

CS 188: Artificial Intelligence

Markov Decision Processes

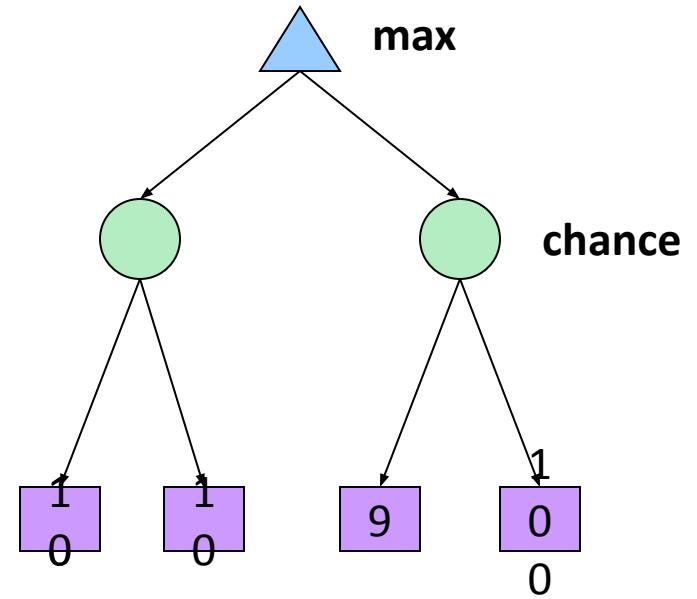


Instructors: Peyrin Kao and Emma Pierson

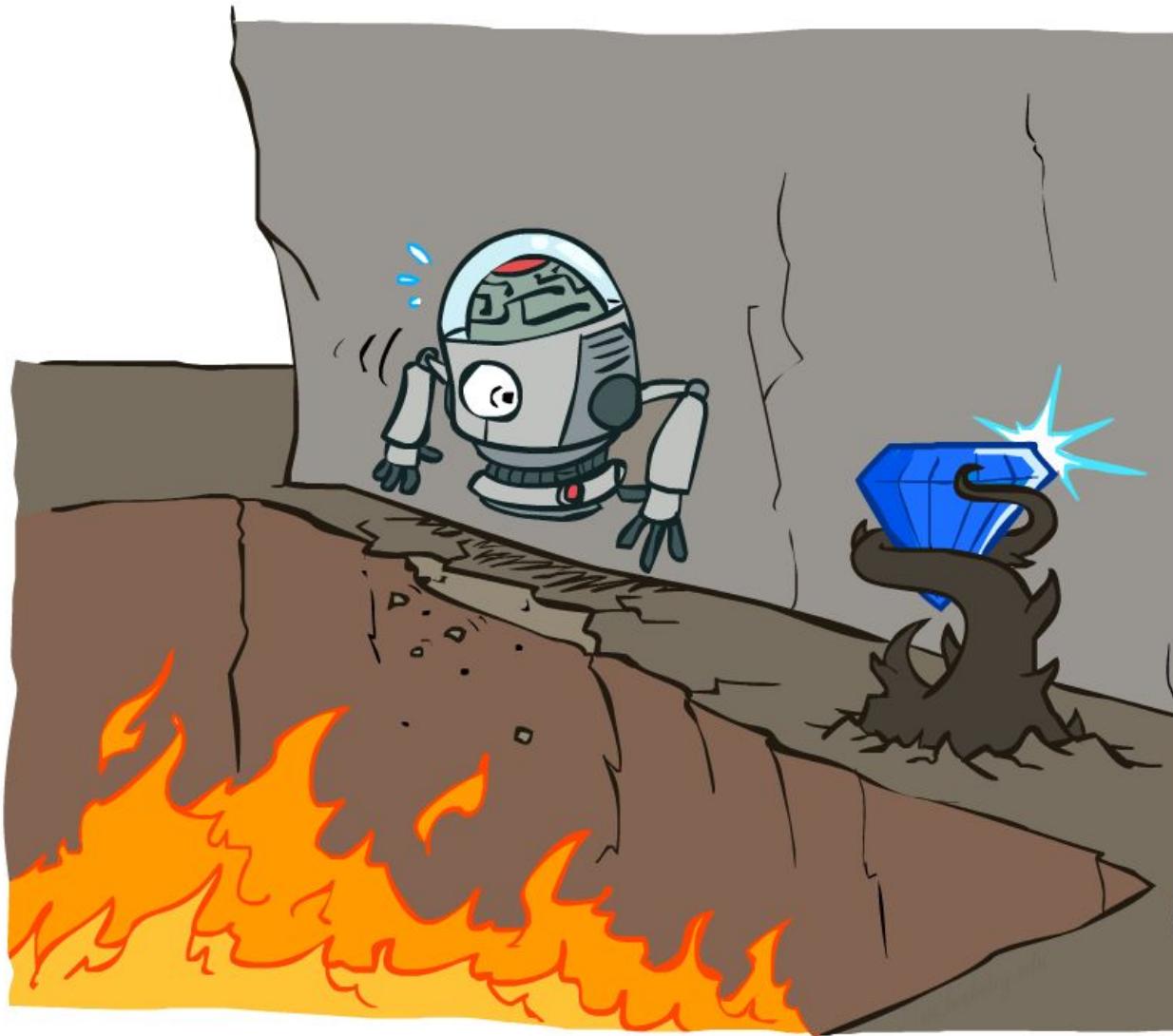
University of California, Berkeley

Review: Expectimax Search

- Uncertain outcomes controlled by chance, not an adversary
- Why wouldn't we know what the result of an action will be?
 - Explicit randomness: rolling dice
 - Unpredictable opponents: the ghosts respond randomly
 - Actions can fail: when moving a robot, wheels might slip
- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes
- **Expectimax search:** compute the average score under optimal play
 - Max nodes as in minimax search
 - Chance nodes are like min nodes but the outcome is uncertain
 - Calculate their **expected utilities**
 - I.e. take weighted average (expectation) of children
- Today, we'll learn how to formalize the underlying uncertain-result problems as **Markov Decision Processes**



Non-Deterministic Search



Reinforcement learning



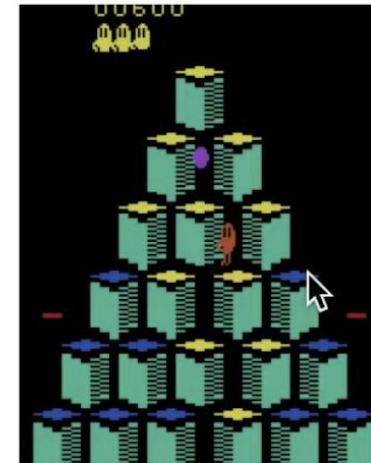
Pong



Enduro



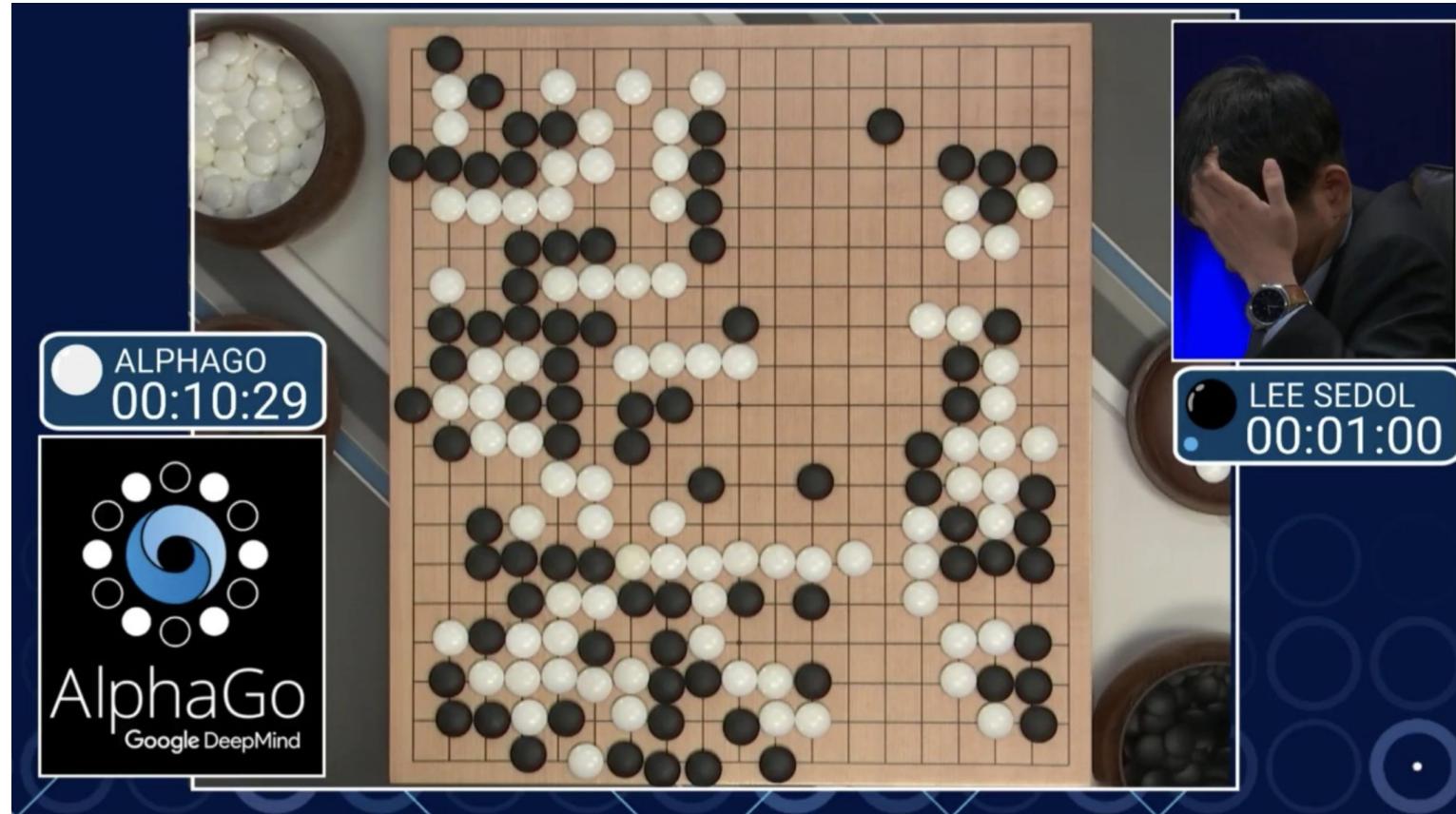
Beamrider



Q*bert

Atari (Minh et al, *Nature* 2015)

