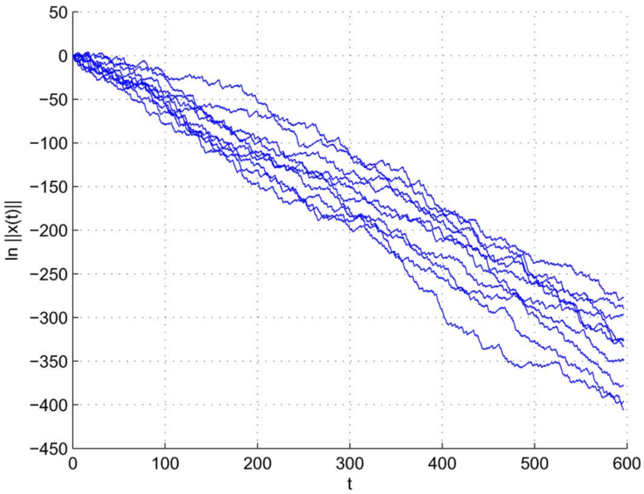

Fig. 4. Ten realizations of $\ln \|x(t)\|$ for the SD-MJLS of Example 2.

the impossibility of resorting to ergodic arguments, and, as such, is an open and challenging research topic.

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Convergence of Type-Symmetric and Cut-Balanced Consensus Seeking Systems

Julien M. Hendrickx and John N. Tsitsiklis

***Abstract*—We consider continuous-time consensus seeking systems whose time-dependent interactions are cut-balanced, in the following sense: if a group of agents influences the remaining ones, the former group is also influenced by the remaining ones by at least a proportional amount. Models involving symmetric interconnections and models in which a weighted average of the agent values is conserved are special cases. We prove that such systems always converge. We give a sufficient condition on the evolving interaction topology for the limit values of two agents to be the same. Conversely, we show that if our condition is not satisfied, then these limits are generically different. These results allow treating systems where the agent interactions are a priori unknown, being for example random or determined endogenously by the agent values.**

***Index Terms*—Multiagent systems, systems engineering and theory.**

I. INTRODUCTION

We consider continuous-time consensus seeking systems of the following kind: each of n agents, indexed by $i = 1, \ldots, n$, maintains a value $x_i(t)$, which is a continuous function of time and evolves according to the integral equation version of

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{n} a_{ij}(t)\left(x_j(t) - x_i(t)\right). \tag{1}$$

Throughout we assume that each $a_{ij}(\cdot)$ is a *nonnegative* and measurable function. We introduce the following assumption which plays a central role in this paper.

Assumption 1: (Cut-balance) There exists a constant $K \geq 1$ such that for all t, and any nonempty proper subset S of $\{1, \ldots, n\}$, we have[1]

$$K^{-1}\sum_{i\in S, j\notin S} a_{ji}(t) \leq \sum_{i\in S, j\notin S} a_{ij}(t) \leq K \sum_{i\in S, j\notin S} a_{ji}(t). \tag{2}$$

Intuitively, if a group of agents influences the remaining ones, the former group is also influenced by the remaining ones by at least a proportional amount. This condition may seem hard to verify in general. But, several important particular classes of consensus seeking systems automatically satisfy it. These include symmetric systems ($a_{ij}(t) = a_{ji}(t)$), type-symmetric systems ($a_{ij}(t) \leq K a_{ji}(t)$), and, as will be

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[1]Note that the second inequality, added to emphasize the symmetry of the condition, is redundant.

seen later, any system whose dynamics conserve a weighted average (with positive weights) of the agent values.

Under the cut-balance condition (2), and without any further assumptions, we prove that each value x_i converges to a limit. Moreover, we show that x_i and x_j converge to the same limit if i and j belong to the same connected component of the "unbounded interactions graph," i.e., the graph whose edges correspond to the pairs (j, i) for which $\int_0^\infty a_{ij}(\tau)\,d\tau$ is unbounded. (As we will show, while this is a directed graph, each of its weakly connected components is also strongly connected.) Conversely, we prove that x_i and x_j generically converge to different limits if i and j belong to different connected components of that graph. (This latter result involves an additional technical assumption that $\int_0^T a_{ij}(\tau)\,d\tau < \infty$ for every $T < \infty$.)

