Taylor's Theorem

Taylor's Theorem is a fundamental result that we'll use often in MACM 316!!

Text §1.1

Theorem: *Let $f(x)$ and its first $n+1$ derivatives be continuous on $[a,b]$, and let $x_o \in [a,b]$ be some given point. Then for every $x \in [a,b]$, there is some real number c between x_o and x such that*

$$f(x) = P_n(x) + E_n(x)$$

where the n^{th} order Taylor polynomial is

$$P_n(x) = f(x_o) + f'(x_o)(x-x_o) + \frac{f''(x_o)}{2!}(x-x_o)^2 + \cdots + \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_o)}{k!}(x-x_o)^k$$

and the remainder (or truncation error) term is

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_o)^{n+1}$$

Alternate formulation **(let $x = x_o + h$, then replace $x_o \to x$):**

$$f(x+h) = f(x) + f'(x)\,h + \frac{f''(x)}{2!}h^2 + \cdots + \frac{f^{(n)}(x)}{n!}h^n + \frac{f^{(n+1)}(c)}{(n+1)!}h^{n+1}$$

for c between x and $x+h$

Taylor Polynomials: Example 1

Approximate the function $f(x) = \cos(x)$ for $0 \leqslant x \leqslant 2\pi$ with a Taylor polynomial of degree 4 near the point $x_o = 0$:

- **Compute derivatives of f:**

$f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$

$f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$

- **Evaluate derivatives at $x_o = 0$:**

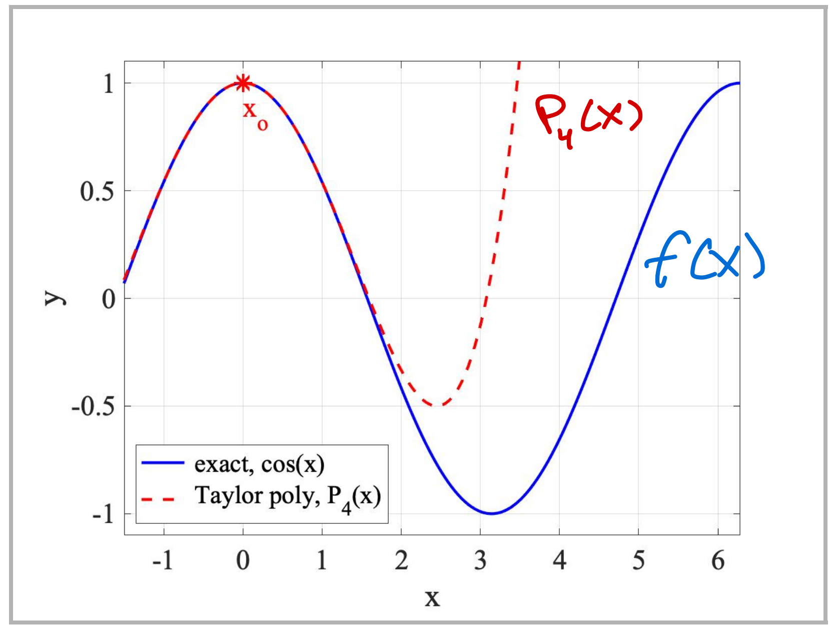
$f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$

$f'''(0) = 0$, $f^{(4)}(0) = 1$

- **Substitute into Taylor's formula:**

$$\cos(x) = \underbrace{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4}_{P_4(x)} \quad \underbrace{\frac{-\sin c}{5!}x^5}_{E_4(x)}$$