Reinforcement learning



AlphaGo Silver et al, Nature 2015

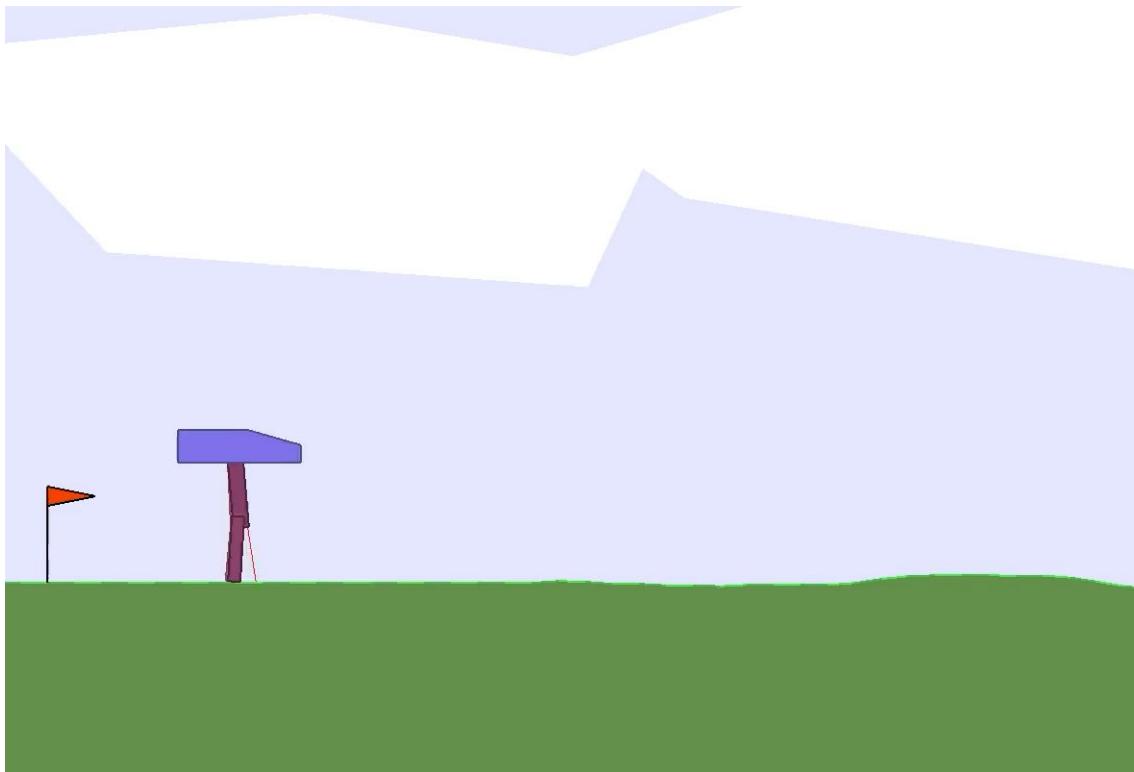
AlphaGoZero Silver et al, Nature 2017

AlphaZero Silver et al, 2017

Tian et al, 2016; Maddison et al, 2014; Clark et al, 2015

Reinforcement learning

Step 0



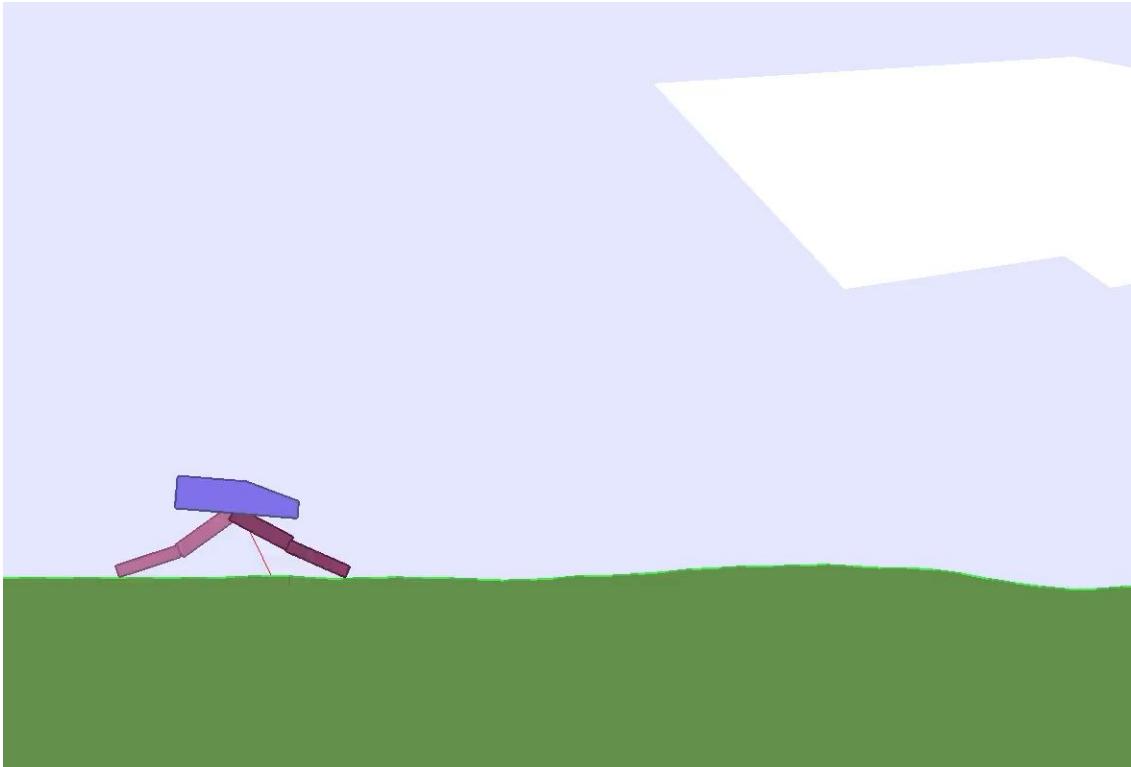
Schulman, Wolski, Dhariwal, Radford, and Klimov, 2017

