UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

# Problem Set 8

Fall 2021

1. **Geometric Sum of Exponentials**

   Let $X_1, X_2, \ldots$ be iid exponentials with parameter $\lambda$. If $N \sim \text{Geom}(p)$ taking values on $\{1, 2, \ldots\}$, then show that
   $$\sum_{i=1}^{N} X_i$$
   is exponential and determine its parameter. *Hint:* Consider Poisson thinning.

   **Solution:** By thinning a Poisson process with rate $\lambda$ with probability $p$, we see that the first arrival is precisely the sum of $N$ iid exponentials. Since thinning produces a $p\lambda$ process, it follows that
   $$\sum_{i=1}^{N} X_i \sim \text{Exponential}(p\lambda).$$

2. **Bus Arrivals at Cory Hall**

   Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate $\lambda$. Students arrive at the stop according to an independent Poisson process of rate $\mu$. Every time the bus arrives, all students waiting get on.

   (a) Given that the interarrival time between bus $i-1$ and bus $i$ is $x$, find the distribution for the number of students entering the $i$th bus. (Here, $x$ is a given number, not a random quantity.)

   (b) Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

   (c) Find the distribution of the number of students getting on the next bus to arrive after 9:30 AM, assuming that time 0 was infinitely far in the past.

   **Solution:**

   (a) We note that the student arrival process is independent of the bus arrival process and thus, the number of arrivals to the student arrival process in the interval of size $x$ is a Poisson random variable with parameter $\mu x$.

   (b) **Solution 1:** We note that the student arrival process and the bus arrival process are independent Poisson processes, and we can thus consider the merged Poisson process with parameter $\lambda+\mu$. Each arrival for the combined process is a bus with probability $\lambda/(\lambda+\mu)$ and likewise each arrival for the combined process is a student with probability $\mu/(\lambda+\mu)$. The sequence of bus/student choices is an IID sequence, so starting immediately after bus of 9:30AM, the number of combined arrivals until we see a bus arrival for the first time is a geometric random variable with parameter $\lambda/(\lambda+\mu)$. Thus, if we let $N$ give the number of students entering the next bus after 9:30AM, we see that:

$$P(N = k) = \Big(\frac{\mu}{\lambda+\mu}\Big)^k \frac{\lambda}{\lambda+\mu},\ k \in \mathbb{N}.$$

**Solution 2:** Let $T \sim \text{Exponential}(\lambda)$ be the interarrival time between the bus that arrived at 9:30AM and the next bus. Let $N$ be the number of students who arrived between 9:30AM and 9:30AM + $T$. Then we know that $N \mid T = t \sim \text{Poisson}(\mu t)$. So using the total probability theorem we have that

$$\begin{aligned}
P(N = k) &= \int_0^\infty P(N = k \mid T = t) f_T(t)\,\mathrm{d}t \\
&= \int_0^\infty \epsilon^{-\mu t} \frac{(\mu t)^k}{k!} \lambda \epsilon^{-\lambda t}\,\mathrm{d}t \\
&= \frac{\mu^k}{k!} \frac{\lambda}{\lambda+\mu} \int_0^\infty t^k (\lambda+\mu) \epsilon^{-(\lambda+\mu)t}\,\mathrm{d}t \\
&= \frac{\mu^k}{k!} \frac{\lambda}{\lambda+\mu} \mathbb{E}[X^k],\ \ [X \sim \text{Exponential}(\lambda+\mu)] \\
&= \frac{\mu^k}{k!} \frac{\lambda}{\lambda+\mu} \frac{k!}{(\lambda+\mu)^k}, \\
&= \Big(\frac{\mu}{\lambda+\mu}\Big)^k \frac{\lambda}{\lambda+\mu},\ \ k \in \mathbb{N}.
\end{aligned}$$

(c) This part is based on the random incidence property of the Poisson process.

