UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 1
Fall 2021

1. **Coin Flipping & Symmetry**

   Alice and Bob have $2n + 1$ fair coins (where $n \geq 1$), each coin with probability of heads equal to $1/2$. Bob tosses $n+1$ coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that, after all coins have been tossed, Bob will have gotten more heads than Alice is $1/2$.

   Hint: Consider the event $A = \{$more heads in the first $n + 1$ tosses than the last n tosses$\}$.

   Solution:

   If we let Ω be the sample space consisting of all possible $2n + 1$ tosses, then Ω is a uniform probability space by assumption. Define the events

   $$A = \{\text{there are more heads in the first } n + 1 \text{ tosses than the last } n \text{ tosses}\},$$
   $$B = \{\text{there are more tails in the first } n + 1 \text{ tosses than the last } n \text{ tosses}\}.$$

   By symmetry, $P(A) = P(B)$, and we note that $A \cap B = \varnothing$ since it is impossible for the first $n + 1$ tosses to have more heads *and* more tails than the last n tosses, and $A \cup B = \Omega$. So, $P(A) + P(B) = 1$ and hence $P(A) = 1/2$.

   Alternatively, if the probability that Bob has more heads than Alice in the first n tosses is p, then the probability that Bob has fewer heads than Alice in the first n tosses is also p, and the probability that they are tied after n tosses is $1 - 2p$. So, the probability that Bob wins is $p + (1/2)(1 - 2p) = 1/2$ since Bob can either win by having more heads than Alice in the first n tosses, or by having the same number of heads as Alice in the first n tosses and then flipping heads on the last toss.

2. **Passengers on a Plane**

   There are N passengers in a plane with N assigned seats (N is a positive integer), but after boarding, the passengers take the seats randomly. Assuming all seating arrangements are equally likely, what is the probability that no passenger is in their assigned seat? Compute the probability when $N \to \infty$.

   Hint: Use the inclusion-exclusion principle and the power series $e^x = \sum_{j=0}^{\infty} x^j/j!$.

   Solution:

   First, let us calculate the probability that at least one passenger sits in his or her assigned seat using inclusion-exclusion. Let A_i, $i = 1, \ldots, N$, be the event that passenger i sits in his or her assigned seat. We first add the probabilities of the single events (of which there are N), and the probability of each event is $(N - 1)!/N!$ (indeed there are $(N - 1)!$ ways to permute the remaining passengers once a specific passenger is fixed, and $N!$ total permutations, so the probability is $(N - 1)!/N!$); next, we subtract the probabilities of the pairwise intersections of

events (of which there are $\binom{N}{2}$), and the probability of each event is $(N-2)!/N!$ (there are $(N-2)!$ ways to permute the passengers other than the fixed two); continuing on, we see that

$$P\Big(\bigcup_{i=1}^{N} A_i\Big) = \sum_{j=1}^{N}(-1)^{j+1}\binom{N}{j}\frac{(N-j)!}{N!} = \sum_{j=1}^{N}(-1)^{j+1}\frac{1}{j!}.$$

Now, the event that no passenger sits in his or her assigned seat is the complement of the event just discussed:

$$1 - P\Big(\bigcup_{i=1}^{N} A_i\Big) = 1 - \sum_{j=1}^{N}(-1)^{j+1}\frac{1}{j!} = \sum_{j=0}^{N}\frac{(-1)^j}{j!}.$$

Taking the limit as $N \to \infty$, the expression converges to $\sum_{j=0}^{\infty}(-1)^j/j!$, and using the expression for the power series of the exponential function, we conclude that the probability converges to $\exp(-1) \approx 0.368$.

3. **Expanding the NBA**

The NBA is looking to expand to another city. In order to decide which city will receive a new team, the commissioner interviews potential owners from each of the N potential cities (N is a positive integer), one at a time. Unfortunately, the owners would like to know immediately after the interview whether their city will receive the team or not. The commissioner decides to use the following strategy: she will interview the first m owners and reject all of them ($m \in \{1, \ldots, N\}$). After the mth owner is interviewed, she will pick the first city that is better than all previous cities. The cities are interviewed in a uniformly random order. What is the probability that the best city is selected? Assume that the commissioner has an objective method of scoring each city and that each city receives a unique score.

You should arrive at an exact answer for the probability in terms of a summation. Approximate your answer using $\sum_{i=1}^{n} i^{-1} \approx \ln n$ and find the optimal value of m that maximizes the probability that the best city is selected. You can also say $\ln(n-1) \approx \ln n$.

