Christoffel Symbols: Analytical and Numerical Approach

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Abstract:

This notebook explores Christoffel symbols, fundamental in differential geometry and general relativity. It first derives them analytically from the metric tensor with examples in different coordinate systems, then numerically computes them using Mathematica.

Introduction

Christoffel symbols are fundamental in differential geometry and general relativity, serving as the connection coefficients of the Levi-Civita connection. They describe how basis vectors change across a coordinate system, enabling the definition of covariant derivatives and parallel transport on manifolds. Physically, Christoffel symbols represent fictitious forces arising in non-inertial reference frames.

Derivation

$$\vec{e}_\lambda \Gamma^\lambda_{\mu\nu} = \partial_\mu \vec{e}_\nu$$

this is simply a definition, meaning that the Christoffel symbol is defined as whatever the coefficient you get when acting with the derivative on the basis vector.

Next, we consider the derivative of the dot product of two basis vectors:

$$\partial_\alpha(\vec{e}_\mu \cdot \vec{e}_\nu) = \vec{e}_\nu \cdot \partial_\alpha \vec{e}_\mu + \vec{e}_\mu \cdot \partial_\alpha \vec{e}_\nu$$

Rearranging the terms, we obtain:

$$\vec{e}_\mu \cdot \partial_\alpha \vec{e}_\nu = \partial_\alpha(\vec{e}_\mu \cdot \vec{e}_\nu) - \vec{e}_\nu \cdot \partial_\alpha \vec{e}_\mu$$

Inserting the definition of the Christoffel symbol, $\partial_\mu \vec{e}_\nu = \vec{e}_\lambda \Gamma^\lambda_{\mu\nu}$, we get:

$$\vec{e}_\mu \cdot \vec{e}_\lambda \Gamma^\lambda_{\alpha\nu} = \partial_\alpha(\vec{e}_\mu \cdot \vec{e}_\nu) - \vec{e}_\nu \cdot \vec{e}_\lambda \Gamma^\lambda_{\alpha\mu}$$

Next, we write three permutations of this equation:

$$\vec{e}_\mu \cdot \vec{e}_\lambda \Gamma^\lambda_{\alpha\nu} = \partial_\alpha(\vec{e}_\mu \cdot \vec{e}_\nu) - \vec{e}_\nu \cdot \vec{e}_\lambda \Gamma^\lambda_{\alpha\mu} \quad (1)$$
$$\vec{e}_\alpha \cdot \vec{e}_\lambda \Gamma^\lambda_{\nu\mu} = \partial_\nu(\vec{e}_\alpha \cdot \vec{e}_\mu) - \vec{e}_\mu \cdot \vec{e}_\lambda \Gamma^\lambda_{\nu\alpha} \quad (2)$$
$$\vec{e}_\alpha \cdot \vec{e}_\lambda \Gamma^\lambda_{\mu\nu} = \partial_\mu(\vec{e}_\alpha \cdot \vec{e}_\nu) - \vec{e}_\nu \cdot \vec{e}_\lambda \Gamma^\lambda_{\mu\alpha} \quad (3)$$

Adding equations (1) and (2), and subtracting (3), we obtain:

$$2\,\vec{e}_\mu \cdot \vec{e}_\lambda \Gamma^\lambda_{\alpha\nu} + \vec{e}_\alpha \cdot \vec{e}_\lambda \Gamma^\lambda_{\nu\mu} - \vec{e}_\alpha \cdot \vec{e}_\lambda \Gamma^\lambda_{\mu\nu} = \partial_\alpha(\vec{e}_\mu \cdot \vec{e}_\nu) + \partial_\nu(\vec{e}_\alpha \cdot \vec{e}_\mu) - \partial_\mu(\vec{e}_\alpha \cdot \vec{e}_\nu)$$

Using the symmetry property $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, we simplify:

$$2\,\vec{e}_\mu \cdot \vec{e}_\lambda \Gamma^\lambda_{\alpha\nu} = \partial_\alpha(\vec{e}_\mu \cdot \vec{e}_\nu) + \partial_\nu(\vec{e}_\alpha \cdot \vec{e}_\mu) - \partial_\mu(\vec{e}_\alpha \cdot \vec{e}_\nu)$$

