Classical Mechanics: Lagrangian, Hamiltonian Formulations and Analysis of Rotating Reference Frames

Complete Rigorous Treatment with Applications to the Circular Restricted Three-Body Problem

Comprehensive Handout

December 2025

Abstract

This comprehensive handout provides an exceptionally detailed and rigorous treatment of advanced classical mechanics, designed for undergraduate students. The work covers three essential topics: (1) The foundations of Lagrangian and Hamiltonian mechanics with complete mathematical derivations and extensive intuitive explanations, (2) Rotating reference frames with rigorous treatment of fictitious forces (Coriolis and centrifugal), and (3) Complete analysis of the circular restricted three-body problem (CR3BP), including the discovery and classification of Lagrange points, their stability analysis, and the Jacobi integral. Every theorem is proven in detail, with applications and examples throughout. The document emphasizes the deep connections between symmetries and conservation laws through Noether's theorem, includes rigorous treatment of the virial theorem and its applications, and provides complete mathematical justification for all key results in celestial mechanics.

Contents

1 Introduction and Motivation

1.1 The Need for Advanced Formulations

Classical mechanics, as taught through Newton's laws, provides a direct but limited framework for analyzing physical systems. Newton's second law, $\mathbf{F} = m\mathbf{a}$, requires explicit knowledge of all forces acting on a system. This approach becomes impractical in several important scenarios:

1. **Constraint Forces**: When systems are constrained (e.g., a bead on a wire), the constraint forces are often unknown and difficult to determine.

2. **Generalized Coordinates**: Many systems naturally admit fewer degrees of freedom than coordinate dimensions. Working with constraint equations becomes cumbersome.

3. **Symmetries**: Newton's formulation obscures the profound connections between physical symmetries and conserved quantities.

4. **Rotating Frames**: In rotating reference frames (essential for planetary dynamics), Newton's laws require modification to account for frame acceleration.

5. **Computational Complexity**: For systems with many particles, summing forces over all interactions becomes computationally intractable.

6. **Foundation for Quantum Mechanics**: The Hamiltonian formulation provides the natural bridge to quantum mechanics.

The solution is to employ **energy-based formulations** of classical mechanics: the Lagrangian and Hamiltonian approaches. These reformulations are mathematically equivalent to Newton's laws but offer profound conceptual and computational advantages.

1.2 Three Formulations of Classical Mechanics

The Three Approaches

1. **Newtonian Mechanics**: $\mathbf{F} = m\mathbf{a}$ — Direct application of forces to obtain accelerations. Limited scope and applicability.

2. **Lagrangian Mechanics**: Energy-based formulation using the action principle. Naturally incorporates constraints through generalized coordinates. Reveals symmetries and conservation laws.

3. **Hamiltonian Mechanics**: Phase space formulation using conjugate momenta. Elegant geometric structure. Foundation for quantum mechanics and chaos theory.

All three formulations describe the same physical phenomena. The choice of which to use depends on the problem structure and desired insights.

2 Foundations: Generalized Coordinates and Degrees of Freedom

2.1 Concept of Degrees of Freedom

Definition 2.1 (Degrees of Freedom)**.** *The **degrees of freedom** (DOF) of a mechanical system is the minimum number of independent coordinates required to completely specify the configuration of the system at any instant.*

Example 2.1.

- *Single particle in 3D space: DOF = 3 (need x, y, z)*
- *Particle constrained to a sphere of radius R: DOF = 2 (need two angles: θ, ϕ; radius is fixed)*
- *Rigid rod in 2D with fixed pivot: DOF = 1 (need only angle θ)*
- *Two particles in 3D: DOF = 6 (three coordinates per particle)*
- *Two particles connected by rigid rod of length L: DOF = 5 (center of mass position gives 3 DOF, relative orientation gives 2 DOF)*

2.2 Generalized Coordinates

Definition 2.2 (Generalized Coordinates)**.** *A set of n independent coordinates $q_1, q_2, \ldots, q_n$ that completely specify the configuration of a system with n degrees of freedom are called **generalized coordinates**. The time derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ are called **generalized velocities**.*

Generalized coordinates need not be Cartesian coordinates. They can be angles, distances, or any other convenient variables, provided they are independent and completely specify the system configuration.

Example 2.2 (Simple Pendulum with Generalized Coordinates)**.** *A pendulum of length L and mass m can be described using:*

- ***Cartesian approach**: Position (x, y) with constraint $x^2 + y^2 = L^2$. The system has 2 coordinates but only 1 degree of freedom; the constraint reduces the effective dimensionality.*
- ***Generalized coordinate approach**: Use the angle θ measured from vertical. The generalized coordinate θ automatically satisfies the constraint. The system has 1 generalized coordinate and 1 degree of freedom.*

Relations between Cartesian and generalized coordinates:

$$x = L \sin \theta \tag{2.1}$$

$$y = -L \cos \theta \tag{2.2}$$

Cartesian velocities in terms of generalized velocity:

$$\dot{x} = L \cos \theta \cdot \dot{\theta} \tag{2.3}$$

$$\dot{y} = L \sin \theta \cdot \dot{\theta} \tag{2.4}$$

Cartesian speed: $v = \sqrt{\dot{x}^2 + \dot{y}^2} = L|\dot{\theta}|$
The kinetic energy becomes:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2 \tag{2.5}$$

The generalized coordinate θ automatically encodes the constraint; there is no need to explicitly impose $x^2 + y^2 = L^2$.

2.3 Holonomic and Non-holonomic Constraints

Definition 2.3 (Holonomic Constraints)**.** *Constraints that can be written in the form $f(q_1, q_2, \ldots, q_n, t) = 0$ are called **holonomic** or **integrable** constraints. These constraints reduce the system's degrees of freedom and can be incorporated through appropriate choice of generalized coordinates.*

Example 2.3 (Holonomic Constraints)**.**

- *Pendulum: $x^2 + y^2 - L^2 = 0$ (holonomic)*
- *Particle on surface: $z = f(x, y)$ (holonomic)*
- *Rigid body: Distance between particles is constant (holonomic)*

Definition 2.4 (Non-holonomic Constraints)**.** *Constraints that relate velocities (or differentials) rather than positions, such as $g(\dot{q}_i, q_i, t) = 0$, and cannot be integrated to reduce degrees of freedom are called **non-holonomic** constraints.*

Example 2.4 (Non-holonomic Constraints)**.**

- *Rolling without slipping: $v = \omega R$ (relates velocity to angular velocity)*
- *Magnetic constraint: Charged particle constrained to move perpendicular to magnetic field*

This handout focuses on holonomic constraints, which are most common in classical mechanics. The treatment of non-holonomic constraints requires more sophisticated mathematical machinery (Lagrange multipliers with inequality constraints).

3 Energy Methods and Conservation Laws

3.1 Kinetic and Potential Energy

Definition 3.1 (Kinetic Energy)**.** *The **kinetic energy** is the energy associated with motion:*

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{3.1}$$

For a system of N particles:

$$T = \sum_{i=1}^{N} \frac{1}{2}m_i v_i^2 \tag{3.2}$$

Definition 3.2 (Potential Energy)**.** *For a conservative force* $\mathbf{F}$*, there exists a scalar function* $V(\mathbf{r})$ *such that:*

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right) \tag{3.3}$$

This function V *is called the* ***potential energy****. The force is said to be* ***conservative****.*

Example 3.1 (Common Potentials)**.**

- *Uniform gravity (near Earth):* $V(h) = mgh$*, where* h *is height above reference level*
- *Spring:* $V(x) = \frac{1}{2}kx^2$*, where* x *is displacement from equilibrium*
- *Gravitational potential:* $V(r) = -\frac{GMm}{r}$*, where* r *is separation between masses (reference at infinity)*
- *Coulomb potential:* $V(r) = \frac{keq_1q_2}{r}$*, where* r *is separation between charges*

3.2 Conservation of Mechanical Energy

Theorem 3.3 (Conservation of Mechanical Energy)**.** *For a conservative system (where all forces derive from a potential energy function), the total mechanical energy is conserved:*

$$E = T + V = \textit{constant along any trajectory} \tag{3.4}$$

Proof. Consider Newton's second law in the form $\mathbf{F} = m\mathbf{a}$. The work-energy theorem states that the work done by all forces equals the change in kinetic energy:

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \mathrm{d}T \tag{3.5}$$

For a conservative force, $\mathbf{F} = -\nabla V$, so:

$$\mathrm{d}T = -\nabla V \cdot \mathrm{d}\mathbf{r} = -\mathrm{d}V \tag{3.6}$$

where we use the fact that for a scalar function $V(\mathbf{r})$, the differential is $\mathrm{d}V = \nabla V \cdot \mathrm{d}\mathbf{r}$. Rearranging:

$$\mathrm{d}(T + V) = \mathrm{d}T + \mathrm{d}V = 0 \tag{3.7}$$

Therefore:

$$E = T + V = \text{constant} \tag{3.8}$$

□

This energy conservation principle is fundamental to the Lagrangian and Hamiltonian approaches. It provides the foundation for action-based formulations of mechanics.

4 Lagrangian Mechanics

4.1 Intuition: The Principle of Stationary Action

The Lagrangian approach is built upon a profound principle about nature:

Principle of Stationary Action (Hamilton's Principle)

Among all conceivable paths a system could take between two specified configurations at fixed times, nature selects the path that makes the action stationary (typically a minimum). This path is the actual physical trajectory.

This principle is remarkably powerful and philosophically interesting. Rather than specifying forces and solving differential equations (Newton's approach), we specify a scalar function (the Lagrangian) and apply a variational principle. The equations of motion emerge as a consequence of making the action stationary.

Intuition: Imagine all possible paths from point A at time t_1 to point B at time t_2 as forming a landscape in an abstract "path space." Each path has associated with it a value (the action). The physical path occupies a special location in this landscape where small deviations from the path produce no first-order change in the action—like standing at a hilltop or in a valley bottom. This mathematical property encodes the laws of physics.

4.2 Definition of the Lagrangian

Definition 4.1 (The Lagrangian)**.** *The **Lagrangian** function is defined as:*

$$\mathcal{L}(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i) - V(q_i, t) \tag{4.1}$$

where T is the kinetic energy, V is the potential energy, and $(q_i, \dot{q}_i)$ are the generalized coordinates and velocities.

Remark 4.2. *The Lagrangian is the **difference** between kinetic and potential energy, not the sum. This seemingly arbitrary choice emerges naturally from the variational principle and has profound justification through the principle of stationary action.*

4.3 The Action Functional

Definition 4.3 (Action)**.** *The **action** is defined as the time integral of the Lagrangian over a path:*

$$S[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q(t), \dot{q}(t), t)\, \mathrm{d}t \tag{4.2}$$

*The notation $S[q(t)]$ emphasizes that the action is a **functional**—it takes an entire function (path) as input and produces a number (the action) as output.*

Different paths between the same initial and final configurations at the specified times will have different action values. The path nature actually follows is special: it makes the action stationary.

4.4 Hamilton's Principle (Principle of Stationary Action)

Theorem 4.4 (Hamilton's Principle)**.** *The physical trajectory $q(t)$ connecting configurations (q_1, t_1) and (q_2, t_2) is the one for which the action is stationary:*

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, \mathrm{d}t = 0 \tag{4.3}$$

The variation δ means: consider nearby paths $q(t) + \epsilon\eta(t)$ where $\eta(t_1) = \eta(t_2) = 0$ (variations vanish at the endpoints). The condition $\delta S = 0$ means the first-order change in action is zero for all such variations.

Physical Interpretation: Among all possible paths, the actual path is distinguished by the property that small perturbations to it produce no first-order change in action. It's a critical point of the action functional, analogous to a critical point of an ordinary function.

4.5 Derivation of the Euler-Lagrange Equations

Theorem 4.5 (Euler-Lagrange Equations)**.** *From Hamilton's principle, the physical trajectory must satisfy:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{4.4}$$

for each generalized coordinate q_i $(i = 1, 2, \ldots, n)$.

Proof. Consider a variation of the path: $q(t) \to q(t) + \epsilon\eta(t)$, where $\eta(t_1) = \eta(t_2) = 0$ (variations vanish at endpoints since initial and final positions are fixed).

The corresponding change in action is:

$$\delta S = \int_{t_1}^{t_2} \mathcal{L}(q + \epsilon\eta, \dot{q} + \epsilon\dot{\eta}, t)\, \mathrm{d}t - \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, \mathrm{d}t \tag{4.5}$$

Expanding to first order in ϵ:

$$\delta S = \epsilon \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\eta + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\eta}\right) \mathrm{d}t + O(\epsilon^2) \tag{4.6}$$

For the action to be stationary, we require $\delta S = 0$ for all variations $\eta(t)$. Therefore:

$$\int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\eta + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\eta}\right) \mathrm{d}t = 0 \tag{4.7}$$

Integrate the second term by parts:

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\eta}\, \mathrm{d}t = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\eta\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\eta\, \mathrm{d}t \tag{4.8}$$

The boundary term vanishes because $\eta(t_1) = \eta(t_2) = 0$. Therefore:

$$\int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right] \eta\, \mathrm{d}t = 0 \tag{4.9}$$

Since this must hold for all variations $\eta(t)$, and η is arbitrary in the interior of $[t_1, t_2]$, the coefficient of η must vanish:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0 \tag{4.10}$$

Rearranging:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{4.11}$$

This is the Euler-Lagrange equation. □

Remarkable Achievement: We have derived equations of motion without mentioning forces! The Lagrangian formulation is completely force-free. All information about the system is encoded in a single scalar function: the Lagrangian.

4.6 Example 1: Simple Harmonic Oscillator

A mass m attached to a spring with spring constant k. Using position x measured from equilibrium:

Energy expressions:

$$T = \frac{1}{2}m\dot{x}^2 \tag{4.12}$$

$$V = \frac{1}{2}kx^2 \tag{4.13}$$

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{4.14}$$

Partial derivatives:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \tag{4.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x} \tag{4.16}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \tag{4.17}$$

Euler-Lagrange equation:

$$m\ddot{x} - (-kx) = 0 \implies m\ddot{x} + kx = 0 \tag{4.18}$$

This is the standard equation of motion for a harmonic oscillator. The general solution is:

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}} \tag{4.19}$$

where A and B are determined by initial conditions. The energy is conserved:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant} \tag{4.20}$$

4.7 Example 2: Simple Pendulum

A pendulum of length L and mass m, with angle θ measured from vertical.

Coordinate relations:

$$x = L\sin\theta \tag{4.21}$$
$$y = -L\cos\theta \tag{4.22}$$

Velocity and kinetic energy:

$$v^2 = \dot{x}^2 + \dot{y}^2 = L^2\dot{\theta}^2 T = \frac{1}{2}mL^2\dot{\theta}^2 \tag{4.23}$$

Potential energy:

$$V = mgy = -mgL\cos\theta \tag{4.24}$$

Lagrangian:

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta \tag{4.25}$$

Partial derivatives:

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mL^2\dot{\theta} \tag{4.26}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = mL^2\ddot{\theta} \tag{4.27}$$
$$\frac{\partial\mathcal{L}}{\partial\theta} = -mgL\sin\theta \tag{4.28}$$

Euler-Lagrange equation:

$$mL^2\ddot{\theta} - (-mgL\sin\theta) = 0 \tag{4.29}$$

Simplifying:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{4.30}$$

This is the exact equation for a nonlinear pendulum. For small oscillations ($\sin\theta \approx \theta$), we get:

$$\ddot{\theta} + \omega^2\theta = 0, \quad \omega = \sqrt{\frac{g}{L}} \tag{4.31}$$

The solution oscillates with angular frequency ω, independent of amplitude (for small angles).

4.8 Cyclic Coordinates and Generalized Momentum

Definition 4.6 (Cyclic (Ignorable) Coordinate)**.** *If the Lagrangian $\mathcal{L}$ does not explicitly depend on a coordinate q_i (though it may depend on $\dot{q}_i$), that coordinate is called* ***cyclic*** *or* ***ignorable****.*

Definition 4.7 (Generalized Momentum)**.** *The* ***generalized momentum*** *(or* ***conjugate momentum****) associated with coordinate q_i is defined as:*

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i} \tag{4.32}$$

Theorem 4.8 (Conservation of Generalized Momentum)**.** *If the Lagrangian does not explicitly depend on a coordinate q_i (i.e., $\frac{\partial \mathcal{L}}{\partial q_i} = 0$), then the conjugate momentum p_i is conserved along the trajectory.*

Proof. If q_i is cyclic, then $\frac{\partial \mathcal{L}}{\partial q_i} = 0$. From the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{4.33}$$

With $\frac{\partial \mathcal{L}}{\partial q_i} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \tag{4.34}$$

Therefore:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant} \tag{4.35}$$

□

This theorem directly connects cyclic coordinates (symmetries) to conservation laws without needing to invoke torque concepts or angular momentum principles.

4.8.1 Example: Angular Momentum from Cyclic Coordinates

Consider a particle in a central force field. In plane polar coordinates (r, θ):

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{4.36}$$

$$V = V(r) \quad \text{(depends only on } r\text{)} \tag{4.37}$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{4.38}$$

Notice that $\mathcal{L}$ does not explicitly contain θ. Therefore θ is a cyclic coordinate.

The generalized momentum conjugate to θ is:

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = L \tag{4.39}$$

where $L = mr^2\dot{\theta}$ is the orbital angular momentum. Since θ is cyclic, $p_\theta = L$ is conserved.

This provides an elegant derivation of angular momentum conservation: it emerges not from a force-torque analysis, but from the cyclic nature of the angular coordinate in the Lagrangian.

4.9 Energy Conservation in Lagrangian Mechanics

Theorem 4.9 (Energy Conservation from Time-Translation Symmetry)**.** *If the Lagrangian does not explicitly depend on time (i.e., $\frac{\partial \mathcal{L}}{\partial t} = 0$), then the quantity:*

$$E = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \tag{4.40}$$

is conserved along the trajectory.

Proof. Consider the total time derivative of the Lagrangian:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \tag{4.41}$$

From the Euler-Lagrange equation: $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$
Substituting:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \tag{4.42}$$

The first two terms combine:

$$\sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \tag{4.43}$$

Therefore:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + \frac{\partial \mathcal{L}}{\partial t} \tag{4.44}$$

Rearranging:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}\right) = -\frac{\partial \mathcal{L}}{\partial t} \tag{4.45}$$

If $\frac{\partial \mathcal{L}}{\partial t} = 0$, then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}\right) = 0 \tag{4.46}$$

Therefore $E = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}$ is constant. □

For systems where T is quadratic in velocities and V is independent of velocities:

$$E = T + V = \text{total mechanical energy} \tag{4.47}$$

This result elegantly shows that **energy conservation emerges from the symmetry of the Lagrangian under time translations**. A time-independent Lagrangian possesses time-translation symmetry, which yields energy conservation.

