CS 188: Artificial Intelligence

Bayes' Nets: Independence

Fall 2025

[These slides were created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]

Review: Bayes' Net Semantics

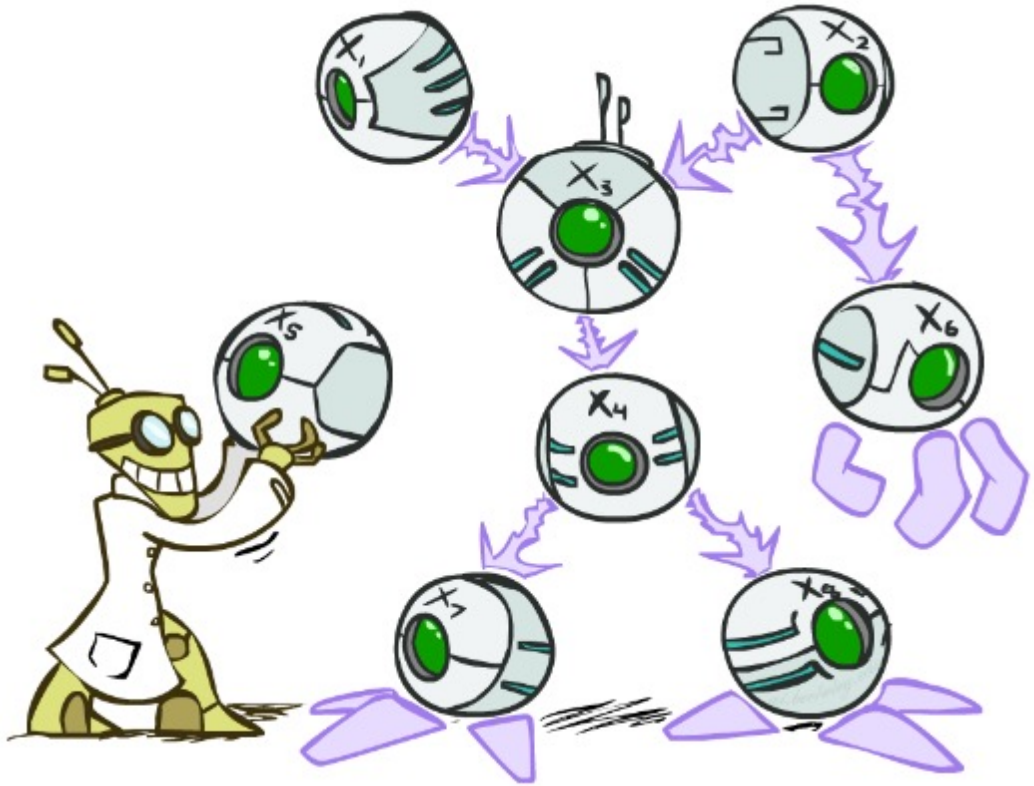
Probability Recap

- Conditional probability $$P(x|y) = \frac{P(x,y)}{P(y)}$$
- Product rule $$P(x,y) = P(x|y)P(y)$$
- Chain rule $$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1}) \end{aligned}$$
- X, Y independent if and only if: $$\forall x, y : P(x,y) = P(x)P(y)$$
- X and Y are conditionally independent given Z if and only if:

$$\forall x, y, z : P(x,y|z) = P(x|z)P(y|z) \qquad X \perp\!\!\!\perp Y|Z$$

Bayes' Nets

- So far: how a Bayes' net encodes a joint distribution
- Next: how to answer queries about that distribution
  - Last Time:
    - First assembled BNs using an intuitive notion of conditional independence as causality
    - Then saw that key property is conditional independence
  - Main goal: answer queries about conditional independence and influence
- Today: how to answer numerical queries (inference)



