Learning the Trading Algorithm in Simulated Markets with Non-stationary Continuum Bandits

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Abstract—The basic Multi-Armed Bandits (MABs) problem is trying to maximize the rewards obtained from bandits with different unknown probability distributions of payoff for pulling different arms, given that only a finite number of attempts can be made. When studying trading algorithms in the market, we are looking at one of the most complex variants of MABs problems, namely the Non-stationary Continuum Bandits (NCBs) problem. The Bristol Stock Exchange (BSE) is a simple simulation of an electronic financial exchange based on a continuous double auction running via a limit order book. The market can be populated by automated trader agents with different trading algorithms. Within them, the PRSH algorithm embodies some basic ideas for solving NCBs problems. However, it faces the difficulty to adjust hyperparameters and adapt to changes in complex market conditions. We propose a new algorithm called PRB, which solves Continuum Bandits problem by Bayesian optimization, and solves Non-stationary Bandits problem by a novel "bandit-overbandit" framework. With BSE, we use as many kinds of trader agents as possible to simulate the real market environment under two different market dynamics. We then examine the optimal hyperparameters of the PRSH algorithm and the PRB algorithm under different market dynamics respectively. Finally, by having trader agents using both algorithms trade in the market at the same time, we demonstrate that the PRB algorithm has better performance than the PRSH algorithm under both market dynamics. In particular, we perform rigorous hypothesis testing on all experimental results to ensure their correctness.

Index Terms—Multi-Armed Bandits, Market Simulation, Nonstationary bandits, Continuum bandits, Softmax Epsilon-Greedy, Bayesian optimization, Hypothesis test

I. INTRODUCTION

In the Multi-Armed Bandits (MABs) problem [1], a gambler needs to make a series of attempts to pull different arms of multi-armed bandits. The payoff for pulling each arm has different unknown probability distributions. Given that only a finite number of attempts can be made, the gambler's objective is to find a sequence of pulls through the arms so that the rewards obtained are maximized. The payoff the gambler gets every time he pulls the arm can be considered feedback, and a lightweight online learning algorithm is needed to quickly adjust the strategy based on the feedback.

Under the most basic setting of MABs problem, there are finite arms to pull, and the distribution of payoff for pulling each arm stays constant. There are also some more complex variants of the MABs problem. For example, when there are an infinite number of arms, it is the Continuum Bandits problem, and when the probability distribution of payoff by pulling each arm changes over time, it is the Non-stationary Bandits problem.

The Bristol Stock Exchange (BSE) [2] is a novel minimal simulation of a centralized financial market based on a con-

tinuous double auction running via a limit order book (LOB). It can be populated by automatic trader agents entering the market randomly at a different time with their limit prices, placing quotes on the LOB, making orders executed as much as possible at a better price to make a profit. It includes a sample of various simple automated trading algorithms, within which an algorithm called Parameterised-Response Zero Intelligence (PRZI) [3] caught our attention.

By changing a certain parameter in PRZI, it can act like several other trading algorithms or their hybrid mix. We can evaluate the performance of a particular value of the parameter by a metric, which is a random variable that conforms to a probability distribution that not only depends on the parameter but also changes with time. Considering the parameter as the arm of bandits in the MABs problem and the metric as the payoff, we transform the online tuning problem of the parameter into a MABs problem. Furthermore, since the parameter is a continuous value and the distribution of the metric changes over time, it is a complex variant of the MABs problem called *Non-stationary Continuum Bandits (NCBs)* problem.

The PRZI-Stochastic-Hillclimb (PRSH) algorithm is an improved version of PRZI, which uses a k-point stochastic hillclimber to adapt the parameter over time, which embodies some basic ideas for solving NCBs problems. However, a function is needed to generate the set of parameters to be explored in the next stage based on the current known optimal parameter, which is set up as a hyperparameter that cannot be dynamically adjusted during the process and is therefore difficult to adapt to complex changes in market conditions.

We propose a new algorithm called PRZI-Bayesian (PRB), which solves the Continuum Bandits problem by Bayesian optimization, and solves the Non-stationary Bandits problem by a novel "bandit-over-bandit" framework. Compared with the PRSH algorithm, the PRB algorithm has a lower cost of hyperparameter selection.

With BSE, we use as many kinds of trader agents as possible to simulate the real market environment under two different market dynamics, i.e. a trending market and a market without trend. We then examine the optimal hyperparameters of the PRSH algorithm and the PRB algorithm under different market dynamics respectively. Finally, by having trader agents using both algorithms trade in the market at the same time, we demonstrate that despite fewer hyperparameters to adjust, the PRB algorithm outperforms the PRSH algorithm under both market dynamics. In particular, we perform rigorous hypothesis testing on all experimental results to ensure their

correctness.

Our contribution is summarized as follows:

- We propose a new algorithm called PRZI-Bayesian (PRB) solving Non-stationary Continuum Bandits (NCBs) problem with fewer hyperparameters than the existing PRZI-Stochastic-Hillclimb (PRSH) algorithm.
- 2) We simulate the real market environment under two different market dynamics with the Bristol Stock Exchange (BSE), which is a novel minimal simulation of a centralized financial market based on a continuous double auction running via a limit order book.
- 3) By simulation, we examine the optimal hyperparameters of the PRSH algorithm and the PRB algorithm under different market dynamics respectively, then demonstrate that the PRB algorithm outperforms the PRSH algorithm under both market dynamics.
- 4) We perform rigorous hypothesis testing on all experimental results to ensure their correctness.

II. RELATED WORKS

In this section we present existing researches related to this work, including existing research on the MABs problem and financial market simulation.

A. MABs problem

Under the most basic problem setting of MABs problem, there are finite arms to pull, and the distribution of payoff for pulling each arm stays constant. Many studies have been done on the basic MABs problem [1], [4]–[6]. Classical algorithms solving the basic MABs problem include *Uniform Exploration* and its improvements *Epsilon-Greedy* and *Softmax Epsilon-Greedy*. There is also *Upper Confidence Bound (UCB)* algorithm that goes further by introducing the concept of confidence interval, as well as *Thompson sampling* [6] which is based on the Bayesian generative model.

There are some studies that focus on more complex variants of the MABs problem. For example, when there are an infinite number of arms, it is the Continuum Bandits problem [7]. Such a problem can be converted to the basic MABs problem using fixed discretization approximately. Auer et al. [8] theoretically proved that the discretization interval need only be less than a threshold at a specified error level to be proven optimal. Kleinberg et al. [9] improved the algorithm on the bestknown upper and lower bounds and adapted an online convex optimization algorithm as the sparser feedback model of the MABs problem. Auer et al. [10] proposed an improvement of an algorithm of Kleinberg et al. by introducing a novel assumption that is complementary to the previous smoothness conditions. A better algorithm in practice is called the zooming algorithm, which was proposed by Kleinberg et al. [11]. In their work, they studied a very general setting for the MABs problem in which the strategies form a metric space and the payoff function satisfies a Lipschitz condition with respect to the metric, then they presented a complete solution for the MABs problem in this setting.

