UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 8
Fall 2021

1. Merging & Splitting

(a) (Splitting/Thinning) Let N_t be a Poisson process with rate λ. Let $0 < p < 1$, and consider the split processes $N_t^{(1)}$ and $N_t^{(2)}$, where each arrival of N_t is routed to $N_t^{(1)}$ with probability p, and $N_t^{(2)}$ with probability $1-p$. Show that $N_t^{(1)}$ is a Poisson process of rate λp, and $N_t^{(2)}$ a Poisson process of rate $\lambda(1-p)$. Also show that these two processes are independent.

(b) (Merging) Let $N_t^{(1)}$ and $N_t^{(2)}$ be two independent Poisson processes of rates λ and μ. Suppose we merge them, i.e. consider N_t the process denoting total arrivals from both processes. Show that N_t is a Poisson process of rate $\lambda + \mu$.

Solution:

(a) Recall the packet routing question from HW 3. It tells us that for each $t \geq 0$, we have $N_t^{(1)} \sim \text{Poisson}(\lambda p t)$ and $N_t^{(2)} \sim (\lambda(1-p)t)$. Therefore $N_t^{(1)}$ and $N_t^{(2)}$ are Poisson processes of rates λp and $\lambda(1-p)$, respectively. Furthermore by part (b) of that problem, these two processes are independent.

(b) It suffices to show that for each $t \geq 0$, $N_t \sim (t(\lambda + \mu))$. In fact the calculations are the same for all t, so we will just do it for $t = 1$. We write

$$\begin{aligned}
P(N_1 = k) &= \sum_{j=0}^{k} P(N_1^{(1)} = j) P(N_1^{(2)} = k - j) \\
&= \sum_{j=0}^{k} \frac{e^{\lambda} \lambda^j}{j!} \cdot \frac{e^{\mu} \mu^{k-j}}{(k-j)!} \\
&= \frac{e^{\lambda+\mu}}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j \mu^{k-j} \\
&= \frac{e^{\lambda+\mu} (\lambda+\mu)^k}{k!}.
\end{aligned}$$

Thus $N_1 \sim \text{Poisson}(\lambda + \mu)$, as desired.

2. Machine

A machine, once in production mode, operates continuously until an alarm signal is generated. The time up to the alarm signal is an exponential random variable with parameter 1. Subsequent to the alarm signal, the machine is tested for an exponentially distributed amount of time with parameter 5. The test results are positive, with probability 1/2, in which case the machine returns to production mode, or negative, with probability 1/2, in which case the machine is taken for repair. The duration of the repair is exponentially distributed with parameter 3.

(a) Let states $1, 2, 3$ correspond to production mode, testing, and repair, respectively. Let $(X(t))_{t\geq 0}$ denote the state of the system at time t. Is $(X(t))_{t\geq 0}$ a CTMC?

(b) Find the rate matrix Q of the CTMC and the transition matrix P of the corresponding jump chain.

(c) Find the stationary distribution of the CTMC.

Solution:

(a) We note that given that the current state is $i \in \{1, 2, 3\}$ in this process, the future is independent of the past.

(b) The transition rates are $\nu_1 = 1, \nu_2 = 5, \nu_3 = 3$ and the transition probabilities are given, so we have:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 1 & 0 \\ 5/2 & -5 & 5/2 \\ 3 & 0 & -3 \end{bmatrix}.$$

(c) We set up the balance equations:

$$\begin{aligned} \pi_1 &= \frac{5}{2}\pi_2 + 3\pi_3 \\ 5\pi_2 &= \pi_1 \\ 3\pi_3 &= \frac{5}{2}\pi_2 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

We thus have:

$$\pi_1 = \frac{30}{41}, \qquad \pi_2 = \frac{6}{41}, \qquad \pi_3 = \frac{5}{41}$$

3. Lazy Server

Customers arrive at a queue in a service facility at the times of a Poisson process of rate λ. The service facility has infinite capacity. There is an infinitely powerful but lazy server who visits the service facility at the times of a Poisson process of rate μ. These two processes are independent. When the server visits the facility she instantaneously serves all the customers that are in the queue and then immediately leaves (until her next visit).

Thus, for instance, at any time, any customers that are waiting in the queue would only be those that arrived after the most recent visit of the server.

(a) Model the queue length as a CTMC and find the stationary distribution.

(b) Suppose tha the CTMC is at stationary, and find the mean number of customers waiting in the queue at any given time.

Solution:

(a) We can model the queue length as a continuous-time Markov chain on the state space $\mathcal{S} := \mathbb{N}$. For each $i \in \mathcal{S} \setminus \{0\}$, the rate at at which a customer arrives is λ, and the rate at which the server arrives is μ, so the rates are $q(i, i+1) = \lambda$, $q(i, 0) = \mu$. Together with $q(0,1) = \lambda$, we have completely specified the chain. For $j \in \mathcal{S} \setminus \{0\}$, the balance equation reads $\pi(j-1)\lambda - \pi(j)(\lambda+\mu) = 0$, or

$$\pi(j) = \Big(\frac{\lambda}{\lambda+\mu}\Big)^j \pi(0).$$

Thus, summing over $j \in \mathcal{S} \setminus \{0\}$, we obtain

$$\sum_{j\in\mathbb{N}} \pi(j) = \sum_{j\in\mathbb{N}} \Big(\frac{\lambda}{\lambda+\mu}\Big)^j \pi(0) = \frac{1}{1-\lambda/(\lambda+\mu)}\pi(0) = \frac{\lambda+\mu}{\mu}\pi(0),$$

so by taking $\pi(0) = \mu/(\lambda+\mu)$ we find that

$$\pi(j) := \frac{\mu}{\lambda+\mu}\Big(\frac{\lambda}{\lambda+\mu}\Big)^j, \qquad j \in \mathcal{S}$$

is a stationary distribution.

(b) If X is a random variable with $P(X=j) = \pi(j)$ for $j \in \mathcal{S}$, then we see that

$$X + 1 \sim \text{Geometric}\Big(\frac{\mu}{\lambda+\mu}\Big).$$

Thus, $\mathbb{E}[X] = (\lambda+\mu)/\mu - 1 = \lambda/\mu$. One way to understand this is that $1/\mu$ is the mean time that a customer spends in the system.