Our method of proof is different from traditional convergence proofs for consensus seeking systems, which rely on either span-norm or quadratic norm contraction properties. It consists of showing that for every $m \le n$, a particular linear combination of the values of the m agents with the smallest values is nondecreasing and bounded, and that its total increase rate eventually becomes bounded below by a positive number if two agents with unbounded interactions were to converge to different limits. The idea of these linear combinations is inspired from and extends a technique used in [3] for a particular average-preserving system. More specifically, [3] analyses in depth a model of opinion dynamics for which the order between the agents is preserved, where the coefficients switch between 0 and 1, are symmetric ($a_{ij} = a_{ji}$), and switch at most finitely often in any finite interval. Convergence is obtained by proving that the average value of the m first agents is nondecreasing and bounded, for any m.

Motivation for our model comes from the fact that there are many systems in which an agent cannot influence the others without being subjected to at least a fraction of the reverse influence. This is, for example, a common assumption in numerous models of social interactions and opinion dynamics [5], [11], or physical systems.

A. Background

Systems of the form (1) have attracted considerable attention [10], [16], [18], [19], [26] (see also [17], [20] for surveys), with motivation coming from decentralized coordination, data fusion [4], [27], animal flocking [6], [9], [25], and models of social behavior [2], [3], [5], [11].

Available results impose some connectivity conditions on the evolution of the coefficients $a_{ij}(t)$, and usually guarantee exponentially fast convergence of each agent's value to a common limit ("consensus"). For example, Olfati-Saber and Murray [18] consider the system

$$\frac{d}{dt}x_i(t) = \sum_{j:(j,i)\in E(t)} (x_j(t) - x_i(t))$$

with a time-varying directed graph $G(t) = (\{1, \ldots, n\}, E(t))$; this is a special case of the model (1), with $a_{ij}(t)$ equal to one if $(j, i) \in E(t)$, and equal to zero otherwise. They show that if the out-degree of every node is equal to its in-degree at all times, and if each graph $G(t)$ is strongly connected, then the system is average-preserving and each $x_i(t)$ converges exponentially fast to $(1/n)\sum_j x_j(0)$. They also obtain similar results for systems with arbitrary but fixed coefficients a_{ij}. Moreau [15] establishes exponential convergence to consensus under weaker conditions: he only assumes that the $a_{ij}(t)$ are uniformly bounded, and that there exist $T > 0$ and $\delta > 0$ such that the directed graph obtained by connecting i to j whenever $\int_t^{t+T} a_{ij}(\tau)\,d\tau > \delta$ has a rooted spanning tree, for every t. Several extensions of such results, involving for example time delays or imperfect communications, are also available.[2]

All of the above described results involve conditions that are easy to describe but difficult to ensure *a priori*, especially when the agent interactions are endogenously determined. This motivates the current work, which aims at an understanding of the convergence properties of the dynamical system (1) under minimal conditions. In the complete absence of any conditions, and especially in the absence of symmetry, it is well known that consensus seeking systems can fail to converge; see e.g., [1, Ch. 6]. On the other hand, it is also known that more predictable behavior and positive results are possible in the following two cases: (i) symmetric (suitably defined) interactions or (ii) average-preserving systems (e.g., in discrete-time models that involve doubly stochastic matrices).

[2]It is common in the literature to treat the system (1) as if the derivative existed for all t, which is not always the case. Nevertheless, such results remain correct under an appropriate reinterpretation of (1).

B. Our Contribution

Our cut-balance condition subsumes the two cases discussed above, and allows us to obtain strong convergence results. Indeed, we prove convergence (not necessarily to consensus) without any additional condition, and then provide sufficient and (generically) necessary conditions for the limit values of any two agents to agree. In contrast, existing results show *convergence to consensus* under some fairly strong assumptions about persistent global connectivity, but offer no insight on the possible behavior when convergence to consensus fails to hold.

The fact that our convergence result requires no assumptions other than the cut-balance condition is significant because it allows us to study systems for which the evolution of $a_{ij}(t)$ is *a priori* unknown, possibly random or dependent on the vector $x(t)$ itself. In the latter type of models, with endogenously determined agent interconnections, it is essentially impossible to check a priori the connectivity conditions imposed in existing results, and such results are therefore inapplicable. In contrast, our results apply as long as the cut-balance condition is satisfied. The advantage of this condition is that it can be often guaranteed *a priori*, e.g., if the system is naturally type-symmetric.

