UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 3
Fall 2021

1. **Triangle Density**

   Consider random variables X and Y which have a joint PDF uniform on the triangle with vertices at $(0,0),(1,0),(0,1)$.

   (a) Find the joint PDF of X and Y.

   (b) Find the marginal PDF of Y.

   (c) Find the conditional PDF of X given Y.

   (d) Find $\mathbb{E}[X]$ in terms of $\mathbb{E}[Y]$.

   (e) Find $\mathbb{E}[X]$.

2. **Conditional Distribution of a Poisson Random Variable with Exponentially Distributed Parameter**

   Let X have a Poisson distribution with parameter $\lambda > 0$. Suppose λ itself is random, having an exponential density with parameter $\theta > 0$.

   (a) Show that

   $$\mathbb{E}(\lambda^k) = \frac{k!}{\theta^k}, \qquad k \in \mathbb{N}$$

   (b) What is the distribution of X?

   (c) Determine the conditional density of λ given $X = k$, where $k \in \mathbb{N}$.

3. **Poisson Merging**

   The Poisson distribution is used to model *rare events*, such as the number of customers who enter a store in the next hour. The theoretical justification for this modeling assumption is that the limit of the binomial distribution, as the number of trials n goes to ∞ and the probability of success per trial p goes to 0, such that $np \to \lambda > 0$, is the Poisson distribution with mean λ.

   Now, suppose we have two independent streams of rare events (for instance, the number of female customers and male customers entering a store), and we do not care to distinguish between the two types of rare events. Can the combined stream of events be modeled as a Poisson distribution?

   Mathematically, let X and Y be independent Poisson random variables with means λ and μ respectively. Prove that $X + Y \sim \text{Poisson}(\lambda + \mu)$. (This is known as **Poisson merging**.) Note that it is **not** sufficient to use linearity of expectation to say that $X + Y$ has mean $\lambda + \mu$. You are asked to prove that the *distribution* of $X + Y$ is Poisson.

   Note: This property will be extensively used when we discuss Poisson processes.