4.2 :- **Reduction of Order.**

Differential equation (find the 2nd sol; given that 1st sol is known)

↙ ↘

Homogeneous (use of formula) — Non homogeneous (use of method).

HOMOGENEOUS

15. $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$ — ①.

$y_1 = x+1$

Using formula.

$$y_2(x) = y_1 \int \frac{e^{-\int p(x)}}{[y_1]^2} dx$$

*Converting into standard form.

$$y'' + \underbrace{\frac{2(1+x)}{(1-2x-x^2)}}_{P(x)} y' - \underbrace{\frac{2}{(1-2x-x^2)}}_{Q(x)} y = 0$$

$\Rightarrow -\int \frac{2(1+x)(-1)}{1-2x-x^2}$ => let $u = 1-2x-x^2$

$\frac{du}{dx} = -2-2x$

$\Rightarrow -\int \frac{1}{u} du$

$\frac{du}{dx} = 2(1+x)(-1)$

$-\ln|u| + c$

$\Rightarrow y_1 \int \frac{e^{\ln|1-2x-x^2|}}{(1+2x+x^2)}$ $\Rightarrow y_1 \int \frac{1-2x-x^2+2-2}{x^2+2x+1}$

$$\int -\frac{(x^2+2x+1)}{(x+1)^2} + \frac{2}{(x+1)^2}$$

$$\int -1\,dx + \int \frac{2}{(x+1)^2}\,dx.$$

$$\left[-x + \left(\frac{-2}{x+1}\right) + c\right]$$

Now finally by formula.

$$(x+1)\left(-x - \frac{2}{(x+1)} + c\right) \Rightarrow \text{Ans!}$$

let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x}\,dx.$$

$$\therefore \int \frac{du}{\cos^2(u)}.$$

$$\int \sec^2(u)\,du$$

14. $x^2y'' - 3xy' + 5y = 0.$

$y_1 = x^2\cos(\ln x)$

$$x^2\cos\ln x\left[\tan u + C\right]$$

$$x^2\cancel{\cos\ln x} \times \frac{\sin\ln x}{\cancel{\cos\ln x}} + C$$

$$\boxed{x^2\sin\ln x + c}$$

Ans!

std form;

$$y'' - \frac{3}{x}y' + \frac{5}{x^2}y = 0.$$

$P(x) = -3/x.$

$$e^{-\int -\frac{3}{x}} \Rightarrow e^{3\int \frac{1}{x}} \Rightarrow e^{\ln x^3}$$
$$\Rightarrow x^3.$$

$$y_2(x) = \int \frac{x^3}{(x^2\cos\ln x)^2} \Rightarrow \int \frac{x^3}{x^4\cos^2\ln x}\,dx$$

$$\Rightarrow \frac{1}{x}\frac{1}{\cos^2\ln x}\,dx.$$

Q: $y'' - 3y' + 2y = 5e^{3x}$; $y_1 = e^x$.

Reduction of order

let

$y(x) = y_1 v \Rightarrow e^x(v)$

$y'(x) = y_1'v + v'y_1$

$y'(x) = e^x v + v'e^x$

$y''(x) = e^x v' + e^x v + v''e^x + v'e^x$

$y''(x) = 2v'e^x + v''e^x + e^x v.$

$y''(x) = v''e^x + 2v'e^x + e^x v.$

$(v''e^x + 2v'e^x + e^x v) - 3(e^x v + v'e^x) + 2(e^x v) = 5e^{3x}.$

$v''e^x + \underline{2v'e^x} + \cancel{e^x}v - 3\cancel{e^x}v + \underline{3v'e^x} + 2\cancel{e^x}v = 5e^{3x}$

$v''e^x + 2v'e^x + 3v'e^x = 5e^{3x}.$

$e^x(v'' + v') = 5e^{3x}.$

$e^x(w' - w) = 5e^{3x}.$

$\frac{dw}{dx} - w = 5e^{2x}.$

Now using integration factor

$e^{\int -1\,dx} \Rightarrow e^{-x}$

Multiplying integration facter on b.s

$e^{-x} \cdot w = \int 5e^x$

$w = \frac{5e^x + c}{e^{-x}}$

$\int u' = \int 5e^{2x} + ce^x$

$v \Rightarrow \frac{5}{2}e^{2x} + c_1e^x + c_2$

$v = ''$

Now $y(x) = y_1(v)$ Ans!

linear independance.

let $y_1, y_2, y_3 \ldots$ be the of a N^{th} order homogeneous linear equ. then linear independance is given by

<u>Wronskian determinant</u>

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \quad \text{or} \quad \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$$

4.3:- HOMOGENEOUS CONST COEEFF D.ES.

$$ay'' + by' + cy = 0$$

let $y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}.$

$$a(m^2 e^{mx}) + b(me^{mx}) + c(e^{mx})$$

$$e^{mx}(am^2 + bm + c) = 0$$

$am^2 + bm + C \Rightarrow$ will give us some roots

case 1: Real & distinct $\rightarrow$ then $y_1 = c_1 e^{m_1 x}$, $y_2 = c_2 e^{m_2 x}.$

case 2: Real & repeating. $\rightarrow$ $y_1 = c_1 e^{mx}$, $y_2 = xc_2 e^{mx}$

case 3: imaginary

$2y'' - 3y' + 4y = 0.$

$y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

$2m^2 - 3m + 4 = 0.$

by quadratic formula.