Applications of our result to systems with endogenously and randomly determined connectivity are presented in [7], [8], where we also analyze discrete-time cut-balanced systems. We also note that our work makes use of a technical result about the evolution of sorted vectors, which is omitted for space reasons, and is available in the extended version of this paper [8].

II. Main Convergence Result and Proof

We now state formally our main theorem, based on an integral formulation of the agent dynamics. The integral formulation avoids issues related to the existence of derivatives, while allowing for discontinuous coefficients $a_{ij}(t)$ and possible Zeno behaviors (i.e., a countable number of discontinuities in a finite time interval).

Without loss of generality, we assume that $a_{ii}(t) = 0$ for all t. We define a directed graph, $G = (\{1, \ldots, n\}, E)$, called the *unbounded interactions graph*, by letting $(j, i) \in E$ if $\int_0^\infty a_{ij}(t)\,dt = \infty$.

Lemma 1: Suppose that Assumption 1 (cut-balance) holds. Every weakly connected component of the unbouned interaction graph G is strongly connected. Equivalently, if there is a directed path from i to j, then there is also a directed path from j to i.

Proof: Consider a weakly connected component W of the graph G. We assume, in order to derive a contradiction, that W is not strongly connected. Consider the decomposition of W into strongly connected components. More precisely, we partition the nodes in W into two or more subsets $C_1, C_2, \ldots$, so that each C_k is strongly connected and so that any edge in W that leaves a strongly connected component leads to a component with a larger label: if $i \in C_k$, $(i, j) \in E$, and $j \notin C_k$, then $j \in C_l$ for some $l > k$. (This decomposition is unique up to certain permutations of the subset labels.) In particular, there is no edge from $C_2 \cup C_3 \cup \cdots$, that leads into C_1. Since W is weakly connected, there must therefore exist an edge from C_1 into $C_2 \cup C_3 \cup \cdots$. However, it is an immediate consequence of the cut-balance condition (applied to $S = C_1$) that if there is an edge that leaves C_1 there must also exist an edge that enters C_1. This is a contradiction, and the proof is complete. ■

The following assumption will be in effect in some of the results.

Assumption 2: (Boundedness) For every i and j, and every $T < \infty$, $\int_0^T a_{ij}(t)\,dt < \infty$.

We now state our main result.

Theorem 1: Suppose that Assumption 1 (cut-balance) holds. Let $x : \Re^+ \to \Re^n$ *be a solution to the system of integral equations*

$$x_i(t) = x_i(0) + \int_0^t \sum_{j=1}^n a_{ij}(\tau)\,(x_j(\tau) - x_i(\tau))\,d\tau \tag{3}$$

for $i = 1, \ldots, n$. *Then:*

(a) *The limit* $x_i^* = \lim_{t\to\infty} x_i(t)$ *exists, and* $x_i^* \in [\min_j x_j(0), \max_j x_j(0)]$, *for all* i.

(b) *For every* i *and* j, *we have* $\int_0^\infty a_{ij}(t)\,|x_j(t) - x_i(t)|\,dt < \infty$. *Furthermore, if* i *and* j *belong to the same connected component of* G, *then* $x_i^* = x_j^*$.

If, in addition, Assumption 2 (boundedness) holds, then:

(a) *If* i *and* j *are in different connected components of* G, *then* $x_i^* \neq x_j^*$, *unless* $x(0)$ *belongs to a particular* $n-1$ *dimensional subspace of* $\Re^n$, *determined by the functions* $a_{ij}(\cdot)$.

The proof relies on the following lemma on the rate of change of certain weighted sums of the components of $x_i(t)$, when the coefficients a_{ij} satisfy the cut-balance assumption. We say that a vector $y \in \Re^n$ is *sorted* if $y_1 \le y_2 \le \cdots \le y_n$.

Lemma 2: For $i, j = 1, \ldots, n$, $i \neq j$, *let* b_{ij} *be nonnegative coefficients that satisfy the cut-balance condition*

$$K^{-1}\sum_{i\in S}\sum_{j\notin S} b_{ji} \le \sum_{i\in S}\sum_{j\notin S} b_{ij} \le K\sum_{i\in S}\sum_{j\notin S} b_{ji},$$

for some $K \ge 1$, *and for every nonempty proper subset* S *of* $\{1, \ldots, n\}$. *Then*

$$\sum_{i=1}^m K^{-i}\left(\sum_{j=1}^n b_{ij}(y_j - y_i)\right) \ge 0$$

for every sorted vector $y \in \Re^n$, *and every* $m \le n$.

