Problems!

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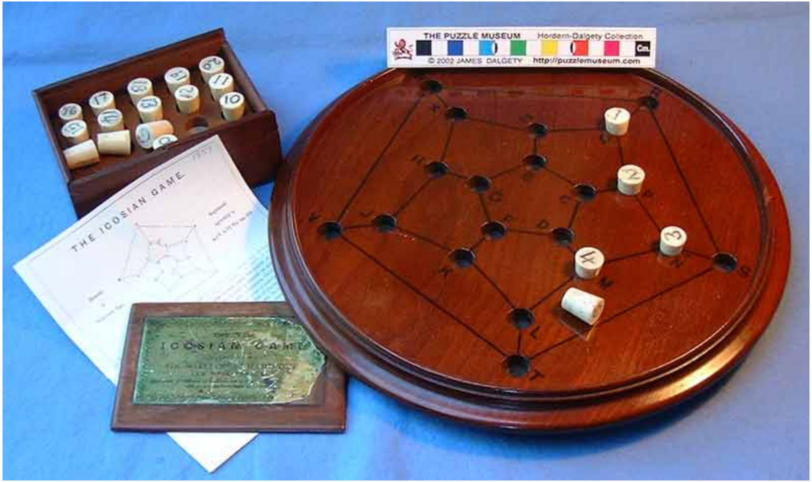
6. We can remove 30 minutes of burn-time from Rope 2 by lighting Rope 1 at both ends and Rope 2 at one end.
7. Now that we have Rope 2 at burn-length 30 minutes, start cooking the egg and light Rope 2 at the other end. When Rope 2 burns up, our egg is done!

Icosian Game

Irish mathematician William Hamilton (Dublin, 1857)



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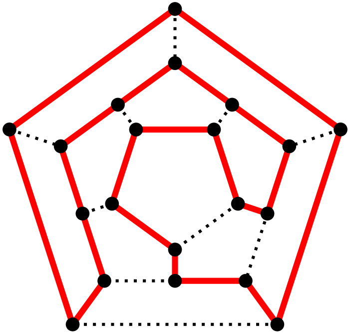

Problem (Hamiltonian Cycle)

Given a graph, decide if it contains a path that visits every node exactly once and terminates at the same starting node.

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Problem (Subset-Sum Problem)

Given a set $S = \{x_1, x_2, \ldots, x_n\}$ of integers, and an integer t (called target) decide if there is a subset of S whose sum is equal to t.

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Problem (Partition Problem)

Given a set $S = \{x_1, x_2, \ldots, x_n\}$ of numbers, decide if it can be partitioned into two sets such that they both have the same sums.

Problem (Satisfiability)

Given an expression that consists of Boolean variables connected by the symbols $\neg, \wedge, \vee, \implies$, decide if there is a way of assigning the values true and false to the variables so that the expression is true.

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A clique in a graph is a set of nodes for which any two are connected.
Given a graph and an integer n, decide if it contains a clique with n nodes.

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Problem (A Diophantine quadratic equation in two variables)

Given three positive integers a, b, c*, decide if the equation*

$$ax^2 + by = c$$

has a solution in positive integers.

Types of problems

- Decision
- Search
- Computation/Construction
- Counting
- Optimization
- …

How hard is a problem?

Here are some example problems – how hard are they to solve?

- What is $1 + 1$?

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Can you say why you feel that the first one is “easier” than the other two?
What about the last two? Is one much harder than the other, or are they both about the same?

How can we measure the hardness of a problem?

- By the type/sophistication of the machine/process required to solve it?

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- By the amount of resources used by the machine?
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  - Memory space
- By the level of difficulty encountered by the (human) solver of the problem?

Problems and problem instances

The 3 examples stated earlier were actually not *problems* but **instances of problems.**

Problems and problem instances

Problems: Generalization of a problem instance.
Not useful to do just $1 + 1$ or $1 + 2$.

Problems and problem instances

For a specific problem instance, we could measure exactly the amount of processor time and memory capacity required to solve it, using some suitable process.
However, when solving a general problem, we cannot always say exactly what resources will be used.

Problems and problem instances

- We normally express resource usage as a function of the **problem's size**.
- Also, when we ask questions about whether a problem is solvable by some machine, it is normal to allow the machine to have unlimited memory capacity and unlimited time as all problems become unsolvable at some point if finite limits are in place.
- It is for this reason, amongst others, that theoretical machines are used in classifying hardness.

O-notation scale

- Polynomial: $1, n, n^2, n^3, \ldots$ (Also, $n^k(\log n)^\ell$)
- Exponential: $2^n, 3^n, \ldots$
- Combinatorial: $n!, n^n, \ldots$

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Problem (Addition of integers)

Find $c = a + b$.

Suggested solution:

1: Select a random number r
2: **if** $r - b = a$ **then**
3: $c \leftarrow$ and stop
4: **else**
5: repeat
6: **end if**

Problems and Processes

When we measure a problem's hardness in terms of resources or machines, we are really measuring the hardness of a process used to solve the problem. The hardness of the problem should be taken to be the hardness of the most efficient process capable of solving it.

Problems and Processes

Interestingly, there exist some problems for which the most efficient processes known or even possible are “guess and check” methods – and something very interesting happens when the number of possible guesses is infinite. . . !

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Given any (finite) haystack H, decide whether H contains a needle.

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This problem is easy, though perhaps a little tedious, to solve: simply search every location within the haystack in some predefined order and terminate with the answer yes should you come across a needle. If you complete the search and no needle has been found, terminate with the answer no.

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Given any (finite) haystack H, decide whether H contains a needle.

This problem is a type of **decision problem**: given some data (the haystack) decide if the data has a certain property (needle containment).

A needle in a haystack

We may divide all possible instances of the problem into yes instances (haystacks with needles) and no instances (haystacks without needles) using our process.

- What happens if the haystacks are infinite?
- Can you still divide them into yes and no haystacks?

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Complexity Onion