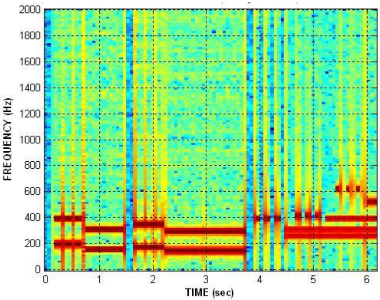
COMS20017 – Algorithms & Data



March 2025
1D Fourier Transform

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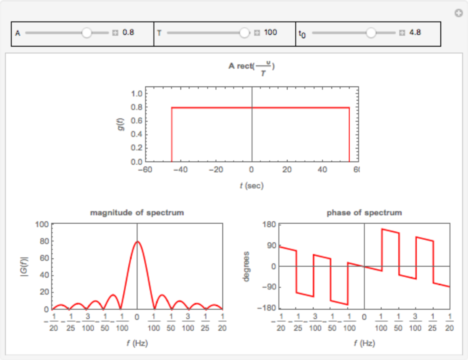
Lecture MM-05

Next in DDCS



https://demonstrations.wolfram.com/RectangularPulseAndItsFourierTransform/

Feature Selection and Extraction

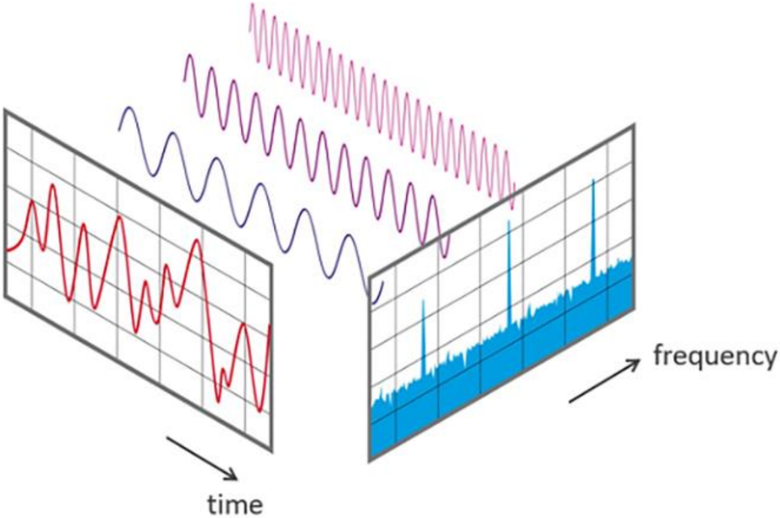
- Signal basics
- **1D Fourier Transform**
- Another look at features
- PCA for dimensionality reduction
- Convolutions

Frequency Analysis for Feature Extraction & more...

- The aim of processing a signal using Fourier analysis is to *manipulate the spectrum of a signal* rather than manipulating the signal itself. Example: *simple compression*

- Functions that are ***not periodic*** can also be expressed as the integral of sines and/or cosines weighted by a coefficient, using the Fourier transform.

Visualising the outcome of the FT



1D Fourier Transform

The Fourier Transform of a single variable continuous function $f(x)$ is:

$$F(u) = \int_{-\infty}^{\infty} f(x) \quad e^{-j2\pi ux} dx$$

Conversely, given $F(u)$, we can obtain $f(x)$ by means of the *inverse* Fourier Transform:

$$f(x) = \int_{-\infty}^{\infty} F(u) \quad e^{j2\pi ux} du$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

1D Fourier Transform: discrete form

The Fourier Transform of a discrete function of one variable, $f(x),\ x=0,1,2\ldots,N\text{-}1$ is:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{\frac{-j2\pi ux}{N}} \qquad \text{for } u = 0,1,2,\ldots,N-1.$$

Conversely, given $F(u)$, we can obtain $f(x)$ by means of the *inverse* Fourier Transform:

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1D Fourier Transform

The concept of the frequency domain follows from *Euler's Formula*:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Thus, each term of the Fourier Transform is composed of the sum of *all* values of the function *f(x)* multiplied by sines and cosines of various frequencies:

