# Betatron Resonances with Space Charge

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**Abstract.** The point is made that betatron resonances do not occur at the incoherent value of the tune, but rather at the frequencies of the appropriate collective modes. This has important implications not only for the design of high intensity, low energy proton synchrotrons, but also for the interpretation of machine studies at existing synchrotrons of this type.

## I INTRODUCTION

In many papers, in proceedings of accelerator schools, and even in textbooks on accelerator physics, we read that the linear part of the space charge force is added to the linear equation of motion, leading to a tune shift, which if large enough can place individual particles on low-order betatron resonance lines. This picture, though in some sense compelling, is misleading and inhibits understanding the transverse intensity limit in low energy proton synchrotrons.

*A Few Historical Notes:* The fact that the incoherent tune is irrelevant when investigating integer resonances was first emphasized in 1962 by Morin [1]. It was re-emphasized by Lapostolle [2] in 1963. However, neither of these two mentioned at the time that the same sort of reasoning applied to higher order resonances. L. Smith [3], in a seminal paper also published in 1963, first pointed out that half-integer, or 'quadrupole' resonances do not occur at the incoherent tune either, and that "a machine designed conservatively regarding aperture and injector emittance could easily handle more beam than the usual space charge limit". However, his analysis was only for round isotropic beams (i.e. the same tune and emittance in each transverse direction). His student F. Sacherer in his Ph.D. thesis [4] (1968) extended the envelope analysis to non-round, anisotropic beams. As well, Sacherer treated the general case for any order resonance in one dimension, for the idealized distribution which gives exactly linear space charge force. This was extended to two-dimensional round isotropic beams by R. Gluckstern [5] (1970). It was further extended to non-round, anisotropic beams by I. Hofmann [8] (1998). These last two did not use the approximation that the tune shift be small compared with the tune, and so are applicable to space charge dominated beams in linacs and transport lines as well.

The case of general distribution is a very difficult problem even for small tune shift. Many people have proposed non-self-consistent models which ignore the non-linear forces inherent in the (non-KV) stationary distribution, and therefore miss an essential feature, namely, Landau damping. Recently, S. Lund and R. Davidson [6] have made good theoretical progress in this direction.

Naturally, the problem can also be studied with computer simulations. This was done by I. Hofmann [9] (1985), who simulated integer and 1/4-integer resonances. He found that neither of these occurred at the incoherent tune. As well, simulations by Machida have firmly established the non-relevance of the maximum incoherent tune shift to the half-integer (quadrupole) resonance [10] (1991) and to the third-integer (sextupole) resonance [19] (1998).

At least two machines have been investigated recently with a view to understanding the space charge limits, namely, the CERN PS [21] and the PSR [22]. In both of these cases, the space charge limits were found to exceed the condition that the incoherent tunes lie far from the lowest order resonances.

Lastly, as emphasized by Rees and Prior [20], effects of images can cloud the issue by contributing driving terms which make it appear that the beam is responding to incoherent resonances.

## II THEORY

In the following analysis, we first ignore the forces due to image charges, and include only the 'direct' space charge term. In terms of locating resonance frequencies, this approximation is valid in proton synchrotrons whose injection energy is less than a few GeV. For example, the approximation is good in the AGS right up to the extraction energy of 30 GeV, since as energy increases, making images relatively more important, the beam size also shrinks, making the direct space charge relatively more important.

We will also be ignoring synchrotron motion. In principle, the results are correct only for coasting beams, but in practice they are applicable if the synchrotron tune is small compared with the space charge tune shift.

*Incorrect theory:* One often thinks of transverse space charge as just another force $F_{\rm sc}$:

$$x'' + \nu_{0x}^2 x + F_{\rm sc} = F(x,\theta) \tag{1}$$

where $\nu_{0x}$ is the unperturbed tune and $F(x,\theta)$ represents lattice errors. Then for linear space charge, $F_{\rm sc} = -\alpha x$ and

$$x'' + (\nu_{0x}^2 - \alpha)x = F(x,\theta) \tag{2}$$

or,

$$x'' + \nu_{{\rm i}x}^2 x = F(x,\theta) \tag{3}$$

and the new tune $\nu_{\mathrm{i}x}$ is shifted downward from the bare tune by the *space charge tune shift* $\Delta\nu_x = \alpha/(2\nu_{0x})$. (We use the sign convention where space charge tune shifts are positive quantities and so the incoherent tune is $\nu_{\mathrm{i}} = \nu_0 - \Delta\nu$.) If desired, nonlinear space charge terms can be added to $F(x, \theta)$, to treat the case of non-KV distributions. The resulting equation of motion can be analyzed using the well-known tools that have been developed for betatron resonances.

But this approach is incorrect. Forces arising from the beam itself are not the same as external forces. As will be shown, any theory which treats the two types of forces in the same way is incorrect and will make incorrect predictions. One often hears it said that although this approach is not self-consistent, it is at least correct for the onset of a betatron resonance. This also is incorrect. It proceeds from the false notion that particles react instantaneously according to their particular (incoherent) frequencies. Actually, particles with different tunes react differently only after a number of turns approximately equal to the reciprocal of the tune difference. Over short time scales, particles respond together; and since they do, the looked-for incoherent motion never gets started. Therefore, the response of a beam to a perturbation proceeds from the more collective to the less collective, not the other way around. In other words, it is the frequencies of the collective modes which are relevant, not the incoherent frequencies.