The Contextual Bandits problem is a variant that introduces additional context feature information and is widely applied to the personalized recommendation scenario. The most common form of this is Linear Contextual Bandit (LinCB), where the desired reward is linear about the context. Li et al. [12] first model personalized recommendation of news articles as a contextual bandit problem and proposed a general contextual bandit algorithm that is computationally efficient and well motivated by learning theory. They later proposed a UCB algorithm for LinCB called LinUCB algorithm [13]. Subsequent work combines the contextual bandit problem with reinforcement learning [14] and neural networks [15], [16], making some breakthroughs.

When the probability distribution of payoff by pulling each arm changes over time, it is the *Non-stationary Bandits* problem. Past studies mainly solve the Non-stationary Bandits problem by considering it the Contextual Bandit problem [17]. However, a recent study by Cheung et al. [18] shows that a novel "bandit-over-bandit framework" that adapts to the latent changes can make nearly optimal results in a surprisingly parameter-free manner.

B. Financial market simulation

Multi-agent simulations are commonly used to simulate financial market [19]–[21]. Financial market simulations have been used to study market microstructure [22], financial market regulation [23] and financial technology education [2].

The Bristol Stock Exchange (BSE) [2] is a novel minimal simulation of a centralized financial market based on a continuous double auction running via a limit order book (LOB). It can be populated by automatic trader agents entering the market at a different time with their limit prices, placing quotes on the LOB, making orders executed as much as possible at a better price to make a profit. The BSE includes a sample of various simple automated trading algorithms that can be used by trader agents. In our work, we use the BSE as the simulation platform, because despite its simplicity, the abstractions embodied within the BSE render it a genuinely useful platform for leading-edge research.

III. THEORETICAL FOUNDATIONS

In this section, the theoretical foundations of our work are presented. We first introduce the basic MABs algorithm and its extensions related to our work, then we detail the BSE framework and the trading algorithms it provides, next, focusing on a particular trading algorithm of research interest, we present how the idea of the MABs problem is reflected in the algorithm and the weakness of the existing solutions.

A. MABs problem

Considering a probability space in which each arm has a corresponding random payoff variable D_a . Each time the algorithm selects an arm a, it observes a sample of D_a . We denote that the expected payoff of each arm as $\mu(a) := \mathbb{E}[D_a]$, and the mean of all samples up to a time t is $\bar{\mu}_t(a)$, then the

best arm is the one with the highest expected payoff μ^* , note $\mu^*:=\max_{\alpha\in A}\mu(a)=\mu(a^*).$

We evaluate any bandit algorithm by **regret**. If a^* is known in advance, it must be chosen at all T times, which brings the theoretically optimal reward μ^*T . Obviously, our algorithm does not know in advance the distribution of D_a corresponding to each arm, so the regret at time t is defined as $R(t) := \mu^*t - \sum_{s=1}^t \mu(a_s)$, which is the difference between the current theoretical optimal reward and the current reward.

Generally speaking, in order to minimize R(T), MABs algorithms aims to balance *exploiting* and *exploring*. The idea of exploiting is to select the arm with the largest known payoff, which maximizes reward in the short run but may lead to a local optimum in the long run, while the idea of exploring is to explore the arm that has not been selected, which may reveal arms with larger payoff but may also waste attempts on arms with smaller payoff.

The simplest algorithm called $Uniform\ Exploration$ tries each arm K times and select the one that gives the most payoff, then sticks to it the rest of the time. The algorithm is a greedy algorithm, essentially separating the acts of explore and exploit completely: it purely explores and then purely exploits.

In practice, there is an algorithm called *Epsilon-Greedy*, which is also greedy but allows the explores to be spread more evenly across the entire timeline. In each timestamp t=1,...,T, a die will be rolled with a success rate of ϵ_t . If it hits, then explore: selects each arm with uniform probability; if it misses, then exploit: selects the arm with the highest current $\bar{\mu}_t(a)$. The *Epsilon-Greedy* algorithm tends to work much better in practice than the previous *Uniform Exploration*.

The result of the continued improvement of Epsilon-Greedy is $Softmax\ Epsilon$ -Greedy. For the exploiting phase, the algorithm acts as Epsilon-Greedy algorithm; for the exploring phase, instead of selecting the arm with uniform probability, the Softmax function is used. The probability of an arm to be chosen at time t is:

$$p_t(a) = e^{\bar{\mu}_t(a)} / \sum_{\alpha \in A} e^{\bar{\mu}_t(\alpha)}$$
 (1)

There is also *Upper Confidence Bound (UCB)* algorithm that goes further by introducing the concept of confidence interval. The confidence interval of the payoff expectation for a given arm will be adjusted according to the number of times it has been tried. In general, the more times an arm is tried, the narrower its confidence interval will be. The UCB algorithm will always choose the arm with highest confidence interval upper bound, since it is either an arm with a high payoff expectation or an arm that has been tried less.

Thompson sampling [6] is based on the Bayesian generative model. It has a prior distribution assumption on the D_a . After an arm has been pulled, the prior will be adjusted to form the posterior distribution \hat{D}_a according to the payoff. Thompson sampling uses the Beta distribution as a prior distribution,

and the corresponding posterior distribution is the Bernoulli distribution.

In the MABs framework we have discussed previously, we have a finite number of arms. Now we think of an infinite number of arms, a special case of which is the so-called Continuum-Armed Bandits (CABs), i.e. we think of all arms as forming a set $X \in [0,1]$ and then the average reward satisfies the Lipschitz continuum $|\mu(x) - \mu(y)| \le L|x-y|, \forall x, y \in X$. Such a problem can be converted to the previous basic MABs problem using the fixed discretization approximately, i.e. choosing a set $S \in X$ that contains only a finite number of elements and performing basic MABs algorithms with this set. Theoretically, the discretization interval need only be less than a threshold at a specified error level to be proven optimal [8]. A better algorithm in practice is called the zooming algorithm [11], where instead of running the basic MABs algorithm on a fixed S at each stage, it modifies the set based on the feedback, and gradually makes S more likely to contain arms with higher rewards

Also, considering the distribution of rewards is constant over time is also a major drawback in practice. A more practical assumption would be to let $\mu_t(a)$ migrate slowly with time. A recent study by Cheung et al. [18] shows that a novel "bandit-over-bandit" framework, using one algorithm to solve the original MABs problem while using another MABs algorithm to tune it, works well in such a changing environment.