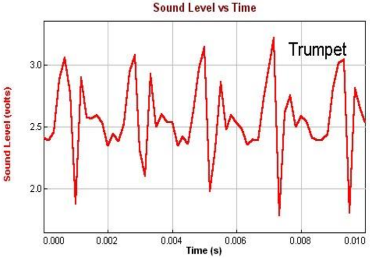
$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\left[\cos\left(\frac{2\pi ux}{N}\right) - j\sin\left(\frac{2\pi ux}{N}\right)\right]$$

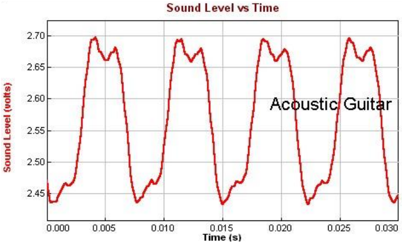
for $u = 0,1,2,\ldots,N-1.$

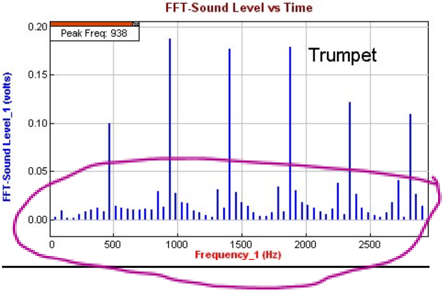
We have transformed from a time domain to a frequency domain representation.

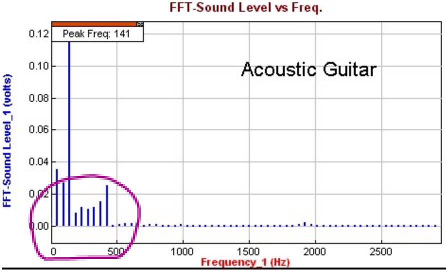
Example: low and high frequencies

Characteristics of sound in audio signals:



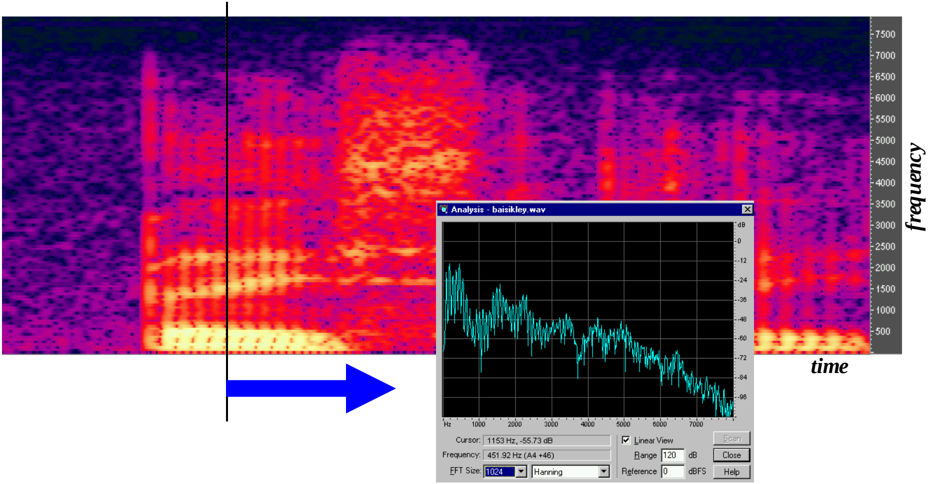




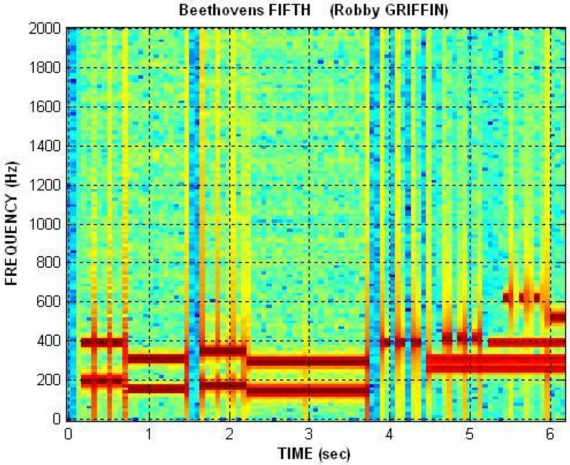


Example: Acoustic Data Analysis

Spectrogram



Can you match the sound to the frequencies?