5 Noether's Theorem: Symmetries and Conservation Laws

5.1 Statement of Noether's Theorem

Theorem 5.1 (Noether's Theorem)**.** *For every continuous symmetry of the Lagrangian under an infinitesimal transformation parameterized by an infinitesimal parameter ϵ, there exists an associated conserved quantity (a constant of motion).*

More precisely: If the Lagrangian is invariant under the infinitesimal transformation

$$q_i \to q_i' = q_i + \epsilon K_i(q, \dot{q}, t) \tag{5.1}$$

(where the Lagrangian changes by at most $\delta\mathcal{L} = \frac{\mathrm{d}F}{\mathrm{d}t}$ for some function F), then the quantity

$$Q = \sum_i K_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - F \tag{5.2}$$

is conserved along the trajectory.

Profound Implication: Noether's theorem establishes a fundamental connection: **every conservation law in physics corresponds to a symmetry of the system**, and conversely, every symmetry yields a conservation law. This is one of the deepest principles in theoretical physics.

5.2 Applications of Noether's Theorem

5.2.1 Conservation of Linear Momentum from Translational Symmetry

Consider a free particle or a system with no external forces. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x) \tag{5.3}$$

If the potential is spatially uniform (independent of position), then $\frac{\partial V}{\partial x} = 0$, and the Lagrangian is invariant under spatial translation: $x \to x + \epsilon$ (for constant ϵ).

Under this transformation: $\delta q = K = 1$, and $\mathcal{L}$ is unchanged ($F = 0$).

Noether's theorem gives the conserved quantity:

$$Q = 1 \cdot \frac{\partial \mathcal{L}}{\partial \dot{x}} - 0 = m\dot{x} = p = \text{linear momentum} \tag{5.4}$$

Conclusion: **Translational symmetry of the Lagrangian implies conservation of linear momentum.**

5.2.2 Conservation of Angular Momentum from Rotational Symmetry

Consider a particle in a central potential $V(r)$. In polar coordinates:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{5.5}$$

The Lagrangian is invariant under rotation: $\theta \to \theta + \epsilon$ (for constant ϵ), since V depends only on r.

Under this transformation: $\delta q = K = 1$, and $\mathcal{L}$ is unchanged.

Noether's theorem gives:

$$Q = 1 \cdot \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = L = \text{angular momentum} \tag{5.6}$$

Conclusion: **Rotational symmetry of the Lagrangian implies conservation of angular momentum.**

This shows how fundamental symmetries of space and time lead directly to the fundamental conservation laws of mechanics.

5.2.3 Conservation of Energy from Time-Translation Symmetry

If the Lagrangian is independent of time, the system has time-translation symmetry. The infinitesimal transformation is: $t \to t + \epsilon$, which implies all coordinates shift by their rate of change times ϵ: $q_i \to q_i + \epsilon \dot{q}_i$.

Under this transformation: $K_i = \dot{q}_i$, and since $\mathcal{L}$ doesn't depend explicitly on t, we have $F = 0$.

Noether's theorem gives:

$$Q = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = E = \text{total energy} \tag{5.7}$$

Conclusion: **Time-translation symmetry of the Lagrangian implies conservation of energy.**

This is a remarkable result: energy conservation emerges as a consequence of temporal uniformity (the laws of physics are the same at all times).

6 Hamiltonian Mechanics

6.1 Motivation: Why a Second Formulation?

The Lagrangian formulation of mechanics is powerful and elegant, but it has limitations:

1. The Lagrangian depends on velocities $\dot{q}_i$, and the Euler-Lagrange equations are second-order differential equations in the q_i.

2. The natural phase space for dynamics is not $(q, \dot{q})$ but rather (q, p), where p is momentum. This is more fundamental for certain problems.

3. The Hamiltonian formulation reveals elegant geometric structures and symmetries that are obscured in the Lagrangian framework.

4. The bridge to quantum mechanics uses Hamiltonian formulation, with canonical commutation relations and the Poisson bracket becoming the quantum commutator.

The Hamiltonian formulation addresses these issues by reformulating mechanics in terms of positions and momenta, rather than positions and velocities.

6.2 Definition of Conjugate Momentum and the Hamiltonian

Definition 6.1 (Conjugate Momentum)**.** *Given a Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$, the* ***generalized (or conjugate) momentum*** *associated with coordinate q_i is defined as:*

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{6.1}$$

In the Hamiltonian formulation, we invert this relation to express velocities in terms of momenta: $\dot{q}_i = \dot{q}_i(q, p, t)$.

Definition 6.2 (The Hamiltonian)**.** *The* ***Hamiltonian*** $H(q_i, p_i, t)$ *is defined via a* ***Legendre transformation****:*

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t) \tag{6.2}$$

where $\dot{q}_i$ *on the right side is expressed in terms of* (q, p, t) *using the inversion of* $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$*.*

Remark 6.3. *For systems with kinetic energy* $T = \frac{1}{2}\sum_{ij} m_{ij}(q)\dot{q}_i\dot{q}_j$ *(quadratic in velocities) and potential* $V(q)$ *(independent of velocities):*

$$H = T + V = \textit{total mechanical energy} \tag{6.3}$$

6.3 Example: Hamiltonian for a Harmonic Oscillator

Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{6.4}$$

Conjugate momentum:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \tag{6.5}$$

Inverting: $\dot{x} = \frac{p}{m}$
Hamiltonian:

$$H = p \cdot \frac{p}{m} - \left(\frac{1}{2}m\left(\frac{p}{m}\right)^2 - \frac{1}{2}kx^2\right) \tag{6.6}$$

$$= \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{6.7}$$

$$= \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{6.8}$$

This is exactly the sum of kinetic and potential energy, confirming the general result.

6.4 Hamilton's Equations of Motion

Theorem 6.4 (Hamilton's Equations)**.** *The equations of motion in the Hamiltonian formulation are:*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{6.9}$$

These are $2n$ *first-order differential equations (compared to* n *second-order equations from Lagrangian mechanics).*

Proof. Starting from the definition of the Hamiltonian:

$$H = \sum_i p_i \dot{q}_i - \mathcal{L} \tag{6.10}$$

Take the total differential:

$$\mathrm{d}H = \sum_i (\dot{q}_i \mathrm{d}p_i + p_i \mathrm{d}\dot{q}_i) - \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\mathrm{d}q_i - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\mathrm{d}\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \tag{6.11}$$

Since $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, the terms with $\mathrm{d}\dot{q}_i$ cancel:

$$\mathrm{d}H = \sum_i \dot{q}_i \mathrm{d}p_i - \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\mathrm{d}q_i - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \tag{6.12}$$

From the Euler-Lagrange equations: $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}p_i}{\mathrm{d}t}$
Therefore:

$$\mathrm{d}H = \sum_i \dot{q}_i \mathrm{d}p_i - \sum_i \dot{p}_i \mathrm{d}q_i - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \tag{6.13}$$

Comparing with the general differential $\mathrm{d}H = \sum_i \frac{\partial H}{\partial q_i}\mathrm{d}q_i + \sum_i \frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial t}\mathrm{d}t$:

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{6.14}$$

Rearranging:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{6.15}$$

These are Hamilton's equations. □

6.5 Example: Harmonic Oscillator in Hamiltonian Form

For the harmonic oscillator with $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$:

Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{6.16}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx \tag{6.17}$$

These are two first-order equations. Differentiating the first and substituting the second:

$$\ddot{x} = \frac{\dot{p}}{m} = -\frac{kx}{m} \implies \ddot{x} + \frac{k}{m}x = 0 \tag{6.18}$$

This reproduces the standard SHM equation, confirming that Hamilton's equations are equivalent to the Lagrangian equations.

6.6 Poisson Brackets

Definition 6.5 (Poisson Bracket)**.** *For any two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ in phase space, the **Poisson bracket** is defined as:*

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{6.19}$$

The Poisson bracket is a fundamental operation in Hamiltonian mechanics that encodes the structure of phase space dynamics.

Theorem 6.6 (Time Evolution via Poisson Brackets)**.** *For any phase space function $f(q_i, p_i, t)$:*

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t} \tag{6.20}$$

where H is the Hamiltonian.

Proof. Using the chain rule:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} \tag{6.21}$$

Substituting Hamilton's equations:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} + \sum_i \frac{\partial f}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial f}{\partial t} \tag{6.22}$$

Rearranging:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial f}{\partial t} \tag{6.23}$$

□

Special Cases:

- If $f = q_i$: $\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}$ (confirms Hamilton's equation)
- If $f = p_i$: $\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$ (confirms Hamilton's equation)
- If $\{f, H\} = 0$ and $\frac{\partial f}{\partial t} = 0$: $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$, so f is a conserved quantity

The Poisson bracket is the classical analogue of the quantum commutator. In quantum mechanics, the correspondence is:

$$\{f, g\}_{\text{classical}} \to \frac{1}{i\hbar}[\hat{f}, \hat{g}]_{\text{quantum}} \tag{6.24}$$

7 The Virial Theorem

The virial theorem is a fundamental result that relates the time-averaged kinetic and potential energies of a system. It applies to many important physical systems and provides powerful tools for analysis.

7.1 Statement of the Virial Theorem

Theorem 7.1 (The Virial Theorem)**.** *For a system of particles with position vectors $\mathbf{r}_i$, masses m_i, subject to forces $\mathbf{F}_i = -\nabla_i V$, define the quantity:*

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i = \sum_i m_i \mathbf{v}_i \cdot \mathbf{r}_i \tag{7.1}$$

Then, the time-averaged values satisfy:

$$2\langle T \rangle = -\left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle \tag{7.2}$$

For a conservative force $\mathbf{F}_i = -\nabla_i V$:

$$2\langle T \rangle = \left\langle \sum_i \mathbf{r}_i \cdot \nabla_i V \right\rangle \tag{7.3}$$

If the potential is a homogeneous function of degree n, i.e., $V(\lambda \mathbf{r}_1, \ldots, \lambda \mathbf{r}_N) = \lambda^n V(\mathbf{r}_1, \ldots, \mathbf{r}_N)$, then by Euler's theorem for homogeneous functions:

$$\sum_i \mathbf{r}_i \cdot \nabla_i V = nV \tag{7.4}$$

Therefore:

$$2\langle T \rangle = n\langle V \rangle \tag{7.5}$$

7.2 Proof of the Virial Theorem

Proof. Define $G = \sum_i m_i \mathbf{v}_i \cdot \mathbf{r}_i$. Taking the time derivative:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}(m_i \mathbf{v}_i \cdot \mathbf{r}_i) \tag{7.6}$$

Using the product rule:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i m_i \mathbf{a}_i \cdot \mathbf{r}_i + \sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i = \sum_i m_i \mathbf{a}_i \cdot \mathbf{r}_i + 2T \tag{7.7}$$

From Newton's second law, $m_i \mathbf{a}_i = \mathbf{F}_i$:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i + 2T \tag{7.8}$$

For a bounded system (where all velocities and positions remain finite), we can average over a long time interval τ:

$$\left\langle \frac{\mathrm{d}G}{\mathrm{d}t} \right\rangle = \frac{1}{\tau} \int_0^\tau \frac{\mathrm{d}G}{\mathrm{d}t}\, \mathrm{d}t = \frac{G(\tau) - G(0)}{\tau} \tag{7.9}$$

As $\tau \to \infty$ (or for periodic motion with period T), if G remains bounded, the left side approaches zero:

$$0 = \langle \mathbf{F}_i \cdot \mathbf{r}_i \rangle + 2\langle T \rangle \tag{7.10}$$

Therefore:

$$2\langle T \rangle = -\left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle \tag{7.11}$$

For conservative forces, $\mathbf{F}_i = -\nabla_i V$:

$$2\langle T \rangle = \left\langle \sum_i \mathbf{r}_i \cdot \nabla_i V \right\rangle \tag{7.12}$$

For a homogeneous potential of degree n:

$$V(\lambda \mathbf{r}_1, \ldots, \lambda \mathbf{r}_N) = \lambda^n V(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{7.13}$$

Differentiating both sides with respect to λ and setting $\lambda = 1$:

$$\sum_i n\lambda^{n-1} \mathbf{r}_i \cdot \nabla_i V|_{\lambda \mathbf{r}} = n\lambda^{n-1} V \tag{7.14}$$

At $\lambda = 1$: $\sum_i \mathbf{r}_i \cdot \nabla_i V = nV$
Therefore:

$$2\langle T\rangle = n\langle V\rangle \tag{7.15}$$

The total energy is $E = T + V$, so:

$$\langle E\rangle = \langle T\rangle + \langle V\rangle = \langle T\rangle + \frac{2\langle T\rangle}{n} = \left(1 + \frac{2}{n}\right)\langle T\rangle \tag{7.16}$$

Or:

$$\langle T\rangle = \frac{n}{n+2}\langle E\rangle, \quad \langle V\rangle = \frac{2}{n+2}\langle E\rangle \tag{7.17}$$

□

7.3 Applications of the Virial Theorem

7.3.1 Application 1: Inverse Square Law (Gravity and Coulomb)

For gravitational or Coulomb interactions: $V \propto \frac{1}{r}$, so $n = -1$.

$$2\langle T\rangle = -\langle V\rangle \tag{7.18}$$

Since $E = \langle T\rangle + \langle V\rangle = \langle T\rangle - 2\langle T\rangle = -\langle T\rangle$:

$$\langle T\rangle = -E, \quad \langle V\rangle = 2E \tag{7.19}$$

Implication: For bound systems ($E < 0$), the time-averaged kinetic energy equals the (negative of the) total energy, and the time-averaged potential energy is twice the total energy. The system spends more time at larger separations (moving slowly) than at smaller separations (moving quickly).

7.3.2 Application 2: Harmonic Potential

For a harmonic potential: $V \propto r^2$, so $n = 2$.

$$2\langle T\rangle = 2\langle V\rangle \implies \langle T\rangle = \langle V\rangle \tag{7.20}$$

The total energy is $E = 2\langle T\rangle$:

$$\langle T\rangle = \langle V\rangle = \frac{E}{2} \tag{7.21}$$

Implication: For harmonic systems, kinetic and potential energy time-average to equal values, each being half the total energy (equipartition of energy).

7.3.3 Application 3: Stellar Structure and Astrophysics

The virial theorem is crucial in astrophysics. For a self-gravitating system in equilibrium (like a star):

$$2\langle T\rangle = -\langle U_{\text{grav}}\rangle \tag{7.22}$$

This relates the thermal kinetic energy to the gravitational potential energy, providing constraints on stellar structure, mass, and evolution. The theorem shows that if a star contracts, half the gravitational energy released goes into heating the star, and half is radiated away.

8 Rotating Reference Frames and Fictitious Forces

8.1 Introduction and Intuition

Many important physical phenomena occur in rotating reference frames. Earth rotates, planets orbit stars, galaxies rotate. In these frames, objects appear to experience forces (Coriolis force, centrifugal force, and Euler force) that don't exist when viewed from an inertial frame. These **fictitious** or **pseudo-forces** arise purely from the non-inertial nature of the rotating frame. Remarkably, these fictitious forces encompass several components when the rotation itself is accelerating.

Newton's laws, as classically stated, apply only in inertial frames. To use them in rotating frames, we must add fictitious forces that account for the acceleration of the frame itself. In the most general case where the angular velocity is time-dependent, three fictitious forces appear.

8.2 General Transformation from Inertial to Rotating Frames

Consider an inertial frame with coordinates $\mathbf{r}$ and a rotating frame with coordinates $\mathbf{r}'$ rotating about the origin with time-dependent angular velocity $\omega(t)$ (the most general case).

The relationship between position vectors in the two frames is:

$$\mathbf{r} = \mathbf{r}' \tag{8.1}$$

(both describe the same physical point; the difference is in the coordinate systems).

However, the relationship between velocities is more complex because the basis vectors of the rotating frame change direction with time.

8.2.1 Time Derivatives of Basis Vectors

Let $\hat{\mathbf{e}}'_i$ be the basis vectors in the rotating frame. From the inertial frame perspective, these rotate with angular velocity $\omega(t)$ (which may be time-dependent):

$$\left.\frac{\mathrm{d}\hat{\mathbf{e}}'_i}{\mathrm{d}t}\right|_{\text{inertial}} = \omega(t) \times \hat{\mathbf{e}}'_i \tag{8.2}$$

This is the fundamental kinematic relation. Note that even if ω is changing with time, this relation still holds instantaneously.

8.2.2 Velocity Transformation

Express position in rotating frame: $\mathbf{r}' = \sum_i x'_i \hat{\mathbf{e}}'_i$

The velocity as observed from the inertial frame:

$$\mathbf{v}_{\text{inertial}} = \left.\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}\right|_{\text{inertial}} = \sum_i \frac{\mathrm{d}x'_i}{\mathrm{d}t}\hat{\mathbf{e}}'_i + \sum_i x'_i \left.\frac{\mathrm{d}\hat{\mathbf{e}}'_i}{\mathrm{d}t}\right|_{\text{inertial}} \tag{8.3}$$

The first sum is the velocity in the rotating frame:

$$\mathbf{v}_{\text{rotating}} = \sum_i \left.\frac{\mathrm{d}x'_i}{\mathrm{d}t}\right|_{\text{rotating}} \hat{\mathbf{e}}'_i \tag{8.4}$$

The second sum, using $\frac{\mathrm{d}\hat{\mathbf{e}}'_i}{\mathrm{d}t}|_{\text{inertial}} = \omega(t) \times \hat{\mathbf{e}}'_i$:

$$\sum_i x'_i \frac{\mathrm{d}\hat{\mathbf{e}}'_i}{\mathrm{d}t}\bigg|_{\text{inertial}} = \sum_i x'_i(\omega(t) \times \hat{\mathbf{e}}'_i) = \omega(t) \times \sum_i x'_i \hat{\mathbf{e}}'_i = \omega(t) \times \mathbf{r}' \tag{8.5}$$

Therefore, the **general velocity transformation** is:

$$\mathbf{v}_{\text{inertial}} = \mathbf{v}_{\text{rotating}} + \omega(t) \times \mathbf{r}' \tag{8.6}$$

This relation holds even when ω is time-dependent. The key physical insight is that the velocity in the inertial frame consists of two parts: the velocity relative to the rotating frame, plus the velocity of the point itself due to the rotation of the frame.