## A Integer Resonance

One of the things that equation 2 leaves out is the fact that space charge forces are centred on the beam, not on the reference orbit through the optics elements. Therefore,

$$x'' + \nu_{0x}^2 x = \alpha(x - \bar{x}) + F(\theta) \tag{4}$$

where $\bar{x}$ is the $x$-coordinate of the centre of charge and $F(\theta)$ is now independent of $x$, as appropriate for the integer resonance case. We take the average and find

$$\bar{x}'' + \nu_{0x}^2 \bar{x} = F(\theta). \tag{5}$$

Thus **the motion of the centre of charge is not affected by space charge** self-forces. This is called the 'coherent' motion. Clearly, this coherent motion will be unstable if the coherent tune, $\nu_{0x}$ (equal here to the bare tune since we are neglecting image charges), is equal to an integer. Or, another way of saying the same thing is that as the coherent tune approaches an integer, the closed orbit distortion increases without limit.

Now to find the 'incoherent' motion, we simply subtract eqn. 5 from eqn. 4.

$$(x - \bar{x})'' + \nu_{0x}^2 (x - \bar{x}) = \alpha(x - \bar{x}) \tag{6}$$

or

$$(x-\bar{x})'' + \nu_{\mathrm{i}x}^2 (x-\bar{x}) = 0 \tag{7}$$

where, as before, $\nu_{\mathrm{i}x}$ is the incoherent tune. Notice that the incoherent equation of motion contains no driving terms for the integer resonance. Therefore, **incoherent motion is not affected by dipole errors**. This means that the incoherent tune can be equal to an integer, with no adverse effects. Conceptually, this is the biggest hurdle to leap. It is therefore worth re-emphasizing in different words: A particle which is shifted by direct space charge to a tune of exactly an integer, turn after turn sees the same dipole errors at the same betatron phase, and yet is not even slightly affected compared with other particles which do not have integer tune. This is not due to space charge stabilizing the resonance, as claimed in ref. [11], since in this completely linear case there is no incoherent tune spread to generate Landau damping. There just isn't any driving term for the incoherent motion.

Although we derived the coherent and incoherent equations of motion for linear space charge, they remain true in the general case, following directly from Newton's third law. For particle $i$, the equation of motion is

$$x_i'' + \nu_{0x}^2 x_i = F_{\mathrm{sc}i} + F(\theta) \tag{8}$$

$F_{\mathrm{sc}i} = \sum_j F_{ij}$, $F_{ij}$ being the force on particle $i$ by particle $j$. Since $F_{ij} = -F_{ji}$, when we sum eqn. 8 over $i$ to obtain the coherent motion, we recover eqn. 5. Subtracting, we now find for the incoherent motion

$$(x_i-\bar{x})'' + \nu_{0x}^2 (x_i-\bar{x}) = F_{\mathrm{sc}i} \tag{9}$$

Since the space charge force is now nonlinear, this cannot be reduced to a new simple harmonic equation with a shifted frequency. Nevertheless, it retains the feature that the incoherent motion does not 'see' dipole errors.

## B Half-Integer Resonance

*Incorrect theory:* Now let us investigate errors of the type $F(x,\theta) = xf(\theta)$. The equation of motion is

$$x'' + \nu_{0x}^2 x = \alpha(x-\bar{x}) + xf(\theta) \tag{10}$$

Coherent motion is found by taking the average

$$\bar{x}'' + \nu_{0x}^2 \bar{x} = \bar{x}f(\theta), \tag{11}$$

and to find the equation for incoherent motion, we again subtract:

$$(x-\bar{x})'' + \nu_{\mathrm{i}x}^2 (x-\bar{x}) = (x-\bar{x})f(\theta). \tag{12}$$

This looks exactly like a single particle equation of motion in the presence of a half-integer resonance driving term. Again, **this approach is incorrect**. The response of a beam to linear errors is to modulate in size, but the beam size is contained in the space charge term $\alpha$. The correct self-consistent approach is to formulate the problem directly in terms of the beam size, or 'envelope'.

In the following analysis, we first assume that the beam has the correct distribution to give only linear space charge forces. However, the results are not restricted to such special distributions, as will be pointed out later in our discussion. No originality is claimed for the analysis, as it can all be found in Sacherer's thesis [4].

## 1 *One Dimension*

We start with the 1-dimensional case, since it exhibits the features we want to emphasize, unencumbered by other considerations such as beam aspect ratio and tune split. For this case the distribution which gives linear space charge is uniform in configuration space between the beam edges at $\pm\hat{x}$.

It can be shown that $\alpha \propto 1/\hat{x}$ and so from the equation of motion (10), it is clear that

$$\alpha = 2\nu_0 \Delta\nu a/\hat{x} \tag{13}$$

where $\Delta\nu$ is the incoherent tune shift, and $a$ is $\sqrt{\beta\epsilon}$, the unperturbed beam size. In terms of the normalized beam size $\tilde{x} = \hat{x}/a$, it can be shown that the envelope equation is

$$\tilde{x}'' + \nu_0^2 \tilde{x} - \frac{\nu_0^2}{\tilde{x}^3} = 2\nu_0\Delta\nu + f(\theta)\tilde{x}. \tag{14}$$

Keeping in mind that $\Delta\nu \ll \nu_0$, we see that the stationary solution to this equation is

$$\tilde{x} = 1 + \frac{\Delta\nu}{2\nu_0} \tag{15}$$

and this can be seen as a change in the $\beta$-function due to space charge: $\beta_x = \beta_{x0}\nu_0/(\nu_0 - \Delta\nu)$.

