The Two-Body Problem: Complete Derivation from First Principles

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1 Kepler's Laws: Introduction

Johannes Kepler, working with Tycho Brahe's precise astronomical observations, formulated three empirical laws describing planetary motion around the Sun. These laws, published between 1609 and 1619, were revolutionary in providing an accurate mathematical description of celestial mechanics:

Kepler's First Law (Law of Ellipses): The orbit of each planet is an ellipse with the Sun at one focus.

Kepler's Second Law (Law of Equal Areas): A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. This reflects the conservation of angular momentum.

Kepler's Third Law (Law of Periods): The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit: $T^2 \propto a^3$.

These empirical laws were later derived rigorously by Newton from his laws of motion and universal gravitation, demonstrating that they are consequences of the inverse-square law of gravity. In this document, we will derive these laws from first principles using classical mechanics.

2 Coordinate Systems

2.1 Spherical Polar Coordinates

In three-dimensional space, a point P can be specified using spherical polar coordinates (r, θ, ϕ), where:

- $r \geq 0$ is the radial distance from the origin
- $\theta \in [0, \pi]$ is the polar angle (angle from the positive z-axis)
- $\phi \in [0, 2\pi)$ is the azimuthal angle (angle from the positive x-axis in the xy-plane)

Transformation from Cartesian to Spherical:

$$x = r \sin\theta \cos\phi \tag{1}$$
$$y = r \sin\theta \sin\phi \tag{2}$$
$$z = r \cos\theta \tag{3}$$

Inverse transformation:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{4}$$
$$\theta = \arccos\left(\frac{z}{r}\right) \tag{5}$$
$$\phi = \arctan\left(\frac{y}{x}\right) \tag{6}$$

Basis Vectors in Spherical Coordinates:
The unit basis vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are defined as:

$$\hat{\mathbf{r}} = \sin\theta \cos\phi\, \hat{\mathbf{i}} + \sin\theta \sin\phi\, \hat{\mathbf{j}} + \cos\theta\, \hat{\mathbf{k}} \tag{7}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta \cos\phi\, \hat{\mathbf{i}} + \cos\theta \sin\phi\, \hat{\mathbf{j}} - \sin\theta\, \hat{\mathbf{k}} \tag{8}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\, \hat{\mathbf{i}} + \cos\phi\, \hat{\mathbf{j}} \tag{9}$$

These form an orthonormal basis: $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{r}} = 0$.

2.2 Plane Polar Coordinates

For motion confined to a plane (which we will show is the case for central forces), we use plane polar coordinates (r, θ):

- $r \geq 0$ is the radial distance from the origin
- $\theta \in [0, 2\pi)$ is the angle measured counterclockwise from the positive x-axis

Transformation from Cartesian to Plane Polar:

$$x = r\cos\theta \tag{10}$$
$$y = r\sin\theta \tag{11}$$

Inverse transformation:

$$r = \sqrt{x^2 + y^2} \tag{12}$$
$$\theta = \arctan\left(\frac{y}{x}\right) \tag{13}$$

2.2.1 Basis Vector Conversion

The plane polar basis vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are related to the Cartesian basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ by:

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} \tag{14}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}} \tag{15}$$

Derivation: The unit vector $\hat{\mathbf{r}}$ points radially outward from the origin. At angle θ, it makes an angle θ with the positive x-axis. Its Cartesian components are therefore $(\cos\theta, \sin\theta)$, giving equation (14).

The unit vector $\hat{\boldsymbol{\theta}}$ is perpendicular to $\hat{\mathbf{r}}$ and points in the direction of increasing θ (counterclockwise). Rotating $\hat{\mathbf{r}}$ by 90 counterclockwise:

$$\hat{\boldsymbol{\theta}} = \cos(\theta + 90)\,\hat{\mathbf{i}} + \sin(\theta + 90)\,\hat{\mathbf{j}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$$

Inverse relations:

$$\hat{\mathbf{i}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \tag{16}$$
$$\hat{\mathbf{j}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} \tag{17}$$

2.2.2 Time Derivatives of Basis Vectors

CRITICAL DISTINCTION: Unlike the Cartesian basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, which are **constant in time** (they always point in the same direction), the polar basis vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are **time-dependent** because their directions change as the particle moves.

Specifically, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ depend on $\theta(t)$, which varies with time. Therefore, we must compute their time derivatives:

$$
\begin{aligned}
\frac{d\hat{\mathbf{r}}}{dt} &= \frac{d}{dt}\left(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}\right) \\
&= -\sin\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{i}} + \cos\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{j}} \\
&= \dot{\theta}\left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\right) \\
&= \dot{\theta}\,\hat{\boldsymbol{\theta}} \qquad (18)
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\frac{d\hat{\boldsymbol{\theta}}}{dt} &= \frac{d}{dt}\left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\right) \\
&= -\cos\theta\,\dot{\theta}\,\hat{\mathbf{i}} - \sin\theta\,\dot{\theta}\,\hat{\mathbf{j}} \\
&= -\dot{\theta}\left(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}\right) \\
&= -\dot{\theta}\,\hat{\mathbf{r}} \qquad (19)
\end{aligned}
$$

Summary of time derivatives:

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \qquad (20)$$

$$\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\,\hat{\mathbf{r}} \qquad (21)$$

These are crucial for deriving velocity and acceleration in polar coordinates.

2.2.3 Velocity in Polar Coordinates

The position vector in polar coordinates is:

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$$

To find the velocity, we differentiate with respect to time:

$$
\begin{aligned}
\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} &= \frac{d}{dt}(r\,\hat{\mathbf{r}}) \\
&= \dot{r}\,\hat{\mathbf{r}} + r\,\frac{d\hat{\mathbf{r}}}{dt} \\
&= \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} \qquad (22)
\end{aligned}
$$

Physical interpretation:

- $v_r = \dot{r}$: **Radial velocity component** — the rate at which the particle moves toward or away from the origin
- $v_\theta = r\dot{\theta}$: **Tangential (transverse) velocity component** — the component perpendicular to the radial direction, representing circular motion

The magnitude of velocity is:

$$v = |\vec{\mathbf{v}}| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

2.2.4 Acceleration in Polar Coordinates

To find acceleration, we differentiate the velocity:

$$\begin{aligned}\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} &= \frac{d}{dt}\left(\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}\right) \\ &= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\theta}\,\frac{d\hat{\boldsymbol{\theta}}}{dt}\end{aligned} \tag{23}$$