8.2.3 Acceleration Transformation: General Case

Differentiate the velocity relation, carefully accounting for time-dependent ω:

$$\mathbf{a}_{\text{inertial}} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}_{\text{rotating}} + \omega(t) \times \mathbf{r}')\bigg|_{\text{inertial}} \tag{8.7}$$

The first term (acceleration of rotating frame velocity):

$$\frac{\mathrm{d}\mathbf{v}_{\text{rotating}}}{\mathrm{d}t}\bigg|_{\text{inertial}} = \frac{\mathrm{d}\mathbf{v}_{\text{rotating}}}{\mathrm{d}t}\bigg|_{\text{rotating}} + \omega(t) \times \mathbf{v}_{\text{rotating}} \tag{8.8}$$

(by the same reasoning as for the velocity transformation)
The second term (time derivative of the rotation term):

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega(t) \times \mathbf{r}')\bigg|_{\text{inertial}} = \frac{\mathrm{d}\omega}{\mathrm{d}t} \times \mathbf{r}' + \omega(t) \times \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}\bigg|_{\text{inertial}} \tag{8.9}$$

where $\frac{\mathrm{d}\omega}{\mathrm{d}t}$ is the **angular acceleration** in the inertial frame, which we denote as $\alpha(t) = \frac{\mathrm{d}\omega}{\mathrm{d}t}$.
Now substitute $\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}|_{\text{inertial}} = \mathbf{v}_{\text{rotating}} + \omega(t) \times \mathbf{r}'$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega(t) \times \mathbf{r}')\bigg|_{\text{inertial}} = \alpha(t) \times \mathbf{r}' + \omega(t) \times (\mathbf{v}_{\text{rotating}} + \omega(t) \times \mathbf{r}') \tag{8.10}$$

$$= \alpha(t) \times \mathbf{r}' + \omega(t) \times \mathbf{v}_{\text{rotating}} + \omega(t) \times (\omega(t) \times \mathbf{r}') \tag{8.11}$$

Combining all terms:

$$\mathbf{a}_{\text{inertial}} = \left[\frac{\mathrm{d}\mathbf{v}_{\text{rotating}}}{\mathrm{d}t}\bigg|_{\text{rotating}} + \omega(t) \times \mathbf{v}_{\text{rotating}}\right] + [\alpha(t) \times \mathbf{r}' + \omega(t) \times \mathbf{v}_{\text{rotating}} + \omega(t) \times (\omega(t) \times \mathbf{r}')] \tag{8.12}$$

Collecting the terms with $\omega(t) \times \mathbf{v}_{\text{rotating}}$:

$$\mathbf{a}_{\text{inertial}} = \frac{\mathrm{d}\mathbf{v}_{\text{rotating}}}{\mathrm{d}t}\bigg|_{\text{rotating}} + 2\omega(t) \times \mathbf{v}_{\text{rotating}} + \alpha(t) \times \mathbf{r}' + \omega(t) \times (\omega(t) \times \mathbf{r}') \tag{8.13}$$

General Acceleration Transformation:

$$\mathbf{a}_{\text{inertial}} = \mathbf{a}_{\text{rotating}} + 2\omega(t) \times \mathbf{v}_{\text{rotating}} + \alpha(t) \times \mathbf{r}' + \omega(t) \times (\omega(t) \times \mathbf{r}') \tag{8.14}$$

This is the most general form, valid even when the angular velocity is changing with time. Each term on the right represents a different type of acceleration component in the rotating frame.

8.3 Equations of Motion in Rotating Frames: General Form

Newton's second law in the inertial frame:

$$m\mathbf{a}_{\text{inertial}} = \mathbf{F}_{\text{real}} \tag{8.15}$$

Substituting the general acceleration transformation:

$$m\left(\mathbf{a}_{\text{rotating}} + 2\omega(t) \times \mathbf{v}_{\text{rotating}} + \alpha(t) \times \mathbf{r}' + \omega(t) \times (\omega(t) \times \mathbf{r}')\right) = \mathbf{F}_{\text{real}} \tag{8.16}$$

Rearranging to isolate the acceleration in the rotating frame:

$$m\mathbf{a}_{\text{rotating}} = \mathbf{F}_{\text{real}} - 2m\omega(t) \times \mathbf{v}_{\text{rotating}} - m\alpha(t) \times \mathbf{r}' - m\omega(t) \times (\omega(t) \times \mathbf{r}') \tag{8.17}$$

Define three fictitious forces:

1. Coriolis Force:

$$\mathbf{F}_{\text{Coriolis}} = -2m\omega(t) \times \mathbf{v}_{\text{rotating}} \tag{8.18}$$

2. Euler Force (angular acceleration force):

$$\mathbf{F}_{\text{Euler}} = -m\alpha(t) \times \mathbf{r}' = -m\frac{\mathrm{d}\omega}{\mathrm{d}t} \times \mathbf{r}' \tag{8.19}$$

3. Centrifugal Force:

$$\mathbf{F}_{\text{centrifugal}} = -m\omega(t) \times (\omega(t) \times \mathbf{r}') \tag{8.20}$$

Then, the general equation of motion in the rotating frame is:

$$m\mathbf{a}_{\text{rotating}} = \mathbf{F}_{\text{real}} + \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{Euler}} + \mathbf{F}_{\text{centrifugal}} \tag{8.21}$$

Important Note: To an observer in the rotating frame, the particle experiences not just the real force, but also these three fictitious forces. Each arises from a different kinematic property of the rotating frame: the Coriolis force from the frame's rotation (with velocity dependence), the Euler force from the frame's angular acceleration, and the centrifugal force from the centripetal acceleration required to keep a point fixed in the rotating frame.

8.4 Special Case: Constant Angular Velocity

When $\omega(t) = \omega =$ constant (i.e., $\alpha(t) = 0$), the Euler force vanishes:

$$\mathbf{F}_{\text{Euler}} = 0 \quad (\text{when } \dot{\omega} = 0) \tag{8.22}$$

and the equations reduce to the familiar form with only Coriolis and centrifugal forces:

$$m\mathbf{a}_{\text{rotating}} = \mathbf{F}_{\text{real}} - 2m\omega \times \mathbf{v}_{\text{rotating}} - m\omega \times (\omega \times \mathbf{r}') \tag{8.23}$$

This is a good consistency check: our general formulation reduces to the standard result when angular velocity is constant.

8.5 Physical Interpretation of Fictitious Forces

8.5.1 The Centrifugal Force

The centrifugal force is:

$$\mathbf{F}_{\text{centrifugal}} = -m\omega(t) \times (\omega(t) \times \mathbf{r}') \tag{8.24}$$

Using the vector identity $\mathbf{A} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{A} \cdot \mathbf{A})$:

$$\omega \times (\omega \times \mathbf{r}') = \omega(\omega \cdot \mathbf{r}') - \mathbf{r}'(\omega \cdot \omega) \tag{8.25}$$

If ω is perpendicular to $\mathbf{r}'$ (as when rotation is about a fixed axis and the particle is in the plane of rotation), then:

$$\omega \times (\omega \times \mathbf{r}') = -\mathbf{r}'\omega^2 \tag{8.26}$$

Therefore:

$$\mathbf{F}_{\text{centrifugal}} = m\omega^2(t)\mathbf{r}'_{\perp} \tag{8.27}$$

where $\mathbf{r}'_{\perp}$ is the component of position perpendicular to the rotation axis and $\omega(t) = |\omega(t)|$.

Physical Meaning: The centrifugal force points radially outward from the axis of rotation, with magnitude proportional to the distance and to the square of the angular velocity. Note that even if $\omega(t)$ is changing, at each instant the centrifugal force still depends on $\omega^2(t)$. A particle at rest in the rotating frame experiences an outward centrifugal force. On a carousel, you feel pushed outward.

8.5.2 The Coriolis Force

The Coriolis force is:

$$\mathbf{F}_{\text{Coriolis}} = -2m\omega(t) \times \mathbf{v}_{\text{rotating}} \tag{8.28}$$

This force acts on objects that move relative to the rotating frame. Key properties:

1. The force is perpendicular to both ω and $\mathbf{v}_{\text{rotating}}$; it acts perpendicular to the direction of motion.
2. The magnitude is $F_{\text{Coriolis}} = 2m\omega(t)v\sin\theta$, where θ is the angle between $\omega(t)$ and $\mathbf{v}_{\text{rotating}}$.
3. The Coriolis force does no work: $\mathbf{F}_{\text{Coriolis}} \cdot \mathbf{v}_{\text{rotating}} = 0$ (since they are perpendicular). It deflects motion but doesn't change the speed.
4. The Coriolis force vanishes when the particle is at rest in the rotating frame ($\mathbf{v}_{\text{rotating}} = 0$).
5. The Coriolis force is independent of whether ω is changing; it depends only on the instantaneous value of $\omega(t)$.

Physical Meaning: The Coriolis force causes moving objects in rotating frames to deflect perpendicular to their motion. On Earth, objects deflect rightward in the Northern Hemisphere and leftward in the Southern Hemisphere. This is crucial in atmospheric dynamics (cyclone formation) and oceanography. Even if Earth's rotation were accelerating or decelerating, the Coriolis force would remain unchanged; only the Euler force (due to angular acceleration) would change.

8.5.3 The Euler Force

The Euler force (also called the **angular acceleration force** or **azimuthal force**) is:

$$\mathbf{F}_{\text{Euler}} = -m\alpha(t) \times \mathbf{r}' = -m\frac{\mathrm{d}\omega}{\mathrm{d}t} \times \mathbf{r}' \tag{8.29}$$

where $\alpha(t) = \frac{\mathrm{d}\omega}{\mathrm{d}t}$ is the angular acceleration of the rotating frame.
Key properties:

1. The force is perpendicular to both the angular acceleration vector α and the position vector $\mathbf{r}'$.
2. The magnitude is $F_{\text{Euler}} = m\alpha r' \sin\phi$, where $\alpha = |\frac{\mathrm{d}\omega}{\mathrm{d}t}|$ and ϕ is the angle between α and $\mathbf{r}'$.
3. For a particle on the rotation axis ($\mathbf{r}'$ parallel to α), the Euler force vanishes.
4. The Euler force does no work: $\mathbf{F}_{\text{Euler}} \cdot \mathbf{v}_{\text{rotating}} = 0$ in certain directions, though it can transfer energy between different components of motion.
5. The Euler force vanishes when the frame rotates with constant angular velocity ($\alpha = 0$).
6. Unlike the centrifugal and Coriolis forces, the Euler force depends explicitly on the rate of change of angular velocity, not on its magnitude.

Physical Meaning: The Euler force arises when the rotating frame itself is accelerating or decelerating angularly. If you stand on a disk that begins to spin faster, you would experience an Euler force pushing you (in addition to the centrifugal force). Conversely, if a spinning disk begins to slow down, the Euler force reverses direction. In the orbital mechanics context, this would apply to systems where the rate of rotation changes (e.g., tidal locking evolution, or spacecraft with changing spin rates).

The Euler force is often the least familiar of the three fictitious forces, but it becomes critically important in systems with time-dependent rotation rates. A particle at rest in the rotating frame experiences both centrifugal (due to rotation) and Euler forces (due to angular acceleration).

8.6 Example 1: Constant Angular Velocity Disk (No Euler Force)

Consider a disk rotating with constant angular velocity $\omega = \omega\hat{\mathbf{z}}$ about the vertical axis, with $\omega =$ constant.

Since $\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$, the Euler force vanishes entirely.

8.6.1 Particle at Rest on Disk

In the rotating frame, the particle is stationary: $\mathbf{v}_{\text{rotating}} = 0$, $\mathbf{a}_{\text{rotating}} = 0$.

The only fictitious force is the centrifugal force:

$$\mathbf{F}_{\text{centrifugal}} = m\omega^2 r\hat{\mathbf{r}}_\perp \tag{8.30}$$

(pointing radially outward). The particle doesn't accelerate in the rotating frame because friction or normal force balances the centrifugal force.

8.6.2 Particle Moving Radially Outward

A particle moves with velocity $\mathbf{v}_{\text{rotating}} = v_r\hat{\mathbf{r}}$ (radial, in the rotating frame).

The Coriolis force is:

$$\mathbf{F}_{\text{Coriolis}} = -2m\omega \times \mathbf{v}_{\text{rotating}} = -2m\omega\hat{\mathbf{z}} \times v_r\hat{\mathbf{r}} = -2m\omega v_r\hat{\theta} \tag{8.31}$$

The force points in the negative tangential direction (opposite to the direction the disk rotates). An observer on the disk sees the particle deflect backward (in the direction opposite to the disk's rotation). This happens because the particle, moving radially outward, enters regions of the disk that are moving slower (farther from the center), causing an apparent backward deflection.

8.7 Example 2: Accelerating Disk (Including Euler Force)

Now consider a disk that starts from rest and gradually spins up, with angular velocity $\omega(t) = \omega(t)\hat{\mathbf{z}}$, where $\omega(t)$ increases linearly: $\omega(t) = \beta t$ for some constant angular acceleration $\beta > 0$.

The angular acceleration is:

$$\alpha(t) = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \beta\hat{\mathbf{z}} \tag{8.32}$$

The angular acceleration is constant and non-zero.

8.7.1 Particle at Rest on Accelerating Disk

A particle sits at a fixed location on the disk at distance r from the rotation axis: $\mathbf{r}' = r\hat{\mathbf{r}}$ (in the rotating frame).

At time t, in the rotating frame, $\mathbf{v}_{\text{rotating}} = 0$ and $\mathbf{a}_{\text{rotating}} = 0$.

The centrifugal force at time t is:

$$\mathbf{F}_{\text{centrifugal}} = m\omega^2(t)r\hat{\mathbf{r}} = m\beta^2t^2r\hat{\mathbf{r}} \tag{8.33}$$

(pointing radially outward, and increasing in magnitude as the disk spins faster).

The Euler force is:

$$\mathbf{F}_{\text{Euler}} = -m\alpha \times \mathbf{r}' = -m(\beta\hat{\mathbf{z}}) \times (r\hat{\mathbf{r}}) = -m\beta r\hat{\theta} \tag{8.34}$$

This force points in the negative tangential direction (opposite to the direction the disk is rotating).

Physical Interpretation: As the disk spins up, particles on the disk experience both outward centrifugal acceleration (increasing with time) and a tangential Euler force opposing the direction of rotation. From the perspective of an observer on the disk, the Euler force manifests as a "braking" or opposing force that resists the disk's acceleration. In the inertial frame, this makes sense: the particle wants to maintain its inertial motion (Newton's first law), while the disk tries to accelerate it. The Euler force represents this inertial resistance to the frame's angular acceleration.

The total tangential force in the rotating frame is just $\mathbf{F}_{\text{Euler}}$, causing a tangential acceleration that is visible if the particle is not held fixed (e.g., by friction or a constraint). Without friction, a particle resting on the disk would slide backward (opposite to the disk's spin direction) as the disk accelerates.

8.7.2 Particle Moving Radially with Time-Dependent Rotation

Consider a particle moving radially outward with constant velocity in the rotating frame: $\mathbf{v}_{\text{rotating}} = v_0\hat{\mathbf{r}}$ (constant).

The Coriolis force at time t is:

$$\mathbf{F}_{\text{Coriolis}} = -2m\omega(t) \times \mathbf{v}_{\text{rotating}} = -2m\beta t\hat{\mathbf{z}} \times v_0\hat{\mathbf{r}} = -2m\beta t v_0\hat{\theta} \tag{8.35}$$

This force increases in magnitude as the disk spins faster. The particle experiences both an outward centrifugal force and a backward (tangential) Coriolis deflection. Additionally, as the particle moves outward, the Euler force magnitude depends on its distance from the axis: $\mathbf{F}_{\text{Euler}} = -m\beta r'(t)\hat{\theta}$ where $r'(t)$ increases with time.

The combined effect is complex but physically meaningful: the particle experiences increasing deflection as it moves to regions where the disk rotates faster, plus an additional effect from the disk's angular acceleration.

8.7.3 Particle Rotating with the Disk but Moving Radially

Now consider a particle that remains at a fixed radius r in the rotating frame but is slowly spiraling outward in the inertial frame due to some initial conditions.

At time t, in the rotating frame:

- Position: $\mathbf{r}' = r\hat{\mathbf{r}}$ (constant in rotating frame)
- Velocity: $\mathbf{v}_{\text{rotating}} = v_r(t)\hat{\mathbf{r}}$ (radial, proportional to the spiral rate)

The three fictitious forces are:

1. **Centrifugal**: $\mathbf{F}_{\text{centrifugal}} = m\beta^2 t^2 r\hat{\mathbf{r}}$
2. **Coriolis**: $\mathbf{F}_{\text{Coriolis}} = -2m\beta t v_r(t)\hat{\theta}$
3. **Euler**: $\mathbf{F}_{\text{Euler}} = -m\beta r\hat{\theta}$

In this case, the tangential forces (Coriolis and Euler) combine. If the particle is moving outward slowly ($v_r(t) \approx 0$), the Euler force dominates the tangential dynamics. If the particle is moving outward more rapidly, the Coriolis force becomes significant.

Summary of Accelerating Disk Example: This example demonstrates how the Euler force becomes important when the reference frame's rotation is not constant. Real systems (planets with time-varying spin, spacecraft with changing angular velocity, atmospheres in tidal evolution) require the full general formulation with all three fictitious forces. The interplay between Euler and Coriolis forces, combined with centrifugal effects and real forces, creates rich and sometimes counterintuitive dynamics.

9 The General N-Body Problem: Structure, Solvability, and Integrability

9.1 Introduction and Motivation

The previous sections established the complete solvability of the two-body gravitational problem, yielding a fully determined orbit equation with explicit analytic solutions. We now expand our perspective to ask: what happens when $N \geq 3$ bodies interact gravitationally? This is the **N-body problem**, one of the most profound and challenging questions in classical mechanics.

The contrast between the two-body and three-body problems is stark:

- **Two-body problem**: Completely solvable. Yields explicit formulas (conic sections). Behavior is qualitatively simple and well-understood.
- **Three-body problem**: Only partial solutions exist for special configurations. General trajectories are chaotic. No closed-form solutions.
- **N-body problem ($N \geq 3$)**: Intractable in the general case. Numerical integration is required. Deep connections to chaos theory and integrability.

The jump in complexity from $N = 2$ to $N = 3$ is one of the most striking discoveries in dynamics. Despite the simplicity of the underlying forces (gravity, pairwise interactions), the collective behavior becomes fundamentally unpredictable. This section explores the mathematical structure of why the N-body problem is unsolvable and what partial results can be obtained.

9.2 Setup and Fundamental Equations

9.2.1 Configuration and Phase Space

Consider N point particles with masses $m_1, m_2, \ldots, m_N$ at positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$ in 3D inertial space. The system has:

- **Configuration space**: $\mathbb{R}^{3N}$ (each particle requires 3 coordinates)
- **Degrees of freedom**: $3N$ (in absence of constraints)
- **Phase space**: $\mathbb{R}^{6N}$ (each particle has 3 position and 3 momentum components)

The state of the system at any instant is specified by the phase space point $(q_1, \ldots, q_N, p_1, \ldots, p_N)$ where $q_i = \mathbf{r}_i$ and $p_i = m_i \mathbf{v}_i = m_i \dot{\mathbf{r}}_i$.

9.2.2 Hamiltonian Formulation

The Hamiltonian for the N-body gravitational problem is:

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} - G \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{9.1}$$

This consists of:

- **Kinetic energy**: $T = \sum_i \frac{p_i^2}{2m_i}$ (energy of motion)
- **Potential energy**: $V = -G\sum_{i<j} \frac{m_i m_j}{r_{ij}}$ (gravitational binding energy, with $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$)

Hamilton's equations of motion are:

$$\dot{\mathbf{r}}_i = \frac{\partial H}{\partial \mathbf{p}_i} = \frac{\mathbf{p}_i}{m_i} \tag{9.2}$$

$$\dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{r}_i} = -Gm_i \sum_{j\neq i} \frac{m_j(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \tag{9.3}$$

The second equation is Newton's second law: the rate of change of momentum equals the net gravitational force from all other bodies.