Rewriting the envelope equation in terms of small perturbations $\delta$ with respect to the stationary solution, i.e. $\tilde{x} = 1 + \frac{\Delta\nu}{2\nu_0} + \delta$, we find

$$\delta'' + (4\nu_0^2 - 6\nu_0\Delta\nu)\delta = f(\theta). \tag{16}$$

Clearly, this is unstable for the $n^{\mathrm{th}}$ Fourier component of $f(\theta)$ when $4\nu_0^2 - 6\nu_0\Delta\nu = n^2$, i.e., when

$$\frac{n}{2} = \nu_0 - \frac{3}{4}\Delta\nu = \nu_{\mathrm{i}} + \frac{1}{4}\Delta\nu. \tag{17}$$

So the incoherent tune $\nu_{\mathrm{i}} = \nu_0 - \Delta\nu$ can be depressed beyond the half-integer by a quarter of the space charge tune shift! Again, this is not due to a 'stabilizing' or 'self-limiting' effect of space charge, but to the fact that betatron resonance occurs at the appropriate collective mode frequency and not at the incoherent frequency.

## 2 Two Dimensions

Linear space charge is provided by the KV distribution: this is a uniformly-filled ellipse in any 2-dimensional projection, and an ellipsoidal shell in the 4 phase space dimensions. Kapchinsky and Vladimirsky [12] showed that the space charge force coefficient $\alpha$ is

$$\alpha_x \propto \frac{1}{\hat{x}(\hat{x}+\hat{y})}, \text{ and } \alpha_y \propto \frac{1}{\hat{y}(\hat{x}+\hat{y})}. \tag{18}$$

By comparison with eqn. 10, it is therefore clear that in terms of incoherent tune shifts,

$$\alpha_x = 2\nu_{0x}\Delta\nu_x \, \frac{a}{\hat{x}} \frac{a+b}{\hat{x}+\hat{y}} \tag{19}$$

$$\alpha_y = 2\nu_{0y}\Delta\nu_y \, \frac{b}{\hat{y}} \frac{a+b}{\hat{x}+\hat{y}} \tag{20}$$

where $a = \sqrt{\beta_x\epsilon_x}$ and $b = \sqrt{\beta_y\epsilon_y}$ are the unperturbed beam sizes.

The envelope equations are the well-known KV equations, here written in terms of tune shifts and the normalized beam sizes $\tilde{x} = \hat{x}/a$, $\tilde{y} = \hat{y}/b$:

$$\tilde{x}'' + \nu_{0x}^2\tilde{x} - \frac{\nu_{0x}^2}{\tilde{x}^3} = 2\nu_{0x}\Delta\nu_x \, \frac{a+b}{a\tilde{x}+b\tilde{y}} \tag{21}$$

$$\tilde{y}'' + \nu_{0y}^2\tilde{y} - \frac{\nu_{0y}^2}{\tilde{y}^3} = 2\nu_{0y}\Delta\nu_y \, \frac{a+b}{a\tilde{x}+b\tilde{y}}. \tag{22}$$

For clarity, we have dropped the gradient error driving terms. As a check, we find by direct substitution that to first order in smallness of $\Delta\nu/\nu$, $\tilde{x} = 1 + \Delta\nu_x/(2\nu_{0x})$ and $\tilde{y} = 1 + \Delta\nu_y/(2\nu_{0y})$, again reflecting the incoherent effect of space charge on the $\beta$-functions.

As before, we can linearize for small perturbations $\delta_x$ and $\delta_y$, i.e. $\tilde{x} = 1 + \frac{\Delta\nu_x}{2\nu_{0x}} + \delta_x$ and $\tilde{y} = 1 + \frac{\Delta\nu_y}{2\nu_{0y}} + \delta_y$.[1] The result is two coupled simple harmonic oscillators. We can find the eigenfrequencies ('eigentunes') of the system, but the general expression is complicated and unenlightening. For the case of a round beam ($a = b$, $\nu_{0x}\Delta\nu_x = \nu_{0y}\Delta\nu_y$), the frequencies are

$$\nu^2 = 2\nu_{0x}^2 + 2\nu_{0y}^2 - 5\nu_{0x}\Delta\nu_x \pm \sqrt{(2\nu_{0x}^2 - 2\nu_{0y}^2)^2 + (\nu_{0x}\Delta\nu_x)^2}. \tag{23}$$

If the tune split is small, i.e. $|\nu_{0x} - \nu_{0y}| \ll \Delta\nu_x/4$, we find two distinct eigenmodes:

$$\nu^2 = \begin{cases} 4\bar{\nu}^2 - 4\nu_{0x}\Delta\nu_x \\ 4\bar{\nu}^2 - 6\nu_{0x}\Delta\nu_x \end{cases} \tag{24}$$

[1] This is not as straightforward as Sacherer implies, since the expansion must be made with the correct order of smallness, namely, $1 \gg \Delta\nu/\nu \gg \delta$.