Bayes' Nets

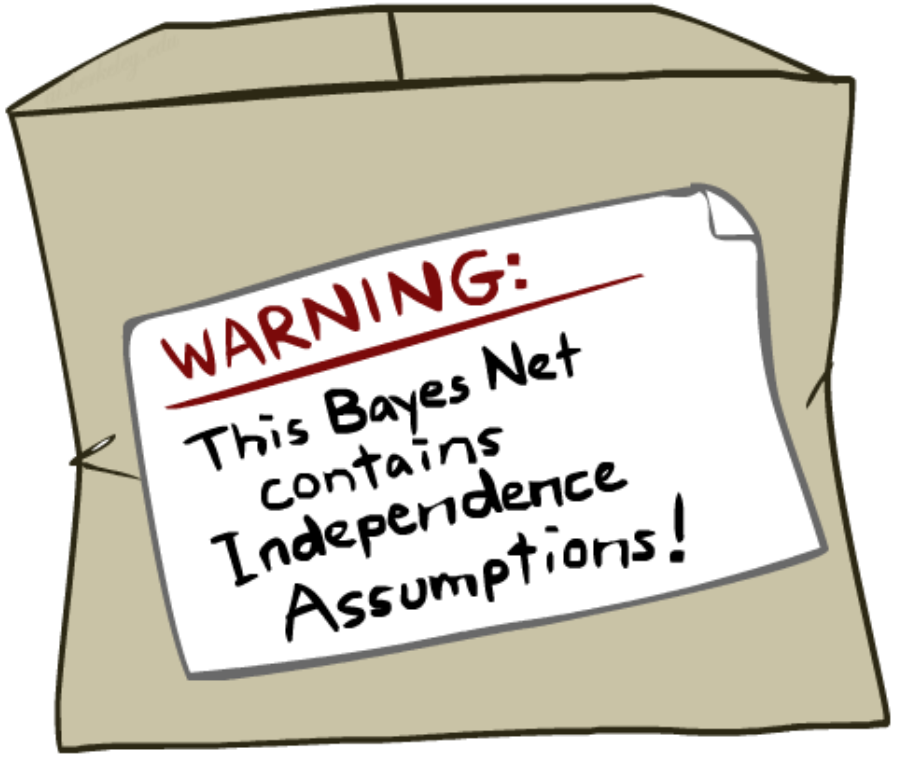
- Representation
- Conditional Independences
- Probabilistic Inference
- Learning Bayes' Nets from Data

Bayes Nets: Assumptions

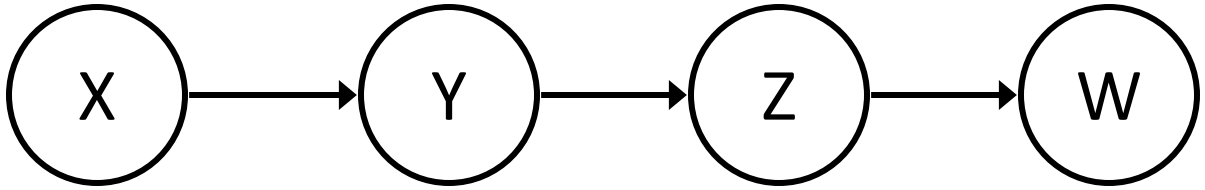
- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i|x_1 \cdots x_{i-1}) = P(x_i|parents(X_i))$$

- Beyond above "chain rule → Bayes net" conditional independence assumptions
  - Often additional conditional independences
  - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph



Example

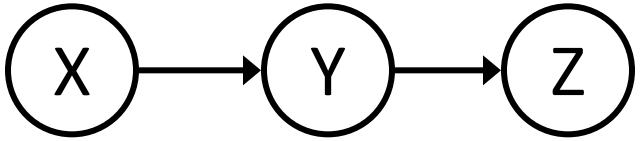


- Conditional independence assumptions directly from simplifications in chain rule:

- Additional implied conditional independence assumptions?

