COL863: Special Topics in Theoretical Computer Science
Rapid Mixing in Markov Chains
II semester, 2016-17
Minor I

Total Marks 100

Due: On moodle at 11:55AM, 6th February 2017

Problem 1.1 (20 marks) *Ex 4.4 of LPW Edition 2.*

Let (X_i, Y_i) be a optimal coupling with marginal distribution μ_i and ν_i respectively. Let,
X = $(X_1, X_2, ..., X_n)$
Y = $(Y_1, Y_2, ..., Y_n)$
Since distribution of X and Y is μ and ν respectively, (X,Y) is a coupling over μ and ν.
$||\mu - \nu||_{TV} \leq$ P(X $\neq$ Y) [follows from Proposition 4.7]
Since,$(X \neq Y) \rightarrow \bigvee_{i=1}^{n}(X_i \neq Y_i)$
P(X $\neq$ Y) $\leq \Sigma_{i=1}^{n} P(X_i \neq Y_i) = \Sigma_{i=1}^{n}[||\mu_i - \nu_i||_{TV}]$
hence prooved.

Problem 1.2 (40 marks) *You have a deck that contains n cards. The shuffling prt ocess is*

Make two independent choices of cards and interchange them.

Note that the two choices may be the same, in which case there is no change in the configuration of the deck. Use a coupling based method to prove an upper bound on $t_{mix}(\epsilon)$ for this chain.

Let X_t and Y_t be 2 random variable defined over space of deck arrangements.
Define coupling as pick 2 cards randomly from the deck X, pick the same first card in deck y and interchange both of them with the position of second card in deck X.
defining **distance(X,Y)** as no. of position with same cards in X and Y. notice distance(X,Y) is increasing w.r.t t.
Since we move same cards to same position, either it remains same if second card's position holds same card and position of first card in each deck is different or if second position contains different cards and first cards are at same position else it increases.
Now consider probability of going from distance k to k+1.
Distance will increase if we pick 2 cards that are not in same position in both decks, let this event be A.
PR(A) = $(\frac{(n-k)}{n})^2$
So, Expected total waiting time = $n^2 * \Sigma_{i=1}^{n}\frac{1}{i^2}$

Problem 1.3 (40 marks) *Again: You have a deck that contains n cards. This time let us call the n positions in the deck $0, 1, \ldots, n-1$. The (lazy) shuffling process is*

> *With probability 1/2 do nothing. Otherwise pick i uniformly at random from $0, 1, \ldots, n-1$ and interchange the cards in positions i and $(i+1) \mod n$.*

Use a coupling based method to prove an upper bound on $t_{mix}(\epsilon)$ for this chain.