UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 3
Fall 2021

1. **Triangle Density**

Consider random variables X and Y which have a joint PDF uniform on the triangle with vertices at $(0,0),(1,0),(0,1)$.

(a) Find the joint PDF of X and Y.
(b) Find the marginal PDF of Y.
(c) Find the conditional PDF of X given Y.
(d) Find $\mathbb{E}[X]$ in terms of $\mathbb{E}[Y]$.
(e) Find $\mathbb{E}[X]$.

Solution:

(a) Note that the joint PDF is uniform on the triangle, which has area $1/2$, so for all valid x, y, $f_{X,Y}(x,y) = 2$.
(a) In order to find the marginal PDF, we integrate out:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \int_0^{1-y} 2\,\mathrm{d}x = 2(1-y)$$

where $0 \le y \le 1$.

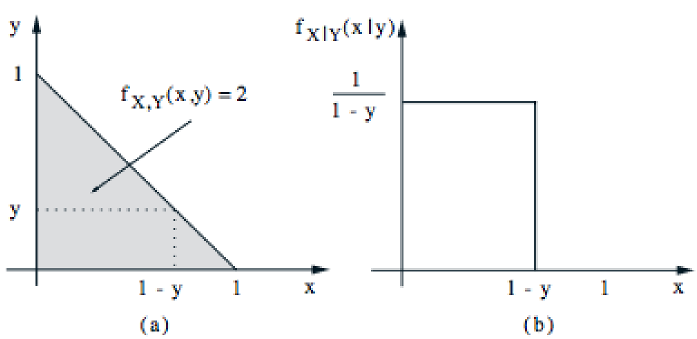

Figure 1: Joint density of (X,Y) (a) and the conditional density $X \mid Y$ (b). Image taken from Bertsekas and Tsitsiklis.

(c) The conditional density is given by, for $0 \le y \le 1$,

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}, \qquad 0 \le x \le 1-y.$$

This should agree with your intuition that given $Y = y$, X should be uniform.

(d) We use the tower property: $\mathbb{E}[\mathbb{E}(X \mid Y)] = \mathbb{E}[X]$. Note that for $0 \le y \le 1$,

$$\begin{aligned}\mathbb{E}[X \mid Y = y] &= \int_0^{1-y} x f_{X|Y}(x \mid y)\,\mathrm{d}x = \int_0^{1-y} x \frac{1}{1-y}\,\mathrm{d}x \\ &= \frac{1}{1-y}\left[\frac{(1-y)^2}{2}\right] = \frac{1-y}{2}.\end{aligned}$$

Thus, we have:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}(X \mid Y)] = \int_0^1 \mathbb{E}[X \mid Y = y] f_Y(y)\,\mathrm{d}y.$$

Note that we are simply trying to find $\mathbb{E}[X]$ in terms of $\mathbb{E}[Y]$, so there is no need to expand out $f_Y(y)$, so we have:

$$\begin{aligned}\mathbb{E}[X] &= \int_0^1 \mathbb{E}[X \mid Y = y] f_Y(y)\,\mathrm{d}y = \int_0^1 \frac{1-y}{2} f_Y(y)\,\mathrm{d}y \\ &= \frac{1}{2} - \frac{1}{2}\int_0^1 y f_Y(y)\,\mathrm{d}y = \frac{1-\mathbb{E}[Y]}{2}.\end{aligned}$$

(e) Finally, we note that by symmetry, $\mathbb{E}[X]$ should be equal to $\mathbb{E}[Y]$, so we have

$$\mathbb{E}[X] = \frac{1-\mathbb{E}[X]}{2},$$

and

$$\mathbb{E}[X] = \frac{1}{3}.$$

2. Conditional Distribution of a Poisson Random Variable with Exponentially Distributed Parameter

Let X have a Poisson distribution with parameter $\lambda > 0$. Suppose λ itself is random, having an exponential density with parameter $\theta > 0$.

(a) Show that

$$\mathbb{E}(\lambda^k) = \frac{k!}{\theta^k}, \qquad k \in \mathbb{N}$$

(b) What is the distribution of X?

(c) Determine the conditional density of λ given $X = k$, where $k \in \mathbb{N}$.

