Question 1

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

Characteristic polynomial of A = $|A - \lambda I| = \left|\begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}\right| = (1-\lambda)^2 - 4 = \boldsymbol{\lambda^2 - 2\lambda - 3}$

The roots of the characteristic polynomial are $\{\mathbf{-1}, \mathbf{3}\}$ (clearly)

The eigenvalues of A are the roots of its characteristic polynomial and hence $\{\mathbf{-1}, \mathbf{3}\}$

We solve for eigenvectors $x^{(i)}$ as $Ax^{(i)} = \lambda_i x^{(i)}$.

For $\lambda_i = -1$:

$$2x_1 + 4x_2 = 0 \Rightarrow \boldsymbol{x} = [\mathbf{-2} \quad \mathbf{1}]^{\boldsymbol{T}} \Rightarrow \left[-\frac{\mathbf{2}}{\sqrt{\mathbf{5}}} \quad \frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\right]^{\boldsymbol{T}}$$

For $\lambda_i = 3$

$$-2x_1 + 4x_2 = 0 \Rightarrow \boldsymbol{x} = [\mathbf{2} \quad \mathbf{1}]^{\boldsymbol{T}} \Rightarrow \left[\frac{\mathbf{2}}{\sqrt{\mathbf{5}}} \quad \frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\right]^{\boldsymbol{T}}$$

Power iteration is straightforward. $\boldsymbol{x_1} = A\boldsymbol{x_0} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. After normalization, $\boldsymbol{x_1} = \begin{bmatrix} \frac{5}{\sqrt{29}} \\ \frac{2}{\sqrt{29}} \end{bmatrix}$

Power iteration will eventually converge to the eigenvector with largest absolute value. In this case, eigenvalue=3. Hence **power iteration will converge to** $\left[\frac{\mathbf{2}}{\sqrt{5}} \quad \frac{\mathbf{1}}{\sqrt{5}}\right]^{\boldsymbol{T}}$

Rayleigh quotient is $\frac{x^T A x}{x^T x}$. For $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have $x^T x = 2$ and $x^T A x = 7$. Hence the Rayleigh quotient is **3.5**

Inverse iteration would converge to the eigenvector with smallest absolute value of eigenvalue. In this case, eigenvalue=-1, and eigenvector is $\left[-\frac{\mathbf{2}}{\sqrt{5}} \quad \frac{\mathbf{1}}{\sqrt{5}}\right]^{\boldsymbol{T}}$

Upon using inverse iteration with shift=2, the algorithm would converge to the eigenvector whose eigenvalue is closest to 2. In this case, eigenvalue=3, hence the eigenvector is $\left[\frac{\mathbf{2}}{\sqrt{5}} \quad \frac{\mathbf{1}}{\sqrt{5}}\right]^{\boldsymbol{T}}$

Since A is not symmetric, QR iteration would converge to a triangular/block triangular matrix.