Fractional Inequalities for Strongly r-Convex Functions on Time Scales

Svetlin G. Georgiev[a], Vahid Darvish[b,c,*], Siyi Lyu[d]

[a] *Sorbonne University, Paris, France*
[b] *Nanjing University of Information Science and Technology, Nanjing, China*
[c] *University of Reading, Reading, UK*
[d] *Nanjing University of Information Science and Technology, Nanjing, China*

Abstract

In this article, we present some inequalities for strongly r-convex functions on time scales using delta-Riemann-Liouville type fractional integral. Finally, we reflect some of our main results on the time scale $\mathbb{R}$.

Keywords and phrases: Strongly r-convex functions, time scales, delta Riemann–Liouville integral, fractional Taylor formula.
AMS Subject Classification: 39A10, 39A11, 39A20.

1 Introduction

The theory of time scales, which has recently received a lot of attention, was initiated by Hilger [8] in his Ph.D. thesis in order to unify discrete and continuous analysis. Since then, many researchers have focused on this subject and presented many interesting results, for instance, we can refer to the monographs [2, 3, 4, 5] and the references therein. Among those, some of the researches (see [1, 6]) are devoted to develop various results concerning fractional calculus on time scales to derive related dynamic inequalities using the fractional Riemann–Liouville integral.
In this manuscript, motivated by the methods in [7], we attempt to prove some upper bounds for the delta-Riemann-Liouville fractional integral of functions which are n-times $r\delta$-continuously Δ-differentiable with strongly r-convexity property on an interval in some time scales.

*v.darvish@reading.ac.uk

The paper is organized as follows. In the next section, we give some preliminary results which are needed for the proof of our main results. In Section 3, we formulate and prove our main results.

2 Preliminaries

Suppose that $\mathbb{T}$ is an unbounded time scale with forward jump operator and delta differentiation operator σ and Δ, respectively. Let also, $a, b \in \mathbb{T}$, $a < b$ and an interval $[a, b]$ in $\mathbb{T}$ means as an intersection of a real interval with the supposed time scale.

For $\alpha \geq 0$, with $h_\alpha : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ we will denote the generalized polynomials on time scales defined as follows

$$\begin{aligned} h_0(t, s) &= 1, \\ h_{\alpha+1}(t, s) &= \int_s^t h_\alpha(t, \sigma(\tau))\Delta\tau, \quad t, s \in \mathbb{T}, \end{aligned} \tag{2.1}$$

where σ is the forward jump operator of the time scale $\mathbb{T}$.

Furthermore, it is established in [5] that for $\alpha, \beta \geq 1$ we have

$$\int_a^t h_{\alpha-1}(t, \sigma(u))h_{\beta-1}(u, a)\Delta u = h_{\alpha+\beta-1}(t, a), \quad t \in [a, b]. \tag{2.2}$$

For example, for the case $\mathbb{T} = \mathbb{R}$, we can define

$$h_\alpha(u, v) = \frac{(u - v)^\alpha}{\Gamma(\alpha + 1)},$$

where $u, v \in \mathbb{R}, \alpha \geq 0$ and Γ is the gamma function.

Since $\sigma(t) = t$ for all $t \in \mathbb{R}$, it can be easily verified that the exampled functions satisfy (2.1) and the relation (2.2) is also valid for them. Indeed,

$$\begin{aligned} \int_a^t h_{\alpha-1}(t, \sigma(u))h_{\beta-1}(u, a)du &= \int_a^t \frac{(t - u)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{(u - a)^{\beta-1}}{\Gamma(\beta)} du \\ &= \frac{(t - a)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} = h_{\alpha+\beta-1}(t, a), \end{aligned}$$

where the second equality is obtained from the same identity in [9].

For $\alpha \geq 1$ and $f \in \mathcal{C}_{rd}(\mathbb{T})$, with D_a^α, we denote the delta-Riemann-Liouville fractional operator defined by

$$\begin{aligned} D_a^\alpha f(t) &= \int_a^t h_{\alpha-1}(t, \sigma(\tau))f(\tau)\Delta\tau, \\ D_a^0 f(t) &= f(t), \quad t \in \mathbb{T}. \end{aligned}$$

We recall that $\mathcal{C}_{rd}(\mathbb{T})$ is the set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$.
We will start with the following useful auxiliary results.

Lemma 2.1. *[5] Let $\alpha, \beta > 1$, $f \in \mathcal{C}_{rd}([a, b])$. Then*

$$\begin{aligned} D_a^{\alpha} D_a^{\beta} f(t) &= D_a^{\alpha+\beta} f(t) \\ &\quad + \int_a^t f(u)\mu(u)h_{\alpha-1}(t, \sigma(u))h_{\beta-1}(u, \sigma(u))\Delta u, \end{aligned}$$

where $t \in [a, b]$ and μ is the graininess function; $\mu(t) = \sigma(t) - t$.

