Linköpings tekniska högskola
IEI
Produktionsekonomi

Finansiell optimering
Laboration 5
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Computer exercise 5 - Estimation of forward rate curves

Aim: You will determine a smooth forward rate curve from market rates and study how they vary over time.

Background: On the interest rate market instruments are traded that give future nominal payments. The price of an interest rate instrument is expressed in terms of *yields* that depend on time. With a yield curve most of the interest rate instruments can be priced. There are several different perspectives from which the interest rate curve can be viewed. The most clarifying perspective is forward rates, since these only contain interest rates for a period once. To assist you an Excel file is provided with real time quotes for Overnight Indexed Swaps (OIS).

Preparation: Repeat yields, zero rates and forward rates in e.g. Luenberger 3.4, 4.1-4.4.

STINA (Stockholm Tomorrow/Next Interbank Average) is a Swedish interest rate that is used for short term borrowing in Sweden. The T/N rate is the base rate that is used in Swedish OIS. For an OIS the overnight (ON) rate, r_t^o, at time t is swapped against a fixed rate, F_m, at time points $t_i \in \mathcal{T}_m = \{t_1, \ldots, t_N\}$, where N is the number of payments with maturity $m \in \mathcal{M} = \{1M, 2M, 3M, 6M, 9M, 1Y, 2Y, \ldots, 10Y\}$. The aggregated floating rates are paid at time points $t_i \in \bar{\mathcal{T}}_m = \{t_1, \ldots, t_K\}$, where K is the number of payments, and the aggregated payment at time t_i is

$$\prod_{t=t_{i-1}}^{t_i-1} (1 + r_t^o \Delta t_t^o) - 1, \tag{1}$$

where Δt_t^o is time measured with the Actual/360 day counting convention. There are several possible ways to deal with this, but to avoid introducing collaterals, it is assumed that there exist a continously compounded instantaneous risk-free rate, $r(t)$, that is equivalent to the ON rate, which gives

$$\prod_{t=t_{i-1}}^{t_i-1} (1 + r_t^o \Delta t_t^o) - 1 = e^{\int_{\bar{T}_{i-1}}^{\bar{T}_i} r(t)dt} - 1. \tag{2}$$

When Δt_t^f is time measured with the 30E/360 convention then the property that the initial value of the OIS is zero and risk neutral valuation of the cash flows gives

$$0 = E^Q\left[\sum_{i=1}^{N} F_m \Delta t_t^f e^{-\int_{T_0}^{T_i} r(t)dt}\right] - E^Q\left[\sum_{i=1}^{K}\left(e^{\int_{\bar{T}_{i-1}}^{\bar{T}_i} r(t)dt} - 1\right) e^{-\int_{\bar{T}_0}^{\bar{T}_i} r(t)dt}\right] \tag{3}$$

$$= \sum_{i=1}^{N} F_m \Delta t_i^f E^Q\left[e^{-\int_{T_0}^{T_i} r(t)dt}\right] - \sum_{i=1}^{K} E^Q\left[\left(e^{-\int_{\bar{T}_0}^{\bar{T}_{i-1}} r(t)dt} - e^{-\int_{\bar{T}_0}^{\bar{T}_i} r(t)dt}\right)\right] \tag{4}$$

$$= \sum_{i=1}^{N} F_m \Delta t_i^f E^Q\left[e^{-\int_{T_0}^{T_i} r(t)dt}\right] - \left(1 - E^Q\left[e^{-\int_{\bar{T}_0}^{\bar{T}_K} r(t)dt}\right]\right). \tag{5}$$

It holds that the discount factor is the risk-neutral expectation,

$$d(T) = E^Q\left[e^{-\int_0^T r(t)dt}\right] = e^{-r_T T}, \tag{6}$$

where r_T is the continuously compounded spot rate. It holds that $T_N = \bar{T}_K$ and when it is assumed that $T_0 = 0$ then it follows that

$$0 \overset{(5)}{=} \sum_{i=1}^{N} F_m \Delta t_i^f E^Q\left[e^{-\int_{T_0}^{T_i} r(t)dt}\right] - \left(1 - E^Q\left[e^{-\int_{\bar{T}_0}^{\bar{T}_K} r(t)dt}\right]\right) \tag{7}$$

$$= \sum_{i=1}^{N} F_m \Delta t_i^f e^{-r_{T_i} T_i} - \left(1 - e^{-r_{T_N} T_N}\right) \tag{8}$$

$$= F_m \sum_{i=1}^{N-1} \Delta t_i^f e^{-r_{T_i} T_i} + \left(1 + F_m \Delta t_N^f\right) e^{-r_{T_N} T_N} - 1 \Leftrightarrow \tag{9}$$

$$r_{T_N} = \frac{1}{T_N} \ln \frac{1 + F_m \Delta t_N^f}{1 - F_m \sum_{i=1}^{N-1} \Delta t_i^f e^{-r_{T_i} T_i}}. \tag{10}$$