Schulman, Levine, Moritz, Jordan, and Abbeel, 2015

Video: Mukilan Krishnakumar

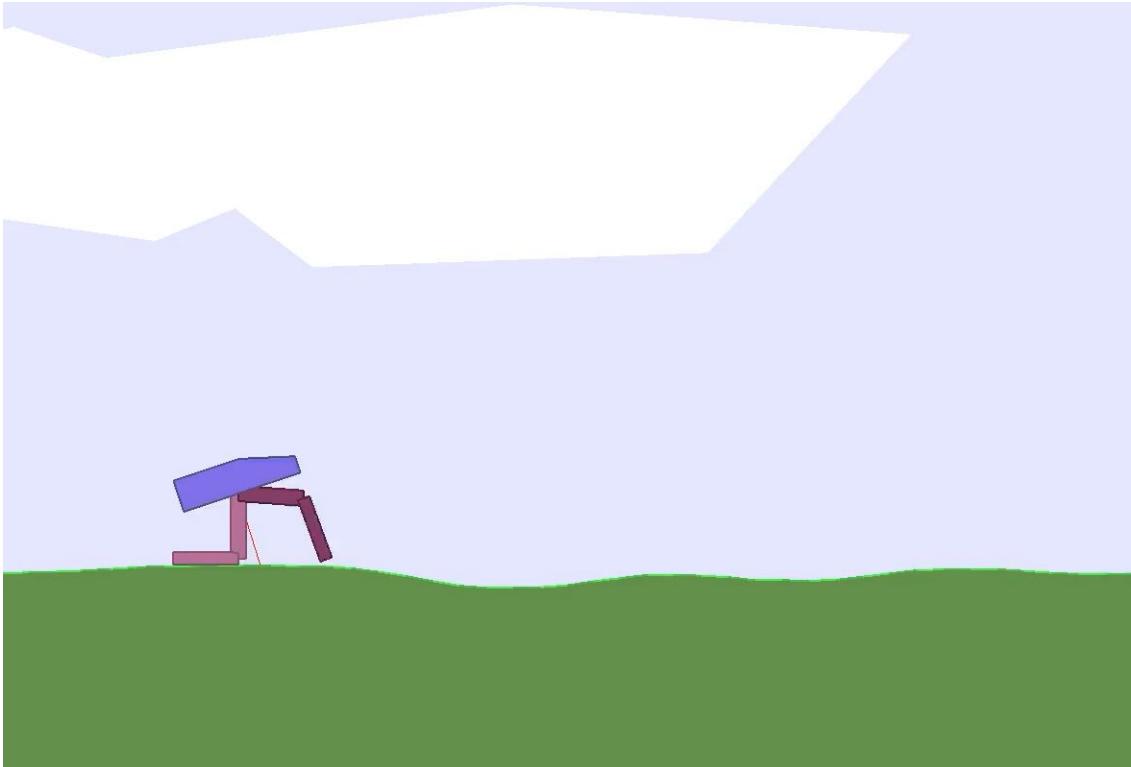
Reinforcement learning

Step 100



Reinforcement learning

Step 500

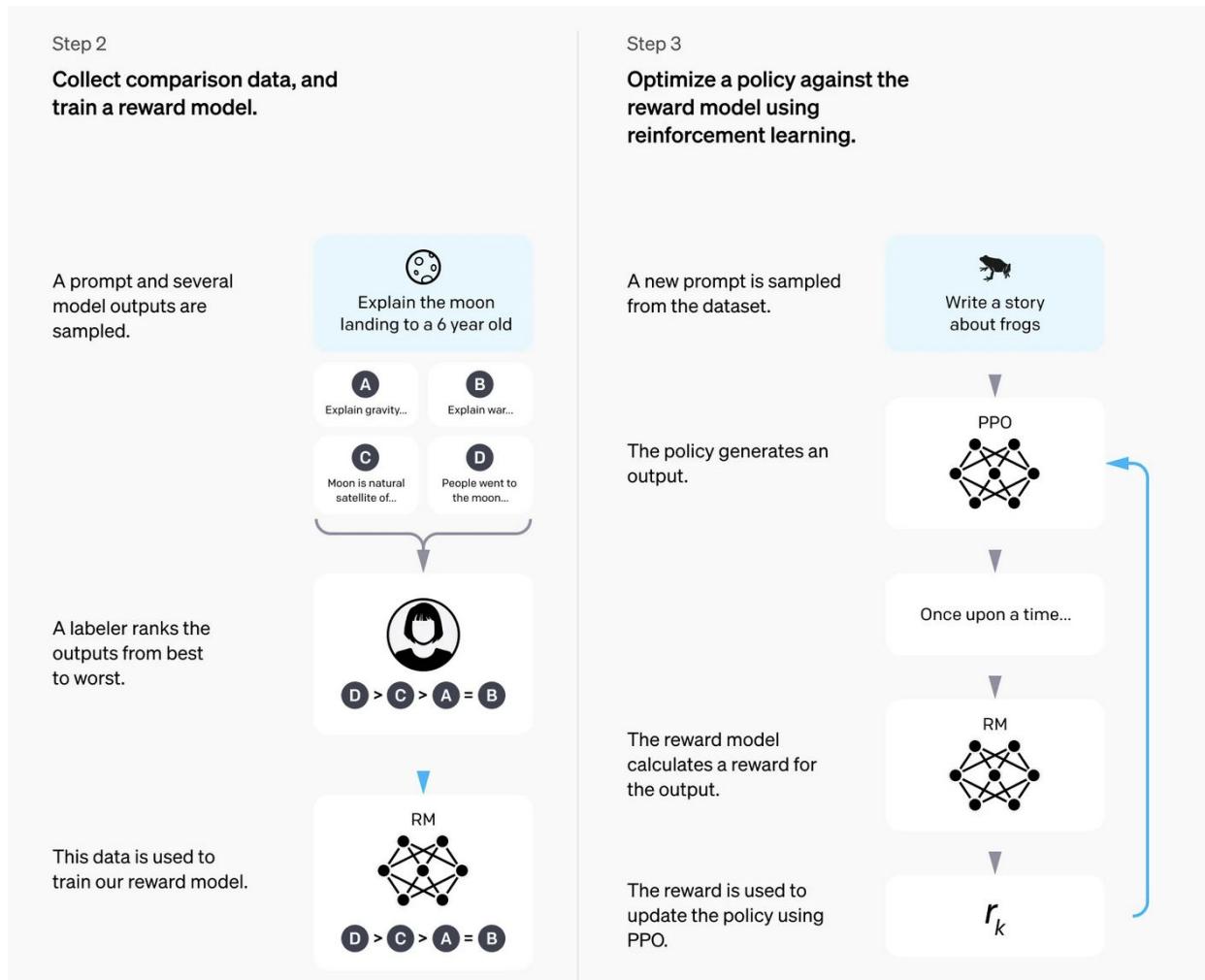


Reinforcement learning



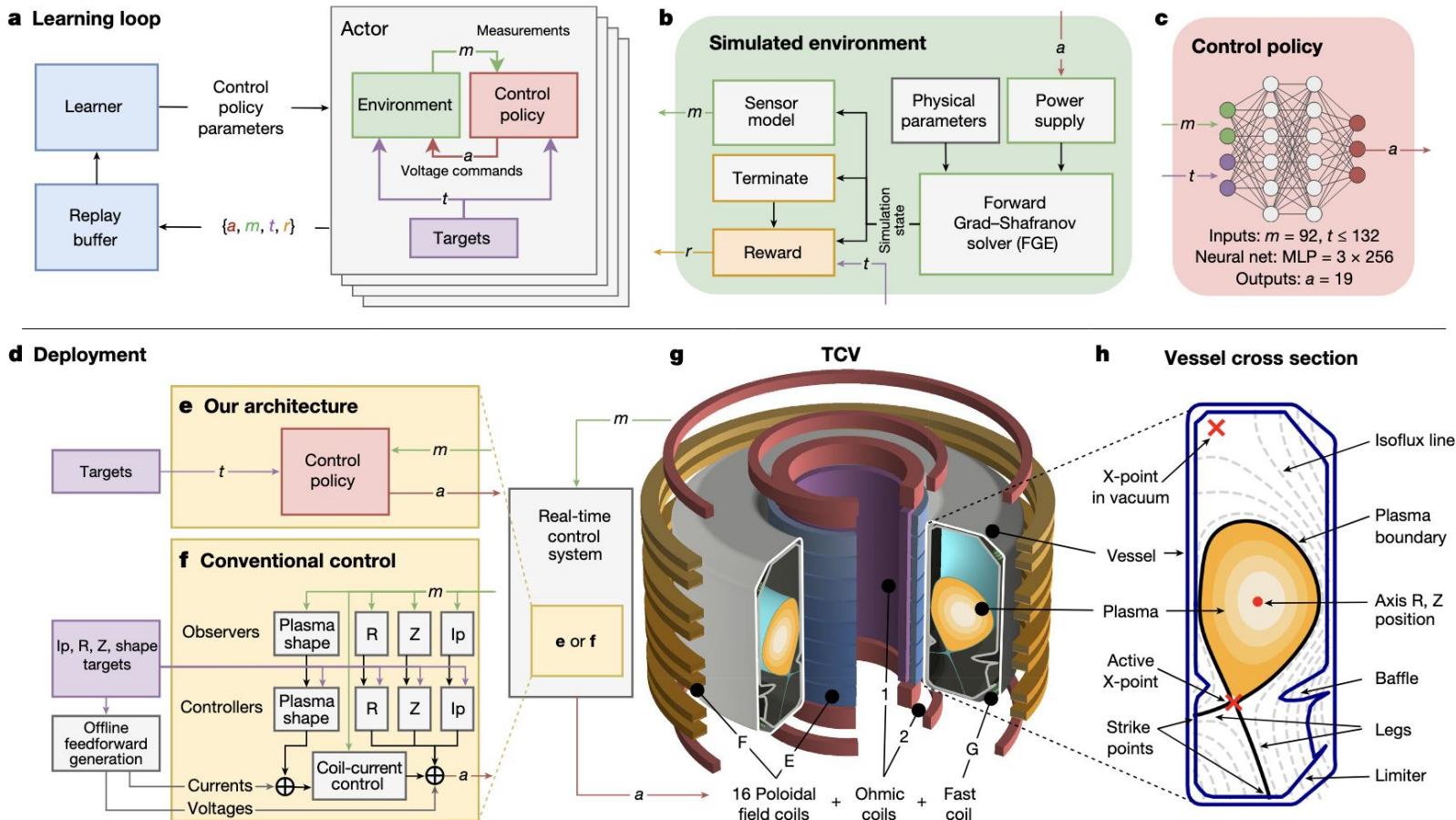
Physical Intelligence, 2024

Reinforcement learning



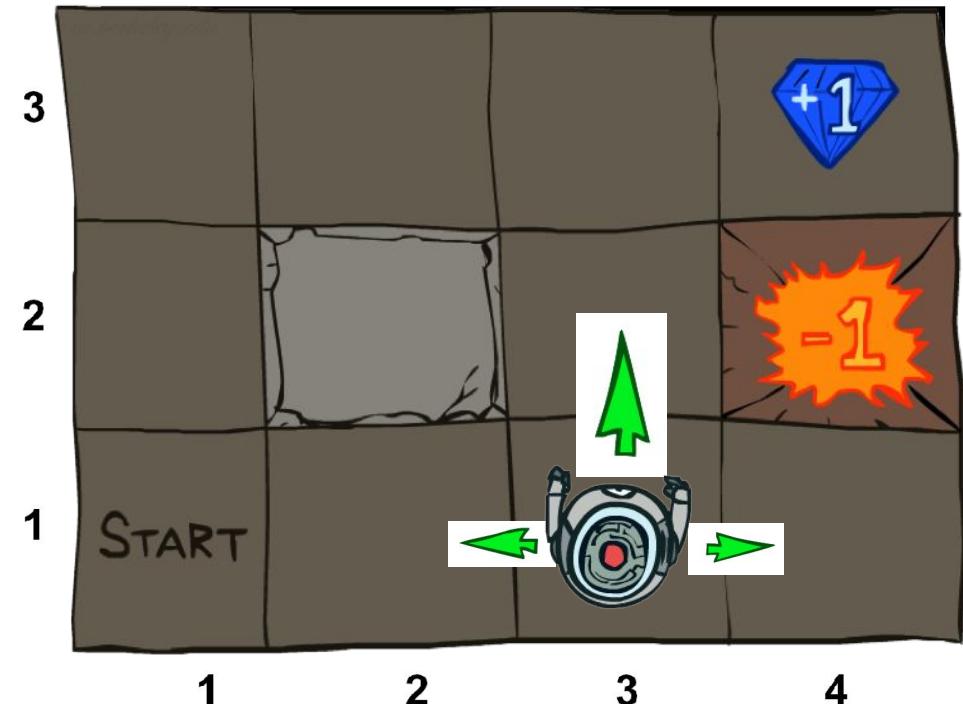
OpenAI, 2022

Reinforcement learning



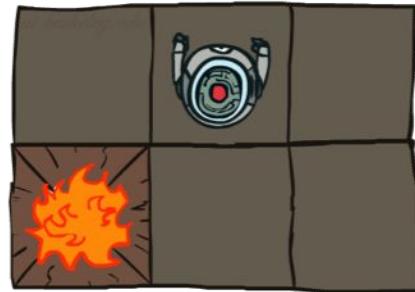
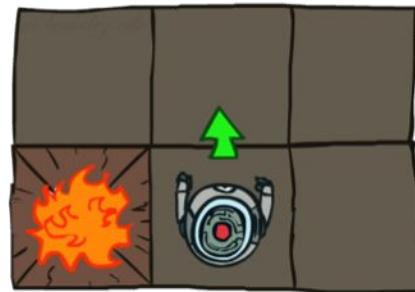
Example: Grid World

- A maze-like problem
 - The agent lives in a grid
 - Walls block the agent's path
- Noisy movement: actions do not always go as planned
 - 80% of the time, the action North takes the agent North (if there is no wall there)
 - 10% of the time, North takes the agent West; 10% East
 - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
 - Small “living” reward each step (can be negative)
 - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

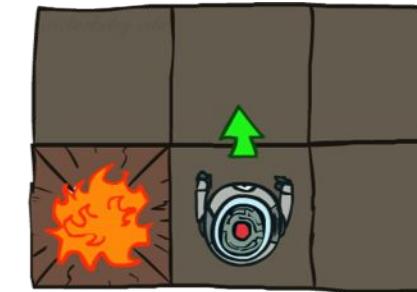


Grid World Actions

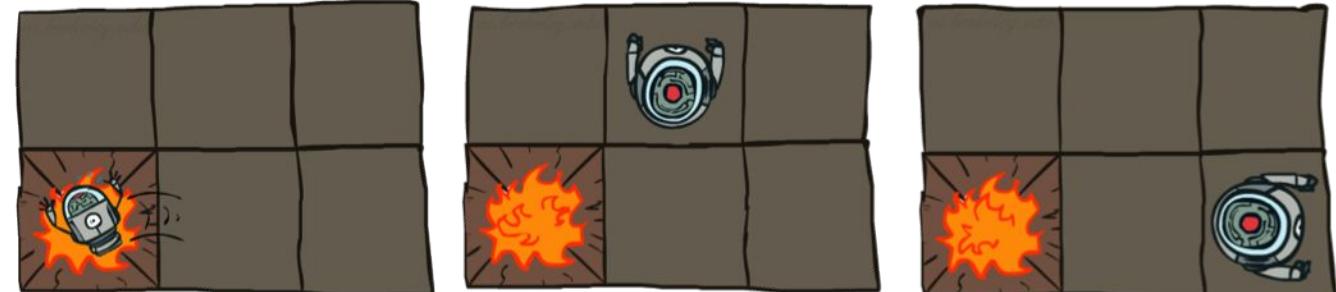
Deterministic Grid World



Stochastic Grid World

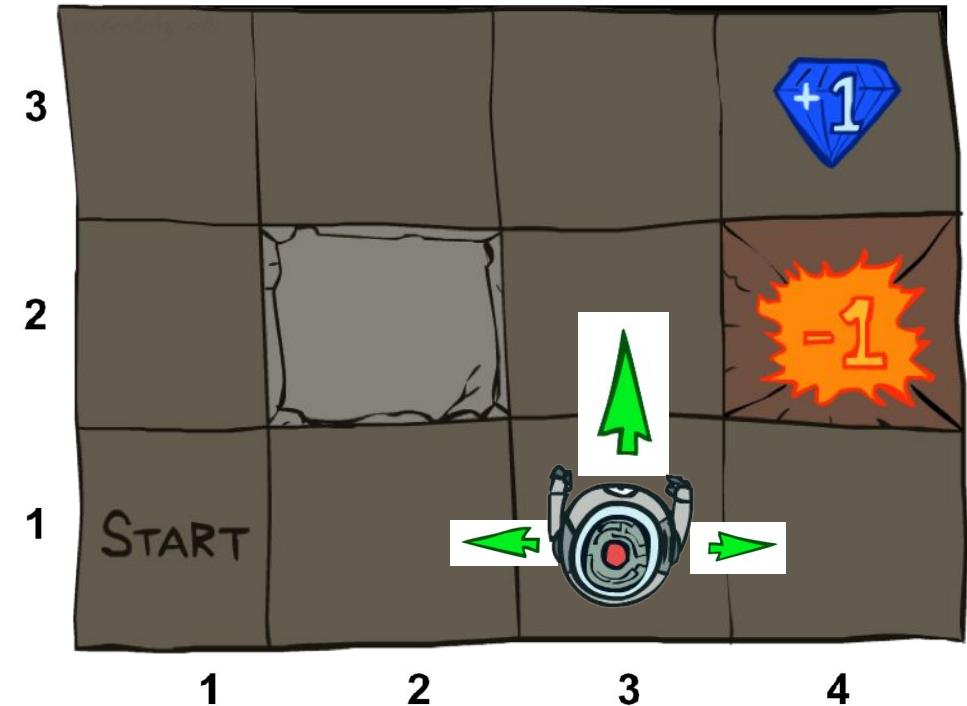


?



Markov Decision Processes

- An MDP is defined by:
 - A set of states $s \in S$
 - A set of actions $a \in A$
 - A transition function $T(s, a, s')$
 - Probability that a from s leads to s' , i.e., $P(s' | s, a)$
 - Also called the model or the dynamics
 - A reward function $R(s, a, s')$
 - Sometimes just $R(s)$ or $R(s')$
 - A start state
 - Maybe a terminal state
- MDPs are non-deterministic search problems
 - One way to solve them is with expectimax search
 - We'll have a new tool soon



Video of Demo Gridworld Manual Intro



What is Markov about MDPs?

- “Markov” generally means that given the present state, the future and the past are independent
- For Markov decision processes, “Markov” means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots, S_0 = s_0)$$

=

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

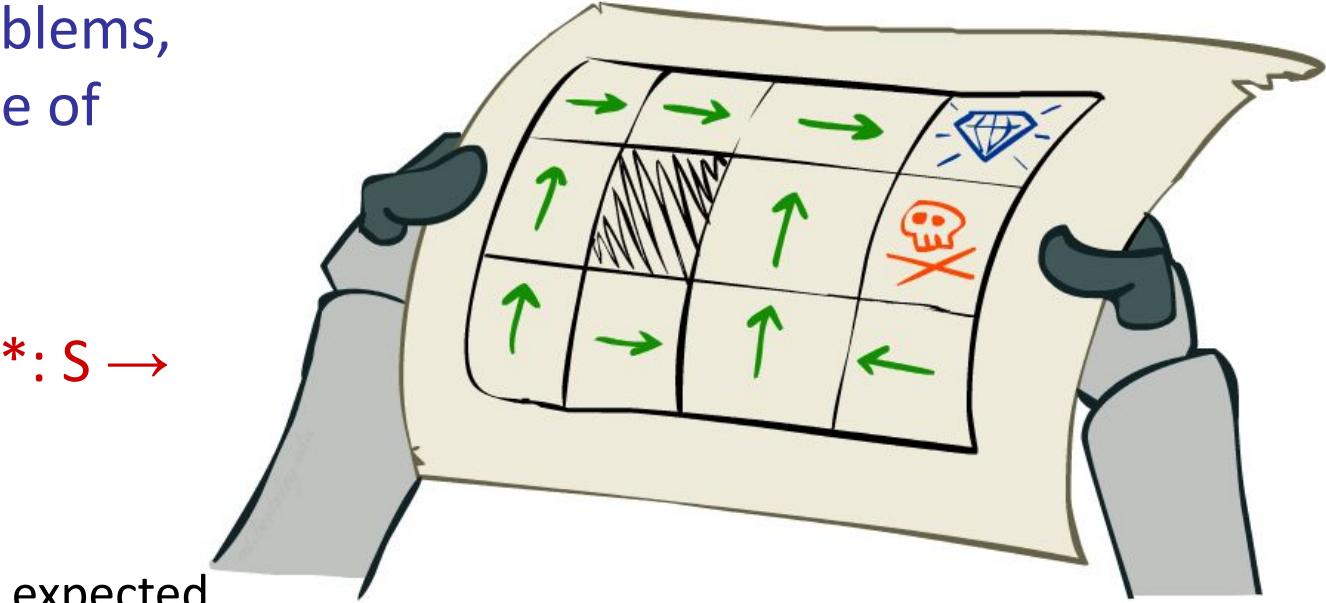


Andrey Markov
(1856-1922)

- This is just like search, where the successor function could only depend on the current state (not the history)

Policies

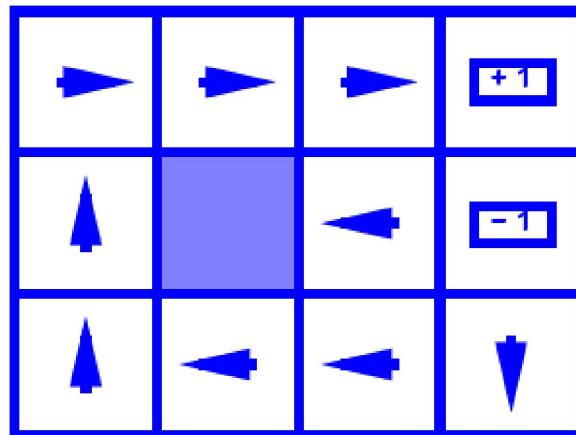
- In deterministic single-agent search problems, we wanted an optimal **plan**, or sequence of actions, from start to a goal
- For MDPs, we want an optimal **policy** $\pi^*: S \rightarrow A$
 - A policy π gives an action for each state
 - An optimal policy is one that maximizes expected utility if followed
 - An explicit policy defines a reflex agent
- Expectimax didn't compute entire policies
 - It computed the action for a single state only