The above integral, i.e,

$$E(f, \alpha, \beta, t) = \int_a^t f(u)\mu(u)h_{\alpha-1}(t, \sigma(u))h_{\beta-1}(u, \sigma(u))\Delta u,$$

$t \in [a, b]$, $\alpha, \beta > 1$ and $f \in \mathcal{C}_{rd}([a, b])$, is called the forward graininess deviation functional of f.
Now we can rewrite Lemma 2.1 in brevity as

$$D_a^{\alpha} D_a^{\beta} f(t) = D_a^{\alpha+\beta} f(t) + E(f, \alpha, \beta, t), \quad t \in [a, b]. \tag{2.3}$$

Definition 2.2. *[5] Let $\alpha > 2$ and $m - 1 < \alpha < m$, $m \in \mathbb{N}$, $\nu = m - \alpha$. For a function $f \in \mathcal{C}_{rd}^m([a, b])$, define*

$$\begin{aligned} \Delta_a^{\alpha-1} f(t) &= D_a^{\nu+1} f^{\Delta^m}(t) \\ &= \int_a^t h_{\nu}(t, \sigma(u)) f^{\Delta^m}(u)\Delta u, \quad t \in [a, b]. \end{aligned}$$

Lemma 2.3. *[5] Suppose all as in Definition 2.2. Then*

$$\begin{aligned} \int_a^t h_{m-1}(t, \sigma(\tau)) f^{\Delta^m}(\tau)\Delta\tau &= -\int_a^t f^{\Delta^m}(u)\mu(u)h_{\alpha-2}(t, \sigma(u))h_{\nu}(u, \sigma(u))\Delta u \\ &\quad + \int_a^t h_{\alpha-2}(t, \sigma(\tau))\Delta_a^{m-1} f(\tau)\Delta\tau, \quad t \in [a, b]. \end{aligned}$$

According to the Taylor formula on time scales and Lemma 2.3, we have the following formula.

Lemma 2.4 (Fractional Taylor Formula)**.** *[5] Under the conditions of Definition* 2.2*, we have*

$$\begin{aligned} f(t) &= \sum_{k=0}^{m-1} h_k(t,a) f^{\Delta^k}(a) \\ &\quad - \int_a^t f^{\Delta^m}(u)\mu(u)h_{\alpha-2}(t,\sigma(u))h_\nu(u,\sigma(u))\Delta u \\ &\quad + \int_a^t h_{\alpha-2}(t,\sigma(\tau))\Delta_a^{\alpha-1} f(\tau)\Delta\tau, \quad t \in [a,b]. \end{aligned}$$

Definition 2.5. *[5] For the same assumptions as above, i.e, for $\alpha > 2$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\nu = m - \alpha$, $f \in \mathcal{C}_{rd}^m([a,b])$, define*

$$B(t) = f(t) + E\left(f^{\Delta^m}, \alpha - 1, \nu + 1, t\right), \quad t \in [a,b].$$

By the fractional Taylor formula it is established the following result.

Lemma 2.6. *[5] Let all as in Definition 2.5. Furthermore, let $f^{\Delta^k}(a) = 0$, $k \in \{0, \ldots, m-1\}$. Then*

$$B(t) = \int_a^t h_{\alpha-2}(t,\sigma(\tau))\Delta_a^{\alpha-1} f(\tau)\Delta\tau, \quad t \in [a,b].$$

Using the fractional Taylor formula we prove the following identity.

Lemma 2.7. *Let $f \in \mathcal{C}_{rd}^{m-1}([a,b])$, $\alpha > 2$, $m-1 < \alpha < m$, $\nu = m - \alpha$. Then*

$$\begin{aligned} \int_a^t f(s)\Delta s &= \sum_{k=0}^{m-1} h_k(t,a) f^{\Delta^k}(a) \\ &\quad - \int_a^t f^{\Delta^{m-1}}(u)\mu(u)h_{\alpha-2}(t,\sigma(u))h_\nu(u,\sigma(u))\Delta u \\ &\quad + \int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau h_\nu(\tau,\sigma(u))f^{\Delta^{m-1}}(u)\Delta u\right)\Delta\tau, \end{aligned}$$

$t \in [a,b]$.

Proof. Let

$$g(t) = \int_a^t f(s)\Delta s, \quad t \in [a,b].$$

Then

$$
\begin{aligned}
g^{\Delta^k}(t) &= f^{\Delta^{k-1}}(t), \quad k \in \{1, \ldots, m\}, \\
\Delta_a^{\alpha-1} g(t) &= \int_a^t h_\nu(t, \sigma(u)) f^{\Delta^{m-1}}(u) \Delta u, \quad t \in [a, b].
\end{aligned}
$$

We apply the fractional Taylor formula for the function g and we get the desired result. This completes the proof. $\square$

