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log functions $(0, \infty)$
exponential $\mathbb{R}$
linear $\mathbb{R}$

Chapter 2

Ex 2.3 :- Notes!

The linear differential equation is given generally as ;

imp condition!

$$(1)\ \frac{dy}{dx} + P(x)y = f(x) \quad — ①$$

The integrating factor is given /solved from the formula.

$$u = e^{\int P(x)\,dx} \quad — ②$$

=> Further steps to solve non-separable linear equations (differential).

① Make form of ①
② Find the integration factor by ②
③ Close/undo product Rule.
④ Take integral on both side.

=> Interval of definition; the interval at which the solution is defined (I)

=> Transcient term; terms that approach zero as $x \to \infty$

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$y = \sqrt{T} + c(\ldots)$

EXERCISE 2.4

A EXACT D.E (2.4)

A differential equ. of the form.

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{—— ①}$$

is called as a exact DE. if.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{—— ②}$$

If ② holds then there exisits a solution $f(x,y) = 0$ of ① such that

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$$\frac{\partial f}{\partial x} = M(x,y) \quad \& \quad \frac{\partial f}{\partial y} = N(x,y)$$

$$2xy\,dx + (x^2-1)\,dy = 0 \quad \text{--- (1)}$$

$$2xy\,dx = -(x^2-1)\,dy \quad \text{--- (ii)}$$

$$\frac{dx}{dy} = -\frac{(x^2-1)}{2xy}$$

$$\frac{dy}{dx} = -\frac{2xy}{(x^2-1)}$$

However we solve by Exact DE method.

let from (1).

$$M(x,y) = 2xy$$
$$N(x,y) = x^2-1$$

$$\frac{\partial M}{\partial y} = 2x \quad ; \quad \frac{\partial N}{\partial x} = 2x$$

Now it obeys condition:-

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

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therefore we can use the exact D.E method.

Now next step finding (f)
we know that;

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$\int \cancel{\partial} f = \int M(x,y)\,\partial x$$

$$f = \int 2xy\,\partial x$$

$$f = 2y\int x\,\partial x$$

$$f = 2y\frac{x^2}{2} + g(y) \quad \text{--- (1)}$$

$\to$ const of integration and unknown.

Using (1)

$$\frac{\partial f}{\partial y} = N(x,y)$$

$$\cancel{x^2} + g'(y) = \cancel{x^2} - 1$$

$$g'(y) = -1$$

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$2xy$

$$\int g'(y)\,dy = \int -1\,dy$$

$$g(y) = -y + C$$

$$\therefore f = x^2y - y + C \text{ Ans!}$$

$$e^{2y} - y\cos xy\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$$

let

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy \quad \text{—(i)}$$

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$\partial f = (2xe^{2y} - x\cos xy + 2y)\,\partial y$$

$$\int \partial f = \int (2xe^{2y} - x\cos xy + 2y)\,dy$$

$$f(x,y) = xe^{2y} - \sin xy + h(x) \quad \text{—(a)}$$

Using (a) in (i).

but $\frac{f(x,y)}{dx}$.

$$\cancel{e^{2y}} - \cancel{y\cos xy} + h'(x) = \cancel{e^{2y}} - \cancel{y\cos xy}$$

$$h'(x) = 0$$

↳ the first derivative of const funct

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Remember

$$M_y = \frac{\partial M}{\partial y} \quad \& \quad N_x = \frac{\partial N}{\partial x}$$

$$h(x) = c.$$

$$\therefore f(x,y) = xe^{2y} - \sin xy + y^2 + c \text{ Ans!}$$

ANOTHER CASE:-

If differential equation is not exact then it can be converted into an exact equation (differential) by multiplying with an integration factor.

let for a equation.

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

If;

$$\frac{M_y - N_x}{N} \Rightarrow \text{funct of } x \text{ only.}$$

then integration factor is $(\mu_x) = e^{\int \frac{M_y - N_x}{N} dx}$

If;

$$\frac{N_x - M_y}{M} = \text{funct of } y \text{ only.}$$

then integration factor is $(\mu_y) = e^{\int \frac{N_x - M_y}{N} dy}$

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$xy\,dx + 2x^2 + 3y^2 + (-20)dy = 0.$

The equation has the form:

$M(x,y) + N(x,y) = 0.$

$M(x,y) = xy$. & $N(x,y) = 2x^2 + 3y^2 - 20$

$\frac{\partial M}{\partial y} / M_y = x$ $\frac{\partial N}{\partial x} / N_x = 4x$

Now $M_y \neq N_x$ therefore the equation is not exact.

