which do not belong to the structurally balanced class $\mathcal{C}_b$ can appear in the sequence of neighbor graphs for only a finite number of times.

The next theorem addresses the case of zero modulus consensus.

Theorem 2: Suppose that all n agents adhere to the update rule (1) and Assumption 1 holds. Suppose that the sequence of neighbor graphs $\mathbb{G}(1), \mathbb{G}(2), \ldots$ is repeatedly jointly strongly connected. Then, the system (2) converges to zero exponentially fast for all initial conditions if, and only if, the graph sequence $\mathbb{G}(1), \mathbb{G}(2), \ldots$ is repeatedly jointly structurally unbalanced.

Remark 3: It is well known that for discrete-time linear consensus processes, repeatedly jointly strong connectivity guarantees exponentially fast consensus [5], [7]. But this is not the case for modulus consensus. Examples can be generated to show that modulus consensus may be reached only asymptotically, but not exponentially fast.

IV. Analysis

In this section, we will present a graphical approach to analyze the discrete-time Altafini's model. As mentioned earlier, the approach we adopt is inspired by a novel idea from [1] which lifts the system to an enlarged system, as follows.

Define a time-dependent $2n$-dimensional vector $z(t)$ such that for each time t,

$$z(t) = \begin{bmatrix} x(t) \\ -x(t) \end{bmatrix}$$

Then, for all $i \in [2n]$,

$$z_i(t+1) = \sum_{j=1}^{2n} \bar{a}_{ij}(t) z_j(t)$$

in which

$$\begin{aligned} \bar{a}_{ij}(t) &= \bar{a}_{i+n,j+n}(t) &= \max\{0, a_{ij}(t)\} \\ \bar{a}_{i+n,j}(t) &= \bar{a}_{i,j+n}(t) &= \max\{0, -a_{ij}(t)\} \end{aligned}$$

It has been shown in [1] that the above enlarged system is equivalent to the discrete-time version of Altafini's model.

It is straightforward to verify that the enlarged system is a discrete-time linear consensus process in which the states are coupled. Thus, it can be written in the form of a state equation

$$z(t+1) = \bar{A}(t)z(t) \tag{3}$$

where each $\bar{A}(t) = [\bar{a}_{ij}(t)]$ is a $2n \times 2n$ stochastic matrix. With this fact, the graph of $\bar{A}(t)$ is an unsigned digraph with $2n$ vertices.

The graph of $\bar{A}(t)$ has the following properties whose proofs are straightforward and are thus omitted.

Lemma 1: For all $i, j \in [n]$, if $a_{ij}(t) > 0$, then the graph of $\bar{A}(t)$ has an arc from vertex j to vertex i and an arc from vertex $j+n$ to vertex $i+n$; if $a_{ij}(t) < 0$, then the graph of $\bar{A}(t)$ has an arc from vertex j to vertex $i+n$ and an arc from vertex $j+n$ to vertex i. In particular, the graph of $\bar{A}(t)$ has self-arcs at all $2n$ vertices.

Lemma 2: Suppose that the graph of $A(t)$ has a directed path from vertex i to vertex j with $i, j \in [n]$. Then, the graph of $\bar{A}(t)$ has a directed path from vertex i to vertex j or $j+n$. In particular, if the directed path from i to j in the graph of $A(t)$ is positive, then the graph of $\bar{A}(t)$ has a directed path from i to j; if the directed path from i to j in the graph of $A(t)$ is negative, then the graph of $\bar{A}(t)$ has a directed path from i to $j+n$.

Lemma 3: If the graph of $\bar{A}(t)$ has a directed path from vertex i to vertex j with $i, j \in [n]$, then it has a directed path from vertex $i+n$ to vertex $j+n$, and vice versa. If the graph of $\bar{A}(t)$ has a directed path from vertex i to vertex $j+n$ with $i, j \in [n]$, then it has a directed path from vertex $i+n$ to vertex j, and vice versa.

In the sequel, we will establish two key relations between the signed digraph of $A(t)$ and the enlarged unsigned digraph of $\bar{A}(t)$, which will play important roles in the proofs of the main results.

Proposition 1: Suppose that the graph of $A(t)$ is strongly connected and structurally balanced with respect to a clustering $b \in \mathcal{I}$. Then, the graph of $\bar{A}(t)$ consists of two disjoint strongly connected components of same size n. In particular, the first component consists of vertices i, $i \in \mathcal{V}_b^+$, and $j+n$, $j \in \mathcal{V}_b^-$, and the other consists of vertices i, $i \in \mathcal{V}_b^-$, and $j+n$, $j \in \mathcal{V}_b^+$.[2]

[1]We are indebted to Lili Wang (Department of Electrical Engineering, Yale University) for pointing out a flaw in the original version of this statement and suggesting how to fix it.

[2]Propositions 1 and 2 are equivalent to Lemmas 1 and 2 in the unpublished work [26]; we became aware of this reference after the first submission of this paper.

