UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 10

Fall 2021

1. **Entropy Warmup** Entropy seems like a very weird concept. We'll walk through a few examples together to build up your intuition. Let's say that we have a random variable X that can take on values from $\{\text{lecture}, \text{midterm}, \text{pop quiz}\}$. (Don't worry, we don't actually have pop quizzes in this class). Each day you go to class, you observe a random value of X, which is determined according to the distribution P_X. If $P(\text{lecture}) = 0.85$, $P(\text{midterm}) = 0.1$, and $P(\text{pop quiz}) = 0.05$, we'd mainly see lectures, occasionally have a midterm, and have a pop quiz very rarely. We can describe how "interesting" it is to see a particular $X = x$ with the notion of the "surprise," which is a function $S(x) = \log_2 \frac{1}{P_X(x)}$. This function is large for low-probability events and small for high-probability events.

   (a) For the probabilities above, calculate $S(\text{lecture})$, $S(\text{midterm})$, and $S(\text{pop quiz})$.

   (b) If $P(\text{lecture}) = \frac{1}{3}$, $P(\text{midterm}) = \frac{1}{3}$, and $P(\text{pop quiz}) = \frac{1}{3}$, calculate the surprises again. Given that $\log_2 \frac{1}{0.85} = 0.234$, $\log_2 \frac{1}{1/3} = 1.58$, $\log_2 \frac{1}{0.1} = 3.32$, and $\log_2 \frac{1}{0.05} = 4.32$, do the relative magnitudes of the values in (a) and (b) make sense intuitively?

   (c) The entropy is the *expected surprise*. Formally,

   $$H(X) = \sum_x P_X(x) S(x) = \sum_x P_X(x) \log_2 \frac{1}{P_X(x)}$$

   We will follow the convention that, if for a particular x, $P_X(x) = 0$, then $P_X(x) \log_2 \frac{1}{P_X(x)} = 0$. Calculate the entropy for the original probability values $(0.85, 0.1, 0.05)$, the entropy of the uniform distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the entropy of the deterministic RV with distribution $(1, 0, 0)$.

   (d) Do the entropies in part (c) make sense to you?

Solution:

(a)

$$S(\text{lecture}) = \log_2 \frac{1}{0.85} = 0.234$$
$$S(\text{midterm}) = \log_2 \frac{1}{0.1} = 3.32$$
$$S(\text{pop quiz}) = \log_2 \frac{1}{0.05} = 4.32$$

(b)

$$S(\text{lecture}) = S(\text{midterm}) = S(\text{pop quiz}) = \log_2 \frac{1}{1/3} = 1.58$$

(c) For $(0.85, 0.1, 0.05)$,

$$\begin{aligned} H(X) &= 0.85 \times S(\text{lecture}) + 0.10 \times S(\text{midterm}) + 0.05 \times S(\text{pop quiz}) \\ &= (0.85)(0.234) + (0.1)(3.32) + (0.05)(4.32) \\ &= 0.747 \end{aligned}$$

For $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,

$$\begin{aligned} H(X) &= \frac{1}{3}S(\text{lecture}) + \frac{1}{3}S(\text{midterm}) + \frac{1}{3}S(\text{pop quiz}) \\ &= \frac{1}{3}(1.58) + \frac{1}{3}(1.58) + \frac{1}{3}(1.58) \\ &= 1.58 \end{aligned}$$

For $(1, 0, 0)$,

$$\begin{aligned} H(X) &= 1 \times S(\text{lecture}) + 0 \times S(\text{midterm}) + 0 \times S(\text{pop quiz}) \\ &= 1 \times 0 \\ &= 0 \end{aligned}$$

(d) The entropy of the deterministic random variable is 0, which makes sense as the outcome should never be a surprise to us. The uniform distribution has the highest entropy, as it contains the most randomness as to which value we will see. The other distribution lies somewhere in the middle.