Dividing by 2 and multiplying by $(\vec{e}^\mu \cdot \vec{e}^\rho)$, we get:

$$(\vec{e}^\mu \cdot \vec{e}^\rho)(\vec{e}_\mu \cdot \vec{e}_\lambda)\Gamma^\lambda_{\alpha\nu} = \frac{1}{2}(\vec{e}^\mu \cdot \vec{e}^\rho)(\partial_\alpha(\vec{e}_\mu \cdot \vec{e}_\nu) + \partial_\nu(\vec{e}_\alpha \cdot \vec{e}_\mu) - \partial_\mu(\vec{e}_\alpha \cdot \vec{e}_\nu))$$

Using the property $(\vec{e}^\mu \cdot \vec{e}^\rho)(\vec{e}_\mu \cdot \vec{e}_\lambda) = \delta^\rho_\lambda$, we solve for $\Gamma^\lambda_{\alpha\nu}$:

$$\delta^\rho_\lambda \Gamma^\lambda_{\alpha\nu} = \Gamma^\rho_{\alpha\nu} = \frac{1}{2}\vec{e}^\mu \cdot \vec{e}^\rho(\partial_\alpha(\vec{e}_\mu \cdot \vec{e}_\nu) + \partial_\nu(\vec{e}_\alpha \cdot \vec{e}_\mu) - \partial_\mu(\vec{e}_\alpha \cdot \vec{e}_\nu))$$

the metric tensor is geometrically defined as the dot product between basis vectors:

$$g_{\mu\nu} = (\vec{e}_\mu \cdot \vec{e}_\nu)$$

Inserting this definition of the metric, we get the formula for the Christoffel symbols in terms of the metric:

$$\Gamma^\rho_{\alpha\nu} = \frac{1}{2} g^{\mu\rho}(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu})$$

The Christoffel symbols $\Gamma^k_{i\,j}$ can be read as follows; the two lower indices, i & j, describe the change in the i^{th} basis vector caused by a change in the j^{th} coordinate. The upper index k then gives the specific direction in which this change occurs in.

How To Calculate Christoffel Symbols From The Metric?

1. **Define a set of coordinates x^μ.** Typically, you will have as many coordinates as there are dimensions in your space.
2. **Write down the components of a specific metric $g_{\mu\nu}$, or equivalently, its line element $d\,s^2$, in these coordinates.** You will also need the inverse metric components.
3. **Write down the formulas for the Christoffel symbols, $\Gamma^\lambda_{\mu\nu}$, in terms of the metric for each value of the index λ separately** (the upper index). You should have, in total, as many equations as there are coordinates.

4. **Based on the properties of the given metric, simplify the Christoffel symbol formulas as much as possible.**
5. **Determine which coordinates the metric depends on.** By doing this, you will be able to tell which derivatives of the metric are zero and, in many cases, directly identify which Christoffel symbols must be zero.
6. **Plug in the different components of the metric and its inverse into each of the Christoffel symbol formulas.**
7. **Calculate the different partial derivatives of the metric and simplify.**

Example

Define a set of coordinates x^μ. In polar coordinates, we have two coordinates:

$$x^1 = r\,,\quad x^2 = \theta$$

The line element in polar coordinates is given by:

$$d\,s^2 = d\,r^2 + r^2\,d\,\theta^2$$

The metric components in matrix form:

$$g_{i\,j} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$$

The inverse metric components:

$$g^{i\,j} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Christoffel symbols are given by:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\,g^{\lambda\sigma}\,(\partial_\mu\,g_{\nu\sigma} + \partial_\nu\,g_{\mu\sigma} - \partial_\sigma\,g_{\mu\nu})$$

For $\lambda = 1$:

$$\Gamma^1_{i\,j} = \frac{1}{2}\,g^{11}\,(\partial_i\,g_{j\,1} + \partial_j\,g_{i\,1} - \partial_1\,g_{i\,j}) + \frac{1}{2}\,g^{12}\,(\partial_i\,g_{j\,2} + \partial_j\,g_{i\,2} - \partial_2\,g_{i\,j})$$

Since $g_{12} = 0$, the second term vanishes:

$$\Gamma^1_{i\,j} = \frac{1}{2}\,g^{11}\,(\partial_i\,g_{j\,1} + \partial_j\,g_{i\,1} - \partial_1\,g_{i\,j})$$