B. BSE

Traders in the BSE are simulations of "sales traders" working in investment banks, who accept limit orders from customers and place quotes on the LOB. The price specified by the customer is the limit price, and if the limit order is executed at a better price than the limit price, the trader can make a profit from the difference between the execution price and the limit price. So traders need their trading algorithms to decide at which price they should quote to make more profit.

We consider some of the trading algorithms currently available in the BSE:

- Giveaway (GVWY): A totally dumb algorithm that issues a quote-price that is identical to its limit price.
- Zero-Intelligence Constrained (ZIC) [24]: A zero-intelligent algorithm that generates random prices for quotes, but the prices are constrained to never generate a random price that would lead to a loss-making deal.
- Shaver (SHVR): Always trying to have the best bid or ask on the LOB, and doing so by shaving a penny off the best price.
- Sniper (SNPR) [25]: Acting only when there are a few seconds left before the market closes and simply cross the spread to get a trade.
- Zero-Intelligence Plus (ZIP) [26]: An improved version of ZIC, which uses simple machine learning and a shallow heuristic decision tree to dynamically alter the margin of constrain.
- Parameterised-Response Zero Intelligence (PRZI) [3]:
 Depending on the setting of a strategy parameter s, which

- is a real number in the range [-1,+1], it acts like SHVR, ZIC, or GVWY or some kind of hybrid mix.
- PRZI-Stochastic-Hillclimber (PRSH): An improved version of PRZI, which uses a k-point stochastic hillclimber to adapt its value of s over time.

One difference between the BSE and the real market is that all orders have a quantity of 1.

In the BSE, the price range of the customer's limit orders is called the supply and demand range, and the prices can be evenly distributed over the supply and demand range to make the supply-demand curve symmetric, or some fine control can be applied to change the supply-demand curve and the position of the equilibrium point, so a buyer's or seller's market can be simulated. The supply and demand range can also be changed over time to simulate the dynamic in the market. Also, the BSE allows the customer orders to arrive in a continuous random stream with random intervals conforming to distributions such as the Poisson distribution, rather than periodically having every single trader being given a new customer order. The market is ticked at t = 1, 2, ..., T. Depending on the settings, the trader agents will receive orders randomly at intervals. Using all these features of the BSE, we can perform sufficiently realistic market simulations.

C. Bridging the MABs problem and the trading algorithm

We are going to focus on the PRZI algorithm in the BSE particularly since it can be turned into the SHVR, ZIC, GVWY or their hybrid mix by changing a parameter, giving it some properties worth investigating. Strategy, or s, is the parameter that determines the behavior of the PRZI algorithm. $s \in [-1.0, +1.0]$ is a scale factor. Once a PRSH trader agent is going to bid or ask, it will decide a minimum and maximum price based on the agent's limit price as well as the best and worst price on the current limit order book. Then, s will determine which value between the minimum and maximum price should be used to bid or ask.

We can evaluate the performance of a particular value of s by *profit-per-second* (pps), which is determined by the profit made by a single transaction divided by the time that a specific value of s exists. Under a certain value of s, the greater the value of pps, the better the value of s.

However, we cannot determine how much pps an s can bring by the result of a single transaction. Since traders who receive limit orders at different prices enter the market at random, pps is affected by the uncertainties in the market, i.e. **stochastic**. At the same time, the market is often dynamic, with a changing demand and supply range, so the probability distribution that pps conforms to is also **changing with time**. Therefore, pps conforms to a stochastic process consisting of a cluster of random variables related to s and t, denoted as $D_t(s)$.

Suppose we can change the value of s at any time, in the whole transaction process. Since each trader can execute transactions a limited number of times, the chances to change the value of s and use a specific value for transactions is also limited. Then we can regard s as the arm of bandits in the MABs problem and pps as the payoff, to transform

the online parameter tuning problem of the PRZI algorithm into the MABs problem. Furthermore, since *s* is a continuous value, and the distribution of *pps* changes over time, it is both a *Continuum Bandits* problem and a *Non-stationary Bandits* problem. Together, we call it a *Non-stationary Continuum Bandits* (*NCBs*) problem.

PRSH is an improved version of PRZI, which uses a k-point stochastic hillclimber to adapt its value of s over time. In general, the PRSH algorithm uses a multi-stage Uniform Exploration to adjust s: if we group the ticks t=1,2,...,T in BSE to a set of stages ph:=0,1,...,P, then conforming to the framework of Continuum Bandits problem, at the stage ph the PRSH algorithm creates a finite set $S_{ph} \in [-1,1]$ of s. In line with Uniform Exploration, at the stage ph, each s is tried N times, from which the s with the highest average pps is selected and denoted as s_{ph} . Next, the PRSH algorithm will generate $S_{ph+1}:=M(s_{ph})$ by a function M. Note that each stage function M generates a new set S according to the optimal s from the previous stage.

Usually, M generates S_{ph+1} by sampling from a distribution with s_{ph} as expectation and a specific variance. This uses the zooming idea of solving the continuum MABs problem: assuming that $|D_{ph}(x) - D_{ph+1}(y)| \le L|x-y|, \forall x, y \in$ $[-1,1], \forall ph \in \{0,1,...,P\}, \text{ i.e., } D_{ph}(s) \text{ is continuous on } s$ and varies smoothly with stage, the distribution with s_{ph} as expectation has a larger possibility of containing s_{ph+1}^* , i.e., the optimal s for the ph+1 stage. The variance is a parameter that needs to be considered. When $D_{ph}(s)$ is constant with stage, according to the idea of zooming, M should sample s from a distribution whose variance decreases with stage, which allows $\lim_{ph\to\infty} P(s^* \in S_{ph}) = 1$, in which s^* is a timeinvariant value. However, when $D_{ph}(s)$ changes with stage, which means that s^* is also changing with stage, even using a decreasing variance, $P(s_{ph}^* \in S_{ph})$ may be a quantity that does not converge at all as ph increases. A better alternative is to use a fixed variance at each stage. When $D_{ph}(s)$ changes slowly with ph, using a smaller variance can increase $P(s_{ph}^* \in S_{ph})$ since s_{ph}^* also changes slowly. However, when $D_{ph}(\hat{s})$ changes rapidly with ph, it is necessary to increase the variance to increase $P(s_{ph}^* \in S_{ph})$.

In PRSH, M is a pre-determined function that is constant with time, so the variance will be considered a hyperparameter, which makes the PRSH algorithm have to make a trade-off between fast-varying and slow-varying market conditions, increasing the cost of its parameter selection.