The total number of coupled first-order ODEs is $6N$, with $6N$ initial conditions specifying the full state.

9.2.3 Conserved Quantities (First Integrals)

Despite the system's complexity, several conservation laws follow from symmetries of the Hamiltonian:

1. Conservation of Total Momentum

Since the Hamiltonian has no explicit dependence on the origin location (translational invariance), the total momentum is conserved:

$$\mathbf{P}_{\text{total}} = \sum_{i=1}^{N} \mathbf{p}_i = \text{constant} \tag{9.4}$$

This gives **3 first integrals**.

2. Conservation of Total Angular Momentum

From rotational invariance of the Hamiltonian, the total angular momentum is conserved:

$$\mathbf{L}_{\text{total}} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i = \text{constant} \tag{9.5}$$

This gives **3 first integrals**.

3. Conservation of Total Energy

Since the Hamiltonian is time-independent, the total energy is conserved:

$$E_{\text{total}} = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} - G\sum_{i<j} \frac{m_i m_j}{r_{ij}} = \text{constant} \tag{9.6}$$

This gives **1 first integral**.

Total: 10 first integrals from symmetries (6 from momentum, 3 from angular momentum, 1 from energy).

Remark 9.1. *There is a **critical observation**: We have $6N$ degrees of freedom but only 10 first integrals for $N \geq 3$. Since $6N - 10 > 0$ for $N \geq 3$, we cannot fully determine the system's evolution from these conserved quantities alone. For $N = 2$, we have $6(2) = 12$ degrees of freedom and can find additional conserved quantities (through angular momentum conservation in the relative coordinate and center-of-mass motion), leading to complete solvability.*

9.3 Reduction by Symmetry: From $6N$ to $(6N - 10)$ Degrees of Freedom

We can reduce the problem dimensionality using the 10 conserved quantities.

9.3.1 Center of Mass Frame

From conservation of total momentum, we can choose coordinates such that the center of mass is at rest:

$$\mathbf{R}_{\text{CM}} = \frac{\sum_i m_i \mathbf{r}_i}{M} = \text{constant} \tag{9.7}$$

where $M = \sum_i m_i$ is the total mass. By choosing an origin at the CM and a frame moving with constant velocity, we eliminate 6 degrees of freedom (3 for CM position, 3 for CM velocity).

Remaining degrees of freedom: $6N - 6$

9.3.2 Angular Momentum as Constraint

From conservation of total angular momentum, the motion can be oriented to align the angular momentum vector with a fixed direction (e.g., the z-axis). This is a rotational constraint but does not reduce degrees of freedom in the typical sense—rather, it specifies the orientation of the orbital plane(s).

For certain restricted configurations (e.g., all bodies in the same plane), this constraint can reduce degrees of freedom by 3.

For planar motion: $6N - 6 - 3 = 6N - 9$ **degrees of freedom**

9.3.3 Energy as a Constraint

The energy equation $E = \text{constant}$ provides one constraint, reducing degrees of freedom by 1:

In the center-of-mass frame: $6N - 10$ **degrees of freedom**

Alternatively, one can think of the restricted phase space where $H = E_0$ (a fixed energy surface), which is a $(6N - 1)$-dimensional manifold embedded in $\mathbb{R}^{6N}$.

9.4 Proof of Unsolvability for $N \geq 3$

The statement "the N-body problem is unsolvable" requires careful clarification. It does not mean no solution exists; rather, it means:

1. **No general closed-form solution** exists in elementary functions or simple quadratures.

2. **For generic initial conditions**, trajectories are chaotic and extremely sensitive to initial conditions.

3. **Numerical integration is required** to determine specific trajectories.

9.4.1 Poincaré's Non-Existence Theorem

Theorem 9.2 (Poincaré, 1890)**.** *For the three-body problem with $N = 3$ bodies, no additional first integrals exist beyond the 10 already known (3 from linear momentum conservation, 3 from angular momentum conservation, 1 from energy conservation, and 3 from rotational freedom). Furthermore, the problem cannot be reduced to a system of one-dimensional quadratures.*

Proof Sketch: Poincaré used a combination of perturbation theory and topological arguments. He showed that:

1. If additional first integrals existed, they would impose further constraints on the phase space.

2. Under the constraints of the gravitational potential and the symmetries already accounted for, the phase space topology prevents the existence of additional smooth, single-valued first integrals.

3. Methods of solving ODEs by quadratures require the existence of enough first integrals to reduce the system to one-dimensional problems; the deficiency of integrals makes this impossible for $N \geq 3$.

Consequence: The N-body problem for $N \geq 3$ is **not integrable in the sense of Liouville**.

9.4.2 Definition of Liouville Integrability

Definition 9.3 (Liouville Integrability)**.** *An N-degree-of-freedom Hamiltonian system is **completely integrable** (or integrable in the Liouville sense) if there exist N independent first integrals $F_1, F_2, \ldots, F_N$ (with $F_1 = H$) such that:*

$$\{F_i, F_j\} = 0 \quad \text{for all } i, j \tag{9.8}$$

*where $\{,\}$ is the Poisson bracket. These first integrals are said to be **in involution**.*

If such integrals exist and the level surfaces $\{(q_i, p_i) : F_i = c_i\}$ are compact and connected, then they are diffeomorphic to N-dimensional tori T^N, and the system can be integrated by quadratures.

Theorem 9.4 (Non-Integrability of the N-Body Problem)**.** *For the gravitational N-body problem with $N \geq 3$:*

1. *In the center-of-mass frame, there are at most $(6N-10)$ independent first integrals.*

2. *For a generic configuration and generic masses, fewer than $(6N-10)$ first integrals are known.*

3. *The system admits no additional first integrals beyond the 10 known from symmetries.*

4. *Therefore, the system is **not completely integrable** in the Liouville sense.*

This theorem implies that no general closed-form solution exists for $N \geq 3$.

9.4.3 Degree-of-Freedom Counting

For the N-body problem:

- **Configuration space dimensionality**: $3N$
- **Phase space dimensionality**: $6N$
- **Known first integrals**: 10 (from symmetries)
- **First integrals needed for complete integrability**: $6N/2 = 3N$ (since we need $3N$ first integrals for $3N$ position-momentum pairs)
- **Deficiency**: $3N - 10$

  For $N = 2$: Deficiency $= 6-10 = -4$. We actually have 6 integrals for 6 dimensions, which over-determines the system (2 particles always move together as a system or around CM). This allows complete solution.

  For $N = 3$: Deficiency $= 9 - 10 = -1$. We are missing 1 integral. This breaks Liouville integrability.

  For $N \geq 4$: Deficiency $= 3N - 10 \geq 2$. We are missing multiple integrals.

9.5 Special Exact Solutions for $N = 3$

Although the general three-body problem is unsolvable, certain special configurations admit exact solutions.

9.5.1 Lagrange Solutions (Relative Equilibria)

Definition 9.5 (Relative Equilibrium)**.** *A configuration is a* ***relative equilibrium*** *if, when viewed from a frame rotating with constant angular velocity ω, the configuration remains stationary.*

Theorem 9.6 (Lagrange, 1772)**.** *For the three-body problem, there are exactly five configurations that form relative equilibria:*

Collinear configurations (3 solutions)*:*

- *L1: Three bodies collinear, with the smallest body between the two larger bodies*
- *L2: Three bodies collinear, with the medium body between the largest and smallest*
- *L3: Three bodies collinear, with the largest body between the medium and smallest*

Triangular configurations (2 solutions)*:*

- *L4: Three bodies form an equilateral triangle*
- *L5: Three bodies form an equilateral triangle (rotated from L4)*

These five configurations generalize to N bodies (giving rise to the concept of Lagrange points in the CR3BP, which we examine in detail in later sections).

In a rotating frame with these relative equilibria, the equations become autonomous, making them solvable. The solutions are periodic in the rotating frame and quasi-periodic in the inertial frame.

9.5.2 Sundman Series and Regularization

Karl Sundman (1912) made a remarkable discovery:

Theorem 9.7 (Sundman)**.** *For the three-body problem, even though no general closed-form solution exists, the solution can be expressed as a power series in $t^{1/3}$:*

$$\mathbf{r}_i(t) = \sum_{k=0}^{\infty} a_{i,k} t^{k/3} \tag{9.9}$$

The series converges for all times $t \geq 0$ (with appropriate regularization near collision singularities).

Significance: While the Sundman series is a formal solution, it converges extremely slowly and is impractical for numerical computation. However, it proves that **the solution exists and is continuous**, despite the system being chaotic for generic initial conditions.

Limitation: The series converges uniformly across all space, but convergence rate is inversely related to proximity to collision events, making it effectively unusable for computing most trajectories accurately over long times.

9.6 Why the Two-Body Problem is Solvable: The Special Case $N = 2$

To understand the contrast, examine the two-body problem in the center-of-mass frame:

9.6.1 Dimensionality Analysis

- **Configuration space**: $3(2) = 6$ dimensions
- **CM constraint**: Reduces by 3 (fixing COM at origin) $\rightarrow$ **3 dimensions**
- **Relative coordinate**: Define $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ (single 3D vector) $\rightarrow$ **3 dimensions**
- **Angular momentum conservation**: Motion confined to a plane $\rightarrow$ **2 dimensions**
- **Energy conservation**: Provides one constraint $\rightarrow$ **1 dimension**
- **Phase space degrees of freedom after constraints**: 2 (radial coordinate r and angular coordinate θ)

9.6.2 Reduction via Central Force

For two bodies, the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ experiences a central force:

$$\mu \ddot{\mathbf{r}} = -\frac{G m_1 m_2}{r^2} \hat{r} \tag{9.10}$$

A central force automatically satisfies two crucial properties:

1. **Angular momentum conservation**: $\mathbf{L} = \mu \mathbf{r} \times \dot{\mathbf{r}} =$ constant. This confines motion to a plane and provides an additional constraint.

2. **Energy conservation in reduced form**: $E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$. Using $L = \mu r^2\dot{\theta}$, this becomes: $E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$, which is effectively a one-dimensional problem.

9.6.3 One-Dimensional Equivalent

The effective one-dimensional problem has Hamiltonian:

$$H_{\text{eff}} = \frac{p_r^2}{2\mu} + U_{\text{eff}}(r) \tag{9.11}$$

where the effective potential is:

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r} = \frac{L^2}{2\mu r^2} - \frac{C}{r} \tag{9.12}$$

This is a **one-degree-of-freedom Hamiltonian system**—a trivial integrable system. The orbit equation is determined by energy conservation alone, yielding the conic sections.

9.6.4 Key Insight

Why N

The two-body problem is solvable because:

1. The relative coordinate experiences a **central force**
2. Central forces have an **additional symmetry**: torque vanishes
3. This forces angular momentum conservation, confining motion to a plane
4. The plane motion is effectively one-dimensional (radial motion)
5. One-dimensional Hamiltonian systems are always integrable

For $N \geq 3$, no single body experiences a purely central force from the others; the pairwise forces combine in complex ways that break the central-force symmetry.

9.7 Why the N-Body Problem is Unsolvable: Loss of Central-Force Structure

For $N \geq 3$, the central-force structure is broken. Consider three bodies:

9.7.1 Non-Central Forces in the Three-Body Problem

The force on body 1 is:

$$\mathbf{F}_1 = -Gm_1m_2\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - Gm_1m_3\frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} \tag{9.13}$$

The gravitational forces from bodies 2 and 3 point in different directions (unless 2 and 3 are aligned with 1, which is a special configuration). The resultant force on body 1 does **not point toward a single center**.

Consequence: The torque about any fixed point does not vanish. The system does not possess the additional symmetry that makes central-force problems solvable.

9.7.2 Breakdown of Dimensional Reduction

Without central-force structure:

- Angular momentum is conserved for the **total system**, but not for individual bodies
- Individual three-dimensional trajectories cannot be confined to planes (in general)
- The problem retains full dimensionality even after symmetry reductions

9.8 Hamiltonian Structure and Phase Space

9.8.1 Canonical Coordinates and Momenta

In the Hamiltonian formalism, the N-body problem is a system of $6N$ first-order ODEs:

$$\dot{q}_{i,\alpha} = \frac{\partial H}{\partial p_{i,\alpha}} \tag{9.14}$$

$$\dot{p}_{i,\alpha} = -\frac{\partial H}{\partial q_{i,\alpha}} \tag{9.15}$$

where $\alpha = 1, 2, 3$ for Cartesian components, $i = 1, \ldots, N$ for bodies, and H is given by (9.1).

9.8.2 Phase Space Structure

The phase space $\mathbb{R}^{6N}$ has a symplectic structure generated by the **symplectic form**:

$$\omega = \sum_{i=1}^{N} \sum_{\alpha=1}^{3} dq_{i,\alpha} \wedge dp_{i,\alpha} \tag{9.16}$$

This is a closed, non-degenerate 2-form that encodes the canonical structure of phase space. Importantly:

- **Liouville's theorem**: The flow preserves volume in phase space
- **Trajectories cannot cross**: Each initial condition determines a unique trajectory
- **Symplectic structure preserved**: The evolution respects canonical transformations

9.8.3 Invariant Structures

Despite non-integrability, the phase space contains important geometric structures:

1. **Equilibrium manifolds**: Fixed points (relative equilibria) and their linearizations
2. **Periodic orbits**: Some special initial conditions yield periodic or quasi-periodic motion (e.g., the Lagrange solutions)

3. **Chaos and strange attractors**: Most trajectories are chaotic in the sense of Lyapunov exponents
4. **Resonance manifolds**: Regions where frequency ratios are rational, leading to periodic behavior

9.9 Hamiltonian Decomposition and Action Variables

For integrable systems, the Hamiltonian can be written in action-angle variables (J_i, ϕ_i):

$$H = H(J_1, \ldots, J_n) \tag{9.17}$$

Then the evolution is trivial: $J_i = \text{constant}$ and $\phi_i = \omega_i(J)t + \phi_{i,0}$.

9.9.1 Action Variables

Definition 9.8 (Action Variables)**.** *For an integrable system with generalized coordinate q_i and conjugate momentum p_i, the action variable is:*

$$J_i = \frac{1}{2\pi} \oint p_i dq_i \tag{9.18}$$

where the integral is taken over one complete period of the i-th degree of freedom.

For the two-body problem in the orbital plane:

- **Radial action**: $J_r = \frac{1}{2\pi} \oint p_r dr$ (integral over one radial oscillation)
- **Angular action**: $J_\theta = \frac{1}{2\pi} \oint p_\theta d\theta = \frac{L}{2\pi}$ (proportional to angular momentum)

The Hamiltonian in action-angle variables is:

$$H(J_r, J_\theta) = E(J_r, J_\theta) \tag{9.19}$$

The angle variables ϕ_r, ϕ_θ are determined such that $\dot{\phi}_i = \partial H / \partial J_i = \omega_i$ (constant angular frequencies).

9.9.2 Breakdown for N-Body Systems

For $N \geq 3$, the action-angle formalism breaks down because:

1. There are not enough conserved quantities to define action variables
2. The frequency vector $\omega(J)$ becomes non-existent for most trajectories
3. Phase space is not foliated by invariant tori (except in special cases)

However, for particular initial conditions (e.g., resonant states or certain periodic orbits), local action-angle coordinates can be defined over finite regions of phase space.

9.10 Numerical Integrability and Practical Solutions

While analytical solutions do not exist for $N \geq 3$, the problem can be solved **numerically** with arbitrary precision.

9.10.1 Direct Integration Methods

Particle-particle methods: Directly integrate the equations of motion:

$$\ddot{\mathbf{r}}_i = \sum_{j \neq i} -Gm_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \tag{9.20}$$

using standard ODE solvers (Runge-Kutta, Adams-Bashforth, etc.).
Advantages:

- Simple to implement
- Works for arbitrary N and arbitrary initial conditions
- Accurate for sufficiently small time steps

Challenges:

- Computational complexity scales as $O(N^2)$ per step (computing all pairwise interactions)
- Large N (e.g., $N > 10^6$ in astrophysical simulations) requires specialized algorithms
- Gravitational singularity at small separations: potential diverges at $r \to 0$
- Chaotic sensitivity: small errors accumulate exponentially over time

9.10.2 Softened Potential

To avoid singularities, the gravitational potential is often softened:

$$V_{\text{softened}} = -G \sum_{i<j} \frac{m_i m_j}{\sqrt{r_{ij}^2 + \epsilon^2}} \tag{9.21}$$

The softening length $\epsilon > 0$ prevents infinite forces at small separations. For $r_{ij} \gg \epsilon$, this recovers the Newtonian potential.

9.10.3 Symplectic Integrators

Importance of structure preservation: Since the N-body system is Hamiltonian, it conserves phase-space volume (Liouville's theorem) and energy (for isolated systems). Standard numerical methods may violate these properties.

Symplectic integrators: Special ODE solvers that preserve the symplectic structure of Hamiltonian systems. These methods:

- Conserve energy exactly (no artificial energy drift)
- Preserve canonical structure
- Generate phase-space trajectories that remain faithful to the true dynamics over very long time scales

A simple example is the **Störmer-Verlet method**:

$$\mathbf{p}_i^{n+1/2} = \mathbf{p}_i^n - \frac{\Delta t}{2}\nabla_i V(\mathbf{r}^n) \tag{9.22}$$

$$\mathbf{r}_i^{n+1} = \mathbf{r}_i^n + \Delta t \frac{\mathbf{p}_i^{n+1/2}}{m_i} \tag{9.23}$$

$$\mathbf{p}_i^{n+1} = \mathbf{p}_i^{n+1/2} - \frac{\Delta t}{2}\nabla_i V(\mathbf{r}^{n+1}) \tag{9.24}$$

This alternates between "kick" (momentum update) and "drift" (position update) in a way that preserves the symplectic structure.

9.11 Conclusion and Physical Interpretation

The fundamental difference between the solvable two-body problem and the unsolvable N-body problem illustrates a profound principle:

Central Lesson: From Order to Chaos

Problem Solvability Depends on Symmetry Structure

The two-body problem is integrable because the relative coordinate satisfies a central-force equation, which possesses an additional conserved quantity (angular momentum in the orbit plane) that enables dimensional reduction to a one-degree-of-freedom system.

For $N \geq 3$, the pairwise superposition of gravitational forces breaks the central-force symmetry. No body orbits a single center; instead, all bodies collectively influence each other. This loss of symmetry destroys the dimensional reduction mechanism, leaving us with a system having more degrees of freedom than available integrals of motion.

Despite the simplicity of the underlying gravitational interaction, the collective behavior becomes fundamentally **chaotic**: sensitive to initial conditions, resistant to prediction, and exhibiting behavior that is qualitatively different from the organized elliptical orbits of the two-body problem.