($2\bar{\nu}^2 = \nu_{0x}^2 + \nu_{0y}^2$), while if the tune split is not small compared with the tune shift, the two modes are

$$\nu^2 = \begin{cases} 4\nu_{0x}^2 - 5\nu_{0x}\Delta\nu_x \\ 4\nu_{0y}^2 - 5\nu_{0x}\Delta\nu_x \end{cases} \tag{25}$$

The physical interpretation is straight-forward. In the small-split case, the two transverse motions are tightly coupled together, and a gradient error in either transverse plane can drive either mode. One mode is symmetric, envelope modulations are in phase so the beam 'breathes' in both directions together; the other mode is antisymmetric, envelope modulations are 180° out of phase. In the large-split case, the envelope modulations in $x$ and $y$ cannot stay in phase, so they are essentially decoupled and act independently.

Resonance occurs for integer values of the eigentune, i.e. $\nu = n$ where $n$ corresponds to the Fourier component of the driving gradient error. All the resonance conditions can be cast into the form

$$\nu_{0x,y} - C\Delta\nu_{x,y} = n/2. \tag{26}$$

For the small-split case, $C = 1/2, 3/4$ (resp. for symmetric and anti-symmetric), and for the large-split case, $C = 5/8$. The analysis can be repeated for non-round beams. For example with $a = 2b$, $C_x = 7/12$ and $C_y = 2/3$ in the large-split case, and $C = 0.273, 0.684$ (sym., anti-sym., $\Delta\nu_{x,y}$ set equal to the larger of the two incoherent tune shifts) in the tightly-coupled case $\nu_{0x} \approx \nu_{0y}$. In the incorrect theory based upon the incoherent tune, the resonance condition in eqn. 26 is $C = 1$.

Those more familiar with formulas for space charge modes as they have been derived for linacs and transport lines (see e.g. Ref. [7, eqn. 10]) will remember the following envelope mode frequencies for isotropic focusing:

$$\nu^2 = \begin{cases} 2\bar{\nu}^2 + 2\nu_{\mathrm{i}}^2 \\ \bar{\nu}^2 + 3\nu_{\mathrm{i}}^2 \end{cases} \tag{27}$$

With the substitution $\nu_{\mathrm{i}} = \bar{\nu} - \Delta\nu$, these are seen to be consistent with eqn. 24. In fact, these are more accurate than eqn. 24, as they do not require $\Delta\nu \ll \nu_0$.

The foregoing analysis is not applicable to coupling resonances, since the envelope equations assume no coupling i.e. no $xy$, $xy'$, etc. correlations. The general case has been treated by Hofmann [8]. There are 4 resonance modes: these can be identified with the betatron resonances $\nu_x = N/2$, $\nu_y = N/2$, $\nu_x - \nu_y = N$, and $\nu_x + \nu_y = N$.

## C Higher Order Resonances

### 1 One Dimension

We have seen that integer resonances can be investigated using the equations of motion of the first moments, and half-integer resonances can be investigated using the equations of motion of the second moments. In general, higher order resonances require the simultaneous solution of the equations of motion of the corresponding higher order beam moments. These equations are not easily solved. However, for the simple case of 1-D motion and linear space charge, the general Vlasov equation can be solved.

For the case of longitudinal motion in a bunch whose length is large compared with the conducting beam pipe, the eigenfrequencies are well-known [17]. In that case, the internal space charge force is proportional to $\hat{x}^{-3}$. In the present case, the space charge force is proportional to $\hat{x}^{-1}$, and so the eigenfrequencies are not the same. However, all other aspects, such as the doubly-infinite number of modes, and the qualitative shapes of these modes, are similar. Sacherer's findings [4] can be summarized in the resonance condition

$$n/m = \nu_0 - C_{mk}\Delta\nu, \tag{28}$$

with the matrix $C$ given in Table 1. There are two indices: $m$ for azimuthal and $k$ for radial. For example, the $(1, 3)$ eigenmode is the first harmonic azimuthally (dipole) and has 2 nodes radially. It is the first non-rigid dipole mode, since only odd-$k$ modes exist for any odd $m$, and only even for even. In general, the mode $(m, k)$ is driven by error driving terms proportional to $x^{k-1}\cos(n\theta)$ where $n \approx m\nu$.

**TABLE 1.** Coherent mode coefficients $C_{mk}$ for 1-D transverse space charge

| | $m$=1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $k$=1 | 0 | | | | |
| 2 | | 3/4 | | | |
| 3 | 9/8 | | 7/8 | | |
| 4 | | 17/16 | | 59/64 | |
| 5 | 65/64 | | 133/128 | | 121/128 |

This agrees with the results already found for the rigid dipole and quadrupole modes in 1-D, namely, $C_{11} = 0$ and $C_{22} = 3/4$.

Remarkably, the frequencies of the non-rigid modes ($k > m$) are shifted *past* the incoherent frequency. However, as explained below, these modes are not seen in simulations, and there is good reason to believe they are not important. Realistic distributions never have perfectly linear space charge and the frequency spread brings with it Landau damping. It is known that in the longitudinal case, Landau thresholds depend mostly upon $k$ [18]; for example, if the lowest sextupole mode is Landau damped, the first non-rigid dipole mode is as well. Recent work by Lund and Davidson [6] using a warm fluid model for the transverse case indicates that these modes are associated with the unphysical aspects of the ideal linear space charge distribution.

## 2 *Two Dimensions*

As already stated, Hofmann [8] has treated the general case up to octupole. It is difficult to make a concise summary of his results because of the additional parameters; tune split and emittance ratio. However, he gives simplified formulas for the case of round beam, equal tunes in both planes. The coefficients $C_{mk}$ extracted from these formulas are shown in Table 2.

