Periodic Sequences Modulated Filter Bank

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***Abstract*—This paper studies nearly perfect reconstruction periodic sequences modulated filter banks, which includes many useful examples, e.g., the discrete Fourier transform (DFT), generalized DFT (GDFT), and cosine modulated ones. A general framework is developed to meet arbitrary but feasible design constraints, e.g., length of filters, system delay, decimation ratio, phase linearity, etc.. An efficient Newton algorithm is proposed, and a Matlab/Octave package with diverse design examples is provided. Numerical design results are reported for performance study.**

***Index Terms*—Filter bank, discrete Fourier transform (DFT) modulation, cosine modulation.**

I. Introduction

Filter banks are important tools in signal processing with wide applications, e.g., audio coding, image compression, feature extraction, adaptive filtering, system identification, etc. [1]–[3]. Filter banks enabling fast implementations, e.g., the discrete Fourier transform (DFT), generalized DFT (GDFT), and cosine/sine modulated ones are particularly useful and attractive. However, their designs and implementations can be quite involved when considering arbitrary but feasible design constraints [4]–[6]. Furthermore, different types of filter banks are treated separately, and a global picture is missing.

This paper considers the design of filter banks modulated by periodic sequences in a single framework. DFT, GDFT, and cosine/sine modulated filter banks all fit into this framework, and they share the same implementations and design methods, except for the differences of parameter settings. Throughout this paper, we adopt zero-based numbering for vectors and matrices, i.e., index 0 for the first element of a vector, and index $(0,0)$ for the most top left element of a matrix.

II. Periodic Sequences Modulated Filter Bank

A. Filter Bank Analysis

Let $x(t)$, $-\infty < t < \infty$, be the original time domain discrete signal, and $h(t)$ be the impulse response of a low pass filter to be designed. This low pass filter is also called the prototype analysis filter as its modulated versions give the bank of filters used for analysis. In practice, it could be a real-valued causal finite impulse response (FIR) filter, but for now, we have no need to impose any assumption on it. We generate K modulated impulse responses using K periodic sequences with minimum integer period T as

$$h_k(t) = w_{k,\,-t-i} h(t), \quad 1 \le k \le K, \; -\infty < t < \infty \tag{1}$$

where $w_{k,t}$ is the kth periodic sequence, integer i is a phase shift to be determined as well. Note that all the modulated analysis filters share the same phase shift, otherwise, efficient implementations via Fast Fourier Transform (FFT) like algorithms might not be possible. Applying these analysis filters on $x(t)$ yields K sub-band signals as

$$x_k(t) = \sum_{p=-\infty}^{\infty} h_k(p) x(t-p) = \sum_{p} w_{k,\,-p-i} h(p) x(t-p) \tag{2}$$

In the rest of this paper, we always omit bounds of the index of a summation if these bounds are $-\infty$ and ∞. In most useful designs, $x_k(t)$ is band-limited, and thus allowing decimation. With decimation ratio B, we resample these results at time $t = nB$ to have decimated signals

$$X(n,k) = x_k(nB) = \sum_{p} w_{k,\,-p-i} h(p) x(nB-p) \tag{3}$$

Here, B also is called the block size since with decimation, we only need to update the sub-band signals when every new B samples have arrived. Since $w_{k,t}$ is a periodic sequence, we can rewrite (3) as

$$\begin{aligned} \bar{x}(n,\ell) &= \sum_{\text{all } p \text{ with } \bmod(p+\ell+1,\,T)=0} h(p) x(nB-p) \\ X(n,k) &= \sum_{\ell=0}^{T-1} w_{k,\,\ell+1-i} \bar{x}(n,\ell) \end{aligned} \tag{4}$$

where $\mathrm{mod}(p,T)$ is the remainder of p divided by T.

Let us define a $K \times T$ modulation matrix $\boldsymbol{W}$, whose (k,ℓ)th element is $w_{k,\ell}$. We further introduce a circular shift matrix $\boldsymbol{S}(i)$, such that $\boldsymbol{W}\boldsymbol{S}(i)$ is the matrix obtained by circularly shifting elements in the row of $\boldsymbol{W}$ to the right by i positions. Then $w_{k,\,\ell+1-i}$ is the $(k,\,\ell)$th element of $\boldsymbol{W}\boldsymbol{S}(i-1)$ by definition. Here, a column index of $\boldsymbol{W}$ should be understood as the remainder of this index divided by T whenever it is out of range $[0, T-1)$. Now, we can rewrite (4) compactly as