Substituting (18) and (19):

$$\begin{aligned}\vec{\mathbf{a}} &= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\theta}\,(-\dot{\theta}\,\hat{\mathbf{r}}) \\ &= \ddot{r}\,\hat{\mathbf{r}} - r\dot{\theta}^2\,\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}\end{aligned} \tag{24}$$

Physical interpretation:
Radial component: $a_r = \ddot{r} - r\dot{\theta}^2$

- $\ddot{r}$: Linear acceleration in the radial direction
- $-r\dot{\theta}^2$: **Centripetal acceleration** — points inward (negative radial direction) and is necessary for circular motion. For uniform circular motion (r = constant, $\dot{r} = \ddot{r} = 0$), this reduces to $a_r = -r\dot{\theta}^2 = -v^2/r$

Tangential component: $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

- $r\ddot{\theta}$: Angular acceleration
- $2\dot{r}\dot{\theta}$: **Coriolis acceleration** — arises when the radial distance is changing ($\dot{r} \neq 0$) while rotating. This can be rewritten as $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$, which relates to angular momentum conservation

3 Newton's Laws and Inertial Frames

3.1 Newton's Laws of Motion

Newton formulated three fundamental laws that form the foundation of classical mechanics:

Newton's First Law (Law of Inertia): A body at rest remains at rest, and a body in uniform motion continues in uniform motion in a straight line, unless acted upon by an external force.

This law defines what we mean by an **inertial frame of reference** — a reference frame in which Newton's laws hold without modification.

Newton's Second Law: The rate of change of momentum of a body is equal to the net force acting upon it:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt} \tag{25}$$

For constant mass:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

Newton's Third Law (Action-Reaction): For every action, there is an equal and opposite reaction. If body 1 exerts a force $\vec{\mathbf{F}}_{12}$ on body 2, then body 2 exerts a force $\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$ on body 1.

3.2 Inertial Frames

An **inertial frame** is a reference frame in which Newton's first law holds — that is, a frame in which a free particle (one experiencing no net force) moves with constant velocity.

Important Note: There is no perfect inertial frame in the universe. The Earth rotates and orbits the Sun, the Sun orbits the galactic center, and so on. However, for many practical purposes:

- A frame fixed to the "fixed stars" (distant galaxies) is approximately inertial
- The Earth's surface is approximately inertial for short-duration experiments
- A frame with origin at the center of mass of the solar system is approximately inertial for planetary motion

The relativity principle states that the laws of physics are the same in all inertial frames. Any two inertial frames are related by a Galilean transformation (constant relative velocity).

3.3 Point Particles and Extension to Extended Bodies

Important Caveat: Newton's laws, as stated above, apply rigorously only to **point particles** — idealized objects with mass concentrated at a single point with no spatial extent.

Real physical bodies have finite size and internal structure. How can we apply Newton's laws to them?

Extension using the Third Law:

We can model any extended body as a collection of a large number of point particles (atoms, molecules). Let the body consist of N particles with positions $\vec{\mathbf{r}}_i$ and masses m_i.

Each particle i experiences:

- External forces $\vec{\mathbf{F}}_i^{\text{ext}}$ from outside the body
- Internal forces $\vec{\mathbf{F}}_{ij}$ from other particles j within the body

By Newton's third law, internal forces satisfy:

$$\vec{\mathbf{F}}_{ij} = -\vec{\mathbf{F}}_{ji}$$

When we sum over all particles, the internal forces cancel in pairs, and only external forces contribute to the net force on the system.

3.4 Newton's Second Law in Momentum Form

For a system of N particles, define the **total momentum**:

$$\vec{\mathbf{P}} = \sum_{i=1}^{N} \vec{\mathbf{p}}_i = \sum_{i=1}^{N} m_i \vec{\mathbf{v}}_i$$

The equation of motion for particle i is:

$$\frac{d\vec{\mathbf{p}}_i}{dt} = \vec{\mathbf{F}}_i^{\text{ext}} + \sum_{j \neq i} \vec{\mathbf{F}}_{ij}$$

Summing over all particles:

$$\begin{aligned} \frac{d\vec{\mathbf{P}}}{dt} &= \sum_{i=1}^{N} \frac{d\vec{\mathbf{p}}_i}{dt} \\ &= \sum_{i=1}^{N} \vec{\mathbf{F}}_i^{\text{ext}} + \sum_{i=1}^{N} \sum_{j \neq i} \vec{\mathbf{F}}_{ij} \end{aligned} \tag{26}$$

The double sum over internal forces vanishes because for every pair:

$$\vec{\mathbf{F}}_{ij} + \vec{\mathbf{F}}_{ji} = 0$$

Therefore:

$$\frac{d\vec{\mathbf{P}}}{dt} = \vec{\mathbf{F}}^{\text{ext}} = \sum_{i=1}^{N} \vec{\mathbf{F}}_i^{\text{ext}} \tag{27}$$

Conclusion: The rate of change of the total momentum of a system equals the total external force. Internal forces do not affect the motion of the center of mass.

4 The Two-Body Problem: Setup and Assumptions

4.1 Assumptions and Simplifications

We now specialize to a system of two bodies interacting gravitationally. We make the following assumptions:

1. **Gravitational forces act between centers:** Each body can be treated as if its entire mass were concentrated at its geometric center (valid for spherically symmetric bodies by Newton's shell theorem)

2. **Point particle approximation:** The separation between the bodies is much larger than their physical sizes, so we can treat them as point masses

3. **Isolated system:** No external forces act on the system ($\vec{\mathbf{F}}^{\text{ext}} = 0$)

4. **Newtonian gravity:** The gravitational force follows Newton's inverse-square law

4.2 The N-Body Problem and Solar System Dynamics

The **N-body problem** seeks to determine the motion of N gravitationally interacting bodies given their initial positions and velocities. For $N \geq 3$, this problem has no general closed-form solution — it is one of the classic unsolved problems in classical mechanics.

However, the **two-body problem** ($N = 2$) *can* be solved exactly. Moreover, it provides an excellent approximation for the solar system because:

- The Sun's mass $M_\odot \approx 1.989 \times 10^{30}$ kg is much greater than any planet's mass

- Jupiter (most massive planet): $M_J \approx 1.898 \times 10^{27}$ kg, so $M_J/M_\odot \approx 10^{-3}$
- Earth: $M_\oplus/M_\odot \approx 3 \times 10^{-6}$

Therefore, **to first approximation**, each planet moves under the gravitational influence of the Sun alone, with negligible effects from other planets. The influences of other planets can be treated as small **perturbations** to the two-body solution.

