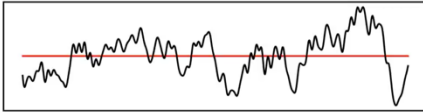
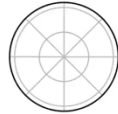


COMS20017 – Algorithms & Data



March 2025 Signals & Frequencies

Majid Mirmehdi

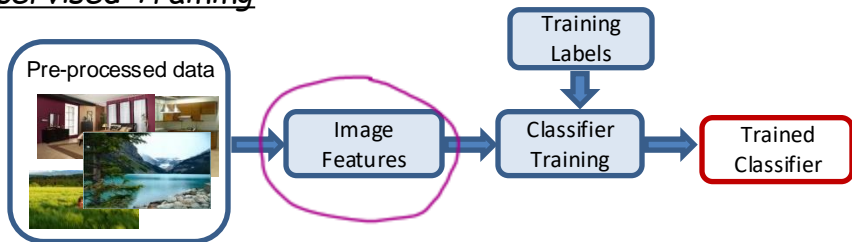
Lecture MM-04

Summary: Typical Data Analysis Problem (Reminder)

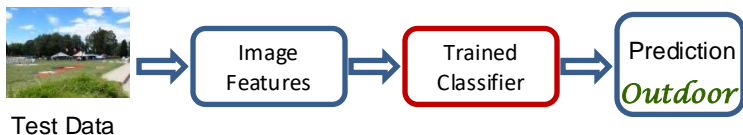
Steps:

1. Pre-processing [Unit - Part 1] → Majid Mirmehdi (~10%)
2. Feature Selection [Unit - Part 3] → Majid Mirmehdi (~40%)
3. Modelling & Classification [Unit - Part 2] → Alin Achim (~50%)

Supervised Training

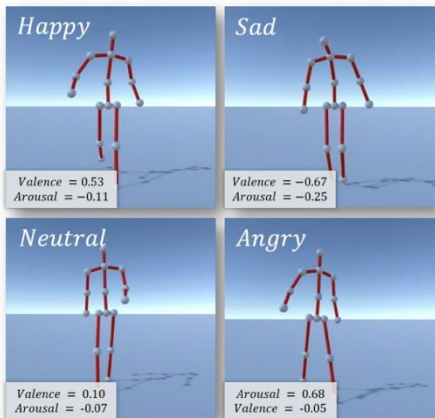


Testing



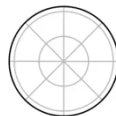
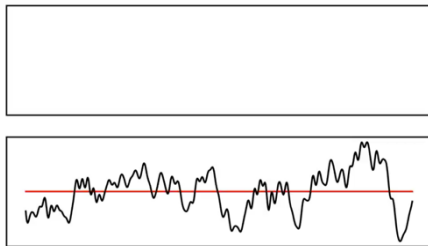
Features help simplify the problem

Patient with mild Parkinson's Disease



- Even “impoverished” motion data can evoke a strong perception

Next in COMS20017



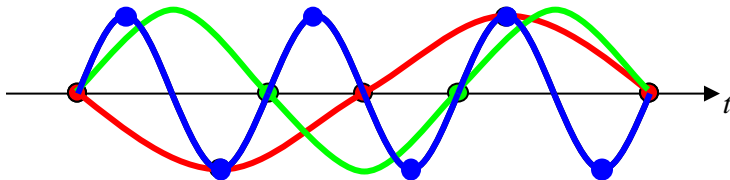
Feature Selection and Extraction

- **Signal basics**
- 1D Fourier Transform
- Characteristics of features
- PCA for dimensionality reduction
- Convolutions

Nyquist-Shannon Sampling Theory - Reminder

"An analogue signal containing components up to some maximum frequency u (Hz) may be completely reconstructed by regularly spread samples, provided the sampling rate is at least $2u$ samples per second"

Also referred to as the Nyquist-Shannon criterion: sampling rate s should be at least twice the highest spatial frequency u .