**TABLE 2.** Coherent mode coefficients $C_{mk}$ for 2-D round beam

| | $m$=1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $k$=1 | 0 | | | |
| 2 | | 1/2, 3/4 | | |
| 3 | 3/4, 5/4 | | 3/4, 11/12 | |
| 4 | | 7/8, 5/4 | | 13/16, 7/8, 31/32 |

Lund and Davidson [6] (LD) give a formula for the simplest of these round beam modes, namely those corresponding to round (symmetric) perturbations with $m = k$. Cast into the form of eqn. 28, their eqn. 103 becomes:

$$C_{mm,\mathrm{sym}} = 1 - \frac{2}{m^2} \tag{29}$$

Since this is for symmetric perturbations, it only applies to even $m$. We see that it agrees with those of Table 2 for $m = 2, 4$. As with the 1-D case, some of the modes in Table 2 are largely irrelevant for realistic distributions. LD argue that the non-appearance of the mode $(m, k) = (2, 4)$ with $C = 5/4$ in their warm-fluid model is due to the non-physical nature of the KV distribution. Hofmann mentions that this mode, which is intrinsically unstable for $\nu/\nu_0 < 0.24$, changes the phase space density of the beam by only a negligible amount beyond this threshold.

Of the other modes in Table 2, it is not clear at this point which are the important ones. Likely, only the diagonal ($m = k$) ones are.

# D Nonlinear space charge

The foregoing was for the case of ideal distributions which give linear space charge. Originally, it seemed that the general case could only be treated with the Vlasov equation. However, in 1971 Sacherer [13] proved that the envelope equations apply to any distribution, provided that the beam parameters are replaced by the appropriate statistical quantities: the beam boundary $\hat{x}$ becomes the rms size and the emittance becomes the rms emittance. Although the resulting equations are still correct, they are no longer closed, since rms emittance is not a conserved quantity in the presence of nonlinear forces. In practice, however, this turns out to only cause difficulty if one wishes to study the evolution of an unstable distribution. Starting with an arbitrary stationary distribution, and approaching a betatron resonance, the envelope equations still correctly predict thresholds. In fact, Sacherer began to study the possibility of more general KV equations because simulations performed by Lapostolle [14] indicated that these equations very accurately described the evolution of rms beam sizes even for non-KV distributions.

This concept of 'linear-space-charge-equivalent beam' (= 'KV-equivalent beam' in 2-D) has been used in studies of higher order resonances as well. It is well-established in the case of space charge dominated beams in transport channels. (See for example, Struckmeier et al. [15].) On the other hand, it has been largely ignored in studies of space charge effects in synchrotrons.

Nonlinear space charge implies that the tune shift is no longer a single quantity, but is a function of amplitude; particles of smaller amplitude have in general a larger tune shift than large-amplitude particles. An important, but often overlooked, implication of the universality of the envelope equations is that the relevant tune shift is not the maximum shift but rather the tune shift of the linear-space-charge-equivalent beam. For example, in the 2-D case, the beam envelope of the KV distribution is twice the rms size and the 100% emittance is 4 times the rms emittance. Therefore, to obtain the tune shift relevant for half-integer resonances, one can use the usual (incoherent) tune shift formula [16]

$$\Delta\nu_x = \frac{r_{\mathrm{p}} N G}{2\pi\beta^2\gamma^3\epsilon_x}, \tag{30}$$

provided that the form factor $G$ is set equal to 1, and the emittance is interpreted as 4 times the rms emittance, i.e. $\epsilon_x = 4\sqrt{\overline{x^2}\,\overline{x'^2} - \overline{xx'}^2}$. It is easily shown that with this definition of emittance, the value of $G$ for maximum incoherent tune in the Gaussian case is 2. Therefore, with quadrupole errors, the maximum incoherent tune shift can exceed that needed for coincidence with a half-integer resonance by as much as a factor of 3.2; a factor of 2 for the Gaussian beam and a further factor of 8/5 to take into account the location of the envelope resonance in the large-split case.

# III SIMULATIONS

## A One Dimension

We performed simulations with up to 50,000 particles and the following equation of motion;

$$x'' + \nu_0^2 x = \alpha x^{m-1}\cos(n\theta) + F_{\mathrm{SC}}. \tag{31}$$

The derivatives are with respect to $\theta$; $m$ and $n$ are integers.

For no space charge, the particle experiences resonant growth when

$$m\nu_0 \approx n. \tag{32}$$

The smaller is the strength $\alpha$, the more accurately does this resonance condition have to be met to see an effect.

The space charge force on the $i^{\mathrm{th}}$ particle in the simulations is simply equal to an intensity parameter multiplied by the difference between the number of particles to its left and to its right. Physically, this corresponds to particles being in the form of planes in free space. A distribution which results in exactly linear space charge force is that which is uniform in $x$. This is the analogue of the 'Kapchinsky-Vladimirsky' distribution for 2-D.