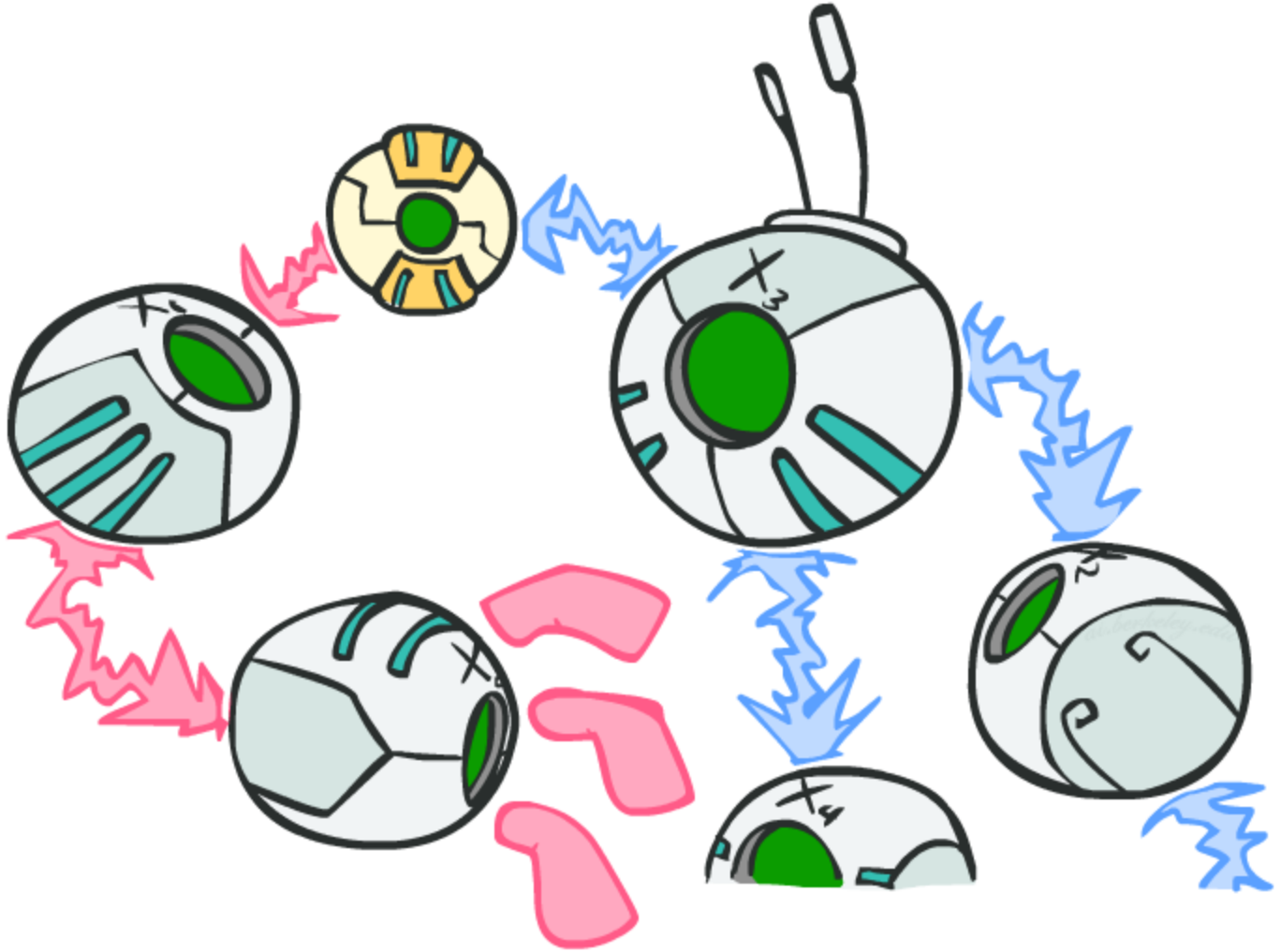
Independence in a BN

- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (tedious in general)
  - If no, can prove with a counter example
  - Example:



  - Question: are X and Z necessarily independent?
    - Answer: no. Example: low pressure causes rain, which causes traffic.
    - X can influence Z, Z can influence X (via Y)
    - Addendum: they *could* be independent: how?

D-separation: Outline



D-separation: Outline

- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

Causal Chains

- This configuration is a "causal chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Example:
    - Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic
    - In numbers:

      P(+y | +x) = 1, P(-y | - x) = 1,
      P(+z | +y) = 1, P(-z | -y) = 1

Causal Chains

- This configuration is a "causal chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)}$$

$$= P(z|y)$$

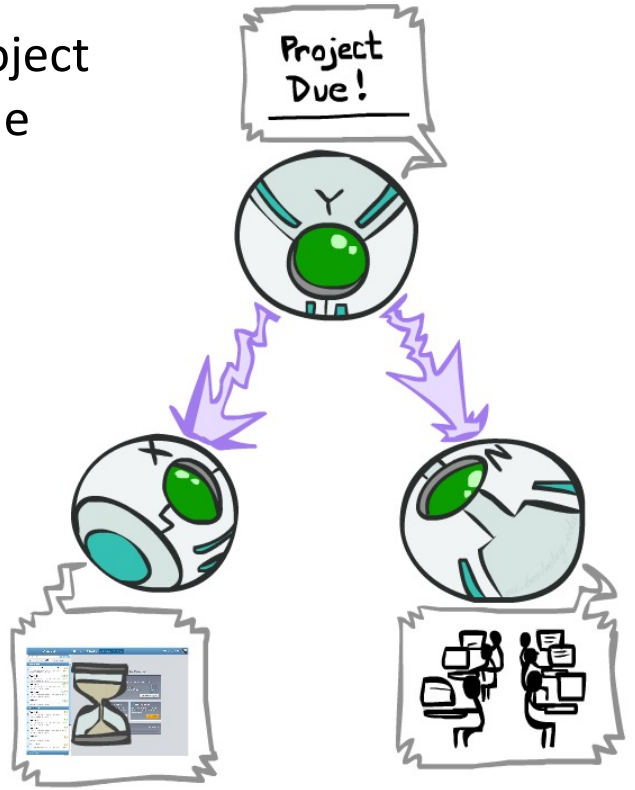
Yes!

