COL863: Special Topics in Theoretical Computer Science
Rapid Mixing in Markov Chains
II semester, 2016-17
Minor II
Total Marks 100
Due: On moodle at 11:55PM, 6th March 2017

Problem 2.1 (30 marks) *Ex 9.1 of LPW Edition 2. Generalize the flow in the upper bound of (9.25) to higher dimensions using an urn with balls of d colours. Use this to show that the resistance between opposite corners of the d-dimensional box of side length n is bounded independent n, when $d \geq 3$. Explain what this means for the simple random walk on $\mathbb{Z}^d$, $d \geq 3$.*

First, we generalize the flow used in the upper bound. Consider an Urn process which uses balls of d colors. Initially, the urn has one ball of each color. At each time step, a ball is drawn randomly and replaced by two balls of the same color. We represent the Urn by a d tuple, $\mathbf{x}$ such that $\mathbf{x}_i$ is the number of balls of color i
Consider the unit flow θ such that $\theta(e) = \Pr\{\text{Urn uses edge } e\}$
Similar to the 2 color Polya Urn, the multicolor Urn process is equally likely to be in any state such that $\sum_i^d \mathbf{x}_i = k + d$ where k is the number of time steps that have passed.
Since there are $\binom{k+d}{d}$ such states (ignoring d's 1-indexing), and each state has d outgoing edges, the flow through each edge is $(\binom{k+d}{d} d)^{-1}$.
We then bound energy of the flow as

$$\varepsilon(\theta) \leq 2 \sum_{k}^{n-1} (\binom{k+d}{d} d)^{-1} \leq c \sum_{k}^{n-1} dk^{-d}$$

The bound holds because $\exists c | (\frac{\binom{k+d}{d}}{k^{-d}}) \leq c \ \forall \ d > 0$. When $d > 1$, $\sum_k^{n-1} k^{-d} \leq c'$. Therefore,

$$\varepsilon(\theta) \leq dc'$$

Thus, the energy of the flow is upper bounded independent of n.

Problem 2.2 (20 marks) *Ex 9.7 of LPW Edition 2. Let B_n be the subset of $\mathbb{Z}^2$ contained in the box of side length $2n$ centred at the origin, 0. Let ∂B_n be the set of vertices on the perimeter of the box. Show that*

$$\lim_{n\to\infty} \mathbf{P}_0\{\tau_{\partial B_n} < \tau_0^+\} = 0.$$

We know $\Pr_a\{\tau_z < \tau_a^+\} = [c(a)\mathcal{R}(a \leftrightarrow z)]^{-1}$. However, the graph doesn't have a z vertex. To apply the tools that we have, we create a z vertex by adding an edge $e_{t,z} \forall t \in \partial B_n$ with conductance on each edge being ∞. Hence, the probability of moving to z upon reaching ∂B_n is 1. Also notice that $\mathbf{P}_0\{\tau_{\partial B_n} < \tau_0^+\}$ is not exactly $\mathbf{P}_0\{\tau_z < \tau_0^+\}$. We have bounds on $\mathcal{R}(a \leftrightarrow z)$ in terms of n. Specifically,

$$\frac{\log(2n-1)}{2} \leq \mathcal{R}(a \leftrightarrow z) \leq 2\log(2n)$$

which means that the escape probability is bounded by the inverse of above bounds. Further, under the limit $n \to \infty$, the escape probability gets sandwiched to 0.

$$\lim_{n\to\infty} \mathbf{P}_0\{\tau_z < \tau_0^+\} = 0$$

Notice that if $n > 2$, then $\{\tau_{\partial B_n} < \tau_0^+\} \implies \{\tau_z < \tau_0^+\}$.
Therefore, $\Pr\{\tau_{\partial B_n} < \tau_0^+\} < \Pr\{\tau_z < \tau_0^+\}$. Also, $\Pr\{\tau_{\partial B_n}\} \geq 0$. Hence, in the limit, $\Pr\{\tau_{\partial B_n} < \tau_0^+\} = 0$

Problem 2.3 (30 marks) *Ex 9.9 of LPW Edition 2. Given a network* $(G = (V, E), \{c(e)\})$, *define the* Dirichlet energy *of a function* $f : V \to \mathbb{R}$ *as*

$$\mathcal{E}_D(f) = \frac{1}{2}\sum_{v,w}[f(v) - f(w)]^2 c(v, w).$$

(a) Prove that

$$\min_{f:f(v)=1,f(w)=0} \mathcal{E}_D(f) = \mathcal{C}(v \leftrightarrow w),$$

and the unique minimizer is harmonic on $V \setminus \{v, w\}$.

