Algorithms & DATA - COMS20017

Tutorial Questions

(Q1) In the context of Minimum Variance Unbiased Estimation (MVUE), define the likelihood function for a random signal x. State the likelihood function for a single random sample $x[0] = A + w[0]$, where A is the DC level and $w \sim \mathcal{N}(0, \sigma^2)$

(Q2) Define the curvature of the log-likelihood function. What information does the curvature provide? Explain why it is convenient to use the log-likelihood function.

(Q3) Define and discuss the Cramer-Rao Lower Bound (CRLB) for scalar parameters.

(Q4) In the context of estimating a DC level in white Gaussian noise, consider N observations

$$x[n] = A + w[n], \quad n = 0, 1, 2, \ldots, N-1,$$

where $w[n] \sim \mathcal{N}(0, \sigma^2)$. Given that

$$p(x; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - A)^2 \right\}$$

determine the CRLB for A.

(Q5) Let X denote a Poisson random variable with probability density function

$$p(x|\lambda) = \frac{\lambda^x}{x!} e^{-\lambda} \quad \text{for} \quad x = 0, 1, \ldots.$$

Assuming that the rate parameter λ is exponentially distributed with

$$f(\lambda) = \frac{1}{\lambda_0} e^{-\frac{\lambda}{\lambda_0}}$$

and the joint density of x and λ is

$$f(x, \lambda) = f(x|\lambda) f(\lambda)$$

determine the maximum a posteriori estimate of λ and comment on the values of λ when λ_0 is much smaller than 1.

(Q6) Let x denote a Rayleigh distributed random variable with probability density function given by

$$f(x|\theta) = \frac{x}{\theta^2} \exp\left\{ -\frac{x^2}{2\theta^2} \right\}$$

Determine the maximum likelihood estimate of θ.

(Q7) Consider the problem of estimating a DC level A in white Gaussian noise, w, where the noisy data are given by

$$x[n] = A + w[n], \quad n = 0, \dots, N-1, \quad w[n] \sim \mathcal{N}(0, \sigma_w^2)$$

(a) Estimate the value of A using the maximum likelihood estimation (MLE) procedure. Discuss briefly the optimality properties of the MLE.

(b) Estimate the value of A using the method of least squares (LS). Discuss briefly the properties of LS estimation. State at least one problem associated with this approach.

(c) Determine the Cramer Rao lower bound (CRLB) of the unknown parameter A. Compare this solution with those from a) and b)

(Q8) (a) Derive the Cramer-Rao lower bound (CRLB) for the estimation of a DC level in white Gaussian noise ($w[n] \sim \mathcal{N}(0, \sigma^2)$), given by

$$x[n] = A + w[n], \quad n = 0, 1, \dots, N-1$$

For which the probability density function is given by

$$p(x; A) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x[n] - A)^2\right]$$

(b) Derive the maximum likelihood estimator (MLE) for the problem in Q8 (a).

(c) Is the MLE (b) unbiased? Is it efficient? Does such an MLE attain the CRLB?

(Q9) Derive the general expression for the maximum a posteriori (MAP) estimator of θ, given the observed variable x.

(Q10) (a) Let $x[n]$ represent N measurements of a constant amplitude signal. The measurement is corrupted by white Gaussian noise with zero mean and variance σ^2. If

$$x[n] = A + w[n] \quad \text{for} \quad n = 0, 1, \dots, N-1,$$

show that the Maximum Likelihood estimate for A is given by

$$\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n].$$

(b) In the same model as above Q10 (a), consider now that A is a zero-mean Gaussian random variable i.e.

$$p(A) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp\left(-\frac{A^2}{2\sigma_n^2}\right)$$

Assuming that the likelihood function is Gaussian as well,

$$p(x|A) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$$

Derive the MAP estimate of A.

(Q11) (a) Assume that the likelihood function is Gaussian, i.e.

$$p(x|\theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)$$

and that the prior pdf is Cauchy:

$$p(\theta) = \frac{\gamma}{\pi(\gamma^2 + \theta^2)}$$

Show that the MAP estimate of x can be obtained in closed-form.

(b) Now assume that the likelihood function is still Gaussian, as in Q11 (a), but this time the prior pdf is Laplace:

$$p(\theta) = \frac{1}{\sqrt{2}\sigma} \exp\left\{ -\frac{\sqrt{2}|\theta|}{\gamma} \right\}$$

Show that the MAP estimate of x can be obtained in closed-form.