$$\text{sampling period } T \leq \frac{1}{2u}$$

$$\text{equivalent to sampling rate } s \geq 2u$$

Basic Signals

AAAAATAAAAA
0000001000000

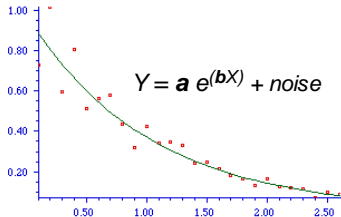
$$\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$

Some basic signals:

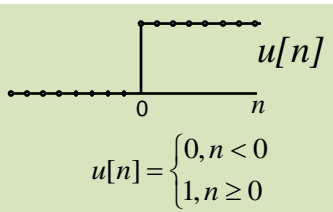
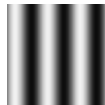
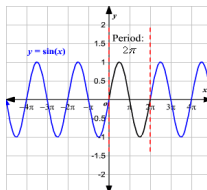
- Unit impulse signal
- Unit step signal
- Exponential signal
- Periodic signal



All signals can be represented by these basic signals!



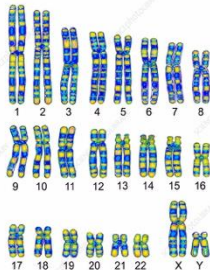
$$y = \sin(x) = \sin(x+2\pi)$$



Signals as Functions

A signal is a physical quantity that is a function of one or more independent variable(s), such as space and/or time.

Data from a *Gene* pool



Position of a car in a video sequence



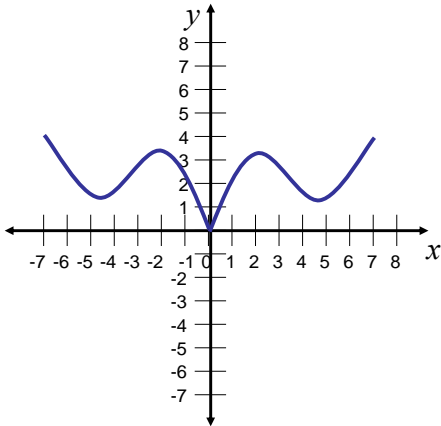
Example signals:

1D signal: $f(x)$

2D signal: $f(x,y)$

3D signal: $f(x,y,t)$ etc.

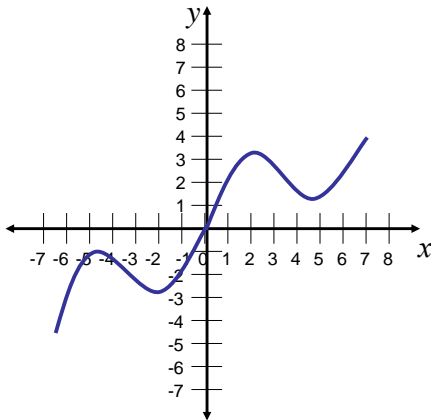
Even Functions have y-axis Symmetry



So a function is **even** when

$$y = f(x) = f(-x)$$

Odd Functions have origin Symmetry



So a function is **odd** when

$$y = f(x) = -f(-x)$$

Examples: odd or even?

$$f(x) = 3x^4 - 7x^2 + 1$$

$$f(-x) = 3(-x)^4 - 7(-x)^2 + 1 = 3x^4 - 7x^2 + 1$$

Even

$$f(x) = 4x^3 - x$$

$$f(-x) = 4(-x)^3 - (-x) = -4x^3 + x$$

Odd

$$f(x) = 3x^5 - x + 1$$

$$f(-x) = 3(-x)^5 - (-x) + 1 = -3x^5 + x + 1$$

Neither Odd,
nor Even!