(b) Deduce that $\mathcal{C}(v \leftrightarrow w)$ *is a convex function of edge conductances.*

(a) The notation is a little confusing. We will show that

$$\min_{f:f(a)=1,f(z)=0} \mathcal{E}_D(f) = \mathcal{C}(a \leftrightarrow z),$$

and that the unique minimizer is harmonic on $V \setminus \{a, z\}$.
Differentiate $\mathcal{E}_D(f)$ wrt $f(x)$ where $x \in V$

$$\frac{\partial \mathcal{E}_D}{\partial f(x)} = \sum_{y:y\ \ x}[f(x) - f(y)]c(x, y) = f(x)c(x) - \sum_{y:y\ \ x} f(y)c(x, y)$$

For f to be a minimizer, $\frac{\partial \mathcal{E}_D}{\partial f(x)} = 0 \implies f(x)c(x) = \sum_y f(y)c(x, y)$. We also know that $P(x, y) = \frac{c(x,y)}{c(x)}$.

$$\therefore f(x) = \sum_y f(y)P(x, y)$$

Hence, the minimizer is harmonic on $V \setminus \{a, z\}$. Hence, the minimizer is a voltage. We will denote the minimizer by W.
Since the minimizer is a voltage, it is uniquely determined by its boundaries. Hence, the minimizer is unique.
Further, the definition of $\mathcal{E}_D$ implies that

$$\mathcal{E}_D = \frac{1}{2}\sum_{v,w}[W(v) - W(w)]I(v\bar{w}) = \frac{1}{2}\sum_v W(v)\sum_w I(v\bar{w})$$

The last equality holds because I is antisymmetric. By the node law, $\sum_w I(v\bar{w}) = 0 \forall v \in V \setminus \{a, z\}$ Hence,

$$\mathcal{E}_D = W(a)\sum_w I(a\bar{w})$$

Notice that $W(z) = 0, W(a) = 1$. Also, $\sum_w I(a\bar{w}) = \|I\|$. We can hence rewrite $\mathcal{E}_D$ as $\frac{\|I\|}{W(a)} = \mathcal{R}(a \leftrightarrow z)^{-1} = \mathcal{C}(a \leftrightarrow z)$

(b) We will deduce that $\mathcal{C}(a \leftrightarrow z)$ is a convex function of edge conductances. Since we have already proved that the minimum value of the Dirichlet energy is $\mathcal{C}(a \leftrightarrow z)$, we have

$$\mathcal{C}(a \leftrightarrow z) = \sum_{e:(v,w)} [W(v) - W(w)]^2 c(e)$$

We recall that a function that is monotonically non-decreasing on a set of variables is convex on that set of variables. We notice that

$$\frac{\partial \mathcal{C}(a \leftrightarrow z)}{\partial c(e : (v, w))} = [W(v) - W(w)]^2 \geq 0$$

Hence we conclude that $\mathcal{C}(a \leftrightarrow z)$ is convex on the conductances of the edges.

Problem 2.4 (20 marks) *Ex 10.3 of LPW Edition 2. Let G be a connected graph on at least three vertices in which the vertex v has only 1 neighbour, w. Show that the simple random walk on G (i.e. $c(e) = 1$ for all edges) satisfied $\mathbf{E}_v\tau_w \neq \mathbf{E}_w\tau_v$.*

First, $\mathbf{E}_v\tau_w = 1P(v,w) + P(v,v)(1 + \mathbf{E}_v\tau_w)$. After some rearrangement, $\mathbf{E}_v\tau_w = \frac{1}{P(v,w)}$ since v has only one neighbour. Since $\forall e, c(e) = 1, \mathbf{E}_v\tau_w = 1$
Next, consider $\mathbf{E}_w\tau_v$. The random walk reaches v in one of two ways:

Transition from w to v

Transition from w to some other state, then back to w

Notice that the random walk cannot reach v through any other state because v has only one neighbour. Thus, $\mathbf{E}_w\tau_v = 1P(w,v) + (1 - P(w,v))(\mathbf{E}_w\tau_w^+ + \mathbf{E}_w\tau_v)$

$$\mathbf{E}_w\tau_v = 1 + (1 - P(w,v))\mathbf{E}_w\tau_w^+ > 1 = \mathbf{E}_v\tau_w$$

Notice that the inequality holds because w must be connected to atleast one vertex other than v because $\mathcal{G}$ is a connected graph.