This approach — solving the two-body problem and adding perturbations — forms the basis of celestial mechanics and has been remarkably successful in predicting planetary motions, satellite orbits, and spacecraft trajectories.

4.3 Equations of Motion for Two Particles

Consider two particles with masses m_1 and m_2 at positions $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ in an inertial frame.

Newton's second law for each particle:

$$m_1\ddot{\vec{\mathbf{r}}}_1 = \vec{\mathbf{F}}_{12} \tag{28}$$

$$m_2\ddot{\vec{\mathbf{r}}}_2 = \vec{\mathbf{F}}_{21} \tag{29}$$

By Newton's third law:

$$\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12} \tag{30}$$

Gravitational force: The force on particle 1 due to particle 2 is:

$$\vec{\mathbf{F}}_{12} = -\frac{Gm_1m_2}{|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|^3}(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) \tag{31}$$

The negative sign indicates attraction (force points from 1 toward 2).

4.4 Conservation of Total Momentum

Adding equations (28) and (29):

$$\begin{aligned} m_1\ddot{\vec{\mathbf{r}}}_1 + m_2\ddot{\vec{\mathbf{r}}}_2 &= \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{21} \\ &= \vec{\mathbf{F}}_{12} - \vec{\mathbf{F}}_{12} = 0 \end{aligned} \tag{32}$$

Therefore:

$$\frac{d}{dt}(m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2) = 0 \tag{33}$$

This states that the **total momentum is conserved**:

$$\vec{\mathbf{P}} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 = \text{constant}$$

Reason: The sum of internal forces vanishes due to Newton's third law. Since there are no external forces ($\vec{\mathbf{F}}^{\text{ext}} = 0$), momentum is conserved.

4.5 Center of Mass Coordinates

Define the **center of mass (COM) position**:

$$\vec{\mathbf{R}} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{M} \tag{34}$$

where $M = m_1 + m_2$ is the total mass.

Define the **relative position vector**:

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 \tag{35}$$

We can invert these relations to express individual positions in terms of COM and relative coordinates:

$$\vec{\mathbf{r}}_1 = \vec{\mathbf{R}} + \frac{m_2}{M}\vec{\mathbf{r}} \tag{36}$$

$$\vec{\mathbf{r}}_2 = \vec{\mathbf{R}} - \frac{m_1}{M}\vec{\mathbf{r}} \tag{37}$$

Derivation: From (34):

$$M\vec{\mathbf{R}} = m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2$$

Substitute $\vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}$:

$$\begin{aligned} M\vec{\mathbf{R}} &= m_1\vec{\mathbf{r}}_1 + m_2(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}) \\ &= (m_1 + m_2)\vec{\mathbf{r}}_1 - m_2\vec{\mathbf{r}} \\ &= M\vec{\mathbf{r}}_1 - m_2\vec{\mathbf{r}} \end{aligned}$$

Solving for $\vec{\mathbf{r}}_1$:

$$\vec{\mathbf{r}}_1 = \vec{\mathbf{R}} + \frac{m_2}{M}\vec{\mathbf{r}}$$

Similarly for $\vec{\mathbf{r}}_2$.

4.6 Dynamics of Center of Mass

From momentum conservation (33):

$$m_1\dot{\vec{\mathbf{r}}}_1 + m_2\dot{\vec{\mathbf{r}}}_2 = \text{constant}$$

But the left side equals $M\dot{\vec{\mathbf{R}}}$, so:

$$\dot{\vec{\mathbf{R}}} = \text{constant} \equiv \vec{\mathbf{V}}_{\text{CM}} \tag{38}$$

Therefore:

$$\ddot{\vec{\mathbf{R}}} = 0 \tag{39}$$

Conclusion: The center of mass moves with constant velocity (or is at rest) in the absence of external forces. This is a direct consequence of momentum conservation.

4.7 Equation for Relative Motion

Subtracting (29) from (28):

$$m_1\ddot{\vec{\mathbf{r}}}_1 - m_2\ddot{\vec{\mathbf{r}}}_2 = \vec{\mathbf{F}}_{12} - \vec{\mathbf{F}}_{21} = 2\vec{\mathbf{F}}_{12}$$

From (36) and (37):

$$\begin{aligned}\ddot{\vec{\mathbf{r}}} &= \ddot{\vec{\mathbf{r}}}_1 - \ddot{\vec{\mathbf{r}}}_2\\ \ddot{\vec{\mathbf{r}}}_1 &= \frac{m_2}{M}\ddot{\vec{\mathbf{r}}}\\ \ddot{\vec{\mathbf{r}}}_2 &= -\frac{m_1}{M}\ddot{\vec{\mathbf{r}}}\end{aligned}$$

Substituting:

$$\begin{aligned}m_1\frac{m_2}{M}\ddot{\vec{\mathbf{r}}} + m_2\frac{m_1}{M}\ddot{\vec{\mathbf{r}}} &= 2\vec{\mathbf{F}}_{12}\\ \frac{2m_1m_2}{M}\ddot{\vec{\mathbf{r}}} &= 2\vec{\mathbf{F}}_{12}\end{aligned}$$

Define the **reduced mass**:

$$\mu = \frac{m_1m_2}{m_1+m_2} = \frac{m_1m_2}{M} \tag{40}$$

Then:

$$\mu\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{F}}_{12} \tag{41}$$

For gravitational force (31):

$$\mu\ddot{\vec{\mathbf{r}}} = -\frac{Gm_1m_2}{r^3}\vec{\mathbf{r}} = -\frac{GM\mu}{r^2}\hat{\mathbf{r}} \tag{42}$$

where $r = |\vec{\mathbf{r}}|$ and $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$.

Remarkable Result: We have reduced the two-body problem to an equivalent **one-body problem** — a single particle of mass μ moving under a central force with effective gravitational parameter GM.

4.8 Transformation to COM Frame vs. COM Coordinates

Important Distinction:

- **Changing to COM coordinates** means expressing the system in terms of $\vec{\mathbf{R}}$ and $\vec{\mathbf{r}}$ instead of $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$. This is just a change of variables.
- **Changing to the COM frame** means choosing a new reference frame with origin at the center of mass and moving with velocity $\vec{\mathbf{V}}_{\text{CM}}$. This is a change of reference frame (Galilean transformation).

