Critical points in 2D

For each problem, we compute ∇f, then find the critical points by setting $\nabla f = 0$. After that, we compute the hessian, H at each critical point (the hessian is not explicitly computed, we only mention the general hessian.

A.

$$f(x,y) = x^2 - 4xy + y^2$$

$$\nabla f = \begin{bmatrix} 2x - 4y \\ 2y - 4x \end{bmatrix} = 0 \Rightarrow (x,y) \in \{(0,0)\}$$

$$H = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$$

$$[x \quad y] \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2(x^2 - 4xy + y^2)$$

Notice that H is neither positive definite, nor negative definite. Hence, $x = 0, y = 0$ is a saddle point.

B.

$$f(x,y) = x^4 - 4xy + y^4$$

$$\nabla f = \begin{bmatrix} 4x^3 - 4y \\ 4y^3 - 4x \end{bmatrix} = 0 \Rightarrow (x,y) \in \{(0,0),(1,1),(-1,-1)\}$$

$$H = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

At 0,0, the hessian is neither positive definite, nor negative definite. Hence, 0,0 is a saddle point

At 1,1 and $-1,-1$, the hessian is $\begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$.

$$[x \quad y] \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 12\left(x^2 - \frac{2}{3}xy + y^2\right) > 0$$

The inequality is because the polynomial has no roots.

Hence, 1,1 and $-1,-1$ are minima.

C. For this question, I checked if matrices were positive definite by looking at the eigenvalues that numpy reports.

$$f(x,y) = 2x^3 - 6x^2y + 6xy^2 - 3x^2 + 6xy$$

$$\nabla f = \begin{bmatrix} 6x^2 - 12xy - 6x + 6y^2 + 6y \\ -6x^2 + 12xy + 6x \end{bmatrix} = 0 \Rightarrow (x,y) \in \{(0,0),(0,-1),(1,0),(-1,-1)\}$$

$$H = \begin{bmatrix} 12x - 12y - 6 & -12x + 12y + 6 \\ -12x + 12y + 6 & 12x \end{bmatrix}$$

At 0,0, $H = [[-6,6],[6,0]]$, NOT positive/negative definite $\Rightarrow$ Saddle point

At $0, -1$, $H = [[6, -6], [-6, 0]]$, NOT positive/negative definite ⇒ Saddle point

At $1,0$, $H = [[6, -6], [-6,12]]$, positive definite ⇒ Minima

At $-1, -1$, $H = [[-6,6], [6, -12]]$, negative definite ⇒ Maxima

D. I looked at definiteness using numpy's eigvals.

$$f(x, y) = (x - y)^4 + x^2 - y^2 - 2x + 2y + 1$$

$$\nabla f = \begin{bmatrix} 4(x-y)^3 + 2x - 2 \\ -4(x-y)^3 - 2y + 2 \end{bmatrix} = 0 \Rightarrow (x, y) \in \{(1,1)\}$$

To solve the system, we simply add the two equations, and get $x - y = 0$, which we then substitute back in either to get their values

$$H = \begin{bmatrix} 12(x-y)^2 + 2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 - 2 \end{bmatrix}$$

At $(1,1)$, $H = [[2,0], [0, -2]]$ which is not positive or negative definite. Hence, $(1,1)$ is a saddle point