Function Decomposition

Any function $f(x) \neq 0$ can be decomposed as the sum of an even function $f_e(x)$ and an odd function $f_o(x)$.

$$f(x) = g(x) + h(x)$$

$$g(x) = \frac{1}{2} [f(x) + f(-x)]$$

Even Part

$$h(x) = \frac{1}{2} [f(x) - f(-x)]$$

Odd Part

Example

$$f(x) = e^{-x}$$

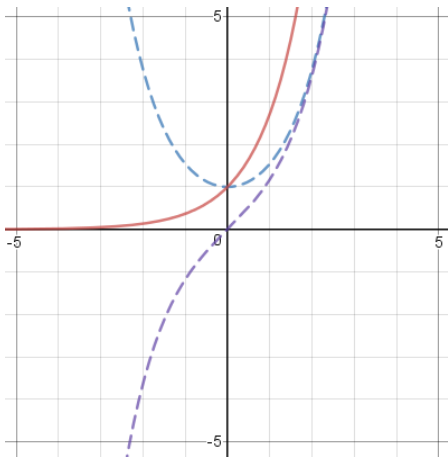
$$f(x) = g(x) + h(x)$$

Even Part

$$g(x) = \frac{1}{2}(e^{-x} + e^x)$$

Odd Part

$$h(x) = \frac{1}{2}(e^{-x} - e^x)$$



Signals as Functions

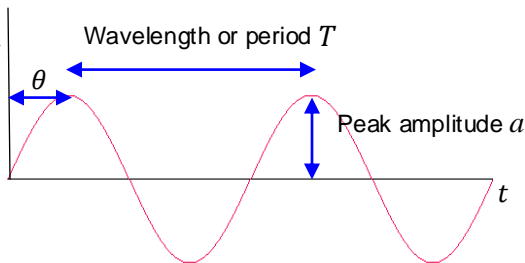
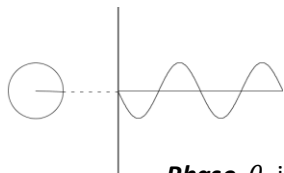
period is the time T it takes to finish one oscillation.

frequency $u = \frac{1}{T}$ is the number of periods per second, measured in Hz.

amplitude a is a measure of how much it changes over a single period.

Example: sine function

$$f(t) = a \sin 2\pi ut$$



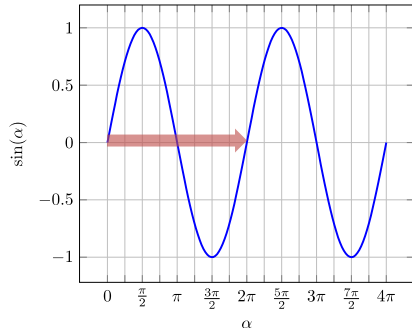
Phase θ is where the signal is in the oscillation (or the angular position).

Periodic Signals

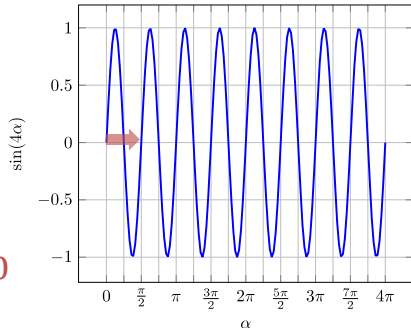
A continuous-time signal $f(t)$ is **periodic** if $f(t + T) = f(t)$

Fundamental period T_0 of $f(t)$ is the smallest T that satisfies the above.