for c between 0 and x



Error is only guaranteed to be small close to x

Note: When x is close to $x_o = 0$

$\frac{1}{5!}$, x^5 is small $\Longrightarrow$ $E_4(x)$ is small

$|\sin(c)| \leq$

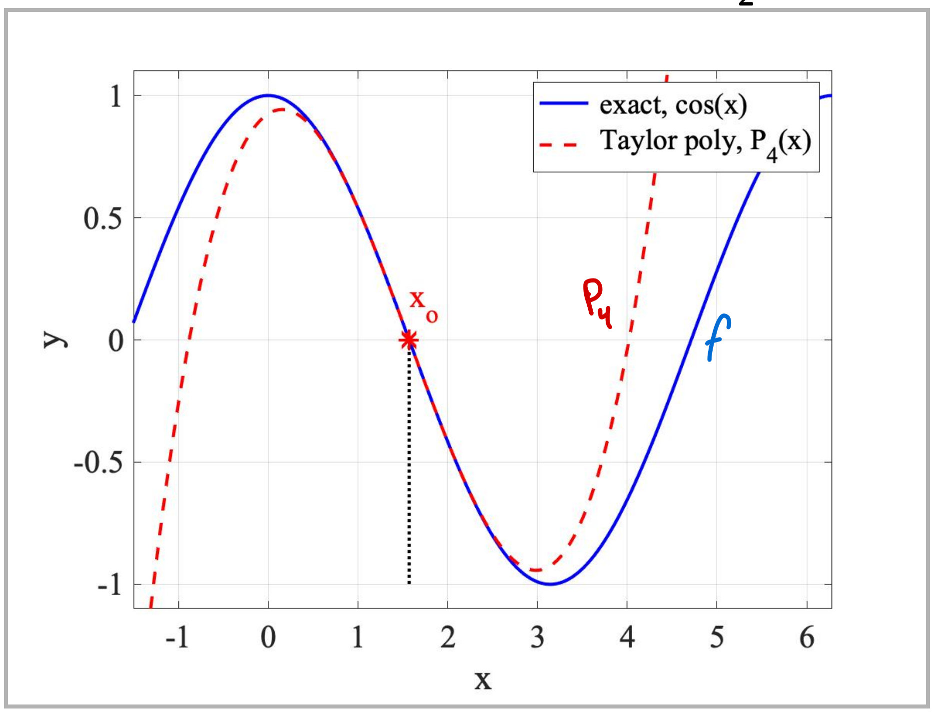
Taylor Polynomials: Example 1b

Repeat the last example, but approximate $\cos(x)$ **at** $x_o = \frac{\pi}{2}$**:**

$f(\frac{\pi}{2}) = 0 \quad f'(\frac{\pi}{2}) = -1 \quad f''(\frac{\pi}{2}) = 0 \quad f^{(3)}(\frac{\pi}{2}) = 1 \quad f^{(4)}(\frac{\pi}{2}) = 0$

$$\cos x = \underbrace{-(x-\tfrac{\pi}{2}) + \tfrac{1}{3!}(x-\tfrac{\pi}{2})^3}_{P_4(x)} - \underbrace{\tfrac{\sin c}{5!}(x-\tfrac{\pi}{2})^5}_{E_4(x)}$$

which holds for some c **between** $\frac{\pi}{2}$ **and** x**.**



Note: **To obtain a useful approximation, you *must* know values of** $f(x_o)$, $f'(x_o)$, $f''(x_o)$, **... analytically. So be strategic in your choice of** x_o**.**

Note: This is 4th order polynomial approximation
is Cubic

Taylor Polynomials: Example 2

Approximate $\sqrt{4.04}$ using Taylor polynomials of degree $n = 1, 2$:

- **Define $f(x) = \sqrt{x}$ and choose $x_o = 4$** → since 4 is close to 4.04

$f(x) = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = \frac{-1}{4}x^{-\frac{3}{2}}$, $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$

- **Taylor polynomial approximation of order $n = 1$ (2 terms) is**

$$P_1(x) = f(4) + f'(4)(x-4) = x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}(x-4)$$

$$P_1(4.04) = 2 + \frac{1}{4}(4.04-4) = 2.01$$

(Exact $\sqrt{4.04} = 2.009975124224\ldots$)