This transition from order (integrable) to chaos (non-integrable) as we increase the number of interacting bodies is one of the most important discoveries in classical mechanics, with far-reaching implications for understanding planetary systems, stellar dynamics, and many other physical phenomena.

The analysis of specific three-body configurations and reduced problems (such as the circular restricted three-body problem treated in the next section) provides tractable special cases where we can gain insight into the general three-body dynamics.

10 The Circular Restricted Three-Body Problem (CR3BP): A Complete Treatment

10.1 Introduction: Motivation and Historical Context

The transition from the solvable two-body problem to the three-body case represents a critical juncture in celestial mechanics. While the general three-body problem is fundamentally unsolvable (as established in Section 3 of this text), the **Circular Restricted Three-Body Problem** (CR3BP) represents a highly tractable special case that captures the essential physics of many real-world systems.

The CR3BP is not merely a mathematical simplification; it arises naturally from practical considerations:

- **Real systems exhibit mass hierarchies**: In the Sun-Earth-spacecraft system, $m_{\text{spacecraft}} \ll m_{\text{Earth}} \ll m_{\text{Sun}}$. The spacecraft's mass is inconsequential to the dynamics of the two primaries.
- **Short-term orbital predictions are accurate**: Over timescales of a few orbits, the assumption of circular primary orbits is excellent. Precession and perturbations are secondary effects.
- **Rich dynamical phenomena emerge**: Despite its simplifying assumptions, the CR3BP exhibits chaotic motion, periodic orbits, and complex transport phenomena absent from simpler models.

The CR3BP has been the workhorse of celestial mechanics for over two centuries, enabling the design of nearly all space missions to libration points (Lagrange points) and providing insights into asteroid dynamics, planetary formation, and stellar binary systems.

10.2 Physical Setup and Assumptions

10.2.1 The Three-Body Configuration

The CR3BP is defined by the following system:

1. **Two massive primaries**: Bodies of mass m_1 and m_2 with $m_1 \geq m_2$ (masses normalized such that $m_1 + m_2 = 1$ in nondimensional units).
2. **Circular orbits of primaries**: The two primaries move in perfect circles around their common center of mass. Their separation r_{12} remains constant in time (no eccentricity).
3. **Infinitesimal test particle**: A third body of negligible mass $m_3 \to 0$ moves under the gravitational influence of m_1 and m_2. The key assumption is that the test particle exerts *no dynamical influence* on the primaries—their orbital parameters are unaffected by m_3.
4. **Planar motion**: All three bodies remain in a single plane throughout their motion. This is justified because:

- For the primaries: their circular orbits define a plane
- For the test particle: the CR3BP admits a planar subspace of solutions, and perturbations perpendicular to this plane are governed by a harmonic restoring force (as we will show with stability analysis)

5. **Newtonian gravitation**: Gravitational forces obey Newton's inverse-square law with no relativistic corrections.

Canonical physical examples:

- **Sun-Earth-spacecraft**: Sun and Earth orbit their center of mass every year; a spacecraft (e.g., SOHO, JWST) moves in this system. The spacecraft's mass is at most 10^3 kg versus Earth at 10^{24} kg.
- **Earth-Moon-satellite**: Earth and Moon orbit their center of mass every 27.3 days; a lunar orbiter traces complex paths. Mass ratio: Moon/Earth ≈ 0.012.
- **Sun-Jupiter-asteroid**: The Sun-Jupiter system hosts thousands of Trojan asteroids at Lagrange points L4 and L5. Mass ratio: Jupiter/Sun ≈ 0.001.

10.2.2 Reference Frames: Inertial versus Rotating

Inertial frame ($\mathcal{I}$): Fixed to the distant stars, non-rotating. Position vectors $\mathbf{R}_i$ for each body in this frame satisfy Newton's laws directly.

Rotating frame ($\mathcal{R}$): Rotates with the primaries. Position vectors $\mathbf{r}_i$ are expressed in this frame. Key advantages:

- The two primaries are *stationary* in this frame—their positions are fixed constants, simplifying the gravitational potential.
- Five equilibrium points (Lagrange points) exist as stationary points.
- Periodic orbits become visible as closed curves (not spirals in the inertial frame).

The transformation between frames is:

$$\mathbf{R} = \mathcal{R}(t)\mathbf{r} \tag{10.1}$$

where $\mathcal{R}(t)$ is a rotation matrix by angle $\theta(t) = \omega t$ (with angular velocity ω of the primaries' orbit).

10.3 Nondimensionalization: Simplifying the Equations

To render the mathematics dimensionally simpler and results universally applicable across all primary mass ratios, we introduce characteristic units:

10.3.1 Choice of Units

Unit of mass:

$$m_1 + m_2 = 1 \tag{10.2}$$

Define the **mass parameter**:

$$\mu = \frac{m_2}{m_1 + m_2} \tag{10.3}$$

By convention (placing the more massive body first), we have $0 < \mu \leq 0.5$. Then:

$$m_1 = 1 - \mu \tag{10.4}$$

$$m_2 = \mu \tag{10.5}$$

Unit of distance:

$$r_{12} = |\mathbf{r}_1 - \mathbf{r}_2| = 1 \tag{10.6}$$

The primaries are separated by unit distance. In dimensional units, for Earth-Sun: 1 unit = 1 AU $\approx 1.496 \times 10^{11}$ m.

Unit of time:

$$T = 2\pi \quad \text{(orbital period of primaries)} \tag{10.7}$$

This implies:

$$n = \frac{2\pi}{T} = 1 \quad \text{(mean motion)} \tag{10.8}$$

So the primaries orbit with unit angular velocity.

Gravitational constant:

$$G = 1 \tag{10.9}$$

These choices ensure that Kepler's third law for circular orbits, $T^2 = 4\pi^2 a^3/(GM)$, is automatically satisfied with our values.

10.3.2 Verification of Consistency

For a two-body circular orbit with $m_1 + m_2 = 1$, separation 1, period 2π:

$$T^2 = 4\pi^2 \frac{a^3}{G(m_1 + m_2)} \tag{10.10}$$

$$(2\pi)^2 = 4\pi^2 \frac{1^3}{1 \cdot 1} \tag{10.11}$$

$$4\pi^2 = 4\pi^2 \quad \checkmark \tag{10.12}$$

10.4 Positions of the Primaries in the Rotating Frame

In the rotating frame with origin at the center of mass, the primaries are located on the x-axis (by convention, the line of primaries).

10.4.1 Center-of-Mass Condition

The center of mass is at the origin:

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = \mathbf{0} \tag{10.13}$$

With $m_1 + m_2 = 1$:

$$(1 - \mu)\mathbf{r}_1 + \mu \mathbf{r}_2 = \mathbf{0} \tag{10.14}$$

10.4.2 Separation and Position Calculation

The separation constraint gives:

$$\mathbf{r}_2 - \mathbf{r}_1 = (1, 0, 0) \tag{10.15}$$

(choosing the x-axis to align with the primary separation).
From the center-of-mass equation:

$$\mathbf{r}_1 = -\mu\mathbf{r}_2/(1-\mu) \tag{10.16}$$

But from the separation: $\mathbf{r}_2 = \mathbf{r}_1 + (1, 0, 0)$. Substituting:

$$\mathbf{r}_1 = -\mu[\mathbf{r}_1 + (1, 0, 0)]/(1-\mu) \tag{10.17}$$
$$\mathbf{r}_1(1-\mu) = -\mu\mathbf{r}_1 - \mu(1, 0, 0) \tag{10.18}$$
$$\mathbf{r}_1[(1-\mu) + \mu] = -\mu(1, 0, 0) \tag{10.19}$$
$$\mathbf{r}_1 = -\mu(1, 0, 0) \tag{10.20}$$

Therefore:

$$\mathbf{r}_1 = (-\mu, 0, 0) \tag{10.21}$$
$$\mathbf{r}_2 = (1-\mu, 0, 0) \tag{10.22}$$

Verification:

- Center of mass: $(1-\mu)(-\mu) + \mu(1-\mu) = -\mu + \mu^2 + \mu - \mu^2 = 0$ ✓
- Separation: $(1-\mu) - (-\mu) = 1 - \mu + \mu = 1$ ✓

In the rotating frame, these positions are *fixed* for all time—the primaries do not move.

10.5 Kinematics: From Inertial to Rotating Frame

This derivation uses the vector calculus developed in Section 1, specialized to the CR3BP geometry.

10.5.1 Angular Velocity Vector

The rotating frame rotates about the z-axis (perpendicular to the orbital plane) with angular velocity:

$$\boldsymbol{\omega} = n\hat{\mathbf{z}} = (0, 0, 1) = \hat{\mathbf{z}} \tag{10.23}$$

(since $n = 1$ in nondimensional units).

10.5.2 Velocity Transformation

From Section 1, the velocity in the inertial frame related to the rotating frame is:

$$\mathbf{v}_{\text{inertial}} = \mathbf{v}_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r} \tag{10.24}$$

Explicitly:

$$\mathbf{v}_{\text{inertial}} = (\dot{x}, \dot{y}, \dot{z}) + (0, 0, 1) \times (x, y, z) \tag{10.25}$$
$$= (\dot{x}, \dot{y}, \dot{z}) + (-y, x, 0) \tag{10.26}$$
$$= (\dot{x} - y, \dot{y} + x, \dot{z}) \tag{10.27}$$

10.5.3 Acceleration Transformation

From Section 1, the full acceleration transformation (with constant $\boldsymbol{\omega}$) is:

$$\mathbf{a}_{\text{inertial}} = \mathbf{a}_{\text{rotating}} + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rotating}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{10.28}$$

Computing each term:
First term: Acceleration in rotating frame $= (\ddot{x}, \ddot{y}, \ddot{z})$.
Second term (Coriolis):

$$2\boldsymbol{\omega} \times \mathbf{v}_{\text{rotating}} = 2(0, 0, 1) \times (\dot{x}, \dot{y}, \dot{z}) \tag{10.29}$$
$$= 2(-\dot{y}, \dot{x}, 0) \tag{10.30}$$

Third term (Centrifugal):

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = (0, 0, 1) \times [(0, 0, 1) \times (x, y, z)] \tag{10.31}$$
$$= (0, 0, 1) \times (-y, x, 0) \tag{10.32}$$
$$= (-x, -y, 0) \tag{10.33}$$

Combining:

$$\mathbf{a}_{\text{inertial}} = (\ddot{x}, \ddot{y}, \ddot{z}) + 2(-\dot{y}, \dot{x}, 0) + (-x, -y, 0) \tag{10.34}$$
$$= (\ddot{x} - 2\dot{y} - x, \ddot{y} + 2\dot{x} - y, \ddot{z}) \tag{10.35}$$

10.6 Gravitational Force on the Test Particle

The test particle at position $\mathbf{r} = (x, y, z)$ experiences gravitational attractions from both primaries.

10.6.1 Force from Primary 1 (at $\mathbf{r}_1 = (-\mu, 0, 0)$)

By Newton's law:

$$\mathbf{F}_1 = -\frac{Gm_1m_3}{|\mathbf{r} - \mathbf{r}_1|^3}(\mathbf{r} - \mathbf{r}_1) \tag{10.36}$$

With $G = 1$, $m_1 = 1 - \mu$, and m_3 canceling:

$$\mathbf{F}_1 = -(1 - \mu)\frac{\mathbf{r} - \mathbf{r}_1}{r_1^3} \tag{10.37}$$

where:

$$r_1 = |\mathbf{r} - \mathbf{r}_1| = \sqrt{(x + \mu)^2 + y^2 + z^2} \tag{10.38}$$

10.6.2 Force from Primary 2 (at $\mathbf{r}_2 = (1 - \mu, 0, 0)$)

Similarly:

$$\mathbf{F}_2 = -\mu\frac{\mathbf{r} - \mathbf{r}_2}{r_2^3} \tag{10.39}$$

where:

$$r_2 = |\mathbf{r} - \mathbf{r}_2| = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2} \tag{10.40}$$

10.6.3 Total Gravitational Force

$$\mathbf{F}_{\text{gravity}} = \mathbf{F}_1 + \mathbf{F}_2 = -(1-\mu)\frac{\mathbf{r}-\mathbf{r}_1}{r_1^3} - \mu\frac{\mathbf{r}-\mathbf{r}_2}{r_2^3} \tag{10.41}$$

10.7 Equations of Motion: Deriving the CR3BP Dynamics

10.7.1 Newton's Second Law in the Inertial Frame

For a test particle of mass m_3:

$$m_3\mathbf{a}_{\text{inertial}} = \mathbf{F}_{\text{gravity}} \tag{10.42}$$

Dividing by m_3:

$$\mathbf{a}_{\text{inertial}} = -(1-\mu)\frac{\mathbf{r}-\mathbf{r}_1}{r_1^3} - \mu\frac{\mathbf{r}-\mathbf{r}_2}{r_2^3} \tag{10.43}$$

10.7.2 Substituting the Acceleration Transformation

From equation (10.35):

$$(\ddot{x}-2\dot{y}-x, \ddot{y}+2\dot{x}-y, \ddot{z}) = -(1-\mu)\frac{(\mathbf{r}-\mathbf{r}_1)}{r_1^3} - \mu\frac{(\mathbf{r}-\mathbf{r}_2)}{r_2^3} \tag{10.44}$$

Component by component:
x-component:

$$\ddot{x}-2\dot{y}-x = -(1-\mu)\frac{x+\mu}{r_1^3} - \mu\frac{x-1+\mu}{r_2^3} \tag{10.45}$$

Rearranging:

$$\ddot{x}-2\dot{y} = x-(1-\mu)\frac{x+\mu}{r_1^3} - \mu\frac{x-1+\mu}{r_2^3} \tag{10.46}$$

y-component: By analogous calculation,

$$\ddot{y}+2\dot{x} = y-(1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} \tag{10.47}$$

z-component:

$$\ddot{z} = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3} \tag{10.48}$$

10.8 The Effective Potential and Reformulation

10.8.1 Recognizing Potential Functions

Observe that the right-hand sides of equations (10.46)–(10.48) can be expressed as gradients of potential functions.

Centrifugal term: The term $-\nabla[x^2+y^2]/2$ yields:

$$-\frac{\partial}{\partial x}\left[\frac{x^2+y^2}{2}\right] = -x \tag{10.49}$$

Gravitational term: The gravitational force is conservative. Define:

$$U_{\text{grav}} = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2} \tag{10.50}$$

Then:

$$-\frac{\partial U_{\text{grav}}}{\partial x} = -(1-\mu)\frac{\partial}{\partial x}(r_1^{-1}) - \mu\frac{\partial}{\partial x}(r_2^{-1}) \tag{10.51}$$

$$= (1-\mu)\frac{1}{r_1^3}\frac{\partial r_1}{\partial x} + \mu\frac{1}{r_2^3}\frac{\partial r_2}{\partial x} \tag{10.52}$$

$$= (1-\mu)\frac{x+\mu}{r_1^3} + \mu\frac{x-1+\mu}{r_2^3} \tag{10.53}$$

(using $\frac{\partial r_i}{\partial x} = (x_{\text{particle}} - x_i)/r_i$).

10.8.2 Definition of the Effective Potential

Define the **effective potential** (also called pseudo-potential):

$$\Omega(x,y,z) = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{10.54}$$

This combines centrifugal and gravitational contributions.

10.8.3 Reformulated Equations of Motion

The equations of motion can now be written compactly as:

$$\ddot{x} - 2\dot{y} = \frac{\partial\Omega}{\partial x} \tag{10.55}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial\Omega}{\partial y} \tag{10.56}$$

$$\ddot{z} = \frac{\partial\Omega}{\partial z} \tag{10.57}$$

These are the **fundamental equations of the circular restricted three-body problem**.

Remarks:

- These are six coupled nonlinear ordinary differential equations (when expanded to first-order form: $\dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}$ as dependent variables).
- They are autonomous (time-independent) in the rotating frame, which is a key advantage for finding equilibrium points and periodic orbits.
- The Coriolis terms appear as explicit coupling between $\dot{x}$ and $\dot{y}$ equations, reflecting the non-inertial nature of the rotating frame.

10.9 The Jacobi Constant: A Conserved Quantity

Although energy is not conserved in the rotating frame (due to the time-dependent coordinate transformation), there exists a remarkable conserved quantity unique to the CR3BP.

10.9.1 Derivation from Dot Products

Multiply equation (10.55) by $\dot{x}$:

$$\dot{x}(\ddot{x} - 2\dot{y}) = \dot{x}\frac{\partial \Omega}{\partial x} \tag{10.58}$$

Multiply equation (10.56) by $\dot{y}$:

$$\dot{y}(\ddot{y} + 2\dot{x}) = \dot{y}\frac{\partial \Omega}{\partial y} \tag{10.59}$$

Multiply equation (10.57) by $\dot{z}$:

$$\dot{z}\ddot{z} = \dot{z}\frac{\partial \Omega}{\partial z} \tag{10.60}$$

10.9.2 Cancellation of Coriolis Terms

Add all three equations:

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} - 2\dot{x}\dot{y} + 2\dot{x}\dot{y} = \dot{x}\frac{\partial \Omega}{\partial x} + \dot{y}\frac{\partial \Omega}{\partial y} + \dot{z}\frac{\partial \Omega}{\partial z} \tag{10.61}$$

The Coriolis terms ($-2\dot{x}\dot{y}$ and $+2\dot{x}\dot{y}$) *cancel exactly*. This is a fundamental property of Coriolis forces: they do no work because they are always perpendicular to velocity.

The left side becomes:

$$\frac{1}{2}\frac{d}{dt}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}\frac{dv^2}{dt} \tag{10.62}$$

The right side is the total time derivative of Ω:

$$\frac{d\Omega}{dt} = \frac{\partial \Omega}{\partial x}\dot{x} + \frac{\partial \Omega}{\partial y}\dot{y} + \frac{\partial \Omega}{\partial z}\dot{z} \tag{10.63}$$

10.9.3 Integration to Find the Constant

Combining:

$$\frac{1}{2}\frac{dv^2}{dt} = \frac{d\Omega}{dt} \tag{10.64}$$

Rearranging:

$$\frac{d}{dt}(2\Omega - v^2) = 0 \tag{10.65}$$

Therefore:

$$2\Omega(x, y, z) - v^2 = \text{constant} \tag{10.66}$$

10.9.4 The Jacobi Constant

By convention, define the **Jacobi constant** as:

$$C_J = -(2\Omega - v^2) = v^2 - 2\Omega(x, y, z) \tag{10.67}$$

(The negative sign is historical convention, making bound orbits have $C_J > 0$.)

Equivalently:

$$C_J = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2\left[\frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right] \tag{10.68}$$

Simplifying:

$$C_J = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - (x^2+y^2) - 2\frac{1-\mu}{r_1} - 2\frac{\mu}{r_2} \tag{10.69}$$

This is the unique first integral of motion for the CR3BP.

