第十二次课程作业

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题目 1. 32. (1) 设 $A=\begin{pmatrix}0 & 0\\ 1 & -2\end{pmatrix}$, 求 $\mathrm{e}^A, \sin A, \cos A$;

(2) 已知 $J=\begin{pmatrix}-2 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 2\end{pmatrix}$, 求 $\mathrm{e}^J, \sin J, \cos J$.

解答. (1) $|\lambda I - A| = \lambda(\lambda+2)$

$\therefore \lambda_1 = 0, \lambda_2 = -2$

$\therefore J = \begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}$

$AP = PJ$

$P = \begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$ $P_{-1} = \begin{pmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 1\end{pmatrix}$

$\therefore \mathrm{e}^A = P\mathrm{e}^J P^{-1} = \begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\mathrm{e}^0 & 0\\ 0 & \mathrm{e}^{-2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ \frac{1-\mathrm{e}^{-2}}{2} & \mathrm{e}^{-2}\end{pmatrix}$

$\sin A = P\sin JP^{-1} = \begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\sin 0 & 0\\ 0 & \sin(-2)\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ \frac{\sin 2}{2} & -\sin 2\end{pmatrix}$

$$\cos A = P\cos JP^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos 0 & 0 \\ 0 & \cos(-2) \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1-\cos 2}{2} & \cos 2 \end{pmatrix}$$

$$(2)\ \mathrm{e}^J = \begin{pmatrix} \mathrm{e}^{-2} & & & \\ & \mathrm{e} & \mathrm{e} & \\ & & \mathrm{e} & \\ & & & \mathrm{e}^2 \end{pmatrix} \sin J = \begin{pmatrix} \sin(-2) & & & \\ & \sin 1 & \cos 1 & \\ & & \sin 1 & \\ & & & \sin 2 \end{pmatrix} \cos J = \begin{pmatrix} \cos(-2) & & & \\ & \cos 1 & -\sin 1 & \\ & & \cos 1 & \\ & & & \cos 2 \end{pmatrix}$$

题目 2. 35. 对下列方阵 A，求矩阵函数 e^{At}：(1) $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

(2) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{pmatrix}$，(3) $A = \begin{pmatrix} -2 & 1 & 3 \\ 0 & -3 & 0 \\ 0 & 2 & -2 \end{pmatrix}$.

解答. $(1)|\lambda I - A| = (\lambda+2)(\lambda-1)(\lambda-3)$

$$\therefore J = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 3 \end{pmatrix}$$

$AP = PJ$

$$P = \begin{pmatrix} 11 & -1 & 1 \\ 1 & 1 & 1 \\ -14 & 1 & 1 \end{pmatrix}$$

$$P^{-1}=\frac{1}{30}\begin{pmatrix}0&2&-2\\-15&25&-10\\15&3&12\end{pmatrix}$$

$$e^{At}=Pe^{Jt}P^{-1}=\frac{1}{30}\begin{pmatrix}11&-1&1\\1&1&1\\-14&1&1\end{pmatrix}\begin{pmatrix}e^{-2t}&&\\&e^{t}&\\&&e^{3t}\end{pmatrix}\begin{pmatrix}0&2&-2\\-15&25&-10\\15&3&12\end{pmatrix}=$$

$$\frac{1}{30}\begin{pmatrix}15e^{t}+15e^{3}t&22e^{-2t}-25e^{t}+3e^{3t}&-22e^{-2t}+10e^{t}+12e^{3t}\\-15e^{t}+15e^{3t}&2e^{-2t}+25e^{t}+3e^{3t}&-2e^{-2t}-10e^{t}+12e^{3t}\\-15e^{t}+15e^{3t}&-28e^{-2t}+25e^{t}+3e^{3t}&28e^{-2t}-10e^{t}+12e^{3t}\end{pmatrix}$$

(2) 解答过程同上 $e^{At}=e^{-2t}\begin{pmatrix}1+2t+2t^2&t+2t^2&\frac{t^2}{2}\\-4t^2&1+2t-4t^2&t-t^2\\-8t+8t^2&-12t+8t^2&1-4t+2t^2\end{pmatrix}$

(3) 解答过程同上 $e^{At}=\begin{pmatrix}e^{-2t}&-5e^{-2t}+6te^{-2t}+5e^{-3t}&3te^{-2t}\\0&e^{-3t}&0\\0&2e^{-2t}-2e^{-3t}&e^{-2t}\end{pmatrix}$

题目 3. 36. 求下列两类矩阵的矩阵函数: $\cos A,\sin A,\mathrm{e}^{A}$:

(1) A 为幂等矩阵;

(2) A 为对合矩阵 (即 $A^2=I$).

解答. (1) $A^2=A$

$\cos A=I-\frac{A^2}{2!}+\frac{A^4}{4!}-\frac{A^6}{6!}\cdots=I+A(-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}\cdots)=I+A(\cos 1-1)$

$\sin A=A-\frac{A^3}{3!}+\frac{A^5}{5!}\cdots=A(1-\frac{1}{3!}+\frac{1}{5!}\cdots)=A\sin 1$

$e^A=I+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\cdots=I+A(1+\frac{1}{2!}+\frac{1}{3!}+\cdots)=I+A(e-1)$

(2) $A^2=I$

$\cos A=I\cos 1$

$\sin A=A\sin 1$

$e^A=A(1+\frac{1}{3!}+\frac{1}{5!}+\cdots)+I(1+\frac{1}{2!}+\frac{1}{4!}+\cdots)$

两边同时左乘 A, 然后二式相加,

$e^A = eI$

题目 4. 37. 设函数矩阵 $A(t) = \begin{pmatrix} \sin t & \cos t & t \\ \frac{\sin t}{t} & \mathrm{e}^t & t^2 \\ 1 & 0 & t^3 \end{pmatrix}$, 其中 $t \neq 0$. 计算 $\lim_{t\to 0} A(t)$, $\frac{\mathrm{d}}{\mathrm{d}t}A(t)$, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}A(t)$.

解答.

$$\lim_{t\to 0} A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\frac{dA(t)}{dt} = \begin{pmatrix} \cos t & -\sin t & 1 \\ \frac{t\cos t - \sin t}{t^2} & e^t & 2t \\ 0 & 0 & 3t^2 \end{pmatrix}$$

$$\frac{d^2}{dt^2}A(t) = \begin{pmatrix} -\sin t & -\cos t & 0 \\ \frac{-t^2\sin t - 2t\cos t + 2\sin t}{t^3} & e^t & 2 \\ 0 & 0 & 6t \end{pmatrix}$$