$$\boldsymbol{X}(n) = \boldsymbol{W}\boldsymbol{S}(i-1)\bar{\boldsymbol{x}}(n) \tag{5}$$

where $\boldsymbol{X}(n)$ and $\bar{\boldsymbol{x}}(n)$ are two column vectors defined by $\boldsymbol{X}(n) = [X(n,0), X(n,1), \ldots, X(n,K-1)]$, and $\bar{\boldsymbol{x}}(n) = [\bar{x}(n,0), \bar{x}(n,1), \ldots, \bar{x}(n,T-1)]$, respectively. For certain modulation matrices, e.g., the DFT and discrete cosine transform (DCT) matrices, (5) can be efficiently calculated.

The extra one sample phase shift in the right side of (4) may look odd at a first glance. It is due to zero-based numbering. In practice, only causal filter is feasible. Thus valid index p in (4) starts from 0. Then this phase shift puts the most recent sample, $x(nB)$, in the summation of term $\bar{x}(n,T-1)$. This arrangement is desired when FFT-like block processing algorithm is used to calculate the $\boldsymbol{X}(n)$ in (5).

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B. Filter Bank Synthesis

After necessary operations in the sub-band domain, we are to convert the modified sub-band signals, $\hat{X}(n,k)$, back to the time domain using another bank of K synthesis filters. These synthesis filters are obtained by modulating the impulse response of another low pass prototype synthesis filter, $g(t)$, with another set of K periodic sequences, $\tilde{w}_{t,k}$, with the same minimum period T, i.e.,

$$g_k(t) = \tilde{w}_{t+j,k}\, g(t), \quad 1 \le k \le K,\ -\infty < t < \infty \tag{6}$$

where integer j is one more phase shift to be determined. Again, the same phase shift is shared across all synthesis filters to make efficient synthesis via FFT-like fast algorithms possible. For the kth sub-band, we first upsample $\hat{X}(n,k)$ to the same sampling rate as $x(t)$'s by zero padding, and then remove the aliasing by passing the upsampled signal through the kth synthesis filter. That is

$$\begin{aligned} y_k(t) =& g_k(t) \otimes \sum_n \delta(t-nB)\hat{X}(n,k) \\ =& \sum_q \tilde{w}_{q+j,k}\, g(q) \sum_n \delta(t-nB-q)\hat{X}(n,k) \\ =& \sum_n \tilde{w}_{t-nB+j,k}\, g(t-nB)\hat{X}(n,k) \end{aligned} \tag{7}$$

where $\otimes$ denotes convolution, and δ is the discrete Dirac delta function defined by

$$\delta(t) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{otherwise} \end{cases}$$

Summation of all these K sub-band signals gives the synthesized time domain signal as,

$$y(t) = \sum_{k=0}^{K-1} \sum_n \tilde{w}_{t-nB+j,\,k}\, g(t-nB)\hat{X}(n,k) \tag{8}$$

In practice, it is convenient to do the summation over index k first and then over index n in (8). Hence we have the following typical "overlap-and-add" operations:

$$\begin{aligned} z_n(nB+\tau) &= g(\tau) \sum_{k=0}^{K-1} \tilde{w}_{\tau+j,\,k}\hat{X}(n,k) \\ y(t) &= \sum_n z_n(t) \end{aligned} \tag{9}$$

Now, let us define a $T \times K$ demodulation matrix $\tilde{\boldsymbol{W}}$ with its (t,k)th element being $\tilde{w}_{t,k}$. Note that $\boldsymbol{S}(j)\tilde{\boldsymbol{W}}$ is the matrix obtained by circularly shifting elements in all columns of $\tilde{\boldsymbol{W}}$ to the top by j positions. Then $\tilde{w}_{\tau+j,\,k}$ is the (τ,k)th element of $\boldsymbol{S}(j)\tilde{\boldsymbol{W}}$. Again, the index $\tau+j$ in $\tilde{w}_{\tau+j,\,k}$ should be understood as $\mathrm{mod}(\tau+j,T)$ since it is a periodic sequence with respect to its first index. Similar to the analysis part, we can rewrite $z_n(t)$ in (9) in matrix-vector form as

$$\boldsymbol{v}(n) = \boldsymbol{S}(j)\tilde{\boldsymbol{W}}\hat{\boldsymbol{X}}(n), \quad z_n(nB+\tau) = g(\tau)v(n,\tau) \tag{10}$$

