[CM8]

To find the MAP estimator, we use Baye's rule to write :

$$P(\lambda | x_1, \ldots x_N) = \frac{P(x_1, \ldots x_N | \lambda)\, P(N)}{P(x_1, \ldots x_N)}$$

From part 1, we know we can write :

$$P(x_1, \ldots, x_N | \lambda) = \prod_{i=1}^{N} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

and we are given that the prior distribution of λ is a normal distribution with known mean μ & known variance σ^2.

$$\lambda \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(\lambda-\mu)^2}{2\sigma^2}}$$

Hence,

$$P(\lambda) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(\lambda-\mu)^2}{2\sigma^2}}$$

Thus, we desire to find the value of λ which maximizes:

$$P(\lambda | x_1 \dots x_N) = \frac{\left(\prod_{i=1}^{N} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}\right) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}}{C}$$

where $C = P(x_1, \dots, x_N)$

We note that because we are simply looking for the value of λ that maximizes this expression, we can take the log of both side and write:

$$\ln(P(\lambda | x_1, \dots x_N)) =$$

$$\left(-N\lambda + \ln\lambda \sum_{i=1}^{N} x_i - \ln \prod_{i=1}^{N} x_i!\right) - \ln\sqrt{2\pi\sigma^2} - \frac{(\lambda-\mu)^2}{2\sigma^2}$$

Taking derivative with respect to λ, we have:

$$\frac{\partial \ln(P(\lambda | x_1, \dots x_N))}{\partial \lambda} = -N + \frac{\sum_{i=1}^{N} x_i}{\lambda} - \frac{(\lambda-\mu)}{\sigma^2}$$

Setting this equal to zero, we have:

$$0 = -N + \frac{\sum_{i=1}^{N} x_i}{\lambda} - \frac{(\lambda - \mu)}{\sigma^2}$$

$$\frac{\lambda - \mu}{\sigma^2} = \frac{\sum_{i=1}^{N} x_i - N\lambda}{\lambda}$$

$$\lambda(\lambda - \mu) = \sigma^2 \sum_{i=1}^{N} x_i - N\lambda\sigma^2$$

$$\lambda^2 - \mu\lambda + N\lambda\sigma^2 = \sigma^2 \sum_{i=1}^{N} x_i$$

$$\boxed{\lambda^2 + \lambda(N\sigma^2 - \mu) - \sigma^2 \sum_{i=1}^{N} x_i = 0}$$

Let $c = -\sigma^2 \sum_{i=1}^{N} x_i$

$$\Rightarrow \quad \lambda^2 + (N\sigma^2 - \mu)\lambda + c = 0$$

$$\lambda = \frac{-(N\sigma^2 - \mu) \pm \sqrt{(N\sigma^2 - \mu)^2 - 4(1)c}}{2}$$

We know the rate λ cannot be negative

$$\Rightarrow \quad \lambda = \frac{1}{2}\left[-(N\sigma^2 - \mu) + \sqrt{(N\sigma^2-\mu)^2 - 4c}\right.$$

$$\lambda = \frac{1}{2}\left[-(N\sigma^2-\mu) + \sqrt{N^2\sigma^4 + \mu^2 - 2N\sigma^2\mu - 4c}\right]$$

Substituting for $c = -\sigma^2 \sum_{i=1}^{N} x_i$

$$\boxed{\lambda = \frac{1}{2}\left[(\mu - N\sigma^2) + \sqrt{N^2\sigma^4 + \mu^2 - 2N\sigma^2\mu + 4\sigma^2 \sum_{i=1}^{N} x_i}\right]}$$