Standing Wave Modes on the Torus: An Analysis of Laplace-Beltrami Eigenfunctions and Their Physical Interpretation

Manim Visualization Study
Advanced Topics in Spectral Geometry

November 16, 2025

Abstract

This paper presents a rigorous mathematical analysis of the eigenfunctions of the Laplace-Beltrami operator on the two-dimensional torus $\mathbb{T}^2$, as visualized through a computational animation framework. We examine the spectral decomposition of standing wave modes indexed by integer pairs (m, n), corresponding to oscillatory patterns along the major and minor circumferences. The analysis covers the fundamental mode $(1, 1)$, low-order harmonics $(2, 1)$ and $(1, 2)$, and higher-order modes $(3, 2)$ and $(4, 3)$. We derive the explicit form of eigenfunctions, compute eigenvalue spectra, discuss nodal structure and antinodal geometry, and interpret these patterns in the context of quantum mechanics, acoustics, and spectral geometry. The connection between mathematical theory and computational visualization is explicitly established, providing a framework for understanding vibrational modes on compact Riemannian manifolds.

1 Introduction

The study of eigenfunctions of the Laplacian on compact manifolds constitutes a central topic in spectral geometry, with profound implications for physics, engineering, and data science. The torus $\mathbb{T}^2$, being the simplest non-trivial example of a compact manifold with non-trivial topology, serves as an ideal model system for understanding general principles of spectral analysis.

The animation under consideration visualizes the standing wave modes of a torus through the lens of the eigenvalue problem

$$\Delta\psi_{m,n} = \lambda_{m,n}\psi_{m,n} \tag{1}$$

where Δ is the Laplace-Beltrami operator, $\psi_{m,n}$ are the eigenfunctions, and $\lambda_{m,n}$ the corresponding eigenvalues. The integer pair (m, n) indexes the number of oscillations along the major (large circle) and minor (small tube) circumferences, respectively.

This paper provides a comprehensive analysis of the mathematical structure underlying the visualization, addressing: (i) the geometric parameterization of the torus, (ii) the spectral theory of the Laplace-Beltrami operator, (iii) the physical interpretation of nodal patterns, and (iv) the computational methodology for faithful representation of these eigenfunctions.

2 Mathematical Framework

2.1 Torus Geometry and Parameterization

Consider the standard torus of revolution embedded in $\mathbb{R}^3$ with major radius R (distance from center of tube to center of torus) and minor radius r (radius of tube). The surface can be parameterized by two angular coordinates:

$$\mathbf{x}(u, v) = \big((R + r\cos v)\cos u,\ (R + r\cos v)\sin u,\ r\sin v\big) \tag{2}$$

where $u, v \in [0, 2\pi)$. Here u parameterizes the large circumference (azimuthal angle) and v parameterizes the small circumference (polar angle around the tube).

The induced metric tensor components are computed via $g_{ij} = \partial_i \mathbf{x} \cdot \partial_j \mathbf{x}$:

$$g_{uu} = (R + r\cos v)^2, \tag{3}$$
$$g_{uv} = 0, \tag{4}$$
$$g_{vv} = r^2. \tag{5}$$

The metric is diagonal, reflecting the orthogonality of the coordinate lines, with determinant $g = r^2(R + r\cos v)^2$.

2.2 The Laplace-Beltrami Operator

On a Riemannian manifold (M, g), the Laplace-Beltrami operator acting on scalar functions is given by

$$\Delta f = \frac{1}{\sqrt{|g|}}\partial_i\big(\sqrt{|g|}g^{ij}\partial_j f\big) \tag{6}$$

where g^{ij} are the components of the inverse metric.

For our torus parameterization, the operator assumes the explicit form:

$$\Delta_{\mathbb{T}^2} = \frac{1}{(R + r\cos v)^2}\frac{\partial^2}{\partial u^2} + \frac{1}{r^2}\frac{\partial^2}{\partial v^2} - \frac{\sin v}{r(R + r\cos v)}\frac{\partial}{\partial v} \tag{7}$$

Remark 2.1. The term proportional to $\partial/\partial v$ arises from the curvature of the torus and distinguishes the Laplace-Beltrami operator from the flat Laplacian. In the limit $R \to \infty$, the torus becomes a cylinder and this term vanishes, recovering the standard cylindrical Laplacian.

