第十次课程作业

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题目 1. 16. 已知正交矩阵 $A = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix}$ 表示一个旋转, 求其旋转轴与旋转角.

解答. 记 A 的旋转轴为 $span(x)$,

则 $Ax = x$,

考虑 $(A - I)x = 0$ 的解空间, 有

$A - I = \frac{1}{3}\begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 2 & -2 & -2 \end{pmatrix}$

$x = (1, 1, 0)$, $span(1, 1, 0)^T$ 为 A 的旋转轴.

设 $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = Q^{-1}AQ$, 则

$tr(B) = tr(A)$

$tr(B) = 1 + 2\cos\theta$

$tr(A) = \frac{5}{3}$

$\cos\theta = \frac{1}{3}$

$\theta = \arccos\frac{1}{3}$

题目 2. 27. 设 $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, b = \begin{pmatrix} 12 \\ 6 \\ 8 \end{pmatrix}$.

(1). 求 $R(A)$ 的标准正交基;

(2). 写出 A 的 QR 分解;

(3). 求 $Ax = b$ 的最小二乘解;

(4). 证明 $u_1 = (0,1,0)^{\mathrm{T}}, u_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$ 也是 $R(A)$ 的标准正交基, 其中 $R(A)$ 为 A 的列空间.

解答. (1). 易得 $R(A) = span(2,1,2)^T, (1,1,1)^T = span\alpha_1, \alpha_2$

设 $R(A)$ 的一组标准正交基为 u_1, u_2, 则

$u_1 = \frac{1}{3}(2,1,2)^T$

做向量 α_2 在向量 u_1 方向上的投影，并做差标准化得:

$u_2 = \alpha_2 - \frac{(u_1,\alpha_2)}{(u_1,u_1)}u_1 = \frac{1}{3\sqrt{2}}(1,-4,1)^T$

(2).$Q = (u_1, u_2)$

$\alpha_1 = 3u_1$

$\alpha_2 = \frac{5}{3}u_1 - \frac{\sqrt{2}}{3}u_2$

$A = (u_1, u_2)\begin{pmatrix} 3 & \frac{5}{3} \\ 0 & -\frac{\sqrt{2}}{3} \end{pmatrix}$

(3). 利用 QR 分解,

$$x = R^{-1}Q^Tb = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\sqrt{2}}{3} & -\frac{5}{3} \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{6} \end{pmatrix}\begin{pmatrix} 12 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

(4).$R(A)^{\perp} = span(1,0,-1)^T$

易知 $u_1, u_2 \perp R(A)^{\perp}$

$u_1^Tu_2 = 0||u_1|| = ||u_2|| = 1$

$u_1=(0,1,0)^{\mathrm{T}}, u_2=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$ 是 $R(A)$ 的标准正交基

题目 3. 28. 求下列矩阵的 QR 分解

(1). $\begin{pmatrix}0&1&1\\1&1&0\\1&0&0\end{pmatrix}$

解答. 解答过程同上题 (2).

$$Q=\begin{pmatrix}0&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&\frac{2}{\sqrt{6}}&-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\end{pmatrix}$$

$$R=\begin{pmatrix}\sqrt{2}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&\frac{3}{\sqrt{6}}&\frac{1}{\sqrt{6}}\\0&0&\frac{1}{\sqrt{3}}\end{pmatrix}$$

题目 4. 30. 计算矩阵 $A=\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}$ 的奇异值分解和相应的四个子空间.

解答. 奇异值分解: 记 A 的奇异值分解: $A=v\cdot D\cdot u^*$

$$A^*=\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}$$

$$A^*A=\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}=\begin{pmatrix}2&1\\1&2\end{pmatrix}=uD^*Du^*$$

$$|\lambda I-A^*A|=\begin{vmatrix}\lambda-2&1\\1&\lambda-2\end{vmatrix}=(\lambda-1)(\lambda-3)$$

可知 $D^*D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma_1 = \sqrt{3}$

$\sigma_2 = 1$

求 $u : A^*A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \alpha_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A^*A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \alpha_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$D = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

记 $\widetilde{V}$ 为 V 前两列构成的矩阵

由 $A = V \cdot \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot u^*$ 得

$A = \widetilde{V} \cdot \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \cdot u^*$

$\therefore \widetilde{V} = A \cdot u \cdot \begin{pmatrix} \sqrt{3} & \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{3}} & \\ & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} & \\ & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & 0 \end{pmatrix}$

将 $\widetilde{V}$ 扩充成 V .

可得 $V = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$

$\therefore A$ 的奇异值分解为: $A = V{\cdot}D{\cdot}u^* = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

A 的四个子空间:

$N(A) = \phi$

$N(A^T) = span\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$

$R(A) = span\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

$R(A^T) = \mathbb{R}^2$

题目 5. 33. 设 $A \in \mathbb{C}^{m\times n}$ 的秩为 $r > 0$,A 的奇异值分解为 $A = U\operatorname{diag}(s_1, \cdots, s_r, 0, \cdots, 0)\, V^*$, 求矩阵 $B = \begin{pmatrix} A \\ A \end{pmatrix}$ 的奇异值分解.

解答. $A = \mathbb{C}^{m\times n}$.

A 的奇异值分解为 $A = U \cdot \operatorname{diag}(s_1, s_2 \cdots s_r, 0, 0 \cdots 0)\, V^*$

求 $B = \begin{pmatrix} A \\ A \end{pmatrix}_{2m\times n}$ 的奇异值分解,

记 $B = U \cdot D \cdot v^*$

$B^* = v \cdot D^* \cdot u^* = (A^*A^*)$

$$B^*B = 2A^*A = V \cdot \operatorname{diag}(2s_1^2, 2s_2^2 \cdots 2sr^2, 0, \cdots 0) \cdot V^* = v \cdot D^*D \cdot v$$

$$\therefore v = V, D = \begin{pmatrix} \sqrt{2}\,\mathrm{diag} \\ 0 \end{pmatrix}_{2m\times n}$$

$$BB^* = u \cdot D \cdot D^* \cdot u^* = u \cdot \begin{pmatrix} 2dd^* & 0 \\ 0 & 0 \end{pmatrix} \cdot u^* = \begin{pmatrix} Udd^*U^* & Udd^*U^* \\ Udd^*U^* & Udd^*U^* \end{pmatrix} = \begin{pmatrix} AA^* & AA^* \\ AA^* & AA^* \end{pmatrix}$$

得 $u = \begin{pmatrix} \frac{U}{\sqrt{2}} & \frac{U}{\sqrt{2}} \\ \frac{U}{\sqrt{2}} & -\frac{U}{\sqrt{2}} \end{pmatrix}_{2n\times 2m}$

$\therefore B = \begin{pmatrix} A \\ A \end{pmatrix}$ 的奇异值分解为: $B = u \cdot D \cdot v^* = \begin{pmatrix} \frac{v}{\sqrt{2}} & \frac{v}{\sqrt{2}} \\ \frac{v}{\sqrt{2}} & -\frac{v}{\sqrt{2}} \end{pmatrix}_{2m\times 2m} \cdot \begin{pmatrix} \sqrt{2}\ \mathrm{diag} \\ 0 \end{pmatrix}_{2m\times n} \cdot V_{n\times n}$