For $\alpha \geq 2$, $\nu \geq 0$, $p \geq 1$, denote

$$
\begin{aligned}
H(\alpha, \nu, t, a) &= \int_a^t h_{\alpha-2}(t, \sigma(\tau)) \int_a^\tau h_\nu(\tau, \sigma(u))(b-u)(u-a) \Delta u \Delta \tau, \\
G(\alpha, \nu, p, t, a) &= \left(\int_a^t (h_\nu(t, \sigma(u)))^p \Delta u \right)^{\frac{1}{p}}, \\
\Psi(\nu, t, a) &= \int_a^t h_\nu(t, \sigma(u))(b-u)(u-a) \Delta u, \quad t \in [a, b].
\end{aligned}
$$

Definition 2.8. *A positive function $f : I \to \mathbb{R}$ is called strongly r-convex function with modulus c on $[a, b]$, if for each $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have*

$$
\begin{aligned}
f(\lambda x + (1-\lambda) y) &\leq (\lambda (f(x))^r + (1-\lambda)(f(y))^r)^{\frac{1}{r}} \\
&\quad - c\lambda(1-\lambda)(x-y)^2, \quad r \neq 0.
\end{aligned}
$$

If we take $c = 0$, we get the definition of r-convexity of the function f.

Note that, if $f : I \to \mathbb{R}$ is positive strongly r-convex function with modulus c, we have

$$
f(t) \leq \left(\frac{b-t}{b-a} (f(a))^r + \frac{t-a}{b-a} (f(b))^r \right)^{\frac{1}{r}} - c(b-t)(t-a),
$$

$t \in [a, b]$.

3 Main Results

Now we are ready to present our results.

Theorem 3.1. *Let $r>0$, $\alpha>2$, $m-1<\alpha<m$, $\nu=m-\alpha$, $q\geq 1$, $f\in\mathcal{C}_{rd}^{m}([a,b])$, $|f^{\Delta^m}|\geq 1$ on $[a,b]$ and $|f^{\Delta^m}|^q$ is strongly r-convex function with modulus c on $[a,b]$, $f^{\Delta^k}(a)=0$, $k\in\{0,1,\dots,m-1\}$. Then*

$$\begin{aligned}|B(t)| &\leq 2^{\frac{1}{r}}\left(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\right)h_{\alpha+\nu}(t,a)\\ &\quad -cH(\alpha,\nu,t,a),\quad t\in[a,b].\end{aligned}$$

Proof. Since $|f^{\Delta^m}|^q$ is strongly r-convex function on $[a,b]$, we have

$$\begin{aligned}|f^{\Delta^m}(t)|^q &\leq \left(\frac{b-t}{b-a}|f^{\Delta^m}(a)|^{qr}+\frac{t-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\\ &\quad -c(b-t)(t-a),\quad t\in[a,b].\end{aligned}$$

Now, using Lemma 2.6 and by considering that $(x+y)^k\leq 2^k(x^k+y^k)$, for all $x,y>0$ and $k>0$, we get

$$\begin{aligned}|B(t)| &= \left|\int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau h_\nu(\tau,\sigma(u))f^{\Delta^m}(u)\Delta u\right)\Delta\tau\right|\\ &\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau))\int_a^\tau h_\nu(\tau,\sigma(u))|f^{\Delta^m}(u)|\Delta u\Delta\tau\\ &\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau))\int_a^\tau h_\nu(\tau,\sigma(u))|f^{\Delta^m}(u)|^q\Delta u\Delta\tau\\ &\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau))\int_a^\tau h_\nu(\tau,\sigma(u))\Bigg(\Bigg(\frac{b-u}{b-a}|f^{\Delta^m}(a)|^{qr}\\ &\quad +\frac{u-a}{b-a}|f^{\Delta^m}(b)|^{qr}\Bigg)^{\frac{1}{r}}-c(b-u)(u-a)\Bigg)\Delta u\Delta\tau\end{aligned}$$

$$
\begin{aligned}
&\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u)) \left(|f^{\Delta^m}(a)|^{qr} + |f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}} \Delta u \Delta\tau \\
&\quad -c \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))(b-u)(u-a)\Delta u\Delta\tau \\
&\leq 2^{\frac{1}{r}} \left(|f^{\Delta^m}(a)|^q + |f^{\Delta^m}(b)|^q\right) \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))\Delta u\Delta\tau \\
&\quad -cH(\alpha,\nu,t,a) \\
&= 2^{\frac{1}{r}} \left(|f^{\Delta^m}(a)|^q + |f^{\Delta^m}(b)|^q\right) \int_a^t h_{\alpha-2}(t,\sigma(\tau)) h_{\nu+1}(\tau,a)\Delta\tau \\
&\quad -cH(\alpha,\nu,t,a) \\
&= 2^{\frac{1}{r}} \left(|f^{\Delta^m}(a)|^q + |f^{\Delta^m}(b)|^q\right) h_{\alpha+\nu}(t,a) \\
&\quad -cH(\alpha,\nu,t,a), \quad t \in [a,b].
\end{aligned}
$$

We notice that the last equality follows from (2.2).This completes the proof. □

The next result reads as follows. But before that, we recall the following well-known inequality.

