UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 9
Fall 2021

1. **Reversibility of CTMCs**

We say a CTMC with rate matrix Q is *reversible* if there is a distribution p satisfying the detailed balance equations:

$$p_i q_{ij} = p_j q_{ji} \quad \forall i, j.$$

Show that if a CTMC is reversible w.r.t. p, then p is a stationary distribution for the chain. Furthermore, show that in this case the embedded chain is also reversible. *Remark.* The converse is true too, i.e. the CTMC is reversible given that the embedded chain is reversible.

Solution: For $1 \le j \le n$, we have

$$(pQ)_j = \sum_{i=1}^{n} p_i q_{ij} = \sum_{i=1}^{n} p_j q_{ji} = p_j \sum_{i=1}^{n} q_{ji} = 0.$$

Hence p is a stationary distribution.

Now, recall the formula for the stationary of the embedded chain given that of the CTMC:

$$\pi_i = \frac{p_i q_i}{\sum_j p_j q_j}.$$

Since $P(i, j) = q_{ij}/q_i$, it follows that

$$\pi_i P(i, j) = \frac{p_i q_{ij}}{\sum_j p_j q_j}.$$

But by reversibility, this is just

$$\frac{p_j q_{ji}}{\sum_j p_j q_j},$$

which is just $\pi_j P(j, i)$. Thus the embedded chain is also reversible. A similar substitution shows that the converse is also true: if the embedded chain is reversible, then the CTMC is as well.

2. **Particles Moving on a Checkerboard**

There are 1278 particles on a 100×100 checkerboard. Each location on the checkerboard can have at most one particle. Each particle, at rate 1, independently over the particles, attempts to jump, and when it does it tries to move in one of the four directions, up, down, left, and right, equiprobably. However, if this movement would either take it out of the checkerboard or onto a location that is already occupied by another particle then the jump is suppressed, and nothing happens.

What is the stationary distribution of the configuration of the particles on the checkerboard?

Solution:

The process evolves as a continuous time Markov chain whose state space is of size $\binom{100\times 100}{1278}$. Each state corresponds to a way of distributing 1278 particles among distinct location on the 100×100 grid. The stationary distribution is the uniform distribution on this state space. Indeed, let $p(i) = \binom{100\times 100}{1278}^{-1}$ for each state i and observe that the detailed balance equations hold true

$$p(i)Q(i,j) = p(j)Q(j,i), \text{ for all states } i \neq j.$$

This is because for all $i \neq j$ we either simultaneously have $Q(i,j) = Q(j,i) = 0$ or $Q(i,j) = Q(j,i) = 1/4$.

3. **Poisson Queues**

A continuous-time queue has Poisson arrivals with rate λ, and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are k customers in the queue ($k \in \mathbb{N}$), k servers are active. Suppose that the service time of each customer is exponentially distributed with rate μ and they are i.i.d.

(a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.

(b) Prove that for all finite values of λ and μ the Markov chain is positive-recurrent and find the invariant distribution.

Solution:

(a) The queue length is a MC as customer arrivals are independent of the current number of customers in the queue. Also, the departures only depend on the current number of customers being served. Next, even when k customers are being served, the completion of their service is independent of one another. Finally, even if one of the k customers has been completely served, the other customer has the same service time distribution as before as the exponential distribution is memoryless.

The only non-zero transition rates are

$$\begin{aligned} Q(k,k+1) &= \lambda, && k \in \mathbb{N}, \\ Q(k,k-1) &= k\mu, && k \in \mathbb{Z}_+. \end{aligned}$$

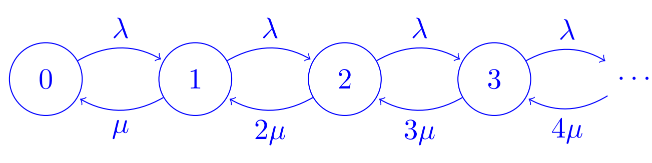

Figure 1: Markov chain of a memoryless queue with infinitely many servers.