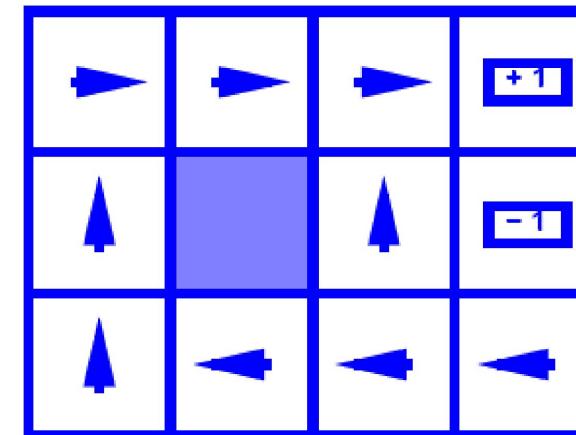
expected

Optimal policy when $R(s, a, s') = -0.03$
for all non-terminals s

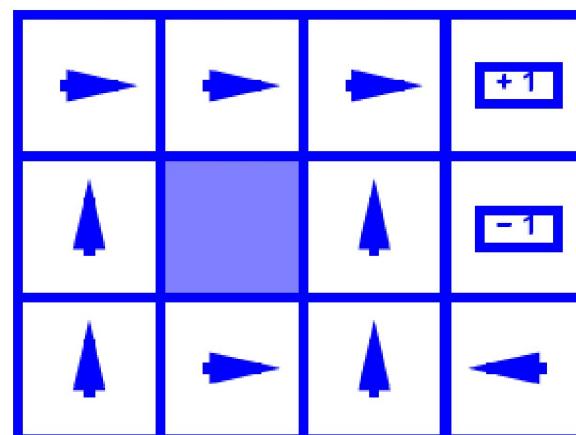
Optimal Policies



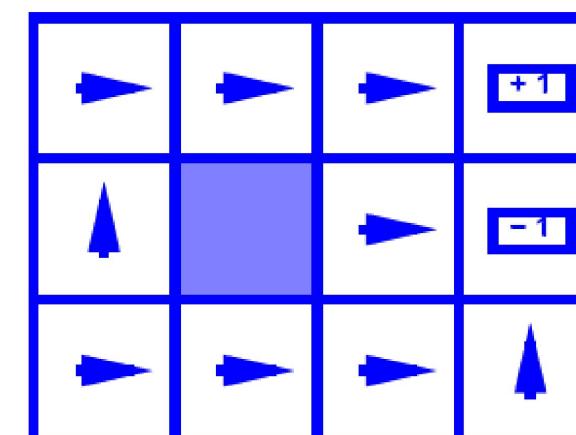
$$R(s) = -0.01$$



$$R(s) = -0.03$$

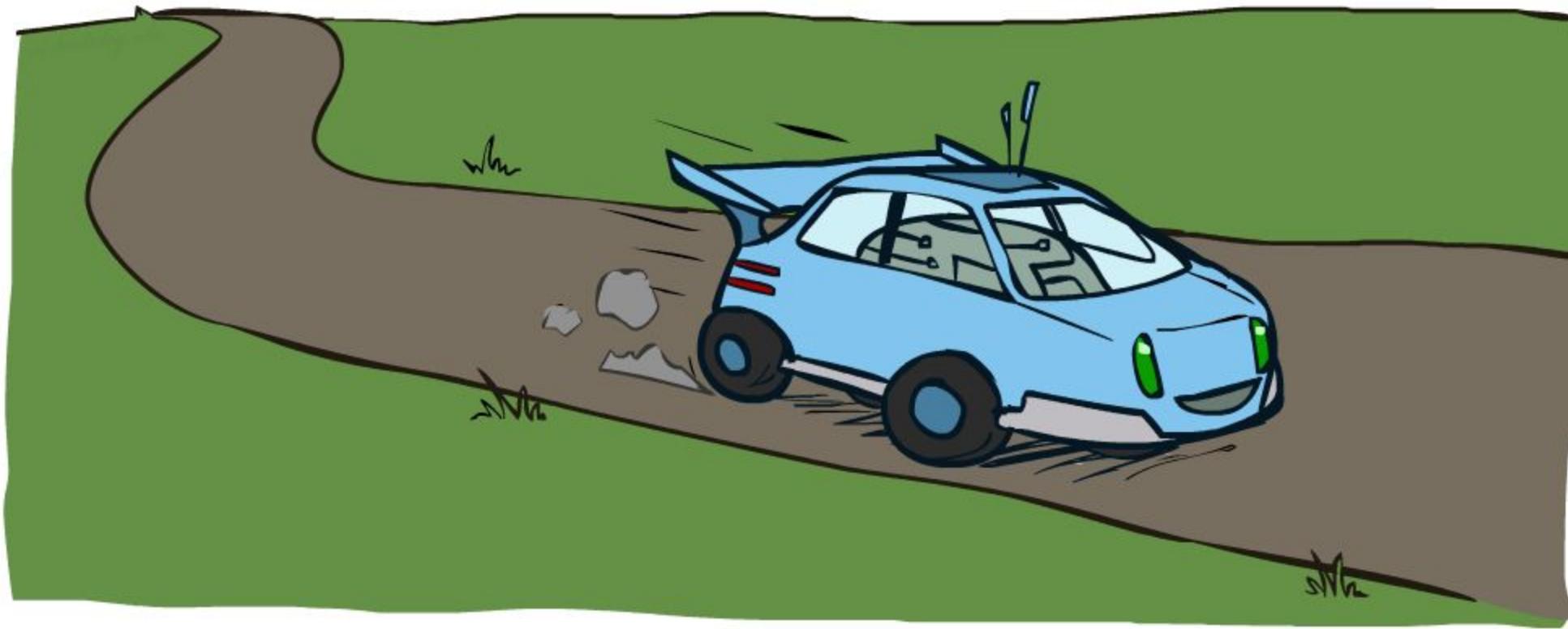


$$R(s) = -0.4$$



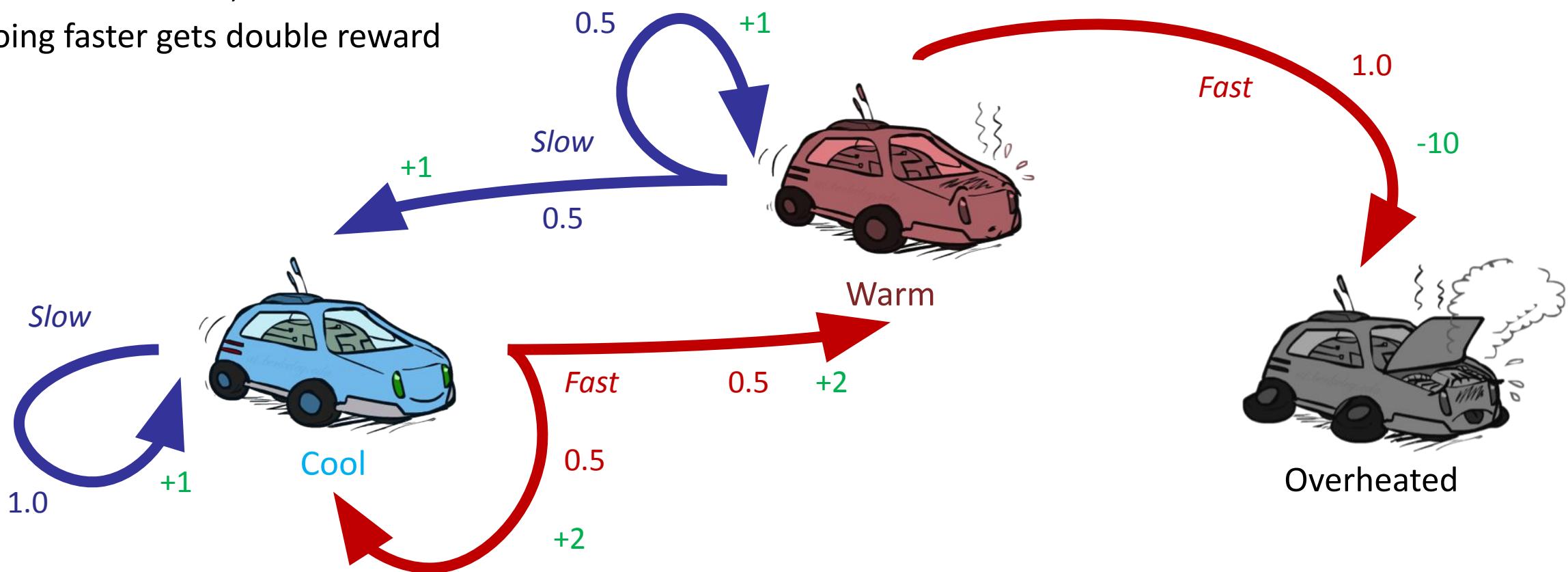
$$R(s) = -2.0$$

Example: Racing

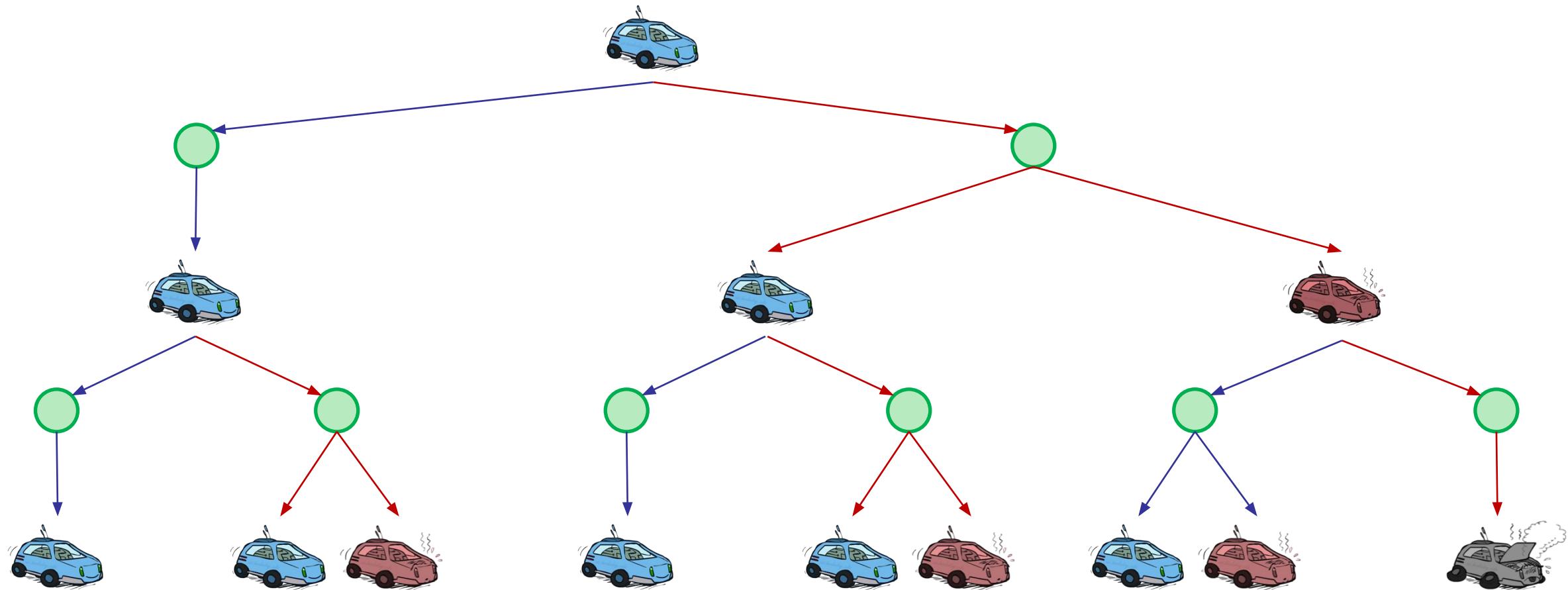


Example: Racing

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward

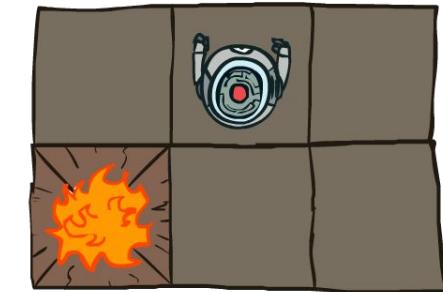
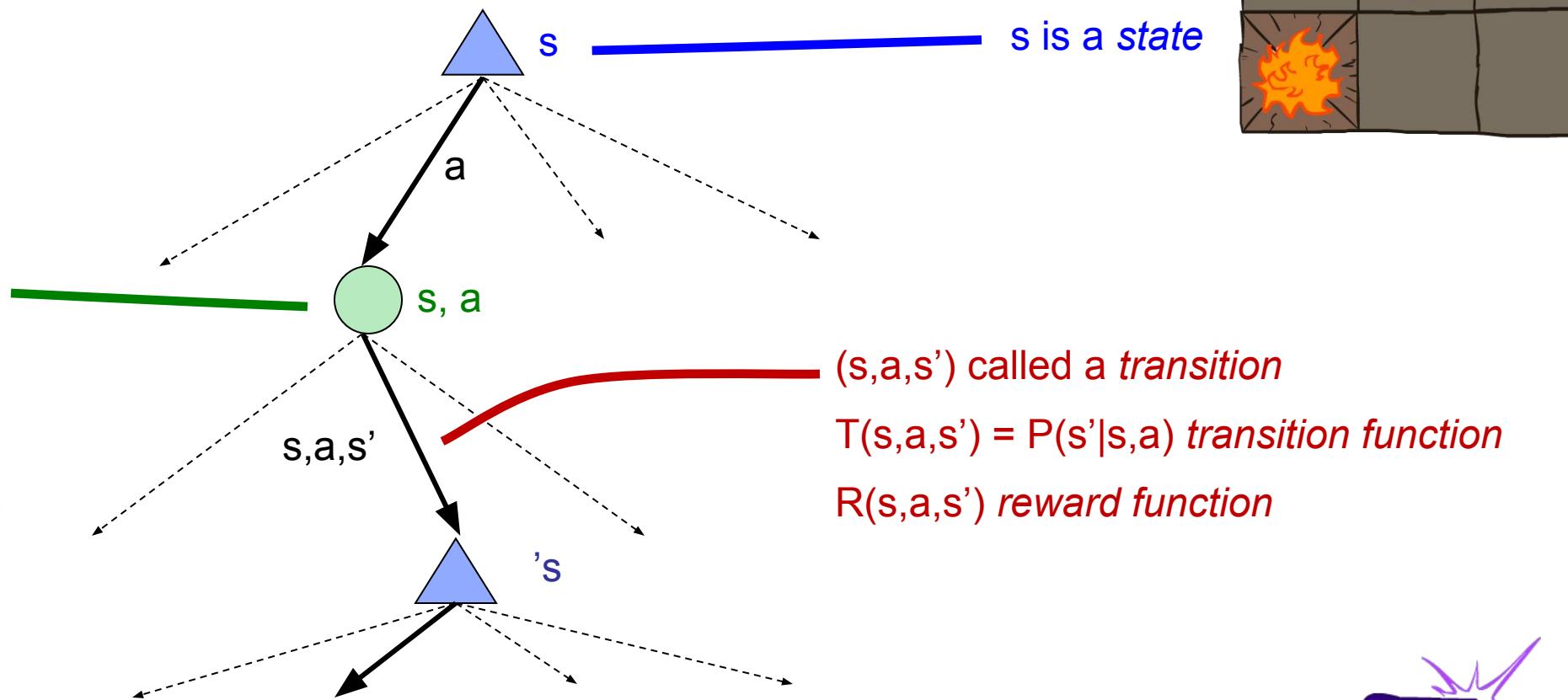
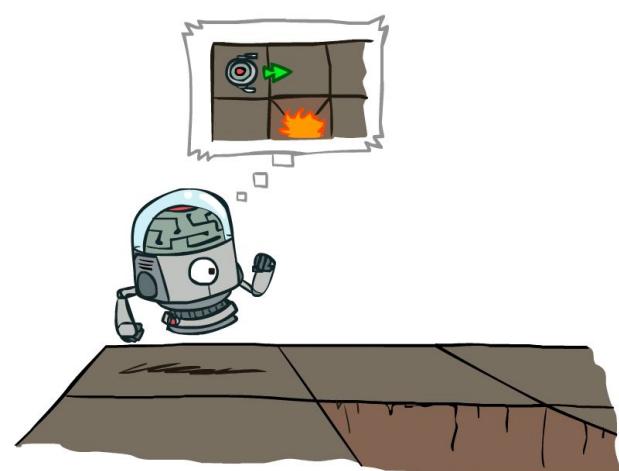


