Euler Method

Created by Haroon Sakhi

Introduction

The Euler method is a numerical technique used to approximate the solutions of ordinary differential equations (ODEs) . Many real - world problems, involve differential equations that cannot be solved analytically . The Euler method provides an iterative approach to approximate the solution at discrete points .

Formula

$$y_{n+1} = y_n + h \times f\ (x_n,\ y_n)$$

where :

- y_n: Current function value
- h : Step size (determines step length)
- $f(x_n, y_n)$: Derivative at the current point
- ✔ Smaller step sizes improve accuracy but increase computation time.

Implementing Euler's Method

Euler's method follows these key steps:

- **Step 1:** Convert the equation into the form $\frac{dy}{dx} = f\ (x,\ y)$.
- **Step 2:** Set the initial condition $(x_0,\ y_0)$.
- **Step 3:** Choose a step size h and the number of steps.
- **Step 4:** Iterate using Euler's formula to compute **y-values.**

Example: Applying Euler’s Method

Defining the Differential Equation

In[20]:=
```
dydx[x_]:=Cos[x]+1
Solution[x_]:=Sin[x]+x
```

☑ The exact solution is $y = \sin x + x$, which we compare against Euler’s approximation.

Writing the Euler Method Function

In[3]:=
```
EulerMethod[xStart_, xEnd_, h_, (x0_ → y0_)] := Module[{x, y, X, Y},
  X = {x0}; Y = {y0}; x = x0; y = y0;
  While[x < xEnd,
    x = x + h;
    y = y + h*dydx[x];
    AppendTo[X, x];
    AppendTo[Y, y];
  ];
  {X, Y}
]
```

Key Features :

- Starts with initial values .
- Iterates using Euler' s formula .
- Stores computed values for visualization .

Running the Method & Comparing with the Exact Solution

In[4]:=
```
xStart = 0; xEnd = 5; h = 0.1;
{X, Y} = EulerMethod[xStart, xEnd, h, (0 → 0)];
xSol = Subdivide[xStart, xEnd, 1000];
ySol = Solution /@ xSol;
```

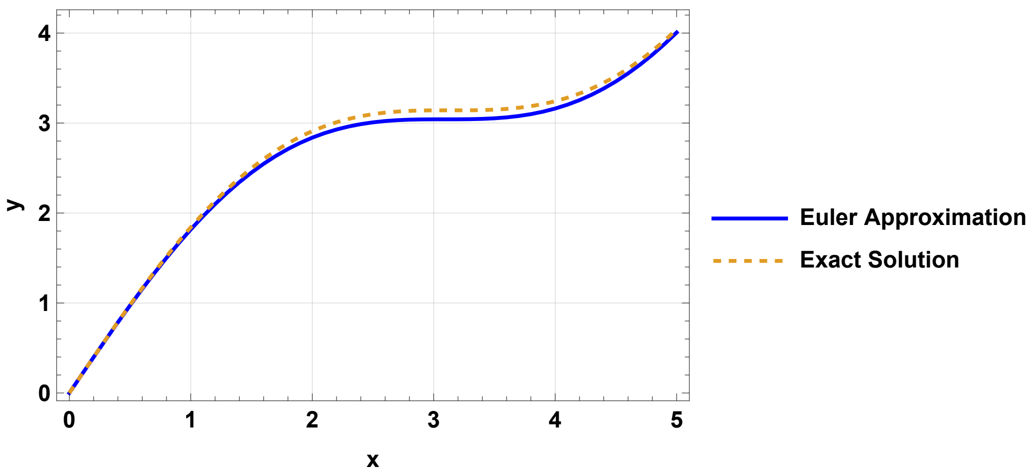
Key Insights :

- **Step size:** 0.1
- **Comparison:** Euler vs.exact solution

Visualizing Results

Euler Approximation vs. Exact Solution

In[8]:=
```
ListLinePlot[{
  Transpose[{X, Y}],
  Transpose[{xSol, ySol}]
  },
  PlotStyle → {Blue, Dashed},
  PlotLegends → {"Euler Approximation", "Exact Solution"},
  GridLines → Automatic,
  Frame → True,
  FrameLabel → {"x", "y"},
  LabelStyle → Directive[Bold, 12]
]
```

Out[8]=


- **Blue Line :** Euler' s approximation
- **Dashed Line :** Exact solution
- **Goal :** Assess accuracy visually

Why Does Euler’s Method Fail?

Euler’s method doesn’t always work well, especially when the function changes rapidly or when a large step size is used. In such cases, the approximation can deviate significantly from the true solution. Let’s take a look at an example where Euler’s method struggles.

Example : Euler' s Method Fails

Consider the differential equation:

In[9]:=
```
dydx[x_, y_] := -2*x*y
Solution[x_] := Exp[-x^2]
```

Implementing Euler's Method

In[11]:=
```
EulerMethod[start_, end_, dx_, {x0_, y0_}] := Module[{X, Y, x, y},
 X = {x0}; Y = {y0}; x = x0; y = y0;
  While[x < end, x = x + dx;
   y = y + dydx[x, y]*dx;
   AppendTo[X, x];
   AppendTo[Y, y];
   ];
  {X, Y}]
```