http://dspfirst.gatech.edu/chapters/03spect/demos/spectrog/real/

1D Fourier Transform

F(u) is a complex number & has real and imaginary parts:

$$F(u) = R(u) + jI(u)$$

Magnitude or *spectrum* of the FT:

$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

Phase angle or phase spectrum:

$$\varphi(u) = \tan^{-1}\frac{I(u)}{R(u)}$$

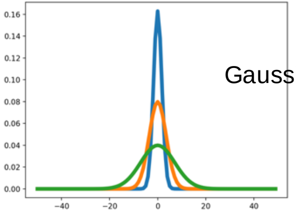
Expressing *F(u)* in polar coordinates:

$$F(u) = |F(u)|e^{j\varphi(u)}$$

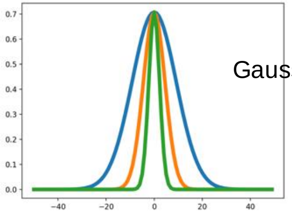
1D Fourier Transform examples

Signal

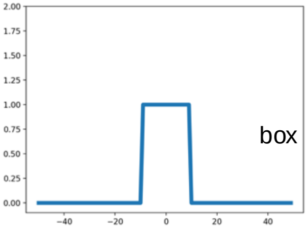
Fourier transform



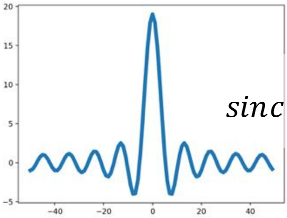
Gaussian



Gaussian



box



$$sinc = \frac{\sin(x)}{x}$$

Some Properties of Fourier Transform

Linearity

$$a\, f(x) + b\, h(x) \quad \Leftrightarrow \quad a\, F(u) + b\, H(u)$$

Zero-frequency
(DC component)

$$f(0) \quad \Leftrightarrow \int_{-\infty}^{\infty} f(x)\; dx$$

Time-shifting

$$f(x - x_0) \quad \Leftrightarrow \quad e^{-j2\pi x_0 u} F(u)$$

Same frequency content, but introduces a phase shift in the frequency domain

Time-scaling

$$f(ax) \quad \Leftrightarrow \quad \frac{1}{|a|}\, F\left(\frac{u}{a}\right) \qquad a \neq 0$$

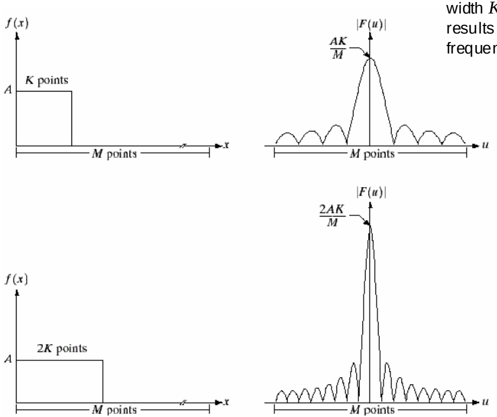
1D Fourier Transform of box signal (example of time-scaling)

$$f(x) = \begin{cases} 1 & 0 \le x \le K \\ 0 & otherwise \end{cases}$$

$A = 1$

$K = 8$

$M = 1024$



The width of the main lobe is inversely proportional to the pulse width K. A wider box signal (larger K) results in a narrower main lobe in the frequency domain, and vice versa.

$$F(u) = \int_{-\infty}^{\infty} f(x) \quad e^{-j2\pi ux} dx$$

$$F(u) = \int_{0}^{K} e^{-j2\pi ux} \, dx$$

$$F(u) = K \left| \frac{\sin(\pi ux)}{\pi ux} \right|$$

The Fourier spectrum of the box signal is a sinc function

The box signal in the time domain has sharp transitions (0 to A and back), which require high-frequency components (infinite sum of sinusoidal components) to represent them accurately in the frequency domain.