Theorem 3.2 ([4]). *(Hölder's inequality) Let $a, b \in \mathbb{T}$, $a < b$. For rd-continuous functions $f, g : [a,b] \to \mathbb{R}$ we have*

$$
\int_a^b |f(t)g(t)|\Delta\tau \leq \left(\int_a^b |f(t)|^p \Delta\tau\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \Delta\tau\right)^{\frac{1}{q}},
$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.3. *Let $r > 0$, $\alpha > 2$, $m-1 < \alpha < m$, $\nu = m - \alpha$, $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in \mathcal{C}_{rd}^m([a,b])$, $|f^{\Delta^m}|^q$ is strongly r-convex function with modulus c on $[a,b]$, $f^{\Delta^k}(a) = 0$, $k \in \{0, 1, \ldots, m-1\}$. Then*

$$
\begin{aligned}
|B(t)| &\leq 2^{\frac{1}{q}} (b-a)^{\frac{1}{q}} G(\alpha,\nu,p,t,a) \bigg(2^{\frac{r+1}{rq}} \left(|f^{\Delta^m}(a)| + |f^{\Delta^m}(b)|\right) \\
&\quad + c^{\frac{1}{q}} (h_2(b,a))^{\frac{1}{q}} \bigg) h_{\alpha-1}(t,a) \quad t \in [a,b].
\end{aligned}
$$

Proof. Since $|f^{\Delta^m}|^q$ is strongly r-convex function on $[a,b]$, we have

$$\begin{aligned} |f^{\Delta^m}(t)|^q &\le \left(\frac{b-t}{b-a}|f^{\Delta^m}(a)|^{qr} + \frac{t-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}} \\ &\quad -c(b-t)(t-a), \quad t\in[a,b]. \end{aligned}$$

Then

$$\begin{aligned} |B(t)| &= \left|\int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau h_\nu(\tau,\sigma(u))f^{\Delta^m}(u)\Delta u\right)\Delta\tau\right| \\ &\le \int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau h_\nu(\tau,\sigma(u))|f^{\Delta^m}(u)|\Delta u\right)\Delta\tau \\ &\le \int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\left(\int_a^\tau (h_\nu(\tau,\sigma(u)))^p\Delta u\right)^{\frac{1}{p}}\left(\int_a^\tau |f^{\Delta^m}(u)|^q\Delta u\right)^{\frac{1}{q}}\right)\Delta\tau \\ &\qquad \text{(by Hölder's inequality),} \\ &\le \int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\left(\int_a^t (h_\nu(t,\sigma(u)))^p\Delta u\right)^{\frac{1}{p}}\left(\int_a^\tau |f^{\Delta^m}(u)|^q\Delta u\right)^{\frac{1}{q}}\right)\Delta\tau \\ &= G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau |f^{\Delta^m}(u)|^q\Delta u\right)^{\frac{1}{q}}\Delta\tau \\ &\le G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Bigg(\int_a^\tau \Bigg(\left(\frac{b-u}{b-a}|f^{\Delta^m}(a)|^{qr} + \frac{u-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}} \\ &\quad -c(b-u)(u-a)\Bigg)\Delta u\Bigg)^{\frac{1}{q}}\Delta\tau \\ &\le G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Bigg(\int_a^\tau \Bigg(\left(|f^{\Delta^m}(a)|^{qr} + |f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}} \\ &\quad +c(b-a)(u-a)\Bigg)\Delta u\Bigg)^{\frac{1}{q}}\Delta\tau \end{aligned}$$

$$
\begin{aligned}
&\leq \quad G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Bigg(2^{\frac{1}{r}}\Bigg(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\Bigg)(b-a)\\
&\quad +c(b-a)\int_a^\tau (u-a)\Delta u\Bigg)^{\frac{1}{q}}\Delta\tau\\
&= \quad G(\alpha,\nu,p,t,a)(b-a)^{\frac{1}{q}}\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Bigg(2^{\frac{1}{r}}\Bigg(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\Bigg)\\
&\quad +ch_2(\tau,a)\Bigg)^{\frac{1}{q}}\Delta\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad 2^{\frac{1}{q}}(b-a)^{\frac{1}{q}}G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Bigg(2^{\frac{1}{rq}}\Bigg(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\Bigg)^{\frac{1}{q}}\\
&\quad +c^{\frac{1}{q}}(h_2(\tau,a))^{\frac{1}{q}}\Bigg)\Delta\tau\\
&\leq \quad 2^{\frac{1}{q}}(b-a)^{\frac{1}{q}}G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Bigg(2^{\frac{r+1}{rq}}\Bigg(|f^{\Delta^m}(a)|+|f^{\Delta^m}(b)|\Bigg)\\
&\quad +c^{\frac{1}{q}}(h_2(\tau,a))^{\frac{1}{q}}\Bigg)\Delta\tau\\
&\leq \quad 2^{\frac{1}{q}}(b-a)^{\frac{1}{q}}G(\alpha,\nu,p,t,a)\Bigg(2^{\frac{r+1}{rq}}\left(|f^{\Delta^m}(a)|+|f^{\Delta^m}(b)|\right)\\
&\quad +c^{\frac{1}{q}}(h_2(b,a))^{\frac{1}{q}}\Bigg)h_{\alpha-1}(t,a). \quad t\in[a,b].
\end{aligned}
$$