Racing Search Tree

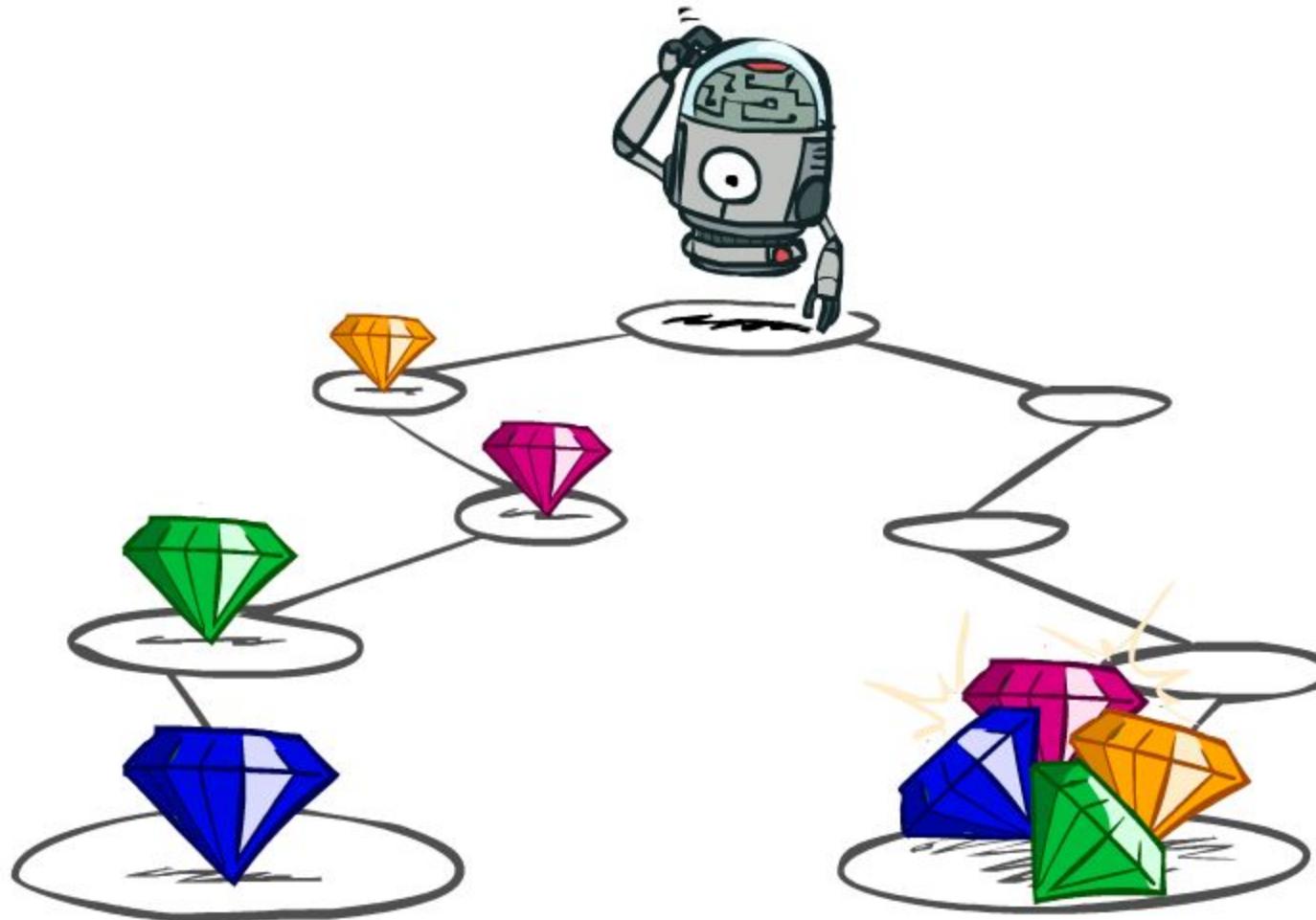


MDP Search Trees

- Each MDP state projects an expectimax-like search tree

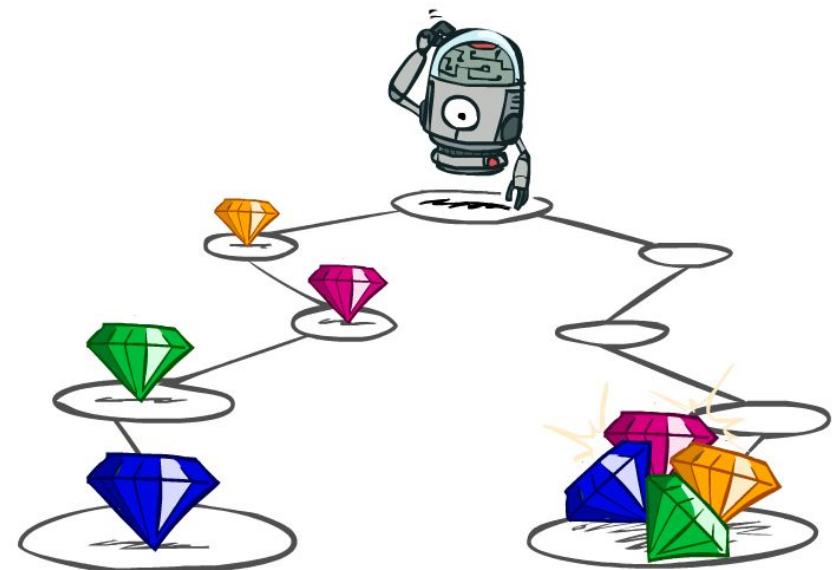


Utilities of Sequences



Utilities of Sequences

- What preferences should an agent have over reward sequences?
- More or less? $[1, 2, 2]$ or $[2, 3, 4]$
- Now or later? $[0, 0, 1]$ or $[1, 0, 0]$



Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially



1

Worth Now



γ



γ^2

Worth In Two Steps

Discounting

- How to discount?

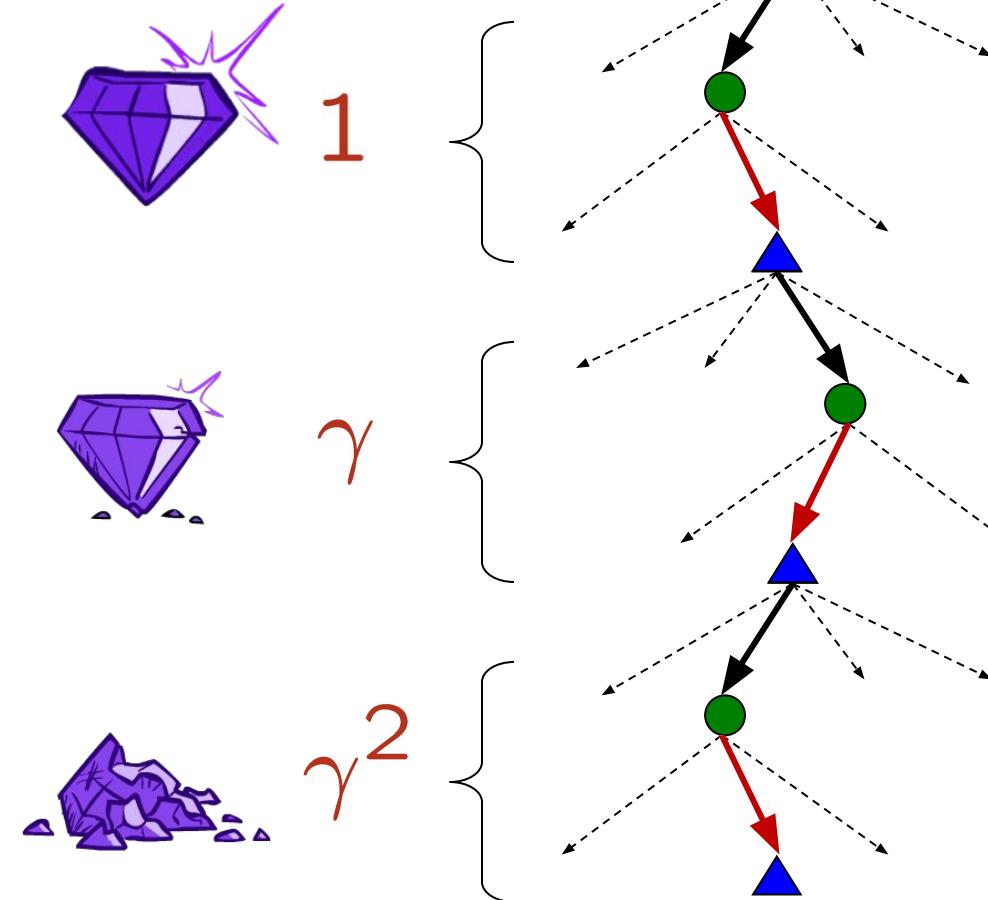
- Each time we descend a level, we multiply in the discount once

- Why discount?

- Sooner rewards probably do have higher utility than later rewards
 - Also helps our algorithms converge

- Example: discount of 0.5

- $U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3$
 - $U([1,2,3]) < U([3,2,1])$



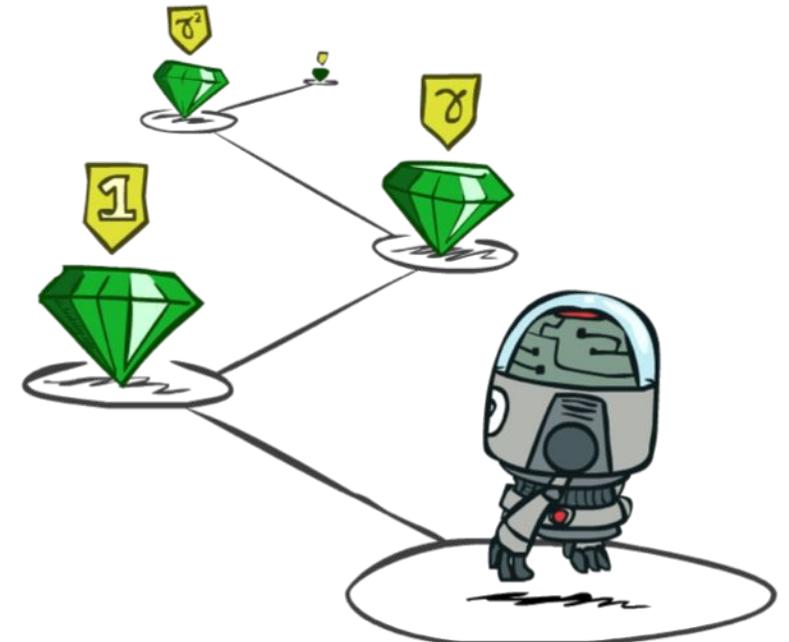
Stationary Preferences

- Theorem: if we assume stationary preferences:

$$[a_1, a_2, \dots] \succ [b_1, b_2, \dots]$$

\Updownarrow

$$[r, a_1, a_2, \dots] \succ [r, b_1, b_2, \dots]$$



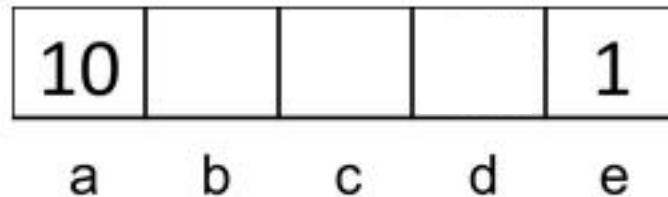
- Then: there are only two ways to define utilities

- Additive utility: $U([r_0, r_1, r_2, \dots]) = r_0 + r_1 + r_2 + \dots$

- Discounted utility: $U([r_0, r_1, r_2, \dots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \dots$

Quiz: Discounting

- Given:



- Actions: East, West, and Exit (only available in exit states a, e)
- Transitions: deterministic

- Quiz 1: For $\gamma = 1$, what is the optimal policy?

10				1
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- Quiz 2: For $\gamma = 0.1$, what is the optimal policy?

10				1
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- Quiz 3: For which γ are West and East equally good when in state d?

Infinite Utilities?!