(b) By flow conservation equations,

$$\pi(k)Q(k,k+1) = \pi(k+1)Q(k+1,k), \qquad k \in \mathbb{N}.$$

Thus,

$$\pi(k) = \pi(0)\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}.$$

Let $\rho = \lambda/\mu$. Then, $\pi(0)\sum_{k=0}^{\infty}\rho^k/k! = 1$. Thus, $\pi(0) = \mathrm{e}^{-\rho}$ and the MC is positive-recurrent for all finite λ and μ.

4. Poisson Process MAP

Customers arrive to a store according to a Poisson process of rate 1. The store manager learns of a rumor that one of the employees is sending 1/2 of the customers to the rival store. Refer to hypothesis $X = 1$ as the rumor being true, that one of the employees is sending every other customer arrival to the rival store and hypothesis $X = 0$ as the rumor being false, where each hypothesis is equally likely. Assume that at time 0, there is a successful sale. After that, the manager observes $S_1, S_2, \ldots, S_n$ where n is a positive integer and S_i is the time of the ith subsequent sale for $i = 1, \ldots, n$. Derive the MAP rule to determine whether the rumor was true or not.

Solution:

Note that both hypotheses are a priori equally likely, so the MAP rule is equivalent to the MLE rule. The interarrival times are independent conditioned on $X = 1$ and $X = 0$. The density of an interarrival interval given $X = 1$ is Erlang of order 2, so for $0 \le s_1 < \cdots < s_n$:

$$f_{S|X}(s_1, s_2, \ldots, s_n \mid 1) = \prod_{i=1}^{n}(s_i - s_{i-1})\mathrm{e}^{-(s_i - s_{i-1})} = \mathrm{e}^{-s_n}\prod_{i=1}^{n}(s_i - s_{i-1})$$

The density of an interarrival interval given $X = 0$ is exponential, so:

$$f_{S|X}(s_1, s_2, \ldots, s_n \mid 0) = \mathrm{e}^{-s_n}$$

We can thus see, by taking the log of both expressions, we declare $X = 1$ if $\sum_{i=1}^{n}\log(S_i - S_{i-1}) \ge 0$, otherwise we declare $X = 0$.

5. Statistical Estimation

Given $X \in \{0, 1\}$, the random variable Y is exponentially distributed with rate $3X + 1$.

(a) Assume $P(X = 1) = p \in (0, 1)$ and $P(X = 0) = 1 - p$. Find the MAP estimate of X given Y.

(b) Find the MLE of X given Y.

Solution:

(a) We know that when $X = 0$, $f_{Y|X}(y \mid 0) = \exp(-y)\mathbf{1}\{y > 0\}$ and when $X = 1$, $f_{Y|X}(y \mid 1) = 4\exp(-4y)\mathbf{1}\{y > 0\}$. The MAP maximizes $f_{X|Y}(x, y)$ over x for the given observation y, which is equivalent to maximizing $f_{X,Y}(x, y)$. Thus,

$$f_{X,Y}(0, y) = (1 - p)\exp(-y)\mathbf{1}\{y > 0\},$$
$$f_{X,Y}(1, y) = 4p\exp(-4y),$$

and

$$\mathrm{MAP}[X \mid Y] = 1 \iff 4p\exp(-4Y) > (1 - p)\exp(-Y)$$

which gives

$$\mathrm{MAP}[X \mid Y] = \mathbf{1}\Big\{Y < \frac{1}{3}\ln\frac{4p}{1 - p}\Big\}.$$

(b) The MLE is the MAP estimate with the prior probability p set to $1/2$.

$$\mathrm{MLE}[X \mid Y] = \mathbf{1}\left\{Y < \frac{1}{3}\ln 4\right\} = \mathbf{1}\{Y < 0.462\}.$$