Proof: We first show that the subgraph induced by vertices i, $i \in \mathcal{V}_b^+$, and $j+n$, $j \in \mathcal{V}_b^-$ is strongly connected. From the definition of structural balance, the arcs connecting any two vertices in $\mathcal{V}_b^+$ (or $\mathcal{V}_b^-$) are all positive, and the arcs connecting one vertex in $\mathcal{V}_b^+$ and one vertex in $\mathcal{V}_b^-$ are all negative. Consider any two vertices i and j in $\mathcal{V}_b^+$. Since the graph of $A(t)$ is strongly connected, there exists a direct path from i to j. Since the graph of $A(t)$ is structurally balanced, the path must be positive. By Lemma 2, there exists a direct path from i to j. Next consider any two vertices i and j such that $i \in \mathcal{V}_b^+$ and $j \in \mathcal{V}_b^-$. Since the graph of $A(t)$ is strongly connected and structurally balanced, there must exist a negative directed path from i to j in the graph of $A(t)$. By Lemma 2, there exists a direct path from i to $j+n$. Thus, the subgraph induced by vertices i, $i \in \mathcal{V}_b^+$, and $j+n$, $j \in \mathcal{V}_b^-$ is strongly connected. Similarly, the subgraph induced by vertices i, $i \in \mathcal{V}_b^-$, and $j+n$, $j \in \mathcal{V}_b^+$ is also strongly connected.

Next we show that the two strongly connected components are disjoint. Let $\mathcal{V}_1$ be the vertex set consisting of vertices i, $i \in \mathcal{V}_b^+$, and $j+n$, $j \in \mathcal{V}_b^-$ and $\mathcal{V}_2$ be the vertex set consisting of vertices i, $i \in \mathcal{V}_b^-$, and $j+n$, $j \in \mathcal{V}_b^+$. Suppose that, to the contrary, there is an undirected path from $i_1 \in \mathcal{V}_1$ to $i_2 \in \mathcal{V}_2$ in the graph of $\bar{A}(t)$. Then, there must exist an arc in the graph of $\bar{A}(t)$ connecting one vertex in $\mathcal{V}_1$ and one vertex in $\mathcal{V}_2$. But, by Lemma 1, such an arc cannot exist in the graph of $\bar{A}(t)$ since any arc in the graph of $A(t)$ connecting one vertex in $\mathcal{V}_b^+$ and one vertex in $\mathcal{V}_b^-$ is negative. Therefore, there does not exist any undirected path between one vertex in $\mathcal{V}_1$ and another vertex in $\mathcal{V}_2$. ■

Proposition 2: Suppose that the graph of $A(t)$ is strongly connected and structurally unbalanced. Then, the graph of $\bar{A}(t)$ is strongly connected.

The proof of this proposition relies on the following results.

Lemma 4: Suppose that a signed digraph $\mathbb{G}$ is strongly connected and structurally unbalanced. Then, there exists a negative directed closed walk in $\mathbb{G}$.

Proof: Suppose that, to the contrary, there does not exist a negative directed walk in $\mathbb{G}$. Since $\mathbb{G}$ is strongly connected, there must exist an undirected cycle $\mathbb{C}$ in $\mathbb{G}$. Label the vertices of $\mathbb{C}$ as $0, 1, \ldots, m-1$, with $[0,1], [1,2], \ldots, [m-1, 0]$ the associated undirected edges. In the proof, we adopt the convention that if an integer i is not in the range $0, 1, \ldots, m-1$ but referring to a vertex of $\mathbb{C}$, then it refers to the vertex (i mod m).

By assumption, the subgraph $\mathbb{C}$ cannot be a directed cycle. Thus, there exists at least a vertex i of $\mathbb{C}$ such that

1) either i is a *source*, i.e., both $(i, i-1)$ and $(i, i+1)$ are arcs of $\mathbb{G}$,
2) or i is a *sink*, i.e., both $(i-1, i)$ and $(i+1, i)$ are arcs of $\mathbb{G}$.

Let $\mathcal{S}$ be the collection of all such vertices. Then, it should be clear that the cardinality of $\mathcal{S}$ is even. Let $i_1, i_2, \ldots, i_{2k}$ be the elements of $\mathcal{S}$, with

$$i_1 < i_2 < \cdots < i_{2k}$$

Without loss of generality, we assume that i_1 is a source. Then, $i_3, i_5, \ldots, i_{2k-1}$ are all sources while $i_2, i_4, \ldots, i_{2k}$ are all sinks. For each $j \in [k]$, let

$$p_j^+ := (i_{2j-1}, i_{2j-1}+1) \cdots (i_{2j}-1, i_{2j})$$

be a directed path of $\mathbb{G}$ on $\mathbb{C}$ from the source i_{2j-1} to the sink i_{2j}. Similarly, let

$$p_j^- := (i_{2j-1}, i_{2j-1}-1) \cdots (i_{2j-2}+1, i_{2j-2})$$

be a directed path of $\mathbb{G}$ on $\mathbb{C}$ from the source i_{2j-1} to the sink i_{2j-2}.