In the **COM frame**:

- The origin is at $\vec{\mathbf{R}} = 0$ always
- The frame moves with constant velocity $\vec{\mathbf{V}}_{\text{CM}}$ relative to the original inertial frame

- Since the COM frame is also inertial (uniform motion), Newton's laws hold in this frame
- The two particles have positions $\vec{\mathbf{r}}_1^*$ and $\vec{\mathbf{r}}_2^*$ related by $m_1\vec{\mathbf{r}}_1^* + m_2\vec{\mathbf{r}}_2^* = 0$

Conservation of Energy and Angular Momentum:

When we change reference frames (Galilean transformation), kinetic energy and angular momentum are *not* invariant in general. However:

Energy: The total kinetic energy can be separated:

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}MV_{\text{CM}}^2 + \frac{1}{2}\mu v^2 \tag{43}$$

where $v = |\dot{\vec{\mathbf{r}}}|$ is the relative speed. The first term is the kinetic energy of COM motion; the second is the kinetic energy in the COM frame.

Proof: Using (36) and (37):

$$\begin{aligned}\vec{\mathbf{v}}_1 &= \dot{\vec{\mathbf{R}}} + \frac{m_2}{M}\dot{\vec{\mathbf{r}}} = \vec{\mathbf{V}}_{\text{CM}} + \frac{m_2}{M}\vec{\mathbf{v}} \\ \vec{\mathbf{v}}_2 &= \vec{\mathbf{V}}_{\text{CM}} - \frac{m_1}{M}\vec{\mathbf{v}}\end{aligned}$$

Then:

$$\begin{aligned}T &= \frac{1}{2}m_1|\vec{\mathbf{V}}_{\text{CM}} + \frac{m_2}{M}\vec{\mathbf{v}}|^2 + \frac{1}{2}m_2|\vec{\mathbf{V}}_{\text{CM}} - \frac{m_1}{M}\vec{\mathbf{v}}|^2 \\ &= \frac{1}{2}m_1 V_{\text{CM}}^2 + \frac{1}{2}m_2 V_{\text{CM}}^2 + m_1\vec{\mathbf{V}}_{\text{CM}} \cdot \frac{m_2}{M}\vec{\mathbf{v}} - m_2\vec{\mathbf{V}}_{\text{CM}} \cdot \frac{m_1}{M}\vec{\mathbf{v}} \\ &\quad + \frac{1}{2}m_1\frac{m_2^2}{M^2}v^2 + \frac{1}{2}m_2\frac{m_1^2}{M^2}v^2\end{aligned}$$

The cross terms cancel:

$$m_1\frac{m_2}{M} - m_2\frac{m_1}{M} = 0$$

The remaining terms give:

$$T = \frac{1}{2}MV_{\text{CM}}^2 + \frac{1}{2}\mu v^2$$

In the COM frame, $V_{\text{CM}} = 0$, so:

$$T^* = \frac{1}{2}\mu v^2 \tag{44}$$

The potential energy $V(r)$ depends only on the relative distance, so it's the same in both frames. Therefore, the **total energy in the COM frame** is:

$$E^* = \frac{1}{2}\mu v^2 + V(r) \tag{45}$$

Angular Momentum: The total angular momentum is:

$$\vec{\mathbf{L}} = m_1\vec{\mathbf{r}}_1 \times \vec{\mathbf{v}}_1 + m_2\vec{\mathbf{r}}_2 \times \vec{\mathbf{v}}_2 \tag{46}$$

This can be decomposed as:

$$\vec{\mathbf{L}} = \vec{\mathbf{R}} \times M\vec{\mathbf{V}}_{\text{CM}} + \vec{\mathbf{L}}^* \tag{47}$$

where $\vec{\mathbf{L}}^* = \mu\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ is the angular momentum in the COM frame.
In the COM frame, the first term vanishes ($\vec{\mathbf{R}} = 0$), leaving:

$$\vec{\mathbf{L}}^* = \mu\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}} \tag{48}$$

Conservation: Since $\vec{\mathbf{F}}_{12}$ is central (parallel to $\vec{\mathbf{r}}$), the torque vanishes, and both E^* and $\vec{\mathbf{L}}^*$ are conserved in the COM frame.

4.9 Example: Sun-Earth System

Consider the Sun-Earth system:

- $m_1 = M_\odot = 1.989 \times 10^{30}$ kg (Sun)
- $m_2 = M_\oplus = 5.972 \times 10^{24}$ kg (Earth)
- $M = M_\odot + M_\oplus \approx M_\odot$ (since $M_\oplus/M_\odot \approx 3 \times 10^{-6}$)

Reduced mass:

$$\begin{aligned}
\mu &= \frac{M_\odot M_\oplus}{M_\odot + M_\oplus} \\
&\approx \frac{M_\odot M_\oplus}{M_\odot} = M_\oplus \left(1 - \frac{M_\oplus}{M_\odot}\right) \\
&\approx M_\oplus \quad \text{(to high accuracy)}
\end{aligned}$$

Center of mass position:

$$\begin{aligned}
\vec{\mathbf{R}} &= \frac{M_\odot \vec{\mathbf{r}}_\odot + M_\oplus \vec{\mathbf{r}}_\oplus}{M_\odot + M_\oplus} \\
&\approx \vec{\mathbf{r}}_\odot + \frac{M_\oplus}{M_\odot}(\vec{\mathbf{r}}_\oplus - \vec{\mathbf{r}}_\odot)
\end{aligned}$$

The COM is displaced from the Sun's center by:

$$\Delta r = \frac{M_\oplus}{M_\odot} \times \text{(Earth-Sun distance)} \approx 3 \times 10^{-6} \times 1.5 \times 10^{11}\ \text{m} \approx 450\ \text{km}$$

This is less than the Sun's radius ($R_\odot \approx 696,000$ km), so the COM is *inside* the Sun!
Approximation for $m_1 \gg m_2$:
When one mass dominates ($m_1/m_2 \gg 1$):

$$\begin{aligned}
\mu &\approx m_2 \\
\vec{\mathbf{R}} &\approx \vec{\mathbf{r}}_1 \\
\vec{\mathbf{r}} &\approx \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1
\end{aligned}$$

In this limit, the heavy body remains essentially at rest at the COM, and the lighter body orbits around it. This justifies treating the Sun as fixed when analyzing planetary orbits.