where $\hat{\boldsymbol{X}}(n)$ and $\boldsymbol{v}(n)$ are two column vectors defined by $\hat{\boldsymbol{X}}(n) = [\hat{X}(n,0), \hat{X}(n,1), \ldots, \hat{X}(n,K-1)]$, and $\boldsymbol{v}(n) = [v(n,0), v(n,1), \ldots, v(n,T-1)]$, respectively, and the index τ in $v(n,\tau)$ should be understood as $\mathrm{mod}(\tau,T)$. It is clear that FFT-like fast algorithms can be used to calculate the matrix-vector multiplication in (10) for certain demodulation matrices.

C. Perfect Reconstruction Conditions

Perfect reconstruction is possible only if $\hat{X}(n,k) = X(n,k)$ for all n and k. Hence we replace the $\hat{X}(n,k)$ in (8) with the $X(n,k)$ in (3) to have

$$\begin{aligned} y(t) = \sum_{k=0}^{K-1} \sum_n \sum_p & w_{k,-p-i}\tilde{w}_{t-nB+j,\,k} \\ & h(p)g(t-nB)x(nB-p) \end{aligned} \tag{11}$$

By letting $p = nB - t + \tau$ and introducing matrix $\boldsymbol{\Gamma} = \tilde{\boldsymbol{W}}\boldsymbol{W}$, we can rewrite (11) as

$$\begin{aligned} y(t) = \sum_n \sum_\tau & \gamma_{t-nB+j,\,t-\tau-nB-i} \\ & h(nB+\tau-t)g(t-nB)x(t-\tau) \end{aligned} \tag{12}$$

where $\gamma_{t-nB+j,\,t-\tau-nB-i}$ is the $(t-nB+j,\, t-\tau-nB-i)$th element of $\boldsymbol{\Gamma}$. Again, both the column and row indices of $\boldsymbol{\Gamma}$ should be understood as their remainders divided by T whenever they are not in range $[0,T)$.

Assuming we are to design a filter bank with system delay τ_0, i.e., $y(t) = x(t-\tau_0)$, (12) reveals the following perfect reconstruction conditions:

$$\begin{aligned} &\sum_n \gamma_{t-nB+j,\,t-\tau-nB-i} h(nB+\tau-t)g(t-nB) \\ &= \delta(\tau-\tau_0), \quad \text{for all } \tau \text{ and } 0 \le t < B \end{aligned} \tag{13}$$

Note that the left side of (13) is a periodical function of t with period B. Thus it suffices to consider those t's in range $[0,B)$ in (13). Interestingly, these conditions are related to $\boldsymbol{\Gamma}$, but irrelevant of the detailed forms of $\boldsymbol{W}$ and $\tilde{\boldsymbol{W}}$.

III. Filter Bank Design

A. Two Optimization Problems

1) Phase Shift Optimization: Phase shift pair (i,j) take discrete values. There is no simple yet general strategy to determine their optimal values except for exhaustive search in range $[0,T)$. Fortunately, for commonly used filter banks like the DFT, GDFT, cosine/sine modulated ones, $\boldsymbol{\Gamma}$ takes a simple form, and we can easily find high quality sub-optimal solutions as shown in our design examples.

2) Prototype Filter Optimization: We assume that both the analysis and synthesis prototype filters are causal, have real-valued and finite impulse responses with length L_h and L_g, respectively. The cut off angular frequency for both prototype filters is set to ω_c. Then the total stop band energy is given by

$$\begin{aligned} &\int_{\omega_c}^{\pi} \left| \sum_{t=0}^{L_h-1} h(t)e^{-\jmath\omega t} \right|^2 d\omega + \zeta \int_{\omega_c}^{\pi} \left| \sum_{t=0}^{L_g-1} g(t)e^{-\jmath\omega t} \right|^2 d\omega \\ &= \boldsymbol{h}^{\mathrm{T}}\boldsymbol{\Pi}_h\, \boldsymbol{h} + \zeta \boldsymbol{g}^{\mathrm{T}}\boldsymbol{\Pi}_g\, \boldsymbol{g} \end{aligned}$$

where $\jmath = \sqrt{-1}$, superscript T denotes transpose, $0 \le \zeta \le 1$ is a relative weight on synthesis filter design, $\boldsymbol{h} =$