Fundamental frequency is then $\omega_0 = \frac{1}{T_0}$



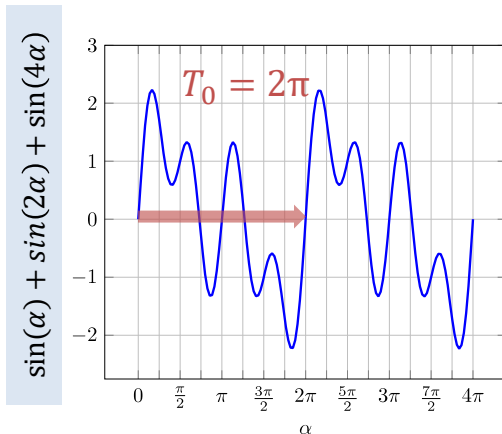
T_0



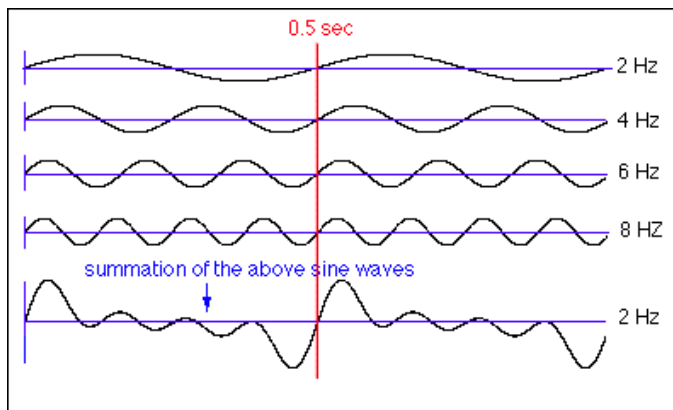
Periodic Signals

$$f(t + T) = f(t)$$

What is T_0 for this example?

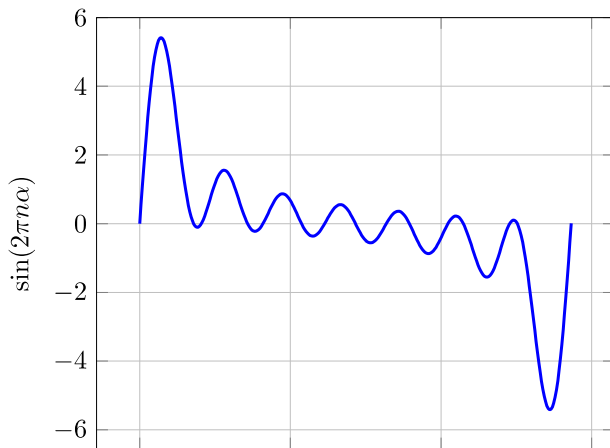


Example I: a simple signal



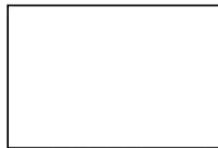
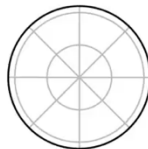
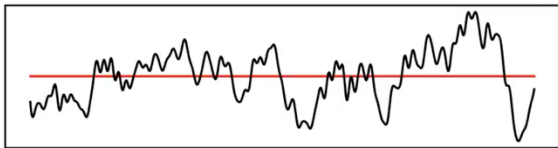
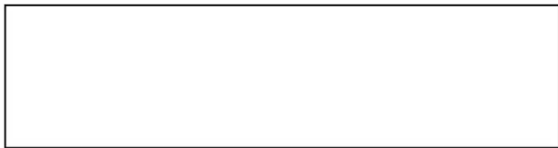
The fundamental period or combined frequency is the highest common factor.

Example I: a simple signal



$$f(\alpha) = \sum_{n=1}^8 \sin(2\pi n\alpha)$$

Example II: another signal

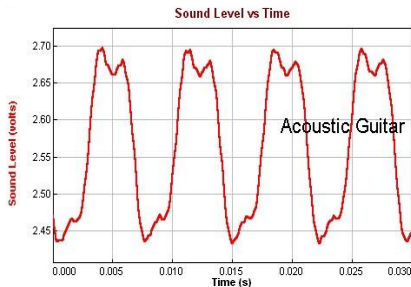
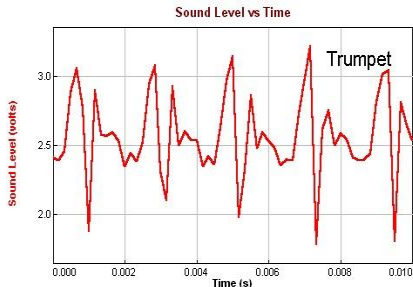


Example III: Visible light



How should we interpret these musical instrument signals?