GIF animations of the simulations can be found at

`http://decu10.triumf.ca:8080/ht/`

In each of the animations the intensity is changing slowly to sweep first the incoherent tune through the resonance, and then the coherent tune. This closely mimics the situation in rings which accumulate a large tune shift over many turns. Since animation snapshots are taken at intervals of $m$ turns, it is easy to verify that nothing happens when the incoherent tune is equal to $n/m$: for the stationary distribution, all the particles look as though they have stopped rotating in phase space and, even though the error driving term is significant, there is no hint of amplitude growth. However, when the frequency of the coherent mode $m$ is equal to $n/m$, one can see that the $m$-fold distortion of the beam boundary is stationary. In that case, with an appropriate driving term, the amplitude of the distortion grows steadily. For example, for $m = 2$, $\nu_{\mathrm{i}} = \frac{n}{2} - \frac{1}{4}\Delta\nu$ (i.e. $\nu_0 = \frac{n}{2} + \frac{3}{4}\Delta\nu$), one sees a growing mismatch which is at a constant orientation in phase space, and individual particles moving around the edge of the mismatched phase space ellipse at a rate equal to the difference between the coherent quadrupole mode tune and the incoherent tune ($= \frac{n}{2} - \nu_{\mathrm{i}} = \frac{1}{4}\Delta\nu$).

We have not been able to find the coherent modes with $k \neq m$.

For any other distribution, particles at different betatron amplitudes rotate in phase space at different rates. We have run simulations for the Gaussian distribution. As the incoherent frequency of the small amplitude particles reach the resonance, nothing happens. As the intensity continues to increase, a stage is reached where there is a barely discernable amplitude growth at an amplitude which contains roughly 50% of the particles. For larger intensities yet, the incoherent tune of the particles in the tail of the Gaussian coincide with the resonance, and show a dramatic emittance increase along with an appearance of an $m$-fold symmetric island structure. In summary, we see the core is affected only by coherent core modes, and the tail affected by incoherent resonance. The reason for this behaviour is that halo particles feel space charge forces from the core of the beam, and these act as external forces because they depend mainly upon the number of particles in the core and not so much on the shape of the core. This picture qualitatively bridges the two (seemingly contradictory) concepts of coherent space charge modes on the one hand, and the core-halo modes on the other.

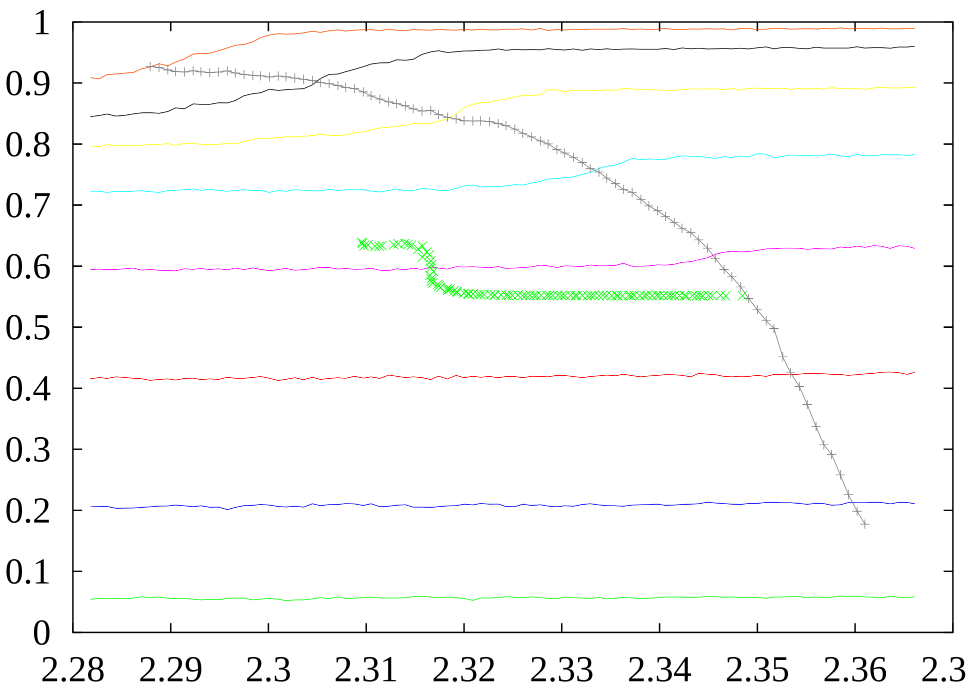


**FIGURE 1.** The plot contains 3 types of curves all plotted against the incoherent tune of the equivalent stationary distribution (ITESD). The data are from simulations in which the bare tune is 2.45 and the intensity is slowly raised to bring the incoherent tune past the 7/3 resonance; so the curves should be thought of as proceeding from right to left. The thick curve in the centre is the rms size of the stationary distribution. A threshold at the coherent mode frequency 2.3167 is evident. The horizontal curves are the fraction of particles inside a fixed emittance for the Gaussian distribution. They step downwards as particles are driven to larger amplitude. The curve composed of connected + symbols is the emittance at which the incoherent tune is on resonance. So for example, when the ITESD is 2.36, small-amplitude particles have tune equal to 7/3; when the ITESD is 2.345, the emittance containing 60% of the particles has a tune of 7/3.

An example is shown in Fig. 1, where we have plotted for the gaussian distribution the fraction of particles inside a given betatron amplitude as a function of the incoherent tune of the stationary beam of same rms size. The resonance is $3\nu = 7$, the bare tune is 2.45, so the value of the incoherent tune of the equivalent stationary beam that puts the coherent $(m, k) = (3, 3)$ mode on resonance is $(2.45 - (2.45 - 2.333)/C_{33} =)$ 2.31667. The rms beam size of the stationary distribution has also been plotted, and there is clearly a sharp threshold at the expected location. On the other hand, in the Gaussian case, the effect is much broader, but still centred upon the same location. This verifies that the relevant comparison between beams of different distribution is not one where the peak incoherent tune is the same, but where the rms beam size is the same.

