COL863: Special Topics in Theoretical Computer Science
Rapid Mixing in Markov Chains
II semester, 2016-17
Minor I
Total Marks 100
Due: On moodle at 11:55AM, 6th February 2017

Problem 1.1 (20 marks) *Ex 4.4 of LPW Edition 2.*

We know that $\exists$ a coupling $(X,Y) \| \mathbb{P}[X \neq Y] = \|\mu - \nu\|_{TV} \forall \mu, \nu$
Specifically, let (X^i, Y^i) be such a coupling for μ^i, ν^i. Since each X^i is independent and so is each Y^i, we can define $(\mathbf{X} \equiv X^1, X^2 ..., X^n, \mathbf{Y} \equiv Y^1, Y^2 ..., Y^n)$ as a general coupling on μ, ν.
We also know that $\forall$ couplings $(X,Y) on \mu, \nu$, $\|\mu - \nu\|_{TV} \leq \mathbb{P}[X \neq Y]$
Hence,

$$\|\mu - \nu\|_{TV} \leq \mathbb{P}[X \neq Y]$$

$$\mathbb{P}[\mathbf{X} \neq \mathbf{Y}] \leq \sum_{i=0}^{n} \mathbb{P}[X_i \neq Y_i]$$

$$\sum_{i=0}^{n} \mathbb{P}[X \neq Y] = \sum_{i=0}^{n} \|\mu - \nu\|_{TV}$$

All that is left to show is that $(\mathbf{X}, \mathbf{Y})$ is a valid coupling.

$$\mathbb{P}[\mathbf{X} = x] = \sum_{\mathbf{Y}} \mathbb{P}[\mathbf{X} = x, \mathbf{Y}]$$

$$= \sum_{\mathbf{Y}} \prod_{j=0}^{n} \Pr[X_j = x_j, Y_j]$$

$$= \prod_{j=0}^{n} \sum_{\mathbf{Y}} \Pr[X_j = x_j, Y_j]$$

$$= \prod_{j=0}^{n} \mu_j$$

$$= \mu$$

Problem 1.2 (40 marks) *You have a deck that contains n cards. The shuffling process is*

> *Make two independent choices of cards and interchange them.*

Note that the two choices may be the same, in which case there is no change in the configuration of the deck. Use a coupling based method to prove an upper bound on $t_{mix}(\epsilon)$ for this chain.

**** REPLACE THIS WITH YOUR ANSWER ****

Problem 1.3 (40 marks) *Again: You have a deck that contains n cards. This time let us call the n positions in the deck $0, 1, \ldots, n-1$. The (lazy) shuffling process is*

> *With probability 1/2 do nothing. Otherwise pick i uniformly at random from $0, 1, \ldots, n-1$ and interchange the cards in positions i and $(i+1) \mod n$.*

Use a coupling based method to prove an upper bound on $t_{mix}(\epsilon)$ for this chain.

**** REPLACE THIS WITH YOUR ANSWER ****