$[h(0), h(1), \ldots, h(L_h-1)]$ and $\boldsymbol{g} = [g(0), g(1), \ldots, g(L_g-1)]$ are two column vectors, and the (p,q)th element of $\boldsymbol{\Pi}_h$ and $\boldsymbol{\Pi}_g$ is given by

$$\begin{cases} \pi - \omega_c, \ \text{if } p = q, \\ -\dfrac{\sin[(p-q)\omega_c]}{p-q}, \ \text{otherwise} \end{cases}$$

The target is to minimize the total stop band energy while maintaining those perfect reconstruction conditions in (13). A convenient way to solve this constrained optimization problem is to reformulate it as an unconstrained minimization problem with cost function

$$\begin{aligned} c(\boldsymbol{\theta}) =& 0.5\boldsymbol{\theta}^{\mathrm{T}}\mathrm{diag}(\boldsymbol{\Pi}_h, \zeta\boldsymbol{\Pi}_g)\boldsymbol{\theta} + 0.5\eta\sum_{t=0}^{B-1}\sum_{\tau} e^2(t,\tau) \\ &+ 0.5\lambda\boldsymbol{\theta}^{\mathrm{H}}\boldsymbol{\theta} \end{aligned} \tag{14}$$

where $\boldsymbol{\theta} = [\boldsymbol{h}, \boldsymbol{g}]$ is a column vector containing both impulse responses, $\eta > 0$ is a penalty coefficient, $\lambda \geq 0$ is a regularization factor, reconstruction error $e(t,\tau)$ is defined by

$$e(t,\tau) = \boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{M}(t,\tau)\boldsymbol{\theta} - \delta(\tau - \tau_0) \tag{15}$$

and $\boldsymbol{M}(t,\tau)$ is a symmetric matrix such that

$$\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{M}(t,\tau)\boldsymbol{\theta} = \sum_n \gamma_{t-nB+j,\, t-\tau-nB-i} h(nB+\tau-t)g(t-nB)$$

In this way, we are searching for nearly perfect reconstruction solutions.

Cost (14) looks complicated, but many $e(t,\tau)$'s are zeros, and thus can be ignored since $\boldsymbol{h}$ and $\boldsymbol{g}$ have finite lengths and typically $\boldsymbol{\Gamma}$ is sparse. One set of necessary conditions to ensure $e(t,\tau) \neq 0$ are

$$\begin{aligned} & 0 \leq \tau \leq L_h + L_g - 2 \\ & n \geq \max\{\lceil (t-\tau)/B \rceil,\ \lfloor (t-L_g)/B \rfloor + 1\} \\ & n \leq \min\{\lfloor t/B \rfloor,\ \lceil (L_h + t - \tau)/B \rceil - 1\} \end{aligned}$$

where operations $\lceil\,\rceil$ and $\lfloor\,\rfloor$ mean rounding towards plus infinity and minus infinity, respectively.

B. Newton Algorithm for Prototype Filter Optimization

The gradient and Hessian of $c(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ are

$$\begin{aligned} \frac{\partial c}{\partial \boldsymbol{\theta}} =& \mathrm{diag}(\boldsymbol{\Pi}_h, \zeta\boldsymbol{\Pi}_g)\boldsymbol{\theta} + 2\eta\sum_{t=0}^{B-1}\sum_{\tau} e(t,\tau)\boldsymbol{M}(t,\tau)\boldsymbol{\theta} + \lambda\boldsymbol{\theta} \\ \frac{\partial^2 c}{\partial \boldsymbol{\theta}^{\mathrm{T}}\partial \boldsymbol{\theta}} =& \mathrm{diag}(\boldsymbol{\Pi}_h, \zeta\boldsymbol{\Pi}_g) + 4\eta\sum_{t=0}^{B-1}\sum_{\tau}\boldsymbol{M}(t,\tau)\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{M}(t,\tau) \\ &+ \lambda\boldsymbol{I} + 2\eta\sum_{t=0}^{B-1}\sum_{\tau} e(t,\tau)\boldsymbol{M}(t,\tau) \end{aligned} \tag{16}$$

where $\boldsymbol{I}$ is the identity matrix of proper dimension. The Hessian is the sum of four terms as shown in (16). The first three terms are nonnegative definite, and the last one is indefinite. Thus the Hessian could be indefinite, and we cannot use the Newton method directly. However, when perfect reconstruction conditions are nearly satisfied, we have $e(t,\tau) \approx 0$ for all t and τ. Thus the contribution of this term to the Hessian could be small enough, and negligible. Hence we can use line search with search direction, $-\left(\text{nonnegative definite part of } \frac{\partial^2 c}{\partial \boldsymbol{\theta}^T \partial \boldsymbol{\theta}}\right)^{-1}\frac{\partial c}{\partial \boldsymbol{\theta}}$, to minimize $c(\boldsymbol{\theta})$.

