UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 5
Fall 2021

1. **Midterm**

   Solve all of the problems on the midterm again (including the ones you got correct).

   Solution:

   See midterm solutions.

2. **Convergence in Probability**

   Let $(X_n)_{n=1}^{\infty}$, be a sequence of i.i.d. random variables distributed uniformly in $[-1, 1]$. Show that the following sequences $(Y_n)_{n=1}^{\infty}$ converge in probability to some limit.

   (a) $Y_n = \prod_{i=1}^{n} X_i$.
   (b) $Y_n = \max\{X_1, X_2, \ldots, X_n\}$.
   (c) $Y_n = (X_1^2 + \cdots + X_n^2)/n$.

   Solution:

   (a) By independence of the random variables,

   $$\mathbb{E}[Y_n] = \mathbb{E}[X_1] \cdots \mathbb{E}[X_n] = 0,$$
   $$\operatorname{var} Y_n = \mathbb{E}[Y_n^2] = (\operatorname{var} X_1)^n = \left(\frac{1}{3}\right)^n.$$

   Now since $\operatorname{var} Y_n \to 0$ as $n \to \infty$, by Chebyshev's Inequality the sequence converges to its mean, that is, 0, in probability.

   (b) Consider $\epsilon \in [0, 1]$. We see that:

   $$\begin{aligned} P(|Y_n - 1| \geq \epsilon) &= P(\max\{X_1, \ldots, X_n\} \leq 1 - \epsilon) \\ &= P(X_1 \leq 1 - \epsilon, \ldots, X_n \leq 1 - \epsilon) \\ &= P(X_1 \leq 1 - \epsilon)^n = \left(1 - \frac{\epsilon}{2}\right)^n \end{aligned}$$

   Thus, $P(|Y_n - 1| \geq \epsilon) \to 0$ as $n \to \infty$ and we are done.

   (c) The expectation is

   $$\mathbb{E}[Y_n] = \frac{1}{n} \cdot n\mathbb{E}[X_1^2] = \frac{1}{3}.$$

   Then, we bound the variance.

   $$\operatorname{var} Y_n = \frac{1}{n} \operatorname{var} X_1^2 \leq \frac{1}{n} \to 0 \qquad \text{as } n \to \infty,$$

   since $X_1^2 \leq 1$. Hence, we see that $Y_n \to 1/3$ in probability as $n \to \infty$.

3. **Gambling Game**

Let's play a game. You stake a positive initial amount of money w_0. You toss a fair coin. If it is heads you earn an amount equal to three times your stake, so you quadruple your wealth. If it comes up tails you lose everything. There is one requirement though, you are not allowed to quit and have to keep playing, by staking all your available wealth, over and over again.

Let W_n be a random variable which is equal to your wealth after n plays.

(a) Find $\mathbb{E}[W_n]$ and show that $\lim_{n\to\infty} \mathbb{E}[W_n] = \infty$.

(b) Since $\lim_{n\to\infty} \mathbb{E}[W_n] = \infty$, this game sounds like a good deal! But wait a moment!! Where does the sequence of random variables $\{W_n\}_{n\geq 0}$ converge almost surely (i.e. with probability 1) to?

Solution:

(a) $\mathbb{E}[W_n \mid W_{n-1} = w_{n-1}] = \frac{1}{2} \cdot 4w_{n-1} + \frac{1}{2} \cdot 0$, hence

$$\mathbb{E}[W_n] = 2\mathbb{E}[W_{n-1}].$$

Unfolding the recursion, with $\mathbb{E}[W_0] = w_0$, yields $\mathbb{E}[W_n] = 2^n w_0$, and so $\lim_{n\to\infty} \mathbb{E}[W_n] = \infty$.

(b) The probability that all the first coin flips will come up heads is 2^{-n}, therefore the probability that the coin flips will keep on coming up heads forever is zero. This means that with probability one at some point you will lose your entire wealth, i.e. $\mathbb{P}(W_n = 0) \to 1$ as $n \to \infty$.

4. **Twitch Plays Pokemon**

You wake up one day and are surprised to see that it is 2014, when the world was peaceful. You immediately start playing Twitch Plays Pokemon. Suddenly, you realized that you may be able to analyze Twitch Plays Pokemon.

You		
		Stairs

Figure 1: Part (a)

(a) The player in the top left corner performs a random walk on the 8 checkered squares and the square containing the stairs. At every step the player is equally likely to move to any of the squares in the four cardinal directions (North, West, East, South) if there is a square in that direction. Find the expected number of moves until the player reaches the stairs in Figure 1.
[Hint: Use symmetry to reduce the number of states in your Markov chain]

You		
Stairs		Stairs

Figure 2: Part (b)

(b) The player randomly walks in the same way as in the previous part. Find the probability that the player reaches the stairs in the bottom right corner in Figure 2.

[Hint: For each squared box, assign a variable that denotes the probability of reaching the "good" stairs. Use symmetry again to reduce the number of such variables.]

Hint: For both problems use symmetry to reduce the number of states and variables. The equations are very reasonable to solve by hand.

Solution:

(a) Using symmetry, the 9 states can be grouped as follows.

a	b	c
b	d	e
c	e	f

Now, observe that state d is equivalent to state c.

a	b	c
b	c	e
c	e	f

With the above states, one can write down the following first-step equations.

$$\begin{aligned}
T_a &= 1 + T_b \\
T_b &= 1 + \frac{1}{3}T_a + \frac{2}{3}T_c \\
T_c &= 1 + \frac{1}{2}T_b + \frac{1}{2}T_e \\
T_e &= 1 + \frac{2}{3}T_c + \frac{1}{3}T_f \\
T_f &= 0
\end{aligned}$$

Solving the above equations gives:

$$T_a = 18, T_b = 17, T_c = 15, T_e = 11$$

Thus, the player has to make 18 moves to go downstairs on average.