2. **Mutual Information and Noisy Typewriter**

The **mutual information** of X and Y is defined as

$$I(X;Y) := H(X) - H(X \mid Y)$$

Here, $H(X \mid Y)$ denotes the **conditional entropy** of X given Y, which is defined as:

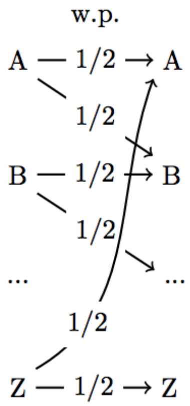
$$\begin{aligned} H(X \mid Y) &= \sum_{y \in \mathcal{Y}} p_Y(y) H(X \mid Y = y) \\ &= \sum_{y \in \mathcal{Y}} p_Y(y) \sum_{x \in \mathcal{X}} p_{X|Y}(x \mid y) \log_2 \frac{1}{p_{X|Y}(x \mid y)} \end{aligned}$$

The interpretation of conditional entropy is the average amount of uncertainty remaining in the random variable X after observing Y. The interpretation of mutual information is therefore the amount of information about X gained by observing Y.

(a) Show that $H(X,Y) = H(Y) + H(X \mid Y) = H(X) + H(Y \mid X)$. This is often called the **Chain Rule**. Interpret this rule.

(b) Show that $I(X;Y) = H(X) + H(Y) - H(X,Y)$. Note that this shows that $I(X;Y) = I(Y;X)$, i.e., mutual information is symmetric.

(c) Consider the noisy typewriter.



Each symbol gets sent to one of the adjacent symbols with probability 1/2. Let X be the input to the noisy typewriter, and let Y be the output (X is a random variable that takes values in the English alphabet). What is the distribution of X that maximizes $I(X;Y)$?

Note
It turns out that $I(X;Y) \geq 0$ with equality if and only if X and Y are independent. The mutual information is an important quantity for channel coding.

Solution:

(a)

$$\begin{aligned} H(X,Y) &= \mathbb{E}[\log \frac{1}{p(X,Y)}] \\ &= \mathbb{E}[\log \frac{1}{p(Y)p(X|Y)}] \\ &= \mathbb{E}[\log \frac{1}{p(Y)}] + \mathbb{E}[\log \frac{1}{p(X|Y)}] \\ &= H(Y) + H(X|Y) \end{aligned}$$

(b) Using the previous part, we get

$$I(X;Y) = H(X) - H(X \mid Y) = H(X) + H(Y) - H(X,Y).$$

(c) Since $I(X;Y) = H(Y) - H(Y \mid X)$ and $H(Y \mid X) = 1$ (regardless of the distribution of X), then $I(X;Y) = H(Y) - 1$ and this is maximized by letting Y be uniform over the English alphabet; this is achieved by letting X be uniformly distributed on the English alphabet as well.

3. Huffman Questions

Consider a set of n objects. Let $X_i = 1$ or 0 accordingly as the i-th object is good or defective. Let $X_1, X_2, \ldots, X_n$ be independent with $P(X_i = 1) = p_i$; and $p_1 > p_2 > \cdots > p_n > 1/2$. We are asked to determine the set of all defective objects. Any yes-no question you can think of is admissible.

(a) Propose an algorithm based on Huffman coding in order to identify all defective objects.

(b) Suppose the worst case scenario happens and we have to ask the maximimum number of questions. What (in words) is the last question we should ask? And what two sets are we distinguishing with this question?

Solution:

(a) Let $x \in \{0,1\}^n$ be a possible configuration of whether each object is good or defective. Because of the independence assumption, we can calculate the joint probabilities as follows

$$P(X = x) = P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i}.$$

Now according to those joint probabilities we use Huffman coding to encode all possible configurations $x \in \{0,1\}^n$.

The naive strategy is to try to determine directly the true configuration $x_t \in \{0,1\}^n$ by identifying each bit of x_t, which results in n yes-no questions.

Instead our strategy is to try to determine the Huffman code $C(x_t) \in \{0,1\}^+$ that corresponds to the true configuration x_t. We are going to do so by identifying each bit of $C(x_t) \in \{0,1\}^+$.

Using this strategy the expected number of questions that we are going to ask is going to be between $H(X_1, \ldots, X_n)$ and $H(X_1, \ldots, X_n) + 1$. Because of independence $H(X_1, \ldots, X_n) = H(X_1) + \cdots + H(X_n)$, and this quantity can be way smaller than n.

(b) If the longest sequence of questions is required, then the last question would try to distinguish whether the true configuration is the one with lowest probability or the one with the second lowest probability. So according to the information $p_1 > p_2 > \cdots > p_n > 1/2$, the last question would try to distinguish if the true configuration is $(0, \ldots, 0, 0)$ or $(0, \ldots, 0, 1)$, and the actual question could be

"Is the n-th object defective?"