10.9.5 Physical Interpretation of the Jacobi Constant

The Jacobi constant has several interpretations:

Energy-like quantity: In the rotating frame, it relates position and velocity. Unlike true energy, it is not a sum of kinetic plus potential energies in the usual sense. Instead, it encodes the balance between:

- Kinetic energy in the rotating frame
- Centrifugal potential
- Gravitational potential

Configuration space constraint: Rearranging C_J = constant:

$$v^2 = C_J + x^2 + y^2 + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} \tag{10.70}$$

Since $v^2 \geq 0$ (velocities are real), we have:

$$x^2 + y^2 + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} \geq -C_J \tag{10.71}$$

This defines the accessible region of configuration space for a given energy (Jacobi constant).

10.9.6 Relationship to Total Energy

In the inertial frame, the total energy (conserved) is:

$$E = \frac{1}{2}(v_{\text{inertial}})^2 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \tag{10.72}$$

The relationship between E and C_J involves the kinetic energy of the rotating frame motion. For low-velocity particles, they approximately coincide, but for high-velocity particles, they diverge. This is why C_J is specifically suited to the rotating frame dynamics.

10.10 Zero-Velocity Surfaces and Hill Regions

10.10.1 Definition and Physical Meaning

The **zero-velocity surface** is the locus of points where $v = 0$ (velocity vanishes in the rotating frame):

$$2\Omega(x, y, z) = C_J \tag{10.73}$$

Equivalently:

$$x^2 + y^2 + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} = C_J \tag{10.74}$$

Physical interpretation: The zero-velocity surface is a boundary in position space that the test particle cannot cross. If it reaches the surface, its velocity becomes zero at that instant. On the other side (where $v^2 < 0$, which is forbidden), the particle cannot venture.

The accessible region where $v^2 \geq 0$ is called the **Hill region** (named after George William Hill, who pioneered this analysis).

10.10.2 Topology of Hill Regions

As the Jacobi constant C_J varies, the topology of the zero-velocity surfaces changes qualitatively. For the Earth-Moon system ($\mu \approx 0.012$), there are five distinct cases:

Case 1: Very High C_J (Low Energy)

The zero-velocity surface is a closed surface in three-dimensional space surrounding Earth. The particle is trapped in the Earth's potential well. Three disconnected Hill regions exist:

- A region around the primary m_1 (Earth)
- A region around the primary m_2 (Moon)
- An exterior region (to infinity)

A particle near Earth cannot reach the Moon or escape to infinity.

Case 2: $C_J = C_{L_1}$ (Energy equals Lagrange Point L_1)

At this critical energy, the zero-velocity surfaces from Earth and Moon touch at the L_1 Lagrange point (located between the two bodies). The bottleneck "opens" slightly.

Case 3: $C_{L_1} > C_J > C_{L_2}$ (Intermediate Energy)

The zero-velocity surface opens further. A particle can now transit from Earth's region to the Moon's region by passing through the L_1 point. However, Earth and Moon are still separate from the exterior region.

Case 4: $C_J = C_{L_2}$ (Energy equals Lagrange Point L_2)

The zero-velocity surface opens at the L_2 point (on the opposite side of Moon from Earth). Now the Moon's region connects to the exterior region.

Case 5: Very Low C_J (High Energy)

For $C_J < C_{L_3}$, nearly all of space is accessible, except two small "tadpole" regions that encircle the L_4 and L_5 Lagrange points.

10.10.3 Practical Significance for Space Missions

These energy boundaries are crucial for mission design:

- **Earth escape**: To leave Earth's sphere of influence, a spacecraft must have enough energy to pass through the L_1 or L_2 bottleneck.
- **Lunar transfer**: A minimum-energy transfer orbit from Earth to Moon exploits the topology change at $C_J = C_{L_1}$.
- **Trojan asteroids**: Asteroids at the L_4 and L_5 points (Trojans) remain trapped in the tadpole regions by the topology of the zero-velocity surfaces.

10.11 Lagrange Points (Equilibrium Points)

10.11.1 Definition

A **Lagrange point** (or libration point) is a location in the rotating frame where the test particle can remain stationary *relative to the rotating frame*:

$$\dot{x} = \dot{y} = \dot{z} = 0, \quad \ddot{x} = \ddot{y} = \ddot{z} = 0 \tag{10.75}$$

At such points, all forces balance in the rotating frame.
From equations (10.55)–(10.57), the equilibrium conditions become:

$$0 = \frac{\partial \Omega}{\partial x} \tag{10.76}$$

$$0 = \frac{\partial \Omega}{\partial y} \tag{10.77}$$

$$0 = \frac{\partial \Omega}{\partial z} \tag{10.78}$$

These are three equations in three unknowns (x, y, z).

10.11.2 Vertical (Out-of-Plane) Equilibrium

For the z-component:

$$\frac{\partial \Omega}{\partial z} = (1-\mu)\frac{\partial}{\partial z}(r_1^{-1}) + \mu \frac{\partial}{\partial z}(r_2^{-1}) = 0 \tag{10.79}$$

Computing the partial derivatives:

$$\frac{\partial}{\partial z}\left[\frac{1}{r_1}\right] = -\frac{1}{r_1^3}\frac{\partial r_1}{\partial z} = -\frac{z}{r_1^3} \tag{10.80}$$

$$\frac{\partial}{\partial z}\left[\frac{1}{r_2}\right] = -\frac{z}{r_2^3} \tag{10.81}$$

Therefore:

$$-z\left[(1-\mu)r_1^{-3} + \mu r_2^{-3}\right] = 0 \tag{10.82}$$

The bracketed term is always positive (masses and distances are positive), so:

$$z = 0 \tag{10.83}$$

Conclusion: All five Lagrange points lie in the orbital plane ($z = 0$).

10.11.3 Planar Equilibrium Conditions

With $z = 0$, we have:

$$r_1 = \sqrt{(x+\mu)^2 + y^2} \tag{10.84}$$

$$r_2 = \sqrt{(x-1+\mu)^2 + y^2} \tag{10.85}$$

The equilibrium equations become:

$$0 = x - (1-\mu)\frac{x+\mu}{r_1^3} - \mu\frac{x-1+\mu}{r_2^3} \tag{10.86}$$

$$0 = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} = y\left[1 - (1-\mu)r_1^{-3} - \mu r_2^{-3}\right] \tag{10.87}$$

Equation (10.87) has two solutions:
Solution 1: $y = 0 \rightarrow$ Collinear points L_1, L_2, L_3
Solution 2: $1 - (1-\mu)r_1^{-3} - \mu r_2^{-3} = 0 \rightarrow$ Triangular points L_4, L_5

10.11.4 Triangular (Equilateral) Lagrange Points L_4, L_5

Condition from $y \neq 0$ case:

$$\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} = 1 \tag{10.88}$$

Multiply equation (10.86) by $(1-\mu)r_1^{-3} + \mu r_2^{-3}$:

$$x\left[(1-\mu)r_1^{-3} + \mu r_2^{-3}\right] = (1-\mu)\frac{x+\mu}{r_1^3} + \mu\frac{x-1+\mu}{r_2^3} \tag{10.89}$$

This leads to:

$$0 = (1-\mu)\frac{\mu}{r_1^3} - \mu\frac{1-\mu}{r_2^3} = (1-\mu)\mu\left(r_1^{-3} - r_2^{-3}\right) \tag{10.90}$$

Since $(1-\mu)\mu > 0$:

$$r_1 = r_2 \tag{10.91}$$

From equation (10.88) with $r_1 = r_2 = r$:

$$\frac{1}{r^3} = 1 \quad \Rightarrow \quad r = 1 \tag{10.92}$$

So the test particle is at unit distance from *both* primaries. Since the primaries are also unit distance apart, the three bodies form an **equilateral triangle**.

Solving for Coordinates:
We need (x, y) such that:

$$(x+\mu)^2 + y^2 = 1 \tag{10.93}$$

$$(x-1+\mu)^2 + y^2 = 1 \tag{10.94}$$

Subtracting the second from the first:

$$(x+\mu)^2 - (x-1+\mu)^2 = 0 \tag{10.95}$$

$$[(x+\mu) + (x-1+\mu)][(x+\mu) - (x-1+\mu)] = 0 \tag{10.96}$$

$$[2x - 1 + 2\mu][1] = 0 \tag{10.97}$$

$$x = \frac{1}{2} - \mu \tag{10.98}$$

Substituting back:

$$\left(\frac{1}{2}\right)^2 + y^2 = 1 \tag{10.99}$$

$$y^2 = \frac{3}{4} \tag{10.100}$$

$$y = \pm\frac{\sqrt{3}}{2} \tag{10.101}$$

Therefore:

$$L_4 : \quad \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right) \tag{10.102}$$

$$L_5 : \quad \left(\frac{1}{2} - \mu, -\frac{\sqrt{3}}{2}\right) \tag{10.103}$$

These points lead (ahead in the direction of motion) and trail (behind) the secondary body by 60°.

Real examples: Jupiter has two large groups of asteroids (Trojan asteroids) at its L_4 and L_5 points relative to the Sun.

10.11.5 Collinear Lagrange Points L_1, L_2, L_3

For $y = 0$, equation (10.86) becomes:

$$x = (1-\mu)\frac{x+\mu}{|x+\mu|^3} + \mu\frac{x-1+\mu}{|x-1+\mu|^3} \tag{10.104}$$

This is a quintic (degree-5 polynomial) equation in x after clearing denominators, and has three real roots corresponding to:

L_1 **(Between the primaries)**: $-\mu < x < 1-\mu$. Located approximately distance $(\mu/3)^{1/3}$ from the secondary.

L_2 **(Beyond the secondary)**: $x > 1-\mu$. Located approximately distance $(\mu/3)^{1/3}$ beyond the secondary (opposite side from primary).

L_3 **(Beyond the primary)**: $x < -\mu$. Located on the opposite side of the primary, approximately distance $1 + \frac{5\mu}{12}$ from the center of mass.

These collinear points have no simple closed-form solutions; they must be found numerically. However, the approximations are accurate for small μ.

Real-world examples:

- **Sun-Earth** L_1: Located ≈ 1 million km from Earth toward the Sun. Hosts SOHO spacecraft for solar observations.
- **Sun-Earth** L_2: Located ≈ 1.5 million km from Earth away from Sun. Hosts JWST for infrared astronomy.
- **Earth-Moon** L_2: Located $\approx 60,000$ km from the Moon. A candidate location for lunar gateway stations.
- **Jupiter-Trojan** L_4 **and** L_5: Contain thousands of Trojan asteroids, some with diameters exceeding 100 km.

11 Linear Stability Analysis of Lagrange Points

Understanding whether Lagrange points are stable is essential for mission design. Small perturbations from equilibrium lead to different behaviors: stable points retain the particle nearby; unstable points cause exponential divergence.

11.1 Definition of the Hessian Matrix

11.1.1 Introduction

The **Hessian matrix** (also called the Hessian) is a square matrix of second partial derivatives of a scalar function. For a function $f(x_1, x_2, \ldots, x_n)$, the Hessian is defined as:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix} \tag{11.1}$$

For a function of two variables $f(x, y)$, the Hessian is a 2×2 matrix:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \tag{11.2}$$

11.1.2 Properties of the Hessian

- **Symmetry**: Under reasonable smoothness conditions (continuous second derivatives), the mixed partial derivatives are equal: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Therefore, the Hessian is *symmetric*.
- **Taylor expansion**: When expanding f to second order about a point (x_0, y_0):

$$f(x_0 + \xi, y_0 + \eta) \approx f(x_0, y_0) + \nabla f \cdot (\xi, \eta) + \frac{1}{2}(\xi, \eta) \cdot H(f) \cdot (\xi, \eta)^T + O(3) \tag{11.3}$$

  The quadratic form $(\xi, \eta) \cdot H \cdot (\xi, \eta)^T$ determines the local curvature of the function.
- **Critical point classification**: At a critical point where $\nabla f = 0$, the Hessian's eigenvalues determine whether the critical point is a local minimum, maximum, or saddle point.

11.2 Explicit Linearization Procedure for CR3BP

11.2.1 Perturbation Ansatz

Let (x_0, y_0, z_0) be a Lagrange point (equilibrium). Introduce small perturbations from equilibrium:

$$x(t) = x_0 + \xi(t) \tag{11.4}$$
$$y(t) = y_0 + \eta(t) \tag{11.5}$$
$$z(t) = z_0 + \zeta(t) \tag{11.6}$$

where $\xi(t)$, $\eta(t)$, $\zeta(t)$ are small deviations with $|\xi|, |\eta|, |\zeta| \ll 1$.
The velocities transform as:

$$\dot{x} = \dot{\xi} \tag{11.7}$$
$$\dot{y} = \dot{\eta} \tag{11.8}$$
$$\dot{z} = \dot{\zeta} \tag{11.9}$$

(since x_0, y_0, z_0 are constants).
The accelerations transform as:

$$\ddot{x} = \ddot{\xi} \tag{11.10}$$
$$\ddot{y} = \ddot{\eta} \tag{11.11}$$
$$\ddot{z} = \ddot{\zeta} \tag{11.12}$$

11.2.2 Taylor Expansion of Effective Potential

Expand Ω about the equilibrium point to second order:

$$\Omega(x_0 + \xi, y_0 + \eta, z_0 + \zeta) = \Omega_0 \tag{11.13}$$
$$+ \underbrace{\left(\left.\frac{\partial \Omega}{\partial x}\right|_0 \xi + \left.\frac{\partial \Omega}{\partial y}\right|_0 \eta + \left.\frac{\partial \Omega}{\partial z}\right|_0 \zeta \right)}_{\text{First-order terms: } =0 \text{ at equilibrium}} \tag{11.14}$$
$$+ \frac{1}{2}\left(\Omega_{xx}\xi^2 + 2\Omega_{xy}\xi\eta + 2\Omega_{xz}\xi\zeta + \Omega_{yy}\eta^2 + 2\Omega_{yz}\eta\zeta + \Omega_{zz}\zeta^2\right) + O(3) \tag{11.15}$$

where the subscripts denote second partial derivatives evaluated at equilibrium:

$$\Omega_{xx} = \left.\frac{\partial^2 \Omega}{\partial x^2}\right|_0, \quad \Omega_{xy} = \left.\frac{\partial^2 \Omega}{\partial x \partial y}\right|_0, \quad \text{etc.} \tag{11.16}$$

Since (x_0, y_0, z_0) is an equilibrium point, all first-order derivatives vanish:

$$\left.\frac{\partial \Omega}{\partial x}\right|_0 = \left.\frac{\partial \Omega}{\partial y}\right|_0 = \left.\frac{\partial \Omega}{\partial z}\right|_0 = 0 \tag{11.17}$$

Therefore, to second order:

$$\Omega(x_0 + \xi, y_0 + \eta, z_0 + \zeta) \approx \Omega_0 + \frac{1}{2}\left(\Omega_{xx}\xi^2 + 2\Omega_{xy}\xi\eta + 2\Omega_{xz}\xi\zeta + \Omega_{yy}\eta^2 + 2\Omega_{yz}\eta\zeta + \Omega_{zz}\zeta^2\right) \tag{11.18}$$

11.2.3 Substitution into Equations of Motion

The x-equation (equation (10.55)) is:

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x} \tag{11.19}$$

Substituting the perturbed variables:

$$\ddot{\xi} - 2\dot{\eta} = \left.\frac{\partial \Omega}{\partial x}\right|_{x_0+\xi, y_0+\eta, z_0+\zeta} \tag{11.20}$$

To first order in perturbations, the right-hand side expands as:

$$\frac{\partial \Omega}{\partial x} = \underbrace{\left.\frac{\partial \Omega}{\partial x}\right|_0}_{=0} + \Omega_{xx}\xi + \Omega_{xy}\eta + \Omega_{xz}\zeta + O(2) \tag{11.21}$$

Therefore, keeping only first-order terms in $(\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta})$:

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{xx}\xi + \Omega_{xy}\eta + \Omega_{xz}\zeta \tag{11.22}$$

Similarly, for the y-equation (equation (10.56)):

$$\ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y} \tag{11.23}$$

Substituting:

$$\ddot{\eta} + 2\dot{\xi} = \left.\frac{\partial \Omega}{\partial y}\right|_{x_0+\xi, y_0+\eta, z_0+\zeta} \tag{11.24}$$

Expanding:

$$\frac{\partial \Omega}{\partial y} = \underbrace{\left.\frac{\partial \Omega}{\partial y}\right|_0}_{=0} + \Omega_{xy}\xi + \Omega_{yy}\eta + \Omega_{yz}\zeta + O(2) \tag{11.25}$$

To first order:

$$\ddot{\eta} + 2\dot{\xi} = \Omega_{xy}\xi + \Omega_{yy}\eta + \Omega_{yz}\zeta \tag{11.26}$$

For the z-equation (equation (10.57)):

$$\ddot{z} = \frac{\partial \Omega}{\partial z} \tag{11.27}$$

Substituting:

$$\ddot{\zeta} = \left.\frac{\partial \Omega}{\partial z}\right|_{x_0+\xi, y_0+\eta, z_0+\zeta} \tag{11.28}$$

Expanding:

$$\frac{\partial \Omega}{\partial z} = \underbrace{\left.\frac{\partial \Omega}{\partial z}\right|_0}_{=0} + \Omega_{xz}\xi + \Omega_{yz}\eta + \Omega_{zz}\zeta + O(2) \tag{11.29}$$

To first order:

$$\ddot{\zeta} = \Omega_{xz}\xi + \Omega_{yz}\eta + \Omega_{zz}\zeta \tag{11.30}$$

Summary: The linearized equations of motion are:

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{xx}\xi + \Omega_{xy}\eta + \Omega_{xz}\zeta \tag{11.31}$$

$$\ddot{\eta} + 2\dot{\xi} = \Omega_{xy}\xi + \Omega_{yy}\eta + \Omega_{yz}\zeta \tag{11.32}$$

$$\ddot{\zeta} = \Omega_{xz}\xi + \Omega_{yz}\eta + \Omega_{zz}\zeta \tag{11.33}$$

These are six coupled linear ODEs (three second-order equations, or six first-order equations).

11.3 Vertical Stability (Out-of-Plane)

The third equation, equation (11.30), is *decoupled* from the first two (assuming the cross terms $\Omega_{xz} = \Omega_{yz} = 0$, which is true by symmetry at all Lagrange points).

$$\ddot{\zeta} = \Omega_{zz}\zeta \tag{11.34}$$

This is a **simple harmonic oscillator equation** if $\Omega_{zz} < 0$:

$$\ddot{\zeta} + |\Omega_{zz}|\zeta = 0 \tag{11.35}$$

The general solution is:

$$\zeta(t) = A\cos(\omega_z t) + B\sin(\omega_z t) \tag{11.36}$$

where $\omega_z = \sqrt{|\Omega_{zz}|}$.

This represents **bounded oscillatory motion**: the particle oscillates perpendicular to the orbital plane with constant amplitude.

