UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

**Discussion 7**
Fall 2021

**1. Recurrence and Transience of Random Walks**

"A drunk man will eventually find his way home but a drunk bird may get lost forever."

Consider the symmetric random walk $S_n = X_1 + \cdots + X_n$ in dimension $d$, where we start at the origin, and with uniform probability we jump to an adjacent point on the $d$-dimensional lattice $\mathbb{Z}^d$. That is, $X_i \overset{iid}{\sim} \text{Uniform}\{\pm e_1, \dots, \pm e_d\}$, where $\{e_1, \dots, e_d\}$ are the unit coordinate vectors in $\mathbb{R}^d$.

(a) Show that if $\sum_{n=0}^{\infty} P(S_n = 0) = \infty$ then the random walk is recurrent. *Hint:* Let $N$ be the number of times the random walk visits the origin. It may help to notice that $\mathbb{E}[N] = \infty$ is equivalent to recurrence of the random walk.

(b) Use this to show that the random walk for $d = 1$ is recurrent. You may use Stirling's approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

(c) Use part (b) to show that the random walk for $d = 2$ is recurrent.

(d) (Optional) Show that the random walk for $d = 3$ is transient.

(e) Use part (d) to show that the random walk for any $d > 3$ is also transient.

**Solution:**

(a) Let $N$ be the number of times the origin is visited by the random walk. Then we may write

$$\begin{aligned}\sum_{n=0}^{\infty} P(S_n = 0) &= \mathbb{E}[\sum_{n=0}^{\infty} \mathbb{1}_{S_n=0}] \\ &= \mathbb{E}[N] \\ &= \sum_{k=1}^{\infty} P(N \geq k),\end{aligned}$$

where we used tail sum in the third line. Now, let $\tau_k$ be the time index at which the origin was visited for the $k$-th time. Then our sum becomes

$$\sum_{k=1}^{\infty} P(\tau_k < \infty) = \sum_{k=1}^{\infty} P(\tau_1 < \infty)^k.$$

where we used the (strong) markov property in $P(\tau_k < \infty) = P(\tau_1 < \infty)^k$. Since this summation diverges, we must have $P(\tau_1 < \infty) = 1$. Hence $S_n$ is recurrent. In fact the statement we have shown is true in the converse direction as well.

(b) It suffices to show that $\sum_{n=0}^{\infty} P(S_{2n}=0)$ diverges. We have

$$P(S_{2n}=0)=\binom{2n}{n}\frac{1}{2^{2n}}\sim\frac{\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}{n\left(\frac{n}{e}\right)^{2n}}\cdot\frac{1}{2^{2n}}=\frac{\sqrt{2}}{\sqrt{n}}.$$

Since $\sum_n n^{-1/2}$ diverges, we deduce from part (a) that $S_n$ is reccurent in dimension 1.

(c) We may do a similar combinatorial computation as we did in part (b) to deduce recurrence. However, let's try a nicer approach. Consider two independent one-dimensional random walks

$$R_n := (S_n^{(1)}, S_n^{(2)}).$$

By rotating the coordinate plane by 45 degrees, we note that this is equivalent to a symmetric random walk $S_n$ on $\mathbb{Z}^2$. In particular, note that

$$P(S_{2n}=[0,0])=P(S_{2n}^{(1)}=0)P(S_{2n}^{(2)}=0)\sim\left(\frac{1}{\sqrt{n}}\right)^2=\frac{1}{n}$$

by part (b). As $\sum_n n^{-1}$ is still divergent, we conclude that $S_n$ is recurrent for $d=2$ as well.

(d) (Optional) We have (where the first sum is over $0\le j,k\le 2n$ satisfying $j+k\le 2n$)

$$\begin{aligned}P(S_{2n}=0)&=6^{-2n}\sum_{j,k}\frac{(2n)!}{(j!k!(n-j-k)!)^2}\\&=2^{-2n}\binom{2n}{n}\sum_{j,k}\left(3^{-n}\frac{n!}{j!k!(n-j-k)!}\right)^2\\&\sim\frac{1}{\sqrt{n}}\sum_{j,k}\left(3^{-n}\frac{n!}{j!k!(n-j-k)!}\right)^2\end{aligned}$$

At this point, we may use Stirling's approximation to further simplify the summation, and eventually we will get that $P(S_{2n}=0)\sim O(n^{-3/2})$. Since $\sum_k n^{-3/2}<\infty$, we see that $S_n$ is transient for $d=3$.

(e) Suppose $d>3$. Let $S_n^{(i)}$ denote the $i$-th coordinate of our random walk $S_n$. Then we can generate a 3-dimensional random walk from the process $(S_n^{(1)}, S_n^{(2)}, S_n^{(3)})$. Since random walks in $d=3$ are transient, note that

$$P(S_n=0\text{ i.o.})\le P((S_n^{(1)},S_n^{(2)},S_n^{(3)})=0\text{ i.o})<1.$$

Hence $S_n$ is also transient for any $d>3$.