$m_{1,2} = \dfrac{3 \pm \sqrt{23}i}{4}.$

Now imaginary roots.

$\therefore \alpha = 3/4 \qquad \beta = \pm \dfrac{\sqrt{23}i}{4}$

$y(x) = e^{\alpha x}\left[c_1(\cos\beta + i\sin\beta) + c_2(\cos\beta - i\sin\beta)\right]$

$y(x) = e^{3/4 x}\left[c_1\left(\cos\dfrac{\sqrt{23}}{4} + i\sin\dfrac{\sqrt{23}}{4}\right) + c_2\left(\cos\dfrac{\sqrt{23}}{4} - i\sin\dfrac{\sqrt{23}}{4}\right)\right]$

Ans!

28. $2x^5 - 7x^4 + 12x^3 + 8x^2 = 0.$

$2m^5 - 7m^4 + 12m^3 + 8m^2 = 0$

$m^2(2m^3 - 7m^2 + 12m + 8) = 0$

Now by considering

$2m^3 - 7m^2 + 12m + 8. \quad —(a)$

already $m^2 = 0$

$m_{1,2} = 0.$

→ let (a)

Coefficients of last term $\Rightarrow 8$

coefficient of 1st term $\Rightarrow 2.$

factors of 8 divided by factors of 2.

$\pm 1, \pm 2, \pm 4 \pm 8 \pm \frac{1}{2}.$

Now by hit & trial.

$m_3 = -\frac{1}{2}$

$$\begin{array}{c|cccc} & 2 & -7 & +12 & 8 \\ -\frac{1}{2} & \downarrow & -1 & 4 & \\ \hline & 2 & -8 & 16 & \boxed{0} \end{array}$$

$(m + \frac{1}{2})(\underline{2m^2 - 8m + 16})$

Now by quadratic formula.

$m_{4,5} = 2 \pm 2i$

$y(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-1/2 x} + c_4 e^{2+2ix} + c_5 e^{2-2ix}$

$e^{2x}(\cos 2x - i\sin 2x) \quad c_4 e^{2x}(\cos 2x + i\sin 2x)$

$y(0) = 0$
$y(\pi) = 0.$

$y'' + 4y = 0$

$m^2 + 4 = 0$

$m_{1,2} = \pm 2i$

$y = c_1 e^0 (\cos 2x - i\sin 2x) + c_2 e^0 (\cos 2x + i\sin 2x).$

More compact form

$y = k_1 \cos 2x + k_2 \sin 2x.$

$y(0) = k_1 \cos 2(0) + k_2 \sin(2 \times 0)$

$k_1 = 0$

$y(\pi) = k_1 \cos 2(\pi) + k_2 \sin(2\pi)$

$k_2 = 0!$

$y(x) = k_2 \sin 2x.$ Ans!

4.4 NON HOMOGENEOUS CONST COEFF (OF VAR) D.E.

=> UNDETERMINED COEFF METHOD.

$\underline{\text{H D.E}} = g(x)$

↓

this can be a const, polynomial, $\sin \beta x$, $\cos \beta x$, e^{ax}, finite sum or product of these.

$y = y_c + y_p$.

For the non-homogeneous part we convert $g(x)$ into y_p.

like.

1	A
$5x+1$	$Ax+B$.
$3x^2-2$	Ax^2+Bx+C.
x^3-x+1	Ax^3+Bx^2+Cx+C.
$\sin 4x$	$A\sin 4x + B\cos 4x$.
$\cos 4x$	$A\cos 4x + B\sin 4x$.
e^{5x}	Ae^{5x}
$(9(x)-2)e^{5x}$	$(Ax-B)e^{5x}$.
x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$.
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$.
$xe^{4x}\cos 4x$	$(Ax^2+Bx+B)e^{4x}\cos 4x + (Cx^2+Gx+D)e^{4x}\sin 4x$
$5x^2\sin 4x$.	$(Ax^2+Bx+C)\sin 4x + (Ex^2+Fx+C)\cos 4x$.

11. $y''+4y=g(x)$ $\qquad$ $y(0)=1\ ;\ y'(0)=2$

$$g(x)=\begin{cases}\sin x, & 0\le x\le \pi/2\\ 0, & x>\pi/2\end{cases}$$

Finding the homonogeneous solution.

$y''+4y=0$

$m^2+4m=0$

$m(m+4)=0$

$m=0.$

$m=-4.$

$y_c = c_1 e^{0x} + c_2 e^{-4x}$

$y_c = c_1 + c_2 e^{-4x}$

Now finding non homogeneous sol.

Using interval $0\le x\le \pi/2$

$g(x)=\sin x.$

$y_p = A\sin x + B\cos x.$

or

$y_p = A\cos x + B\sin x$ — ②

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$ — ①

Using both in the original equ.