Proof: We prove the stronger result that $\sum_{i=1}^n w_i\left(\sum_{j=1}^n b_{ij}(y_j - y_i)\right) \ge 0$ for any nonnegative weights w_i such that $w_i \ge K w_{i+1}$ for $i = 1, \ldots, n-1$. Observe that the expression above can be rewritten as

$$\sum_{i=1}^n y_i\left(\sum_{j=1}^n w_j b_{ji} - \sum_{j=1}^n w_i b_{ij}\right) = \sum_{i=1}^n y_i q_i$$

where the last equality serves as the definition of q_i. We observe that

$$\sum_{i=1}^n y_i q_i = y_1\sum_{i=1}^n q_i + \sum_{k=1}^{n-1}\left((y_{k+1} - y_k)\sum_{i=k+1}^n q_i\right).$$

It follows that the desired inequality $\sum_{i=1}^n y_i q_i \ge 0$ holds for every sorted vector y if and only if (i) $\sum_{i=1}^n q_i = 0$ and (ii) $\sum_{i=k+1}^n q_i \ge 0$, for $k = 1, \ldots, n-1$. We have

$$\sum_{i=1}^n q_i = \sum_{i=1}^n\left(\sum_{j=1}^n w_j b_{ji} - \sum_{j=1}^n w_i b_{ij}\right) = \sum_{i=1}^n\sum_{j=1}^n w_j b_{ji} - \sum_{j=1}^n\sum_{i=1}^n w_i b_{ij} = 0$$

which establishes property (i). To establish property (ii), we observe that

$$\sum_{i=k+1}^n\left(\sum_{j=1}^n w_j b_{ji} - \sum_{j=1}^n w_i b_{ij}\right) = \sum_{i=k+1}^n\sum_{j=1}^k w_j b_{ji} + \sum_{i=k+1}^n\sum_{j=k+1}^n w_j b_{ji} - \sum_{i=k+1}^n\sum_{j=1}^k w_i b_{ij} - \sum_{i=k+1}^n\sum_{j=k+1}^n w_i b_{ij}.$$

The second and the fourth terms cancel each other. We use the inequality $w_j \ge w_k$ for $j \le k$ in the first term, and the inequality $w_i \le w_{k+1}$ for $i \ge k+1$ in the third term, to obtain

$$\sum_{i=k+1}^n q_i = \sum_{i=k+1}^n\left(\sum_{j=1}^n w_j b_{ji} - \sum_{j=1}^n w_i b_{ij}\right) \ge w_k\sum_{i=k+1}^n\sum_{j=1}^k b_{ji} - w_{k+1}\sum_{i=k+1}^n\sum_{j=1}^k b_{ij}.$$

Using the property $w_k \ge K w_{k+1}$, and the cut-balance assumption, we conclude that the right-hand side in the above inequality is nonnegative, which completes the proof of property (ii). We now let $w_i = K^{-i}$, for $i = 1, \ldots, m$, and $w_i = 0$ for $i > m$, to obtain the desired result. ■

Proof (of Theorem 1): For every t, we define a permutation $p(t)$ of the indices $\{1, \ldots, n\}$ which sorts the components of the vector $x(t)$. (More precisely, it sorts the pairs $(x_i(t), i)$ in lexicographic order.) In particular, $p_i(t)$ is the index of the ith smallest component of $x(t)$, with ties broken according to the original indices of the components of $x(t)$. Formally, if $i<j$, then either $x_{p_i(t)}(t) < x_{p_j(t)}(t)$ or $x_{p_i(t)}(t) = x_{p_j(t)}(t)$ and $p_i(t) < p_j(t)$. For every i, we then let $y_i(t) = x_{p_i(t)}(t)$. The vector $y(t)$ is thus sorted, so that $y_i(t) \le y_j(t)$ for $i<j$. Let also $b_{ij}(t) = a_{p_i(t)p_j(t)}(t)$. (This coefficient captures an interaction between the ith smallest and the jth smallest component of $x(t)$.) As proved in [8, Appendix], $y(t)$ satisfies an equation of the same form as (3)

$$y_i(t) = y_i(0) + \int_0^t \sum_{j=1}^n b_{ij}(\tau)\,(y_j(\tau) - y_i(\tau))\,d\tau \tag{4}$$

for $i = 1, \ldots, n$. The definition of the functions b_{ij} implies that they also satisfy the cut-balance condition (2). We now define $S_m(t)$ to be a weighted sum of the values of the first m (sorted) agents

$$S_m(t) = \sum_{i=1}^m K^{-i} y_i(t) = S_m(0) + \int_0^t \sum_{i=1}^m K^{-i}\sum_{j=1}^n b_{ij}(\tau)\,(y_j(\tau) - y_i(\tau))\,d\tau.$$