We can convert the equation into an exact equation by multiplying with an integration factor.

first check

$\frac{M_y - N_x}{N(x,y)} \Rightarrow \frac{x - 4x}{2x^2+3y^2-20}$

$\frac{x(\cancel{x}-4)}{2x^2+3y^2-20}$

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It is a function of (x) & (y)

2nd check.

$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} \Rightarrow \frac{3x}{xy} \Rightarrow \frac{3}{y}.$

hence it satisfies by being a funct of y.
Now using this to find the integration factor.

$u = e^{\int 3/y \, dy}.$

$u = e^{3\int \frac{1}{y} dy}$

$u = \cancel{e}^{3\ln(y) + c}. \Rightarrow y^3.$

Multiplying integration factor on b.s of the equation

$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$

The above equation is now an exact equ.

$\frac{\partial f}{\partial x} = M(x,y)$ & $\frac{\partial f}{\partial y} = N(x,y).$

$\partial f = xy^4\,\partial x \Rightarrow f \Rightarrow \frac{y^4 x^2}{2} + g(y)$

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$\frac{df}{dy} = N(x,y)$

$df = 2x^2y^3 + 3y^5 - 20y^3\, dy$

$\frac{1}{2}(y^4x^2)' + g'(y) = 2x^2y^3 + 3y^5 - 20y^3$

$\frac{1}{2}(\cancel{4y^3x^2}) + g'(y) = \cancel{2x^2y^3} + 3y^5 - 20y^3$

$\int g'(y) = \int 3y^5 - \int 20y^3$

$g(y) = \frac{\cancel{3}^{1}y^6}{\cancel{6}_2} - \frac{\cancel{20}^{5}y^4}{\cancel{4}_1}$

$g(y) = y^6/2 - 5y^4 + C$

using for $f(x,y)$:

$\frac{x^2y^4}{2} + y^6/2 - 5y^4 = -C$ Ans!

We can find value of C by $y(1) = 1$

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(EXERCISE 2.5) Solution by substitution!

Homogeneous equations:-

If a function 'f' possesses the property.

$$f(tx, ty) = t^{\alpha} f(x,y) \qquad \alpha \in \mathbb{R}$$

then the function is known as a homogeneous function of degree α.

Example:-

$f(x,y) = x^3 + y^3$.

$f(t^c x, ty) = t^3x^3 + t^3y^3$

$= t^3 f(x,y)$

① $M(x,y)\,dx + N(x,y)\,dy = 0$. let this be a first order linear differential equation then it is said to be homogeneous if M and N are both homogeneous of the same degree.

$M(x,y) \Rightarrow M(tx, ty) = t^{\alpha} M(x,y)$.

$N(x,y) \Rightarrow N(tx, ty) = t^{\alpha} N(x,y)$.

★ appropriate substitutions for x & y will reduce a homogeneous equation to a first order separable differential equation i.e. $x = vy$ or $y = ux$.

Question.

$$(x^2+y^2)dx + (x^2-xy)dy = 0.$$

$M(x,y) = x^2+y^2$.
$N(x,y) = x^2-xy$.

not exact
not linear
not separable.

$M(tx,ty) = t^2(x^2+y^2)$
$N(t, ty) = t^2(x^2-xy)$.

M & N are homogeneous of degree 2!

let $y = ux$. — ①

$\frac{dy}{dx} = u(1) + x\frac{du}{dx}$.

$dy = x\,du + u\,dx$. — ⑪

Substituting ① & ⑪ in the original equ.

$(x^2+u^2x^2)dx + (x^2 - x(ux))(x\,du + u\,dx)$

$(x^2+u^2x^2)dx + (x^2-x^2u)(x\,du+u\,dx)$

$dx(x^2) + dx(u^2x^2) + x^3du - x^3u\,du$
$+ x^2u\,dx - u^2x^2dx$.

$x^2dx + x^3du - x^3u\,du + x^2u\,dx = 0$

$x^2dx + x^2u\,dx + x^3du - x^3u\,du. = 0$

$dx(x^2+x^2u) + du(x^3 - x^3) = 0$

$x^2(1+u)dx + x^3(1-u)du. = 0$

$\frac{x^2(1+u)}{(1-u)}dx = -x^3du$

$\frac{1+u}{1-u} = -x\frac{du}{dx}$

$dx\left(\frac{-1}{x}\right) = \left(\frac{1-u}{1+u}\right)du$.

Converting into proper fraction.