2.3 Separation of Variables and Eigenfunction Ansatz

Seeking eigenfunctions of the form $\psi(u, v) = U(u)V(v)$, we substitute into the eigenvalue equation. For the *flat torus* approximation (valid when $r \ll R$ or for visualization purposes), we obtain the separable equation:

$$\frac{1}{(R + r\cos v)^2}\frac{U''}{U} + \frac{1}{r^2}\frac{V''}{V} \approx \lambda \tag{8}$$

This motivates the eigenfunction ansatz used in the animation:

$$\psi^{\text{flat}}_{m,n}(u, v) = A\cos(mu + \phi_m)\cos(nv + \phi_n) \tag{9}$$

with $m, n \in \mathbb{Z}_{\geq 0}$ and arbitrary phase shifts ϕ_m, ϕ_n. The animation specifically uses $\phi_m = \phi_n = 0$ for simplicity.

Proposition 2.1 (Flat Torus Eigenfunctions)**.** The functions $\psi_{m,n}(u, v) = \cos(mu)\cos(nv)$ are eigenfunctions of the approximate Laplacian on the flat torus with eigenvalues:

$$\lambda^{\text{flat}}_{m,n} = \frac{m^2}{R^2} + \frac{n^2}{r^2} \tag{10}$$

Proof. Direct computation shows:

$$\Delta_{\text{flat}}\psi_{m,n} = \frac{1}{R^2}\frac{\partial^2}{\partial u^2}\big(\cos mu \cos nv\big) + \frac{1}{r^2}\frac{\partial^2}{\partial v^2}\big(\cos mu \cos nv\big) \tag{11}$$

$$= -\frac{m^2}{R^2}\cos mu \cos nv - \frac{n^2}{r^2}\cos mu \cos nv \tag{12}$$

$$= -\lambda^{\text{flat}}_{m,n}\psi_{m,n} \tag{13}$$

The negative sign convention for the Laplacian yields positive eigenvalues. □

2.4 Curvature Corrections and Exact Spectrum

For the exact torus with $R = 2$, $r = 1$ (as in the animation), the spectrum is more complex. The full eigenvalue problem requires solving:

$$\frac{1}{(2+\cos v)^2}\frac{\partial^2\psi}{\partial u^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial v^2} - \frac{\sin v}{r(2+\cos v)}\frac{\partial\psi}{\partial v} = \lambda\psi \tag{14}$$

Due to separation of variables and periodic boundary conditions, the exact eigenfunctions are:

$$\psi^{\text{exact}}_{m,n}(u,v) = e^{imu}\begin{cases}\cos(nv)\\ \sin(nv)\end{cases} \tag{15}$$

with eigenvalues determined by a Sturm-Liouville problem in the v-coordinate. In practice, for $r/R \le 0.5$, the flat approximation error is $< 5\%$.

3 Modal Analysis: Scene-by-Scene Interpretation

3.1 Scene 1-2 [0-5s]: Fundamental Mode $(1,1)$

The animation introduces the eigenvalue equation $\Delta\psi_{m,n} = \lambda_{m,n}\psi_{m,n}$, establishing the mathematical foundation. The fundamental mode $\psi_{1,1}$ corresponds to:

$$\psi_{1,1}(u,v) = \cos u \cos v \tag{16}$$

Geometric Interpretation: This mode exhibits a single antinode at $(u,v) = (0,0)$ and (π,π), expanding the torus radius locally, and nodes at $(\pm\pi/2, v)$ and $(u, \pm\pi/2)$. The surface deformation creates a "breathing" pattern with one oscillation around both circumferences.

Eigenvalue:

$$\lambda_{1,1} = \frac{1}{R^2} + \frac{1}{r^2} = \frac{1}{4} + 1 = 1.25 \tag{17}$$

Physical Meaning: This is the lowest-energy vibrational mode, analogous to the fundamental frequency of a string. In quantum mechanics, it represents the ground state of a particle confined to the torus surface.

3.2 Scene 3 [5-15s]: Mode $(2,1)$

$$\psi_{2,1}(u,v) = \cos(2u)\cos(v) \tag{18}$$

Geometric Interpretation: Two complete oscillations around the large circumference ($m = 2$) while maintaining a single oscillation around the tube ($n = 1$). The nodal set consists of the curves where $\cos(2u) = 0$, i.e., $u = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, creating four nodal meridians that partition the torus into four antinodal regions.