It follows from Lemma 2, that the integrand is always nonnegative, so that $S_m(t)$ is nondecreasing. Moreover, since all $b_{ij}(t)$ are nonnegative, (4) can be used to show that $y_1(0) \le y_i(t) \le y_n(0)$, for all i and t. In particular, each $S_m(t)$ is bounded above and therefore converges. This implies, using induction on i, that every $y_i(t)$ converges to a limit $y_i^* = \lim_{t\to\infty} y_i(t) \in [y_1(0), y_n(0)] = [\min_j x_j(0), \max_j x_j(0)]$. Using the continuity of x and the definition of y, it is then an easy exercise to show that each $x_i(t)$ must also converge to one of the values y_j^*. This concludes the proof of part (a) of the theorem.

We now prove part (b). For every $m = 1, \ldots, n$, since $S_m(t)$ converges to some S_m^*, we have

$$\int_0^\infty \sum_{i=1}^m K^{-i}\sum_{j=1}^n b_{ij}(t)\,(y_j(t) - y_i(t))\,dt = S_m^* - S_m(0) < \infty. \tag{5}$$

The integrand in this expression can be rewritten as

$$\sum_{i=1}^m K^{-i}\left(\sum_{j=1}^m b_{ij}(t)\,(y_j(t) - y_i(t)) + \sum_{j=m+1}^n b_{ij}(t)\,(y_m(t) - y_i(t))\right) + \sum_{i=1}^m K^{-i}\sum_{j=m+1}^n b_{ij}(t)\,(y_j(t) - y_m(t)). \tag{6}$$

It follows from Lemma 2 applied to the coefficients b_{ij} and the sorted vector $(y_1(t), y_2(t), \ldots, y_m(t),\ y_m(t), \ldots, y_m(t))$ that the first term in the sum above is nonnegative, and thus that

$$\sum_{i=1}^m K^{-i}\sum_{j=m+1}^n b_{ij}(t)\,(y_j(t) - y_m(t)) \le \sum_{i=1}^m K^{-i}\sum_{j=1}^n b_{ij}(t)\,(y_j(t) - y_i(t)).$$

Equation (5) then implies that

$$\int_0^\infty \sum_{i=1}^m\sum_{j=m+1}^n K^{-i} b_{ij}(t)\,(y_j(t) - y_m(t))\,dt < \infty.$$

Since $K^{-i}b_{ij}(t) \geq 0$ and $y_j(t) \geq y_m(t)$ for $j > m$, every term of the sum in the integrand above is nonnegative, for every t. Then, the boundedness of the integral implies the boundedness of every $\int_0^\infty b_{ij}(t)\,(y_j(t) - y_m(t))\,dt = \int_0^\infty b_{ij}(t)\,|y_j(t) - y_m(t)|\,dt$ when $m < j$. A symmetrical argument shows that $\int_0^\infty b_{ij}(t)\,|y_j(t) - y_m(t)|\,dt$ is also bounded when $j < m$, so that

$$\int_0^\infty \sum_{i=1}^n \sum_{j=1}^n b_{ij}(t)\,|y_j(t) - y_i(t)|\,dt < \infty. \tag{7}$$

Because of the definitions $y_i(\tau) = x_{p_i(\tau)}(\tau)$ and $b_{ij}(\tau) = a_{p_i(\tau)p_j(\tau)}(\tau)$, and the fact that $p(t)$ is a permutation, the equality

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij}(t)\,|x_j(t) - x_i(t)| = \sum_{i=1}^n \sum_{j=1}^n b_{ij}(t)\,|y_j(t) - y_i(t)|$$

holds for all t, which together with the nonnegativity of all $a_{ij}(t)\,|x_j(t) - x_i(t)|$ and (7) implies that $\int_0^\infty a_{ij}(t)\,|x_j(t) - x_i(t)|\,dt < \infty$ for all i, j.

