UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 11
Fall 2021

1. BSC: MLE & MAP

You are testing a digital link that corresponds to a BSC with some error probability $\epsilon \in [0, 0.5]$.

(a) Assume you observe the input and the output of the link. How do you find the MLE of ϵ?

(b) You are told that the inputs are i.i.d. bits that are equal to 1 with probability 0.6 and to 0 with probability 0.4. You observe n outputs (n is a positive integer). How do you calculate the MLE of ϵ?

(c) The situation is as in the previous case, but you are told that ϵ has PDF $4-8x$ on $[0, 0.5)$. How do you calculate the MAP of ϵ given n outputs? You may leave your answer in terms of quadratic equation to be solved.

Solution:

(a) We observe the input X and the output Y. Thus, if P_ϵ denotes the probability distribution when the error probability of the BSC is ϵ, then for $(x, y) \in \{0, 1\}^2$,

$$\epsilon_{\text{MLE}} = \arg\max_{\epsilon \in [0,0.5]} P_\epsilon(X = x, Y = y) = \arg\max_{\epsilon \in [0,0.5]} \epsilon^{\mathbf{1}\{y \neq x\}} (1-\epsilon)^{\mathbf{1}\{y = x\}}.$$

Now if $x \neq y$, the expression is clearly maximized on the largest possible value of ϵ which is $\epsilon = 0.5$. If $x = y$, the expression is maximized for smallest value of ϵ which is 0.

(b) Suppose that we observe the outputs $y_1, \ldots, y_n$. Thus,

$$\epsilon_{\text{MLE}} = \arg\max_{\epsilon \in [0,0.5]} P_\epsilon(Y_1 = y_1, \ldots, Y_n = y_n).$$

Since every use of the channel is independent we have,

$$\begin{aligned}
&P_\epsilon(Y_1 = y_1, \ldots, Y_n = y_n) \\
&\quad = \prod_{i=1}^{n} P_\epsilon(Y_i = y_i) \\
&\quad = \prod_{i=1}^{n} \big[\big(0.6(1-\epsilon) + 0.4\epsilon\big)\mathbf{1}\{y_i = 1\} + \big(0.4(1-\epsilon) + 0.6\epsilon\big)\mathbf{1}\{y_i = 0\}\big] \\
&\quad = \prod_{i=1}^{n} (0.6 - 0.2\epsilon)^{y_i} (0.4 + 0.2\epsilon)^{1-y_i} \\
&\quad = (0.6 - 0.2\epsilon)^{\sum_{i=1}^{n} y_i} (0.4 + 0.2\epsilon)^{n - \sum_{i=1}^{n} y_i}.
\end{aligned}$$

Let $t = \sum_{i=1}^{n} y_i$. As we can see, what matters for estimating ϵ is t. To find the maximizer of the expression, we first take the log and then set the derivative to 0. Thus,

$$\frac{-0.2t}{0.6 - 0.2\epsilon} + \frac{0.2(n-t)}{0.4 + 0.2\epsilon} = 0.$$

Solving the equation, we get

$$\epsilon_{\text{MLE}} = 3 - \frac{5t}{n}.$$

Of course, since we know that $0 \le \epsilon \le 0.5$, if ϵ_{MLE} is not in the interval $[0, 0.5]$ we should pick the closest point to it which will be either 0 or 0.5.

(c) This time we want to maximize $P(Y_1 = y_1, \dots, Y_n = y_n \mid \epsilon = \cdot) f_\epsilon(\cdot)$. Similar to the calculations of previous part, we want to maximize,

$$(4 - 8\epsilon)(0.6 - 0.2\epsilon)^t (0.4 + 0.2\epsilon)^{(n-t)}.$$

Taking the log and setting the derivative equal to 0 we have

$$\frac{-8}{4 - 8\epsilon} + \frac{-0.2t}{0.6 - 0.2\epsilon} + \frac{0.2(n-t)}{0.4 + 0.2\epsilon} = 0.$$

Then, we get the following quadratic equation.

$$\begin{aligned} 0 = &-8(0.6 - 0.2\epsilon)(0.4 + 0.2\epsilon) - 0.2t(4 - 8\epsilon)(0.4 + 0.2\epsilon) \\ &+ 0.2(n-t)(4 - 8\epsilon)(0.6 - 0.2\epsilon). \end{aligned}$$

One can solve the long quadratic equation analytically, and find ϵ_{MAP}.