There are also other hyperparameters in PRSH. k represents the number of s in S_{ph} and N is the number of attempts per s. Since the trader agents will receive orders randomly at intervals, the total number of quotes placed by a single trader agent is uncertain throughout the trading process. Therefore, N is replaced by a time window W. Each stage ph = 1, 2, ..., P will contain $k \times W$ ticks t. The total number of phases will be $P := \lceil \frac{T}{k \times W} \rceil$. Strategy wait time, noted as v, will determine W together with T and k, i.e. $W := \lfloor \frac{v}{k} \rfloor$. So, in summary, in the PRSH algorithm, we have three hyperparameters k, v,

IV. METHOD

Address to the shortcomings of the existing PRSH algorithm, we propose the PRZI-Bayesian (PRB) algorithm, which uses a Bayesian optimization approach to solve the Continuum Bandits problem and the "bandit-over-bandit" framework to solve the Non-stationary Bandits problem. In this section, the PRB algorithm is described in detail.

A. Bayesian optimization

Thompson sampling [6] is a basic form of Bayesian optimization using Beta distribution as the prior distribution. But the Beta distribution is a discrete binary probability distribution that only applies to the MABs problem in which the payoff is binary distributed. In contrast, in the market trading problem we are studying, the payoff *pps* is a continuous value.

The Bayesian optimization algorithm is based on *Gaussian Process*, using the normal distribution as a prior distribution, which supports the case where the rewards are continuous values. Furthermore, we do not need to generate a set of *s* beforehand like what must be done under the classic Continuum Bandits framework. In each step, the classical Bayesian optimization can consider the tradeoff of exploiting and exploring automatically and generate continuous "advice" of the next *s* to explore, therefore naturally suitable for solving Continuum Bandits problems.

We make the *pps* mean-variance normalized to get y, note the vector consisting of y up to time t as y_t , and the vector consisting of all corresponding s up to time t as s_t . Then the Gaussian Process up to time t can be noted as $y_t \sim \mathbb{N}_t(m(s_t), \kappa(s_t, s_t))$. $\mathbb{N}_t(\mu, \Sigma)$ is a t-dimensional normal distribution, and each dimension in it represents a s explored. $m(s_t)$ gives the expectation vector of y_t corresponding to s_t . $\kappa(s_t, s_t) = exp(-\frac{(s_t - s_t)^2}{2})$ is the Gaussian kernel. Using the Gaussian kernel as covariance matrix can smooth value across dimensions, which is based on an assumption that similar s will yield similarly distributed y.

Suppose we have observed a set of random variables y_1 on the dimensional ensemble s_1 and next want to require the posterior probabilities of the random variables y_2 on the dimensional ensemble s_2 , i.e. it is known that:

$$\mathbf{y}_1 \sim P(\mathbf{Y}_1) = \mathbb{N}(m(\mathbf{x}_1), \kappa(\mathbf{s}_1, \mathbf{s}_1))$$
 (2)

and

$$\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} \sim P(\mathbf{Y}_1, \mathbf{Y}_2) = \mathbb{N} \left(m \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}, \kappa \begin{pmatrix} \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \right) \right)$$
(3)

what we require is:

$$\mathbf{y}_2 \sim P(\mathbf{Y}_2 | \mathbf{Y}_1) = \mathbb{N}(\boldsymbol{\mu}, \Sigma)$$
 (4)

it gives that:

$$\begin{cases} \mu = m(s_1) + \kappa(s_2, s_1)\kappa^{-1}(s_1, s_1)(y_1 - m(s_1)) \\ \Sigma = \kappa(s_2, s_2) - \kappa(s_2, s_1)\kappa^{-1}(s_1, s_1)\kappa(s_1, s_2) \end{cases}$$
(5)

The expectation vector of y is altered by the observation and the variance of y is reduced around the s explored. Bayesian optimization selects the s to be explored next directly by the current posterior distribution of y over the value range of s. s corresponding to a distribution of y with a large mean are likely to achieve better results and have the value of exploiting, whereas s corresponding to a distribution of y with a large variance have been tried less often and have the value of exploring. Including a similar idea to the UCB algorithm, the Bayesian optimization follows three basic rules:

- 1) Priority is given to s corresponding to a distribution of y with both large mean and large variance.
- 2) When time allows, choose s that corresponds to a distribution of y with a large variance.
- 3) When time is tight, choose s that corresponds to a distribution of y with a large mean value.

B. Bandit-over-bandit

We can use Bayesian optimization to solve the Continuum Bandits problem, but the Non-stationary Bandits problem is still left to be handled. We will use the "bandit-over-bandit" approach address to the problem.

By bandit-over-bandit, we mean using one algorithm to solve the MABs problem while using another MABs algorithm to adjust it. Bandit-over-bandit still has the concept of stages that we mentioned in Sec. 3. An algorithm is used at one stage and then at the end of the stage the algorithm is adjusted and applied to the next stage. The adjustment may take various forms, such as pre-setting a set of parameters for the MABs algorithm that is directly applied to the target problem, then treating these parameters as arms of bandits, using another MABs algorithm to select the parameters.

The bandit-over-bandit applied to Bayesian optimization will take another form. We know that the s_t of the Gaussian process records all s so far since its mean vector and covariance matrix are obtained stage-by-stage from the observations of s. If the environment changes, however, the posterior distribution obtained from the observations so far may become inaccurate. In this case, rather than continuing to adjust the distribution on the current basis, it would be better to abandon all previous observations and start from scratch. Ideally, we would maintain several different Gaussian processes, each starting to observe s and making an adjustment at a different time, i.e. having different lengths of memory.

To achieve this, we maintain k Gaussian processes simultaneously and use the Softmax Epsilon-Greedy algorithm to selectively drop the observations of certain Gaussian processes at each stage. Let the set of Gaussian processes be G, then each Gaussian process is denoted as $g_i \in G, i = 1, 2, ..., k$. W, ph determined by means in Sec. 3. Then the flow of our algorithm is shown in Algorithm 1.

