UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 3
Fall 2021

1. Graphical Density

Figure 1 shows the joint density $f_{X,Y}$ of the random variables X and Y.

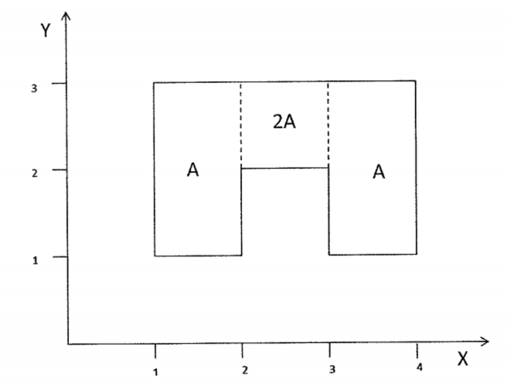

Figure 1: Joint density of X and Y.

(a) Find A and sketch f_X, f_Y, and $f_{X|X+Y\le 3}$.
(b) Find $\mathbb{E}[X \mid Y = y]$ for $1 \le y \le 3$ and $\mathbb{E}[Y \mid X = x]$ for $1 \le x \le 4$.
(c) Find $\operatorname{cov}(X, Y)$.

Solution:

(a) The integration over the total shown area should be 1 so $2A + 2A + 2A = 1$ so $A = 1/6$. To spell this out in more detail,

$$\begin{aligned} 1 &= \int_1^3 \int_1^2 A \, \mathrm{d}x \, \mathrm{d}y + \int_2^3 \int_2^3 2A \, \mathrm{d}x \, \mathrm{d}y + \int_1^3 \int_3^4 A \, \mathrm{d}x \, \mathrm{d}y \\ &= 2A + 2A + 2A = 6A. \end{aligned}$$

We find the densities as follows. X is clearly uniform in intervals $(1, 2)$, $(2, 3)$, and $(3, 4)$. The probability of X being in any of these intervals is $2A = 1/3$ so

$$f_X(x) = \frac{1}{3}\mathbf{1}\{1 \le x \le 4\}.$$

Y is uniform in intervals $(1, 2)$ and $(2, 3)$. The probability of the first interval is $1/3$ and the probability of being in second one is $2/3$. So

$$f_Y(y) = \frac{1}{3}\mathbf{1}\{1 \le y \le 2\} + \frac{2}{3}\mathbf{1}\{2 < y \le 3\}.$$

Finally, given that $X+Y \leq 3$, (X,Y) is chosen randomly in the triangle constructed by $(1,1),(1,2),(2,1)$. Thus,

$$f_{X|X+Y\leq 3}(x) = \int_1^{3-x} 2\,\mathrm{d}y = 2(2-x)\mathbf{1}\{1 \leq x \leq 2\}.$$

Sketching the densities is then straightforward.

(b) Given any value of $y \in [1,3]$, X has a symmetric distribution with respect to the line $x = 2.5$. Thus, $\mathbb{E}[X \mid Y = y] = 2.5$ for all y, $1 \leq y \leq 3$. To calculate $\mathbb{E}[Y \mid X = x]$, we consider two cases:

(a) $2 \leq x \leq 3$, then $\mathbb{E}[Y \mid X = x] = 2.5$,

(b) $1 \leq x < 2$ or $3 < x \leq 4$, then $\mathbb{E}[Y \mid X = x] = 2$.

(c) Since $\mathbb{E}[X \mid Y = y] = \mathbb{E}[X]$ we have

$$\begin{aligned}\mathbb{E}[XY] &= \int_1^3 \mathbb{E}[XY \mid Y = y] f_Y(y)\,\mathrm{d}y = \int_1^3 y f_Y(y)\mathbb{E}[X]\,\mathrm{d}y \\ &= \mathbb{E}[X]\mathbb{E}[Y].\end{aligned}$$

So the covariance is 0.

2. Joint Density for Exponential Distribution

(a) If $X \sim$ Exponential(λ) and $Y \sim$ Exponential(μ), X and Y independent, compute $\mathbb{P}(X < Y)$.

(b) If X_k, $1 \leq k \leq n$ are independent and exponentially distributed with parameters $\lambda_1, \ldots, \lambda_n$, show that $\min_{1\leq k\leq n} X_k \sim$ Exponential$(\sum_{j=1}^n \lambda_j)$.

