Haroun Habeeb 2013CS10225

COL726: Numerical Algorithms

Minor 1 – Q6

We will use homogenous coordinates for a little bit. The vector spaces are assumed to be $\mathbb{R}^3$ for $\{\boldsymbol{A}_i\}$ and $\mathbb{R}^2$ for $\{\boldsymbol{a}_i, \boldsymbol{a}'_i\}$. We will define the basis as we go along. Vectors are column vectors.

From my graphics class, I know that orthographic projections can be represented by a matrix.

Let the top view matrix be M_T and front view matrix be M_F. There are 3x4 matrices since we are operating in homogenous coordinates.

Hence, $\boldsymbol{a}_k = M_T\boldsymbol{A}_k$ and $\boldsymbol{a}'_k = M_F\boldsymbol{A}_k$. The form of these matrices is dependent on the basis that we choose.

Part (a)

Let us shift the origin to A_0, a_0 and a'_0 for each vector space appropriately. Notice that this shift of origin is an affine transformation for each vector space and only results in the multiplication of an additional matrix to C.

Then, we choose the basis to be $\{\boldsymbol{A}_i - \boldsymbol{A}_0\}_{i=1}^{3}$ for $\mathbb{R}^3$. We also scale our axes such that the basis vectors are unit norm. Note that these transformations can also be represented by a 4x4 matrix C.

The equations then become $\boldsymbol{a}_k - \boldsymbol{a}_0 = M_T C^{-1}(\boldsymbol{A}_k - \boldsymbol{A}_0)$ and $\boldsymbol{a}'_k - \boldsymbol{a}'_0 = M_F C^{-1}(\boldsymbol{A}_k - \boldsymbol{A}_0)$

A quick look at $i = 1,2,3$ tells us that $M_T C^{-1}$ is $\begin{bmatrix} \boldsymbol{a}_1 - \boldsymbol{a}_0 & \boldsymbol{a}_2 - \boldsymbol{a}_0 & \boldsymbol{a}_3 - \boldsymbol{a}_0 & \begin{bmatrix}1\\1\end{bmatrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and similarly, $M_F C^{-1}$ is $\begin{bmatrix} \boldsymbol{a}'_1 - \boldsymbol{a}'_0 & \boldsymbol{a}'_2 - \boldsymbol{a}'_0 & \boldsymbol{a}'_3 - \boldsymbol{a}'_0 & \begin{bmatrix}1\\1\end{bmatrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ since $A_1 - A_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ in the new basis. Similar argument for $A_2 - A_0$ and $A_3 - A_0$.

Since we are reasonable people, we will drop the dimensions due to homogenous coordinates because the vectors have a value of 0 in the homogenous coordinate. We'll write the above equations as:

$$\begin{bmatrix} \boldsymbol{a}_k - \boldsymbol{a}_0 \\ \boldsymbol{a}'_k - \boldsymbol{a}'_0 \end{bmatrix} = \begin{bmatrix} [\boldsymbol{a}_1 - \boldsymbol{a}_0 & \boldsymbol{a}_2 - \boldsymbol{a}_0 & \boldsymbol{a}_3 - \boldsymbol{a}_0] \\ [\boldsymbol{a}'_1 - \boldsymbol{a}'_0 & \boldsymbol{a}'_2 - \boldsymbol{a}'_0 & \boldsymbol{a}'_3 - \boldsymbol{a}'_0] \end{bmatrix} (\boldsymbol{A}_k - \boldsymbol{A}_0) \qquad eq1$$

The problem of 3D estimation is to find $\boldsymbol{A}_k$ given $\{\boldsymbol{a}_i, \boldsymbol{a}'_i\}$, which is the same as solving $eq1$.

Do note that $eq1$ solves for vectors in the space spanned by the basis $\{\boldsymbol{A}_i - \boldsymbol{A}_0\}_{i=1}^{3}$.

We call $\begin{bmatrix} [\boldsymbol{a}_1 - \boldsymbol{a}_0 & \boldsymbol{a}_2 - \boldsymbol{a}_0 & \boldsymbol{a}_3 - \boldsymbol{a}_0] \\ [\boldsymbol{a}'_1 - \boldsymbol{a}'_0 & \boldsymbol{a}'_2 - \boldsymbol{a}'_0 & \boldsymbol{a}'_3 - \boldsymbol{a}'_0] \end{bmatrix}$ as L. We have dropped dimensions from homogenous coordinates. An important thing to note is that we will still need a way to figure out what $\boldsymbol{A}_0$ in a different basis.

Part (b)

If the equation that we wrote is consistent, 3D reconstruction will be possible. Essentially, the rank of L must be 3, or less.

Part (c)

It will be unique if the matrix L has rank 3. It is a system of linear equations in 3 variables and we are looking for solutions (for $\boldsymbol{A}_k - \boldsymbol{A}_0$).

Part (d)

The basis, as mentioned earlier, is $\{\boldsymbol{A}_i - \boldsymbol{A}_0\}_{i=1}^{3}$.

Part (e)

L represents the projection matrices of space spanned by $\{\boldsymbol{A}_i - \boldsymbol{A}_0\}_{i=1}^{3}$ into the space of $\{\boldsymbol{a}_i - \boldsymbol{a}_0\}$ and $\{\boldsymbol{a}'_i - \boldsymbol{a}'_0\}$ respectively.

Part (f)

$0 \leq rank \leq 3$ since we have 3 columns and 4 rows.

$1 \leq nullity \leq 4$ since max rank is 3 and there are 4 rows.

Part (g)

If $\{\boldsymbol{A}_i - \boldsymbol{A}_0\}_{i \in Selected\ Points}$ spans $\mathbb{R}^3$, we should be good.

Part (h)

Yes. L will look a lot uglier, but it will work since we have made no use of our views being top and bottom views in treating the 3D estimation problem as a system of linear equations.

Part (i)

To prove: $\forall\ V_0\ \exists\ V_1, V_2 \mid V\ is\ linear\ combination\ of\ V_1, V_2$. Where V_i is an orthographic projection view.

Let V_0 be defined by vectors (u, v, n) where n is the normal outward to the screen and u, v are vectors along the screen, orthogonal to each other. u is the vector to the right of the screen and v is the vector that represents "up".

We will show that V_0 is a linear combination of two other views defined by $(-n, u, v)$ and (n, v, u)

For example, when object is in the first octant, if V_0 is a projection onto XZ plane, defined by $(\hat{x}, \hat{z}, \hat{y})$, then the other two views are the projection onto XY and YZ such that the component along Y cancels out. These views would be defined by $(-\hat{y}, \hat{x}, \hat{z})$ and $(\hat{z}, \hat{y}, \hat{x})$

Notice that under the ordered basis $\{u, v, n\}$, the projection matrices without dimensions for homogeneous coordinates are

$$V_0 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_1: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$V_2: \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly, V_0 is a linear combination of V_1 and V_2 under the ordered basis (u, v, n). However, a change of basis is achieved by multiplying all three views by the same matrix. Hence, V_0 is a combination of V_1 and V_2