# Real Analysis: Definitions and Theorems

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## 1 Measure Theory

## 2 Lebesgue Integral

## 3 Differentiation and Integral

## 4 Hilbert Space: An Introduction

### 4.1 $L^2$ space

**Proposition 4.1.1** The Space $L^2(\mathbb{R}^d)$ has the following properties:

**(i)** $L^2(\mathbb{R}^d)$ is a vector space.

**(ii)** $f(x)\overline{g(x)}$ is integrable whenever $f, g \in L^2(\mathbb{R}^d)$, and the Cauchy-Schwarz inequality holds: $|(f,g)| \leq ||f||\ ||g||$.

**(iii)** If $g \in L^2(\mathbb{R}^d)$ is fixed, the map $f \mapsto (f,g)$ is linear in $f$, and also $(f,g) = \overline{(g,f)}$.

**(iv)** The triangle inequality holds: $||f+g|| \leq ||f|| + ||g||$

**Theorem 4.1.2** The space $L^2(\mathbb{R}^d)$ is complete in its metric.

**Theorem 4.1.3** The space $L^2(\mathbb{R}^d)$ is **separable**, int the sense that there exists a countable collection $\{f_k\}$ of elements in $L^2(\mathbb{R}^d)$ such that their linear combinations are dense in $L^2(\mathbb{R}^d)$

### 4.2 Hilbert space

**Definition 4.2.1** A set $\mathcal{H}$ is a **Hilbert Space** if it satisfies the following:

**(i)** $\mathcal{H}$ is a vector space over $\mathbb{C}$ (or $\mathbb{R}$).

**(ii)** $\mathcal{H}$ is equipped with an inner product $(\cdot,\cdot)$, so that

**1.** $f \mapsto (f,g)$ is linear on $\mathcal{H}$ for every fixed $g \in \mathcal{H}$

**2.** $(f,g) = \overline{(g,f)}$

**3.** $(f,f) \geq 0$ for all $f \in \mathcal{H}$

We let $||f|| = (f,f)^{1/2}$.

**(iii)** $||f|| = 0$ if and only if $f = 0$.

**(iv)** The Cauchy-Schwarz and triangle inequalities hold

$$|(f,g)| \leq ||f||\ ||g|| \quad \textit{and} \quad ||f+g|| \leq ||f|| + ||g||$$

**(v)** $\mathcal{H}$ is complete in the metric $d(f,g) = ||f-g||$.

**(vi)** $\mathcal{H}$ is seperable.

**Definition 4.2.2 (Orthogonality)** Two element $f$ and $g$ in a Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ are **orthogonal** or **perpendicular** if $(f,g) = 0$, and we write $f \perp g$.

**Proposition 4.2.3** If $f \perp g$, then $||f+g||^2 = ||f||^2 + ||g||^2$.

**Proposition 4.2.4** If $\{e_k\}_{k=1}^{\infty}$ is orthonormal, and $f = \sum a_k e_k \in \mathcal{H}$ where the sum is finite, then

$$||f||^2 = \sum |a_k|^2.$$

**Theorem 4.2.5** The following properties of an orthonormal set $\{e_k\}_{k=1}^{\infty}$ are equivalent.

**(i)** Finite linear combinations of elements in $\{e_k\}$ are dense in $\mathcal{H}$.

**(ii)** If $f \in \mathcal{H}$ and $(f,e_j) = 0$ for all $j$, then $f = 0$.

**(iii)** If $f \in \mathcal{H}$, and $S_N(f) = \sum_{k=1}^{N} a_k e_k$, where $a_k = (f,e_k)$, then $S_N(f) \to f$ as $N \to \infty$ in the norm.