This completes the proof. □

Theorem 3.4. *Let $r>0$, $\alpha>2$, $m-1<\alpha<m$, $\nu=m-\alpha$, $q\geq 1$, $f\in\mathcal{C}_{rd}^m([a,b])$, $|f^{\Delta^m}|\geq 1$ and $|f^{\Delta^m}|^q$ is strongly r-convex function with modulus c on $[a,b]$. Then*

$$
\begin{aligned}
|\Delta_a^{\alpha-1}f(t)| \quad &\leq \quad 2^{\frac{1}{r}}\left(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\right)h_{\nu+1}(t,a)\\
&\quad -c\Psi(\nu,t,a), \quad t\in[a,b].
\end{aligned}
$$

Proof. Since $|f^{\Delta^m}|^q$ is strongly r-convex function on $[a,b]$, we have

$$
\begin{aligned}
|f^{\Delta^m}(t)|^q &\leq \left(\frac{b-t}{b-a}|f^{\Delta^m}(a)|^{qr}+\frac{t-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\\
&\quad -c(b-t)(t-a), \quad t\in[a,b].
\end{aligned}
$$

Then, using that $|f^{\Delta^m}|\geq 1$ on $[a,b]$, we have

$$
\begin{aligned}
&|\Delta_a^{\alpha-1}f(t)| = \left|\int_a^t h_\nu(t,\sigma(u))f^{\Delta^m}(u)\Delta u\right|\\
&\leq \int_a^t h_\nu(t,\sigma(u))|f^{\Delta^m}(u)|\Delta u\\
&\leq \int_a^t h_\nu(t,\sigma(u))|f^{\Delta^m}(u)|^q\Delta u\\
&\leq \int_a^t h_\nu(t,\sigma(u))\Bigg(\left(\frac{b-u}{b-a}|f^{\Delta^m}(a)|^{qr}+\frac{u-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\\
&\quad -c(b-u)(u-a)\Bigg)\Delta u\\
&= \int_a^t h_\nu(t,\sigma(u))\left(\frac{b-u}{b-a}|f^{\Delta^m}(a)|^{qr}+\frac{u-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\Delta u\\
&\quad -c\int_a^t h_\nu(t,\sigma(u))(b-u)(u-a)\Delta u\\
&= \int_a^t h_\nu(t,\sigma(u))\left(\frac{b-u}{b-a}|f^{\Delta^m}(a)|^{qr}+\frac{u-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\Delta u\\
&\quad -c\Psi(\nu,t,a)
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_a^t h_\nu(t,\sigma(u))\left(|f^{\Delta^m}(a)|^{qr}+|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\Delta u\\
&\quad -c\Psi(\nu,t,a)\\
&\le 2^{\frac{1}{r}}\left(|f^{\Delta^m}(a)|^{q}+|f^{\Delta^m}(b)|^{q}\right)\int_a^t h_\nu(t,\sigma(u))\Delta u\\
&\quad -c\Psi(\nu,t,a)\\
&= 2^{\frac{1}{r}}\left(|f^{\Delta^m}(a)|^{q}+|f^{\Delta^m}(b)|^{q}\right)h_{\nu+1}(t,a)-c\Psi(\nu,t,a),
\end{aligned}
$$

$t\in[a,b]$. This completes the proof. □

Corollary 3.5. *Suppose that all conditions of Theorem 3.4 hold and* $f^{\Delta^k}(a)=0$*,* $k\in\{0,1,\ldots,m-1\}$*. Then*

$$
\begin{aligned}
|B(t)| &\le 2^{\frac{1}{r}}\left(|f^{\Delta^m}(a)|^{q}+|f^{\Delta^m}(b)|^{q}\right)h_{\alpha+\nu}(t,a)\\
&\quad -c\Psi(\nu,t,a)h_{\alpha-1}(t,a),\quad t\in[a,b].
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
|B(t)| &= \left|\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Delta_a^{\alpha-1}f(\tau)\Delta\tau\right|\\
&\le \int_a^t h_{\alpha-2}(t,\sigma(\tau))|\Delta_a^{\alpha-1}f(\tau)|\Delta\tau\\
&\le 2^{\frac{1}{r}}\left(|f^{\Delta^m}(a)|^{q}+|f^{\Delta^m}(b)|^{q}\right)\int_a^t h_{\alpha-2}(t,\sigma(\tau))h_{\nu+1}(\tau,a)\Delta\tau\\
&\quad -c\Psi(\nu,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Delta\tau\\
&= 2^{\frac{1}{r}}\left(|f^{\Delta^m}(a)|^{q}+|f^{\Delta^m}(b)|^{q}\right)h_{\alpha+\nu}(t,a)\\
&\quad -c\Psi(\nu,t,a)h_{\alpha-1}(t,a),\quad t\in[a,b].
\end{aligned}
$$