- Evidence along the chain "blocks" the influence

Common Cause

- This configuration is a "common cause"

Y: Project due



X: Forums busy

Z: Lab full

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

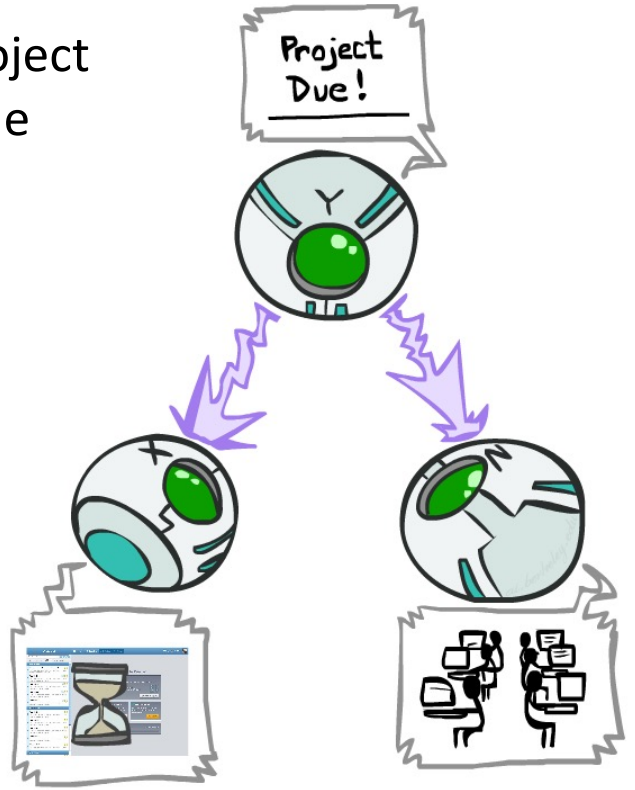
- Guaranteed X independent of Z ? *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Example:
    - Project due causes both forums busy and lab full
    - In numbers:

      P(+x | +y) = 1, P(-x | -y) = 1,
      P(+z | +y) = 1, P(-z | -y) = 1

Common Cause

- This configuration is a "common cause"

Y: Project due



X: Forums busy

Z: Lab full

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X and Z independent given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

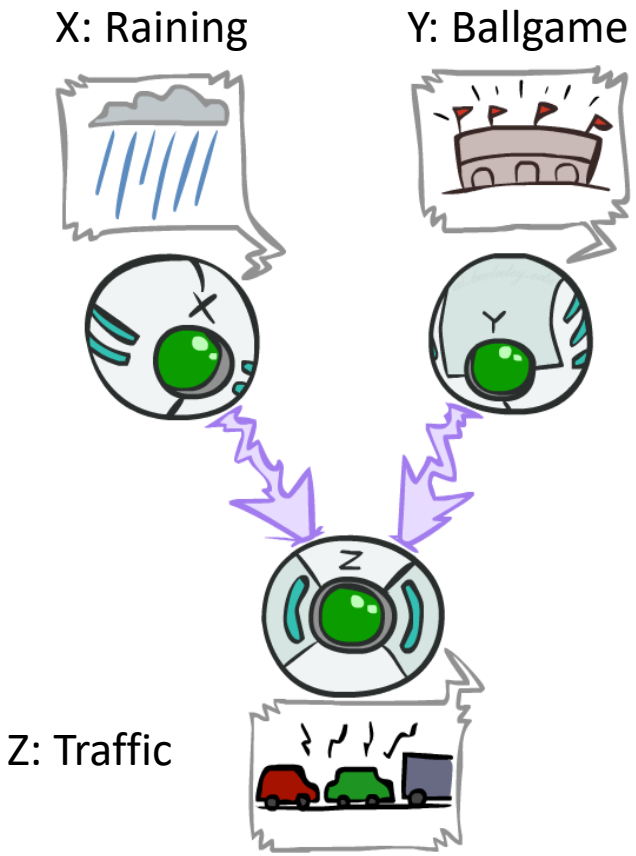
$$= P(z|y)$$

Yes!

- Observing the cause blocks influence between effects.

Common Effect

- Last configuration: two causes of one effect (v-structures)

