

Algorithms & DATA - COMS20017

Tutorial Questions

(Q1) In the context of Minimum Variance Unbiased Estimation (MVUE), define the likelihood function for a random signal x . State the likelihood function for a single random sample $x[0] = A + w[0]$, where A is the DC level and $w \sim \mathcal{N}(0, \sigma^2)$

(Q2) Define the curvature of the log-likelihood function. What information does the curvature provide? Explain why it is convenient to use the log-likelihood function.

(Q3) Define and discuss the Cramer-Rao Lower Bound (CRLB) for scalar parameters.

(Q4) In the context of estimating a DC level in white Gaussian noise, consider N observations

$$x[n] = A + w[n], \quad n = 0, 1, 2, \dots, N-1,$$

where $w[n] \sim \mathcal{N}(0, \sigma^2)$. Given that

$$p(x; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - A)^2 \right\}$$

determine the CRLB for A .

(Q5) Let X denote a Poisson random variable with probability density function

$$p(x|\lambda) = \frac{\lambda^x}{x!} e^{-\lambda} \quad \text{for } x = 0, 1, \dots$$

Assuming that the rate parameter λ is exponentially distributed with

$$f(\lambda) = \frac{1}{\lambda_0} e^{-\frac{\lambda}{\lambda_0}}$$

and the joint density of x and λ is

$$f(x, \lambda) = f(x|\lambda)f(\lambda)$$

determine the Maximum Likelihood estimate for λ and comment on the values of λ when λ_0 is much smaller than 1.

(Q6) Let x denote a Rayleigh distributed random variable with probability density function given by

$$f(x|\theta) = \frac{x}{\theta^2} \exp \left\{ -\frac{x^2}{2\theta^2} \right\}$$

Determine the maximum likelihood estimate of θ .

- (Q7) Consider the problem of estimating a DC level A in white Gaussian noise, w , where the noisy data are given by

$$x[n] = A + w[n], \quad n = 0, \dots, N-1, \quad w[n] \sim \mathcal{N}(0, \sigma_w^2)$$

- (a) Estimate the value of A using the maximum likelihood estimation (MLE) procedure. Discuss briefly the optimality properties of the MLE.
 - (b) Estimate the value of A using the method of least squares (LS). Discuss briefly the properties of LS estimation. State at least one problem associated with this approach.
 - (c) Determine the Cramer Rao lower bound (CRLB) of the unknown parameter A . Compare this solution with those from a) and b)
- (Q8) (a) Derive the Cramer-Rao lower bound (CRLB) for the estimation of a DC level in white Gaussian noise ($w[n] \sim \mathcal{N}(0, \sigma^2)$), given by

$$x[n] = A + w[n], \quad n = 0, 1, \dots, N-1$$

For which the probability density function is given by

$$p(x; A) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2\sigma^2} (x[n] - A)^2 \right]$$

- (b) Derive the maximum likelihood estimator (MLE) for the problem from Q8 (a).
 - (c) Is the MLE (b) unbiased? Is it efficient? Does such an MLE attain the CRLB?
- (Q9) Derive the general expression for the maximum a posteriori (MAP) estimator of θ , given the observed variable x .
- (Q10) (a) Let $x[n]$ represent N measurements of a constant amplitude signal. The measurement is corrupted by white Gaussian noise with zero mean and variance σ^2 . If

$$x[n] = A + w[n] \quad \text{for} \quad n = 0, 1, \dots, N-1,$$

show that the Maximum Likelihood estimate for A is given by

$$\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n].$$

- (b) In the same model as above Q10 (a), consider now that A is a zero-mean Gaussian random variable i.e.

$$p(A) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp \left(-\frac{A^2}{2\sigma_n^2} \right)$$

Assuming that the likelihood function is Gaussian as well,

$$p(x|A) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$$

Derive the MAP estimate of A .

(Q11) (a) Assume that the posterior pdf is Gaussian, i.e.

$$p(x|\theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)$$

and that the prior pdf is Cauchy:

$$p(\theta) = \frac{\gamma}{\pi(\gamma^2 + \theta^2)}$$

Show that the MAP estimate of x can be obtained in closed-form.

(b) Now assume that the posterior pdf is still Gaussian, as in Q11 (a), but this time the prior pdf is Laplace:

$$p(\theta) = \frac{1}{\sqrt{2}\sigma} \exp\left\{-\frac{\sqrt{2}|\theta|}{\gamma}\right\}$$

Show that the MAP estimate of x can be obtained in closed-form.