5 Central Forces

5.1 Definition

A **central force** is a force that:

1. Points along the line joining the two particles (radial direction)
2. Depends only on the distance r between the particles

Mathematically:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = f(r)\hat{\mathbf{r}} \tag{49}$$

where $f(r)$ is a scalar function and $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$.
Example: Newton's gravitational force is central:

$$\vec{\mathbf{F}} = -\frac{GM\mu}{r^2}\hat{\mathbf{r}}$$

Here $f(r) = -GM\mu/r^2 < 0$ (attractive).

5.2 Properties of Central Forces

5.2.1 Property 1: Conservation of Angular Momentum

Theorem: For a central force, the angular momentum $\vec{\mathbf{L}} = \mu\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ is conserved.
Proof: The torque about the origin is:

$$\begin{aligned}\vec{\boldsymbol{\tau}} &= \vec{\mathbf{r}} \times \vec{\mathbf{F}} \\ &= \vec{\mathbf{r}} \times (f(r)\hat{\mathbf{r}}) \\ &= f(r)(\vec{\mathbf{r}} \times \hat{\mathbf{r}}) \\ &= f(r)\left(\vec{\mathbf{r}} \times \frac{\vec{\mathbf{r}}}{r}\right) \\ &= \frac{f(r)}{r}(\vec{\mathbf{r}} \times \vec{\mathbf{r}}) = 0\end{aligned} \tag{50}$$

Since $\vec{\mathbf{r}} \times \vec{\mathbf{r}} = 0$ (parallel vectors), the torque vanishes.
By the rotational equation of motion:

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\boldsymbol{\tau}} = 0 \implies \vec{\mathbf{L}} = \text{constant} \tag{51}$$

In polar coordinates, the angular momentum is:

$$\vec{\mathbf{L}} = \mu\vec{\mathbf{r}} \times \vec{\mathbf{v}} = \mu(r\hat{\mathbf{r}}) \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) \tag{52}$$

Using $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0$ and $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$ (unit vector perpendicular to the plane):

$$\vec{\mathbf{L}} = \mu r^2\dot{\theta}\,\hat{\mathbf{k}} \tag{53}$$

The magnitude is:

$$L = \mu r^2\dot{\theta} = \text{constant} \tag{54}$$

5.2.2 Property 2: Motion in a Plane

Theorem: Under a central force, the motion is confined to a plane.

Proof: Since $\vec{\mathbf{L}}$ is constant, it defines a fixed direction in space. At any time:

$$\vec{\mathbf{r}} \cdot \vec{\mathbf{L}} = 0 \quad \text{and} \quad \vec{\mathbf{v}} \cdot \vec{\mathbf{L}} = 0$$

Therefore, both $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ lie in the plane perpendicular to $\vec{\mathbf{L}}$.

Exception: If $\vec{\mathbf{r}} \parallel \vec{\mathbf{v}}$ initially (e.g., radial motion), then $\vec{\mathbf{L}} = 0$, and the motion is along a straight line through the origin.

Consequence: We can choose coordinates so that the motion lies in the xy-plane and use plane polar coordinates (r, θ).

5.3 Equation of Motion in Polar Coordinates

Using (24), Newton's second law $\mu\vec{\mathbf{a}} = \vec{\mathbf{F}}$ becomes:

$$\mu \left[(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \right] = f(r)\hat{\mathbf{r}} \tag{55}$$

Equating components:

Radial equation:

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r) \tag{56}$$

Tangential equation:

$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{57}$$

The tangential equation can be rewritten as:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$\implies r^2\dot{\theta} = \text{constant} = \frac{L}{\mu} \tag{58}$$

This is just angular momentum conservation (54).

5.4 Areal Velocity (Kepler's Second Law)

Define the **areal velocity** as the rate at which the position vector sweeps out area.

Consider a small time interval dt. The particle moves from position $\vec{\mathbf{r}}$ to $\vec{\mathbf{r}} + d\vec{\mathbf{r}}$. The area swept is approximately a triangle with base r and height $|d\vec{\mathbf{r}}| \sin\alpha$, where α is the angle between $\vec{\mathbf{r}}$ and $d\vec{\mathbf{r}}$.

More precisely, the area swept is:

$$dA = \frac{1}{2}|\vec{\mathbf{r}} \times d\vec{\mathbf{r}}| = \frac{1}{2}|\vec{\mathbf{r}} \times \vec{\mathbf{v}}|dt \tag{59}$$

Therefore:

$$\frac{dA}{dt} = \frac{1}{2}|\vec{\mathbf{r}} \times \vec{\mathbf{v}}| = \frac{L}{2\mu} \tag{60}$$

Since L is constant, dA/dt is constant. This is **Kepler's Second Law**: The radius vector sweeps out equal areas in equal times.

In polar coordinates:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu} = \text{constant} \tag{61}$$

6 Energy and Conservative Forces

6.1 Conservative Forces

A force is **conservative** if:

1. It can be expressed as the negative gradient of a scalar potential energy function: $\vec{\mathbf{F}} = -\nabla U$
2. The work done by the force is independent of path
3. The work done around any closed loop is zero: $\oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$

These conditions are equivalent.

Property of Conservative Forces: The total mechanical energy $E = T + U$ is conserved.

Proof: The work-energy theorem states:

$$dT = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\nabla U \cdot d\vec{\mathbf{r}} = -dU$$

Therefore:

$$d(T+U) = 0 \implies E = T + U = \text{constant}$$

6.2 Central Forces are Conservative

Theorem: Any central force $\vec{\mathbf{F}} = f(r)\hat{\mathbf{r}}$ is conservative.

Proof: Define:

$$U(r) = -\int_{r_0}^{r} f(r')\,dr' \tag{62}$$

where r_0 is a reference point (often taken as $r_0 = \infty$).

Then:

$$\begin{aligned}
\nabla U &= \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}} \\
&= \frac{dU}{dr}\hat{\mathbf{r}} \quad (\text{since } U \text{ depends only on } r) \\
&= -f(r)\hat{\mathbf{r}}
\end{aligned} \tag{63}$$

Therefore $\vec{\mathbf{F}} = -\nabla U$, confirming that the force is conservative.

6.3 Total Energy in Polar Coordinates

The kinetic energy in polar coordinates is (from (22)):

$$T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) \tag{64}$$

We can separate this into:

- **Radial kinetic energy:** $T_r = \frac{1}{2}\mu\dot{r}^2$
- **Tangential (rotational) kinetic energy:** $T_\theta = \frac{1}{2}\mu r^2\dot{\theta}^2$

Using (54), we can express the rotational term in terms of angular momentum:

$$T_\theta = \frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{L^2}{2\mu r^2} \tag{65}$$

This term acts like a potential energy that repels the particle from the origin — the **centrifugal barrier**.