Note that this implies that r_{T_N} can be determined if $r_{T_1}, \ldots, r_{T_{N-1}}$ are known, this method for computing spot rates is known as bootstrapping.
If the fixed leg of a OIS with maturity less than one year is paid only at maturity, then the spot rates for the 1, 2, 3, 6, 9 and 12 month maturities can be determined directly by

$$r_T \overset{(10)}{=} \frac{1}{T} \ln\left(1 + F_m \Delta t^f\right), \tag{11}$$

where it in the laboration is simplified that $\Delta t^f = \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{6}{12}, \frac{9}{12}, 1$. Given that the fixed rate is paid anually for the maturities that are longer than one year, (10) can be used to determine the other spot rates. In the laboration it can for simplicity be used that $\Delta t_N^f = 1$ for the longer OIS contracts.

Preparation: Download the forwardRates.zip from lisam and study the files.

Exercise: Add bootrapping of continuously compounded spot rates, r_T given the OIS rates with (10) and (11) to forwardRates.m.

The price of a zero coupon bond with principal 1 is

$$P_\tau = e^{-r_\tau \tau}.$$

Instead of zero rates, forward rates, f_t where $t = 0, \dots, T-1$, will be used that define the continuous rate between time t and $t+1$. If the money is invested to a fixed rate from 0 to T the growth will be $e^{\sum_{t=0}^{T-1} f_t \Delta t}$. For the time discretization Δt the price of a zero coupon bond can also be written as

$$P_\tau = e^{-\sum_{t=0}^{\frac{\tau}{\Delta t}-1} f_t \Delta t}.$$

Since the prices must be equivalent the following has to hold,

$$\sum_{t=0}^{\frac{\tau}{\Delta t}-1} f_t \Delta t = r_\tau \tau \qquad \tau \in \mathcal{T}.$$

This is an under determined equation system since the number of forward rates are more than the number of zero rates. Therefore it is possible to adapt the forward rate curve to economic reasoning. The future rate reflect the payment that actors on financial markets demand to lend money in the future. It is reasonable to assume that these demands should be stable over time. E.g. the return for February and March 5 years from now should be similar. One way to control this is to determine a forward rate curve which have small derivatives, $(f_{t+1} - 2f_t + f_{t-1})/\Delta t^2$. The following optimization problem can be formulated to minimize the sum of squares of the second order derivatives,

$$\begin{aligned} \min \quad & \sum_{t=1}^{T-2} \left(\frac{f_{t+1} - 2f_t + f_{t-1}}{\Delta t^2} \right)^2, \\ \text{s.t.} \quad & \sum_{t=0}^{\frac{\tau}{\Delta t}-1} f_t \Delta t = r_\tau \tau \qquad \tau \in \mathcal{T}. \end{aligned} \tag{12}$$

Exercise: Determine the forward curve according to (12) with AMPL. Use the file forwardRates.mod. Are the curves plausible for all dates?

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The prices can contain noise, and one way to deal with this is to allow for price deviations, z_τ, and penalize the squared error with weight p. This gives the least squares formulation

$$\begin{aligned} \min \quad & \sum_{t=1}^{T-2} \left(\frac{f_{t+1} - 2f_t + f_{t-1}}{\Delta t^2} \right)^2 + p \sum_{\tau \in \mathcal{T}} z_\tau^2, \\ \text{s.t.} \quad & \sum_{t=0}^{\frac{\tau}{\Delta t}-1} f_t \Delta t + z_\tau \tau = r_\tau \tau \qquad \tau \in \mathcal{T}. \end{aligned} \tag{13}$$

Exercise: Determine the forward curve according to (13) with AMPL with the file forwardRatesLS.mod. Change LeastSquare to true in forwardRates.m to switch to the least squares formulation.

Exercise: How should p be selected to obtain plausible forward rate curves? What happens when p is too large/small?

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Exercise: Compute the eigenvectors (fV) and eigenvalues (fE) for the covariance matrix of changes in forward rates.

Exercise: How does the properties of the measurement using (12) translate into the eigenvectors?

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Exercise: What are the properties of the eigenvectors using the measurement method (13)?

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