UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 12
Fall 2021

1. **Hypothesis Testing for Bernoulli Random Variables**

Assume that

- If $X = 0$, then $Y \sim \text{Bernoulli}(1/4)$.
- If $X = 1$, then $Y \sim \text{Bernoulli}(3/4)$.

Using the Neyman-Pearson formulation of hypothesis testing, find the optimal randomized *decision rule* $r : \{0,1\} \to \{0,1\}$ with respect to the criterion

$$\min_{\text{randomized } r:\{0,1\}\to\{0,1\}} P\big(r(Y) = 0 \bigm| X = 1\big)$$
$$\text{s.t. } P\big(r(Y) = 1 \bigm| X = 0\big) \le \beta,$$

where $\beta \in [0,1]$ is a given upper bound on the false positive probability.

Solution:

Here, the likelihood ratio is

$$\frac{f_{Y|X}(y \mid 1)}{f_{Y|X}(y \mid 0)} = \begin{cases} 3, & \text{if } y = 1 \\ \frac{1}{3}, & \text{if } y = 0 \end{cases}$$

If $\beta \le P(Y = 1|X = 0)$, then the optimal decision rule is to have $r(0) = 0$ and have $r(1) = 1$ with probability $\gamma = \frac{\beta}{1/4}$. Otherwise, the optimal decision rule is $r(1) = 1$ and have $r(0) = 1$ with probability γ, chosen to make $P\big(r(Y) = 1 \bigm| X = 0\big) = \beta$. Then,

$$\frac{1}{4} + \frac{3}{4}\gamma = \beta$$

so $\gamma = \frac{4}{3}\beta - \frac{1}{3}$.

2. **Joint Gaussian Probability**

Let $X \sim \mathcal{N}(1,1)$ and $Y \sim \mathcal{N}(0,1)$ be jointly Gaussian with covariance ρ. What is $P(X > Y)$?

Solution:

Let $\bar{X} = X - 1$.

We can write $Y = \rho\bar{X} + \sqrt{1-\rho^2}Z$, where $Z \sim \mathcal{N}(0,1)$ is independent of $\bar{X}$. (To check that this is correct, observe that $\text{cov}(\bar{X}, \rho\bar{X} + \sqrt{1-\rho^2}Z) = \rho$ and also $\text{var}(\rho\bar{X} + \sqrt{1-\rho^2}Z) = \rho^2 + (1-\rho^2) = 1$ as required.)

So, $P(X > Y) = P(\bar{X} > Y - 1) = P((1-\rho)\bar{X} - \sqrt{1-\rho^2}Z > -1)$. But

$$(1-\rho)\bar{X} - \sqrt{1-\rho^2}Z \sim \mathcal{N}\big(0, (1-\rho)^2 + 1 - \rho^2\big) = \mathcal{N}\big(0, 2(1-\rho)\big)$$

by independence so

$$P(X > Y) = P\Big(\mathcal{N}(0,1) > -\frac{1}{\sqrt{2(1-\rho)}}\Big) = \Phi\Big(\frac{1}{\sqrt{2(1-\rho)}}\Big).$$