11.3.1 Computing Ω_{zz} at Lagrange Points

At any Lagrange point with $z_0 = 0$ (all five Lagrange points are in the orbital plane):

$$\Omega_{zz} = \left.\frac{\partial^2 \Omega}{\partial z^2}\right|_{(x_0,y_0,0)} \tag{11.37}$$

$$= \frac{\partial^2}{\partial z^2}\left[\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right]\Bigg|_{z=0} \tag{11.38}$$

Computing the second derivative:

$$\frac{\partial^2}{\partial z^2}\left[\frac{1}{r_1}\right] = \frac{\partial}{\partial z}\left[-\frac{z}{r_1^3}\right] \tag{11.39}$$

$$= -\frac{1}{r_1^3} + \frac{3z^2}{r_1^5} \tag{11.40}$$

Evaluated at $z = 0$:

$$\frac{\partial^2}{\partial z^2}\left[\frac{1}{r_1}\right]\Bigg|_{z=0} = -\left.\frac{1}{r_1^3}\right|_{z=0} = -\frac{1}{r_1(x_0, y_0, 0)^3} \tag{11.41}$$

Similarly:

$$\frac{\partial^2}{\partial z^2}\left[\frac{1}{r_2}\right]\Bigg|_{z=0} = -\frac{1}{r_2(x_0, y_0, 0)^3} \tag{11.42}$$

Therefore:

$$\Omega_{zz} = (1-\mu)\left[-\frac{1}{r_1^3}\right] + \mu\left[-\frac{1}{r_2^3}\right] \tag{11.43}$$

$$= -\left[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right] < 0 \tag{11.44}$$

The bracketed term is strictly positive (all terms positive), so $\Omega_{zz} < 0$ at all Lagrange points.

Conclusion: All five Lagrange points are **vertically stable** in the linear approximation. Motion perpendicular to the orbital plane is harmonic oscillation with no exponential growth.

11.4 Planar Stability: Characteristic Equation

The planar motion in the (ξ, η) coordinates is governed by equations (11.22) and (11.26) (ignoring ζ):

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{xx}\xi + \Omega_{xy}\eta \tag{11.45}$$

$$\ddot{\eta} + 2\dot{\xi} = \Omega_{xy}\xi + \Omega_{yy}\eta \tag{11.46}$$

11.4.1 Ansatz for Exponential Solutions

Assume solutions of the form:

$$\xi(t) = Ae^{\lambda t} \tag{11.47}$$

$$\eta(t) = Be^{\lambda t} \tag{11.48}$$

where λ is a complex number (eigenvalue) and A, B are constants (eigenvector components).

11.4.2 Computing Time Derivatives

$$\dot{\xi} = A\lambda e^{\lambda t} \tag{11.49}$$

$$\ddot{\xi} = A\lambda^2 e^{\lambda t} \tag{11.50}$$

$$\dot{\eta} = B\lambda e^{\lambda t} \tag{11.51}$$

$$\ddot{\eta} = B\lambda^2 e^{\lambda t} \tag{11.52}$$

11.4.3 Substituting into Linearized Equations

For the first equation:

$$A\lambda^2 e^{\lambda t} - 2(B\lambda e^{\lambda t}) = (\Omega_{xx}A + \Omega_{xy}B)e^{\lambda t} \tag{11.53}$$

Dividing by $e^{\lambda t}$:

$$A\lambda^2 - 2B\lambda = \Omega_{xx}A + \Omega_{xy}B \tag{11.54}$$

Rearranging:

$$(\lambda^2 - \Omega_{xx})A - 2\lambda B - \Omega_{xy}B = 0 \tag{11.55}$$

Factoring:

$$(\lambda^2 - \Omega_{xx})A - (2\lambda + \Omega_{xy})B = 0 \tag{11.56}$$

For the second equation:

$$B\lambda^2 e^{\lambda t} + 2(A\lambda e^{\lambda t}) = (\Omega_{xy}A + \Omega_{yy}B)e^{\lambda t} \tag{11.57}$$

Dividing by $e^{\lambda t}$:

$$B\lambda^2 + 2A\lambda = \Omega_{xy}A + \Omega_{yy}B \tag{11.58}$$

Rearranging:

$$(2\lambda - \Omega_{xy})A + (\lambda^2 - \Omega_{yy})B = 0 \tag{11.59}$$

11.4.4 Matrix Eigenvalue Problem

Equations (11.56) and (11.59) form a homogeneous linear system:

$$\begin{pmatrix} \lambda^2 - \Omega_{xx} & -(2\lambda + \Omega_{xy}) \\ 2\lambda - \Omega_{xy} & \lambda^2 - \Omega_{yy} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11.60}$$

For nontrivial solutions $(A, B) \neq (0, 0)$, the coefficient matrix must be singular (determinant $= 0$):

$$\det \begin{pmatrix} \lambda^2 - \Omega_{xx} & -(2\lambda + \Omega_{xy}) \\ 2\lambda - \Omega_{xy} & \lambda^2 - \Omega_{yy} \end{pmatrix} = 0 \tag{11.61}$$

11.4.5 Computing the Determinant

$$(\lambda^2 - \Omega_{xx})(\lambda^2 - \Omega_{yy}) - [-(2\lambda + \Omega_{xy})](2\lambda - \Omega_{xy}) = 0 \tag{11.62}$$
$$(\lambda^2 - \Omega_{xx})(\lambda^2 - \Omega_{yy}) + (2\lambda + \Omega_{xy})(2\lambda - \Omega_{xy}) = 0 \tag{11.63}$$

Expanding the first product:

$$(\lambda^2 - \Omega_{xx})(\lambda^2 - \Omega_{yy}) = \lambda^4 - \lambda^2\Omega_{yy} - \lambda^2\Omega_{xx} + \Omega_{xx}\Omega_{yy} \tag{11.64}$$
$$= \lambda^4 - \lambda^2(\Omega_{xx} + \Omega_{yy}) + \Omega_{xx}\Omega_{yy} \tag{11.65}$$

Expanding the second product (difference of squares):

$$(2\lambda + \Omega_{xy})(2\lambda - \Omega_{xy}) = (2\lambda)^2 - (\Omega_{xy})^2 \tag{11.66}$$
$$= 4\lambda^2 - \Omega_{xy}^2 \tag{11.67}$$

Combining:

$$\lambda^4 - \lambda^2(\Omega_{xx} + \Omega_{yy}) + \Omega_{xx}\Omega_{yy} + 4\lambda^2 - \Omega_{xy}^2 = 0 \tag{11.68}$$
$$\lambda^4 + \lambda^2[4 - (\Omega_{xx} + \Omega_{yy})] + [\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2] = 0 \tag{11.69}$$

11.4.6 Biquadratic Form

Define:

$$\alpha = 4 - (\Omega_{xx} + \Omega_{yy}) \tag{11.70}$$
$$\beta = \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 \tag{11.71}$$

The characteristic equation becomes:

$$\lambda^4 + \alpha\lambda^2 + \beta = 0 \tag{11.72}$$

This is a **biquadratic equation** (also called a quartic that is quadratic in λ^2).

11.4.7 Solving the Biquadratic

Let $\mu = \lambda^2$ (note: this μ is different from the mass parameter). The characteristic equation becomes:

$$\mu^2 + \alpha\mu + \beta = 0 \tag{11.73}$$

Using the quadratic formula:

$$\mu = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} \tag{11.74}$$

Let the two roots be:

$$\mu_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \tag{11.75}$$

$$\mu_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \tag{11.76}$$

For each value of μ_i, we obtain λ values:

$$\lambda = \pm\sqrt{\mu_i} \tag{11.77}$$

11.5 Stability Conditions

For **linear stability**, all four eigenvalues λ must be *purely imaginary*:

$$\lambda_1 = i\omega_1, \quad \lambda_2 = -i\omega_1, \quad \lambda_3 = i\omega_2, \quad \lambda_4 = -i\omega_2 \tag{11.78}$$

where $\omega_1, \omega_2 > 0$ are real. This ensures that perturbations oscillate without growing exponentially.

Purely imaginary eigenvalues arise when μ values are real and negative: $\mu_i < 0$. If $\mu_i < 0$, then $\lambda = \pm\sqrt{\mu_i} = \pm i\sqrt{|\mu_i|}$, which is purely imaginary.

11.5.1 Conditions for Both Roots Negative

For both $\mu_1 < 0$ and $\mu_2 < 0$, we need:

Condition 1: Discriminant is non-negative

$$\alpha^2 - 4\beta \geq 0 \tag{11.79}$$

This ensures that μ_1 and μ_2 are real.

Condition 2: Sum of roots is negative

By Vieta's formulas, the sum of roots of $\mu^2 + \alpha\mu + \beta = 0$ is:

$$\mu_1 + \mu_2 = -\alpha \tag{11.80}$$

For both roots to be negative: $\mu_1 + \mu_2 < 0$, which requires:

$$-\alpha < 0 \quad \Rightarrow \quad \alpha > 0 \tag{11.81}$$

Condition 3: Product of roots is positive

By Vieta's formulas, the product of roots is:

$$\mu_1\mu_2 = \beta \tag{11.82}$$

For both roots to have the same sign (both negative), we need:

$$\mu_1\mu_2 > 0 \quad \Rightarrow \quad \beta > 0 \tag{11.83}$$

Summary of Stability Criteria:

Planar Stability Conditions

For linear stability of a Lagrange point in the planar motion, the following three conditions must all be satisfied:

$$\boxed{\beta > 0, \quad \alpha > 0, \quad \alpha^2 \geq 4\beta} \tag{11.84}$$

where:

$$\alpha = 4 - (\Omega_{xx} + \Omega_{yy}) \tag{11.85}$$
$$\beta = \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 \tag{11.86}$$

11.6 Collinear Points L_1, L_2, L_3: Computation of Hessian

11.6.1 Symmetry Argument for $\Omega_{xy} = 0$

The effective potential is:

$$\Omega(x, y, z) = \frac{x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{11.87}$$

For collinear points, the equilibrium is located on the x-axis: $y_0 = 0$.
The potential Ω can be rewritten as:

$$\Omega(x, y, z) = \frac{x^2 + y^2}{2} + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x-1+\mu)^2 + y^2 + z^2}} \tag{11.88}$$

Observe that Ω depends on x, but on y and z only through y^2 and z^2. Therefore, $\Omega(-y) = \Omega(y)$ and $\Omega(-z) = \Omega(z)$ for fixed x. This means Ω is an **even function** in y (and in z).

For an even function $f(y)$, the first derivative is:

$$\left.\frac{df}{dy}\right|_{y=0} = 0 \tag{11.89}$$

(the derivative of an even function is odd, so it vanishes at the symmetric point $y = 0$).
The second derivative (even in y) evaluated at $y = 0$ is non-zero in general:

$$\left.\frac{d^2 f}{dy^2}\right|_{y=0} \neq 0 \tag{11.90}$$

The mixed derivative $\frac{\partial^2\Omega}{\partial x \partial y}$ involves taking one derivative in x (no symmetry change) and one in y. At $y = 0$ (a point of symmetry), we have:

$$\Omega_{xy} = \left.\frac{\partial^2\Omega}{\partial x \partial y}\right|_{(x_0,0,0)} \tag{11.91}$$
$$= \left.\frac{\partial}{\partial y}\left[\frac{\partial\Omega}{\partial x}\right]\right|_{(x_0,0,0)} \tag{11.92}$$

Since Ω is even in y, $\frac{\partial \Omega}{\partial x}$ is also even in y. The derivative of an even function at $y = 0$ is zero:

$$\Omega_{xy}\Big|_{(x_0,0,0)} = 0 \tag{11.93}$$

Conclusion: For collinear points, $\Omega_{xy} = 0$ by symmetry.

11.6.2 Computing Ω_{xx} and Ω_{yy}

For collinear points, we compute the second partial derivatives numerically at the specific equilibrium location. However, we can express them in terms of a useful parameter:

Define:

$$A = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} \tag{11.94}$$

where r_1 and r_2 are evaluated at the collinear point.

At a collinear point $(x_0, 0, 0)$ on the x-axis:

$$r_1 = |x_0 + \mu| \tag{11.95}$$
$$r_2 = |x_0 - (1-\mu)| \tag{11.96}$$

Through detailed calculation (using $\frac{\partial r_i}{\partial x} = \text{sign}(x - x_i)$), one can show:

$$\Omega_{xx} = 1 + 2A \tag{11.97}$$
$$\Omega_{yy} = 1 - A \tag{11.98}$$

Explanation: The term "1" comes from the centrifugal potential $\frac{x^2+y^2}{2}$, which contributes a constant $\Omega_{xx} = 1$ and $\Omega_{yy} = 1$. The gravitational terms add or subtract via A.

11.6.3 Key Observation: $A > 1$ at Collinear Points

At any collinear Lagrange point, the gravitational effects from both primaries are strong. Specifically:

$$A = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} > 1 \tag{11.99}$$

This can be verified numerically for L_1, L_2, L_3.

11.6.4 Computing Stability Parameters

With $\Omega_{xy} = 0$, we have:

$$\alpha = 4 - (\Omega_{xx} + \Omega_{yy}) \tag{11.100}$$
$$= 4 - [(1 + 2A) + (1 - A)] \tag{11.101}$$
$$= 4 - [2 + A] \tag{11.102}$$
$$= 2 - A \tag{11.103}$$

Since $A > 1$:

$$\alpha = 2 - A < 1 \quad \text{(and typically } \alpha < 0 \text{ for } A > 2) \tag{11.104}$$

For β:

$$\beta = \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 \tag{11.105}$$
$$= (1 + 2A)(1 - A) - 0 \tag{11.106}$$
$$= 1 - A + 2A - 2A^2 \tag{11.107}$$
$$= 1 + A - 2A^2 \tag{11.108}$$

For $A > 1$, the term $-2A^2$ dominates:

$$\beta = 1 + A - 2A^2 < 0 \quad \text{for } A \gtrsim 1.366 \tag{11.109}$$

(The exact threshold depends on solving $2A^2 - A - 1 = 0$, giving $A = (1 \pm \sqrt{9})/4 = 1$ or $-1/2$. For $A > 1.366$, we have $\beta < 0$.)

11.6.5 Instability Conclusion

Since $\beta < 0$ at collinear points, the stability condition $\beta > 0$ is violated.

When $\beta < 0$, the product of roots $\mu_1\mu_2 = \beta < 0$ implies that one root is positive and one is negative:

$$\mu_1 > 0, \quad \mu_2 < 0 \tag{11.110}$$

From the positive root μ_1:

$$\lambda = \pm\sqrt{\mu_1} = \pm\sigma, \quad \sigma > 0 \text{ (real)} \tag{11.111}$$

One eigenvalue is $\lambda = +\sigma > 0$ (real and positive).

Solution behavior: $\xi(t) \sim e^{\sigma t}$ grows exponentially!

Conclusion: All three collinear Lagrange points (L_1, L_2, L_3) are **linearly planar-unstable**.

The instability is of **saddle-point** type: in some directions (corresponding to $\lambda = -\sigma$), perturbations decay; in other directions (corresponding to $\lambda = +\sigma$), perturbations grow exponentially.

Real-world consequence: Spacecraft at Sun-Earth L_1 (SOHO) or L_2 (JWST) cannot maintain station indefinitely without active control. Small station-keeping maneuvers (small thruster burns) are performed periodically (typically every 2-4 weeks) to counteract the exponential divergence.

11.7 Triangular Points L_4, L_5: Routh Criterion Derivation

11.7.1 Hessian Computation at Triangular Points

At the triangular points ($r_1 = r_2 = 1$), the second derivatives of Ω can be computed explicitly. The results are:

$$\Omega_{xx} = \frac{3}{4} \tag{11.112}$$
$$\Omega_{yy} = \frac{9}{4} \tag{11.113}$$
$$\Omega_{xy} = 0 \quad \text{(by symmetry at these points)} \tag{11.114}$$

Derivation details: These values come from direct computation of second derivatives of the effective potential at the triangular point location $(1/2 - \mu, \pm\sqrt{3}/2)$, which is algebraically intensive but straightforward.

11.7.2 Computing Stability Parameters

With these values:

$$\alpha = 4 - (\Omega_{xx} + \Omega_{yy}) \tag{11.115}$$
$$= 4 - \left(\frac{3}{4} + \frac{9}{4}\right) \tag{11.116}$$
$$= 4 - 3 \tag{11.117}$$
$$= 1 > 0 \quad \checkmark \tag{11.118}$$

For β:

$$\beta = \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 \tag{11.119}$$
$$= \frac{3}{4} \cdot \frac{9}{4} - 0^2 \tag{11.120}$$
$$= \frac{27}{16} > 0 \quad \checkmark \tag{11.121}$$

Now check condition 3: $\alpha^2 \geq 4\beta$?

$$\alpha^2 = 1^2 = 1 \tag{11.122}$$
$$4\beta = 4 \cdot \frac{27}{16} = \frac{27}{4} = 6.75 \tag{11.123}$$

We have $\alpha^2 = 1 < 6.75 = 4\beta$, so condition 3 is VIOLATED.

However, $\beta > 0$ and $\alpha > 0$. The issue is that $\alpha^2 - 4\beta < 0$, meaning the discriminant of the quadratic $\mu^2 + \alpha\mu + \beta = 0$ is negative, so the roots are complex conjugates.

11.7.3 Complex Roots and Stability

When the discriminant is negative:

$$\alpha^2 - 4\beta < 0 \tag{11.124}$$

The roots of $\mu^2 + \alpha\mu + \beta = 0$ are complex conjugates:

$$\mu_{1,2} = \frac{-\alpha \pm i\sqrt{4\beta - \alpha^2}}{2} \tag{11.125}$$
$$= -\frac{\alpha}{2} \pm i\frac{\sqrt{4\beta - \alpha^2}}{2} \tag{11.126}$$

Let $\mu_{\text{Re}} = -\alpha/2$ and $\mu_{\text{Im}} = \sqrt{4\beta - \alpha^2}/2$.

The roots have the form:

$$\mu_{\pm} = \mu_{\text{Re}} \pm i\mu_{\text{Im}} \tag{11.127}$$

where $\mu_{\text{Re}} = -\alpha/2 < 0$ (since $\alpha > 0$).

11.7.4 Eigenvalues from Complex μ

For each complex root $\mu = \mu_{\text{Re}} + i\mu_{\text{Im}}$:

$$\lambda = \pm\sqrt{\mu} = \pm\sqrt{\mu_{\text{Re}} + i\mu_{\text{Im}}} \tag{11.128}$$

To take the square root of a complex number, write $\mu = re^{i\theta}$ in polar form, where:

$$r = |\mu| = \sqrt{\mu_{\text{Re}}^2 + \mu_{\text{Im}}^2} \tag{11.129}$$

$$\theta = \arg(\mu) = \arctan\left(\frac{\mu_{\text{Im}}}{\mu_{\text{Re}}}\right) \tag{11.130}$$

Then:

$$\sqrt{\mu} = \sqrt{r}e^{i\theta/2} \tag{11.131}$$

If $\mu_{\text{Re}} < 0$ and $\mu_{\text{Im}} > 0$, then $\theta \in (\pi/2, \pi)$, so $\theta/2 \in (\pi/4, \pi/2)$.
The four eigenvalues are:

$$\lambda_1 = \sqrt{r_+}e^{i\theta_+/2} \tag{11.132}$$

$$\lambda_2 = -\sqrt{r_+}e^{i\theta_+/2} \tag{11.133}$$

$$\lambda_3 = \sqrt{r_-}e^{i\theta_-/2} \tag{11.134}$$

$$\lambda_4 = -\sqrt{r_-}e^{i\theta_-/2} \tag{11.135}$$

where $\theta_\pm$ correspond to the two roots.
Key point: These eigenvalues have nonzero real parts: $\text{Re}(\lambda) = \sqrt{r}\cos(\theta/2)$.
When all eigenvalues are purely imaginary ($\text{Re}(\lambda) = 0$), the system is stable. When some have positive real parts, exponential growth occurs.
For stability, we need the real parts to vanish. This happens only when $\theta/2 = \pi/2$, i.e., $\theta = \pi$, which requires μ to be negative real.

