UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 1

Fall 2021

1. Miscellaneous Review

(a) Show that the probability that exactly one of the events A and B occurs is $\mathbb{P}(A)+\mathbb{P}(B)-2\mathbb{P}(A\cap B)$.

(b) If A is independent of itself, show that $\mathbb{P}(A)=0$ or 1.

Solution:

(a) The probability of the event that exactly one of A and B occur is

$$\begin{aligned} P(A\cap B^{\mathsf{c}})+P(A^{\mathsf{c}}\cap B) &= P(A)-P(A\cap B)+P(B)-P(A\cap B) \\ &= P(A)+P(B)-2P(A\cap B). \end{aligned}$$

(b) $P(A\cap A)=P(A)P(A)$, so $P(A)=P(A)^2$; this implies that $P(A)\in\{0,1\}$.
Alternatively, suppose for the sake of contradiction that $0<P(A)<1$. Then, $P(A\mid A)=1\neq P(A)$, which contradicts the supposed independence of A with itself. Hence, $P(A)\in\{0,1\}$.

2. Choosing from Any Jar Makes No Difference

Each of k jars contains w white and b black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $w/(w+b)$.

Solution:

We derive a recursion for the probability p_i that a white ball is chosen from the ith jar. We have, using the total probability theorem,

$$p_{i+1}=\frac{w+1}{w+b+1}p_i+\frac{w}{w+b+1}(1-p_i)=\frac{1}{w+b+1}p_i+\frac{w}{w+b+1},$$

starting with the initial condition $p_1=w/(w+b)$. Thus, we have

$$p_2=\frac{1}{w+b+1}\cdot\frac{w}{w+b}+\frac{w}{w+b+1}=\frac{w}{w+b}.$$

More generally, this calculation shows that if $p_{i-1}=w/(w+b)$, then $p_i=w/(w+b)$. Thus, we obtain $p_i=w/(w+b)$ for all i.

3. **Colored Sphere**

Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest is colored red. Show that, no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

Hint: Carefully define some relevant events.

Solution:

Pick an inscribed cube uniformly at random, enumerate its vertices, and let B_i be the event that vertex i is blue. Note that:

$$P(B_1 \cup \cdots \cup B_8) \leq \sum_{i=1}^{8} P(B_i) = \sum_{i=1}^{8} \frac{1}{10} = \frac{8}{10} < 1$$

In other words, the probability of at least one vertex being blue is *less* than 1, so there must exist an inscribed cube where each vertex is red.

Note: This is an example of a powerful tool known as the probabilistic method.

4. **Balls & Bins**

Let $n \in \mathbb{Z}_{>1}$. You throw n balls, one after the other, into n bins, so that each ball lands in one of the bins uniformly at random. What is an appropriate sample space to model this scenario? What is the probability that exactly one bin is empty?

Solution:

An appropriate sample space is to take $\Omega = \{1, \ldots, n\}^n$, the set of n-tuples where each coordinate is a number in $\{1, \ldots, n\}$. An outcome $\omega \in \Omega$ represents a scenario as follows: the first coordinate gives the label of the bin into which the first ball fell; the second coordinate gives the label of the bin into which the second ball fell; and so on.

Notice that this choice of sample space treats all of the balls as distinguishable and all of the bins as distinguishable. The reason for making this choice is that the sample space is *uniform*, that is, all outcomes have the same probability.

In contrast, if we chose a sample space corresponding to *indistinguishable balls* (and distinguishable bins), then the sample space would *not* be uniform, which makes the problem harder to analyze. The reason why the sample space is no longer uniform is that some outcomes can happen in more ways, so they have higher probabilities. Concretely, the outcome that all balls land in the first bin will have a smaller probability than the outcome that half the balls land in the first bin and the other half land in the second bin, because in the latter outcome you have the freedom to change *which* balls land in first bin (because the balls are indistinguishable).

Now, we return to our uniform sample space with distinguishable balls. The probability of each outcome is n^{-n}, so we must count how many outcomes correspond to exactly one empty bin. There are n ways to choose which bin is empty; then $n-1$ ways to choose which of the remaining bins will have two balls; then, there are $\binom{n}{2}$ ways to choose *which* two of the n balls will land in the bin with two balls; finally, there are $(n-2)!$ ways to throw the remaining $n-2$ balls into the $n-2$ other bins. Therefore, the total number of outcomes is $n(n-1)(n-2)!\binom{n}{2} = n!\binom{n}{2}$, so the desired probability is $n!\binom{n}{2}/n^n$.