## B Two Dimensions

Machida [10] has performed simulations for quadrupole resonance in the SSC LEB. With $\nu_{0x} = 11.87$, $\nu_{0y}$ was varied in the range 11.95 to 11.55. The beam distribution was Gaussian, the maximum incoherent tune shift was kept at $\Delta\nu_y = 0.33$, and the half-integer stopband was 0.02. Machida found the threshold for emittance growth to be approximately 11.63, or, in other words, with an incoherent tune shift of 0.33, the bare tune could be placed as close as 0.13 from the 1/2-integer. Equation 23 predicts that the resonance $2\nu_y = 23$ causes an envelope instability when $\nu_{0y} = 11.61$. Agreement is even better when the stopband width is taken into account.

Hofmann [9] has simulated the case for an octupole driving term near a 1/4-integer resonance. He found that emittance growth occurred when the incoherent frequency was 10% beyond the resonance. This is within his uncertainty of $1 - C_{mm,\mathrm{sym}} = \frac{1}{8}$ of Table 2.

# IV EXPERIMENT

A number of observers have noted that the incoherent space charge tune shift can exceed the distance of the bare tune to the nearest low order resonance. For example, in the CERN PS with vertical tune 6.22, Cappi et al. [21] observed losses when the tune shift was larger than 0.27. When they moved the vertical tune to 6.28, losses occurred only for tune shift larger than about 0.35.

Experiments performed on the LANL PSR (Nueffer et al. [22]) test the theory more quantitatively since both beam sizes and beam distributions were accurately measured. However, the calculated peak incoherent tune shifts contained in the report are a factor of 2 too small, since their tune shift formula (1) is incorrect [23]. In fact, their formula gives the tune shift of the KV-equivalent beam, which is a factor of 2 smaller than the peak tune shift of the Gaussian beam. Their Table I should therefore read as follows.

Note that the experiments were performed on bunched beams. Therefore, $\nu_0 > 2$ with $\nu_\mathrm{i} < 2$ indeed means that some slice of the beam bunch has an incoherent tune of exactly 2. Resonances of all orders are expected at $\nu_y = 2$, since there was no correction of the driving terms. The order $m$ of the intensity-limiting resonance was not known. An experiment at low intensity had been performed in 1987 [24], in which $\nu_y$ was lowered towards 2. It was found that losses began at $\nu_y \approx 2.03$. The second last entry in the table indicates that the coherent mode responsible for emittance growth is shifted by $((2.059 - 2.03)/0.043 =)$ 0.67 of the KV-equivalent tune shift. We guess that the mode responsible is the quadrupole mode $2\nu_y = 4$, since $\Delta\nu_\mathrm{c}/\Delta\nu_\mathrm{KV} = 5/8$ (for round beams), and the stopband of the quadrupole mode is likely the largest.

As well, there is a qualitative result from the PSR experiments in support of the coherent mode theory. According to the incoherent resonance theory, only

**TABLE 3.** LANL PSR data $\nu_{0x} = 3.155$, varying $\nu_{0y}$

| $\nu_{0y}$ | $N/10^{13}$ | $2\sigma_y$ | $2\Delta p/p$ | $\epsilon_x$ | $\epsilon_y$ | $\Delta\nu_{y\mathrm{i}}$ | $\Delta\nu_{y\mathrm{KV}}$ |
|---|---|---|---|---|---|---|---|
| 2.193 | 0.6 | 8.5 | 0.38 | 20.1 | 8.6 | 0.150 | 0.075 |
| | 1.18 | 9.9 | 0.41 | 22.4 | 11.6 | 0.230 | 0.115 |
| | 2.3 | 13.3 | 0.38 | 26.6 | 20.0 | 0.296 | 0.148 |
| 2.142 | 0.6 | 8.4 | 0.41 | 19.7 | 8.4 | 0.150 | 0.075 |
| | 1.18 | 11.5 | 0.45 | 16.7 | 15.7 | 0.190 | 0.095 |
| | 2.3 | 15.5 | 0.41 | 25.4 | 28.6 | 0.226 | 0.113 |
| 2.100 | 0.6 | 9.5 | 0.47 | 15.5 | 10.7 | 0.122 | 0.061 |
| | 1.18 | 14.5 | 0.45 | 16.4 | 25.0 | 0.136 | 0.068 |
| | 2.3 | 21.5 | 0.31 | 44.0 | 55.0 | 0.128 | 0.064 |
| 2.059 | 0.6 | 12.6 | 0.46 | 15.0 | 18.9 | 0.086 | 0.043 |
| | 1.18 | 20.6 | 0.45 | 16.0 | 50.5 | 0.078 | 0.039 |

those particles whose tunes are on resonance will experience amplitude growth. In the particular case of a Gaussian beam approaching a resonance from above, this would imply that the smallest-amplitude particles would grow in amplitude, thus de-populating the centre of the Gaussian, making it flatter. Fig. 1c of [22] (which corresponds with the third-last line of Table 3) shows that in at least one case, the opposite occurred.

## V IMAGES

As mentioned above, a signature of the integer resonance $\nu = n$ is an $n$-fold closed orbit distortion. As the tune approaches the integer, the steering of the orbit around the ring must more and more accurately correct the $n^{\text{th}}$ Fourier component. According to the theory emphasized in the present report, the integer resonance is not approached by increasing the incoherent tune shift while leaving the bare tune unchanged, since the bare tune is the frequency of the coherent mode. Nevertheless, it was observed at ISIS [20] that with a tune above 4, the $4^{\text{th}}$ harmonic of the closed orbit needed to be corrected as the intensity was raised. Rees et al. concluded that this was due to the (usually neglected) closed-orbit-dependent terms of the image force expansion. Let us briefly derive this expansion.