- **Error estimate: how accurate is $P_1(4.04)$?** c is some number between 4 and 4.04 (we don't know it is, thus, it is estimated)
  - * Take remainder term $E_1(4.04) = \frac{1}{2}f''(c)(4.04-4)^2$
  - * Holds for $c \in [4, 4.04]$, but estimate $f''(c) \approx f''(4) = -\frac{1}{32}$
  - * Then absolute error is $|E_1(4.04)| \approx \frac{1}{2}(-\frac{1}{32})(4.04-4)^2 \approx 0.25 \times 10^{-4}$
    corresponding to a relative error $\approx \frac{0.25 \times 10^{-4}}{2} = 0.125 \times 10^{-4}$ real value is roughly 2
- **Compare using exact value for relative error:**

$$\left| \frac{2.01 - \sqrt{4.04}}{\sqrt{4.04}} \right| \approx 0.1238 \times 10^{-4}$$

← Error is roughly within 5 sig digits

- **The 3-term Taylor polynomial ($n = 2$, $x_o = 4$, $x = 4.04$) can be found using the previous result and adding one more term:**

Exercise

$$P_2(x) = \underbrace{f(x_o) + f'(x_o)(x - x_o)} + \frac{f''(x_o)}{2}(x - x_o)^2$$

$$P_2(4.04) = f(4) + f'(4)(4.04-4) + \frac{f''(4)}{2}(4.04-4)^2$$

$$= 4^{\frac{1}{2}} + \frac{1}{2}(4)^{-\frac{1}{2}}(0.04) - \frac{1}{8}(4)^{-\frac{3}{2}}(0.04)^2$$

$$= 2 + \frac{1}{4}(0.04) - \frac{1}{16}(0.04)^2 = 2.0099$$

Exercise: **Estimate error using the remainder term $E_2(x)$.**

$$E_2(x) = \frac{1}{3!} f'''(x_0)(x-x_0)^3$$

$$= \frac{1}{9}\left(\frac{3}{8}(4)^{-\frac{5}{2}}\right)(4.04-4)^3$$

$$= 8.33\bar{3} \times 10^{-8} \approx 0.83 \times 10^{-9}$$

Limiting Behaviour of the Remainder

- Consider "alternate form" of Taylor's formula:

$$f(x+h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!} h^{n+1}}_{E_n(x)}$$

- For small h, this form has two advantages:
  - ∗ it emphasizes that you're approximating f at a point close to x
    (if $|h| > 1$ then $E_n(x)$ grows with n)
  - ∗ the remainder term $E_n(x)$ is clearly $O(h^{n+1})$
- When viewed as an "algorithm", the Taylor polynomial approximation can be improved (accuracy increased) by either:
  - ∗ reducing h: smaller interval, but more restrictive $\Longrightarrow O\left(h^{n+1}\right)$
  - ∗ increasing n: more terms, also more expensive $\Longrightarrow O\left(\frac{1}{(n+1)!}\right)$

. . . there is an obvious trade-off in accuracy vs. efficiency

(such trade-offs are a common theme in MACM 316)

Taylor Polynomials: Example 3

Text p.36

1. (a) Approximate π using the Taylor polynomial for $\arctan x$ at $x_o = 0$:

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{2n-1}}{2n-1}$$
$$= \sum_{k=1}^{n} (-1)^{k-1} \left(\frac{x^{2k-1}}{2k-1} \right) = P_n(x)$$

and recalling $\tan\left(\frac{\pi}{4}\right) = 1 \implies \pi = 4\arctan(1)$ $= 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right]$

(b) Use the first neglected term to estimate the order n needed so that π is correct to 5 significant digits.

Recall from Calc II:
for alternating series only, the first neglected term is a bound for the remainder!

2. Then use the amazing formula

$$\pi = 4\left(\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right)\right)$$

combined with the Taylor polynomial to approximate π.

