

**Tutorial Note 11: Growth**

Solution to Exercises

Teaching Assistant: Harlly Zhou

**Example 1** (4) No. You can only take out  $\log m$ .(5) No. You can only take out  $m^{\frac{5}{6}}$ . This is the case of Cobb Douglas with  $\alpha + \beta < 1$ .

(6) Yes.

$$\begin{aligned}
F(mK, mN_1, mN_2) &= \left( \frac{1}{3}(mN_1)^{\frac{1}{3}} + \frac{2}{3}(mN_2)^{\frac{1}{3}} \right)^2 (mK)^{\frac{1}{3}} \\
&= \left[ m^{\frac{1}{3}} \left( \frac{1}{3}N_1^{\frac{1}{3}} + \frac{2}{3}N_2^{\frac{1}{3}} \right) \right]^2 m^{\frac{1}{3}} K^{\frac{1}{3}} \\
&= m^{\frac{2}{3}} \left( \frac{1}{3}N_1^{\frac{1}{3}} + \frac{2}{3}N_2^{\frac{1}{3}} \right)^2 m^{\frac{1}{3}} K^{\frac{1}{3}} \\
&= m \left( \frac{1}{3}N_1^{\frac{1}{3}} + \frac{2}{3}N_2^{\frac{1}{3}} \right)^2 K^{\frac{1}{3}} \\
&= mF(K, N_1, N_2).
\end{aligned}$$

**Exercises**

1. At steady state,

$$\frac{K_t}{A_t N_t} = \frac{K_{t+1}}{A_{t+1} N_{t+1}}.$$

Consider the growth rate of  $\frac{K}{N}$ . We have

$$\begin{aligned}
\frac{K_{t+1}}{N_{t+1}} &= \frac{K_{t+1}}{A_{t+1} N_{t+1}} A_{t+1} \\
&= \frac{K_t}{A_t N_t} A_{t+1} \\
&= \frac{K_t}{N_t} \frac{A_{t+1}}{A_t} \\
&= \frac{K_t}{N_t} (1 + g_A),
\end{aligned}$$

where the second line is by substituting the steady state condition. The same

process applies for  $\frac{Y}{N}$ . Similarly, consider the growth rate of  $K_t$ .

$$\begin{aligned} K_{t+1} &= \frac{K_{t+1}}{A_{t+1}N_{t+1}} A_{t+1}N_{t+1} \\ &= \frac{K_t}{A_t N_t} A_{t+1}N_{t+1} \\ &= K_t \frac{A_{t+1}}{A_t} \frac{N_{t+1}}{N_t} \\ &= K_t(1 + g_A)(1 + g_N). \end{aligned}$$

Taking logarithm gives you the growth rates.

2. (1) Note that

$$\begin{aligned} \frac{Y_{t+1}}{Y_t} &= 1 + g_Y = \frac{A_{t+1}}{A_t} \left( \frac{K_{t+1}}{K_t} \right)^{\frac{1}{3}} \left( \frac{N_{t+1}}{N_t} \right)^{\frac{2}{3}} \\ &= (1 + g_A)(1 + g_K)^{\frac{1}{3}}(1 + g_N)^{\frac{2}{3}}. \end{aligned}$$

Taking log approximation, we get

$$g_Y = g_A + \frac{1}{3}g_K + \frac{2}{3}g_N.$$

When  $g_Y = g_K$ , we have

$$g_Y = g_K = \frac{3}{2}g_A + g_N.$$

(2) Dividing both sides of the LoM by  $K_t$ , we get

$$\frac{K_{t+1}}{K_t} = 1 - \delta + \frac{I_t}{K_t}.$$

Rearranging the terms, we get

$$\frac{I}{K} = g_K + \delta.$$

This means that the investment should compensate for the depreciation and catch up with the capital growth rate.

(3) 8%.

3. (1) Rewriting the production function using per effective labour terms yields

$$\hat{y} = \hat{k}^{1-\alpha}.$$

At steady state,

$$s(\hat{k}_{ss})^{1-\alpha} = (\delta + g_A + g_N)\hat{k}_{ss}.$$

Solving the equation yields

$$\hat{k}_{ss} = \left( \frac{s}{\delta + g_A + g_N} \right)^{\frac{1}{\alpha}}.$$

(2) The steady state consumption per effective labour is

$$\hat{c}_{ss} = (1-s)\hat{k}_{ss}^{1-\alpha} = (1-s) \left( \frac{s}{\delta + g_A + g_N} \right)^{\frac{1-\alpha}{\alpha}}.$$

The golden rate of saving should solve the following problem:

$$\max_{s \in (0,1)} (1-s) \left( \frac{s}{\delta + g_A + g_N} \right)^{\frac{1-\alpha}{\alpha}}.$$

Taking derivative with respect to  $s$  yields the first-order condition for the maximization problem:

$$-\left( \frac{s}{\delta + g_A + g_N} \right)^{\frac{1-\alpha}{\alpha}} + \frac{1-s}{\delta + g_A + g_N} \left( \frac{s}{\delta + g_A + g_N} \right)^{\frac{1-\alpha}{\alpha}-1} = 0.$$

Dividing both sides by  $\left( \frac{s}{\delta + g_A + g_N} \right)^{\frac{1-\alpha}{\alpha}-1}$ , we obtain

$$\frac{1-s}{\delta + g_A + g_N} = \frac{s}{\delta + g_A + g_N}.$$

Solving the equation yields:

$$s_G = \frac{1}{2}.$$

(3) Note that  $g_K = g_A + g_N$ . So in period  $t$ , the growth rate is 5%. After this, the growth rate is 10%. The saving rate is lower than the golden rule. The increase in  $g_A$  has no effect on  $s_G$ .