(c) Deduce that

$$\mathbb{P}(X_i = \min_{1\leq k\leq n} X_k) = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}$$

Solution:

(a)

$$\mathbb{P}(X < Y) = \int_{y=0}^{\infty} \mathbb{P}(X < y|Y = y) f_Y(y) dy$$

Since X and Y are independent, $\mathbb{P}(X < y|Y = y) = \mathbb{P}(X < y)$, and since $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$, $\mathbb{P}(X < y) = 1 - e^{-\lambda y}$ and $f_Y(y) = \mu e^{-\mu y}$. Plugging in, we get, $\mathbb{P}(X < Y) = \frac{\lambda}{\lambda+\mu}$.

(b) We need to verify a nice fact about a collection of independent exponentially distributed random variable. Given a collection of random variables, $Y_i \sim Exp(\mu_i), 1 \leq i \leq n$, $\min(Y_i, 1 \leq i \leq n)$ is exponentially distributed with parameter $\sum_{i=1}^n \mu_i$. This follows from checking the cdf of $\min(Y_i)$, i.e.

$$P(\min(Y_i) \geq y) = P(\cap_{i=1}^n Y_i \geq y) = \prod_{i=1}^n P(Y_i \geq y) = \prod_{i=1}^n e^{-\mu_i y} = e^{-y\sum_{i=1}^n \mu_i}.$$

(c) Now, $\mathbb{P}(X_i = \min_{1\leq k\leq n} X_k) = \mathbb{P}(X_i \leq \min_{1\leq k\leq n, k\neq i} X_k)$. From the previous argument, $\min_{1\leq k\leq n, k\neq i} X_k \sim \sum_{j=1, j\neq i} \lambda_j$. Using the result of part (a), the claim follows.

3. **Packet Routing**

Packets arriving at a switch are routed to either destination A (with probability p) or destination B (with probability $1-p$). The destination of each packet is chosen independently of each other. In the time interval $[0,1]$, the number of arriving packets is Poisson(λ).

(a) Show that the number of packets routed to A is Poisson distributed. With what parameter?

(b) Are the number of packets routed to A and to B independent?

Solution:

(a) Let X, Y be random variables which are equal to the number of packets routed to the destinations A, B respectively. Let $Z = X + Y$. We are given that $Z \sim \text{Poisson}(\lambda)$. We prove that X has the Poisson distribution with mean $p\lambda$.

$$\begin{aligned}
P(X = x) &= \sum_{z=x}^{\infty} P(X = x, Z = z) \\
&= \sum_{z=x}^{\infty} P(Z = z) P(X = x \mid Z = z) \\
&= \sum_{z=x}^{\infty} \frac{e^{-\lambda}\lambda^z}{z!} \binom{z}{x} p^x (1-p)^{z-x} \\
&= e^{-\lambda} \sum_{z=x}^{\infty} \frac{\lambda^z}{z!} \frac{z!}{x!(z-x)!} p^x (1-p)^{z-x} \\
&= \frac{e^{-\lambda}(\lambda p)^x}{x!} \sum_{z=x}^{\infty} \frac{(\lambda(1-p))^{z-x}}{(z-x)!} \\
&= \frac{e^{-\lambda}(\lambda p)^x}{x!} e^{\lambda(1-p)} \\
&= \frac{e^{-\lambda p}(\lambda p)^x}{x!}.
\end{aligned}$$

(b) We prove that X and Y are independent.

$$\begin{aligned}
P(X = x, Y = y) &= \sum_{z=0}^{\infty} P(X = x, Y = y, Z = z) \\
&= \sum_{z=0}^{\infty} P(X = x, Y = y \mid Z = z) P(Z = z) \\
&= P(X = x, Y = y \mid Z = x + y) P(Z = x + y) \\
&= \frac{\cancel{(x+y)!}}{x!y!} p^x (1-p)^y \frac{e^{-\lambda}\lambda^{x+y}}{\cancel{(x+y)!}} \\
&= \frac{e^{-\lambda p}(\lambda p)^x}{x!} \cdot \frac{e^{-\lambda(1-p)}(\lambda(1-p))^y}{y!} \\
&= P(X = x) P(Y = y).
\end{aligned}$$

4. **Gaussian Densities**

(a) Let $X_1 \sim \mathcal{N}(0,1)$, $X_2 \sim \mathcal{N}(0,1)$, where X_1 and X_2 are independent. Convolve the densities of X_1 and X_2 to show that $X_1 + X_2 \sim \mathcal{N}(0,2)$. *Remark.* Note that this property is similar to the one shared by independent Poisson random variables.

(b) Let $X \sim \mathcal{N}(0,1)$. Compute $\mathbb{E}[X^n]$ for all integers $n \geq 1$.