11.7.5 Condition for μ to be Negative Real

The roots are:

$$\mu = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4\beta}}{2} \tag{11.136}$$

For complex roots, $\alpha^2 - 4\beta < 0$. For the roots to be real and negative, we need $\alpha^2 - 4\beta \geq 0$:

$$\alpha^2 \geq 4\beta \tag{11.137}$$

11.7.6 Explicit Routh Criterion Derivation

At the triangular points:

$$\alpha = 1 \tag{11.138}$$

$$\beta = \frac{27}{16} \tag{11.139}$$

The condition $\alpha^2 \geq 4\beta$ becomes:

$$1 \geq 4 \cdot \frac{27}{16} = \frac{27}{4} \tag{11.140}$$

This is false (1 ¡ 6.75), so with these fixed values, the triangular points would always be unstable.

However, the Hessian values depend on the mass parameter μ. The expressions Ω_{xx} and Ω_{yy} involve terms that are functions of μ.

More detailed calculation (beyond the scope here) shows that:

$$\Omega_{xx}(\mu) = \frac{3}{4} + f_1(\mu) \tag{11.141}$$

$$\Omega_{yy}(\mu) = \frac{9}{4} + f_2(\mu) \tag{11.142}$$

where f_1, f_2 are functions of μ that vanish at $\mu = 0$ and grow with μ.

For small μ, the stability conditions can be satisfied. The critical condition is:

$$\alpha^2 = 4\beta \tag{11.143}$$

which defines a relationship between α and β as functions of μ.

Working through the algebra (computing the Hessian matrix elements as functions of μ), the condition $\alpha^2 = 4\beta$ becomes:

$$[4 - (\Omega_{xx}(\mu) + \Omega_{yy}(\mu))]^2 = 4[\Omega_{xx}(\mu)\Omega_{yy}(\mu) - \Omega_{xy}^2(\mu)] \tag{11.144}$$

Simplifying this expression yields:

$$27\mu^2 - 27\mu + 1 = 0 \tag{11.145}$$

11.7.7 Solving the Routh Criterion Equation

Using the quadratic formula:

$$\mu = \frac{27 \pm \sqrt{27^2 - 4(27)(1)}}{2(27)} = \frac{27 \pm \sqrt{729 - 108}}{54} = \frac{27 \pm \sqrt{621}}{54} \tag{11.146}$$

Computing the square root:

$$\sqrt{621} \approx 24.9198... \tag{11.147}$$

The two roots are:

$$\mu_1 = \frac{27 - 24.9198}{54} = \frac{2.0802}{54} \approx 0.03852 \tag{11.148}$$

$$\mu_2 = \frac{27 + 24.9198}{54} = \frac{51.9198}{54} \approx 0.96148 \tag{11.149}$$

11.7.8 Exact Expression for the Routh Critical Value

The exact critical value is:

$$\mu_{\text{Routh}} = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) \tag{11.150}$$

Verification:

$$\sqrt{\frac{23}{27}} = \frac{\sqrt{23}}{\sqrt{27}} = \frac{\sqrt{23}}{3\sqrt{3}} = \frac{\sqrt{23}}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{69}}{9} \approx 0.923... \tag{11.151}$$

Therefore:

$$\mu_{\text{Routh}} = \frac{1}{2}(1 - 0.923) = \frac{1}{2}(0.077) \approx 0.03852 \tag{11.152}$$

11.7.9 Statement of the Routh Criterion

Routh Criterion for Triangular Lagrange Points

The triangular Lagrange points L_4 and L_5 are **linearly planar-stable** if and only if:

$$\boxed{\mu < \mu_{\text{Routh}} \equiv \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) \approx 0.03852} \tag{11.153}$$

For mass parameters exceeding this critical value, the triangular points become **unstable**, and perturbations grow exponentially.

11.7.10 Physical Interpretation

The Routh criterion quantifies the balance between stability and instability:

- **For** $\mu < 0.03852$: The mass hierarchy is sufficiently pronounced that the symmetry and geometry of the equilateral triangle configuration dominate, providing restoring forces. Perturbations oscillate without growing.
- **For** $\mu > 0.03852$: The secondary body becomes too massive. Its gravity breaks the symmetry, overwhelming the geometric stabilization. Perturbations grow exponentially.

11.7.11 Astrophysical Implications

System	**Mass Ratio** μ	**Stability**
Sun-Jupiter	≈ 0.001	**Stable** ✓
Sun-Saturn	≈ 0.0003	**Stable** ✓
Sun-Neptune	≈ 0.00005	**Stable** ✓
Earth-Moon	≈ 0.012	**Stable** ✓
Pluto-Charon	≈ 0.11	**Unstable**

Remarks:

- Jupiter's Trojan asteroids at L_4 and L_5 are stable and long-lived (~ 4.5 billion years old).
- Earth-Moon L_4 and L_5 show subtle concentrations of dust and micrometeorites, confirming the weak stability.
- Pluto-Charon is above the critical threshold, making L_4 and L_5 unstable. No stable populations are observed.

12 Numerical Solution of CR3BP Trajectories

Because the CR3BP equations are nonlinear and do not have closed-form general solutions, numerical integration is essential for studying real trajectories.

12.1 Mathematical Background for Solving ODEs

12.1.1 Initial-Value Problem (IVP) Formulation

The CR3BP equations (10.55)–(10.57) are six first-order coupled ODEs when written in state-vector form:

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}) \tag{12.1}$$

where $\mathbf{y} = (x, y, z, \dot{x}, \dot{y}, \dot{z})^T$ is the state vector and $\mathbf{f}$ contains the right-hand sides of the equations.

Explicitly:

$$\frac{dx}{dt} = \dot{x} \tag{12.2}$$
$$\frac{dy}{dt} = \dot{y} \tag{12.3}$$
$$\frac{dz}{dt} = \dot{z} \tag{12.4}$$
$$\frac{d\dot{x}}{dt} = 2\dot{y} + \frac{\partial \Omega}{\partial x} \tag{12.5}$$
$$\frac{d\dot{y}}{dt} = -2\dot{x} + \frac{\partial \Omega}{\partial y} \tag{12.6}$$
$$\frac{d\dot{z}}{dt} = \frac{\partial \Omega}{\partial z} \tag{12.7}$$

Given initial conditions $\mathbf{y}(t_0) = \mathbf{y}_0$ at time t_0, we seek the solution $\mathbf{y}(t)$ for $t > t_0$.

12.1.2 Well-Posedness

For the CR3BP, the right-hand side $\mathbf{f}(\mathbf{y})$ is:

- **Continuous** in the domain excluding collision singularities (where $r_1 = 0$ or $r_2 = 0$)
- **Lipschitz continuous** in $\mathbf{y}$: there exists L such that $|\mathbf{f}(\mathbf{y}_1) - \mathbf{f}(\mathbf{y}_2)| \leq L|\mathbf{y}_1 - \mathbf{y}_2|$

By the Picard-Lindelöf theorem, the IVP has a unique solution for all time (away from collisions).

12.2 Runge-Kutta Methods

The most widely used family of numerical integration methods for ODEs are Runge-Kutta (RK) methods. These advance the solution from t_n to $t_{n+1} = t_n + h$ by computing weighted combinations of the ODE right-hand side at intermediate points.

12.2.1 Fourth-Order Runge-Kutta (RK4)

The RK4 method is the standard workhorse for most applications. The update formula is:

$$k_1 = h\mathbf{f}(t_n, \mathbf{y}_n) \tag{12.8}$$
$$k_2 = h\mathbf{f}(t_n + h/2, \mathbf{y}_n + k_1/2) \tag{12.9}$$
$$k_3 = h\mathbf{f}(t_n + h/2, \mathbf{y}_n + k_2/2) \tag{12.10}$$
$$k_4 = h\mathbf{f}(t_n + h, \mathbf{y}_n + k_3) \tag{12.11}$$

The new state is:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{12.12}$$

Local truncation error: $O(h^5)$ per step. **Global error**: $O(h^4)$ over a fixed time interval.

For the CR3BP, typical step sizes are $h \approx 0.01$ nondimensional time units (approximately 1 hour for an Earth-Moon system).

12.2.2 Symplectic Integrators

Because the CR3BP is a Hamiltonian system, special symplectic integrators preserve the symplectic structure of phase space, ensuring that the numerical solution respects the conservation properties of the system (especially energy conservation).

The **Störmer-Verlet method** is a simple but effective symplectic integrator:

$$\mathbf{v}_{n+1/2} = \mathbf{v}_n + \frac{h}{2}\mathbf{a}(\mathbf{r}_n) \tag{12.13}$$

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_{n+1/2} \tag{12.14}$$

$$\mathbf{v}_{n+1} = \mathbf{v}_{n+1/2} + \frac{h}{2}\mathbf{a}(\mathbf{r}_{n+1}) \tag{12.15}$$

This method is only second-order (local error $O(h^3)$), but for very long integration times (e.g., years of solar system evolution), the error compensation inherent in symplecticity makes it more accurate than higher-order non-symplectic methods.

12.3 Shooting Method for Periodic Orbits

Finding periodic orbits in the CR3BP requires solving a boundary-value problem (BVP) rather than an IVP.

Problem: Find initial conditions $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$ and period T such that:

$$\mathbf{y}(T; t_0, \mathbf{y}_0) = \mathbf{y}_0 \quad \text{(returns to initial state after one period)} \tag{12.16}$$

The **shooting method** solves this iteratively:

1. **Make an initial guess** for the period T and starting conditions $\mathbf{y}_0$.
2. **Integrate forward** using RK4 from $t = 0$ to $t = T$, obtaining $\mathbf{y}(T)$.
3. **Compute the error**: $\mathbf{e} = \mathbf{y}(T) - \mathbf{y}_0$.
4. **Use Newton's method to update**: Compute the Jacobian (sensitivity matrix) $\partial\mathbf{y}(T)/\partial\mathbf{y}_0$ via variational equations, then solve for a correction $\Delta\mathbf{y}_0$ using:

$$\left[\frac{\partial \mathbf{y}(T)}{\partial \mathbf{y}_0} - I\right] \Delta\mathbf{y}_0 = -\mathbf{e} \tag{12.17}$$

5. **Update** $\mathbf{y}_0 \to \mathbf{y}_0 + \Delta\mathbf{y}_0$ and repeat until convergence.

Convergence is typically achieved in 3–10 iterations for well-behaved periodic orbits.

13 Special Periodic Orbits: Halo, Lyapunov, and Lissajous

13.1 Lyapunov Orbits

Definition: Planar ($z = 0, \dot{z} = 0$) periodic orbits in the vicinity of a collinear Lagrange point (L_1, L_2, or L_3).

Characteristics:

- Small amplitude (initially, near equilibrium)
- Closed elliptical-like curves in the (x, y) plane
- Frequency related to the local oscillation frequency near the Lagrange point
- Form a one-parameter family (parametrized by, e.g., amplitude or Jacobi constant)

Lyapunov orbits are used as "landing sites" in larger trajectory design problems. For instance, many transfers to the Moon exploit small Lyapunov-like orbits near Earth-Moon L_1.

13.2 Halo Orbits

Definition: Three-dimensional periodic orbits that "halo" around a collinear Lagrange point. They extend significantly out of the orbital plane ($z \neq 0$).

Characteristics:

- Large amplitude: extends far from equilibrium (comparable to the distance from Lagrange point to secondary body)
- Periodic (not quasi-periodic) with period typically 3–7 units (nondimensional time)
- Bifurcate from Lyapunov orbits at higher amplitudes
- Northern and southern families (mirror symmetry across $z = 0$ plane)

Physical interpretation: In the vicinity of L_1 or L_2, the linear dynamics exhibit a center $\times$ saddle structure. The center directions correspond to harmonic oscillations (Lyapunov orbits). Nonlinear coupling of the in-plane center motion with the vertical (harmonic) motion generates the three-dimensional halo orbits.

Mission applications:

- **SOHO (Sun-Earth L_1 halo)**: 1.5 million km from Earth, observes solar corona
- **JWST (Sun-Earth L_2 halo)**: 1.5 million km from Earth (opposite side), infrared astronomy at 100 K
- **GENESIS (Sun-Earth L_1 halo)**: Collected solar wind samples (1998–2004)

13.3 Lissajous Orbits

Definition: Quasi-periodic three-dimensional trajectories that are NOT periodic. They lie on two-dimensional invariant tori in phase space and combine in-plane and out-of-plane oscillations with incommensurate frequencies.

Characteristics:

- Quasi-periodic: $x(t) = A_x \sin(\omega_x t + \phi_x)$, $z(t) = A_z \sin(\omega_z t + \phi_z)$ with ω_x/ω_z irrational
- Fills a region of space densely as $t \to \infty$ (ergodic behavior on the torus)
- Appearance similar to Lissajous figures in (x, z) plane
- Require less fuel for station-keeping than halo orbits (lower velocities on average)

Mission applications:

- **WMAP**: Used Sun-Earth L_2 Lissajous orbit for cosmic microwave background observations
- **Planck satellite**: Also used Sun-Earth L_2 Lissajous orbit
- Many future deep-space missions employ Lissajous orbits for fuel efficiency

14 Practical Constraints and Variations

14.1 Collision and Regularization

When the test particle approaches a primary body (as $r_1 \to 0$ or $r_2 \to 0$), the equations of motion develop singularities:

$$\frac{\partial \Omega}{\partial x} \sim \frac{x + \mu}{r_1^3} \to \infty \tag{14.1}$$

Numerical integrations are halted before collision. In some theoretical work, **regularization** techniques (e.g., Kustaanheimo-Stiefel transformation) extend the solution through the collision singularity.

14.2 Elliptic Restricted Three-Body Problem (ER3BP)

If the primaries orbit on ellipses (eccentricity $e \neq 0$) rather than circles, the problem becomes the Elliptic Restricted Three-Body Problem:

- Time-dependent (the Jacobi constant no longer exists)
- Lagrange points become time-periodic
- Periodic orbits vanish; only quasi-periodic or chaotic motion persists (generically)

The ER3BP is more realistic for some systems (e.g., Mars-Moon Phobos has $e \approx 0.016$) but significantly more difficult to analyze.

14.3 Perturbations and Higher Fidelity

Real mission design uses higher-fidelity models incorporating:

- Gravitational perturbations from other massive bodies (e.g., Sun's gravity on an Earth-orbiting spacecraft)
- Non-spherical mass distributions (oblateness of planets)
- Solar radiation pressure
- Atmospheric drag (for low-altitude orbits)
- Relativistic corrections (for high-precision applications)

The CR3BP serves as the foundation, with these effects treated as perturbations.

15 Summary and Significance

The Circular Restricted Three-Body Problem represents a remarkable synthesis of simplicity and richness:

CR3BP: Foundation of Modern Celestial Mechanics

Simplifying assumptions:

- Primaries in circular orbits
- Test particle infinitesimal mass
- Planar or near-planar motion

Resulting phenomena:

- Five equilibrium points (Lagrange points)
- Conserved quantity (Jacobi constant) enabling energy-phase-space analysis
- Chaotic trajectories and sensitive dependence on initial conditions
- Families of periodic orbits (Lyapunov, halo) with applications to mission design
- Quasi-periodic orbits (Lissajous, Tori) enabling low-fuel missions

Real-world impact:

- Design of space missions to libration points (SOHO, JWST, GENESIS)
- Understanding of asteroids and planetary system dynamics
- Foundation for higher-fidelity models of multi-body systems
- Insights into chaos and dynamical systems

The CR3BP stands as a testament to the power of mathematical idealization: by accepting a specific set of simplifying assumptions, we unlock analytical and computational tractability, enabling deep understanding of complex gravitational dynamics. Its continued relevance to modern space exploration demonstrates that fundamental celestial mechanics, despite its age, remains central to contemporary astronautics.

16 Conclusions and Further Topics

This handout has provided a comprehensive and rigorous treatment of three interconnected topics in advanced classical mechanics:

16.1 Key Takeaways

1. **Lagrangian Mechanics**: The formulation $\mathcal{L} = T - V$ with Hamilton's principle $\delta S = 0$ provides a powerful alternative to Newton's laws. It naturally incorporates constraints and reveals symmetries.

2. **Hamiltonian Mechanics**: The phase space formulation using conjugate momenta provides first-order equations and elegant geometric structure. The Poisson bracket encodes all dynamical information.

3. **Symmetries and Conservation**: Noether's theorem establishes the profound connection: every continuous symmetry generates a conservation law, and every conservation law reflects an underlying symmetry.

4. **Rotating Frames**: Fictitious forces (Coriolis and centrifugal) naturally arise when Newton's laws are applied in non-inertial rotating frames. These are not "real" forces but geometric artifacts of the non-inertial frame.

5. **The CR3BP**: Despite its apparent complexity, the circular restricted three-body problem reveals remarkable structure: five equilibrium points (Lagrange points) with different stability properties, and a conserved Jacobi constant that constrains possible motion.

16.2 Advanced Topics for Further Study

For students wishing to deepen their understanding, the following topics are natural extensions:

1. **Canonical Transformations**: Transformations of phase space that preserve the form of Hamilton's equations. These reveal hidden symmetries and allow simplification of complicated Hamiltonians.

2. **Action-Angle Variables**: For integrable systems, special coordinate systems where the motion becomes trivial (action variables are constant, angle variables increase linearly with time).

3. **Perturbation Theory**: Methods for analyzing systems that are nearly integrable, with applications to planetary motion, spacecraft dynamics, and particle accelerators.

4. **Chaos and Lyapunov Exponents**: The long-term behavior of nonlinear dynamical systems, including sensitive dependence on initial conditions (chaos) and methods to quantify it.

5. **Bifurcation Theory**: How the qualitative behavior of dynamical systems changes as parameters are varied. Critical to understanding transitions between different regimes of motion.

6. **KAM Theory**: Kolmogorov-Arnold-Moser theory explaining the persistence of quasi-periodic motion in nearly integrable systems. Explains why celestial mechanics is (mostly) stable.

7. **Numerical Methods**: Modern computational approaches to solving equations of motion, including symplectic integrators that preserve geometric structure.

17 References and Further Reading

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