**2. Arrival Times of a Poisson Process**

Consider a Poisson process $(N_t, t\ge 0)$ with rate $\lambda=1$. For $i\in\mathbb{Z}_{>0}$, let $T_i$ be a random variable which is equal to the time of the $i$-th arrival.

(a) Find $\mathbb{E}[T_3 \mid N_1=2]$.

(b) Given $T_3=s$, where $s>0$, find the joint distribution of $T_1$ and $T_2$.

(c) Find $\mathbb{E}[T_2 \mid T_3=s]$.

**Solution:**

(a) By the memoryless property, $\mathbb{E}[T_3 \mid N_1 = 2] = 1 + \mathbb{E}[T_1] = 2$.

(b) We know the distribution of sum of IID exponential random variables is Erlang. So, since the inter-arrival times of Poisson process are exponentially distributed we have

$$f_{T_i}(s) = \frac{s^{i-1}e^{-s}}{(i-1)!}\mathbf{1}\{s \geq 0\}.$$

$$\begin{aligned} f_{T_1,T_2|T_3}(s_1, s_2 \mid T_3 = s) &= \frac{f_{T_1,T_2,T_3}(s_1, s_2, s)}{f_{T_3}(s)} \\ &= \frac{e^{-s_1}e^{-(s_2-s_1)}e^{-(s-s_2)}}{s^2e^{-s}/2!}\mathbf{1}\{0 \leq s_1 \leq s_2 \leq s\} \\ &= \frac{2}{s^2}\mathbf{1}\{0 \leq s_1 \leq s_2 \leq s\}. \end{aligned}$$

Thus, $T_1$ and $T_2$ are uniformly distributed on the feasible region $\{0 \leq s_1 \leq s_2 \leq s\}$. In particular, the joint distribution is precisely that of the order statistics of two i.i.d. Uniform$[0, 2]$ random variables.

(c) By part (b), $T_2$ is the maximum of two uniform random variables between 0 and $s$. Thus, if $0 \leq x \leq s$,

$$F_{T_2|T_3=s}(x) = P(T_2 \leq x \mid T_3 = s) = \left(\frac{x}{s}\right)^2$$

and

$$f_{T_2|T_3=s}(x) = \frac{2x}{s^2}\mathbf{1}\{0 \leq x \leq s\}.$$

Finally,

$$\mathbb{E}[T_2 \mid T_3 = s] = \int_0^s \frac{2x^2}{s^2}\,\mathrm{d}x = \frac{2s}{3}.$$

3. **Illegal U-Turns**

Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate $\lambda$. You decide to make a U-turn once you see that the road has been clear of police cars for $\tau > 0$ units of time. Let $N$ be the number of police cars you see before you make a U-turn.

(a) Find $\mathbb{E}[N]$.

(b) Let $n$ be a positive integer $\geq 2$. Find the conditional expectation of the time elapsed between police cars $n-1$ and $n$, given that $N \geq n$.

(c) Find the expected time that you wait until you make a U-turn.

**Solution:**

(a) The random variable $N$ is equal to the number of successive interarrival intervals that are smaller than $\tau$. Interarrival intervals are independent and each one is smaller than $\tau$ with probability $1 - e^{-\lambda\tau}$. So $P(N = k) = e^{-\lambda\tau}(1 - e^{-\lambda\tau})^k$, and $N$ is a shifted geometric random variable with parameter $p = e^{-\lambda\tau}$ that starts from 0 (i.e., $N + 1 \sim$ Geometric$(p)$). Thus, $\mathbb{E}[N] = 1/p - 1 = e^{\lambda\tau} - 1$.

(b) Let $S_n$ be the $n$th interarrival time. The event $\{N \geq n\}$ indicates that the time between cars $n-1$ and $n$ is less than or equal to $\tau$. So we want to compute

$$\mathbb{E}[S_n \mid S_n < \tau] = \frac{\int_0^\tau t\lambda e^{-\lambda t}\,\mathrm{d}t}{\int_0^\tau \lambda e^{-\lambda t}\,\mathrm{d}t}.$$

Using integration by part for the integral in the numerator, we find that the answer is

$$= \frac{1/\lambda - (\tau + 1/\lambda)e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.$$

(c) You make the U-turn at time $T = S_1 + S_2 + \cdots + S_N + \tau$ and $S_i \leq \tau$ for $i \in \{1, \ldots, N\}$. Then, using Parts (a) and (b),

$$\begin{aligned}
\mathbb{E}[T] &= \tau + \sum_{n=0}^{\infty} P(N = n)\mathbb{E}[S_1 + \cdots + S_N \mid N = n] \\
&= \tau + \sum_{n=0}^{\infty} P(N = n) n \mathbb{E}[S_i \mid S_i \leq \tau] \\
&= \tau + (e^{\lambda\tau} - 1) \times \frac{1/\lambda - (\tau + 1/\lambda)e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.
\end{aligned}$$