Algorithm 1 PRB

```
1: Initialization:
 2: G_{ph} := \{g_{ph,i}\}, i = 1, 2, ..., k
 3: t \leftarrow 1
 4: ph \leftarrow 1
 5: R_i \leftarrow 0, i = 1, 2, ..., k
 6: n_i \leftarrow 0, i = 1, 2, ..., k
 7: for ph \in \{1, 2, ..., P\} do
       for i \in \{1, 2, ..., k\} do
           for t \in \{(p-1) \times k \times W, (p-1) \times k \times W + 1, ..., p \times k \times W \}
 9:
           k \times W do
              if been chosen to place an order then
10:
                 Using g_{ph,i} to sample a s
11:
                 Using s to bid or ask
12:
                 n_i \leftarrow n_i + 1
13:
                 t_{buf} \leftarrow t
14:
              end if
15:
              if the order is executed then
16:
                 Get reward r, i.e. profit
17:
                 R_i \leftarrow R_i + r
18:
                 for j \in \{1, 2, ..., k\} do
19:
20:
                    Update g_{ph,j} with the pair s, r/(t-t_{buf})
                 end for
21:
              end if
22:
           end for
23:
       end for
24:
25:
       \bar{\mu}_i = R_i/n_i, i = 1, 2, ..., k
26:
       Sample k-1 samples of g with p(g_{ph,i})
                           e^{\bar{\mu}_i} without replacement and discard the
              i \in \{1, 2, ..., k\}
       remaining one q
       Generate a new g, forming G_{ph+1} together with the
27:
       k-1 g above
       R_i \leftarrow 0, i = 1, 2, ..., k
28:
       n_i \leftarrow 0, i = 1, 2, ..., k
29:
30: end for
```

Using the Softmax Epsilon-Greedy algorithm to randomly drop a Gaussian process at each stage, we can obtain k Gaussian processes with different memory lengths after enough stages have been performed.

Compared to PRSH, PRB has only two hyperparameters, k and v, making its hyperparameter selection much less time costly. In the subsequent experiments, we will also demonstrate the performance superiority of the PRB algorithm compared to the PRSH algorithm.

V. EXPERIMENT DESIGN

In this section, we present our experimental design. We will first introduce the setup of the market simulation, then introduce the method of hyperparameter selection for PRSH and PRB, and finally introduce the experiment used to compare the performance of PRSH and PRB.

A. Market Simulation

We use the BSE to generate experimental data with 1000 seconds simulation. New orders arrive at intervals modeled with a Poisson distribution like a real market. We generate symmetrical supply-demand curves, but supply and demand ranges are changing over time. According to different market dynamics, the ways of change are also different. For trending markets, our supply and demand range is $[0.1 \times t + \mathbb{N}(0.5) +$ $100, 0.1 \times t + \mathbb{N}(0, 5) + 300$ as shown in Fig.1. For markets without trend, our supply and demand range is $[\mathbb{N}(0,20)]$ $100, \mathbb{N}(0,20) + 300$] as shown in Fig.2 (In both figures, red lines represent the upper limit of the supply and demand range and the blue line represent the lower limit of it). The $\mathbb{N}(\mu, \sigma)$ indicates a white Gaussian noise whose mean is μ , standard deviation is σ . Therefore, we simulate a trending market in which the supply and demand range changes linearly with time and has less volatility and a market in which the supply and demand range does not change with time but has greater volatility. Note market dynamics as $e \in \{e_{trend}, e_{trendless}\}$.

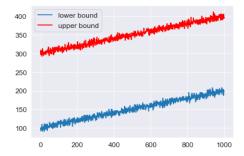


Fig. 1. Supply and demand range of a trending market

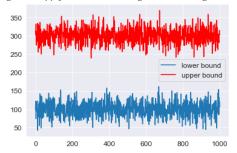


Fig. 2. Supply and demand range of a trend-less market

We populate the market by as many types of traders as we can to make the market real. When performing hyperparameters selection, both buyers and sellers are 20 GVWY traders, 20 ZIC traders, 20 ZIP traders, 20 SNPR traders, 20 SHVR traders, and 20 traders with algorithms either PRSH or PRB. When comparing the performance of the two algorithms, we include both traders using PRSH and PRB, which means both buyers and sellers are 20 GVWY traders, 20 ZIC traders, 20 ZIP traders, and 20 SNPR traders, 20 SHVR traders, 20 PRSH traders, and 20 PRB traders.

B. Hyperparameters Selection

For PRSH, we are going to consider $\{2,4,6,8,10,12,14,16\}$ and v \in $\{32, 64, 128, 256\}.$ For mutation function we have 3 kinds:

- 1) $s_i = M(s) = s_0 + \mathbb{N}(0, 0.05), i = 1, 2, ..., k.$
- 2) $s_i = M(s) = s_0 + \mathbb{N}(0, 0.15), i = 1, 2, ..., k.$

$$s_i = M(s, i) = \begin{cases} s_0 + \mathbb{U}(0, 0.1), i = 1, 3, ..., k/2 - 1\\ s_0 - \mathbb{U}(0, 0.1), i = 2, 4, ..., k/2 \end{cases}$$

, so that half the mutants are exploring the value of increasing s while the other half are exploring the effect of decreasing s. Note the three mutation function as $m \in \{m_1, m_2, m_3\}$.

We repeated 100 experiments in each market dynamic e with each combination of parameters, record the total profit per PRSH trader made in each experiment as a sample $x_{e,k,v,m}^i$, i = 1, 2, ..., 100. Note the i.i.d. samples as $x_{e,k,v,m} = (x_{e,k,v,m}^1, x_{e,k,v,m}^2, ..., x_{e,k,v,m}^{100})$, which are sampled from a distribution X(e,k,v,m). We will estimate $\mathbb{E}[X(e,k,v,m)]$ by $\mathbb{E}[X(e,k,v,m)] = \boldsymbol{x_{e,k,v,m}}$, and observe the distribution of $x_{e,k,v,m}$ for different k,v,m. Then we are going to find $k^*,v^*,m^*=\arg\max_{k,v,m^*}\hat{\mathbb{E}}[X(e,k,v,m)]$, this is the possible optimal parameter obtained from the sample by estimating $\mathbb{E}[X(e, k^*, v^*, m^*)]$. However, to prove the optimality of the parameters we obtained, we need to perform hypothesis testing. All hypothesis tests in this paper will take the significance level $\alpha = 0.05$.