Define the **centrifugal potential energy**:

$$U_{\text{centrifugal}}(r) = \frac{L^2}{2\mu r^2} \tag{66}$$

The total energy is:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + U(r) \tag{67}$$

Example: Force from Centrifugal Potential

The centrifugal "force" is:

$$\begin{aligned} F_{\text{centrifugal}} &= -\frac{dU_{\text{centrifugal}}}{dr} \\ &= -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) \\ &= \frac{L^2}{\mu r^3} = \mu r\dot{\theta}^2 \end{aligned} \tag{68}$$

This is the familiar centrifugal term in (56).

6.4 Effective Potential

Define the **effective potential**:

$$U_{\text{eff}}(r) = U(r) + \frac{L^2}{2\mu r^2} \tag{69}$$

Then:

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) \tag{70}$$

Solving for $\dot{r}$:

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}[E - U_{\text{eff}}(r)]} \tag{71}$$

This can be integrated to find $r(t)$:

$$t = \int \frac{dr}{\sqrt{\frac{2}{\mu}[E - U_{\text{eff}}(r)]}} \tag{72}$$

However, we typically want the orbit shape $r(\theta)$ rather than $r(t)$.

Using $dt = d\theta/\dot{\theta} = (r^2/L)d\theta\mu$:

$$\theta = \int \frac{L/(\mu r^2)}{\sqrt{\frac{2}{\mu}[E - U_{\text{eff}}(r)]}}dr \tag{73}$$

6.5 Gravitational Potential Energy

For Newton's gravitational force:

$$f(r) = -\frac{GM\mu}{r^2}$$

From (62) with $r_0 = \infty$ (where $U(\infty) = 0$):

$$\begin{aligned}
U(r) &= -\int_{\infty}^{r} \left(-\frac{GM\mu}{r'^2}\right) dr' \\
&= -GM\mu \int_{\infty}^{r} \frac{dr'}{r'^2} \\
&= -GM\mu \left[-\frac{1}{r'}\right]_{\infty}^{r} \\
&= -GM\mu \left(-\frac{1}{r} + 0\right) \\
&= -\frac{GM\mu}{r}
\end{aligned} \tag{74}$$

Convention: We choose $U(\infty) = 0$, which means bound orbits have $E < 0$.

6.6 Effective Potential for Gravity

Combining (74) with (69):

$$U_{\text{eff}}(r) = -\frac{GM\mu}{r} + \frac{L^2}{2\mu r^2} \tag{75}$$

For convenience, define:

$$C = GM\mu \tag{76}$$

Then:

$$U_{\text{eff}}(r) = -\frac{C}{r} + \frac{L^2}{2\mu r^2} \tag{77}$$

6.7 Analysis of Effective Potential

To find critical points, set $dU_{\text{eff}}/dr = 0$:

$$\begin{aligned}
\frac{dU_{\text{eff}}}{dr} &= \frac{C}{r^2} - \frac{L^2}{\mu r^3} = 0 \\
\frac{C}{r^2} &= \frac{L^2}{\mu r^3} \\
r^* &= \frac{L^2}{\mu C}
\end{aligned} \tag{78}$$

This is the location of the **minimum** of U_{eff}, corresponding to a **circular orbit**. The value at the minimum is:

$$\begin{aligned}
U_{\min} = U_{\text{eff}}(r^*) &= -\frac{C}{r^*} + \frac{L^2}{2\mu (r^*)^2} \\
&= -\frac{\mu C^2}{L^2} + \frac{\mu C^2}{2L^2} \\
&= -\frac{\mu C^2}{2L^2}
\end{aligned} \tag{79}$$

Classification of orbits by energy:

- **Circular orbit** ($E = U_{\min}$): The particle remains at $r = r^*$
- **Elliptical orbits** ($U_{\min} < E < 0$): Bound orbits with $r_{\min} < r < r_{\max}$
- **Parabolic orbit** ($E = 0$): Marginally unbound, particle escapes to infinity with zero velocity
- **Hyperbolic orbits** ($E > 0$): Unbound, particle escapes with nonzero velocity at infinity

7 Solving the Orbit Equation

7.1 Deriving the Orbit Equation

From (70):

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)$$

where:

$$U_{\text{eff}}(r) = -\frac{C}{r} + \frac{L^2}{2\mu r^2}$$

We want to find $r(\theta)$. Use the chain rule:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot \frac{L}{\mu r^2}$$

Substituting into the energy equation:

$$E = \frac{1}{2}\mu\left(\frac{dr}{d\theta}\right)^2 \frac{L^2}{\mu^2 r^4} - \frac{C}{r} + \frac{L^2}{2\mu r^2}$$

Multiply through by $\frac{2\mu}{L^2}$:

$$\frac{2\mu E}{L^2} = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 - \frac{2\mu C}{L^2 r} + \frac{1}{r^2}$$

Substitution 1: Change variable from r to $u = 1/r$
Let:

$$u = \frac{1}{r} \implies r = \frac{1}{u} \tag{80}$$

Then:

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

Substituting:

$$\frac{2\mu E}{L^2} = u^4 \cdot \frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2 - \frac{2\mu C}{L^2}u + u^2$$

Simplifying:

$$\frac{2\mu E}{L^2} = \left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2\mu C}{L^2}u$$

Substitution 2: Complete the square
Let:

$$a = -\frac{\mu C}{L^2} \tag{81}$$

Then:

$$\left(\frac{du}{d\theta}\right)^2 + u^2 + 2au = \frac{2\mu E}{L^2}$$

Complete the square on the left:

$$\left(\frac{du}{d\theta}\right)^2 + (u+a)^2 = \frac{2\mu E}{L^2} + a^2$$

Define:

$$\frac{1}{b^2} = \frac{2\mu E}{L^2} + a^2 = \frac{2\mu E}{L^2} + \frac{\mu^2 C^2}{L^4} \tag{82}$$

Then:

$$\left(\frac{du}{d\theta}\right)^2 + (u+a)^2 = \frac{1}{b^2}$$

Substitution 3: Let $s = u + a$
Then:

$$\left(\frac{ds}{d\theta}\right)^2 + s^2 = \frac{1}{b^2}$$

Multiply by b^2:

$$b^2\left(\frac{ds}{d\theta}\right)^2 + (bs)^2 = 1$$

Substitution 4: Let $w = bs$
Then:

$$\left(\frac{dw}{d\theta}\right)^2 + w^2 = 1$$

This is the equation for simple harmonic motion! The general solution is:

$$w = \cos(\theta - \theta_0) \tag{83}$$

where θ_0 is a constant of integration.