Splitting into cases:

$$\Gamma^1_{11} = \frac{1}{2}\,g^{11}\,(\partial_1\,g_{11} + \partial_1\,g_{11} - \partial_1\,g_{11}) = 0$$

$$\Gamma^1_{22} = \frac{1}{2}\,g^{11}\,(\partial_1\,g_{22}) = -\frac{1}{2}\,g^{11}\,\partial_1\,g_{22}$$

Since $g_{22} = r^2$, we get:

$$\Gamma^1_{22} = -\frac{1}{2}(1)(2r) = -r$$

For $\lambda = 2$:

$$\Gamma^2_{ij} = \frac{1}{2} g^{22} \left(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij}\right)$$

Splitting into cases:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22} \partial_1 g_{22}$$

Since $g_{22} = r^2$, we get:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\frac{1}{r^2}(2r) = \frac{1}{r}$$

Thus, the nonzero Christoffel symbols are:

$$\Gamma^1_{22} = -r\,, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$

Now, we'll write out the geodesic equations (we'll have two geodesic equations, one for $k = 1$ and one for $k = 2$):

$$\ddot{x}^k + \Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j$$

This is the geodesic equation for $k = 1$, where I've written the sum over i and j in full (since there are two coordinates, we have four terms).

$$\ddot{x}^1 + \Gamma^1_{11}\,\dot{x}^1\,\dot{x}^1 + \Gamma^1_{12}\,\dot{x}^1\,\dot{x}^2 + \Gamma^1_{21}\,\dot{x}^2\,\dot{x}^1 + \Gamma^1_{22}\,\dot{x}^2\,\dot{x}^2 = 0$$

This is the geodesic equation for $k = 2$, where I've once again written the sum over i and j in full.

$$\ddot{x}^2 + \Gamma^2_{11}\,\dot{x}^1\,\dot{x}^1 + \Gamma^2_{12}\,\dot{x}^1\,\dot{x}^2 + \Gamma^2_{21}\,\dot{x}^2\,\dot{x}^1 + \Gamma^2_{22}\,\dot{x}^2\,\dot{x}^2 = 0$$

We can now plug in our coordinates ($x^1 = r$ & $x^2 = \theta$). Also, by symmetry of the Christoffel symbols, we can combine the cross-terms and get a factor of 2:

$$\ddot{x}^1 + \Gamma^1_{11}\,\dot{x}^1\,\dot{x}^1 + 2\,\Gamma^1_{12}\,\dot{x}^1\,\dot{x}^2 + \Gamma^1_{22}\,\dot{x}^2\,\dot{x}^2 = 0$$

$$\Rightarrow\ \ddot{r} + \Gamma^1_{11}\,\dot{r}^2 + 2\,\Gamma^1_{12}\,\dot{r}\,\dot{\theta} + \Gamma^1_{22}\,\dot{\theta}^2 = 0$$

$$\ddot{x}^2 + \Gamma^2_{11}\,\dot{x}^1\,\dot{x}^1 + 2\,\Gamma^2_{12}\,\dot{x}^1\,\dot{x}^2 + \Gamma^2_{22}\,\dot{x}^2\,\dot{x}^2 = 0$$

$$\Rightarrow\ \ddot{\theta} + \Gamma^2_{11}\,\dot{r}^2 + 2\,\Gamma^2_{12}\,\dot{r}\,\dot{\theta} + \Gamma^2_{22}\,\dot{\theta}^2 = 0$$

By substituting Christoffel symbols we get the geodesic equations.