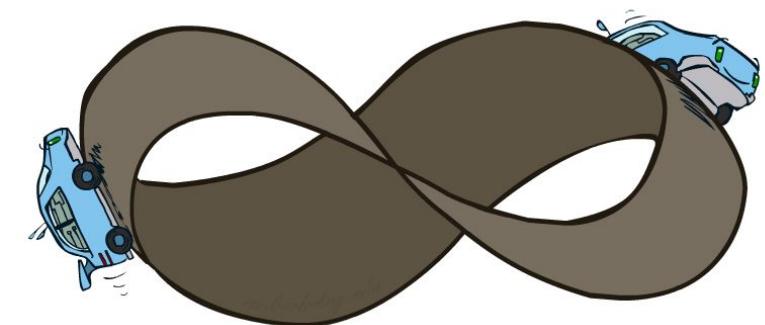
- Problem: What if the game lasts forever? Do we get infinite rewards?
 - E.g. racing game

- Solutions:

- Finite horizon: (similar to depth-limited search)
 - Terminate episodes after a fixed T steps (e.g. life)
 - Gives nonstationary policies (π depends on time left)
- Discounting: use $0 < \gamma < 1$

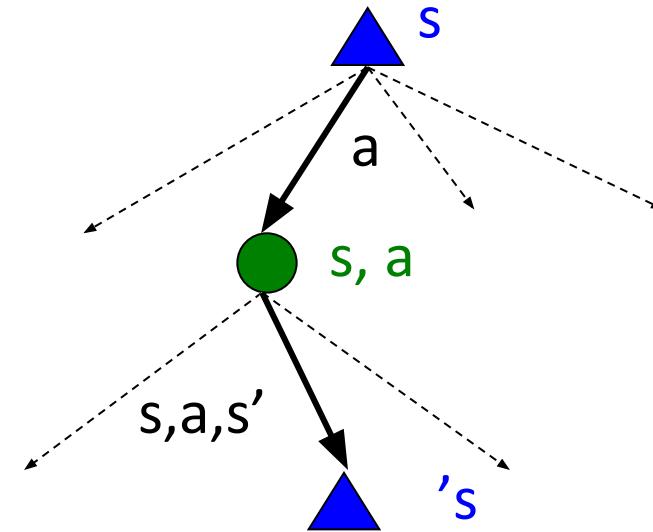
$$U([r_0, \dots, r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1 - \gamma)$$

- Smaller γ means smaller “horizon” – shorter term focus
- Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like “overheated” for racing)

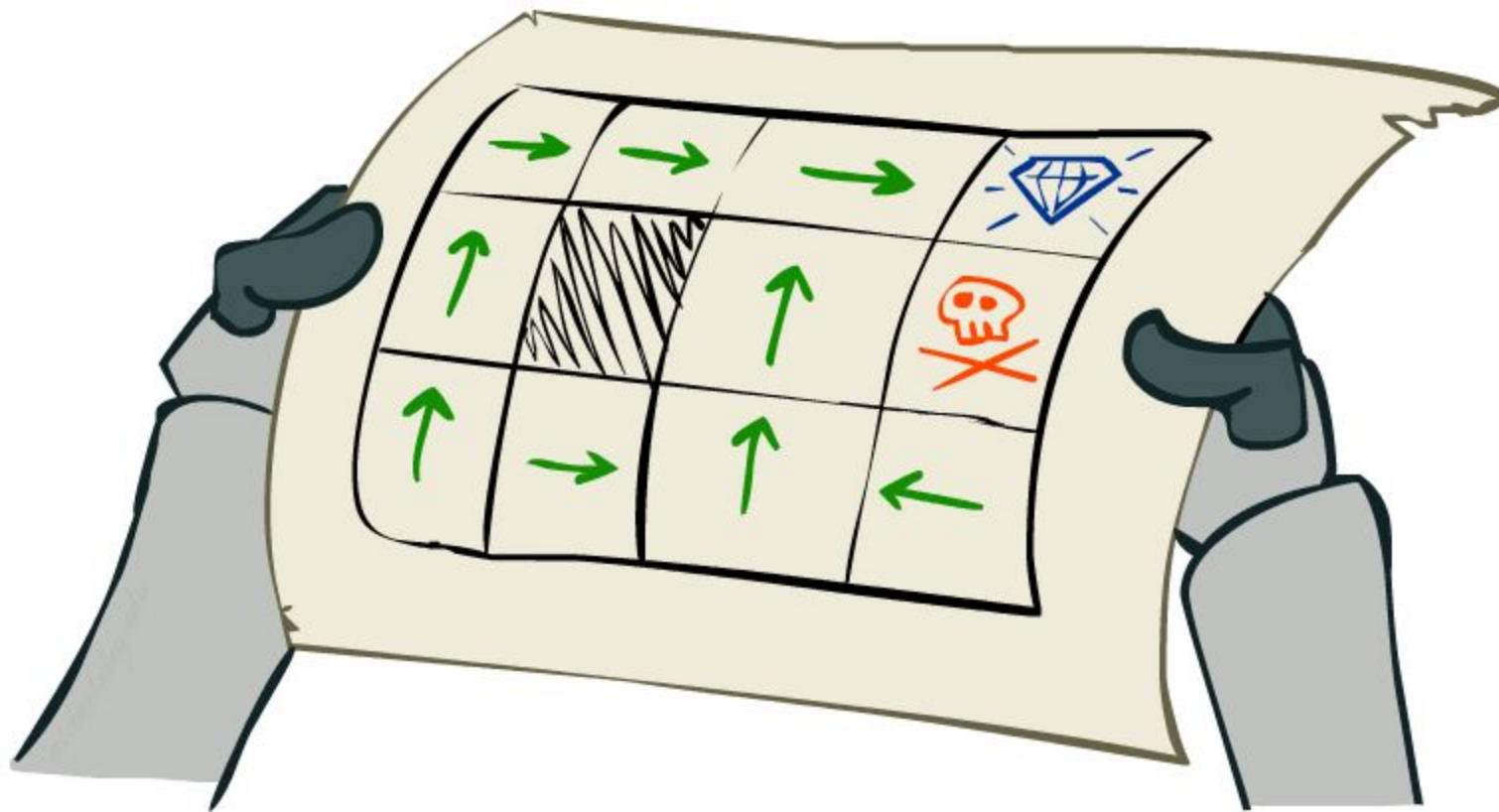


Recap: Defining MDPs

- Markov decision processes:
 - Set of states S
 - Start state s_0
 - Set of actions A
 - Transitions $P(s'|s,a)$ (or $T(s,a,s')$)
 - Rewards $R(s,a,s')$ (and discount γ)
- MDP quantities so far:
 - Policy = Choice of action for each state
 - Utility = sum of (discounted) rewards

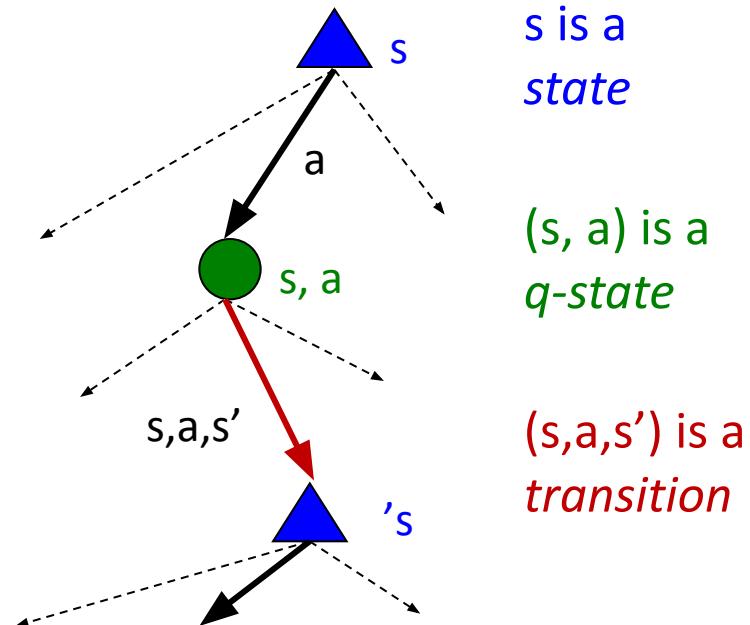


Solving MDPs



Optimal Quantities

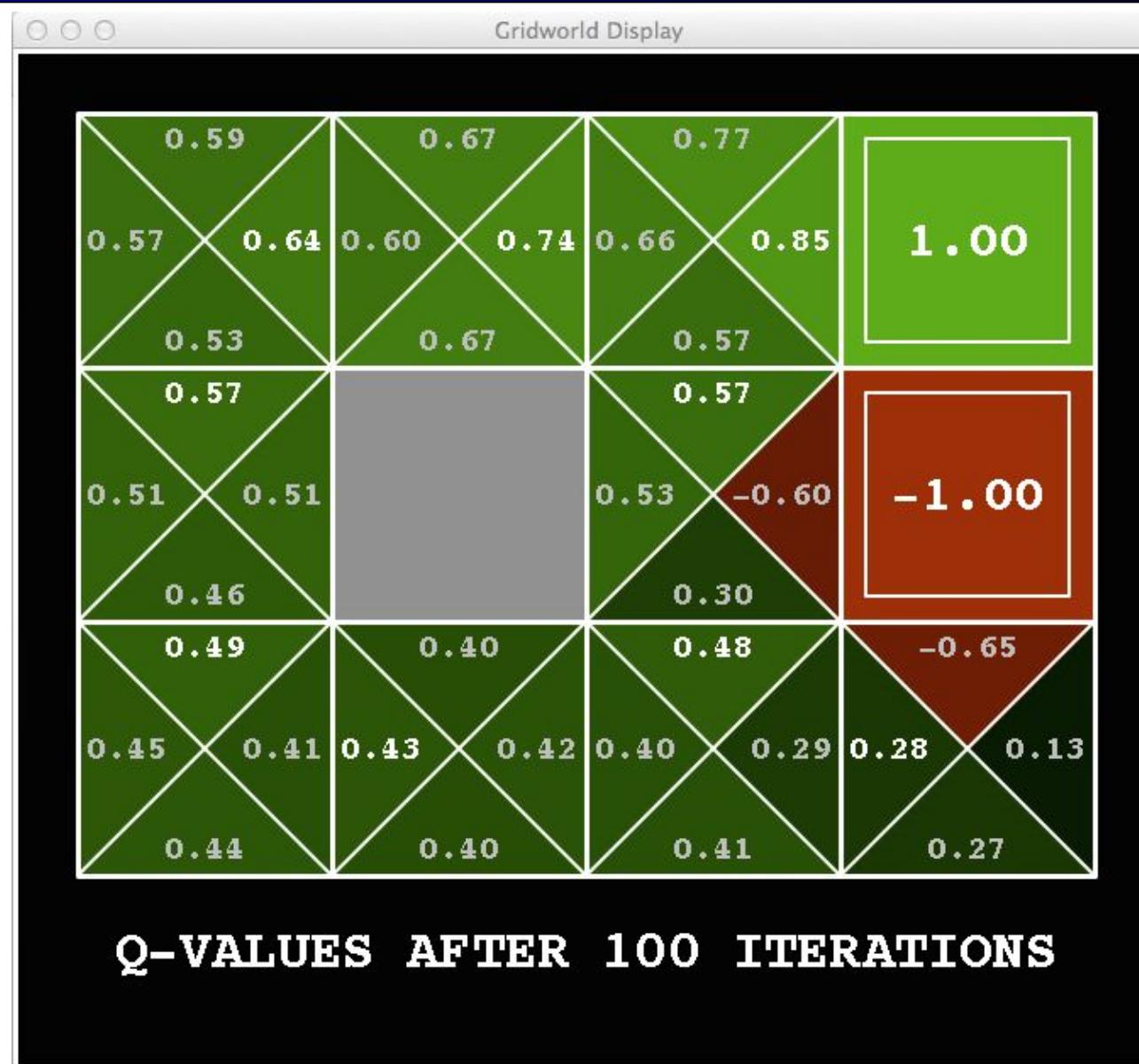
- The value (utility) of a state s :
 $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a) :
 $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
 $\pi^*(s)$ = optimal action from state s



Snapshot of Demo – Gridworld V Values



Snapshot of Demo – Gridworld Q Values



Values of States

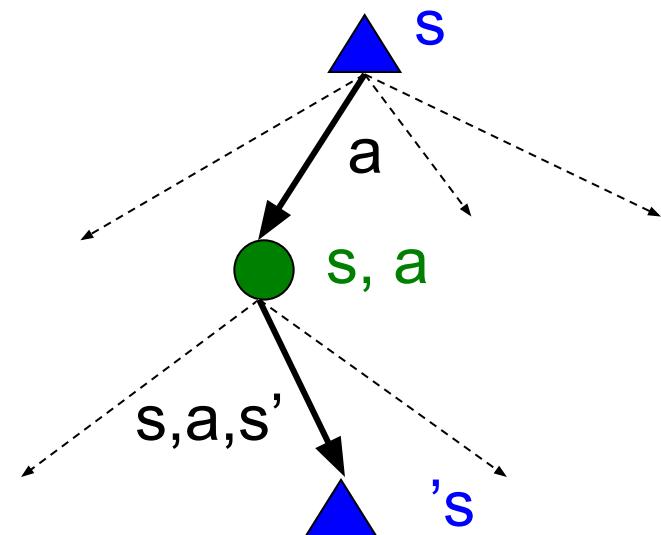
- Fundamental operation: compute the (expectimax) value of a state
 - Expected utility under optimal action
 - Average sum of (discounted) rewards
 - This is just what expectimax computed!