**Solution 1:** We note that in Part (b), we found the number of future student arrivals before the next bus. What we are looking for is the sum of the number of students waiting at 9:30AM and the number of future student arrivals before the next bus. Let $N_1$ be the number of students already at the bus stop at 9:30AM, and let $N_2$ be the number of students arriving after 9:30AM until the next bus. First observe that $N_1$ and $N_2$ are independent by definition of the Poisson process. Also note that $N_1$ and $N_2$ have the same PMF, the one found in Part (b), because we assume that time 0 was infinitely far in the past, and so $N_1$ can be considered as $N_2$ if we consider the Poisson process backwards in time. So in order to find the PMF of $N = N_1 + N_2$, the number of students getting on the next bus to arrive after 9:30AM we just need to convolve the PMFs of $N_1$ and $N_2$

$$\begin{aligned}
P(N = k) &= P(N_1 + N_2 = k) \\
&= \sum_{i=0}^{k} P(N_1 = i \mid N_2 = k-i) P(N_2 = k-i) \\
&= \sum_{i=0}^{k} P(N_1 = i) P(N_2 = k-i) \\
&= \sum_{i=0}^{k} \Big(\frac{\mu}{\lambda+\mu}\Big)^i \frac{\lambda}{\lambda+\mu} \Big(\frac{\mu}{\lambda+\mu}\Big)^{k-i} \frac{\lambda}{\lambda+\mu} \\
&= (k+1) \Big(\frac{\mu}{\lambda+\mu}\Big)^k \Big(\frac{\lambda}{\lambda+\mu}\Big)^2,\ \ k \in \mathbb{N}.
\end{aligned}$$

**Solution 2:** Let $T_1$ be the time from the bus arrival right before 9:30AM until 9:30AM (we know that such a bus arrival exists, because we assume that time 0 was infinitely far in the past), and let $T_2$ be the time from 9:30AM until the next bust arrival. We know that $T_1, T_2$ are i.i.d. Exponential($\lambda$), and $E = T_1 + T_2 \sim$ Erlang$(2, \lambda)$. Let $N$ be the number of students who get on the next bus after 9:30AM, then $N \mid E = t \sim$ Poisson$(\mu t)$. So using the total probability theorem

$$
\begin{aligned}
P(N = k) &= \int_0^\infty P(N = k \mid T = t) f_E(t)\,\mathrm{d}t \\
&= \int_0^\infty \epsilon^{-\mu t} \frac{(\mu t)^k}{k!} \lambda^2 t \epsilon^{-\lambda t}\,\mathrm{d}t \\
&= \frac{\mu^k}{k!} \frac{\lambda^2}{\lambda+\mu} \int_0^\infty t^{k+1} (\lambda+\mu) \epsilon^{-(\lambda+\mu)t}\,\mathrm{d}t \\
&= \frac{\mu^k}{k!} \frac{\lambda^2}{\lambda+\mu} \mathbb{E}[X^{k+1}], \quad [X \sim \text{Exponential}(\lambda+\mu)] \\
&= \frac{\mu^k}{k!} \frac{\lambda^2}{\lambda+\mu} \frac{(k+1)!}{(\lambda+\mu)^{k+1}}, \quad [\text{Discussion 4, Problem 2(b)}] \\
&= (k+1) \Big(\frac{\mu}{\lambda+\mu}\Big)^k \Big(\frac{\lambda}{\lambda+\mu}\Big)^2, \ k \in \mathbb{N}.
\end{aligned}
$$

3. **Frogs**

Three frogs are playing near a pond. When they are in the sun they get too hot and jump in the lake at rate 1. When they are in the lake they get too cold and jump onto the land at rate 2. The rates here refer to the rate in exponential distribution. Let $X_t$ be the number of frogs in the sun at time $t \geq 0$.

(a) Find the stationary distribution for $(X_t)_{t\geq 0}$.

(b) Check the answer to (a) by noting that the three frogs are independent two-state Markov chains.

**Solution:**

(a) Let the states $S = \{0, 1, 2, 3\}$ be the number of frogs in the sun. The Markov chain has $\lambda_0 = 6$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\mu_3 = 3$, $\mu_2 = 2$, and $\mu_1 = 1$. Here $\lambda_i$ and $\mu_i$ are the rates of jumping forward and backward from state $i \in S$, respectively. Using detailed balance, we compute the stationary distribution to be

$$\pi = \frac{1}{27} \begin{bmatrix} 1 & 6 & 12 & 8 \end{bmatrix}.$$

(b) The individual frogs follow independent Markov chains, each with stationary distribution

$$\pi = \frac{1}{3} \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

The probability of being in state $i \in S$ is therefore

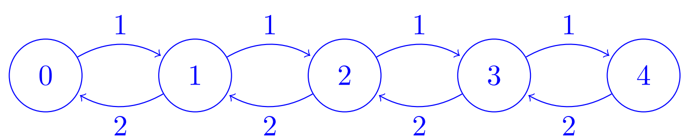
$$P(X_t = i) = \binom{3}{i} \Big(\frac{1}{3}\Big)^{3-i} \Big(\frac{2}{3}\Big)^i, \qquad i \in S.$$