For a parallel plate configuration, we simply extend the derivation first given by Laslett [25, Appendix B]. The origin is midway between the plates, which are situated at $y = \pm h$. A line charge $\lambda$ is located at $\bar{y}$ ($y_1$ in Laslett's notation). Then the potential at other locations $y$ is given by

$$U = -2\lambda \log \left| \frac{\sin\left(\frac{\pi y}{2h}\right) - \sin\left(\frac{\pi \bar{y}}{2h}\right)}{1 + \cos\left(\frac{\pi (y+\bar{y})}{2h}\right)} \right| . \tag{33}$$

From this we subtract the direct line charge field $-2\lambda \log \left| \frac{\pi(y-\bar{y})}{4h} \right|$ and expand up to $4^{\text{th}}$ order in $y$ and $\bar{y}$, to get:

$$-\frac{U_{\text{image}}}{\lambda} = \frac{\pi^2}{24\, h^2}\left(y^2 + 4y\bar{y} + \bar{y}^2\right) + \frac{\pi^4}{11520\, h^4}\left(7y^4 + 32y^3\bar{y} + 42y^2\bar{y}^2 + 32y\bar{y}^3 + 7\bar{y}^4\right) \tag{34}$$

We can find the force by differentiating w.r.t. $y$. However, since the beam is held at $y = \bar{y}$ by external dipole errors, and we are interested in perturbations about this point, we express the result in terms of $\check{y} = y - \bar{y}$:

$$\frac{E_{y\text{image}}}{4\lambda} \approx \frac{\pi^2}{48}\frac{\check{y}}{h^2} + \frac{\pi^2}{16}\frac{\bar{y}}{h^2} + \frac{\pi^4}{192}\frac{\bar{y}^3}{h^4} + \frac{\pi^4}{128}\frac{\check{y}\bar{y}^2}{h^4} + \frac{\pi^4}{256}\frac{\check{y}^2\bar{y}}{h^4} + \frac{7\pi^4}{11520}\frac{\check{y}^3}{h^4} \tag{35}$$

The factor of 4 on the left is in order to make the usual image coefficient as defined by Laslett [25] appear explicitly in the expansion. We recover the usual 'incoherent' space charge image coefficient $\epsilon_1 = \frac{\pi^2}{48}$ and the 'coherent' image coefficient $\xi_1 = \frac{\pi^2}{16}$.

The case of the circular boundary (radius $h$) is much simpler mathematically, since there is only one image:

$$\frac{E_{y\mathrm{image}}}{4\lambda} = \frac{1}{2}\,\frac{\bar{y}}{h^2 - y\bar{y}} \approx \frac{1}{2}\,\frac{\bar{y}}{h^2} + \frac{1}{2}\,\frac{\bar{y}^3}{h^4} + \frac{1}{2}\,\frac{\check{y}\bar{y}^2}{h^4}, \tag{36}$$

and this yields as usual, $\epsilon_1 = 0$ and $\xi_1 = \frac{1}{2}$.

Let us introduce new 'higher order' image coefficients $\kappa$ for the general boundary:

$$\frac{E_{y\mathrm{image}}}{4\lambda} = \epsilon_1\,\frac{\check{y}}{h^2} + \xi_1\,\frac{\bar{y}}{h^2} + \kappa_{30}\,\frac{\bar{y}^3}{h^4} + \kappa_{21}\,\frac{\check{y}\bar{y}^2}{h^4} + \kappa_{12}\,\frac{\check{y}^2\bar{y}}{h^4} + \kappa_{03}\,\frac{\check{y}^3}{h^4} + \cdots \tag{37}$$

The $\kappa_{21}$ term was investigated by Rees and Prior [20]: it represents a quadrupole, or half-integer, driving term whose strength is proportional to the square of the closed orbit distortion. Therefore, if $\nu_y$ is depressed toward $n$, and there is an $n$-fold distortion of the closed orbit, there will be a $2n$-fold quadrupole driving term, and a $2\nu_y = 2n$ stopband. We expect this to occur at the quadrupole coherent mode (not at the incoherent tune as suggested in [20]), and this has been confirmed in simulations [26]. Similarly, the $\kappa_{12}$ term will open a sextupole stopband for $n$-fold distortions of the closed orbit where $3\nu_y = n$. As well, the modulations of the vacuum chamber can drive octupole ($\kappa_{03}$) and quadrupole ($\epsilon_1$) resonances.

For large space charge tune shift, the higher order image terms are not small compared with driving terms due to lattice errors, and they can have a significant impact. It is important to realize that they are all intensity-dependent and so correction schemes derived by experimenting with a low intensity beam will not work as intended at high intensity.

## VI CONCLUSION

The main point of the present paper is that the practice of restricting the incoherent tune spread to lie between the lowest order betatron resonances is too conservative. Conversely, the assumption that in a well-tuned machine, the incoherent tune spread does not overlap any low-order resonances, is also not warranted. The error incurred is especially large for centrally-peaked distributions. For example, the Gaussian distribution in 2 dimensions can have a peak incoherent tune shift over twice as large as the distance of the bare tune from the low-order resonance.

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