- Recursive definition of value:

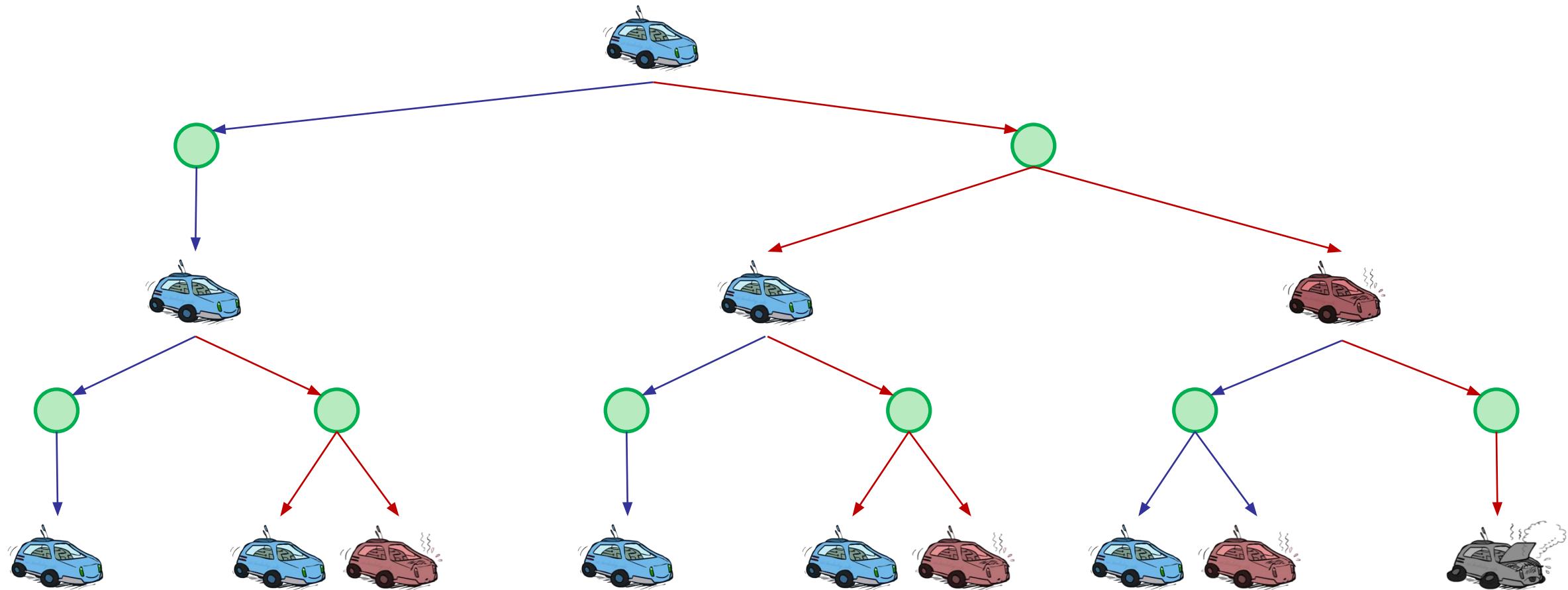
$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

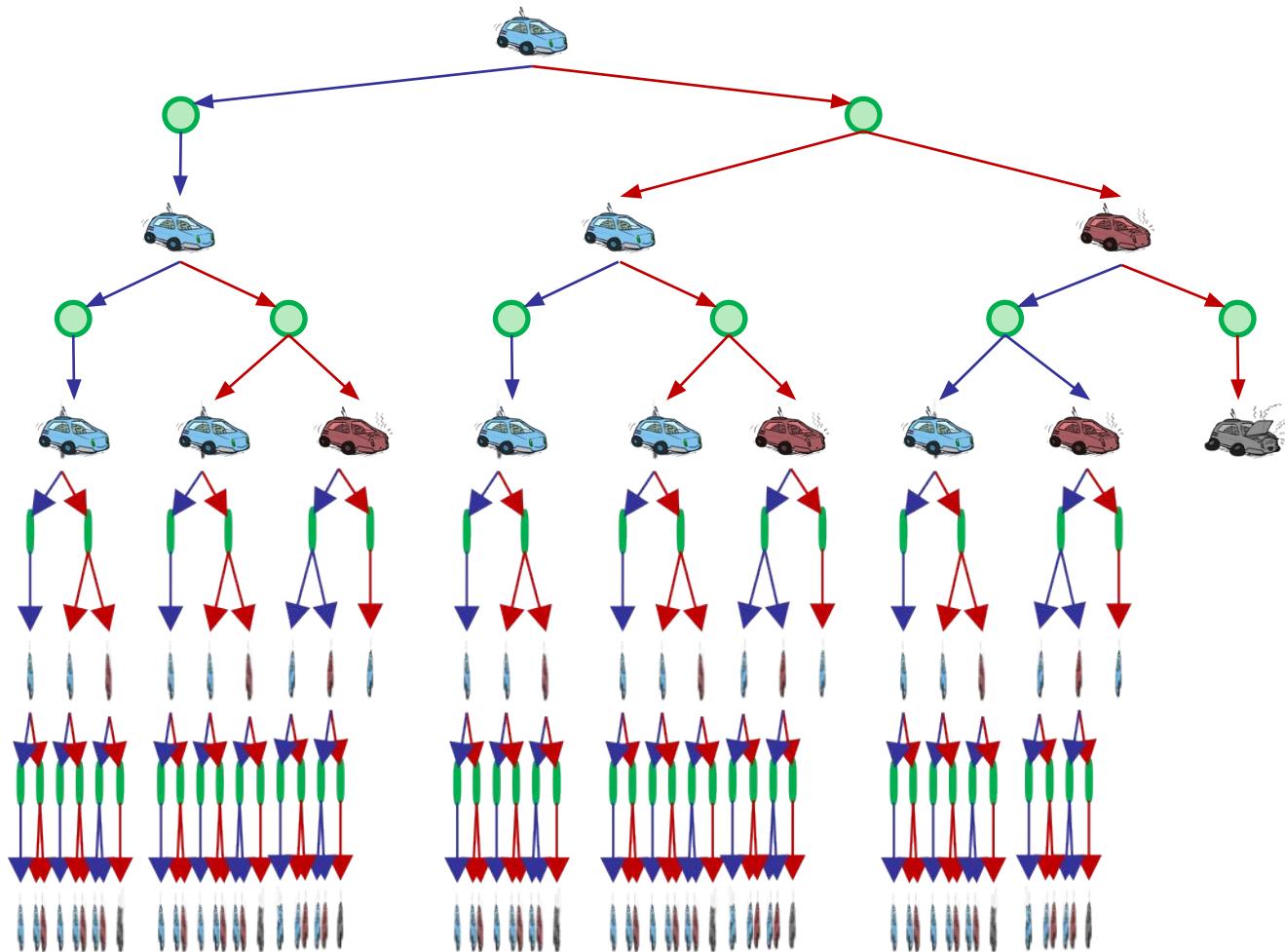
$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$



Racing Search Tree

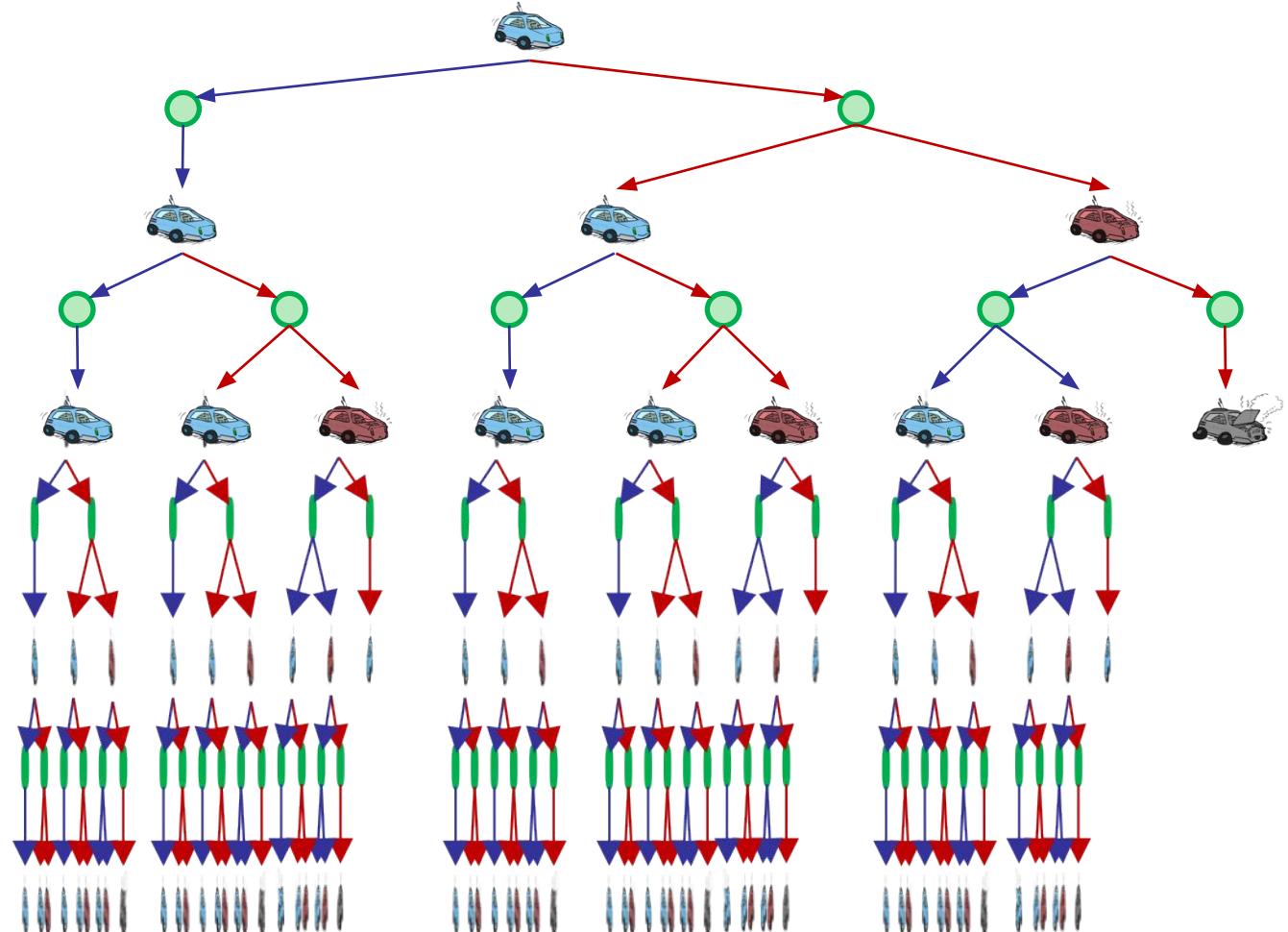


Racing Search Tree



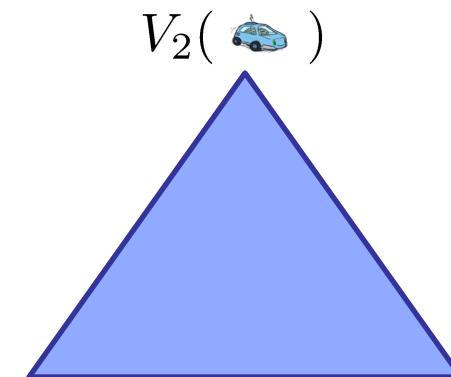
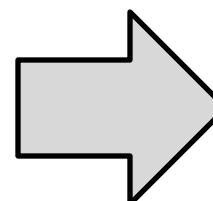
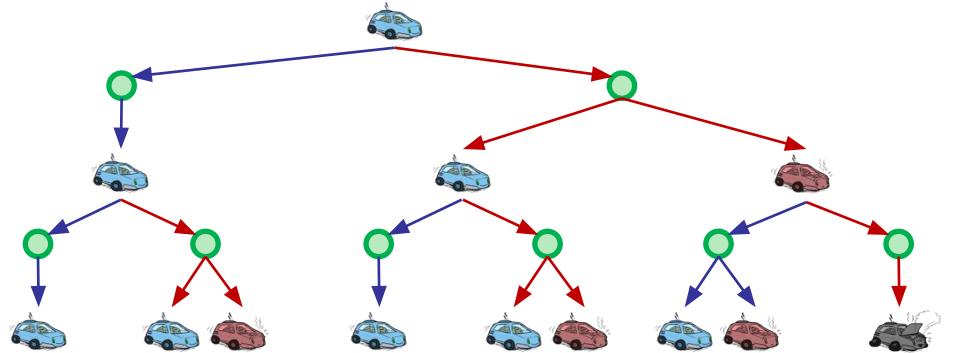
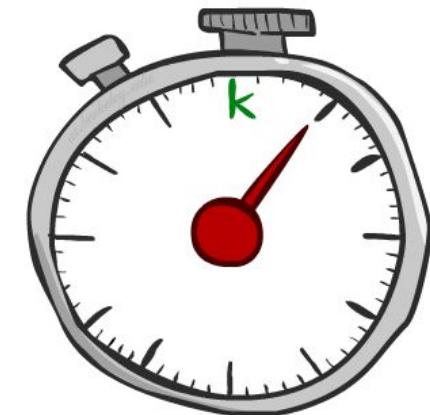
Racing Search Tree

- We're doing way too much work with expectimax!
- Problem: States are repeated
 - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
 - Idea: Do a depth-limited computation, but with increasing depths until change is small
 - Note: deep parts of the tree eventually don't matter if $\gamma < 1$
- So perhaps we should cache and also ignore very deep parts of the tree?