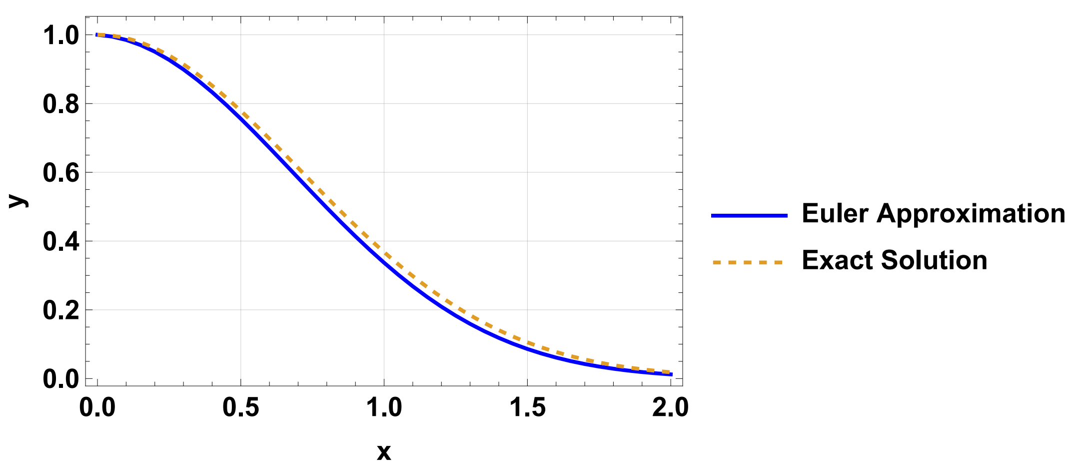
Running Euler's Method and Comparing Results

In[12]:=
```
start = 0;
end= 2;
dx = 0.05;
{X, Y} = EulerMethod[start, end, dx, {0, 1}];

(* Compute exact solution for comparison *)
xSol = Subdivide[start, end, 1000];
ySol = Exp[-xSol^2];

(* Plot results *)
ListLinePlot[{Transpose[{X, Y}], Transpose[{xSol, ySol}]},
 PlotStyle → {Blue, Dashed},
 PlotLegends → {"Euler Approximation", "Exact Solution"},
 GridLines → Automatic, Frame → True, FrameLabel → {"x", "y"},
 LabelStyle → Directive[Bold, 14]]
```

Out[18]=

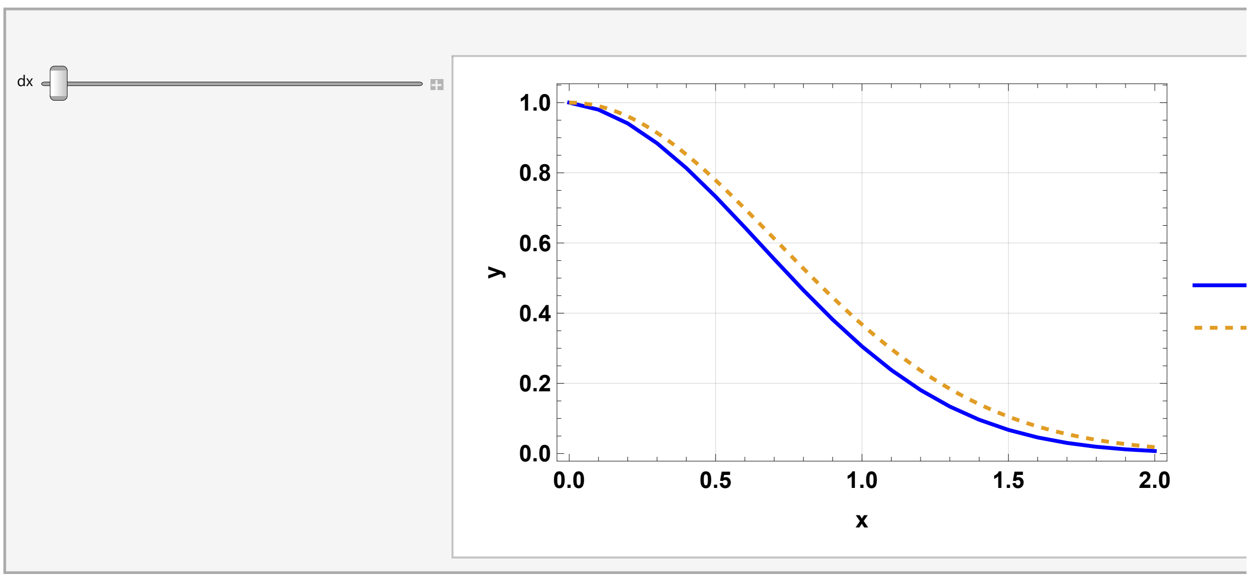


Making the Method Interactive

In[22]:=

```
Manipulate[
  {X, Y} = EulerMethod[0, 2, dx, {0, 1}];
  ListLinePlot[{
    Transpose[{X, Y}],
    Transpose[{xSol, ySol}]
    },
    PlotStyle → {Blue, Dashed},
    PlotLegends → {"Euler Approximation", "Exact Solution"},
    GridLines → Automatic,
    Frame → True,
    FrameLabel → {"x", "y"},
    LabelStyle → Directive[Bold, 12]
  ],
  {dx, 0.1, 0.5, 0.01}
]
```

Out[22]=



Adjust dynamically to observe how step size affects accuracy!

What Went Wrong?

Euler's method struggles because the function decreases rapidly, and the approximation lags behind the exact solution.

The errors accumulate as we move forward, making the results unreliable.

This highlights the need for more accurate methods, such as Runge-Kutta (RK4), which provides much better accuracy with similar computational effort.

Conclusion

- ✅ Euler’s method is simple and effective but accumulates errors.
- 🔬 Smaller step sizes yield better accuracy but increase computation.
- 🚀 Runge-Kutta methods offer improved precision over fewer steps.

🚀 Next Steps: In the next section, we’ll explore how Runge-Kutta (RK4) significantly improves accuracy over Euler’s method.