(b) Consider 9 initial states and corresponding probabilities of reaching the "good" stairs as follows. Using symmetry, one can obtain the following table.

p	$\frac{1}{2}$	$1-p$
q	$\frac{1}{2}$	$1-q$
0	$\frac{1}{2}$	1

With the above probabilities, one can write down the following first-step equations.

$$p = \frac{1}{2}q + \frac{1}{2} \cdot \frac{1}{2}$$

$$q = \frac{1}{3}p + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0$$

Solving the above equations gives:

$$p = 0.4, q = 0.3$$

Thus, we are going to reach the good stairs with probability 0.4.

5. **Discrete Uniform Records**

Consider a Markov chain (X_n, Y_n) that moves on $\mathbb{N}_0^2$ (where by $\mathbb{N}_0$ we mean $\mathbb{N} \cup \{0\}$) as follows. From (i, j) the chains moves to either $(i+1, j)$ or (i, k) for some $0 \le k < j$, with each of these $j + 1$ possibilities chosen uniformly at random. Let $T = \min\{n \ge 0 : Y_n = 0\}$ be the first time the chain hits the x-axis.

(a) Find a recurrence for $\mathbb{E}[T]$ for any initial position $(X_0, Y_0) = (i, j)$.

(b) Find the distribution of X_T for any initial position (i, j). *Hint:* Develop first step equations for the moment generating function $M_{X_T}(s) = \mathbb{E}_{i,j}[e^{sX_T}]$. Later we will learn about characteristic functions $\varphi_X(t) = \mathbb{E}[e^{itX}]$, which are essentially the Fourier transforms of our random variables (whereas the m.g.f. is the Laplace transform). The m.g.f. and characteristic functions both have the property of carrying complete information about the distribution of a random variable. For the purposes of this class, we may think of these two transforms as equivalent.

Solution:

(a) Let $t_{i,j} = \mathbb{E}[T]$ for starting position (i, j). Clearly, we have $t_{i,0} = 0$ for all i. When $j = 1$, we have a geometric random variable, so $t_{i,1} = 2$. For $j = 2$, we have the first step equation

$$t_{i,2} = 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot t_{i+1,2}.$$

Since $t_{i,2} = t_{i+1,2}$, we have $t_{i,2} = \frac{5}{2}$ for all i. Continuing in this way, we have the general equation

$$t_{i,k} = 1 + \frac{1}{k+1} t_{i+1,k} + \frac{1}{k+1} \sum_{m=0}^{k} t_{i,m},$$

which we can turn into the recurrence

$$t_{i,k} = \frac{k+1}{k} \left(1 + \frac{1}{k+1} \sum_{m=1}^{k} t_{i,m} \right) = \frac{k+1}{k} + \frac{1}{k} \sum_{m=1}^{k} t_{i,m}.$$

(b) Denote by $g_{i,j}(z) = \mathbb{E}_{i,j}[z^{X_T}]$. Then note that for any i,

$$g_{i,0} = z^i.$$

From first step analysis, we have

$$g_{i,j}(z) = \frac{1}{j+1} \left(g_{i+1,j}(z) + \sum_{k=0}^{j-1} g_{i,k}(z) \right).$$

Noting that $g_{i+1,j} = z \cdot g_{i,j}$, we can then see either by inspection or by doing a simple induction that the solution to these equations is

$$g_{i,j}(z) = \frac{z^i}{2-z} \quad \forall i \geq 0, j \geq 1.$$

Thus, for any $j \geq 1$, the p.g.f. tells us that

$$\sum_{k=0}^{\infty} P(X_T = k) z^k = \mathbb{E}_{i,j}[z^{X_T}] = \frac{z^i}{2-z}.$$

But as this is the formula for a geometric series, we deduce that the distribution of X_T is given by for $j > 0$.

$$P_{i,j}(X_T = k) = \begin{cases} 2^{-k+i-1} & k \geq i, \\ 0 & k < i. \end{cases}$$

For $j = 0$, it is clearly just $P_{i,0}(X_T = i) = 1$.

6. **Noisy Guessing**

Let X, Y, and Z be i.i.d. with the standard Gaussian distribution. Find $\mathbb{E}[X \mid X+Y, X+Z, Y-Z]$.

Hint: Argue that the observation $Y - Z$ is redundant.

Solution:

Since $Y - Z = X + Y - (X + Z)$, we have

$$\mathbb{E}(X \mid X+Y, X+Z, Y-Z) = \mathbb{E}(X \mid X+Y, X+Z).$$

First, we calculate $\mathbb{E}(X \mid X+Y) = (X+Y)/2$ by symmetry. Also,

$$\mathbb{E}(X + Z \mid X+Y) = \mathbb{E}(X \mid X+Y) = \frac{X+Y}{2},$$

so the innovation is $X + Z - (X+Y)/2 = (X - Y + 2Z)/2$. Thus,

$$\begin{aligned} \operatorname{cov}\Big(X, \frac{X-Y+2Z}{2}\Big) &= \frac{1}{2}, \\ \operatorname{var} \frac{X-Y+2Z}{2} &= \frac{3}{2}, \end{aligned}$$

and so $\mathbb{E}(X \mid (X-Y+2Z)/2) = (X-Y+2Z)/6$. Hence,

$$\begin{aligned} \mathbb{E}(X \mid X+Y, X+Z) &= \frac{X+Y}{2} + \frac{X-Y+2Z}{6} = \frac{2}{3}X + \frac{1}{3}Y + \frac{1}{3}Z \\ &= \frac{1}{3}(X+Y+X+Z). \end{aligned}$$