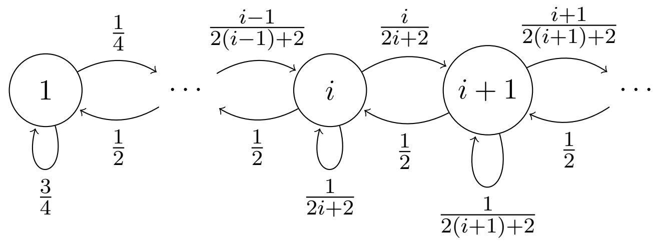
UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 7
Fall 2021

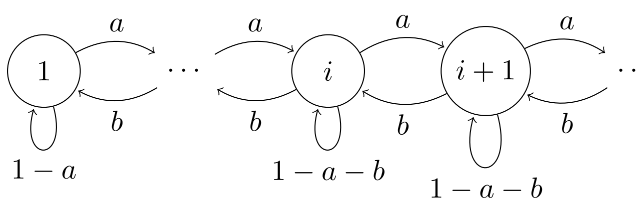
1. Markov Chains with Countably Infinite State Space

(a) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:



Show that this Markov chain has no stationary distribution.

(b) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:



Assume that $0 < a < b$ and $0 < a + b \leq 1$. Show that the probability distribution given by

$$\pi(i) = \left(\frac{a}{b}\right)^{i-1}\left(1 - \frac{a}{b}\right), \text{ for } i \in \mathbb{Z}_{>0},$$

is a stationary distribution of this Markov chain.

Solution:

(a) The given Markov chain is a birth-death chain, so its stationary distribution, if it exists, would satisfy the detailed balance equations

$$\pi(i)P(i, i+1) = \pi(i+1)P(i+1, i), \text{ for } i \geq 1.$$

Substituting the given transition probabilities, this means

$$\pi(i+1) = \frac{i}{i+1}\pi(i) = \frac{i}{i+1}\frac{i-1}{i}\pi(i-1) = \cdots = \frac{1}{i+1}\pi(1), \text{ for } i \geq 0.$$

The stationary distribution will exist if there is a solution to these equations that satisfies $\sum_{i=1}^{\infty} \pi(i) = 1$. Since

$$\sum_{i=1}^{\infty} \pi(i) = \pi(1) \sum_{i=1}^{\infty} \frac{1}{i},$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i} = \infty,$$

it is not possible to choose $\pi(1)$ in such a way that we get $\sum_{i=1}^{\infty} \pi(i) = 1$. Therefore this Markov chain does not admit a stationary distribution.

(b) First of all observe that

$$\left(\frac{a}{b}\right)^{i-1} \left(1 - \frac{a}{b}\right) \geq 0, \text{ for all } i \geq 1,$$

and

$$\sum_{i=1}^{\infty} \left(\frac{a}{b}\right)^{i-1} \left(1 - \frac{a}{b}\right) = \frac{1}{1 - \frac{a}{b}} \left(1 - \frac{a}{b}\right) = 1,$$

which means that $\pi(i)$ is a valid probability distribution.

This Markov chain is a birth-death chain as well, so we are left to verify that π satisfies the detailed balance equations

$$\begin{aligned} \pi(i)P(i, i+1) &= \left(\frac{a}{b}\right)^{i-1} \left(1 - \frac{a}{b}\right) a = \left(\frac{a}{b}\right)^{i} \left(1 - \frac{a}{b}\right) b \\ &= \pi(i+1)P(i+1, i). \end{aligned}$$

2. Poisson Branching

Consider a branching process such that at generation n, each individual in the population survives until generation $n+1$ with probability $0 < p < 1$, independently, and a Poisson number (with parameter λ) of immigrants enters the population. Let X_n denote the number of people in the population at generation n.

(a) Suppose that at generation 0, the number of people in the population is a Poisson random variable with parameter λ_0. What is the distribution at generation 1? What is the distribution at generation n?

(b) What is the distribution of X_n as $n \to \infty$? What if at generation 0, the number of individuals is an arbitrary probability distribution over the non-negative integers; does the distribution still converge? (Justify the convergence rigorously.)

Solution:

(a) Suppose that at generation n, X_n is Poisson with parameter λ_n. The number of individuals who survive to generation $n+1$ is Poisson with parameter $\lambda_n p$ by Poisson splitting; and then, the number of individuals at generation $n+1$ is Poisson with parameter $\lambda_n p + \lambda$ by Poisson merging.

