Chapter no. 8:-
→ distinct Real Eigen
→ Repeated Eigen Values
→ Complex Eigen Values.

8.2:-

⇒ general solution ⇒?

2. $\frac{dx}{dt} = 2x + 2y$. $\frac{dy}{dt} = x + 3y$

The system can be written as.

$$X' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} X.$$

where $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

let

$\det(A - \lambda I)$

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} \Rightarrow (2-\lambda)(3-\lambda) - (2)(1)$$

$\Rightarrow 6 - 2\lambda - 3\lambda + \lambda^2 - 2$

$\Rightarrow 6 - 5\lambda + \lambda^2 - 2$

$= \lambda^2 - 5\lambda + 4 = 0$

$= \lambda^2 - 4\lambda - 1\lambda + 4 = 0$

$= \lambda(\lambda - 4) - 1(\lambda - 4) = 0$

$= (\lambda - 1)(\lambda - 4)$

$\Rightarrow \left.\begin{array}{l} \lambda = 1 \\ \lambda = 4 \end{array}\right\}$ Real and distinct.

Now using $\lambda = 4$ in the augmented $A - \lambda I$ matrix.

$$\left(\begin{array}{cc|c} 2-4 & +2 & 0 \\ 1 & 3-4 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right)$$

Now by apply row operations

$R_2 + \frac{1}{2}R_1$ ⇒ given below

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$-2k_1 + 2k_2 = 0$.

$k_1 = k_2$.

$\therefore k_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$.

Similarly with $\lambda = 1$ on the augmented matrix of $A - \lambda I$

$$\left(\begin{array}{cc|c} 1 & +2 & 0 \\ 1 & 2 & 0 \end{array}\right)$$

$R_2 - R_1$

$\Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$1k_1 + 2k_2 = 0$

$k_1 = -2k_2$

let $k_2 = 1$.

$\therefore k_1 = -2$.

$K_{02} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$X_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{1t}$.

Now the general solution of the matrix is given as

$X = c_1 X_1 + c_2 X_2$.

$X = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{1t}$.

Q4. $\frac{dx}{dt} = -\frac{5}{2}x + 2y$. ✓

$\frac{dy}{dt} = 3x + (-2y)$

→ Real & distinct eigen values.

Similar to the previous Question.

Q6. $X' = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix} X$

The producure is similar

here $A = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix}$;

follow the remaining steps as it is.

Q8. $\frac{dx}{dt} = 2x - 7y$

$\frac{dy}{dt} = 5x + 10y + 4z$

$\frac{dz}{dt} = 5y + 2z$.

converting into matrix form

$A = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$ ✓

$(A - \lambda I) = \begin{pmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{pmatrix}$ ✓

finding det.
expanding by R1.

$2-\lambda[(10-\lambda)(2-\lambda) - 20] - (-7)$
$[(5)(2-\lambda) - 0] + 0$.

$\Rightarrow 2-\lambda\,[20 - 2\lambda - 10\lambda + \lambda^2 - 20$

$7(10-5\lambda)$

$\Rightarrow 2-\lambda(\lambda^2-10\lambda-2\lambda)+70-35\lambda.$

$\Rightarrow 2\lambda^2-20\lambda-4\lambda-\lambda^3+10\lambda^2+2\lambda^2$
$+70-35\lambda.$

$\Rightarrow 2\lambda^2+19\lambda^2-\lambda^3-53\lambda(+6\lambda)+70.$

$\Rightarrow$ OR simply.

$2\lambda^2+19\lambda^2-\lambda^3-59\lambda+70=0.$

$-2\lambda^2+\lambda^3-19\lambda^2+59\lambda-70=0.$

$\lambda_1=2$

$\lambda_2=7$

$\lambda_3=5.$

Using $\lambda_1=2$ in the $(A-\lambda I)$ augmented matrix.

$$\left(\begin{array}{ccc|c} 2-2 & -7 & 0 & 0 \\ 5 & 10-2 & 04 & 0 \\ 0 & 5 & 5-3-2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 0 & -7 & 0 & 0 \\ 5 & 8 & 04 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right)$$