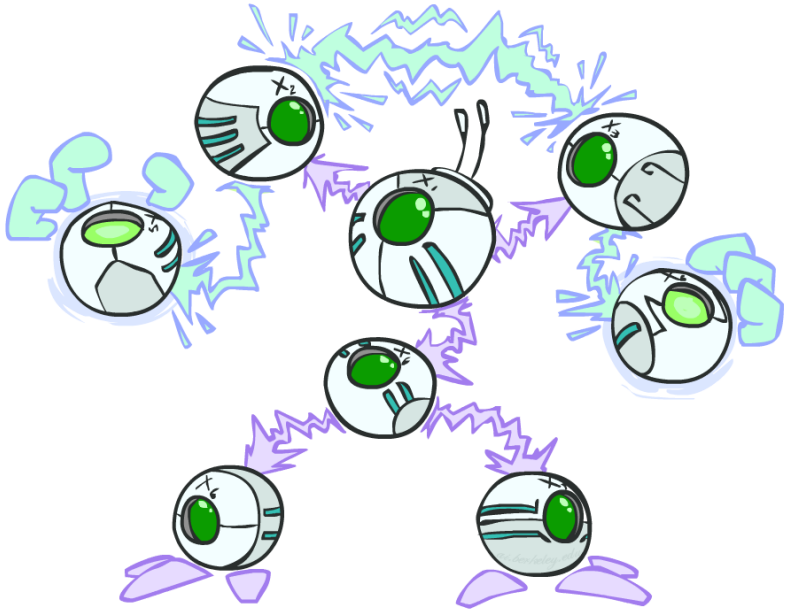


- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases
  - Observing an effect activates influence between possible causes.

The General Case

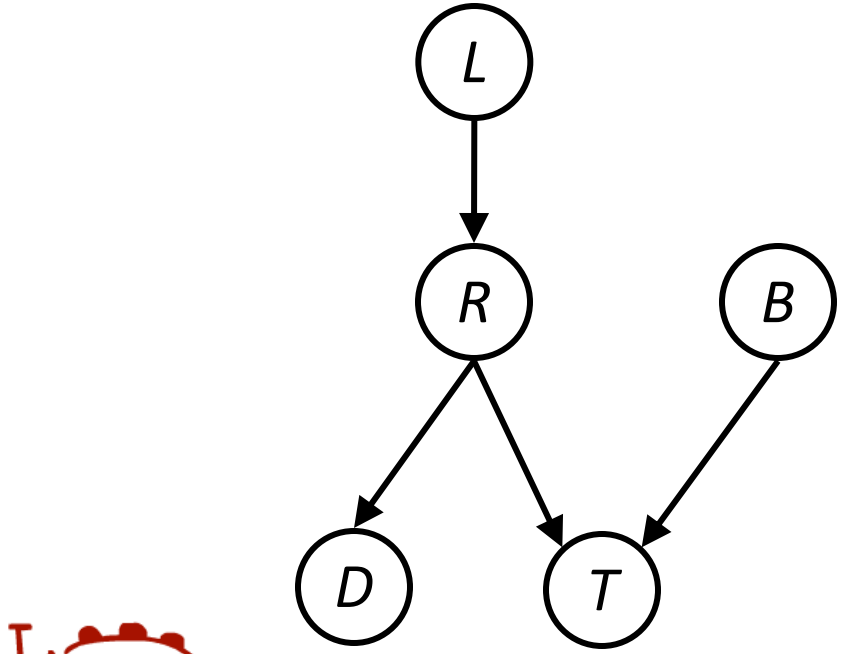
The General Case

- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases



Reachability

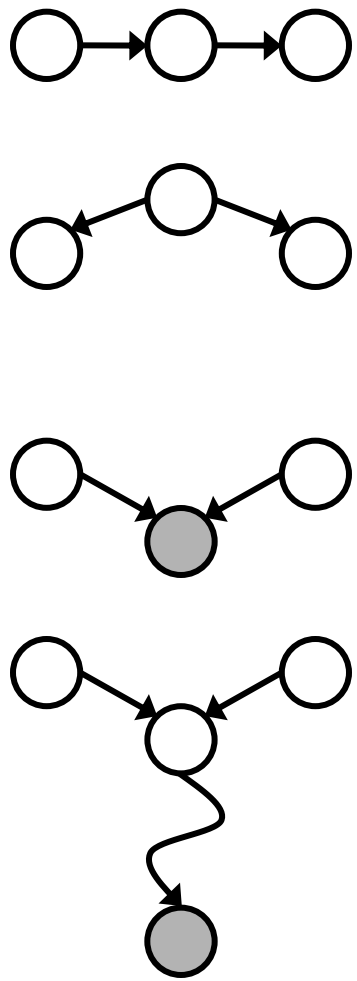
- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"



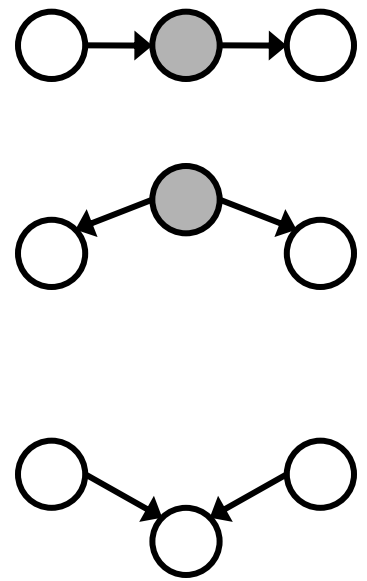
Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y "d-separated" by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!
- A path is active if each triple is active:
  - Causal chain A → B → C where B is unobserved (either direction)
  - Common cause A ← B → C where B is unobserved
  - Common effect (aka v-structure)
    A → B ← C where B *or one of its descendents* is observed
- All it takes to block a path is a single inactive segment
- Note: These triples are all active (and similar for the other cases, i.e. the variables on either end of the triple can be observed or unobserved)