This completes the proof. □

Theorem 3.6. *Let $r>0$, $\alpha>2$, $m-1<\alpha<m$, $\nu=m-\alpha$, $q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$, $f\in\mathcal{C}^m_{rd}([a,b])$, $|f^{\Delta^m}|^q$ is strongly r-convex function with modulus c on $[a,b]$. Then*

$$
\begin{aligned}
|\Delta_a^{\alpha-1}f(t)| \quad &\leq \quad 2^{\frac{1}{q}}G(\alpha,\nu,p,t,a)(b-a)^{\frac{1}{q}}\Bigg(2^{\frac{r+1}{rq}}\left(|f^{\Delta^m}(a)|+|f^{\Delta^m}(b)|\right)\\
&\qquad +c^{\frac{1}{q}}(h_2(t,a))^{\frac{1}{q}}\Bigg),\quad t\in[a,b].
\end{aligned}
$$

Proof. Since $|f^{\Delta^m}|^q$ is strongly r-convex function on $[a,b]$, we have

$$
\begin{aligned}
|f^{\Delta^m}(t)|^q \quad &\leq \quad \left(\frac{b-t}{b-a}|f^{\Delta^m}(a)|^{qr}+\frac{t-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\\
&\qquad -c(b-t)(t-a),\quad t\in[a,b].
\end{aligned}
$$

Then

$$
\begin{aligned}
|\Delta_a^{\alpha-1}f(t)| &= \left|\int_a^t h_\nu(t,\sigma(u))f^{\Delta^m}(u)\Delta u\right|\\
&\leq \int_a^t h_\nu(t,\sigma(u))|f^{\Delta^m}(u)|\Delta u\\
&\leq \left(\int_a^t (h_\nu(t,\sigma(u)))^p\Delta u\right)^{\frac{1}{p}}\left(\int_a^t |f^{\Delta^m}(u)|^q\Delta u\right)^{\frac{1}{q}}\\
&= G(\alpha,\nu,p,t,a)\left(\int_a^t |f^{\Delta^m}(u)|^q\Delta u\right)^{\frac{1}{q}}\\
&\leq G(\alpha,\nu,p,t,a)\Bigg(\int_a^t\left(\frac{b-u}{b-a}|f^{\Delta^m}(a)|^{qr}+\frac{u-a}{b-a}|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\\
&\qquad -c(b-u)(u-a)\Bigg)\Delta u\Bigg)^{\frac{1}{q}}\\
&\leq G(\alpha,\nu,p,t,a)\Bigg(\int_a^t\left(|f^{\Delta^m}(a)|^{qr}+|f^{\Delta^m}(b)|^{qr}\right)^{\frac{1}{r}}\Delta u\\
&\qquad +c\int_a^t(b-u)(u-a)\Delta u\Bigg)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq G(\alpha,\nu,p,t,a)\Bigg(2^{\frac{1}{r}}\int_a^t\Big(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\Big)\Delta u \\
&\quad +c(b-a)\int_a^t(u-a)\Delta u\Bigg)^{\frac{1}{q}} \\
&= G(\alpha,\nu,p,t,a)\Bigg(2^{\frac{1}{r}}\Big(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\Big)(b-a) \\
&\quad +c(b-a)h_2(t,a)\Bigg)^{\frac{1}{q}} \\
&= G(\alpha,\nu,p,t,a)(b-a)^{\frac{1}{q}}\Bigg(2^{\frac{1}{r}}\Big(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\Big) \\
&\quad +ch_2(t,a)\Bigg)^{\frac{1}{q}} \\
&\leq 2^{\frac{1}{q}}G(\alpha,\nu,p,t,a)(b-a)^{\frac{1}{q}}\Bigg(2^{\frac{1}{rq}}\Big(|f^{\Delta^m}(a)|^q+|f^{\Delta^m}(b)|^q\Big)^{\frac{1}{q}} \\
&\quad +c^{\frac{1}{q}}(h_2(t,a))^{\frac{1}{q}}\Bigg) \\
&\leq 2^{\frac{1}{q}}G(\alpha,\nu,p,t,a)(b-a)^{\frac{1}{q}}\Bigg(2^{\frac{r+1}{rq}}\left(|f^{\Delta^m}(a)|+|f^{\Delta^m}(b)|\right) \\
&\quad +c^{\frac{1}{q}}(h_2(t,a))^{\frac{1}{q}}\Bigg), \quad t\in[a,b].
\end{aligned}
$$