$-7k_2=0$; $5k_1+8k_2 \overset{+4k_3}{} =0$; $5k_2=0.$

Now $k_2=0.$

$\therefore 5k_1+8k_2+4k_3=0$

$5k_1+4k_3=0.$

$5k_1=-4k_3$

let

$k_1=4$

$\therefore k_3=-5.$

$$k_1\begin{pmatrix}4\\0\\-5\end{pmatrix}$$

$$X_1=\begin{pmatrix}4\\0\\-5\end{pmatrix}e^{2t}$$

Similarly, we need to find it for the remaining eigenvalues and then give the general solution

$X=c_1X_1+C_2X_2+C_3X_3\ldots$

10.

$$X' = \begin{pmatrix} 1.0 & 0 & 1 \\ 0.0 & 1 & 0 \\ 1.0 & 0 & 1 \end{pmatrix} X$$

$$(A-\lambda I) = \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$$

finding determinant; expanding by R_1.

$$\Rightarrow (1-\lambda)\left[(1-\lambda)^2\right] - 0 + 1\left[(1-\lambda)+0\right]$$

$$\Rightarrow 1-\lambda\,(1-2\lambda+\lambda^2) + 1(-1+\lambda)$$

$$\Rightarrow \cancel{1} - 2\lambda + \lambda^2 - \cancel{\lambda} + 2\lambda^2 + (-\lambda^3) + \cancel{(-1)} + \cancel{\lambda}$$

$$-\lambda^3 + 3\lambda^2 - 2\lambda = 0$$

$\lambda_1 = 0$
$\lambda_2 = 1$
$\lambda_3 = 2$.

Similarly solve it using the eigen values; but remember never let all 3 values of a eigen vector to be 0.

12.

$$X' = \begin{pmatrix} -1 & +4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix} X$$

$$A-\lambda I = \begin{pmatrix} -1-\lambda & +4 & 2 \\ 4 & -1-\lambda & -2 \\ 0 & 0 & 6-\lambda \end{pmatrix}$$

finding the determinant:

$$\Rightarrow (-1-\lambda)\left[(-1-\lambda)(6-\lambda) - 0\right] = 0$$

$$\Rightarrow -(+(4))\left[4(6-\lambda) - 0\right]$$

$$+2[0].$$

$$\Rightarrow (-1-\lambda)(-6+\lambda-6\lambda+\lambda^2)$$

$$-4(24-4\lambda) + 0.$$

$$\Rightarrow (-1-\lambda)(\lambda^2-5\lambda-6)$$

$$-\lambda^2+5\lambda+6-\lambda^3+5\lambda^2+6\lambda$$

$$(-96+16\lambda)$$

$$-\lambda^3+4\lambda^2+11\lambda+6-96+16\lambda$$

$$-\lambda^3+4\lambda^2+27\lambda-90=0.$$

$$\lambda^3+(-4)\lambda^2-27\lambda+90=0.$$

$\lambda_1 = 6,\ \lambda_2 = 3\ \lambda_3 = -5$.

Solve by similar steps!

$$X' = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} X.$$

$$X(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 1-\lambda[(2-\lambda)(1-\lambda)]$$

$$-1[0] + 4[-(2-\lambda)]$$

$$\Rightarrow (1-\lambda)^2(2-\lambda) - 1 + (4(\lambda-2))$$

$$= (+\lambda^2 - 2\lambda + 1)(2-\lambda) - 1 + 4\lambda - 8$$

$$= 2\lambda^2 - \cancel{4\lambda} + 2 - \lambda^3 + 2\lambda^2 - 01\lambda - 1 + \cancel{4\lambda} - 8$$

$$= -\lambda^3 + 4\lambda^2 - \lambda - 6 \Rightarrow 0$$

$\lambda_1 = -1$

$\lambda_2 = 3$

$\lambda_3 = 2.$

✓ Now using $\lambda_1 = 1.$ in the augment matrix

$$\Rightarrow \left(\begin{array}{ccc|c} 02 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right)$$

$$2k_1 + k_2 + 4k_3 = 0$$

$$k_2 = 0.$$

$$2k_1 + 4k_3 = 0.$$

$$2k_1 = -4k_3$$

$$k_1 = -2k_3$$

Now let $k_3 = (1)$

$\therefore k_1 = -2.$

$$k_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}; \quad X_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