Characteristics of sound in audio signals:
High pitch - rapidly varying signal
Low pitch - slowly varying signal

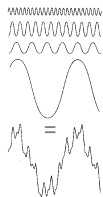


Fourier Series



Trigonometric Fourier Series: Any *periodic* function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient. → *Jean Baptiste Joseph Fourier (1822).*

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)$$



A function with period T is represented by two infinite sequences of coefficients. n is the no. of cycles/period.

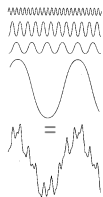
- The sines and cosines are the **Basis Functions** of this representation. a_n and b_n are the **Fourier Coefficients**.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.

Fourier Series



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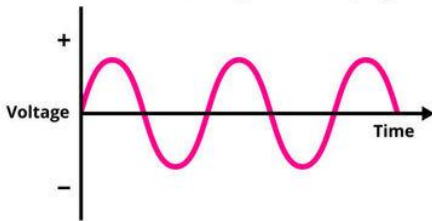
- The sines and cosines are the **Basis Functions** of this representation. a_n and b_n are the **Fourier Coefficients**.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.
- **Note the function has an even part and an odd part!**
- a_0 is often referred to as the **DC term** or the **average of the signal**.

AC ⚡ DC

Direct Current (DC)



Alternating Current (AC)



Fourier Series Solution

A *Fourier series* provides an equivalent representation of the function:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)$$

The coefficients are:

$$a_0 = \frac{1}{T} \int_{-T/2}^{+T/2} f(x) dx$$

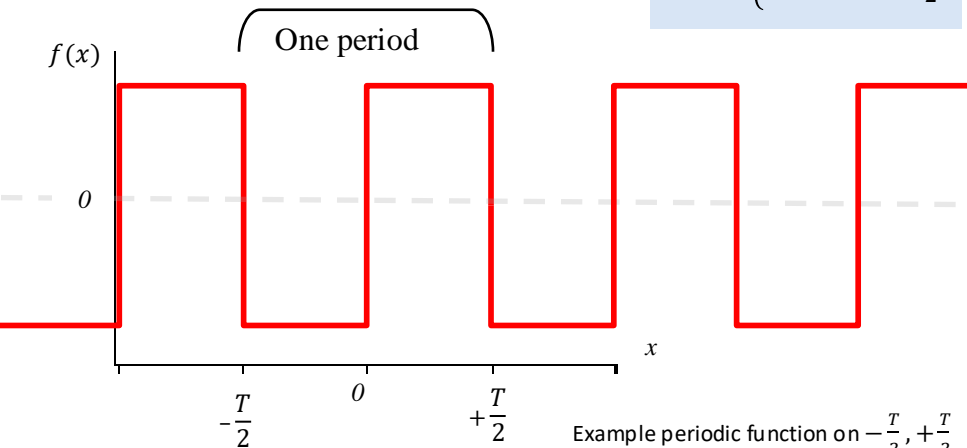
$$a_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \cos\left(\frac{2\pi nx}{T}\right) dx$$

$$b_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \sin\left(\frac{2\pi nx}{T}\right) dx$$

Fourier Series Example: Square Wave

$f(x) \rightarrow$ a square wave

$$f(x) = \begin{cases} -1 & -\frac{T}{2} \leq x < 0 \\ +1 & 0 \leq x < \frac{T}{2} \end{cases}$$



Fourier Series Example: Square Wave

$f(x) \rightarrow$ a square wave

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)$$

$$f(x) = \begin{cases} -1 & -\frac{T}{2} \leq x < 0 \\ +1 & 0 \leq x < \frac{T}{2} \end{cases}$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left(\frac{2\pi nx}{T}\right) dx = \mathbf{0}$$

The integral of an odd function over a symmetric interval is zero!