Numerical Approach

Example :

The Schwarzschild metric is given by:

$$d s^2 = \left(1 - \frac{2 G M}{r c^2}\right) c^2 d t^2 - \frac{d r^2}{1 - \frac{2 G M}{r c^2}} - r^2 d \theta^2 - r^2 \sin^2 \theta d \phi^2$$

Defining the Metric tensor

In[1]:=
```
ClearAll[t,r,theta,phi,M,G,c,g,dydx,InverseMetric,Christoffel]

coordinates={t,r,theta,phi};

g[i_,j_]:=Module[{metric},metric={{1-(2 G M)/(r c^2),0,0,0},{0,-1/(1-(2 G M)/(r c^2)),0,0},{0,0
 metric⟦i+1,j+1⟧ (*Adjusting for Mathematica's 1-based indexing*)];
```

Partial derivative of the Metric tensor

In[4]:=
```
dydx[i_,j_,k_]:=Simplify[D[g[i,j],coordinates⟦k+1⟧]];
```

Inverse Metric Computation

In[5]:=
```
InverseMetric[i_,j_]:=Module[{metricMatrix},metricMatrix=Table[g[m,n],{m,0,3},{n,0,3}];
 Simplify[Inverse[metricMatrix]⟦i+1,j+1⟧]];
```

Christoffel Symbols formula

In[6]:=
```
Christoffel[i_,j_,k_]:=Simplify[Sum[(1/2) InverseMetric[i,l] (dydx[k,l,j]+dydx[l,j,k]-dydx[j,k,l
```

Displaying Nonzero Christoffel Symbols

In[7]:=
```
Do[Module[{value=Christoffel[i,j,k]},If[value=!=0,Print["[",coordinates⟦i+1⟧," ",coordinates⟦j+1
```

[t t r] $\frac{G M}{r \left(-2 G M + c^2 r\right)}$

[t r t] $\frac{G M}{r \left(-2 G M + c^2 r\right)}$

[r t t] $\frac{G M \left(-2 G M + c^2 r\right)}{c^4 r^3}$

[r r r] $\frac{G M}{2 G M r - c^2 r^2}$

[r theta theta] $\frac{2 G M}{c^2} - r$

[r phi phi] $\left(-1 + \frac{2 G M}{c^2 r}\right) r \, \mathrm{Sin}[\mathrm{theta}]^2$

[theta r theta] $\frac{1}{r}$

[theta theta r] $\frac{1}{r}$

[theta phi phi] $-\mathrm{Cos}[\mathrm{theta}] \times \mathrm{Sin}[\mathrm{theta}]$

[phi r phi] $\frac{1}{r}$

[phi theta phi] Cot[theta]

[phi phi r] $\frac{1}{r}$

[phi phi theta] Cot[theta]

Results

The metric components can be written in matrix form as:

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2GM}{c^2 r} & 0 & 0 & 0 \\ 0 & -(1 - \frac{2GM}{c^2 r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

The nonzero Christoffel symbols for the Schwarzschild metric are:

$$\Gamma^{0}_{\mu\nu} = \begin{pmatrix} 0 & \frac{GM}{r(-2GM+c^2 r)} & 0 & 0 \\ \frac{GM}{r(-2GM+c^2 r)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{1}_{\mu\nu} = \begin{pmatrix} \frac{GM(-2GM+c^2 r)}{c^4 r^3} & 0 & 0 & 0 \\ 0 & \frac{GM}{2GMr-c^2 r^2} & 0 & 0 \\ 0 & 0 & \frac{2GM}{c^2} - r & 0 \\ 0 & 0 & 0 & \left(-1 + \frac{2GM}{c^2 r}\right) r \sin^2\theta \end{pmatrix}$$

$$\Gamma^{2}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix}$$

$$\Gamma^{3}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & \cot\theta \\ 0 & 1/r & \cot\theta & 0 \end{pmatrix}$$

Geodesic Equation:

This will give us the geodesic equations

In[10]:=
```
GeodesicEq[mu_] :=
  Simplify[D[x[mu][λ], {λ, 2}] +
    Sum[Christoffel[mu, rho, sigma]×D[x[rho][λ], λ]×D[x[sigma][λ], λ],
      {rho, 0, 3}, {sigma, 0, 3}]];

(* Compute and display geodesic equations for each coordinate *)
Do[
  Print["Geodesic equation for ", coordinates⟦mu + 1⟧, " : ",
    GeodesicEq[mu] == 0],
  {mu, 0, 3}
]
```

⚠ Conclusion

- **We computed Christoffel symbols and Geodesic equations using Mathematica for the Schwarzschild metric.**
- **By changing the metric tensor in the code, this method can be extended to compute Christoffel symbols for other space-time geometries.**

- **This method provides a systematic way to study connections in curved space.**