

Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Secret Santa

Imagine you are throwing a party and you want to play Secret Santa. Thus you would like to assign to every person at the party another person to whom they must anonymously give a gift. However, there are some restrictions on who can give gifts to who. For instance, nobody should be assigned to give a gift to themselves or to their spouse. Since you are the host, you know all of these restrictions. Give an efficient algorithm that determines if you and your guests can play Secret Santa.

Solution: Let n be the number of guests. For guest i , make two vertices u_i and v_i . Let $U = \{u_i : i = 1, \dots, n\}$ and $V = \{v_i : i = 1, \dots, n\}$. Construct a graph $G = (U \cup V, E)$, where there is an edge between u_i and v_j if guest i can give a gift to guest j . You can play Secret Santa iff G has a perfect matching. Run max-flow on G augmented with a source vertex s connected to every vertex in U and a target vertex t connected to every vertex in V , with edge capacities $c_e = 1$ for every edge, to get a flow f . $\text{size}(f)$ is the size of the largest matching, so G has a perfect matching iff $\text{size}(f) = n$.

Zero Sum Games: In this game, there are two players: a maximizer and a minimizer. We generally write the payoff matrix M in perspective of the maximizer, so every row corresponds to an action that the maximizer can take, every column corresponds to an action that the minimizer can take, and a positive entry corresponds to the maximizer winning. M is a n by m matrix, where n is the number of choices the maximizer has, and m is the number of choices the minimizer has.

A linear program that represents fixing the maximizer's choices to a probabilistic distribution where the maximizer has n choices, and the probability that the maximizer chooses choice i is p_i is the following:

$$\begin{aligned} \max(z) \\ M_{1,1}(p_1) + \dots + M_{n,1}(p_n) &\geq z \\ M_{1,2}(p_1) + \dots + M_{n,2}(p_n) &\geq z \\ &\vdots \\ M_{1,m}(p_1) + \dots + M_{n,m}(p_n) &\geq z \\ p_1 + p_2 + \dots + p_n &= 1 \\ p_1, p_2, \dots, p_n &\geq 0 \end{aligned}$$

The dual represents fixing the minimizers choices to a probabilistic distribution.

By strong duality, the optimal value of the game is the same if you fix the minimizer's distribution first or the maximizer's distribution first.

2 Weighted Rock-Paper-Scissors

You and your friend used to play rock-paper-scissors, and have the loser pay the winner 1 dollar. However, you then learned in CS170 that the best strategy is to pick each move uniformly at random, which took all the fun out of the game.

Your friend, trying to make the game interesting again, suggests playing the following variant: If you win by beating rock with paper, you get 5 dollars from your opponent. If you win by beating scissors with rock, you get 3 dollars. If you win by beating paper with scissors, you get 1 dollar.

- Draw the payoff matrix for this game.
- Write a linear program to find the optimal strategy.

Solution:

		Your Friend:		
		rock	paper	scissors
(a) You:	rock	0	-5	3
	paper	5	0	-1
	scissors	-3	1	0

- Let r , p , s be the probabilities that you play rock, paper, scissors respectively. Let z stand for the expected payoff, if your opponent plays optimally as well.

$$\begin{aligned}
 \max \quad & z \\
 5p - 3s & \geq z && \text{(Opponent chooses rock)} \\
 s - 5r & \geq z && \text{(Opponent chooses paper)} \\
 3r - p & \geq z && \text{(Opponent chooses scissors)} \\
 r + p + s & = 1 \\
 r, p, s & \geq 0
 \end{aligned}$$

3 Domination

In this problem, we explore a concept called *dominated strategies*. Consider a zero-sum game with the following payoff matrix for the row player:

		Column:		
		A	B	C
Row:	D	1	2	-3
	E	3	2	-2
	F	-1	-2	2

- If the row player plays optimally, can you find the probability that they pick D without directly solving for the optimal strategy? (Hint: Notice that the payoff for E is always greater than the payoff for D . When this happens, we say that E *dominates* D , i.e. D is a *dominated strategy*).
- Given the answer to part a, if the both players play optimally, what is the probability that the column player picks A ?
- Given the answers to part a and b, what are both players' optimal strategies? (You might be able to figure this out without writing or solving any LP).

Solution:

- (a) 0. Regardless of what option the column player chooses, the row player always gets a higher payoff picking E than D , so any strategy that involves a non-zero probability of picking D can be improved by instead picking E .
- (b) 0. We know that the row player is never going to pick D , i.e. will always pick either E or F . But in this case, picking B is always better for the column player than picking A (A is only better if the row player picks D). That is, conditioned on the row player playing optimally, B dominates A .
- (c) Based on the previous two parts, we only have to consider the probabilities the row player picks E or F and the column player picks B or C . Looking at the 2-by-2 submatrix corresponding to these options, it follows that the optimal strategy for the row player is to pick E and F with probability $1/2$, and similarly the column player should pick B , C with probability $1/2$.