Very Simple Application Example

Naïve approach:

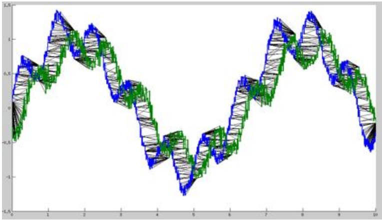
Automatic speech recognition between two speech utterances $x(n)$ and $y(n)$:

$$E = \sum_{\forall n} (x(n) - y(n))^2$$

Problems with this approach?

$$\begin{cases} x(n) \; = \; K\, y(n), \text{yet } E \neq 0 \\ \quad K \textit{ being a scaling parameter} \\ \\ x(n) \; = \; y(n-m), \text{yet } E \neq 0 \\ \quad m \textit{ causing a delay shift} \end{cases}$$

One solution could be Dynamic Time Warping? (recall from earlier lecture)



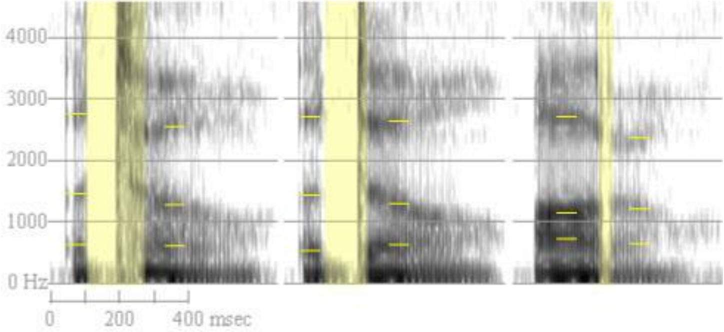
Use Frequency Domain Features!

- Take the Fourier transform of both utterances to get $X(u)$ and $Y(u)$.
- Then consider the Euclidean distance between their magnitude spectrums: $|X(u)|$ and $|Y(u)|$:

$$d_E = \sum_{\forall u} (|X(u)| - |Y(u)|)^2$$

Use Frequency Domain Features!

Still a difficult task even in the frequency domain.

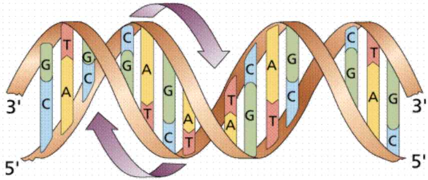


a tot *a dot* *otto*

DNA Sequence Example

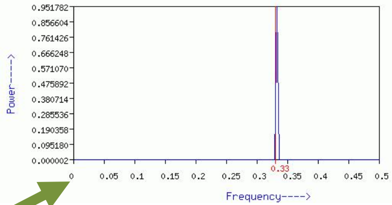
- The analysis of correlations in DNA sequences is used to identify protein coding genes in genomic DNA.
- Locating and characterizing repeats and periodic clusters provides certain information about the structural and functional characteristics of the molecule.
- DNA sequences are represented by letters, A, C, G or T, and –

- e.g. ACAATG-GCCATAAT-ATGTGAAC--GCTCA…



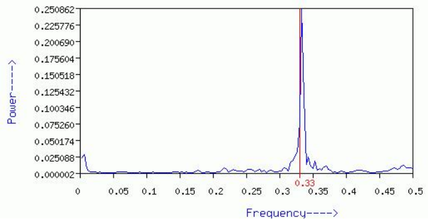
DNA Sequence Example

Consider the periodic sequence A--A--A--A--.... where blanks can be filled randomly by A, C, G or T. This shows a periodicity of 3.



The spectral density of such a sequence is significantly non-zero only at one frequency (0.33) which corresponds to the perfect periodicity of base A ($1/0.333 = 3.0$).

Destroy the perfect repetition by randomly replacing the A's with any of the letters…



DNA Sequence Analysis

The computation of Fourier & other linear transforms of *symbolic data* is a big problem.

The simplest solution is to map each symbol to a number.