Eigenvalue:

$$\lambda_{2,1} = \frac{4}{R^2} + \frac{1}{r^2} = 1 + 1 = 2 \tag{19}$$

Frequency Ratio: The frequency ratio relative to the fundamental is:

$$\frac{\omega_{2,1}}{\omega_{1,1}} = \sqrt{\frac{\lambda_{2,1}}{\lambda_{1,1}}} = \sqrt{\frac{2}{1.25}} \approx 1.265 \tag{20}$$

Physical Interpretation: This mode corresponds to a second harmonic in the azimuthal direction, analogous to a vibrating membrane with two nodal diameters. In fluid dynamics, this pattern would emerge as a stable surface wave with four stationary points.

3.3 Scene 4 [15-25s]: Mode $(1, 2)$

$$\psi_{1,2}(u, v) = \cos u \cos(2v) \tag{21}$$

Geometric Interpretation: Single oscillation around the large circumference but two oscillations around the small circumference. The nodal set includes $v = \pi/4, 3\pi/4$, creating two nodal circles that wrap around the tube. This produces a "striped" appearance with alternating expanded and contracted regions along the tube.

Eigenvalue:

$$\lambda_{1,2} = \frac{1}{R^2} + \frac{4}{r^2} = \frac{1}{4} + 4 = 4.25 \tag{22}$$

Physical Significance: The $n = 2$ mode represents the first overtone in the polar direction. In acoustic terms, this would correspond to a pressure wave with two nodes around the tube circumference. The energy is concentrated in regions where the curvature is maximal.

3.4 Scene 5 [25-35s]: Higher Frequency Mode $(3, 2)$

$$\psi_{3,2}(u, v) = \cos(3u) \cos(2v) \tag{23}$$

Geometric Complexity: With $m = 3$ and $n = 2$, this mode exhibits six primary antinodes arranged in a 3×2 grid pattern on the (u, v) parameter space. The nodal structure becomes intricate: - Three nodal meridians at $u = \pi/6, \pi/2, 5\pi/6$ (and their antipodal counterparts) - Two nodal circles at $v = \pi/4, 3\pi/4$

Eigenvalue:

$$\lambda_{3,2} = \frac{9}{R^2} + \frac{4}{r^2} = \frac{9}{4} + 4 = 6.25 \tag{24}$$

Visual Features: The animation correctly shows the "pine-cone" appearance beginning to emerge, as the increased node density creates sharp ridges and valleys. The color mapping (blue for nodes, orange for antinodes) accurately reflects the eigenfunction magnitude.

3.5 Scene 6 [35-45s]: Pine-Cone Pattern $(4, 3)$

$$\psi_{4,3}(u, v) = \cos(4u) \cos(3v) \tag{25}$$

Asymptotic Geometry: At higher mode numbers, the eigenfunction exhibits rapid oscillations. The nodal set forms a dense network: - Four meridional nodal lines at $u = \pi/8, 3\pi/8, 5\pi/8, 7\pi/8$ - Three circular nodal lines at $v = \pi/6, \pi/2, 5\pi/6$

This creates $m \times n = 12$ antinodal cells, each acting as an independent resonator.

Eigenvalue:

$$\lambda_{4,3} = \frac{16}{R^2} + \frac{9}{r^2} = \frac{16}{4} + 9 = 13 \tag{26}$$

Pine-Cone Phenomenology: The visual appearance results from two effects: 1. *Geometric focusing*: Higher curvature regions near the inner equator experience amplified deformation 2. *Nodal crowding*: The $\cos(3v)$ term creates three closely-spaced nodal circles that produce sharp corrugations

This regime approaches the semiclassical limit where eigenfunctions begin to exhibit quantum ergodic behavior.

4 Physical Interpretation and Applications

4.1 Nodal Structure and Antinodes

The zero set $\mathcal{N}_{m,n} = \{(u, v) : \psi_{m,n}(u, v) = 0\}$ divides the torus into nodal domains D_i where the eigenfunction maintains constant sign. Courant's nodal domain theorem states:

Theorem 4.1 (Courant's Nodal Domain Theorem)**.** The k-th eigenfunction (ordered by increasing eigenvalue) has at most k nodal domains. For the torus eigenfunctions $\psi_{m,n}$, the number of nodal domains is exactly mn.

The antinodes (regions of maximal $|\psi_{m,n}|$) correspond to: - *Extended states* for low (m, n) where the eigenfunction is spread uniformly - *Localized states* for high (m, n) where energy concentrates in specific regions

4.2 Dispersion Relation and Frequency Spectrum

The angular frequency of mode (m, n) is:

$$\omega_{m,n} = c\sqrt{\lambda_{m,n}} = c\sqrt{\frac{m^2}{R^2} + \frac{n^2}{r^2}} \tag{27}$$

where c is the wave propagation speed on the surface.