So, if X_0 is Poisson with parameter λ_0, then X_1 is Poisson with parameter $\lambda_0 p + \lambda$, and in general, X_n is Poisson with parameter

$$\lambda_n = \lambda_0 p^n + \lambda(p^{n-1} + \cdots + 1).$$

(b) Note that as $n \to \infty$, $\lambda_n \to \lambda/(1-p)$, so we are tempted to guess that X_n will converge to a Poisson distribution with parameter $\lambda/(1-p)$.

To justify formally why the distribution converges, observe that $\{X_n\}_{n=0}^{\infty}$ is a Markov chain on state space $\mathcal{X} = \mathbb{N}$. Suppose at some generation, there are a Poisson (with parameter $\lambda/(1-p)$) number of people. In the next generation, there will be a Poisson number of people with parameter $p\lambda/(1-p) + \lambda = \lambda/(1-p)$, which implies that the Poisson distribution with rate $\lambda/(1-p)$ is stationary for this chain.

The existence of the stationary distribution means the chain is positive-recurrent; it is certainly irreducible and aperiodic. So, the chain converges to its stationary distribution for any initial distribution.

3. **Balls and Bins: Poisson Convergence**

Consider throwing m balls into n bins uniformly at random. In this question, we will show that the number of empty bins converges to a Poisson limit, under the condition that $n\exp(-m/n) \to \lambda \in (0, \infty)$.

(a) Let $p_k(m, n)$ denote the probability that exactly k bins are empty after throwing m balls into n bins uniformly at random. Show that

$$p_0(m, n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(1 - \frac{j}{n}\right)^m.$$

(*Hint*: Use the Inclusion-Exclusion Principle.)

(b) Show that

$$p_k(m, n) = \binom{n}{k} \left(1 - \frac{k}{n}\right)^m p_0(m, n-k). \tag{1}$$

(c) Show that

$$\binom{n}{k} \left(1 - \frac{k}{n}\right)^m \le \frac{\lambda^k}{k!} \tag{2}$$

as $m, n \to \infty$ (such that $n\exp(-m/n) \to \lambda$).

(d) Show that

$$\binom{n}{k} \left(1 - \frac{k}{n}\right)^m \ge \frac{\lambda^k}{k!} \tag{3}$$

as $m, n \to \infty$ (such that $n\exp(-m/n) \to \lambda$). This is the hard part of the proof. To help you out, we have outlined a plan of attack:

i. Show that

$$\binom{n}{k} \left(1 - \frac{k}{n}\right)^m \ge \left(1 - \frac{k}{n}\right)^{k+m} \frac{n^k}{k!}.$$

ii. Show that

$$\ln\left\{n^k \left(1 - \frac{k}{n}\right)^m\right\} \to k \ln \lambda$$

as $m, n \to \infty$ (such that $n\exp(-m/n) \to \lambda$). You may use the inequality $\ln(1-x) \ge -x - x^2$ for $0 \le x \le 1/2$.

iii. Show that

$$\left(1-\frac{k}{n}\right)^k \to 1$$

as $m, n \to \infty$ (such that $n\exp(-m/n) \to \lambda$). Conclude that (3) holds.

(e) Now, show that

$$p_0(m,n) \to \exp(-\lambda).$$

(Try using the results you have already proven.) Conclude that

$$p_k(m,n) \to \frac{\lambda^k \exp(-\lambda)}{k!}.$$

Solution:

(a) The probability that there are no empty bins is, by the Inclusion-Exclusion Principle,

$$\begin{aligned}
p_0(m,n) &= \sum_{j=0}^{n}(-1)^j P(\text{some } j \text{ bins are empty}) \\
&= \sum_{j=0}^{n}(-1)^j \binom{n}{j} P(\text{a specific set of } j \text{ bins are empty}) \\
&= \sum_{j=0}^{n}(-1)^j \binom{n}{j}\left(1-\frac{j}{n}\right)^m.
\end{aligned}$$

The last equality is justified by the following reasoning: if a specific set of j bins are empty, then each of the m balls must land in one of the $n-j$ bins, which occurs with probability $(1-j/n)^m$.

