UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 3
Fall 2021

1. Graphical Density

Figure 1 shows the joint density $f_{X,Y}$ of the random variables X and Y.

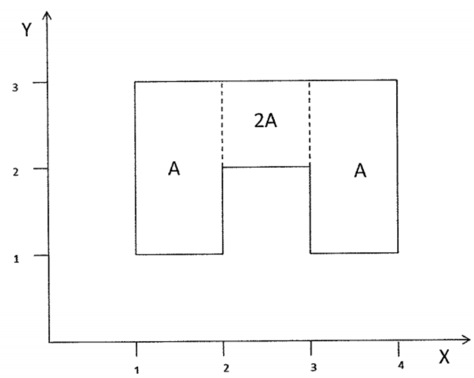

Figure 1: Joint density of X and Y.

(a) Find A and sketch f_X, f_Y, and $f_{X|X+Y\leq 3}$.

(b) Find $\mathbb{E}[X \mid Y = y]$ for $1 \leq y \leq 3$ and $\mathbb{E}[Y \mid X = x]$ for $1 \leq x \leq 4$.

(c) Find $\text{cov}(X, Y)$.

2. Joint Density for Exponential Distribution

(a) If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, X and Y independent, compute $\mathbb{P}(X < Y)$.

(b) If X_k, $1 \leq k \leq n$ are independent and exponentially distributed with parameters $\lambda_1, \ldots, \lambda_n$, show that $\min_{1\leq k\leq n} X_k \sim \text{Exponential}(\sum_{j=1}^n \lambda_j)$.

(c) Deduce that

$$\mathbb{P}(X_i = \min_{1\leq k\leq n} X_k) = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}$$

3. Packet Routing

Packets arriving at a switch are routed to either destination A (with probability p) or destination B (with probability $1-p$). The destination of each packet is chosen independently of each other. In the time interval $[0, 1]$, the number of arriving packets is $\text{Poisson}(\lambda)$.

(a) Show that the number of packets routed to A is Poisson distributed. With what parameter?

(b) Are the number of packets routed to A and to B independent?

4. Gaussian Densities

(a) Let $X_1 \sim \mathcal{N}(0,1)$, $X_2 \sim \mathcal{N}(0,1)$, where X_1 and X_2 are independent. Convolve the densities of X_1 and X_2 to show that $X_1 + X_2 \sim \mathcal{N}(0,2)$. *Remark.* Note that this property is similar to the one shared by independent Poisson random variables.

(b) Let $X \sim \mathcal{N}(0,1)$. Compute $\mathbb{E}[X^n]$ for all integers $n \geq 1$.

5. Moving Books Arround

You have N books on your shelf, labelled $1, 2, \ldots, N$. You pick a book j with probability $1/N$. Then you place it on the left of all others on the shelf. You repeat the process, independently. Construct a Markov chain which takes values in the set of all $N!$ permutations of the books.

(a) Find the transition probabilities of the Markov chain.

(b) Find its stationary distribution.

Hint: You can guess the stationary distribution before computing it.