Note that $\boldsymbol{M}(t,\tau)$ is sparse when $B \gg 1$. Our Matlab/Octave implementation package exploits this sparsity to speedup the evaluations of gradient and Hessian.

C. Extra Useful Constraints

Constraint $\boldsymbol{h}^{\mathrm{T}}\boldsymbol{h} = \boldsymbol{g}^{\mathrm{T}}\boldsymbol{g}$ removes the scaling ambiguity between $\boldsymbol{h}$ and $\boldsymbol{g}$. It always helps, and we use it throughout our designs. It might be desirable to consider these following optional symmetry constraints:

- type I: $g(t) = h(L_h - t)$ when $L_h = L_g$;
- type II: $g(t) = h(t)$ when $L_h = L_g$;
- type III: $h(t) = h(L_h - t)$;
- type IV: $g(t) = g(L_g - t)$.

The last two symmetry constraints lead to linear phase prototype filters. However, with certain modulation sequences, the analysis and synthesis filters might lose linear phase property even if the prototype filters are symmetric.

IV. Design Examples

We consider three types of widely used filter banks: DFT, GDFT, and odd-stacked cosine modulated filter banks. Their numbers of sub-bands, modulation sequences and $\boldsymbol{\Gamma}$ matrices are summarized in Table I.

A. DFT and GDFT Modulation

DFT modulated filter bank, including its special case, discrete short time Fourier transform (STFT), is considered. Since $\boldsymbol{\Gamma} = \boldsymbol{I}$, from (13) we notice that the phase shift pair (i,j) must satisfy

$$\mathrm{mod}(i + j + \tau_0, T) = 0 \tag{17}$$

to make signal reconstruction possible. Also, all pairs (i,j) with relationship (17) are equally optimal. For any decimation ratio B in range $[1, T]$, it can be shown that when both $h(t)$ and $g(t)$ are sequences of B successive ones and zeros at other positions, such $h(t)$ and $g(t)$ are one feasible solution satisfying (13) with τ_0 given by the delay associated with the maximum spike in $h(t) \otimes g(t)$. Such sequences provide good random initial guesses for our Newton algorithm. GDFT modulated filter banks are closely related to DFT ones, and have similar properties.

One critically sampled DFT filter bank with design parameters $B = 8, L_h = L_g = 128, \tau_0 = 127, \omega_c = 1.3\pi/B, \zeta = 0, \eta = 10^6$, and $\lambda = 0.01$, is shown in Fig. 1. The resultant average square reconstruction error, $\sum_{t=0}^{B-1}\sum_\tau e^2(t,\tau)/B$, is 5.6×10^{-10}. It is difficult to design critically sampled DFT filter banks with sharp stop band rejection using traditional methods. However, if we only care about the analysis part by setting $\zeta = 0$, the proposed method gives reasonably good design as shown in Fig. 1.

TABLE I
COMPARISONS AMONG THREE TYPES OF FILTER BANKS. $\boldsymbol{J}$ IS AN ANTI-DIAGONAL MATRIX WITH UNITARY NON-ZERO ENTRIES

	DFT	GDFT	Cosine
K	$K=T$	$K=T/2$	$K=T/4$
$w_{k,t}$	$\sqrt{1/K}\exp(-\jmath 2\pi kt/K)$	$\sqrt{1/K}\exp[-\jmath 2\pi(k+0.5)(t+0.5)/K]$	$\sqrt{2/K}\cos[\pi(k+0.5)(t+0.5)/K]$
$\tilde{w}_{t,k}$	$\sqrt{1/K}\exp(\jmath 2\pi tk/K)$	$\sqrt{1/K}\exp[\jmath 2\pi(t+0.5)(k+0.5)/K]$	$\sqrt{2/K}\cos[\pi(t+0.5)(k+0.5)/K]$
$\boldsymbol{\Gamma}$	$\boldsymbol{I}_K$	$\begin{bmatrix} \boldsymbol{I}_K & -\boldsymbol{J}_K \\ -\boldsymbol{J}_K & \boldsymbol{I}_K \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{I}_{2K}-\boldsymbol{J}_{2K} & \boldsymbol{J}_{2K}-\boldsymbol{I}_{2K} \\ \boldsymbol{J}_{2K}-\boldsymbol{I}_{2K} & \boldsymbol{I}_{2K}-\boldsymbol{J}_{2K} \end{bmatrix}$