Firstly, the normality of X(e, k, v, m) needs to be tested using the Kolmogorov–Smirnov test (K-S test). The hypothesis of the K-S test is:

- $\mathbb{H}_0: X(e,k,v,m)$ conforms to the normal distribution.
- $\mathbb{H}_1: X(e,k,v,m)$ doesn't conform to the normal distri-

If X(e, k, v, m) conforms to the normal distribution we can then perform Z-test to test whether $\mathbb{E}[X(e, k^*, v^*, m^*)] >$ $\mathbb{E}[X(e,k,v,m)], \forall k,v,m$. We will perform Z-test on each $x_{e,k,v,m}$ that we have obtained one-by-one with x_{e,k^*,v^*,m^*} . The hypothesis of the Z-test is:

- $\mathbb{H}_0 : \mathbb{E}[X(e, k^*, v^*, m^*)] \le \mathbb{E}[X(e, k, v, m)]$
- $\mathbb{H}_1 : \mathbb{E}[X(e, k^*, v^*, m^*)] > \mathbb{E}[X(e, k, v, m)]$

If at the significance level $\alpha = 0.05$, for some combinations of $\{k, v, m\}$, we cannot reject \mathbb{H}_0 , then record those $\{k, v, m\}$. Eventually all recorded $\{k,v,m\}$ together with $\{k^*,v^*,m^*\}$ will form $\{K_X^*(e), V_X^*(e), M_X^*(e)\}.$

For PRB, we are going to explore $k \in \{2,3,4\}$ and $v \in$ $\{32, 64, 128, 256\}.$

We repeated 100 experiments in each market dynamic e with each combination of parameters, and record the total profit per PRB trader made in each experiment as a sample $y_{e,k,v}^i$, i=1,2,...,100. Note the i.i.d. samples as $y_{e,k,v} = (y_{e,k,v}^1, y_{e,k,v}^2, ..., y_{e,k,v}^{100}),$ which are sampled from a distribution Y(e, k, v). We will estimate $\mathbb{E}[Y(e, k, v)]$ by $\mathbb{E}[Y(e,k,v)] = y_{e,k,v}^{-}$, and observe the distribution of $y_{e,k,v}$ for different k, v. Then we are going to find $k^*, v^* =$

 $\arg\max_{v} \mathbb{E}[Y(e,k,v)]$, this is the possible optimal parameter obtained from the sample by estimating $\mathbb{E}[Y(e, k^*, v^*)]$. Also, to prove the optimality of k^*, v^* , we need to perform hypothesis testing.

Testing the normality of Y(e, k, v) using the K-S test. The hypothesis of the K-S test is:

- \mathbb{H}_0 : Y(e, k, v) conforms to the normal distribution.
- ullet $\mathbb{H}_1:Y(e,k,v)$ doesn't conform to the normal distribu-

If Y(e, k, v) conforms to the normal distribution we can then perform Z-test to test whether $\mathbb{E}[Y(e, k^*, v^*)] >$ $\mathbb{E}[Y(e, k, v)], \forall k, v$. We will perform Z-test on each $\boldsymbol{y_{e,k,v}}$ that we have examined one-by-one with y_{e,k^*,v^*} . The hypothesis

- $\mathbb{H}_0: \mathbb{E}[Y(e, k^*, v^*)] \le \mathbb{E}[Y(e, k, v)]$
- $\mathbb{H}_1 : \mathbb{E}[Y(e, k^*, v^*)] > \mathbb{E}[Y(e, k, v)]$

If at the significance level $\alpha = 0.05$, for some combinations of $\{k, v\}$, we cannot reject \mathbb{H}_0 , then record those $\{k, v\}$. Eventually all recorded $\{k, v\}$ together with $\{k^*, v^*\}$ will form $\{K_Y^*(e), V_Y^*(e)\}.$

C. Comparison Experiment Design

To compare the performance of PRSH traders and PRB traders, we put both traders into the market. Since $\{K_X(e)^*, V_X(e)^*, M_X(e)^*\}$ and $\{K_Y(e)^*, V_Y(e)^*\}$ contain the optimal hyperparameter combinations for the PRSH and PRB, respectively, and the hyperparameter combinations that cannot be identified as inferior to the optimal by hypothesis testing, each PRSH trader will randomly choose $\{k, v, m\}$ from $\{K_X(e)^*, V_X(e)^*, M_X(e)^*\}$ and each PRB trader will randomly choose $\{k, v\}$ from $\{K_Y(e)^*, V_Y(e)^*\}$.

We repeated 100 experiments in each market dynamic e. In each experiment, we record the difference between the total profit per PRB trader made and the total profit per PRSH trader made as a sample d_e^i , i = 1, 2, ..., 100. Note the i.i.d. samples as $d_e = (d_e^1, d_e^2, ..., d_e^{100})$, which are sampled from a distribution D(e). We will estimate $\mathbb{E}[D(e)]$ by $\mathbb{E}[D(e)] = \bar{d}_e$. What we will be interested in is whether $\mathbb{E}[D(e)] > 0$. If we can show by hypothesis testing that $\mathbb{E}[D(e)] > 0$, then we have good reason to believe that PRB outperforms PRSH.

In this case, testing the normality of D(e) using the K-S test. The hypothesis of the K-S test is:

- $\mathbb{H}_0: D(e)$ conforms to the normal distribution.
- $\mathbb{H}_1: D(e)$ doesn't conform to the normal distribution.

If D(e) conforms to the normal distribution we can then perform Z-test to test whether $\mathbb{E}[D(e)] > 0$. The hypothesis of the Z-test is:

- $\mathbb{H}_0: \mathbb{E}[D(e)] \leq 0$
- $\mathbb{H}_1 : \mathbb{E}[D(e)] > 0$

If we can reject the null hypothesis \mathbb{H}_0 at the significance level $\alpha = 0.05$, we can believe that the PRB outperforms the PRSH.

 $TABLE\ I$ K-S Test result (p-value) of the profit made by PRSH traders under trending market

M	m1			m2			m3					
K	32	64	128	256	32	64	128	256	32	64	128	256
2	0.61	0.38	0.98	0.84	0.85	0.54	0.42	0.73	0.7	0.96	0.25	0.97
4	0.96	0.61	0.9	0.98	0.92	0.76	0.66	0.99	0.81	0.95	0.97	0.99
6	0.85	0.54	0.85	0.84	0.93	0.2	1.0	0.76	0.99	0.95	0.72	0.94
8	0.73	0.94	0.98	1.0	0.9	0.95	0.73	0.49	0.51	1.0	0.7	0.58
10	0.55	0.8	0.55	0.4	0.84	0.86	1.0	0.42	0.98	0.24	0.99	0.5
12	0.99	0.86	0.31	0.62	0.9	0.34	0.58	0.88	0.92	0.83	0.84	0.64
14	0.78	0.98	0.63	1.0	0.68	0.98	0.41	0.61	0.87	0.97	0.72	0.65
16	0.95	0.85	0.97	0.83	0.61	0.97	0.94	0.97	0.88	0.99	0.39	0.97

 $TABLE \ II \\ K-S \ Test \ result \ (P-value) \ of the profit made by PRSH \ traders \ under \ trend-less \ market$