Solution:

(a) $\mathbb{E}(\lambda^k) = \int_0^\infty x^k \theta \epsilon^{-\theta x}\,\mathrm{d}x$. Integrating by parts, with proper limits,

$$\begin{aligned}\mathbb{E}[\lambda^k] &= \int_0^\infty x^k \theta \exp(-\theta x)\,\mathrm{d}x \\ &= -x^k \exp(-\theta x)\Big|_{x=0}^{\infty} + k\int_0^\infty x^{k-1}\exp(-\theta x)\,\mathrm{d}x \\ &= \frac{k}{\theta}\int_0^\infty x^{k-1}\theta \exp(-\theta x)\,\mathrm{d}x,\end{aligned}$$

so

$$\mathbb{E}(\lambda^k) = \frac{k}{\theta}\mathbb{E}(\lambda^{k-1}).$$

Continuing, and with the base case

$$\mathbb{E}(\lambda) = \frac{1}{\theta},$$

we get

$$\mathbb{E}(\lambda^k) = \frac{k!}{\theta^k}.$$

(b) The PDF of λ is: $f(\lambda) = \theta\exp(-\theta\lambda)\mathbf{1}\{\lambda > 0\}$.
The PMF of X conditioned on λ is

$$\mathbb{P}(X = k \mid \lambda) = \frac{\epsilon^{-\lambda}\lambda^k}{k!}, \qquad k \in \mathbb{N}.$$

Applying the total law of probability yields, for $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}(X = k) &= \int_0^\infty \frac{\epsilon^{-\lambda}\lambda^k}{k!}\theta\exp(-\theta\lambda)\,\mathrm{d}\lambda \\ &= \frac{\theta}{(1+\theta)k!}\int_0^\infty \lambda^k(1+\theta)\exp\bigl(-(1+\theta)\lambda\bigr)\,\mathrm{d}\lambda = \frac{\theta}{(1+\theta)^{k+1}},\end{aligned}$$

because the last integral is $\mathbb{E}[Y^k]$ when $Y \sim \text{Exponential}(1+\theta)$, which is $k!/(1+\theta)^k$.

(c)

$$f(\lambda \mid X = k) = \frac{\mathbb{P}(X = k \mid \lambda)f(\lambda)}{\mathbb{P}(X = k)} = \frac{\epsilon^{-(1+\theta)\lambda}\lambda^k(1+\theta)^{k+1}}{k!}, \qquad \lambda > 0.$$

To understand the above equation, think about the analogy to Bayes Law. Remember here that θ is fixed and λ is the argument. You should check that the integral of this over $[0, \infty)$ is 1.

3. **Poisson Merging**

The Poisson distribution is used to model *rare events*, such as the number of customers who enter a store in the next hour. The theoretical justification for this modeling assumption is that the limit of the binomial distribution, as the number of trials n goes to ∞ and the probability of success per trial p goes to 0, such that $np \to \lambda > 0$, is the Poisson distribution with mean λ.

Now, suppose we have two independent streams of rare events (for instance, the number of female customers and male customers entering a store), and we do not care to distinguish between the two types of rare events. Can the combined stream of events be modeled as a Poisson distribution?

Mathematically, let X and Y be independent Poisson random variables with means λ and μ respectively. Prove that $X + Y \sim \text{Poisson}(\lambda + \mu)$. (This is known as **Poisson merging**.) Note that it is **not** sufficient to use linearity of expectation to say that $X + Y$ has mean $\lambda + \mu$. You are asked to prove that the *distribution* of $X + Y$ is Poisson.

Note: This property will be extensively used when we discuss Poisson processes.

Solution:

For $z \in \mathbb{N}$,

$$
\begin{aligned}
P(X+Y=z) &= \sum_{j=0}^{z} P(X=j, Y=z-j) = \sum_{j=0}^{z} \frac{\epsilon^{-\lambda}\lambda^j}{j!}\frac{\epsilon^{-\mu}\mu^{z-j}}{(z-j)!} \\
&= \frac{\epsilon^{-(\lambda+\mu)}}{z!}\sum_{j=0}^{z}\frac{z!}{j!(z-j)!}\lambda^j\mu^{z-j} \\
&= \frac{\epsilon^{-(\lambda+\mu)}}{z!}\sum_{j=0}^{z}\binom{z}{j}\lambda^j\mu^{z-j} = \frac{\epsilon^{-(\lambda+\mu)}(\lambda+\mu)^z}{z!}.
\end{aligned}
$$

Here is some intuition for why Poisson merging holds. If we are interested in the number of customers entering a store in the next hour, we can discretize the hour into n time intervals, where n is a positive integer. In each time interval, independently of other time intervals, the probability that a female customer enters the store is λ/n and the probability that a male customer enters the store is μ/n. Since the two types of customers are assumed to be independent, the probability that a customer, disregarding gender, enters the store is $\lambda/n + \mu/n - \lambda\mu/n^2$. As $n \to \infty$, the number of customers who enter the store in the hour is Poisson with mean $\lim_{n\to\infty} n[\lambda/n + \mu/n - \lambda\mu/n^2] = \lambda + \mu$.

We will be able to give a much easier proof of this result after we introduce transforms of random variables.