This completes the proof. $\square$

Corollary 3.7. *Suppose that all conditions of Theorem 3.6 hold and additionally* $f^{\Delta^k}(a)=0$, $k\in\{0,1,\ldots,m-1\}$. *Then*

$$
\begin{aligned}
|B(t)| &\leq 2^{\frac{2r+1}{rq}}(b-a)^{\frac{1+q}{q}}\left(|f^{\Delta^m}(a)|+|f^{\Delta^m}(b)|\right)G(\alpha,\nu,p,b,a)h_{\alpha-1}(t,a) \\
&\quad +2^{\frac{1}{q}}(b-a)c^{\frac{1}{q}}G(\alpha,\nu,p,b,a)G(\alpha,\alpha-2,p,b,a)(h_3(t,a))^{\frac{1}{q}}, \quad t\in[a,b].
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
|B(t)| &= \left|\int_a^t h_{\alpha-2}(t,\sigma(u))\Delta_a^{\alpha-1} f(u)\Delta u\right| \\
&\leq \int_a^t h_{\alpha-2}(t,\sigma(u))|\Delta_a^{\alpha-1} f(u)|\Delta u \\
&\leq 2^{\frac{2r+1}{rq}}(b-a)^{\frac{1}{q}}\left(|f^{\Delta^m}(a)|+|f^{\Delta^m}(b)|\right)\int_a^t h_{\alpha-2}(t,\sigma(u))G(\alpha,\nu,p,\tau,a)\Delta\tau \\
&\quad +2^{\frac{1}{q}}(b-a)^{\frac{1}{q}}c^{\frac{1}{q}}\int_a^t h_{\alpha-2}(t,\sigma(u))G(\alpha,\nu,p,\tau,a)(h_2(\tau,a))^{\frac{1}{q}}\Delta\tau \\
&\leq 2^{\frac{2r+1}{rq}}(b-a)^{\frac{1}{q}}\left(|f^{\Delta^m}(a)|+|f^{\Delta^m}(b)|\right)G(\alpha,\nu,p,b,a)h_{\alpha-1}(t,a) \\
&\quad +2^{\frac{1}{q}}(b-a)^{\frac{1}{q}}c^{\frac{1}{q}}G(\alpha,\nu,p,b,a)\left(\int_a^t (h_{\alpha-2}(t,\sigma(u)))^p\Delta\tau\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_a^t h_2(\tau,a)\Delta\tau\right)^{\frac{1}{q}} \qquad \text{(by Hölder's inequality)} \\
&\leq 2^{\frac{2r+1}{rq}}(b-a)^{\frac{1}{q}}\left(|f^{\Delta^m}(a)|+|f^{\Delta^m}(b)|\right)G(\alpha,\nu,p,b,a)h_{\alpha-1}(t,a) \\
&\quad +2^{\frac{1}{q}}(b-a)^{\frac{1}{q}}c^{\frac{1}{q}}G(\alpha,\nu,p,b,a)G(\alpha,\alpha-2,p,b,a)(h_3(t,a))^{\frac{1}{q}}, \quad t\in[a,b].
\end{aligned}
$$

This completes the proof. □

In order to illustrate our results more understandable, we consider the case $\mathbb{T}=\mathbb{R}$, hence $\sigma(t)=t$ for all $t\in\mathbb{R}$. So, the delta-Riemann-Liouville fractional integral's definition will be as

$$D_a^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t f(\tau)(t-\tau)^{\alpha-1}d\tau.$$

On the other hand, based on the definition of the forward graininess deviation functional, $E(f,\alpha,\beta,t)=0$ which by Definition 2.5 yields $f(t)=B(t)$. So, with these in mind and based on our main results, we will derive some inequalities in the sequel.

Theorem 3.8. *Let $r > 0$, $\alpha > 2$, $m-1 < \alpha < m$, $\nu = m - \alpha$, $q \geq 1$, $f \in \mathcal{C}^m([a,b])$, $a, b \in \mathbb{R}$, $|f^{(m)}| \geq 1$ on $[a,b]$ and $|f^{(m)}|^q$ is strongly r-convex function with modulus c on $[a,b]$, $f^{(k)}(a) = 0$, $k \in \{0,1,\ldots,m-1\}$. Then*

$$\int_a^b \left(|f(t)| + cD_a^m[(b-t)(t-a)]\right) dt \leq 2^{\frac{1}{r}} \left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right) \frac{(b-a)^{\alpha+\nu+1}}{\Gamma(\alpha+\nu+2)},$$

for $t \in [a,b]$.

Proof. According to Theorem 3.1, and by (2.3) which yields $D_a^\alpha D_a^\beta f(t) = D_a^{\alpha+\beta} f(t)$, we get

$$|f(t)| \leq 2^{\frac{1}{r}} \left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right) \frac{(t-a)^{\alpha+\nu}}{\Gamma(\alpha+\nu+1)} - cD_a^m[(b-t)(t-a)], \quad t \in [a,b],$$

Next by integrating the above inequality, we are proving the claim. □

Another inequality reads as follows.