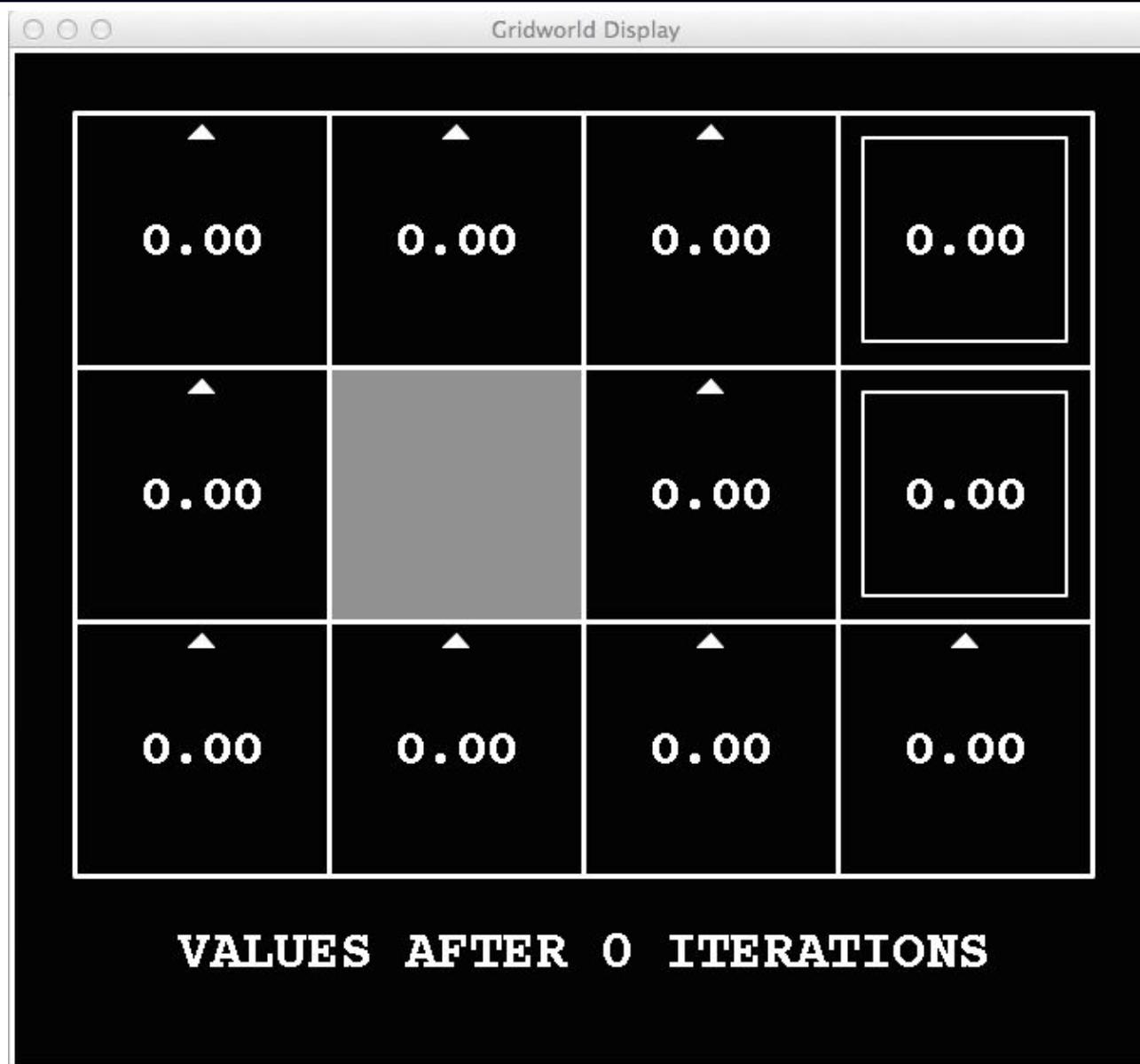
Time-Limited Values

- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
 - Equivalently, it's what a depth- k expectimax would give from s



[Demo – time-limited values (L8D6)]

$k=0$



$k=1$



VALUES AFTER 1 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

$k=2$



k=3



k=4



Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



k=6



k=7



k=8



k=9



VALUES AFTER 9 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=10



Noise = 0.2
Discount = 0.9
Living reward = 0

k=11



k=12



VALUES AFTER 12 ITERATIONS

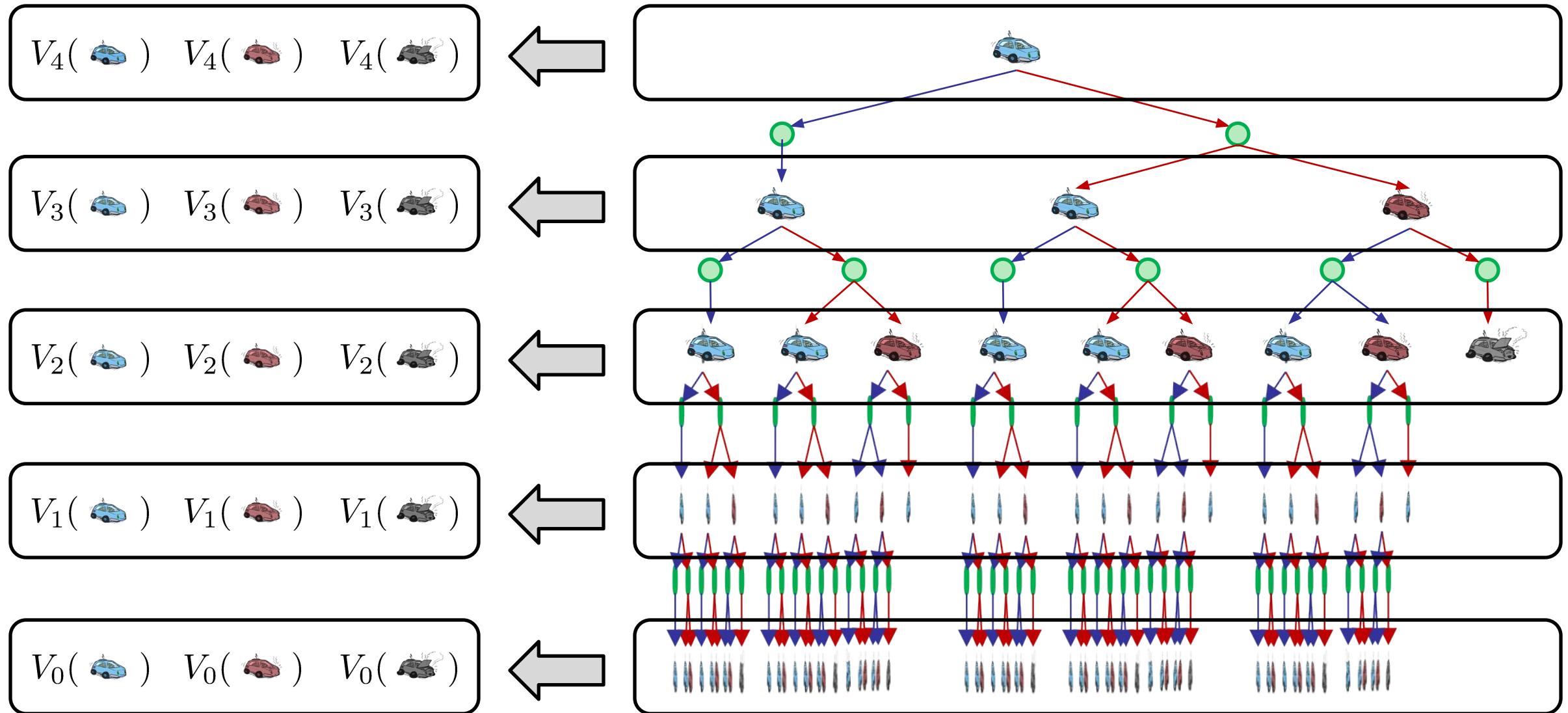
Noise = 0.2
Discount = 0.9
Living reward = 0

k=100

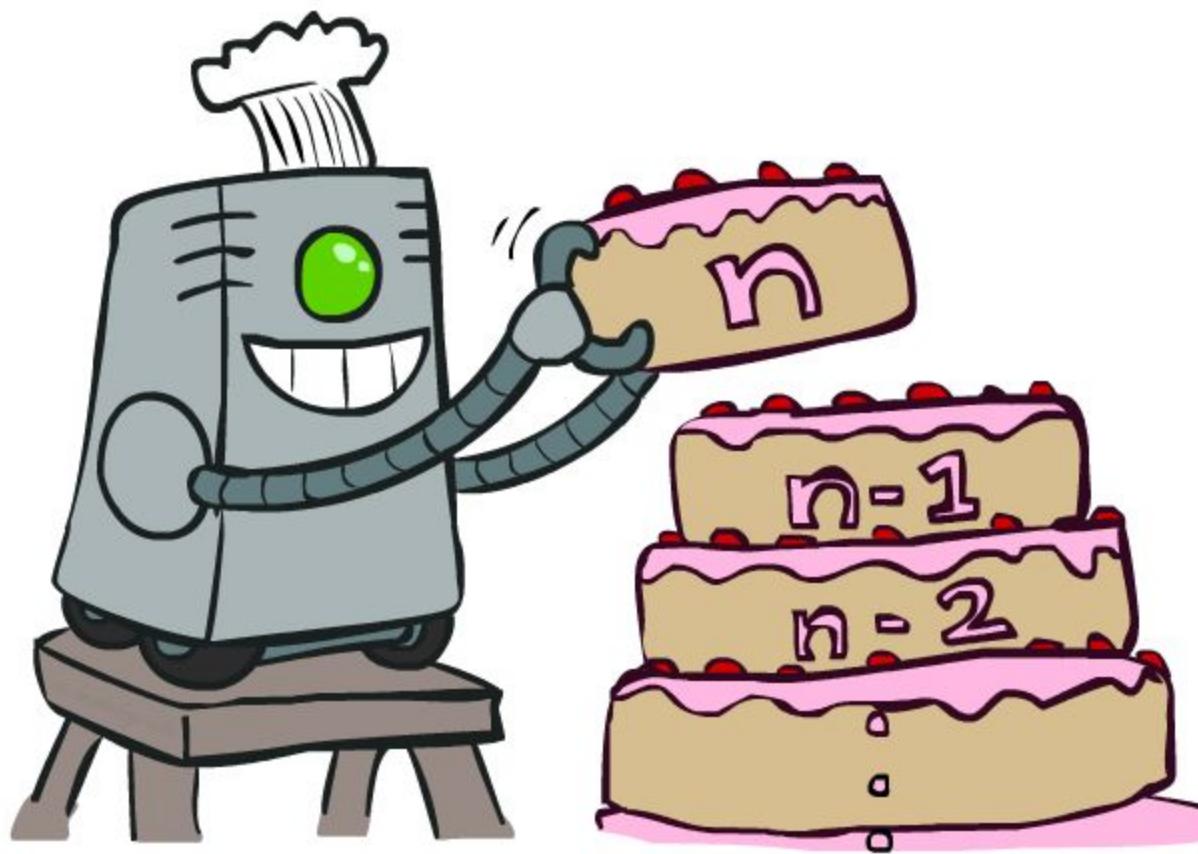


Noise = 0.2
Discount = 0.9
Living reward = 0

Computing Time-Limited Values



Value Iteration

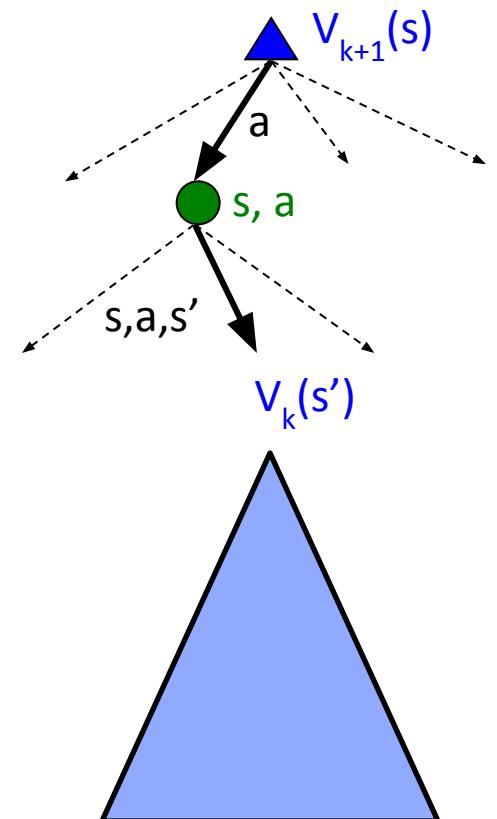


Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

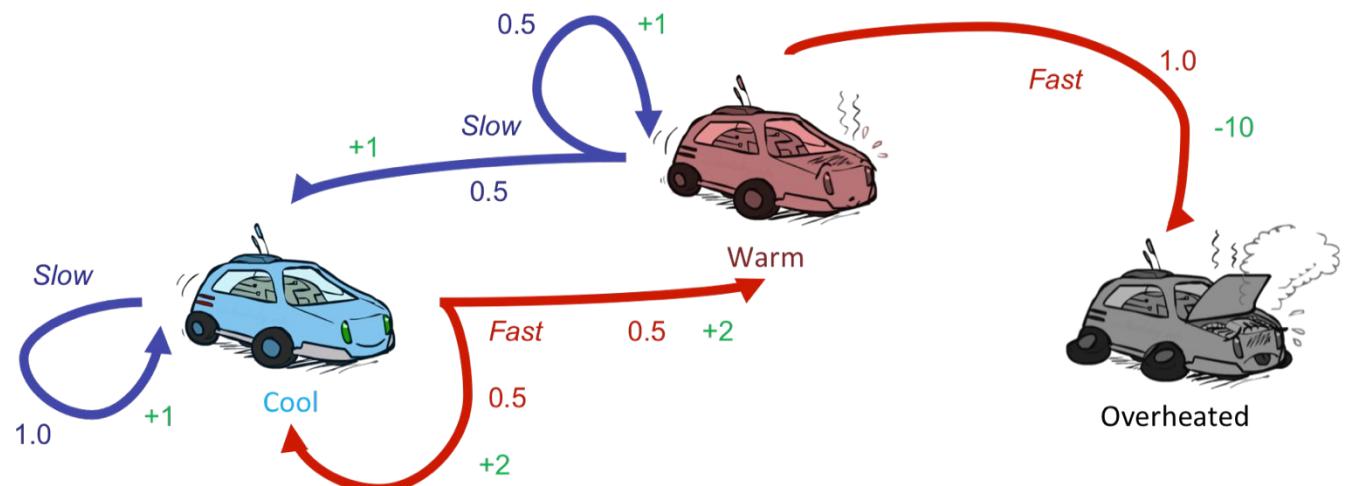
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Repeat until convergence
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
 - Basic idea: approximations get refined towards optimal values
 - Policy may converge long before values do



Example: Value Iteration

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

Next Time: Policy-Based Methods