Suppose now that the edge (j, i) is in the graph G, i.e., that $\int_0^\infty a_{ij}(t)\,dt = \infty$. From part (a), we know that $|x_i(t) - x_j(t)|$ converges to a constant value for every i, j. Assumption 2 ($\int_0^{t'} a_{ij}(t)\,dt < \infty$ for all t') and the fact that $\int_0^\infty a_{ij}(t)\,|x_j(t) - x_i(t)|\,dt < \infty$ imply that the value to which $|x_i(t) - x_j(t)|$ converges must be 0, and thus that $x_i^* = x_j^*$. If i and j are not directly connected, i.e., (j, i) is not and edge in G, but belong to the same connected component of G, the equality $x_i^* = x_j^*$ follows by using transitivity along a path from i to j.

It remains to prove part (c). Consider a partition of the agents in two groups, V_1 and V_2, that are disconnected in G, i.e., $\lim_{t\to\infty} \int_0^t a_{ij}(\tau)\,d\tau < \infty$ for all $i \in V_1$ and $j \in V_2$, and also for all $i \in V_2$ and $j \in V_1$. Thus, there exists some $t_{1/4}$ such that $\int_{t_{1/4}}^\infty \sum_{i\in V_1, j\in V_2} (a_{ij}(\tau) + a_{ji}(\tau))\,d\tau < (1/4)$. We will first show that there exists a full-dimensional set of initial vectors $x(0)$ for which $\lim_{t\to\infty} x_i(t) \neq \lim_{t\to\infty} x_j(t)$, when $i \in V_1$ and $j \in V_2$.

Since $\int_0^t a_{ij}(\tau)\,d\tau < \infty$, it can be proved that the system in (3) admits a unique solution and that the state transition (or fundamental) matrix, which maps the initial conditions $x(0)$ to $x(t)$, has full rank for any finite t; see [21], for example (specifically, Theorem 54, Proposition C3.8, appendix C3, and appendix C4). In particular, we can chose $x(0)$ such that $x_i(t_{1/4}) = 0$ if $i \in V_1$, and $x_i(t_{1/4}) = 1$ if $i \in V_2$. Let m be the number of agents in V_1, and let y be the sorted version of x as above. There holds $y_1(t_{1/4}) = \cdots = y_m(t_{1/4}) = 0$ and $y_{m+1}(t_{1/4}) = \cdots = y_n(t_{1/4}) = 1$.

Consider now a $t^* \geq t_{1/4}$ such that $y_m(t) < y_{m+1}(t)$ holds for all $t \in [t_{1/4}, t^*]$. The continuity of x and the definition of y implies that for all $t \in [t_{1/4}, t^*]$, we have $x_i(t) \leq y_m(t)$ for every $i \in V_1$, and $x_j(t) \geq y_{m+1}(t)$ for every $j \in V_2$. In the same time interval, the permutation $p(t)$ that maps the indices of $x(t)$ to the corresponding indices of $y(t)$ takes thus values smaller than or equal to m for indices $i \in V_1$ and larger than m for indices $j \in V_2$. As a result, $\sum_{i\leq m, j>m} (b_{ij}(t) + b_{ji}(t)) = \sum_{i\in V_1, j\in V_2} (a_{ij}(t) + a_{ji}(t))$ for all $t \in [t_{1/4}, t^*]$. The definition of $t_{1/4}$ and the nonnegativity of the $a_{ij}(t)$ imply that, for all t in that interval,

$$\begin{aligned}\int_{t_{1/4}}^t \sum_{i=1}^m \sum_{j=m+1}^n (b_{ij}(\tau) + b_{ji}(\tau))\,d\tau \\ = \int_{t_{1/4}}^t \sum_{i\in V_1, j\in V_2} (a_{ij}(\tau) + a_{ji}(\tau))\,d\tau < \frac{1}{4}.\end{aligned} \tag{8}$$

We now fix an arbitrary $t \in [t_{1/4}, t^*]$, and show that $y_m(t)$ and $y_{m+1}(t)$ remain separated by at least $1/2$. Using (4) and $y_m(t_{1/4}) = 0$, we see that

$$\begin{aligned}y_m(t) = &\int_{t_{1/4}}^t \sum_{j=1}^m b_{mj}(\tau)\,(y_j(\tau) - y_m(\tau))\,d\tau \\ &+ \int_{t_{1/4}}^t \sum_{j=m+1}^n b_{mj}(\tau)\,(y_j(\tau) - y_m(\tau))\,d\tau.\end{aligned} \tag{9}$$