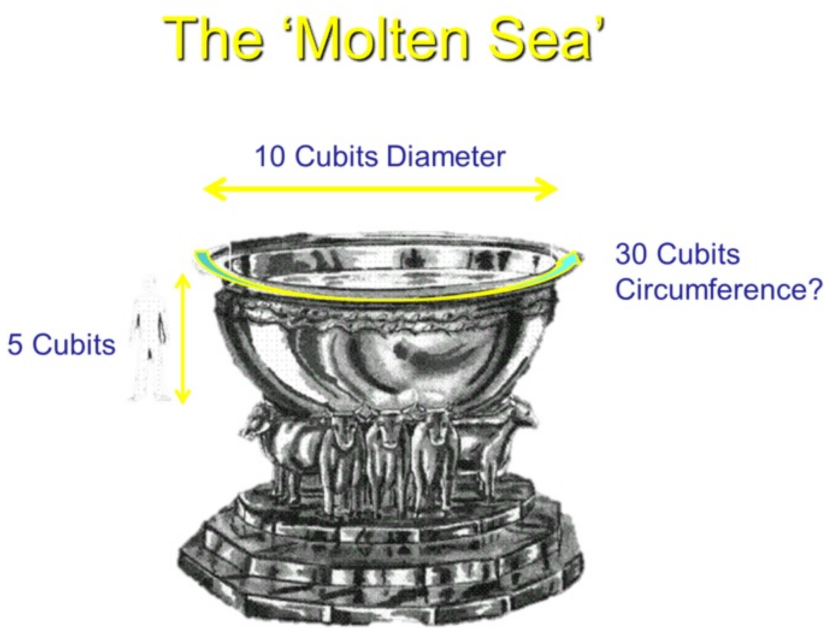
MATLAB code: `piapprox.m` implements both formulas – play with it!

Question: Why is approximation #2 so much better than #1?

Text p.25,Q11

Estimating Pi Throughout History

People have actually been trying to estimate π at least since the days of King Solomon and the prophet Jeremiah:



"And he made a molten sea,
ten cubits from the one brim to the other.
It was round all about,
and its height was five cubits.
And a line of thirty cubits
did compass it round about."

(1 Kings 7:23)

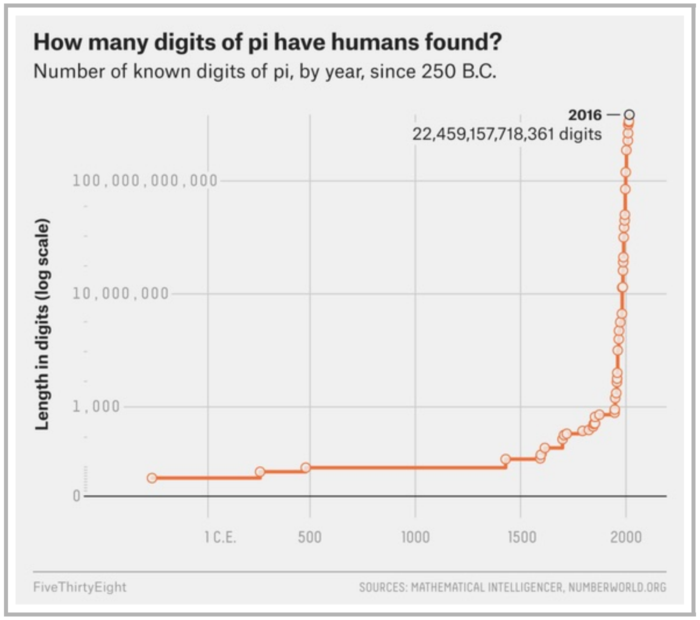
Corollary: $\pi = \frac{30 \text{ cubits}}{10 \text{ cubits}} = 3$ **?**

Computing Pi's Digits

- These arctan series are the simplest of many formulas for approximating π.
- The most famous of all is the Bailey–Borwein[SFU]–Plouffe (BBP) formula:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left[\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right]$$

- BBP can compute binary or hex digits of π starting at any position ... without computing any preceding digits!
- Every so often, newspapers report on someone who has computed π to some ridiculously huge number of digits ... the current record is **31 trillion!** (set in March 2019)
- For most practical purposes 16 decimal digits are sufficient – actually, the best that you can hope for (think ε_M).
- BUT BBP remains an important benchmark algorithm for testing speed and correctness of new super-computers.