$-A\cos x - B\sin x + 4(A\cos x + B\sin x) = \sin x$

$+3\cos x + B\sin x = \sin x$

comparing ~~ø~~ coefficients

$A \Rightarrow 0$

$B = 1/3.$

$\therefore\ y_p = \frac{1}{3}\sin x.$

$\therefore\ y = c_1 + c_2 e^{-4x} + \frac{1}{3}\sin x.$

now

$y(0)=1$

$y'(0)=2$ are in this interval.

$\boxed{1 = c_1 + c_2}$

$\therefore\ c_2 = 5/6$ & $c_1 = 1/3$

$\Rightarrow y = \frac{1}{3} + \frac{5}{6}e^{-4x} + \frac{1}{2}\sin x.$

for $x > \pi/2$

Now we are to consider continuity to exisit

$\therefore$

$\lim\limits_{x \to \pi/2^-}$ = $\lim\limits_{x \to \pi/2^+}$

start of this interval is end of

By the funct $g(x) = 0$.

$\therefore$

$y = y_p + y_c$

$y_p \to 0$

$\Rightarrow C_3 \cos 2x + C_4 \sin 2x$

$0 \le x \le \pi/2$

$\cos 2x + \frac{5}{6} \sin 2x + \frac{1}{3} \sin x = C_3 \cos 2(\pi/2) + C_4 \sin(2 \times \pi/2)$

$-C_3 + 0 = -1 + \frac{1}{3}$

$\Rightarrow C_3 \Rightarrow 2/3.$

similarly,

derivative = derivative

$C_4 = 5/6.$

4.6 :- Variation of parameters!

↳ here $y_p = u_1 y_1 + u_2 y_2$.

14. $y'' - 2y' + y = e^t \tan^{-1} t$.

alway convert into standard form in variation of parametres

$y_c \Rightarrow$ By same 4.3 method.

$m_{1,2} = 1$.

$y_c = c_1 \underbrace{e^t}_{y_1} + \underbrace{tc_2 e^t}_{y_2}$

Now developing Wronskian det.

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix}$$

$(e^t)((t+1)e^t) - (e^t)(te^t)$.

$e^t((t+1)e^t - te^t)$

OR.

$e^{2t}(t+1) - te^{2t}$

$e^{2t}(t+1-t)$

$e^{2t}(1) \Rightarrow e^{2t}$

Now by cramers Rul.

$$\Rightarrow \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$y_1 u_1' + y_2 u_2' = 0$

$y_1' u_1' + y_2' u_2' = f(x)$

$$\Rightarrow \begin{vmatrix} 0 & te^t \\ e^t \tan^{-1} t & (t+1)e^t \end{vmatrix}$$

$W_1 \Rightarrow -te^{2t} \tan^{-1}(t)$

$$W_2 \begin{vmatrix} e^t & 0 \\ e^t & e^t \tan^{-1} t \end{vmatrix}$$

$e^{2t} \tan^{-1}(t)$

$u_1' = \frac{W_1}{W}$, $u_2' = \frac{W_2}{W}$.

simila

$u_1' = -t\tan^{-1}(t)$

$\int u_1' = \int \underline{-t\tan^{-1}(t)}$

↓

integration by parts.

let.

$s = \tan^{-1}(t)$ $\quad dv = dt.$

$ds = \frac{1}{1+t^2}.$ $\quad v = t.$

$[s \times v - \int v\,ds.]$

4.7:- Cauchy Euler Equ:

HOMO & NON-HOMO BUT WITH NON CONST COEFFICENTS / TWO VARIABLES!

$\boxed{y = x^m}$

$y' = m x^{m-1}$

$y'' = (m-1) m x^{m-2}$

Back substitute and gather roots.

(i) Real roots

e.g $m_1 = -1$; $m_2 = 4$

$y_c = c_1 x^{-1} + c_2 x^4.$

(ii) Repeated roots.

e.g $m_{1,2} = -1/2$

$y_c = c_1 x^{-1/2} \; c_2 y_2$

by reduct of order method.

$\rightarrow y_c = c_1 x^{-1/2} - c_2 \ln x^{-1/2}.$

Non homogen's equ Cauchy Euler equs involve variation of parameter method as well.

Q no. 32.

—x—

Pending!

Complex Roots.

$x^{\alpha}[c_1\cos(\beta \ln x) + c_2\sin(\beta \ln x)]$

Q16

→ use substitution $x = e^t$ to convert into a DE. with const coefficients.

err $x^2y'' - 4xy' + 6y = \ln x^2$

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = \ln x^2.$$

$x = e^t.$

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}.$$

$$\frac{dt}{dx} = \frac{1}{e^t}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\left(\frac{1}{x}\right)$$

$$\frac{dt}{dx} = \frac{1}{x}.$$

Now taking derivative.
ProductRule.

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

Now this alone is not possible to be solved

$\therefore$

let $\frac{dt}{dt}$ → and rearrange

$$\frac{1}{x}\frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dt}\right) + \frac{dy}{dx}\left(-\frac{1}{x^2}\right)$$

$$\frac{1}{x}\left(\frac{1}{x}\right)\frac{d^2y}{dt^2} + ''$$

Now back substitute both in the original equation. and solve normally.