(b) If there are exactly k empty boxes, then there are $\binom{n}{k}$ ways to choose which boxes are empty; $(1-k/n)^m$ is the probability that these boxes are empty; and $p_0(m, n-k)$ is the probability that none of the other $n-k$ boxes are empty.

(c) One has

$$\binom{n}{k}\left(1-\frac{k}{n}\right)^m \le \frac{n!}{k!(n-k)!}\exp\left(-\frac{km}{n}\right) \le \frac{[n\exp(-m/n)]^k}{k!} \to \frac{\lambda^k}{k!}.$$

(d) i. Observe that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \ge \frac{(n-k)^k}{k!} = \frac{n^k}{k!}\left(1-\frac{k}{n}\right)^k,$$

which implies the desired result.

ii. Although we did not ask you to prove $\ln(1-x) \ge -x-x^2$, we include a proof for completeness. Assume $0 \le x \le 1/2$. Using the power series expansion for $\ln(1-x)$,

$$\ln(1-x) = -\sum_{i=1}^{\infty}\frac{x^i}{i} \ge -x - \frac{x^2}{2}\sum_{i=2}^{\infty}\frac{2x^{i-2}}{i}.$$

Since $x^i \leq 2^{-i}$, $\sum_{i=2}^{\infty} i^{-1} 2x^{i-2} \leq \sum_{i=2}^{\infty} i^{-1} 2^{-i-1} \leq \sum_{i=2}^{\infty} 2^{-i-2} = 2$. Hence (the negative sign reverses the direction of the inequality) $\ln(1-x) \geq -x - x^2$. Applying the inequality, we have

$$\ln\left\{n^k \left(1 - \frac{k}{n}\right)^m\right\} \geq k \ln n - \frac{km}{n} - \frac{k^2 m}{n^2}.$$

Now, it's time to estimate the order of m. Since $n \exp(-m/n) \to \lambda$, one has $\ln n - m/n \to \ln \lambda$, or

$$m = n \ln n - n \ln \lambda + o(n),$$

where $o(n)$ is a term such that $o(n)/n \to 0$ as $n \to \infty$. With this estimate, we see that $k^2 m/n^2 \to 0$ as $n \to \infty$. Therefore, we ignore this term and obtain

$$\ln\left\{n^k \left(1 - \frac{k}{n}\right)^m\right\} \to k \ln n - k \ln n + k \ln \lambda = k \ln \lambda, \tag{4}$$

as desired.

iii. k is a constant, so clearly

$$\left(1 - \frac{k}{n}\right)^k \to 1 \qquad \text{as } n \to \infty. \tag{5}$$

Putting together all of the various parts together, we have

$$\binom{n}{k}\left(1 - \frac{k}{n}\right)^m \geq \left(1 - \frac{k}{n}\right)^m \left(1 - \frac{k}{n}\right)^k \frac{n^k}{k!} \xrightarrow{(4)} \left(1 - \frac{k}{n}\right)^k \frac{\lambda^k}{k!}$$
$$\xrightarrow{(5)} \frac{\lambda^k}{k!}.$$

(e) One has

$$p_0(m,n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(1 - \frac{j}{n}\right)^m \to \sum_{j=1}^{\infty} (-1)^j \frac{\lambda^j}{j!} = \exp(-\lambda).$$

Therefore, for any fixed k, $p_0(m, n-k) \to \exp(-\lambda)$. Hence, from the previous results and $p_0(m, n-k) \to \exp(-\lambda)$, we have our desired result:

$$p_k(m,n) \to \frac{\lambda^k \exp(-\lambda)}{k!}.$$

4. Poisson Practice

Let $(N(t), t \geq 0)$ be a Poisson process with rate λ. Let T_k denote the time of k-th arrival, for $k \in \mathbb{N}$, and given $0 \leq s < t$, we write $N(s,t) = N(t) - N(s)$. Compute:

(a) $\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0)$.

(b) $\mathbb{E}(N(1,3) \mid N(1,2) = 3)$.

(c) $\mathbb{E}(T_2 \mid N(2) = 1)$.

