"Careful" LU Decomposition

Alternate matrix view of GE – called LU decomposition or LU factorization:

- **Row operations reduce A to upper triangular form:**

Aim: Show $A = LU$

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & -2 \end{bmatrix} \xRightarrow{\text{[slide 13]}} U = \begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{1}{2} & -5 \\ 0 & 0 & 45 \end{bmatrix}$$

Faster for Large matrices

- **Trick:** **Write elementary row operations as matrix multiplications:**

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}}_{M_1} \underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & -2 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{1}{2} & -5 \\ 0 & \frac{9}{2} & 0 \end{bmatrix}}_{M_1A}$$

← First step

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}}_{M_2} \underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{1}{2} & -5 \\ 0 & \frac{9}{2} & 0 \end{bmatrix}}_{M_1A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -9 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{1}{2} & -5 \\ 0 & \frac{9}{2} & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{1}{2} & -5 \\ 0 & 0 & 45 \end{bmatrix}}_{M_2M_1A}$$

← second step

- **Then:** $M_2M_1A = U \implies A = \underbrace{M_1^{-1}M_2^{-1}}_{L}U$ or $A = LU$

- **In general:** $U = M_{n-1} \dots M_2M_1A \implies A = \underbrace{M_1^{-1}M_2^{-1} \dots M_{n-1}^{-1}}_{L}U$

Question: How to compute M_i^{-1} and $\prod_{i=1}^{n-1} M_i^{-1}$?

- **Inverting any M_i is actually very simple! Notice that**

$$\begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix}$$

Exactly same M_{2} exept the negativ switched to positi

- **Similarly for M_2^{-1}. Then computing $M_1^{-1}M_2^{-1}$ is also easy:**

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix}}_{M_1^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{bmatrix}}_{M_2^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}}_{\prod_{i=1}^{n-1} M_i^{-1}} = L$$

Important: **We NEVER actually compute the matrices M_i or M_i^{-1}! The above derivation justifies how we can construct L using multipliers for no extra work:**

$$L = M_1^{-1}M_2^{-1}\ldots M_{n-1}^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & 1 \end{bmatrix}$$

and

$$A = LU = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

- **There are two reasons for storing the multipliers in the lower triangular part:**
  * **It saves on storage since these matrix entries are zeroed out and subsequently "unused".**
  * **It's a convenient way to organize sub-diagonal entries of L since they are in the exact same locations ($\ell_{ij} = m_{ij}$).**
- **Return to the row-reduction result from Example 3 (ignore RHS):**

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 4 \\ (\frac{3}{2}) & \frac{1}{2} & -5 \\ (-\frac{1}{2}) & (9) & 45 \end{bmatrix}$$

→ GE with multipliers stored.

values for U

- **It's then easy to write the LU decomposition of A:**

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 9 & 0 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{1}{2} & -5 \\ 0 & 0 & 45 \end{bmatrix}}_{U} \qquad (*)$$

Exercise: **Verify $(*)$ by matrix multiplication.**

Bonus: **The determinant is trivial to calculate using the LU factors:**

$$\det(A) = \det(LU) = \det(L) \cdot \det(U) = \quad 1^3$$

Solving $Ax = b$ with LU Decomposition

1. Determine LU decomposition: $Ax = b \implies L\underbrace{Ux}_{z} = b$

2a. Solve lower triangular system $Lz = b$ for z (forward substitution)

2b. Solve upper triangular system $Ux = z$ for x (backward substitution)

} cheap $O(n^2)$

Example 3 (again): This will be on Final

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 9 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}}_{b} \implies \begin{aligned} z_1 &= 7 \\ z_2 &= -\tfrac{19}{2} \\ z_3 &= 90 \end{aligned}$$

Fx

$$\underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{1}{2} & -5 \\ 0 & 0 & 45 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 7 \\ -19/2 \\ 90 \end{bmatrix}}_{z} \implies \begin{aligned} x_3 &= 2 \\ x_2 &= 1 \\ x_1 &= -1 \end{aligned} \implies x = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Solving another system with same A but different b: can reuse the LU decomposition from Step 1, and repeat Steps 2a-b.

$\implies$ avoids redoing the LU decomposition and saves a lot of work!

Exercise: Repeat with $b = [3,\ 3,\ 7.5]^T$.