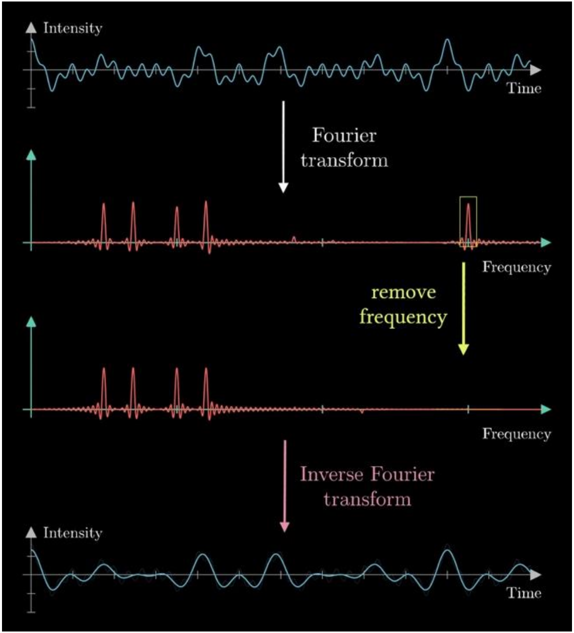
Consider, for example, the following symbolic periodic sequence:
s = (ATAGACATAGACATAGAC . . .).

The mapping:
A → 1,
T → 0,
G → 0,
C → 0,
→ Period = 2

The mapping:
A → 1,
T → 2,
G → 3,
C → 4,
→ Period = 6

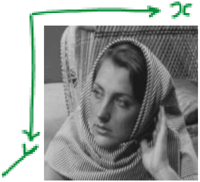
This clearly shows that some of the relevant harmonic structure can be exposed by the symbolic-to-numeric labelling.

Another Example: Noise Removal



https://www.3blue1brown.com/lessons/fourier-transforms

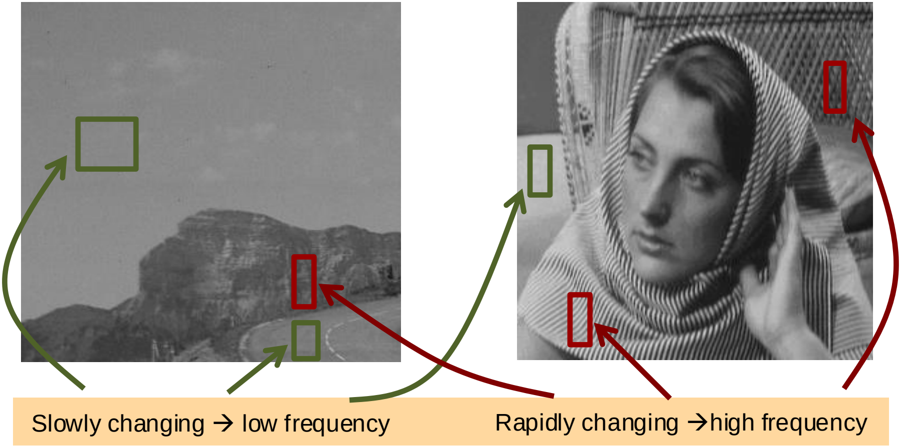
The 2D Fourier Transform (NOT IN EXAM)



FT → straightforward extension to 2D:

- Images are functions of two variables → e.g. $f(x, y)$
- Defined in terms of *spatial frequency* → 2D frequency.
- Fourier Transform is particularly useful for characterising intensity variations across an image.
- FT identifies the *Rate of change of intensity* along each dimension.

Examples: 2D Spatial Frequency (NOT IN EXAM)



More on the 2D FT next year in the *Image Processing and Computer Vision* unit

Frequency Spectrum – Summary

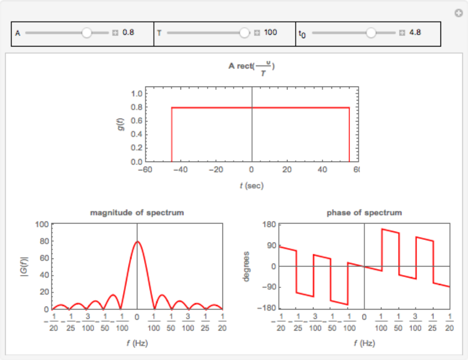
$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

- Distribution of $|F(u)|$ → frequency spectrum/space of signal.
- Features often extracted from the ***Power Spectrum***:

$$\omega(f) = |F(u)|^2$$

- Slowly changing signals → spectrum concentrated around low frequencies.
- Rapidly changing signals → spectrum concentrated around high frequencies.
- Also bandlimited signals → frequency content confined within some frequency band.

Next…



Feature Selection and Extraction

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- **Another look at features**
- PCA for dimensionality reduction
- Convolutions