The first term is nonpositive by the definition of y. Consider now the second term. Since $y_i(t_{1/4}) \in [0, 1]$ for all i, we obtain $y_i(\tau) \in [0, 1]$ for all $t \geq t_{1/4}$, so that $y_j(\tau) - y_m(\tau) \leq 1$ for every j. Equations (9) and (8) then imply that

$$y_m(t) \leq \int_{t_{1/4}}^t \sum_{j=m+1}^n b_{mj}(\tau)\,d\tau < \frac{1}{4}$$

for every $t \in [t_{1/4}, t^*]$. A similar argument shows that $y_{m+1}(t) > 3/4$ for all $t \in [t_{1/4}, t^*]$. Recalling the definition of t^*, we have essentially proved that, after time $t_{1/4}$ and as long as $y_m(t) < y_{m+1}(t)$, we must have $y_{m+1}(t) - y_m(t) > 1/2$. Because $y(t)$ is continuous, it follows easily that the inequality $y_{m+1}(t) - y_m(t) > 1/2$ must hold for all times. Since we have seen that, for $t \in [t_{1/4}, t^*]$, there holds $x_j(t) > 3/4$ for all $j \in V_2$ and $x_i(t) < 1/4$ for all $i \in V_1$, this implies that $x_i^* = \lim_{t\to\infty} x_i(t) \leq 1/4$ for $i \in V_1$ and $x_j^* = \lim_{t\to\infty} x_j(t) \geq 3/4$ for $j \in V_2$.

Note that the function that maps the initial condition $x(0)$ to $x^* = \lim_{t\to\infty} x(t)$ is linear; let L be the matrix that represents this linear mapping. We use e_i to denote the ith unit vector in $\Re^n$. We have shown above that if i and j belong to different connected components of G, there exists at least one $x(0)$ for which $x_i^* - x_j^* = (e_i - e_j)^T Lx(0) \neq 0$. Therefore, $(e_i - e_j)^T L \neq 0$. In particular, the set of initial conditions $x(0)$ for which $(e_i - e_j)^T Lx(0) = 0$ is contained in an $n - 1$ dimensional subspace of $\Re^n$, which establishes part (c) of the theorem. ■

We note that Theorem 1 has an analog for the case where each agent's value $x_i(t)$ is actually a multi-dimensional vector, obtained by applying Theorem 1 separately to each component.

The key assumption in Theorem 1, which allows us to prove the convergence of the x_i, is that the aggregate influence of a group of agents on the others remains within a constant factor of the reverse aggregate influence. It is shown in [8] that the result does not hold if the bound K on the ratio is allowed to grow, even arbitarily slowly. On the other hand, our condition could be relaxed by allowing K to grow at a rate that depends on the integral of the $a_{ij}(t)$, as in [12].

III. Particular Cases of Cut-Balanced Dynamics

The cut-balance condition is a rather weak assumption, but may be hard to check. The next proposition provides five specific cases of cut-balanced systems that often arise naturally. It should however be understood that the class of cut-balanced systems is not restricted to these five particular cases.

Proposition 1: A collection of nonnegative coefficients $a_{ij}(\cdot)$ that satisfies any of the following five conditions also satisfies the cut-balance condition (Assumption 1).

(a) *Symmetry: $a_{ij}(t) = a_{ji}(t)$, for all i, j, t.*
(b) *Type-symmetry: There exists $K \geq 1$ such that $K^{-1}a_{ji}(t) \leq a_{ij}(t) \leq Ka_{ji}(t)$, for all i, j, t.*
(c) *Average-preserving dynamics: $\sum_j a_{ij}(t) = \sum_j a_{ji}(t)$, for all i, t.*
(d) *Weighted average-preserving dynamics: There exist $w_i > 0$ such that $\sum_j w_i a_{ij}(t) = \sum_j w_j a_{ji}(t)$, for all i, t.*
(e) *Bounded coefficients and set-symmetry: There exist M and α with $M \geq \alpha > 0$ such that for all i, j, t either $a_{ij}(t) = 0$ or $a_{ij}(t) \in [\alpha, M]$; and, for any subset S of $\{1, \ldots, n\}$, there exist $i \in S$ and $j \notin S$ with $a_{ij}(t) > 0$ if and only if there exist $i' \in S$ and $j' \notin S$ with $a_{j'i'}(t) > 0$.*

Proof: Condition (a) implies condition (b), with $K = 1$. If condition (b) holds, then by summing over all i in some set of nodes S and all $j \notin S$, we obtain the cut-balance condition.