Similary by λ_2 & λ_3

$$X_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

$$X_3 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} e^{3t}$$

$$X = c_1 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} e^{3t}.$$

at $X(0) = \begin{pmatrix}1\\3\\0\end{pmatrix}$

$\therefore$

$$\begin{pmatrix}1\\3\\0\end{pmatrix} = c_1\begin{pmatrix}-2\\0\\1\end{pmatrix} + c_2\begin{pmatrix}2\\0\\1\end{pmatrix} + c_3\begin{pmatrix}5\\-3\\2\end{pmatrix}$$

✓ $1 = -2c_1 + 2c_2 + 5c_3$ — (1)

✓ $3 = -3c_3$ — (2)

✓ $0 = c_1 + c_2 + 2c_3$ — (3).

$\Rightarrow \left[c_3 = -1\right]$ ✓

$0 = c_1 + c_2 + (-1)\times 2$. ✓

$c_1 + c_2 = +1(2)$

$c_1 = \left[+2 - c_2\right]$ ✓

using in (1).

$1 = -2(2 - c_2) + 2c_2 + 5(-1)$

$1 = -4 + 2c_2 + 2c_2 - 5$.

$1 \Rightarrow -9 + 4c_2$

$10 = +4c_2$.

$c_2 = 5/2$.

$c_1 = +2 - 5/2$

$c_1 = \frac{4-5}{2}$

$c_1 = \frac{-1}{2}$.

$\therefore$

✓ $$X = -\frac{1}{2}\begin{pmatrix}-2\\0\\1\end{pmatrix}e^{-t} + \frac{5}{2}\begin{pmatrix}2\\0\\1\end{pmatrix}e^{3t} - \begin{pmatrix}5\\-3\\2\end{pmatrix}e^{2t}$$ Ans!

2. $\frac{dx}{dt} = -6x + 5y$

$\frac{dy}{dt} = -5x + 4y.$

$A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$

$(A - \lambda I) = \begin{pmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{pmatrix}$

$|A - \lambda I| = (-6-\lambda)(4-\lambda) - (5)(-5)$

$= -24 + 6\lambda - 4\lambda + \lambda^2 + 25.$

$= \lambda^2 + 2\lambda + 1.$

$= \lambda^2 + 1\lambda + 1\lambda + 1$

$0 = \lambda(1+\lambda) + 1(1+\lambda).$

$0 \Rightarrow (\lambda+1)(\lambda+1)$

$\left.\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -1 \end{matrix}\right\}$ Repeated eigen values! ✓

OR eigen value of multiplicity of two.

$(A - \lambda I)k = 0.$

$\begin{bmatrix} -6-(-1) & 5 \\ -5 & -4-(-1) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$

$R_2 + (R_1)$

$\begin{pmatrix} -5 & +5 \\ 0\cancel{5} & 0\cancel{5} \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0.$

$-5k + 5k_2 = 0.$

$k_1 = k_2.$

let $k_1 = 1$

$k_2 = 1.$

✓ $k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}.$

Now as the second eigen vector would yield ⇒ but is repeat.

$\therefore \quad (A - \lambda I)P = k.$

$\begin{pmatrix} -5 & +5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$-5p_1 + 5p_2 = 1.$

$p_2 \Rightarrow p_1 + \frac{1}{5}.$

$\therefore$ let $p_1 = 0$ $p_2 = 1/5$

and ans. $P = \begin{pmatrix} 0 \\ 1/5 \end{pmatrix}$; $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 0 \\ 1/5 \end{pmatrix}e^{-t}$

Thus the overall solution will be.

$$X = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 1/5 \end{pmatrix} e^{-t} \right]$$ Ans!

24. $X' = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} X.$

Similar to the previous one!