Active Triples



Inactive Triples



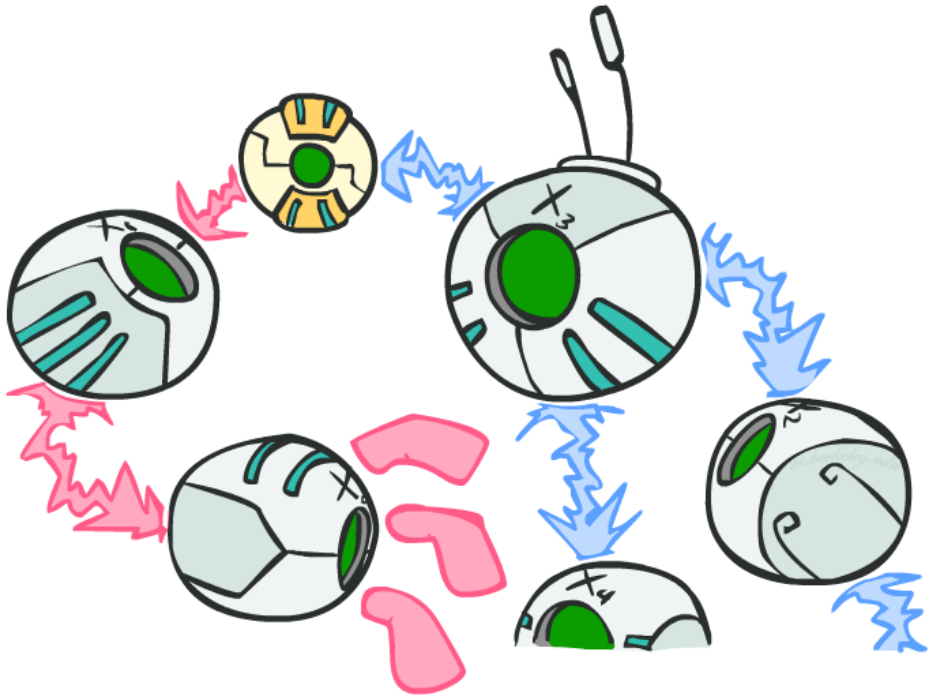
D-Separation

- Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$?
- Check all (undirected!) paths between X_i and X_j
  - If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

  - Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

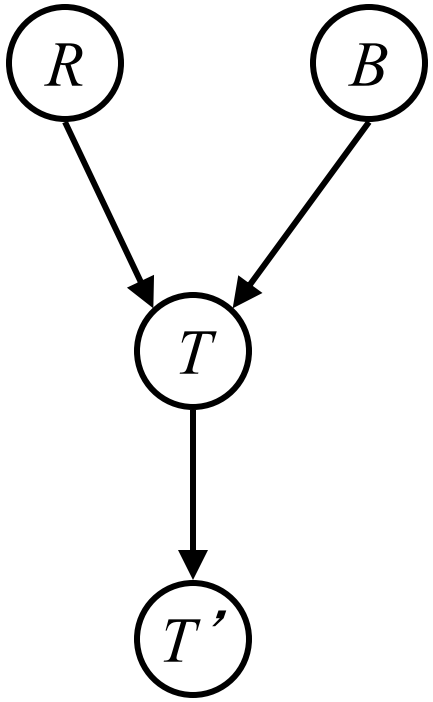


Example

$R \perp\!\!\!\perp B$ *Yes*

$R \perp\!\!\!\perp B | T$

$R \perp\!\!\!\perp B | T'$



Example

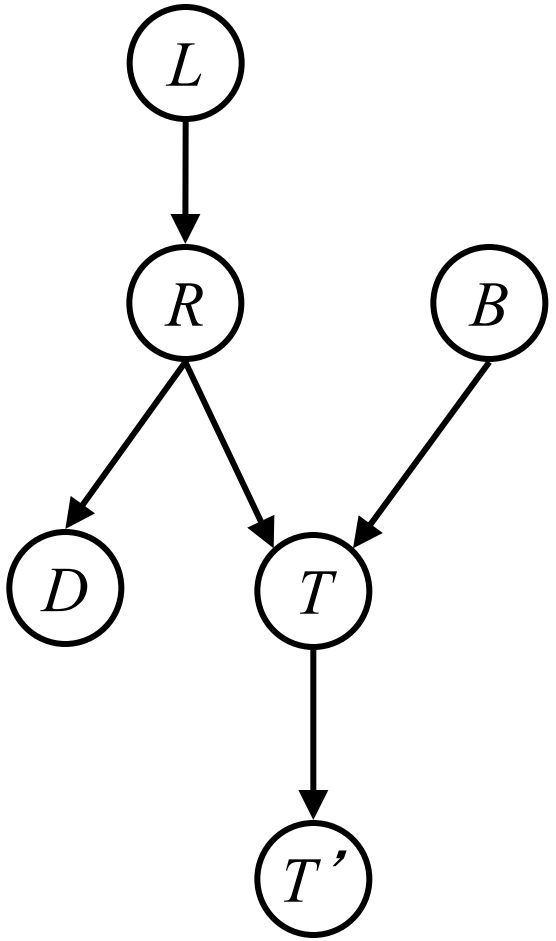
$L \perp\!\!\!\perp T' | T$ *Yes*

$L \perp\!\!\!\perp B$ *Yes*

$L \perp\!\!\!\perp B | T$

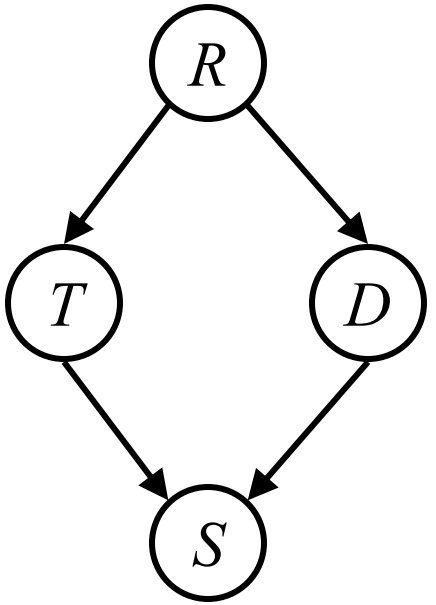
$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$ *Yes*



Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- Questions:



$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R \qquad \textit{Yes}$$

$$T \perp\!\!\!\perp D | R, S$$

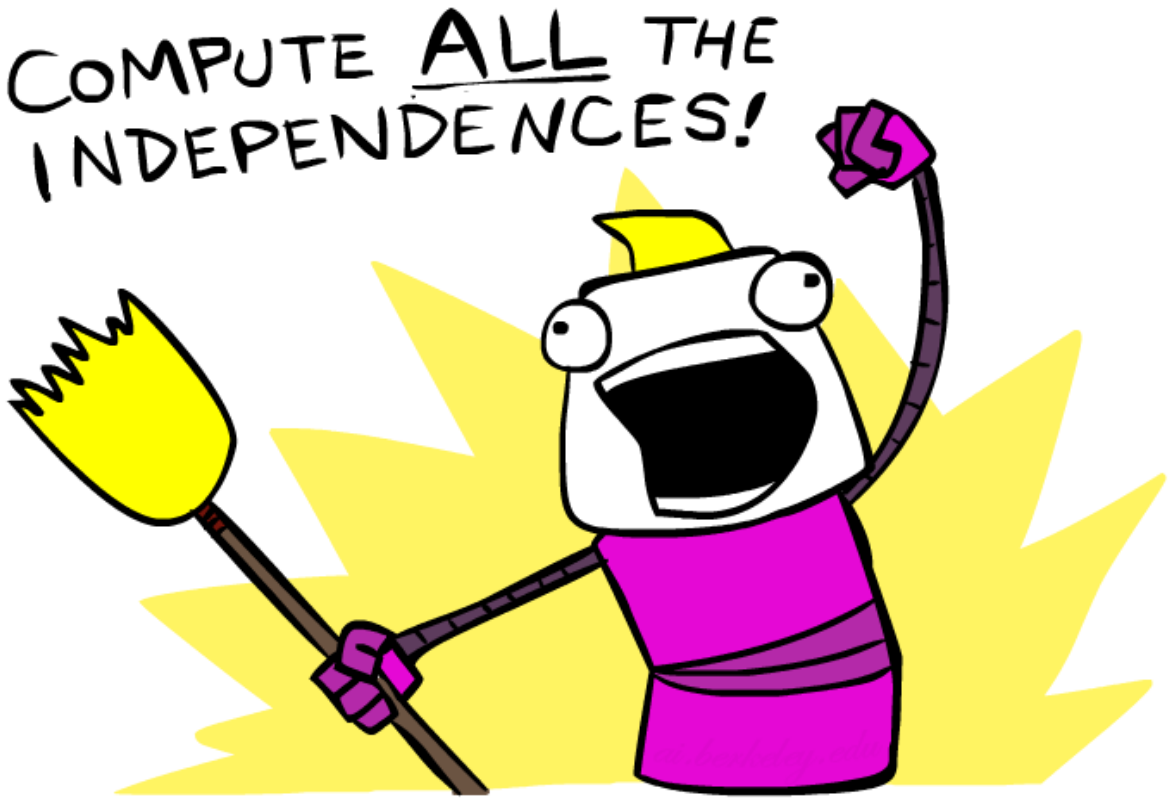
Structure Implications

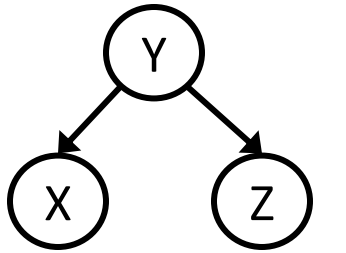
- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

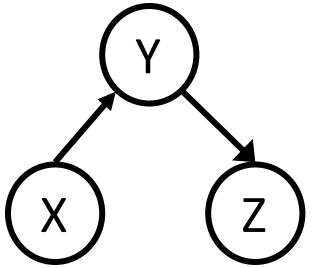
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

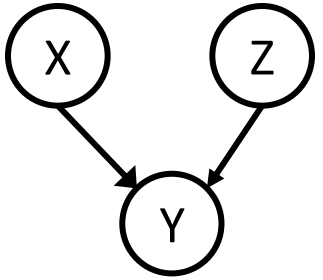
- This list determines the set of probability distributions that can be represented

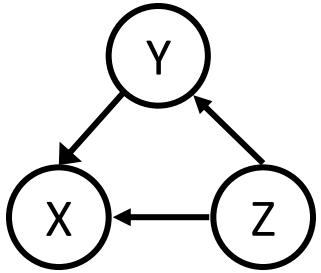
Computing All Independences







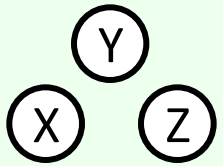


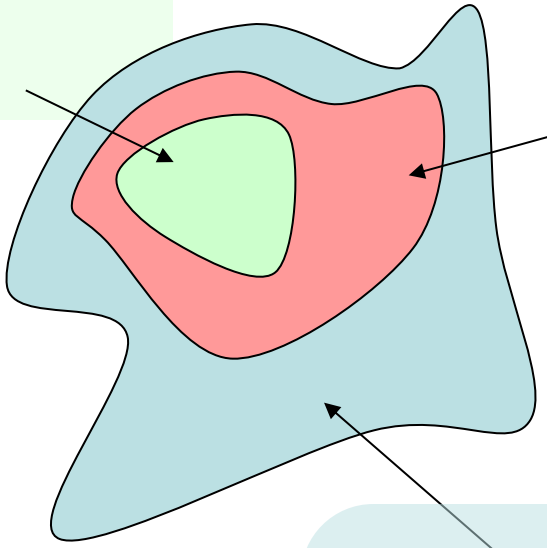


Topology Limits Distributions

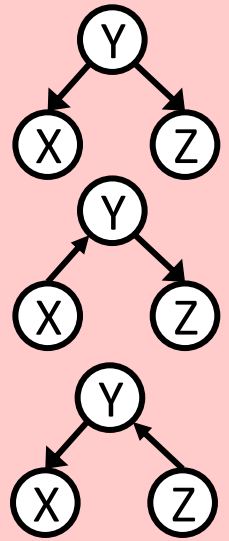
- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution

$\{X \perp\!\!\!\perp Y, X \perp\!\!\!\perp Z, Y \perp\!\!\!\perp Z,$
$X \perp\!\!\!\perp Z \mid Y, X \perp\!\!\!\perp Y \mid Z, Y \perp\!\!\!\perp Z \mid X\}$





$\{X \perp\!\!\!\perp Z \mid Y\}$



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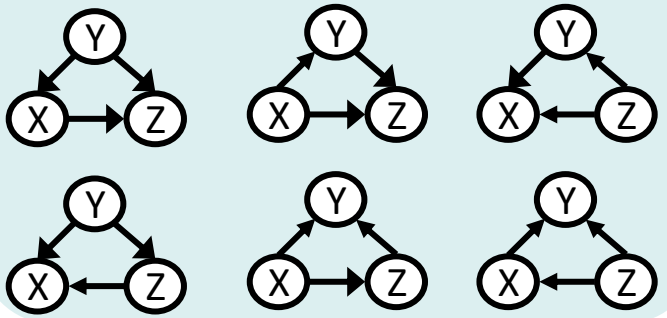

Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

Bayes' Nets

- ✔ Representation
- ✔ Conditional Independences
- Probabilistic Inference
  - Enumeration (exact, exponential complexity)
  - Variable elimination (exact, worst-case exponential complexity, often better)
  - Probabilistic inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data