Solution:

(a) The event $\{N(1) + N(2,4) + N(3,5) = 0\}$ is the same as the intersection of 2 events, $\{N(1) = 0\}$ and $\{N(2,5) = 0\}$. These are independent with probabilities $\exp(-\lambda)$ and $\exp(-3\lambda)$. Hence

$$\mathbb{P}[N(1) + N(2,4) + N(3,5) = 0] = \exp(-4\lambda).$$

(b) $N(1,3) = N(1,2) + N(2,3)$. We know $N(2,3)$ is independent of $N(1,2)$. Hence, $\mathbb{E}(N(1,3) \mid N(1,2) = 3) = 3 + \lambda$.

(c) Since $N(2) = 1$, the second interarrival time T_2 hasn't lapsed yet at $t = 2$. From the memoryless property of the exponential distribution:

$$\mathbb{E}\big(T_2 - 2 \bigm| N(2) = 1\big) = \frac{1}{\lambda}.$$

Hence the answer is $2 + \lambda^{-1}$.

5. Basketball II

Team A and Team B are playing an untimed basketball game in which the two teams score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team.

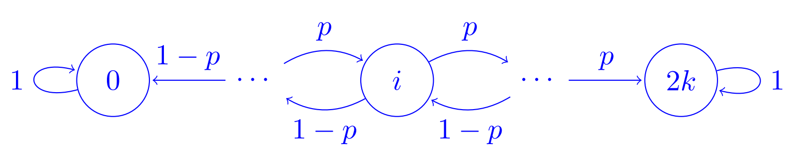
(a) Suppose $\lambda_A = \lambda_B$, and Team A has a head start of $m < k$ points. Find the probability that Team A wins.

(b) Keeping the assumptions from part (a), find the expected time $\mathbb{E}[T]$ it will take for the game to end.

Solution:

(a) We consider the merged process with rate $\lambda_A + \lambda_B$ and notice that each point is a point for A with probability $p = \lambda_A/(\lambda_A + \lambda_B)$ and a point for B with probability $1 - p = \lambda_B/(\lambda_A + \lambda_B)$. Now, we consider the embedded Markov chain, i.e. let the state of the chain be the number of additional points Team B needs to score to win. Thus, we have the transition probabilities:

$$\begin{aligned}
&P_{0,0} = 1\\
&P_{i,i+1} = p, \text{ where } 0 < i < 2k\\
&P_{i,i-1} = 1 - p, \text{ where } 0 < i < 2k\\
&P_{2k,2k} = 1
\end{aligned}$$



But since $\lambda_A = \lambda_B$, this is just the case of the symmetric gambler's ruin problem. See the first page and a half of [click here]. Since $\lambda_A = \lambda_B$, we know then that $P(A \text{ wins}) = (m + k)/2k$.

(b) Note that the waiting time X_i of each jump is iid distributed as $X \sim \text{Exponential}(2\lambda)$ where $\lambda = \lambda_A = \lambda_B$. If N_m is the number of jumps made until the game ends (starting from state m), then Wald's identity tells us that $\mathbb{E}[T] = \mathbb{E}[\sum_{i=1}^{N_m} X_i] = \mathbb{E}[N_m]\mathbb{E}[X] = \mathbb{E}[N]/2\lambda$. Or, more directly, we can see this by writing

$$\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{N_m} X_i | N_m]] = \mathbb{E}[N_m \mathbb{E}[X_i]] = \mathbb{E}[N_m]\mathbb{E}[X].$$

To compute $\mathbb{E}[N_m]$, let $r_m = \mathbb{E}[N_{m+1}] - \mathbb{E}[N_m]$. Hitting time equations give us

$$\mathbb{E}[N_m] = 1 + \frac{1}{2}\mathbb{E}[N_{m-1}] + \frac{1}{2}\mathbb{E}[N_{m+1}]$$

which can be rewritten as $r_m = r_{m-1} - 2$. In particular, we have

$$r_{2k-1} = r_0 - 2(2k-1).$$

Therefore

$$-\mathbb{E}[N_{2k-1}] = r_{2k-1} = r_0 - 2(2k-1) = \mathbb{E}[N_1] - 2(2k-1).$$

But $\mathbb{E}[N_{2k-1}] = \mathbb{E}[N_1]$ by symmetry, so we must have $\mathbb{E}[N_1] = 2k-1$. Now we can plug this base case into our recurrence ($r_m = r_{m-1} - 2$) to get that $\mathbb{E}[N_m] = (k+m)(k-m)$. Thus

$$\mathbb{E}[T] = \frac{(k+m)(k-m)}{2\lambda}.$$