**(iv)** If $a_k = (f,e_k)$, then $||f||^2 = \sum_{k=1}^{\infty} |a_k|^2$

**Theorem 4.2.6** Any Hilbert space has an orthonormal basis.

**Definition 4.2.7** Give two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ with respective inner products $(\cdot,\cdot)_{\mathcal{H}}$ and $(\cdot,\cdot)_{\mathcal{H}'}$. A mapping $U : \mathcal{H} \to \mathcal{H}'$ between these space is called **unitary** if:

**(i)** $U$ is linear, that is, $U(\alpha f + \beta g) = \alpha U(f) + \beta U(g)$.

**(ii)** $U$ is a bijection.

**(iii)** $||Uf||_{\mathcal{H}'} = ||f||_{\mathcal{H}}$ for all $f \in \mathcal{H}$

**Corollary 4.2.8** Any two infinte-dimesional Hilbert spaces are unitarily equivalent.

**Corollary 4.2.9** Any two finite-dimensional Hilbert spaces are unitarily equivalent if and only if they have the same dimension.

**Definition 4.2.10** **Pre-Hilbert space** is a space $\mathcal{H}_0$ that satisfies all the defining properties of a Hilbert space except (v).

**Proposition 4.2.11** Suppose we are given a pre-Hilbert space $\mathcal{H}_0$ with inner product $(\cdot,\cdot)_0$. Then we can find a Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ such that

**(i)** $\mathcal{H}_0 \subset \mathcal{H}$.

**(ii)** $(f,g)_0 = (f,g)$ whenever $f, g \in \mathcal{H}_0$.

**(iii)** $\mathcal{H}_0$ is dense in $\mathcal{H}$.

## 4.3 Fourier series and Fatou's theorem

**Theorem 4.3.1** Suppose $f$ is integrable on $[-\pi,\pi]$.

**(i)** If $a_n = 0$ for all $n$, then $f(x) = 0$ for a.e. x.

**(ii)** $\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx}$ tends to $f(x)$ for a.e. x, as $r \to 1$, $r < 1$.

In the theorem above, $a_n$ is the n-th Fourier coefficient of $f$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

**Theorem 4.3.2** Suppose $f \in L^2([-\pi,\pi])$. Then:

**(i)** We have Parseval's relation

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|dx$$

.

**(ii)** The mapping $f \mapsto \{a_n\}$ is a unitary correspondence between $L^2([-\pi,\pi])$ and $l^2(\mathbb{Z})$.

**(iii)** The Fourier series of $f$ converges to $f$ in the $L^2$-norm, that is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2 dx \to 0 \quad as\ N \to \infty$$

where $S_N(f) = \sum_{|n|\le N} a_n e^{inx}$.

**Definition 4.3.3** If $F$ is a function defined in the unit disc $\mathbb{D}$, we say that $F$ has a radial limit at the point $-\pi \leq \theta \leq \pi$ on the circle, if the limit

$$\lim_{r<1,\ r\to 1} F(re^{i\theta})$$

exists.

**Theorem 4.3.4** A bounded holomorphic function $F(re^{i\theta})$ on the unit disc has radial limits at almost every $\theta$.

**Definition 4.3.5** We define the **Hardy Space** $H^2(\mathbb{D})$ to consist of all holomorphic functions $F$ on the unit disc $\mathbb{D}$ that satisfy

$$\sup_{0\leq r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{i\theta})|^2 d\theta < \infty$$

we also define the "norm" for functions $F$ in this class, $||F||_{H^2(\mathbb{D})}$, to be the square root of the above quantity.

## 4.4 Closed subspaces and orthogonal projections

**Definition 4.4.1** A **linear subspace** $\mathcal{S}$ of $\mathcal{H}$ is a subset of $\mathcal{H}$ that satisfies $\alpha f + \beta g \in \mathcal{S}$ whenever $f, g \in \mathcal{S}$ and $\alpha$, $\beta$ are scalars. The subspace $\mathcal{S}$ is **closed** if whenever $\{f_n\} \subset \mathcal{S}$ converges to some $f \in \mathcal{H}$, then $f$ also belongs to $\mathcal{S}$.

