UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 6
Fall 2021

1. Finite Boundary Times

Consider the random walk $S_n = \sum_{i=1}^{n} X_i$, where the X_i are iid with mean zero and variance 1 (note that they do not have to be discrete). Show that almost surely the random walk will leave the interval $[-a, a]$ in finite time.

Hint: Let T be the first time that the walk leaves the interval $[-a, a]$, and show that $\lim_{n\to\infty} P(T > n) = 0$.

2. Confidence Interval Comparisons

In order to estimate the probability of a head in a coin flip, p, you flip a coin n times, where n is a positive integer, and count the number of heads, S_n. You use the estimator $\hat{p} = S_n/n$.

(a) You choose the sample size n to have a guarantee

$$P(|\hat{p} - p| \geq \epsilon) \leq \delta.$$

Using Chebyshev Inequality, determine n with the following parameters. Note that you should not have p in your final answer.

(i) Compare the value of n when $\epsilon = 0.05$, $\delta = 0.1$ to the value of n when $\epsilon = 0.1, \delta = 0.1$.

(ii) Compare the value of n when $\epsilon = 0.1, \delta = 0.05$ to the value of n when $\epsilon = 0.1, \delta = 0.1$.

(b) Now, we change the scenario slightly. You know that $p \in (0.4, 0.6)$ and would now like to determine the smallest n such that

$$P\Big(\frac{|\hat{p} - p|}{p} \leq 0.05\Big) \geq 0.95.$$

Use the CLT to find the value of n that you should use. *Recall that the CLT states that the sum of IID random variables tends to a normal distribution with the sample mean and variance as it's parameters for n large enough.*

3. Characteristic Function Basics

The definition of the characteristic function for random variable X is $\varphi_X(t) = \mathbb{E}[e^{itX}]$. It has many important properties - most notably that there is a bijection between the CDF (and therefore also PDF) of a random variable and its characteristic function. This problem goes over some of its basic properties.

(a) Let X be a Rademacher random variable, i.e. takes values ± 1 each with probability 1/2. Show that $\varphi_X(t) = \cos(t)$.

(b) Let X be a uniform random variable on the interval $[a, b]$. Show that

$$\varphi_X(t) = \frac{e^{itb} - e^{ita}}{it(b-a)}.$$

. What happens if $b = -a$?

(c) Show that $\varphi_X(-t) = \overline{\varphi_X(t)}$, where the bar means take the complex conjugate. Use this fact to argue that if the distribution of X is symmetric about the origin, then the characteristic function is strictly real.

(d) Show that

$$\varphi_X^{(k)}(t)\Big|_{t=0} = i^k \mathbb{E}[X^k].$$

This can be particularly useful for computing higher moments of random variables.

(e) Show that that for independent $X_1, ..., X_n$ and scalars $a_1, ..., a_n$,

$$\varphi_{a_1X_1+...+a_nX_n}(t) = \varphi_{X_1}(a_1t) \cdot ... \cdot \varphi_{X_n}(a_nt).$$

This can be particularly useful for finding the distribution of $X + Y$ without having to deal with a convolution (in particular it tells us that convolution corresponds to multiplication in the fourier domain, a concept which may be familiar if you've taken some signals courses).