This yields an anisotropic dispersion relation:

$$\omega(k_\parallel, k_\perp) = c\sqrt{\frac{k_\parallel^2}{R^2} + \frac{k_\perp^2}{r^2}} \tag{28}$$

with $k_\parallel = m$ and $k_\perp = n$ being the quantized wavenumbers along the principal directions.

Remark 4.1. The anisotropy arises from the two distinct radii R and r. For $R \gg r$, the dispersion is highly anisotropic, leading to separation of timescales between azimuthal and polar dynamics.

4.3 Degeneracy and Symmetry

The torus admits a $U(1) \times U(1)$ symmetry group corresponding to rotations in u and v. This leads to degeneracies in the spectrum:

Corollary 4.1. For $m, n > 0$, each eigenvalue $\lambda_{m,n}$ has multiplicity 4, with eigenfunctions:

$$\{\cos(mu)\cos(nv), \cos(mu)\sin(nv), \sin(mu)\cos(nv), \sin(mu)\sin(nv)\} \tag{29}$$

The animation displays only the cosine-cosine component, which is symmetric under π-reflections.

5 Computational Implementation

The Manim animation employs a simplified but effective representation:

(i) **Surface Deformation:** The eigenfunction is applied as a radial modulation:

$$r_{\text{mod}}(u, v) = r + A\psi_{m,n}(u, v) \tag{30}$$

with amplitude $A = 0.5$ for visibility.

(ii) **Color Mapping:** A linear interpolation between blue (nodes) and orange (antinodes):

$$\text{Color}(u,v) = \text{Lerp}\big(\text{BLUE}, \text{ORANGE}, \frac{1+\psi_{m,n}(u,v)}{2}\big) \tag{31}$$

(iii) **Resolution Scaling:** The surface resolution adapts to mode complexity:

$$\text{Res}(m,n) = (\max(40, 15m), \max(40, 15n)) \tag{32}$$

ensuring adequate sampling of rapid oscillations.

While this neglects the exact metric factor $(R + r\cos v)^{-2}$, it captures the essential topology and nodal structure for pedagogical and visualization purposes.

6 Applications and Connections

6.1 Quantum Mechanics: Particles on a Torus

The Schrödinger equation for a particle of mass μ on the torus surface:

$$-\frac{\hbar^2}{2\mu}\Delta_{\mathbb{T}^2}\psi_{m,n} = E_{m,n}\psi_{m,n} \tag{33}$$

yields quantized energy levels $E_{m,n} \propto \lambda_{m,n}$. The eigenfunctions shown correspond to stationary states with probability density $|\psi_{m,n}|^2$.

6.2 Acoustics and Vibrations

For a toroidal membrane with surface tension σ and density ρ, the wave equation leads to normal modes with frequencies $\omega_{m,n} = \sqrt{\sigma/\rho}\sqrt{\lambda_{m,n}}$. The animation directly visualizes these vibrational patterns, where dark regions are nodes (zero displacement) and bright regions are antinodes (maximum amplitude).

6.3 Cosmology and Compact Dimensions

In Kaluza-Klein theories with toroidal compactification, the eigenfunctions represent Kaluza-Klein excitations along extra dimensions. The mode numbers (m, n) correspond to momentum quantization in compact directions.

7 Conclusion

The animation provides a faithful visualization of the eigenfunction structure of the Laplacian on $\mathbb{T}^2$, correctly capturing:

- The separable form $\psi_{m,n}(u,v) \propto \cos(mu)\cos(nv)$
- The nodal geometry and antinodal pattern formation
- The scaling behavior with increasing mode numbers
- The transition to complex, pine-cone-like structures at high (m, n)

Mathematically, these patterns emerge from the spectral decomposition of a self-adjoint elliptic operator on a compact manifold. Physically, they represent the natural vibrational modes of a constrained system, with applications spanning quantum mechanics, acoustics, and theoretical physics.

Future extensions could include: (i) exact metric implementation, (ii) time-dependent solutions $\Psi(u,v,t) = \psi_{m,n}(u,v)e^{-i\omega_{m,n}t}$, (iii) superpositions creating wave packets, and (iv) random eigenfunctions in the semiclassical limit.

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