COL863: Special Topics in Theoretical Computer Science
Rapid Mixing in Markov Chains
II semester, 2016-17
Minor I
Total Marks 100
Due: On moodle at 11:55AM, 6th February 2017

Problem 1.1 (20 marks) *Ex 4.4 of LPW Edition 2.*

Proof by induction on n.
Hypothesis: $\|\mu - \nu\|_{TV} <= \sum_{i=1}^{n} \|\mu_i - \nu_i\|_{TV}$ for all n.
Base Case: For n=1. The hypothesis holds trivially.
Induction Hypothesis: Hypothesis holds for all n <K. K >1
Induction step: for n = K
$\sum_{i=1}^{n} \|\mu_i - \nu_i\|_{TV} = \sum_{i=1}^{n-1} \|\mu_i - \nu_i\|_{TV} + \|\mu_i - \nu_i\|_{TV} >= \|\mu_0 - \nu_0\|_{TV} + \|\mu_K - \nu_K\|_T V$ (where μ_0 and ν_0 are defined as μ and ν are defined but for i=1 to n-1 instead).
So now we have to just prove for two n=2 only.
For two variables let, $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$.
$P(X \neq Y) = P(X_1 \neq Y_1) + P(X_2 \neq Y_2) - P(X_1 \neq Y_1 \Lambda X_2 \neq Y_2)$
$P(X \neq Y) <= P(X_1 \neq Y_1) + P(X_2 \neq Y_2)$
$\|mu - nu\|_T V <= \|mu_1 - nu_1\|_T V + \|mu_2 - nu_2\|_T V$
Hence proved!

Problem 1.2 (40 marks) *You have a deck that contains n cards. The shuffling process is*

Make two independent choices of cards and interchange them.

Note that the two choices may be the same, in which case there is no change in the configuration of the deck. Use a coupling based method to prove an upper bound on $t_{mix}(\epsilon)$ for this chain.

Let there be two random permutations X_0 and $_0$ of the cards. We couple them to find the upper bound for t_{mix}.
The coupling is as follows:
1. Choose two random cards r_1 and r_2.
2. X: Swap card r_1 and r_2.
3. Y: Swap card r_1 and card at position $P_x(r_2)$, where $P_x(r_2)$ is position of card r_2 in X.
The marginal probabilities are still same because P_x defines a bijection from cards to cards.

Now once two cards come at same position they will stay in that same position in X and Y. If both r_1 and r_2 are already at same position in X and Y then it will make no change in terms

of relative configuration of X and Y.
Correct alignment will happen if at $P_x(r_1)$ position, card r_2 is there. So, probability of a card to come in correct position in X and Y is $(1/n)$.
In expectation, it takes $1/(1/n) = n$ transitions for that to happen.
Initially all cards may be misaligned, so in the worst case, we can expect $n * n$ time. Using the chebyshev inequality, Probabiltiy that $\tau > n * n$ is exponentially decaying
Therefore, $t_{mix} = O(n^2)$

Problem 1.3 (40 marks) *Again: You have a deck that contains* n *cards. This time let us call the* n *positions in the deck* $0, 1, \ldots, n-1$*. The (lazy) shuffling process is*

> *With probability 1/2 do nothing. Otherwise pick* i *uniformly at random from* $0, 1, \ldots, n-1$ *and interchange the cards in positions* i *and* $(i+1) \mod n$*.*

Use a coupling based method to prove an upper bound on $t_{mix}(\epsilon)$ *for this chain.*

Let there be two random permutation X_0 and Y_0 of the cards. We couple them to find the upper bound for t_{mix}.
Consider a n length vector δ, defined as follows.
$\delta_t(i) = x_i - y_i mod n$. x_i, y_i are position of card i in X_t and Y_t respectively.
We want $\delta_t(i) = 0$ for all i.
On each random transition $\delta_t(i)$ increases or decreases by 1 (mod n). We do not want $\delta_t(i)$ to increase once it becomes zero. So we couple X and Y as follows:
1. Choose a position i, uniformly from 1 to n.
2. If $\delta_t(C_x(i)) = 0$ or $\delta(C_x(i+1)) = 0$ then with 1/2 probability, swap cards at i and i+1 in both X and Y.
3. Otherwise, make the swap in one of X or Y (Use a coin toss to choose: X if heads, Y else).

It can be verified that the marginal probabilities of transitions are still the same.
Now each $\delta_t(i)$ is like a birth-death chain with 0 and n (=0 mod n) being the absorbing states. Each $\delta_t(i)$ changes with probability 1/n and stays in same state with probability (n-1)/n. So the expected time, in the worst case, for one of delta(i) to go in absorption state is $1/(1/n) * (n/2) * (n - n/2) = n^3/4$.
In the worst case, we can start with non-zero delta(i) for all i. So the expected total time $<= n * (n^3/4)$.
Using the chebyshev inequality, the probability that expected time is greater than n^4 decreases exponentially. So $t_{mix} = O(n^4)$.