2. Hypothesis Testing for Uniform Distribution

Assume that

- If $X = 0$, then $Y \sim \text{Uniform}[-1, 1]$.
- If $X = 1$, then $Y \sim \text{Uniform}[0, 2]$.

Using the Neyman-Pearson formulation of hypothesis testing, find the optimal randomized *decision rule* $r : [-1, 2] \to \{0, 1\}$ with respect to the criterion

$$\begin{aligned} \min_{\text{randomized } r:[-1,2]\to\{0,1\}} \quad & P\big(r(Y) = 0 \bigm| X = 1\big) \\ \text{s.t. } & P\big(r(Y) = 1 \bigm| X = 0\big) \le \beta, \end{aligned}$$

where $\beta \in [0, 1]$ is a given upper bound on the false positive probability.

Solution:

Here, the likelihood ratio is

$$\frac{f_{Y|X}(y \mid 1)}{f_{Y|X}(y \mid 0)} = \frac{\mathbf{1}\{0 \le y \le 2\}}{\mathbf{1}\{-1 \le y \le 1\}}.$$

Thus, $\hat{X} = 1$ if $Y > 1$ and $\hat{X} = 0$ if $Y < 0$. If $Y \in [0, 1]$ we need randomization, so $\hat{X} = 1$ with some probability γ. We choose γ such that

$$P(\hat{X} = 1 \mid X = 0) = \beta.$$

That is,

$$\gamma P(Y \in [0,1] \mid X = 0) = \frac{\gamma}{2} = \beta.$$

Thus, $\gamma = 2\beta$.

3. **Bayesian Hypothesis Testing for Gaussian Distribution**

Assume that X has prior probabilities $P(X = 0) = P(X = 1) = 1/2$. Further

- If $X = 0$, then $Y \sim \mathcal{N}(\mu_0, \sigma_0^2)$.
- If $X = 1$, then $Y \sim \mathcal{N}(\mu_1, \sigma_1^2)$.

Assume $\mu_0 < \mu_1$ and $\sigma_0 < \sigma_1$.

Using the Bayesian formulation of hypothesis testing, find the optimal *decision rule* $r : \mathbb{R} \to \{0, 1\}$ with respect to the minimum expected cost criterion

$$\min_{r:\mathbb{R}\to\{0,1\}} \mathbb{E}[I\{r(Y) \neq X\}].$$

Solution:

We can write

$$\begin{aligned} E[I(r(Y) \neq X)] &= P(r(Y) \neq X) \\ &= P(r(Y) = 1 \mid X = 0) \cdot \frac{1}{2} + P(r(Y) = 0 \mid X = 1) \cdot \frac{1}{2} \end{aligned}$$

We can write $P(r(Y) = 1 \mid X = 0)$ as

$$P(r(Y) = 1 \mid X = 0) = \int \begin{cases} f(y \mid X = 0) & \text{if } r(y) = 1 \\ 0 & \text{otherwise} \end{cases} dy$$

and do something similar for $P(r(Y) = 0 \mid X = 1)$. Combining everything together, we get

$$E[I(r(Y) \neq X)] = \frac{1}{2} \int \begin{cases} f(y \mid X = 0) & \text{if } r(y) = 1 \\ f(y \mid X = 1) & \text{if } r(y) = 0 \end{cases} dy$$

Since we're free to choose $r(y)$ as 0 or 1, trying to minimize this leads to

$$\begin{cases} 0, & \text{if } f(y \mid X = 0) > f(y \mid X = 1) \\ 1, & \text{if } f(y \mid X = 0) < f(y \mid X = 1). \end{cases}$$

The condition $f(y \mid X = 0) < f(y \mid X = 1)$ can be written as

$$\Big(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\Big)y^2 - 2\Big(\frac{\mu_0}{\sigma_0^2} - \frac{\mu_1}{\sigma_1^2}\Big)y + \Big(\frac{\mu_0^2}{\sigma_0^2} - \frac{\mu_1^2}{\sigma_1^2} - 2\ln\frac{\sigma_1}{\sigma_0}\Big) > 0,$$

and if we let $a < b$ be the two roots of this quadratic, then the optimal decision rule can be written as

$$r(y) = \begin{cases} 0, & \text{if } y \in (a, b) \\ 1, & \text{if } y \in (-\infty, a) \cup (b, \infty). \end{cases}$$