TABLE II
COMPARISONS AMONG DIFFERENT STRATEGIES FOR SELECTING (i, j) IN A COSINE MODULATED FILTER BANK DESIGN

	random search	$(-\tau_0,\, 0)$	$(0,\, -\tau_0)$	$(T/8-\tau_0,\, -T/8)$	$(-T/8-\tau_0,\, T/8)$
$10^9\times$ (best cost in 100 trials)	1.71	2.03	1.86	2.99	2.99

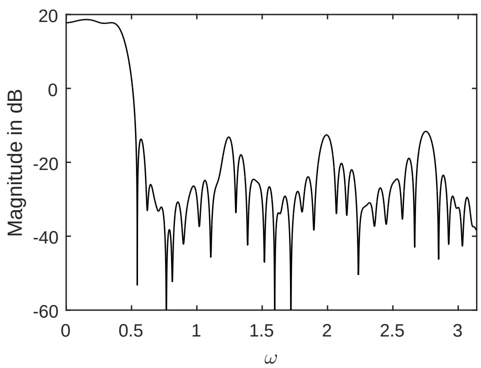

Fig. 1. Low pass analysis filter frequency response of the designed critically sampled DFT filter bank.

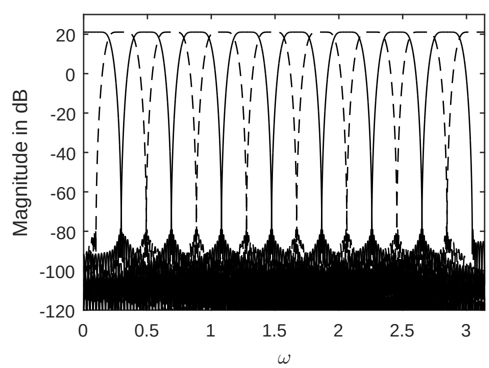

Fig. 2. Frequency response of the designed critically sampled cosine modulated filter bank (the same for analysis and synthesis filters).

B. Cosine Modulation

Among many variations of cosine/sine modulated filter banks, we consider the one with modulation sequences given by the real part of GDFT modulation sequences. Note that modulation sequences for this type of cosine modulated filter bank given in many papers have more complicated forms. Since $\mathrm{diag}(\boldsymbol{\Gamma}) = \boldsymbol{I}$ and most off-diagonal entries of $\boldsymbol{\Gamma}$ are zeros, phase shifts i and j should be selected according to (17) such that reconstruction is possible. Particularly, choices

$$i = \pm\lfloor T/8\rfloor - \tau_0, \quad j = -\tau_0 - i \tag{18}$$

are good since they lead to a time domain aliasing cancellation structure for even K, as shown in modified DCT (MDCT). Such a setting is adopted in [6] as well. The time domain aliasing cancellation works only when $\mathrm{mod}(K, B) = 0$. Hence, it might not be easy to design good oversampled cosine modulated filter banks with $\mathrm{mod}(K, B) \neq 0$.

A cosine modulated filter bank with settings $B = 16, L_h = L_g = 256, \tau_0 = 255, \omega_c = \pi/B, \zeta = 1, \eta = 0.1, \lambda = 0$, and type I symmetry, is designed. Table II compares different strategies for selecting (i, j). The best cost is the minimum one in 100 trials started from random initial guesses. As expected, the best strategy is to randomly search for the optimal i in range $[0, T)$. The strategy given in (18) is optimal for perfect reconstruction designs, but turns out to be sub-optimal, yet good enough, for nearly perfect reconstruction designs. Fig. 2 shows frequency responses of the best design, and the average square reconstruction error is 7×10^{-10}. Comparing with the design with the same requirements given in Fig. 4 in [4], our design has a much lower side lobes, and significantly less aliasing between successive odd/even sub-bands.

V. CONCLUSIONS

The family of periodic sequences modulated filter banks is studied. A Newton algorithm is developed for prototype filter design, and sparsity in the design problem enables efficient implementation. Numerical results are presented for performance study. A Matlab/Octave design package supporting any valid design requirements, e.g., length of prototype filters, system delay, decimation ratio, symmetry constraints, is provided.

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