Solution:

(a) We know that the pdf $f_Z(z)$ of $Z = X + Y$ is given by $f_X * f_Y$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx \\ &= \int_{-\infty}^{\infty} \Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Big)\Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}}\Big)\, dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left(x^2 + z^2 - 2xz + x^2\right)\right\} dx \end{aligned}$$

Now the trick here is to complete the square for x in $x^2 + z^2 - 2xz + x^2$. We can write it as

$$\begin{aligned} x^2 + z^2 - 2xz + x^2 &= 2(x^2 - xz) + z^2 \\ &= 2(x^2 - xz) + \frac{1}{2}z^2 + \frac{1}{2}z^2 \\ &= 2(x^2 - xz + \frac{1}{4}z^2) + \frac{1}{2}z^2 \\ &= 2(x - \frac{1}{2}z)^2 + \frac{1}{2}z^2 \\ &= \frac{(x-\frac{1}{2}z)^2}{\frac{1}{2}} + \frac{z^2}{2} \end{aligned}$$

Substituting it back in, we get

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left[\frac{\left(x-\frac{1}{2}z\right)^2}{\frac{1}{2}} + \frac{z^2}{2}\right]\right\} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{-\frac{\left(x-\frac{1}{2}z\right)^2}{2\cdot\frac{1}{2}}\right\} \exp\left\{-\frac{z^2}{2\cdot 2}\right\} dx \\ &= \frac{1}{\sqrt{2\pi\cdot 2}} \exp\left\{-\frac{z^2}{2\cdot 2}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\cdot\frac{1}{2}}} \exp\left\{-\frac{\left(x-\frac{1}{2}z\right)^2}{2\cdot\frac{1}{2}}\right\} dx \end{aligned}$$

The integral is integrating the PDF of a $N(\frac{1}{2}z, \frac{1}{2})$ RV from $-\infty$ to ∞, so it is equal to 1. Therefore, our answer is

$$f_Z(z) = \frac{1}{\sqrt{2\pi\cdot 2}} \exp\left\{-\frac{z^2}{2\cdot 2}\right\}$$

which is the PDF of a $N(0,2)$ RV.

(b) For odd n, the integrand is odd, so $\mathbb{E}[X^n] = 0$. So suppose n is even. We proceed using integration by parts:

$$\begin{aligned}
\sqrt{2\pi}\mathbb{E}[X^n] &= \int x^n e^{-x^2/2} dx \\
&= \underbrace{(-x^{n-1}e^{-x^2/2})_{-\infty}^{\infty}}_{0} + \int (n-1)x^{n-2}e^{-x^2/2} dx \\
&= (n-1)\sqrt{2\pi}\mathbb{E}[X^{n-2}].
\end{aligned}$$

Therefore, we deduce that for even $n = 2k$:

$$\mathbb{E}[X^{2k}] = \prod_{i=1}^{k}(2i-1).$$

5. Moving Books Arround

You have N books on your shelf, labelled $1, 2, \ldots, N$. You pick a book j with probability $1/N$. Then you place it on the left of all others on the shelf. You repeat the process, independently. Construct a Markov chain which takes values in the set of all $N!$ permutations of the books.

(a) Find the transition probabilities of the Markov chain.

(b) Find its stationary distribution.
Hint: You can guess the stationary distribution before computing it.

Solution:

(a) The state space consists of all the $N!$ permutations on N symbols. Then the transition probabilities can be written as

$$P((\sigma_1, \ldots, \sigma_{j-1}, \sigma_j, \sigma_{j+1}, \ldots, \sigma_N), (\sigma_j, \sigma_1, \ldots, \sigma_{j-1}, \sigma_{j+1}, \sigma_N)) = \frac{1}{N}, \quad \text{for } j = 1, \ldots, N.$$

(b) Because of symmetry all the states should have the same stationary probabilities, i.e.

$$\pi(\sigma) = \frac{1}{N!}, \quad \text{for all } \sigma \in S_N.$$

We can verify that this probability distribution satisfies the balance equations. Let $\sigma^{(1)} = (\sigma_1, \sigma_2 \ldots, \sigma_{j-1}, \sigma_j, \ldots, \sigma_N)$ be a permutation, and for $j = 2, \ldots, n$ let $\sigma^{(j)} = (\sigma_2, \ldots, \sigma_{j-1}, \sigma_1, \sigma_j, \ldots, \sigma_N)$ be the same permutation with $\sigma^{(1)}$ except that σ_1 has been moved in the jth position. With this notation

$$\begin{aligned}
\pi(\sigma^{(1)}) &= \sum_{j=1}^{N} \pi(\sigma^{(j)})P(\sigma^{(j)}, \sigma^{(1)}) \\
&= \sum_{j=1}^{N} \frac{1}{N!} \cdot \frac{1}{N} \\
&= \frac{1}{N!}.
\end{aligned}$$