Now fix a $j \in [k]$ and choose a directed path q_j^+ of $\mathbb{G}$ from i_{2j-2} to i_{2j-1}. Such a path exists since $\mathbb{G}$ is strongly connected. Let $n(p_j^-)$ and $n(q_j^+)$ be the numbers of negative signs contained in paths p_j^- and q_j^+, respectively. Then, $n(p_j^-) \equiv n(q_j^+) \bmod 2$ since otherwise, by concatenating the two paths p_j^- and q_j^+, we have a negative directed closed walk. Note that this argument applies for all j. Now consider a directed closed walk by concatenating paths

$$w := q_1^+ p_1^+ q_3^+ p_3^+ \cdots q_{2k-1}^+ p_{2k-1}^+$$

Similarly, we let $n(p_j^+)$ be the number of negative signs contained in p_j^+. Then, by the previous arguments, we have

$$\begin{aligned}&\sum_{j=1}^{k} \left(n(p_{2j-1}^-) + n(p_{2j-1}^+) \right) \\ &\equiv \sum_{j=1}^{k} \left(n(q_{2j-1}^+) + n(p_{2j-1}^+) \right) \bmod 2\end{aligned}$$

On the other hand, the left hand side of the expression is the total number of negative signs in $\mathbb{C}$ which is an odd number. Thus, the right hand side of the expression is also an odd number. In other words, there exists a negative directed closed walk. ■

Following Lemma 4, we have the next result.

Corollary 1: Suppose that a signed digraph $\mathbb{G}$ is strongly connected and structurally unbalanced. Then, there exists a negative directed cycle in $\mathbb{G}$.

Proof: Let w be the negative closed walk in $\mathbb{G}$. Such a closed walk exists by Lemma 4. Choose a vertex in w as a start point and we label it as vertex i_1. Express w as

$$w = (i_1, i_2)(i_2, i_3) \cdots (i_n, i_1)$$

If w is itself a cycle, then the statement is true. Suppose not, then we choose the least integer number j such that

$$i_j = i_{j+k}$$

for some k. In other words, i_j is the first vertex in w (with respect to the start point) which appears at least twice in w. For this vertex i_j, we may choose the integer k to be the least positive number that $i_j = i_{j+k}$ holds. It should be clear that

$$(i_j, i_{j+1}) \cdots (i_{j+k-1}, i_{j+k})$$

is a directed cycle. If this cycle is negative, then the statement is true. Suppose not, we then remove this positive directed cycle out of the closed walk. Then, what remains is still a closed walk, denoted by

$$w' := (i_1, i_2) \cdots (i_{j-1}, i_j)(i_j, i_{j+k+1}) \cdots (i_n, i_1)$$

Moreover, this closed walk is also negative. For convenience, we call such an operation a *cycle reduction* of a closed walk. If w' is a directed cycle, then the statement is true. Suppose not, then we can apply the operation of cycle reduction on w'. Thus, we get a sequence of negative closed walks as

$$w \to w^{(1)} \to w^{(2)} \to \cdots$$

This sequence stops

1) either at certain step, the removed directed cycle is negative.
2) or there is an integer l such that $w^{(l)}$ is itself a negative directed cycle.

Then, in either of the two cases above, we have found a negative directed cycle. This completes the proof. ■

We are now in a position to prove Proposition 2.

Proof of Proposition 2: By Corollary 1, there exists a directed negative cycle in the graph of $A(t)$. For any pair of $i, j \in [n]$, since the graph of $A(t)$ is strongly connected, there must exist a directed path from vertex i to a vertex k on the cycle, and a directed path from vertex k to vertex j. Thus, there is a directed path from i to j, through k, in the graph of $A(t)$. By Lemma 2, there exists a directed path in the graph of $\bar{A}(t)$ from i to j or $j+n$, depending on the sign of the path from i to j in the graph of $A(t)$. Now consider the directed walk from i to j in the graph of $A(t)$ consisting of the above path and a complete round of the cycle. Since the cycle is negative, the walk has a different sign from the above path. Thus, there exist two directed paths in the graph of $\bar{A}(t)$ from i to j and to $j+n$. By Lemma 3, there exist directed paths in the graph of $\bar{A}(t)$ from $i+n$ to j and $j+n$. This completes the proof. ■

The results of Propositions 1 and 2 can be extended to the cases when a finite sequence of neighbor graphs is jointly strongly connected and structurally balanced (or unbalanced).

V. Conclusions

In this paper, the discrete-time version of Altafini's opinion dynamics model has been studied through a graphical approach. Necessary and sufficient conditions for exponential stability of the system with respect to different limit states have been established under appropriate assumptions. The time-varying case without the strong connectivity assumption, which was partially studied in [26], is a direction for future research.

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