UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 4
Fall 2021

1. Reversible Distributions

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathcal{S}$.

(a) Show that for every $m, k \in \mathbb{N}$, with $m \geq 1$, we have

$$P(X_k = i_0 \mid X_{k+1} = i_1, \ldots, X_{k+m} = i_m) = P(X_k = i_0 \mid X_{k+1} = i_1),$$

for all states $i_0, i_1, \ldots, i_m \in \mathcal{S}$. This is the *backwards Markov property.*

(b) In general, is the reversed chain (i.e. the chain $Y_k := X_{-k}$ for $k \in -\mathbb{N}$) a temporally homogeneous Markov chain? If not, provide an example.

(c) Show that if, in addition, the chain is reversible and started from a stationary distribution $X_0 \sim \pi$, then

$$(X_0, \ldots, X_n) \stackrel{d}{=} (X_n, \ldots, X_0).$$

2. Markov Chain Practice

Consider a Markov chain with three states 0, 1, and 2. The transition probabilities are $P(0,1) = P(0,2) = 1/2$, $P(1,0) = P(1,1) = 1/2$, and $P(2,0) = 2/3$, $P(2,2) = 1/3$.

(b) In the long run, what fraction of time does the chain spend in state 1?

(c) Suppose that X_0 is chosen according to the steady state distribution. What is $P(X_0 = 0 \mid X_2 = 2)$?

3. More Almost Sure Convergence

(a) Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

(b) Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

$$P\big(X_n = (y + n^{-1})^{-1} \bigm| Y = y\big) = 1.$$

Does $(X_n)_{n=1}^{\infty}$ converge a.s.?

(c) Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \ldots, 2^{k+1} - 1\}$ and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

(d) Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as $n \to \infty$?