[Source: http://fivethirtyeight.com]

Practical Algorithms via Taylor Polynomial Approximation

- **Taylor polynomials are incredibly useful, and ubiquitous in software**
- **MATLAB provides many built-in functions – you'll study two of these on Hwk #2:**

```
>> type linspace

function y = linspace(d1, d2, n)
% LINSPACE Linearly spaced vector.
%   LINSPACE(d1, d2, n) generates a row vector of
%   n equally spaced points between d1 and d2.
. . .
y = d1 + (0:n-1).*((d2 - d1)./(n-1));
```

```
>> type sin

'sin' is a built-in function.
```

- **MATLAB implements `sin` using a C code from FDLIBM available at NETLIB, which is THE most trusted source of numerical software:**

`http://www.netlib.org/fdlibm` $\longrightarrow$ `k_sin.c`

For fun: Look up other basic math functions in FDLIBM!

```
/* ====================================================
 * Copyright (C) 1993 by Sun Microsystems, Inc.
 * ==================================================== */

/* __kernel_sin( x, y, iy)
 * kernel sin function on [-pi/4, pi/4], pi/4 ~ 0.7854
 * Input x is assumed to be bounded by ~pi/4 in magnitude.
 * Input y is the tail of x.
 * Input iy indicates whether y is 0. (if iy=0, y assume to be 0).
 *
 * ALGORITHM:
 *
 *      1. Since sin(-x) = -sin(x), we need only to consider positive x.
 *      2. if x < 2^-27 (hx<0x3e400000 0), return x with inexact if x!=0.
 *      3. sin(x) is approximated by a polynomial of degree 13 on
 *         [0,pi/4]
 *                                3            13
 *              sin(x) ~ x + S1*x + ... + S6*x
 *         where
 *
 *      |sin(x)         2     4     6     8     10     12  |     -58
 *      |----- - (1+S1*x +S2*x +S3*x +S4*x +S5*x  +S6*x   )| <= 2
 *      |  x                                               |
 */

...

half =  5.00000000000000000000e-01, /* 0x3FE00000, 0x00000000 */
S1   = -1.66666666666666324348e-01, /* 0xBFC55555, 0x55555549 */
S2   =  8.33333333332248946124e-03, /* 0x3F811111, 0x1110F8A6 */
S3   = -1.98412698298579493134e-04, /* 0xBF2A01A0, 0x19C161D5 */
S4   =  2.75573137070700676789e-06, /* 0x3EC71DE3, 0x57B1FE7D */
S5   = -2.50507602534068634195e-08, /* 0xBE5AE5E6, 0x8A2B9CEB */
S6   =  1.58969099521155010221e-10; /* 0x3DE5D93A, 0x5ACFD57C */
```

Intermediate Value Theorem or IVT

Text §1.1

Theorem: *Let $f(x)$ be continuous for all $x \in [a, b]$, and let K be any number between $f(a)$ and $f(b)$. Then there exists some real number $c \in (a, b)$ with $f(c) = K$.*

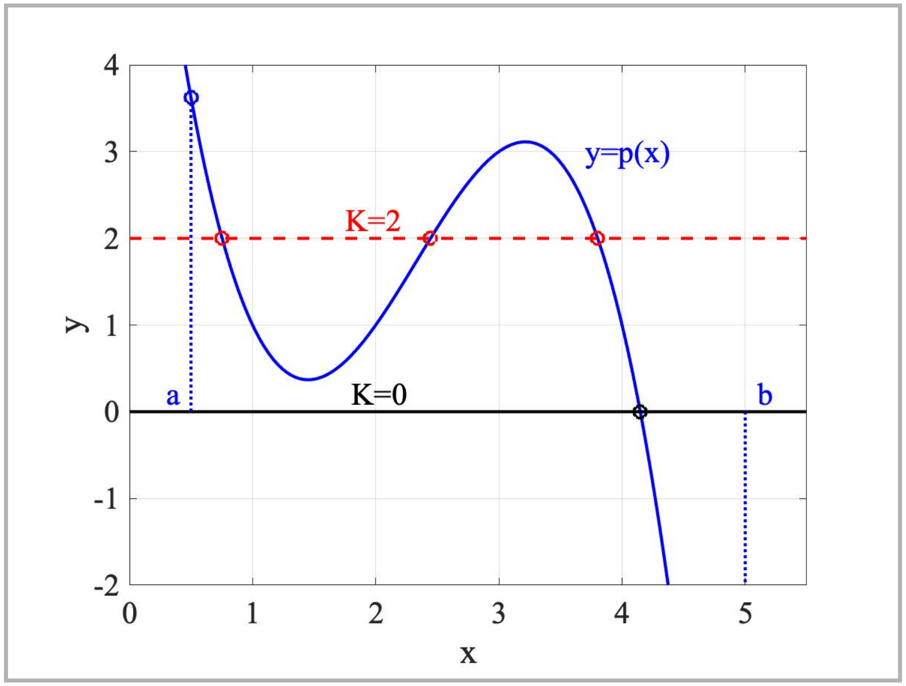
This expresses the intuitive idea that continuous functions have no gaps or jumps.