Condition (c) implies condition (d), with $w_i = 1$. Suppose that condition (d) holds. Then

$$\begin{aligned}\sum_{i\in S, j\notin S} w_j a_{ji}(t) &= \sum_{i\in S} \sum_{j=1}^n w_j a_{ji}(t) - \sum_{i\in S, j\in S} w_j a_{ji}(t) \\ &= \sum_{i\in S} \sum_{j=1}^n w_i a_{ij}(t) - \sum_{i\in S, j\in S} w_i a_{ij}(t).\end{aligned}$$

It follows that $\sum_{i\in S,j\notin S} w_j a_{ji}(t) = \sum_{i\in S,j\notin S} w_i a_{ij}(t)$, and thus that

$$\sum_{i\in S,j\notin S} a_{ji}(t) \geq \frac{\min_i w_i}{\max_i w_i} \sum_{i\in S,j\notin S} a_{ij}(t).$$

A reverse inequality follows from a symmetrical argument. Therefore, the cut-balance condition holds with $K = \max_i w_i / \min_i w_i$, which is no less than 1 because $w_i > 0$ for all i.

Finally, suppose that the condition (e) is satisfied, and consider a set S and a time t. If $a_{ij}(t) = 0$ for all $i \in S$ and $j \notin S$, then (e) implies that $a_{ji}(t) = 0$ for all $i \in S$ and $j \notin S$ so that $\sum_{i\in S,j\notin S} a_{ij}(t) = 0 = \sum_{i\in S,j\notin S} a_{ji}(t)$, and the cut-balance condition is trivially satisfied for that set S and any K. If on the other hand there exists $i \in S, j \notin S$ for which $a_{ij}(t) > 0$, then (e) implies the existence of $i' \in S, j' \notin S$ such that $a_{j'i'}(t) > 0$, and $a_{j'i'}, a_{ij} \in [\alpha, M]$. Let $|S|$ be the cardinality of S. Then

$$|S|(n-|S|)M \geq \sum_{i\in S,j\notin S} a_{ij}(t) \geq \alpha$$
$$\text{and} \quad |S|(n-|S|)M \geq \sum_{i\in S,j\notin S} a_{ji}(t) \geq \alpha$$

so that the cut balance condition holds with $K = \max_S |S|(n-|S|)M/\alpha \leq n^2 M/4\alpha$. ■

Note that condition (d) remains sufficient for cut-balance if the weights w_i change with time, provided that the ratio $(\max_i w_i(t))/(\min_i w_i(t))$ remains uniformly bounded (the same proof applies). We also note that the connectivity condition in (e) is equivalent to requiring every weakly connected component to be strongly connected in the graph obtained by connecting (j, i) if $a_{ij}(t) > 0$, for every t.

IV. Extensions

The results in this paper can be extended or applied in various ways; see [7] and [8] for the details.

(a) **Discrete-time systems.** A cut-balance condition can be proved to imply convergence results (similar to Theorem 1) for discrete-time consensus seeking systems. A special case of this result asserts the unconditional convergence of systems that preserve some weighted average of the states, and thus includes a sample path version of recent results of [23] on stochastic consensus-seeking systems.

(b) **Random interactions.** Convergence for various models of random interactions, as in [13], [14], [22] is a simple consequence of deterministic convergence results: Theorem 1 can be directly applied to systems where the coefficients $a_{ij}(\cdot)$ are modeled as random processes whose sample paths satisfy the cut-balance condition with probability 1 (possibly with a different constant K for different sample paths). Indeed, if this is the case, Theorem 1 implies that each $x_i(t)$ converges, with probability 1. Furthermore, if $\mathrm{P}(\int_0^\infty a_{ij}(t)\,dt = \infty) = 1$, then $x_i^* = x_j^*$, with probability 1.

(c) **Endogenous connectivity.** Parts (a) and (b) of Theorem 1 remain valid, under the cut-balance condition, even if the connectivity coefficients $a_{ij}(t)$ are determined endogenously, as functions of the current state vector $x(t)$.

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