Fourier Series Example: Square Wave

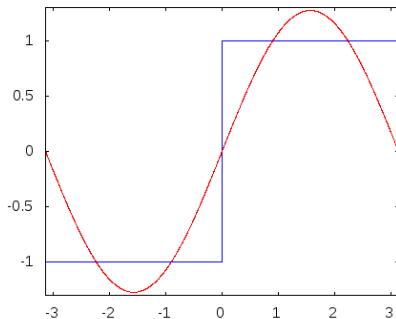
$f(x) \rightarrow$ a square wave

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin\left(\frac{2\pi nx}{T}\right) dx$$

$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

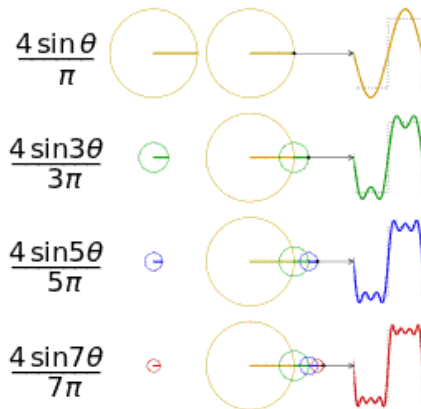
$$f(x) = \begin{cases} -1 & -\frac{T}{2} \leq x < 0 \\ +1 & 0 \leq x < \frac{T}{2} \end{cases}$$

$n = 1, 3, 5, 7, \dots$



$$f(x) = \frac{4}{\pi} \cdot \sin \frac{2\pi x}{T} + \frac{4}{3\pi} \cdot \sin 3 \cdot \frac{2\pi x}{T} + \frac{4}{5\pi} \cdot \sin 5 \cdot \frac{2\pi x}{T} + \frac{4}{7\pi} \cdot \sin 7 \cdot \frac{2\pi x}{T} + \dots$$

Approximating the Square Wave

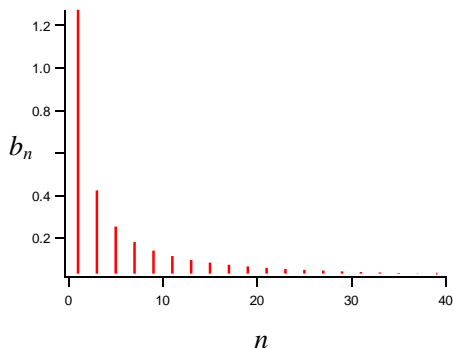


$$\theta = \frac{2\pi x}{T}$$

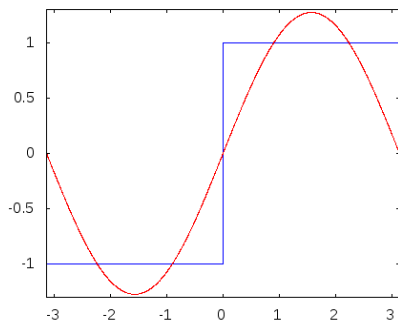
$$f(x) = \frac{4}{\pi} \cdot \sin \frac{2\pi x}{T} + \frac{4}{3\pi} \cdot \sin 3 \cdot \frac{2\pi x}{T} + \frac{4}{5\pi} \cdot \sin 5 \cdot \frac{2\pi x}{T} + \frac{4}{7\pi} \cdot \sin 7 \cdot \frac{2\pi x}{T} + \dots$$

Fourier Space/Domain for the Square Wave

- The set of *Fourier Space* coefficients b_n contain complete information about the function
- Although $f(x)$ is periodic to infinity, b_n is negligible beyond a finite range
- Sometimes the Fourier representation is more convenient to use, or just view



Next...



Feature Selection and Extraction

- Signal basics
- **1D Fourier Transform**
- Another look at features
- PCA for dimensionality reduction
- Convolutions