M		n	n1		m2			m3				
K	32	64	128	256	32	64	128	256	32	64	128	256
2	0.97	0.82	0.24	0.86	0.38	0.32	0.88	0.97	0.98	0.78	0.8	0.74
4	0.37	0.89	0.76	0.91	0.99	0.99	0.91	0.97	0.62	0.85	0.98	0.87
6	0.83	0.94	1.0	0.53	0.88	0.92	0.58	0.98	0.82	0.78	0.63	0.99
8	0.61	0.27	0.51	0.97	0.32	0.7	0.96	0.7	0.94	0.82	0.67	0.24
10	0.97	0.99	0.94	0.99	0.54	0.94	0.74	0.72	0.14	0.51	0.95	0.99
12	0.99	0.74	0.17	0.63	0.45	0.95	0.97	0.51	0.99	0.99	0.99	0.73
14	0.52	0.88	0.52	0.67	0.89	0.99	0.24	0.99	0.5	0.71	0.91	0.99
16	1.0	0.83	0.51	0.75	0.96	0.68	0.93	0.95	0.99	0.77	0.32	0.98

 $TABLE \; III \\ Mean \; Profit \; made \; by \; PRSH \; traders \; under \; trending \; market \;$

M		m1				m2				m3			
K	32	64	128	256	32	64	128	256	32	64	128	256	
2	1.15	1.17	1.19	$1.24_{0.10}$	1.16	1.2	1.2	$1.24_{0.14}$	1.2	1.18	1.2	$1.23_{0.11}$	
4	1.22	$1.27_{0.39}$	$1.23_{0.08}$	1.17	1.2	1.18	1.22	$1.22_{0.06}$	1.21	1.19	1.21	$1.23_{0.09}$	
6	$1.26_{0.28}$	1.21	1.17	1.19	1.16	1.2	1.18	1.22	1.19	1.21	1.28	1.19	
8	1.22	1.19	1.17	1.2	1.2	1.22	1.2	1.17	1.17	1.18	1.21	1.19	
10	$1.23_{0.12}$	1.15	1.19	1.2	1.18	1.2	1.19	1.17	1.18	1.2	1.21	$1.23_{0.08}$	
12	1.21	1.22	1.21	$1.27_{0.38}$	1.2	1.18	$1.26_{0.36}$	$1.25_{0.20}$	1.19	1.18	1.19	$1.24_{0.12}$	
14	1.2	1.2	1.17	$\frac{1.25}{0.19}$	1.16	$\frac{1.24}{0.15}$	1.19	1.2	1.18	$1.22_{0.05}$	$1.23_{0.07}$	1.2	
16	1.2	$\frac{1.25}{0.19}$	1.2	$\frac{1.27}{0.48}$	1.17	1.18	1.18	1.2	$\frac{1.24}{0.14}$	1.2	1.18	$\frac{1.23}{0.10}$	

 ${\small \textbf{TABLE IV}} \\ {\small \textbf{MEAN PROFIT MADE BY PRSH TRADERS UNDER TREND-LESS MARKET}}$

M		m1				m	12		m3			
K	32	64	128	256	32	64	128	256	32	64	128	256
2	1.18	$1.23_{0.09}$	1.19	$1.23_{0.08}$	1.16	$1.23_{0.06}$	1.19	$1.27_{0.46}$	$1.23_{0.11}$	1.18	1.21	$1.23_{0.06}$
4	$1.24_{0.11}$	1.21	$1.25_{0.19}$	$1.27_{0.38}$	1.17	1.19	1.22	$1.27_{0.37}$	1.22	1.21	$1.26_{0.31}$	1.21
6	$1.23_{0.10}$	1.21	1.22	1.22	1.2	1.22	$1.26_{0.37}$	$1.24_{0.09}$	$1.23_{0.10}$	$1.25_{0.16}$	1.27	$1.24_{0.10}$
8	$1.25_{0.21}$	$1.26_{0.30}$	1.2	1.21	1.16	1.21	1.21	1.19	1.19	$1.25_{0.23}$	1.22	1.19
10	$1.24_{0.12}$	$1.22_{0.05}$	1.19	$1.24_{0.13}$	$1.25_{0.18}$	1.21	$1.22_{0.06}$	1.22	1.17	1.22	$\frac{1.24}{0.17}$	$1.24_{0.15}$
12	1.19	$1.25_{0.19}$	1.21	$1.23_{0.10}$	1.14	1.2	1.17	1.22	1.2	1.21	1.21	1.22
14	1.21	1.21	1.2	1.19	$1.23_{0.10}$	1.16	$1.27_{0.48}$	$1.24_{0.13}$	1.19	1.21	$1.24_{0.17}$	$1.26_{0.35}$
16	1.22	1.21	1.18	$\frac{1.23}{0.06}$	1.22	1.21	1.18	$\frac{1.24}{0.15}$	1.2	1.22	1.2	1.21

VI. EXPERIMENT RESULT

In this section, we analyze the experimental results. We will show the results of hyperparameter selection, to determine

the set of hyperparameters to be used for the comparison experiments. Then we will show the results of the comparison experiments, which demonstrate the superiority of the PRB

algorithm over the PRSH algorithm in terms of performance.

A. Hyperparameters Selection

1) PRSH: Table I and Table II show the K-S Test p-values of the profit made by PRSH traders under trending market and trend-less market respectively. All p-values exceed 0.05, which means at the significance level of $\alpha=0.05$, we can't reject the null hypothesis of the K-S test that $X(e,k,v,m), \forall e,k,v,m$ conforms to the normal distribution.

Table III and Table IV shows the mean profit (i.e. $\hat{\mathbb{E}}[X(e,k,v,m)]$) made by PRSH traders under trending market and trend-less market respectively. All the $\hat{\mathbb{E}}[X(e,k,v,m)]$ are divided by 1,000 for clarity. The parameters combination with the highest mean profit in the trending market is $(k^*,v^*,m^*)=\arg\max_{k,v}\hat{\mathbb{E}}[X(e_{trend},k,v,m)]=(6,128,m_3)$ with $\hat{\mathbb{E}}[X(e_{trend},k^*,v^*,m^*)]=1277.68$, while the combination with the highest mean profit under trend-less market is $(k^*,v^*,m^*)=\arg\max_{k,v}\hat{\mathbb{E}}[X(e_{trendless},k,v,m)]=(6,128,m_3)$ with $\hat{\mathbb{E}}[X(e_{trendless},k^*,v^*,m^*)]=1274.51$. The combinations with the highest mean profit are displayed in bold in the table.