Theorem 3.9. *Let $r > 0$, $\alpha > 2$, $m-1 < \alpha < m$, $\nu = m - \alpha$, $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in \mathcal{C}^m([a,b])$, $|f^{(m)}|^q$ is strongly r-convex function with modulus c on $[a,b]$, $f^{(k)}(a) = 0$, $k \in \{0,1,\ldots,m-1\}$. Then*

$$\begin{aligned} \int_a^b |f(t)| dt \quad &\leq \quad \frac{2^{\frac{1}{q}}(b-a)^{\alpha+\frac{1}{q}}}{\Gamma(\nu+1)\Gamma(\alpha+1)} \Bigg(2^{\frac{r+1}{rq}} \left(|f^{(m)}(a)| + |f^{(m)}(b)|\right) \\ &\quad + c^{\frac{1}{q}} \left(\frac{(b-a)^2}{2}\right)^{\frac{1}{q}} \Bigg) \int_a^b \left(\int_a^t (t-u)^{p\nu} du\right)^{\frac{1}{p}} dt. \end{aligned}$$

Proof. Theorem 3.3 and then integrating, deduces the desired result. □

According to Theorem 3.4 we conclude the following result

Theorem 3.10. *Suppose that all conditions of Theorem 3.8 hold. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f(a) \right| \leq \frac{2^{\frac{1}{r}}(b-a)^{\alpha+\nu}}{\Gamma(\alpha+\nu+2)} \left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right)$$

Proof. Let $g(t) = (b-t)(t-a)$ for $t \in [a,b]$. We have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(a)\right| = \frac{1}{b-a}\left|\int_a^b (f(t)-f(a))dt\right|$$

$$\begin{aligned}
&\leq \frac{1}{b-a}\int_a^b |f(t)-f(a)|dt \\
&\leq \frac{1}{b-a}\int_a^b \left|\int_a^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\Delta_a^{\alpha-1}f(\tau)d\tau\right| dt \qquad \text{(by fractional Taylor formula)} \\
&\leq \frac{1}{b-a}\int_a^b \left(\int_a^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}|\Delta_a^{\alpha-1}f(\tau)|d\tau\right) dt \\
&\leq \frac{1}{b-a}\int_a^b \left(\int_a^t \frac{2^{\frac{1}{r}}(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{(\tau-a)^{\nu+1}}{\Gamma(\nu+2)}\left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right)\right) \\
&\quad -\frac{c}{b-a}\int_a^b \left(\int_a^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_a^t \frac{(t-u)^{\nu}}{\Gamma(\nu+1)}(b-u)(u-a)du\right)d\tau\right)dt \\
&\qquad \text{(by Theorem 3.4)} \\
&= \frac{2^{\frac{1}{r}}\left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right)}{b-a}\int_a^b \frac{(t-a)^{\alpha+\nu}}{\Gamma(\alpha+\nu+1)}dt \\
&\quad -\frac{c}{b-a}\int_a^b \left(\int_a^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_a^t \frac{(t-u)^{\nu}}{\Gamma(\nu+1)}(b-u)(u-a)du\right)d\tau\right)dt \\
&= \frac{2^{\frac{1}{r}}(b-a)^{\alpha+\nu}}{\Gamma(\alpha+\nu+2)}\left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right) \\
&\quad -\frac{c}{b-a}\int_a^b \left(\int_a^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}D_a^{\nu+1}g(t)d\tau\right)dt \\
&= \frac{2^{\frac{1}{r}}(b-a)^{\alpha+\nu}}{\Gamma(\alpha+\nu+2)}\left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right) \\
&\quad -\frac{c}{b-a}\int_a^b \left(\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}D_a^{\nu+1}g(t)\right)dt \\
&= \frac{2^{\frac{1}{r}}(b-a)^{\alpha+\nu}}{\Gamma(\alpha+\nu+2)}\left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right) - \frac{c}{b-a}D_a^{\alpha}D_a^{\nu+1}g(b) \\
&= \frac{2^{\frac{1}{r}}(b-a)^{\alpha+\nu}}{\Gamma(\alpha+\nu+2)}\left(|f^{(m)}(a)|^q + |f^{(m)}(b)|^q\right),
\end{aligned}$$

the last equality is due to $g(b) = 0$, and this completes the proof. $\square$

Competing Interests The authors declare that there are no potential conflicts of interest to this work.

Statements and Declarations Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] G. A. Anastassiou, Principles of delta fractional calculus on time scales and inequalities, Mathematical and Computer Modelling 52 (2010) 556-566.

[2] R. Agarwal, M. Bohner, Basic Calculus on time scales and some of its applications, Results in Mathematics 35 (1-2) (1999) 3-22.

[3] R. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: a survey, Mathematical Inequalities & Applications 4 (4) (2001) 535-557.

[4] S. Georgiev, Integral Equations on Time Scales. Atlantis Press 2016.

[5] S. Georgiev, Integral Inequalities on Time Scales, De Gryuter, 2020.

[6] S. Georgiev, Fractional Dynamic Calculus and Fractional Dynamic Equations on Time Scales, Springer, 2018.

[7] S. G. Georgiev, V. Darvish, T. A. Roushan, B. Kaymakçalan, Inequalities for strongly r-convex functions on time scales, submitted.

[8] S. Hilger, Analysis on measure chains: A unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18-56.

[9] E.T. Whittaker, G.N. Watson, A Course in Modern Analysis, Cambridge University Press, 1927.