**Lemma 4.4.2** Suppose $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $f \in \mathcal{H}$. Then:

**(i)** There exists a (unique) element $g_0 \in \mathcal{S}$ which is closest to $f$, in the sense that

$$||f - g_0|| = \inf_{g\in\mathcal{S}} ||f - g||$$

**(ii)** The element $f - g_0$ is perpendicular to $\mathcal{S}$, that is,

$$(f - g_0, g) = 0 \quad for\ all\ g \in \mathcal{S}$$

**Definition 4.4.3** If $\mathcal{S}$ is a subspace of a Hilbert space $\mathcal{H}$, we define the orthogonal complement of $\mathcal{S}$ by

$$\mathcal{S}^{\perp} = \{f \in \mathcal{H} : (f, g) = 0 \quad for\ all\ g \in \mathcal{S}\}$$

**Proposition 4.4.4** If $\mathcal{S}$ is a closed subspace of a Hilbert space $\mathcal{H}$, then

$$\mathcal{H} = \mathcal{S} \bigoplus \mathcal{S}^{\perp}$$

**Definition 4.4.5** The mapping $P_{\mathcal{S}}$ is called the orthogonal projection onto $\mathcal{S}$ and satisfies the following simple properties:

**(i)** $f \mapsto P_{\mathcal{S}}(f)$ is linear.

**(ii)** $P_{\mathcal{S}}(f) = f$ whenever $f \in \mathcal{S}$.

**(iii)** $P_{\mathcal{S}}(f) = 0$ whenever $f \in \mathcal{S}^{\perp}$.

**(iv)** $||P_{\mathcal{S}}(f)|| \leq ||f||$ for all $f \in \mathcal{H}$.

## 4.5 Linear transformation

**Definition 4.5.1** Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces. Amapping $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear transformation (also called linear operator or operator) if

$$T(af + bg) = aT(f) + bT(g) \quad for\ all\ scalars\ a, b\ and\ f, g \in \mathcal{H}_1$$

We can also say that a linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ us bounded if there exists $M > 0$ so that

$$||T(f)||_{\mathcal{H}_2} \leq M||f||_{\mathcal{H}_1}$$

The norm of $T$ is denoted by $||T||_{\mathcal{H}_1 \to \mathcal{H}_2}$ and defined by

$$||T|| = \inf M$$

**Lemma 4.5.2** $||T|| = \sup\{|(Tf, g)| : \ ||f|| \leq 1, \ ||g|| \leq 1\}$, where of course $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$.

**Definition 4.5.3** A linear transformation $T$ is continous if $T(f_n) \to T(f)$ whenever $f_n \to f$.

**Proposition 4.5.4** A linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded if and only if it is continuous.

**Definition 4.5.5** A **linear functional** $l$ is a linear transformation from Hilbert space $\mathcal{H}$ to the underlying field of scalars, which we may assume to be the complex numbers,

$$l : \mathcal{H} \to \mathbb{C}$$

**Definition 4.5.6** Let $l$ be a continuous linear functional on a Hilbert space $\mathcal{H}$. Then, there exists a unique $g \in \mathcal{H}$ such that

$$l(f) = (f, g) \quad for\ all\ f \in \mathcal{H}$$

Moreover, $||l|| = ||g||$

**Proposition 4.5.7** Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear transformation. There exists a unique bounded linear transformation $T^*$on $\mathcal{H}$ so that:

**(i)** $(Tf, g) = (f, T^*, g)$.

**(ii)** $||T|| = ||T^*||$.

**(iii)** $(T^*)^* = T$

The linear operator $T^* : \mathcal{H} \to \mathcal{H}$ satisfting the above conditions is called the **adjoint** of $T$.

**Definition 4.5.8** Suppose $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Then, a linear transformation $T : \mathcal{H} \to \mathcal{H}$ is said to be diagonized with respect to the basis $\{\varphi_k\}$ If

$$T(\varphi_k) = \lambda_k \varphi_k, \quad where\ \lambda_k \in \mathbb{C}\ for\ all\ k.$$

# References