Hint: Consider the events $B_i = \{$i-th city is the best$\}$ and $A = \{$best city is chosen$\}$.

Solution:

Let B_i, $i = 1, \ldots, N$, be the event that the ith city is the best of the N cities, and let A be the event that the best city is picked by the commissioner. Then,

$$P(A) = \sum_{i=1}^{N} P(A \mid B_i)P(B_i) = \frac{1}{N}\sum_{i=1}^{N} P(A \mid B_i)$$

using the law of total probability. Next, $P(A \mid B_i) = 0$ for $i = 1, \ldots, m$ because if the best city is among the first m cities, there is no chance of picking the best city. Also, $P(A \mid B_i) = m/(i-1)$ for $i = m+1, \ldots, N$ because $P(A \mid B_i)$ is the probability that second-best city among the first i cities is within the first m cities. Therefore,

$$P(A) = \frac{m}{N}\sum_{i=m+1}^{N}\frac{1}{i-1}.$$

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Now, we turn towards approximation.

$$P(A) \approx \frac{m}{N}(\ln N - \ln m) = -\frac{m}{N}\ln\frac{m}{N}.$$

Letting $x := m/N$, then $P(A) \approx -x\ln x$, and differentiating with respect to x suggests that the optimal value is $x = 1/\epsilon$, so we should reject the first N/ϵ cities. Plugging in this value for x into $P(A) = -x\ln x$ gives the optimal probability as $P(A) \approx 1/\epsilon$.

Note: This problem is a famous example from optimal stopping theory and is commonly known as the secretary problem (a boss is interviewing secretaries instead of a commissioner interviewing city representatives). In fact, one may use a dynamic programming approach to see why the policy outlined here is in fact the optimal policy. If you are interested, the details of such an approach can be found in *Dynamic Programming and the Secretary Problem* by Beckmann.

4. **Random Walk on a Circle**

Suppose we have n points labeled $\{1, 2, \ldots, n\}$ around a circle. An ant starts at point 1, and at each second has an equal probability of moving clockwise or counterclockwise to an adjacent point. For each point $k \in \{2, 3, \ldots, n\}$, find the probability that the first time the ant lands at k, it has visited all other points already.

Solution: We claim that the probabilities are all $\frac{1}{n-1}$. Fix a point $k \in \{2, 3, \ldots, n\}$. Consider the stopping time τ_k which is the first time the ant arrives at a point adjacent to k. Then define the events

$A_k = \{\tau < \infty\}$
$B_k = \{$starting at time τ, the ant travels the rest of the circle before first arriving at $k\}$.

Then the probability that by the time the ant first hits k it has already visited all other points is given by

$$P(B_k, A_k) = P(B_k|A_k)\underbrace{P(A_k)}_{1}.$$

But note by symmetry that $P(B_k|A_k)$ is the same as the probability of arriving at point 2 after visiting all other points, given that the ant starts at 1. Hence the probabilities are all the same, i.e. $\frac{1}{n-1}$.

5. **Superhero Basketball**

Superman and Captain America are playing a game of basketball. At the end of the game, Captain America scored n points and Superman scored m points, where $n > m$ are positive integers. Supposing that each basket counts for exactly one point, what is the probability that after the start of the game (when they are initially tied), Captain America was always *strictly* ahead of Superman? (Assume that all sequences of baskets which result in the final score of n baskets for Captain America and m baskets for Superman are equally likely.)

Hint: Think about symmetry. First, try to figure out which is more likely: there was a tie and Superman scored the first point, or there was a tie and Captain America scored the first point?

Solution:

Let T be the event that Captain America and Superman were tied at least once after the first point. Let C be the event that Captain America scores the first point, and S be the event that Superman scores the first point. In fact, $P(C \cap T) = P(S \cap T)$ for the following reason: given any sequence of baskets where the first point is scored by Captain America and there is a tie, flip all of the baskets until the first tie. This yields a sequence of baskets where the first point is scored by Superman and there is a tie. Thus, the outcomes in $C \cap T$ are in one-to-one correspondence with the outcomes in $S \cap T$.

However, since Captain America won the game, $P(S \cap T) = P(S)$ (if Superman scored the first point, then there must have been a point when Captain America caught up).

So far, we have $P(T) = P(C \cap T) + P(S \cap T) = 2P(S \cap T) = 2P(S)$. We can calculate $P(S) = m/(m+n)$. Thus, $P(T) = 2m/(m+n)$. Finally, the question is asking for $P(T^c) = 1 - 2m/(m+n) = (n-m)/(m+n)$.

Note: This is yet another famous problem known as the ballot problem.