4. **Taxi Queue**

Empty taxis pass by a street corner according to a Poisson process of rate two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner according to a Poisson process of rate one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

**Solution:**

Consider a continuous time Markov chain with states $X \in \{0, 1, 2, 3, 4\}$ which denotes the number of people waiting. For $n = 0, 1, 2, 3$, the transitions from $n$ to $n + 1$ have rate 1, and the transitions from $n + 1$ to $n$ have rate 2.




The balance equations are then

$$\pi(n) = \frac{1}{2}\pi(n-1), \qquad n = 1, 2, 3, 4.$$

Using the above equations and $\sum_{i=0}^{4} \pi(i) = 1$ we find that $\pi(i) = 2^{-i}\pi(0)$ and $\pi(0) = 16/31$. Since the expected waiting time for a new taxi is 0.5, the expected waiting time of John given that he joins the queue can be computed as follows.

$$\mathbb{E}[T] = \frac{\pi(0) \times 0.5 + \pi(1) \times 1 + \pi(2) \times 1.5 + \pi(3) \times 2}{\pi(0) + \pi(1) + \pi(2) + \pi(3)} = \frac{13}{15}.$$

The denominator represents the fact that we are conditioning on the event that John joins the queue.

5. $M/M/2$ **Queue**

A queue has Poisson arrivals with rate $\lambda$. It has two servers that work in parallel. When there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. exponential random variables with rate $\mu$. Let $X(t)$ be the number of customers either in the queue or in service at time $t$.

(a) Argue that the process $(X(t), t \geq 0)$ is a Markov process.

(b) Draw the state transition diagram.

(c) Find the range of values of $\mu$ for which the Markov chain is positive-recurrent and for this range of values calculate the stationary distribution of the Markov chain.

**Solution:**

(a) The queue length is a MC as customer arrivals are independent of the current number of customers in the queue. Also, the departures only depend on the current number of customers being served. Next, even when $k$ ($k = 1, 2$) customers are being served, the completion of their service is independent of one another. Finally, when $k = 2$, even if one of the customers has been completely served, the other customer has the same service time distribution as before as the exponential distribution is memoryless.

(b) It is shown in the following figure.

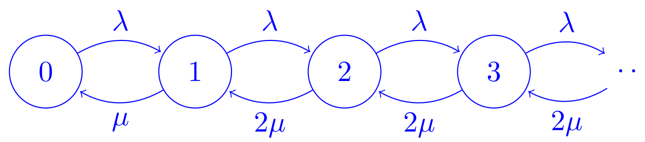



Figure 1: Markov chain for a queue with two servers.

(c) It suffices to solve the detailed balance equations, i.e.

$$\pi(1) = \frac{\lambda}{\mu}\pi(0)$$
$$\pi(i+1) = \frac{\lambda}{2\mu}\pi(i), \qquad i \in \mathbb{Z}_+.$$

Iterating these recurrences yields an expression for the stationary distribution:

$$\pi(i) = \left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^{i-1}\pi(0).$$

Also, we know that the stationary distribution must normalize,

$$\sum_{i=0}^{\infty}\pi(i) = 1.$$

This allows us to solve for $\pi(0)$,

$$\pi(0)\left[1 + \sum_{i=1}^{\infty}\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^{i-1}\right] = 1.$$

The series converges if $\lambda < 2\mu$, and in this case the Markov chain is positive-recurrent. Then, solving the equation we have

$$\pi(0) = \frac{2\mu - \lambda}{2\mu + \lambda}.$$

Hence,

$$\pi(i) = \frac{2\mu - \lambda}{2\mu + \lambda}\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^{i-1}, \qquad i \in \mathbb{Z}_+.$$