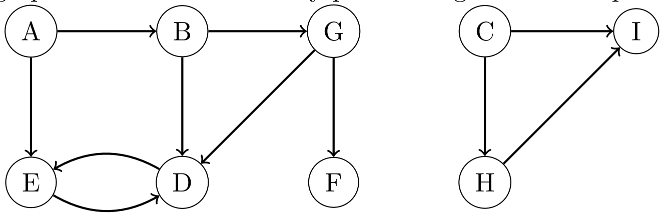
Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Graph Basics

For part (a), refer to the figure below. For parts (b) and (c), please prove only for simple graphs; that is, graphs that do not have any parallel edges or self-loops.



(a) Run DFS at node A, trying to visit nodes alphabetically (e.g. given a choice between nodes D and F, visit D first).

- List the nodes in the order you visit them (so each node should appear in the ordering exactly once).
- List each node with its pre- and post-number. The numbering starts from 1 and ends at 18.
- Label each edge as **T**ree, **B**ack, **F**orward or **C**ross.

(b) Let $|E|$ be the number of edges in a simple graph and $|V|$ be the number of vertices. Show that $|E|$ is in $O(|V|^2)$.

(c) For each vertex v_i, let d_i be the *degree*- the number of edges incident to it. Show that $\sum d_i$ must be even.

2 Short Answer

For each of the following, either prove the statement is true or give a counterexample to show it is false.

(a) If (u, v) is an edge in an undirected graph and during DFS, $\text{post}(v) < \text{post}(u)$, then u is an ancestor of v in the DFS tree.

(b) In a directed graph, if there is a path from u to v and $\text{pre}(u) < \text{pre}(v)$ then u is an ancestor of v in the DFS tree.

(c) In any connected undirected graph G there is a vertex whose removal leaves G connected.

3 Semiconnected DAG

A directed acyclic graph G is *semiconnected* if for any two vertices A and B, there is either a path from A to B or a path from B to A. Show that G is semiconnected if and only if there is a directed path that visits all of the vertices of G.

4 Dijkstra's Algorithm Fails on Negative Edges

Draw a graph with five vertices or fewer, and indicate the source where Dijkstra's algorithm will be started from.

1. Draw a graph with no negative cycles for which Dijkstra's algorithm produces the wrong answer.
2. Draw a graph with at least two negative weight edge for which Dijkstra's algorithm produces the correct answer.

5 Fixing Dijsktra's Algorithm with Negative Weights

Dijkstra's algorithm doesn't work on graphs with negative edge weights. Here is one attempt to fix it:

1. Add a large number M to every edge so that there are no negative weights left.
2. Run Dijkstra's to find the shortest path in the new graph.
3. Return the path found by Dijkstra's, but with the old edge weights (i.e. subtract M from the weight of each edge).

Show that this algorithm doesn't work by finding a graph for which it must give the wrong answer.

6 Updating Labels

You are given a tree $T = (V, E)$ with a designated root node r, and a non-negative integer label $l(v)$. If $l(v) = k$, we wish to relabel v, such that $l_{\text{new}}(v)$ is equal to $l(w)$, where w is the kth ancestor of v in the tree. We follow the convention that the root node, r, is its own parent. Give a linear time algorithm to compute the new label, $l_{\text{new}}(v)$ for each v in V

Slightly more formally, the *parent* of any $v \neq r$, is defined to be the node adjacent to v in the path from r to v. By convention, $p(r) = r$. For $k > 1$, define $p^k(v) = p^{k-1}(p(v))$ and $p^1(v) = p(v)$ (so p^k is the kth ancestor of v). Each vertex v of the tree has an associated non-negative integer label $l(v)$. We want to find a linear-time algorithm to update the labels of all vertices in T according to the following rule: $l_{\text{new}}(v) = l(p^{l(v)}(v))$.