7.2 Back-substituting to Find $r(\theta)$

Working backwards:

$$w = bs = b(u+a) = \cos(\theta - \theta_0) \tag{84}$$

$$u + a = \frac{1}{b}\cos(\theta - \theta_0) \tag{85}$$

$$u = \frac{1}{b}\cos(\theta - \theta_0) - a \tag{86}$$

$$\frac{1}{r} = \frac{1}{b}\cos(\theta - \theta_0) + \frac{\mu C}{L^2} \tag{87}$$

Multiply through by r:

$$1 = \frac{r}{b}\cos(\theta - \theta_0) + \frac{\mu C r}{L^2}$$

Solving for r:

$$r = \frac{1}{\frac{1}{b}\cos(\theta - \theta_0) + \frac{\mu C}{L^2}}$$

Multiply numerator and denominator by $L^2/(\mu C)$:

$$r = \frac{L^2/(\mu C)}{1 + \frac{L^2}{b\mu C}\cos(\theta - \theta_0)}$$

From (82):

$$\frac{1}{b^2} = \frac{2\mu E}{L^2} + \frac{\mu^2 C^2}{L^4}$$

Multiply by $L^4/(\mu C)^2$:

$$\frac{L^4}{b^2\mu^2 C^2} = \frac{2EL^2}{\mu C^2} + 1$$

Let:

$$r^* = \frac{L^2}{\mu C} \tag{88}$$

and

$$e = \frac{L^2}{b\mu C} = \sqrt{1 + \frac{2EL^2}{\mu C^2}} \tag{89}$$

Then the orbit equation becomes:

$$r = \frac{r^*}{1 + e\cos(\theta - \theta_0)} \tag{90}$$

Choice of θ_0: We conventionally choose coordinates so that perihelion (closest approach) occurs at $\theta = 0$. This requires $\theta_0 = \pi$, giving:

$$r(\theta) = \frac{r^*}{1 - e\cos\theta} \tag{91}$$

Physical meaning of $\theta_0 = \pi$: At perihelion, r is minimum. From (90), r is minimum when $\cos(\theta - \theta_0) = +1$, i.e., $\theta = \theta_0$. Setting $\theta_0 = \pi$ means perihelion is at $\theta = \pi$, but we conventionally measure θ from perihelion, so we redefine $\theta \to \theta + \pi$, giving (91).

7.3 Conversion to Cartesian Coordinates

Using $x = r\cos\theta$ and $y = r\sin\theta$:

$$r = \frac{r^*}{1 - e\cos\theta} \implies r(1 - e\cos\theta) = r^*$$

$$r - ex = r^*$$

$$r = r^* + ex$$

Square both sides:

$$x^2 + y^2 = (r^* + ex)^2 = r^{*2} + 2r^*ex + e^2x^2$$

Rearranging:

$$x^2(1 - e^2) - 2r^*ex + y^2 = r^{*2}$$

This is the equation of a **conic section** with one focus at the origin.

7.4 Case-by-Case Analysis

7.4.1 Case 1: Parabola ($e = 1$, $E = 0$)

From (89): $e = 1 \implies 1 + 2EL^2/(\mu C^2) = 1 \implies E = 0$.

The orbit equation is:

$$r = \frac{r^*}{1 - \cos\theta}$$

In Cartesian form:

$$y^2 = r^{*2} + 2r^*x$$

Or:

$$y^2 = 2r^*(x + r^*/2)$$

This is a parabola with vertex at $(-r^*/2, 0)$ and focus at origin.

7.4.2 Case 2: Ellipse ($0 < e < 1$, $E < 0$)

From (89): $0 < e < 1 \implies E < 0$ (bound orbit).

The Cartesian equation is:

$$x^2(1 - e^2) - 2r^*ex + y^2 = r^{*2}$$

Complete the square in x:

$$(1 - e^2)\left[x^2 - \frac{2r^*e}{1 - e^2}x\right] + y^2 = r^{*2}$$

$$(1 - e^2)\left[\left(x - \frac{r^*e}{1 - e^2}\right)^2 - \left(\frac{r^*e}{1 - e^2}\right)^2\right] + y^2 = r^{*2}$$

$$(1 - e^2)\,(x - x_0)^2 + y^2 = r^{*2} + (1 - e^2)\left(\frac{r^*e}{1 - e^2}\right)^2$$

where:

$$x_0 = \frac{r^*e}{1 - e^2} = \frac{eL^2/(\mu C)}{1 - e^2} \tag{92}$$

The right side simplifies to:

$$r^{*2} + \frac{r^{*2}e^2}{1 - e^2} = \frac{r^{*2}(1 - e^2) + r^{*2}e^2}{1 - e^2} = \frac{r^{*2}}{1 - e^2}$$

So:

$$(1 - e^2)(x - x_0)^2 + y^2 = \frac{r^{*2}}{1 - e^2}$$

Divide by $r^{*2}/(1 - e^2)$:

$$\frac{(x - x_0)^2}{r^{*2}/(1 - e^2)^2} + \frac{y^2}{r^{*2}/(1 - e^2)} = 1$$

Define:

$$a = \frac{r^*}{1-e^2} = \frac{L^2}{\mu C(1-e^2)} \tag{93}$$

$$b = \frac{r^*}{\sqrt{1-e^2}} = \frac{L^2}{\mu C\sqrt{1-e^2}} \tag{94}$$

Then:

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{95}$$

This is an **ellipse** with semi-major axis a, semi-minor axis b, and center at $(x_0, 0)$.

Important: The **focus is at the origin** $(0,0)$, not at the center. The distance from center to focus is:

$$c = x_0 = ea$$

From (93) and (89):

$$a = \frac{r^*}{1-e^2} = \frac{L^2}{\mu C(1-e^2)} = \frac{L^2}{\mu C} \cdot \frac{1}{1-\left[1+\frac{2EL^2}{\mu C^2}\right]} = -\frac{C}{2E}$$

Therefore:

$$a = -\frac{C}{2E} = -\frac{GM\mu}{2E} \tag{96}$$

Physical consequence: All elliptical orbits with the same semi-major axis a have the same energy $E = -C/(2a)$, regardless of eccentricity e.