Under e_{trend} , of the full 96 combinations of k, v, m, 23 combinations could not reject the null hypothesis in Z-test at the significant level of $\alpha=0.05$ (including k^*, v^*, m^* itself), while 37 combinations under $e_{trendless}$. All the combinations are underlined in the table, and the p-values of the Z-test are displayed in the subscript. We will use those combinations of parameters to create $K_X(e_{trend})^*, V_X(e_{trend})^*, M_X(e_{trend})^*$ and $K_X(e_{trendless})^*, V_X(e_{trendless})^*, M_X(e_{trendless})^*$ respectively.

TABLE V
K-S TEST RESULT (P-VALUE) OF THE PROFIT MADE BY PRB TRADERS
UNDER TRENDING MARKET

V K	32	64	128	256
2	0.58	0.33	0.47	0.82
3	0.69	1.0	0.96	0.54
4	0.45	0.91	0.97	0.99

TABLE VI
K-S TEST RESULT (P-VALUE) OF THE PROFIT MADE BY PRB TRADERS
UNDER TRENDING MARKET

V K	32	64	128	256
2	0.96	0.95	0.92	0.89
3	0.41	0.47	0.54	0.89
4	0.61	0.88	0.85	0.26

2) PRB: Table V and Table VI show the K-S Test p-values of the profit made by PRB traders under trending market and trend-less market respectively. All p-values exceed 0.05, which means at the significance level of $\alpha=0.05$, we can't reject the null hypothesis of the K-S test that $Y(e,k,v), \forall e,k,v$ conforms to the normal distribution.

TABLE VII
MEAN PROFIT MADE BY PRB TRADERS UNDER TRENDING MARKET

V K	32	64	128	256
2	$2.20_{0.08}$	$2.21_{0.09}$	2.19	$\frac{2.23}{0.18}$
3	$2.19_{0.06}$	$2.27_{0.46}$	$\frac{2.22}{0.15}$	$\frac{2.23}{0.21}$
4	2.28	$\frac{2.21}{0.11}$	2.19	$\frac{2.27}{0.48}$

TABLE VIII
MEAN PROFIT MADE BY PRB TRADERS UNDER TRENDING MARKET

V K	32	64	128	256
2	2.35	$\frac{2.23}{0.18}$	$2.26_{0.33}$	$\frac{2.32}{0.19}$
3	$2.31_{0.29}$	$2.30_{0.36}$	$2.32_{0.19}$	$2.28_{0.48}$
4	$2.27_{0.41}$	$2.32_{0.20}$	$2.32_{0.23}$	$2.30_{0.30}$

Table VII and Table VIII show the mean profit (i.e. $\hat{\mathbb{E}}[Y(e,k,v)]$) made by PRB traders under trending market and trend-less market respectively. All the $\hat{\mathbb{E}}[Y(e,k,v)]$ are divided by 1,000 for clarity. The parameters combination with the highest mean profit in the trending market is $(k^*,v^*)=\arg\max_{k,v}\hat{\mathbb{E}}[Y(e_{trend},k,v)]=(4,32)$ with $\hat{\mathbb{E}}[Y(e_{trend},k^*,v^*)]=2276.37$, while the combination with the highest mean profit under trend-less market is $(k^*,v^*)=\arg\max_{k,v}\hat{\mathbb{E}}[Y(e_{trendless},k,v)]=(2,32)$ with $\hat{\mathbb{E}}[Y(e_{trendless},k^*,v^*)]=2352.83$. The combinations with the highest mean profit are displayed in bold in the table.

Under e_{trend} , of the full 12 combinations of k,v, 10 combinations could not reject the null hypothesis in Z-test at the significant level of $\alpha=0.05$ (including k^*,v^* itself), while all 12 combinations under $e_{trendless}$. All the combinations are underlined in the table, and the p-values of the Z-test are displayed in the subscript. We will use those combinations of parameters to create $K_Y(e_{trend})^*, V_Y(e_{trend})^*$ and $K_Y(e_{trendless})^*, V_Y(e_{trendless})^*$ respectively.

B. Comparing PRSH with PRB

The kernel density plot of $d_{e_{trend}}$ and $d_{e_{trendless}}$ is shown in Fig. 10. It can be seen from the graph that both $\hat{\mathbb{E}}[D(e_{trend})]$ and $\hat{\mathbb{E}}[D(e_{trendless})]$ are much larger than 0. Table IX shows the statistic result of $d_{e_{trend}}$ and $d_{e_{trendless}}$. From the K-S Test p-values we can find that both $D(e_{trend})$ and $D(e_{trendless})$ passed the normality test at the significant level $\alpha=0.05$, since both p-values are greater than 0.05. And in both cases, the p-values of the Z-Test are 0.0, which means the null hypothesis is significantly rejected at a significance level of $\alpha=0.05$ and $\mathbb{E}[D(e)]>0, \forall e$. Therefore, we can conclude that traders who use the PRB algorithm on average make much more profit than those who use PRSH in both markets with and without a trend, which demonstrate the performance superiority of the PRB algorithm compared to the PRSH algorithm.

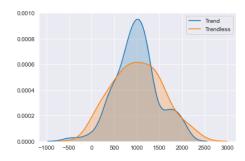


Fig. 3. Kernel density plot of $d_{e_{trend}}$ and $d_{e_{trendless}}$

 $\begin{array}{c} \text{TABLE IX} \\ \text{Statistic results of } d_{e_{trend}} \text{ and } d_{e_{trendless}} \end{array}$

e	Mean	Std	K-S Test p-value	Z-Test p-value
trend	995.20	475.24	0.50	0.0
trend-less	1022.44	550.39	0.98	0.0

VII. DISSCUSSION AND FUTURE WORKS

In our works, we propose a new PRB algorithm solving the Non-stationary Continuum Bandits problem by Bayesian optimization and a "bandit-over-bandit" framework. Despite fewer hyperparameters, our PRB algorithm outperforms the existing PRSH algorithm in both markets with or without trend. This conclusion is demonstrated by the simulation of real market environment and rigorous hypothesis testing.

Despite the rigorous experimental design, the theories in our work do not have rigorous mathematical proofs. In particular, our work is based on the assumptions that similar parameters produce similar payoff and that the change in the payoff distribution is smooth in time. The assumption is difficult to prove due to the complexity of the market. In addition, no rigorous mathematical representation is given in our work for concepts such as the rate of change of the payoff distribution, which involves a series of advanced mathematical concepts such as variational analysis. In future work, we will try to elaborate the theories in our work with more rigorous mathematics.

Moreover, regarding the market simulation aspect, we only consider a trending market in which the supply and demand range changes linearly with time and has less volatility and a market in which the supply and demand range does not change with time but has greater volatility. In future works, we will consider more complex market dynamics, as well as asymmetric and even time-varying supply and demand curves, etc.

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