Real Analysis: Definitions and Theorems

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1 Measure Theory

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4 Hilbert Space: An Introduction

4.1 L^2 space

Proposition 4.1.1 The Space $L^2(\mathbb{R}^d)$ has the following properties:

(i) $L^2(\mathbb{R}^d)$ is a vector space.

(ii) $f(x)\overline{g(x)}$ is integrable whenever $f, g \in L^2(\mathbb{R}^d)$, and the Cauchy-Schwarz inequality holds: $|(f, g)| \leq ||f||\ ||g||$.

(iii) If $g \in L^2(\mathbb{R}^d)$ is fixed, the map $f \mapsto (f, g)$ is linear in f, and also $(f, g) = \overline{(g, f)}$.

(iv) The triangle inequality holds: $||f + g|| \leq ||f|| + ||g||$

Theorem 4.1.2 The space $L^2(\mathbb{R}^d)$ is complete in its metric.

Theorem 4.1.3 The space $L^2(\mathbb{R}^d)$ is **separable**, int the sense that there exists a countable collection $\{f_k\}$ of elements in $L^2(\mathbb{R}^d)$ such that their linear combinations are dense in $L^2(\mathbb{R}^d)$

4.2 Hilbert space

Definition 4.2.1 A set $\mathcal{H}$ is a **Hilbert Space** if it satisfies the following:

(i) $\mathcal{H}$ is a vector space over $\mathbb{C}$ (or $\mathbb{R}$).

(ii) $\mathcal{H}$ is equipped with an inner product $(\cdot, \cdot)$, so that

1. $f \mapsto (f, g)$ is linear on $\mathcal{H}$ for every fixed $g \in \mathcal{H}$

2. $(f, g) = \overline{(g, f)}$

3. $(f, f) \geq 0$ for all $f \in \mathcal{H}$

We let $||f|| = (f, f)^{1/2}$.

(iii) $||f|| = 0$ if and only if $f = 0$.

(iv) The Cauchy-Schwarz and triangle inequalities hold

$$|(f, g)| \leq ||f|| \, ||g|| \quad and \quad ||f + g|| \leq ||f|| + ||g||$$

(v) $\mathcal{H}$ is complete in the metric $d(f, g) = ||f - g||$.

(vi) $\mathcal{H}$ is seperable.

Definition 4.2.2 (Orthogonality) Two element f and g in a Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$ are **orthogonal** or **perpendicular** if $(f, g) = 0$, and we write $f \perp g$.

Proposition 4.2.3 If $f \perp g$, then $||f + g||^2 = ||f||^2 + ||g||^2$.

Proposition 4.2.4 If $\{e_k\}_{k=1}^{\infty}$ is orthonormal, and $f = \sum a_k e_k \in \mathcal{H}$ where the sum is finite, then

$$||f||^2 = \sum |a_k|^2.$$

Theorem 4.2.5 The following properties of an orthonormal set $\{e_k\}_{k=1}^{\infty}$ are equivalent.

(i) Finite linear combinations of elements in $\{e_k\}$ are dense in $\mathcal{H}$.

(ii) If $f \in \mathcal{H}$ and $(f, e_j) = 0$ for all j, then $f = 0$.

(iii) If $f \in \mathcal{H}$, and $S_N(f) = \sum_{k=1}^{N} a_k e_k$, where $a_k = (f, e_k)$, then $S_N(f) \to f$ as $N \to \infty$ in the norm.

(iv) If $a_k = (f, e_k)$, then $||f||^2 = \sum_{k=1}^{\infty} |a_k|^2$

Theorem 4.2.6 Any Hilbert space has an orthonormal basis.

Definition 4.2.7 Give two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ with respective inner products $(\cdot, \cdot)_{\mathcal{H}}$ and $(\cdot, \cdot)_{\mathcal{H}'}$. A mapping $U : \mathcal{H} \to \mathcal{H}'$ between these space is called **unitary** if:

(i) U is linear, that is, $U(\alpha f + \beta g) = \alpha U(f) + \beta U(g)$.

(ii) U is a bijection.

(iii) $||Uf||_{\mathcal{H}'} = ||f||_{\mathcal{H}}$ for all $f \in \mathcal{H}$

Corollary 4.2.8 Any two infinte-dimesional Hilbert spaces are unitarily equivalent.

Corollary 4.2.9 Any two finite-dimensional Hilbert spaces are unitarily equivalent if and only if they have the same dimension.

Definition 4.2.10 **Pre-Hilbert space** is a space $\mathcal{H}_0$ that satisfies all the defining properties of a Hilbert space except (v).

Proposition 4.2.11 Suppose we are given a pre-Hilbert space $\mathcal{H}_0$ with inner product $(\cdot, \cdot)_0$. Then we can find a Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$ such that

(i) $\mathcal{H}_0 \subset \mathcal{H}$.

(ii) $(f, g)_0 = (f, g)$ whenever $f, g \in \mathcal{H}_0$.

(iii) $\mathcal{H}_0$ is dense in $\mathcal{H}$.

4.3 Fourier series and Fatou's theorem

Theorem 4.3.1 Suppose f is integrable on $[-\pi, \pi]$.

(i) If $a_n = 0$ for all n, then $f(x) = 0$ for a.e. x.

(ii) $\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx}$ tends to $f(x)$ for a.e. x, as $r \to 1$, $r < 1$.

In the theorem above, a_n is the n-th Fourier coefficient of f

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Theorem 4.3.2 Suppose $f \in L^2([-\pi, \pi])$. Then:

(1) We have Parseval's relation

References