$dx\left(\frac{-1}{x}\right) = \left[-1 \quad \frac{2}{1+u}\right]du$

∫ on b.s.

$-\ln|x| + C = -u + 2\ln|1+u|$

$\therefore$

$-u + 2\ln|1+u| = -\ln|x| + C.$

$-u + 2\ln|1+u| + \ln|x| = +C$

$-u + 2\ln|(1+u)(x)| = C$

now

as $u = y/x$.

$-y/x + 2\ln|(1+y/x)(x)| = C.$

$y = -[x\{[-2\ln|(x+y)|] + C\}]$

$y = -(-\ln|x+y|^2 + (x))$ Ans)

Bernoulli's Equation!

The DE;

$\frac{dy}{dx} + P(x)y = f(x)y^n$ is known as the Bernoulli's equation!

For bernoulli's equation the particular substitution i.e $u = y^{1-n}$ is made to reduce it to a first order linear equation.

Example:-

$x\frac{dy}{dx} + y = x^2y^2$

$\frac{dy}{dx} + \frac{y}{x} = \frac{x^2y^2}{x}$

$\frac{dy}{dx} + \frac{y}{x} = xy^2$

here $p(x) = y/x$ $n = 2$.

$f(x) = xy^2$.

let $u = y^{1-2}$

$u = y^{-1}$ $\Rightarrow u = 1/y$.

$\frac{du}{dy} = \frac{1}{y}$ $y = \frac{1}{u}$

$\frac{dy}{dx} \Rightarrow ?$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

putting into the original equation!

$\frac{dy}{dx} \Rightarrow \frac{-(u^{-2})}{1} \times \frac{du}{dx}$ $\Rightarrow (-u^{-2}) \frac{du}{dx} + \frac{1}{ux} = \frac{x}{u^2}$

Now solving generally. $\frac{du}{dx} - \frac{u}{x} = -x$

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$= e^{+\int \frac{-y}{x} dx}$

$= e^{-\ln x}$

$= x^{-1}$

Multiplying the integration factor.

$x^{-1}\left(\frac{du}{dx}\right) - \frac{u}{x}(x^{-1}) = (-x^{+1})(x^{-1})$

$\frac{1}{x}\frac{du}{dx} - \frac{u}{x^2} = -1$

$\frac{1}{x}\left(\frac{du}{dx} - \frac{u}{x^2}\right) = -1$

Closing product Rule.

$\frac{d}{dx}\left(\frac{1}{x}u\right) = -1$

$\int \cancel{\frac{d}{dx}}\left(\frac{u}{x}\right) = \int -1 \frac{d}{dx}$

$\therefore \; u/x \Rightarrow -x + c$

since $u = y^{-1}$

$\frac{1}{} = -x + c \Rightarrow y = \frac{1}{-x^2 + c}$

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Notes!

DE of the form;

$\frac{dy}{dx} = f(Ax + By + C)$. can always be reduced to an equ (in) separtible form by substitution.

$u = Ax + Bx + C \; ; \; B \neq 0$

Example:

$\frac{dy}{dx} = (-2x + y)^2 - 7 \; ; \; y(0) = 0$

let $u = (-2x + y) \rightarrow$ put in original equ.

$\frac{du}{dx} = -2 + \frac{dy}{dx}$.

$\frac{du}{dx} + 2 = \frac{dy}{dx} \rightarrow$ putting value in original equ.

$\frac{du}{dx} + 2 = u^2 - 7$

$\frac{du}{dx} = u^2 - 9 \Rightarrow \frac{du}{u^2 - 9} = dx$.

$\frac{du}{(u-3)(u+3)} = dx$.

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$$\left[\frac{1}{(u-3)(u+3)}\right] du = dx.$$

Using partial fractions.

$$\int \frac{1}{6}\left(\frac{1}{u-3} - \frac{1}{u+3}\right) du = \int dx.$$

$$\frac{1}{6} \ln|u-3| - \ln|u+3| = x + C_1$$

$$e^{\ln\left|\frac{u-3}{u+3}\right|} = e^{(6x + 6C_1)}$$

$$\cancel{e}^{\cancel{\ln}} \left|\frac{u-3}{u+3}\right| = e^{6x} \cdot e^{6C_1}.$$

$$\frac{u-3}{u+3} = e^{6x} \cdot e^{6C_1}.$$

$$\frac{u-3}{u+3} = Ce^{6x}$$

$$\frac{-2x+y-3}{-2x+y+3} = Ce^{6x} \text{ Ans!}$$

we canuse $y(0) = 0$ for value of c.

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