26. $\frac{dx}{dt} = 3x + 2y + 4z$

$\frac{dy}{dt} = 2x + 2z$

$\frac{dz}{dt} = 4x + 2y + 3z.$

$A \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

$|A - \lambda I|$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & 0-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

-4

$\Rightarrow (3-\lambda)(-\lambda \times (3-\lambda)) - 2(2(3-\lambda)-8)$
$+4(4-4(-\lambda))$

$\Rightarrow (3-\lambda)(-3\lambda + \lambda^2 - 4)$
$-2(6-2\lambda-8) + 4$
$(4+4\lambda).$

$(+3\lambda^2 - \lambda^3 + 4\lambda)$

$\Rightarrow -9\lambda + 3\lambda^2 - 12 - 2(-2-2\lambda)$
$16 + 16\lambda$

-12

$\Rightarrow -\lambda^3 + 6\lambda^2 - 5\lambda + 4 + 4\lambda$
$+16 + 16\lambda$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 5\lambda - 12 + 4 + 4\lambda$
$+16 + 16\lambda.$

$\Rightarrow -\lambda^3 + 6\lambda^2 + 15\lambda + 8$

$\lambda_1 = 8$
$\lambda_2 = -1$
$\lambda_3 = -1$

using $\lambda_1 = 8$ in the augmented matrix.

$$\left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right)$$

converting into echelon form.

$R_2 \times 5 + R_1 \times 2$, $R_3 \times 5 + R_1 \times 4$.

$$\left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -36 & 18 & 0 \\ 0 & 20 & -9 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 20 & -9 & 0 \end{array}\right)$$

Now,

$\frac{10}{10} R_3 + 10 R_2$

$$\Rightarrow \left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Now

$k_3 = 1 \neq 0$.

$-2k_2 + k_3 = 0$

$\cancel{+}2k_2 = \cancel{+}k_3$ ✓

let $k_2 = 1$

$k_3 = 2$

$-5k_1 + 2k_2 + 4k_3 = 0$.

$-5k_1 + 2(1) + 4(2) = 0$

$5k_1 + 10 \Rightarrow k_1 = -10/-5 \Rightarrow +2$

$\therefore$

$$K = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$X = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}. \checkmark$$

Now $\lambda = -1$

$$\left(\begin{array}{ccc|c} 3-(-1) & 2 & 4 & 0 \\ 2 & 0-1 & 2 & 0 \\ 4 & 2 & 3-(-1) & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right)$$

Now.

$R_2 - \frac{R_1}{2}$; $R_3 - R_1$

$$\left(\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$4k_1 + 2k_2 + 4k_3 = 0$

let $k_1 = 0$

$k_2 = 1$.

$4(0) + 2(1) + 4k_3 = 0$

$0 + 2 + 4k_3 = 0$

$k_3 = -1/2$.

$$X' = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} X.$$

$$(A-\lambda I) = \begin{pmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix}$$

$$\Rightarrow (4-\lambda)(-4-\lambda) - (-5)(5)$$

$$\Rightarrow -16 \cancel{-4\lambda} \cancel{+4\lambda} + \lambda^2 + 25.$$

$$\Rightarrow \lambda^2 + 9$$

$$\lambda^2 \Rightarrow -9$$

$$\boxed{\lambda \Rightarrow \pm 3i}$$

✓

Complex roots

let $\lambda_1 = 3i$

$$\left(\begin{array}{cc|c} 4-3i & -5 & 0 \\ 5 & -4-3i & 0 \end{array}\right) \text{—(a)}$$

$$\left(\begin{array}{cc|c} 1 & -5/4-3i & 0 \\ 5 & -4-3i & 0 \end{array}\right)$$

$R_2 + (5R_1)$

$$\left(\begin{array}{cc|c} 1 & -5/4-3i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

OR at (a) we can also do that;

$(4-3i)R_2 - 5$ Rough work.

$$4-3ik_1 - 5k_2 = 0.$$

$$k_1 = \frac{5}{4-3i} k_2$$

let $k_2 = 4-3i$

$k_1 = 5.$

$$K_1 = \begin{pmatrix} 5 \\ 4-3i \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 5 \\ 4 \end{pmatrix} + i \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$$\downarrow \qquad \downarrow$$

$$B_1 \qquad B_2$$

Now;

$$\lambda_1 = \alpha + \beta i \Rightarrow 0 + 3i$$

$$X_1 = [B_1 \cos\beta t - B_2 \sin\beta t] e^{\alpha t}$$

$$X_1 = \left[\begin{pmatrix} 5 \\ 4 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t\right] e^{0t}$$

