

CS 170 HW 5

Due **2021-02-23**, at **10:00 pm**

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, write “none”.

2 Arbitrage

Shortest-path algorithms can also be applied to currency trading. Suppose we have n currencies $C = \{c_1, c_2, \dots, c_n\}$: e.g., dollars, Euros, bitcoins, dogecoins, etc. For any pair i, j of currencies, there is an exchange rate $r_{i,j}$: you can buy $r_{i,j}$ units of currency c_j at the price of one unit of currency c_i . Assume that $r_{i,i} = 1$ and $r_{i,j} \geq 0$ for all i, j .

The Foreign Exchange Market Organization (FEMO) has hired Oski, a CS170 alumnus, to make sure that it is not possible to generate a profit through a cycle of exchanges; that is, for any currency $i \in C$, it is not possible to start with one unit of currency i , perform a series of exchanges, and end with more than one unit of currency i . (That is called *arbitrage*.)

More precisely, arbitrage is possible when there is a sequence of currencies c_{i_1}, \dots, c_{i_k} such that $r_{i_1, i_2} \cdot r_{i_2, i_3} \cdot \dots \cdot r_{i_{k-1}, i_k} \cdot r_{i_k, i_1} > 1$. This means that by starting with one unit of currency c_{i_1} and then successively converting it to currencies $c_{i_2}, c_{i_3}, \dots, c_{i_k}$ and finally back to c_{i_1} , you would end up with more than one unit of currency c_{i_1} . Such anomalies last only a fraction of a minute on the currency exchange, but they provide an opportunity for profit.

We say that a set of exchange rates is arbitrage-free when there is no such sequence, i.e. it is not possible to profit by a series of exchanges.

- (a) Give an efficient algorithm for the following problem: given a set of exchange rates $r_{i,j}$ which is *arbitrage-free*, and two specific currencies s, t , find the most advantageous sequence of currency exchanges for converting currency s into currency t .

Hint: represent the currencies and rates by a graph whose edge weights are real numbers.

- (b) Oski is fed up of manually checking exchange rates, and has asked you for help to write a computer program to do his job for him. Give an efficient algorithm for detecting the possibility of arbitrage. You may use the same graph representation as for part (a).

3 Money Changing.

Fix a set of positive integers called *denominations* x_1, x_2, \dots, x_n (think of them as the integers 1, 5, 10, and 25). The problem you want to solve for these denominations is the following: Given an integer A , express it as

$$A = \sum_{i=1}^n a_i x_i$$

for some nonnegative integers $a_1, \dots, a_n \geq 0$.

- (a) Under which conditions on the denominations x_i are you able to do this for all integers $A > 0$?
- (b) Suppose that you want, given A , to find the nonnegative a_i 's that satisfy $A = \sum_{i=1}^n a_i x_i$, and such that the sum of all a_i 's is minimal—that is, you use the smallest possible number of coins. Define a *greedy algorithm* for this problem. (Your greedy algorithm may not necessarily solve the problem, i.e., it may give a suboptimal answer on some inputs)
- (c) Show that the greedy algorithm finds the optimum a_i 's in the case of the denominations 1, 5, 10, and 25, and for any amount A .
- (d) Give an example of a denomination where the greedy algorithm fails to find the optimum a_i 's for some A . (Do you know of an actual country where such a set of denominations exists?)

4 Bounded Bellman-Ford (Optional)

You may submit your solution to this problem if you wish it to be graded, but it will be worth no points.

Modify the Bellman-Ford algorithm so that given a graph G , nodes s and t , and an integer k , it finds the weight of the lowest-weight path from s to t with the restriction that the path must have at most k edges.