Example: Apply IVT to the function

$$p(x) = 9 - 14x + 7x^2 - x^3$$

on the interval $[\frac{1}{2}, 5]$.

- Consider $K = 0$ and $K = 2$.
- Endpoints: $p(\frac{1}{2}) = 3.625, \quad p(5) = -11$
- Because $K = 0$ lies in between, there must be a $c \in [\frac{1}{2}, 5]$ with $p(c) = 0$.
- Similarly for $K = 2$ (see plot).



Aside: If interval was $[\frac{1}{2}, 4]$ instead, then $p(4) = 1$ and $K = 2$ still lies between the endpoint values, **BUT** IVT doesn't apply with $K = 0$ anymore.

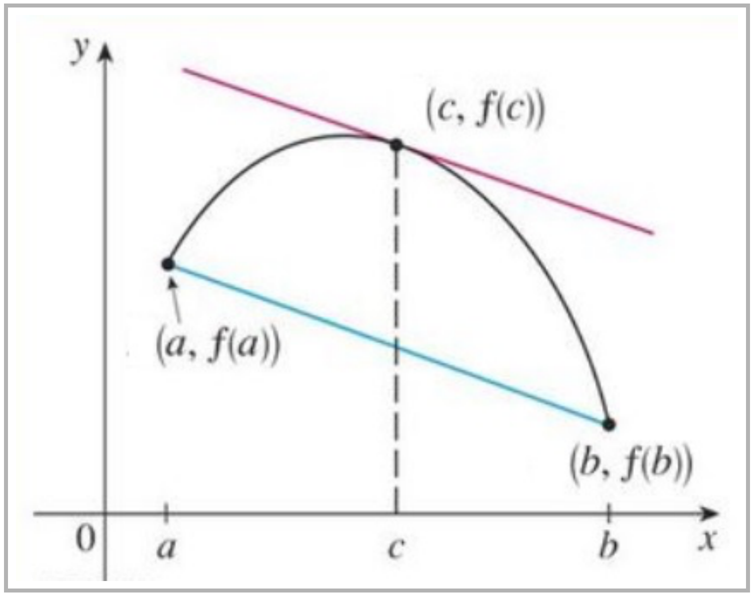
Mean Value Theorem or MVT

Text §1.1

Theorem: *Let* $f(x)$ *be continuous and differentiable for all* $x \in [a,b]$. *Then there exists some real number* $c \in (a,b)$ *with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **This is simply a statement equating:**
  - ∗ **slope of secant line joining** $x = a, b$ **(RHS)**
  - ∗ **slope of tangent line at** $x = c$ **(LHS)**
- **We'll use MVT mostly to prove other useful results.**



MVT Example

Apply MVT to the polynomial

$$f(x) = 1 - x + 3x^3 \quad \text{for } 0.5 \leqslant x \leqslant 2.5$$

where the endpoints are $(0.5,\ 0.875)$ and $(2.5,\ 45.375)$:

- **The *secant line* through these two points is**

$$S(x) = -10.25 + 22.25x \qquad \left(\text{slope} = \tfrac{45.375-0.875}{2.5-0.5} \approx 22.25\right)$$

- **Now, determine which *tangent line* has the same slope:**

$$f'(c) = -1 + 9c^2 \approx 22.25 \implies c \approx 1.607, \quad f(c) \approx 11.85$$

- **Tangent line to $f(x)$ at $x = 1.607$ is**

$$T(x) = -23.91 + 22.25x$$