$$X_1 \begin{pmatrix} 5\cos 3t \\ 4\cos 3t + 3\sin 3t \end{pmatrix}$$

and

$$X_2 = [B_2 \cos\beta t + B_1 \sin\beta t]e^{\alpha t}$$

$$X_2 = \left[\begin{pmatrix}0\\-3\end{pmatrix}\cos 3t + \begin{pmatrix}5\\4\end{pmatrix}\sin 3t\right]e^{ot}$$

$$x = c_1X_1 + X_2$$

$$= c_1\begin{pmatrix}5\cos 3t\\4\sin 3t + 3\sin 3t\end{pmatrix} + c_2\begin{pmatrix}0\cos 3t + 5\sin 3t\\-3\cos 3t + 4\sin 3t\end{pmatrix}$$ Ans!

43. $X' = \begin{pmatrix}1 & -1 & 2\\-1 & 1 & 0\\-1 & 0 & 1\end{pmatrix}$

$A - \lambda I \begin{pmatrix}1-\lambda & -1 & 2\\-1 & 1-\lambda & 0\\-1 & 0 & 1-\lambda\end{pmatrix}$

$|A - \lambda I|$ 2 → find as usual

$\lambda_1 = 1$

$\lambda_2 = 1+i$

$\lambda_3 = 1-i$

$\underline{\lambda_2 = 1+i}$

$$\Rightarrow \left(\begin{array}{ccc|c}-i & -1 & 2 & 0\\-1 & -i & 0 & 0\\-1 & 0 & -i & 0\end{array}\right)$$

$-k_1 - ik_3 = 0$

$k_1 = -ik_3$

let $k_1 = 1$

$k_3 = i$

$-1k_1 - k_2 \cdot 2k_3 = 0$

$k_2 = i$

$\therefore k_2 = \begin{pmatrix}1\\i\\i\end{pmatrix} \Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix} + i\begin{pmatrix}0\\1\\1\end{pmatrix}$

B_1 & B_2

$\lambda = \alpha + \beta i$

$\Rightarrow \alpha = 1$

$\beta = 1$

$$X_2 = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\cos 1t - \begin{pmatrix}0\\1\\1\end{pmatrix}\sin t\right]e^t$$

47. $\begin{pmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix} X$; $X(0) = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}$

$|A - \lambda I|$ $\begin{vmatrix} 1-\lambda & -12 & -14 \\ 1 & 2-\lambda & -3 \\ 1 & 1 & -2-\lambda \end{vmatrix}$

Imp! $\lambda_1 = 1$ is simple.

Using $\lambda_2 = 5i$

$\left|\begin{array}{ccc|c} 1-5i & -12 & -14 & 0 \\ 1 & 2-5i & -3 & 0 \\ 1 & 1 & -2-5i & 0 \end{array}\right|$

$1-\lambda\,[(2-\lambda)(-2-\lambda) - (-3)(1)]$

$+12\,[(1)(-2-\lambda) - (1)(-3)]$

$-14\,[(1)(1) - (2-\lambda)(1)]$

$\Rightarrow (1-\lambda)\,[-4 - \cancel{2\lambda} + \cancel{2\lambda} + \lambda^2 + 3]$

$+12\,[-2-\lambda+3] - 14\,[1-2+\lambda]$

$\Rightarrow (1-\lambda)(\lambda^2 - 1) + 12(1-\lambda)$

$-14(-1+\lambda)$

$\lambda^2 - 1 - \lambda^3 + \lambda + 12 - 12\lambda + 14 - 14\lambda$

$\lambda^2 - \lambda^3 - 25\lambda + 25 \Rightarrow -\lambda^3 + \lambda^2 - 25\lambda + 49 + 1 - 25$

$\Rightarrow \lambda_1 = 1$; $\lambda_2 = 5i$, $\lambda_3 - 5i$

$R_3 - R_2$

$\left[\begin{array}{ccc|c} 1-5i & -12 & -4 & 0 \\ 1 & 2-5i & 3 & 0 \\ 0 & -1+5i & 1-5i & 0 \end{array}\right]$

Now;

$R_3\left(\frac{1}{-1+5i}\right)$

$\Rightarrow \left[\begin{array}{ccc|c} 1-5i & -12 & -4 & 0 \\ 1 & 2-5i & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

Now you can easily solve

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