7.4.3 Case 3: Hyperbola ($e > 1$, $E > 0$)

From (89): $e > 1 \implies E > 0$ (unbound orbit).

The analysis is similar to the ellipse case, but now $e^2 - 1 > 0$. The standard form is:

$$\frac{(x-x_0)^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{97}$$

where:

$$a = \frac{r^*}{e^2-1} = \frac{C}{2E} \tag{98}$$

$$b = \frac{r^*}{\sqrt{e^2-1}} \tag{99}$$

Again, the **focus is at the origin**.

7.4.4 Case 4: Circle ($e = 0$, $E = U_{\min}$)

From (89): $e = 0 \implies 2EL^2/(\mu C^2) = -1 \implies E = -\mu C^2/(2L^2) = U_{\min}$.

The orbit is:

$$r = r^* = \frac{L^2}{\mu C} = \text{constant}$$

This is a **circle** of radius r^*, centered at the origin (where the central body is located).

8 Kepler's Laws Derived

8.1 Kepler's First Law

Statement: The orbit of each planet is an ellipse with the Sun at one focus.

Derivation: We have shown that for an inverse-square law force ($f(r) \propto 1/r^2$), the orbit equation is:

$$r = \frac{r^*}{1 - e\cos\theta}$$

For bound orbits ($E < 0$), we have $0 < e < 1$, and this represents an ellipse. The Sun (central body) is at the origin, which is one focus of the ellipse, as shown in the Cartesian form (95).

Conclusion: Kepler's First Law is an **exact** consequence of Newton's laws and inverse-square gravity.

8.2 Kepler's Second Law

Statement: A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Derivation: We derived this as the areal velocity law (60):

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{constant}$$

This is a consequence of angular momentum conservation, which holds for any central force.

Conclusion: Kepler's Second Law is an **exact** consequence of the central nature of the gravitational force.

8.3 Kepler's Third Law

Statement: The square of the orbital period is proportional to the cube of the semi-major axis:

$$T^2 \propto a^3$$

Derivation: The period T is the time for one complete orbit. The total area of the ellipse is:

$$A_{\text{total}} = \pi ab$$

Using (61):

$$A_{\text{total}} = \frac{L}{2\mu}T$$

Therefore:

$$T = \frac{2\mu\pi ab}{L}$$

From (94):

$$b = a\sqrt{1-e^2} = a\sqrt{1 - \frac{L^2(\mu C)^2}{a^2(\mu C)^2}} = \frac{L}{\mu}\sqrt{\frac{a}{C}}$$

Wait, let me redo this more carefully. From (93) and (94):

$$\frac{b^2}{a^2} = \frac{1-e^2}{1} = 1 - e^2$$

And from (96) and (88):

$$a = \frac{r^*}{1-e^2}, \quad r^* = \frac{L^2}{\mu C}$$

So:

$$b^2 = a^2(1-e^2) = a \cdot r^* = a \cdot \frac{L^2}{\mu C}$$

Therefore:

$$b = \sqrt{\frac{aL^2}{\mu C}}$$

Substituting into the period formula:

$$\begin{aligned} T &= \frac{2\mu\pi ab}{L} = \frac{2\mu\pi a}{L}\sqrt{\frac{aL^2}{\mu C}} \\ &= 2\pi\sqrt{\frac{\mu a^3}{C}} = 2\pi\sqrt{\frac{\mu a^3}{GM\mu}} \\ &= 2\pi\sqrt{\frac{a^3}{GM}} \end{aligned} \tag{100}$$

Squaring:

$$T^2 = \frac{4\pi^2}{GM}a^3 \tag{101}$$

Important Caveat: In our derivation, $M = m_1 + m_2$ is the total mass. Kepler originally stated his third law as $T^2 \propto a^3$ with the constant of proportionality depending only on the Sun's mass $M_\odot$. This is an **approximation** valid when $m_2 \ll m_1$, so $M \approx m_1 = M_\odot$.

For the exact version:

$$T^2 = \frac{4\pi^2}{G(M_\odot + M_{\text{planet}})}a^3$$

For Earth:

$$M_\oplus/M_\odot \approx 3 \times 10^{-6} \implies M_\odot + M_\oplus \approx M_\odot\left(1 + 3 \times 10^{-6}\right)$$

So Kepler's approximation is excellent to better than 0.001%.

Conclusion: Kepler's First and Second Laws are exact consequences of Newton's laws. Kepler's Third Law is approximate, with the true form being:

$$T^2 = \frac{4\pi^2}{G(m_1 + m_2)}a^3$$

8.4 Vis-Viva Equation

For elliptical orbits, we can derive a useful relation between orbital speed and position.

From energy conservation (67):

$$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r}$$

From (96):

$$E = -\frac{GM\mu}{2a}$$

Equating:

$$-\frac{GM\mu}{2a} = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r}$$

Dividing by $\mu/2$:

$$-\frac{GM}{a} = v^2 - \frac{2GM}{r}$$

Rearranging:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{102}$$

This is the **vis-viva equation**. It gives the orbital speed at any point in the orbit in terms of the radial distance r and the semi-major axis a.

Special cases:

- **Circular orbit** ($r = a$): $v_c = \sqrt{GM/a}$
- **Escape velocity** ($a \to \infty$): $v_{\text{esc}} = \sqrt{2GM/r}$
- **Perihelion/Aphelion**: Maximum and minimum speeds occur at r_{min} and r_{max}

9 Conclusion

We have derived the complete solution to the gravitational two-body problem from first principles:

1. Established coordinate systems (spherical and plane polar) and derived velocity/acceleration formulas
2. Stated Newton's laws and defined inertial frames
3. Showed that Newton's laws for point particles extend to systems via the third law
4. Reduced the two-body problem to an equivalent one-body problem using COM coordinates and reduced mass
5. Proved that central forces conserve angular momentum and confine motion to a plane
6. Introduced the effective potential and derived the orbit equation
7. Solved the orbit equation to obtain conic sections

8. Classified orbits by energy and eccentricity
9. Derived Kepler's three laws as consequences of Newton's laws
10. Obtained the vis-viva equation relating speed, position, and orbital parameters

This framework forms the foundation of celestial mechanics and has enabled precise predictions of planetary motion, satellite orbits, and interplanetary trajectories.