* The product Rule applies when we are considering all the possible combinations of ways to complete the task.

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Chapter no. 6

COUNTING

COUNTING

- THE PRODUCT RULE:

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and each of these ways of doing the first tasks, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

n_1 : No. of ways to do task 1.
n_2: No. of ways to do task 2.

$n_1 \times n_2$ = Total no. of ways to peform the procedure (i-e task 1 + task 2).

EXAMPLE 1:-

A new company has just 2 employees and rented an floor of 12 offices, how can each individual / how many ways are there to assign d/f offices to these 2 employees?

let say the 1st employee can be given the office in 12 ways then the 2nd employee can be given in only 11 ways.

$\therefore$ total no. of ways to assign diff offices to both $\Rightarrow 12 \cdot 11$
$\Rightarrow 132$.

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EXAMPLE 2:-

THE Chairs of Audi can be labelled in uppercase English letter followed by a positive integer not exceeding 100. What are the largest no. of chairs that can be labeled?

No. of letters = 26
No. of integers = 100

Total no. of ways ⇒ 26.100 ⇒ 2600.

EXAMPLE 4:-

How many different Bit strings of length seven are there?

A Bit string of 7 length would have each bit either equal to 1 or 0 i.e 2 possible ways for each bit.

$\underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2} \Rightarrow 2^7 \Rightarrow 128$.

EXAMPLE : 5 How many d/f no. plates can be made if a plate contains a sequence of 3 uppercase English letters followed by 3 digits.

$\underbrace{\underline{26}\ \underline{26}\ \underline{26}}_{\text{3 unique independant letters, Uppercase}}\quad \underbrace{\underline{10}\ \underline{10}\ \underline{10}}_{\text{3 unique, independant digit}} \Rightarrow 17,576,000$ ways.

0 - 9

OR 0 → 999 ⇒ 1000 ways.

Product Rule occurs when you have to perform all task to know the possible combinations. While in sum rule you only perform one task out of many possible tasks and that they donot overlap.

CELL NO. COMBINATION POSSIBILITIES IN PAKISTAN.

can only be 0 Total no. of digits = 11

$\underline{1}$ $\underline{1}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ ⇒ 10^9 Ans!

can only be 3

What can be the combination of a 4-digit PINCODE if (i) DIGIT should be integer.
(ii) All digits should be non-replicating.

$\underline{10}$ $\underline{9}$ $\underline{8}$ $\underline{7}$ = 5040 Ans!

Say we picked from 10 possible ways

as non replicating our domain of choice reduced successively.

THE SUM RULE :-

If a task can be done in one of n_1 ways or in one of n_2 ways where none of the set of n_1 ways is the same as any of the set of n_2 ways, then $n_1 + n_2$ ways to do the task.

n_1 ways
OR
n_2 ways

condition that both have no overlap.

Total no. of ways to do the task → $n_1 + n_2$

Product Rule

Outcome → make an outfit

n1 = choose pants

n2 = choose shirts.

$P = n1 \cdot n2$

correspond for the same outcome

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~~no. of ways~~

n1 → you can go the park

n2 → no. of ways you can go to the museum.

Out come = $(n_1 + n_2)$ these are alternative choices.

mutually exclusive

Example 12: Suppose that either a member of the mathematics faculty or a student of math major is chosen as a university committee representative. There are 37 members of the mathematics faculty and 83 mathemathematics majors then how many d/f choices are there?

There are 37 ways to choose a member of the mathematics faculty,

and there are 83 ways to choose a student who is a mathematics major.

Choosing the faculty is never the same as choosing a student

$\Rightarrow 37 + 83 = 120.$

Example: A student is to choose a project from one of three lists. The 3 lists contain 23, 15 and 19 possible projects respectively. No project is on more than one list. How many ways projects can be choosen from?

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$23+15+19=57$. ⇒ when the projects list are mutually exclusive i.e not repeating.

if the projects are not mutually exclusive simply adding them would increase the no. of projects by double counting.

in that case our solution would be:-

A ⇒ 23
B ⇒ 15
C ⇒ 19

we would also require :-
$A \cap B$? $A \cap B \cap C = ?$
$B \cap C = ?$

Answer $= A+B+C - |A \cap B| - |B \cap C| - |A \cap B \cap C|$

EXAMPLE 15:-

given conditions.

(i) Variable name could be 1 character / 2 character.

(ii) variable name should be starting from alphabets.

(iii) Alphabet letters are not distinguished when considered in upper or lower case.

No. of possible ways to make a variable name ⇒?

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Let V be the requirement

$$V = V_1 + V_2$$

$V_1 \Rightarrow$ one letter variable that would have to start with an alphabet.

$V_1 \Rightarrow 26$

because a variable cannot be one character & 2 character at the same time

$V_2 \Rightarrow$ 2 letter variable that starts with a alphabet but can possess alphanumeric characters latter.

$V_2 = 26 \cdot 36$

$(26+10)$

$\therefore$ V_1 and V_2 are mutually exclusive.

also 5 reserve alphanumeric combinations cannot be used.

$$V = V_1 + (V_2 - 5)$$

$V = 26 + 931 \Rightarrow 957$ Answer!

EXAMPLE 16:-

We need to consider the following details.

(i) Password length: Password can either be 7, 6, 8 characters long. (Mutually exclusive).

(ii) Character choices: Each character can be upper case or a digit (26 possibilities) (10 possibilities)

(iii) There should be atleast 1 digit.

Date

Sol

Formulating 6 character password.

if all digits and alphabet possiblities are considered.

6 character long password ⇒ 36 . 36 . 36 . 36 . 36 . 36 ⇒ 36^6

$P_6 = 36^6 - 26^6$

⇒ 1867866560 (to satisfy our condition)

But here could be a possib that all were alphabets i.e as:

$26\ 26\ 26\ 26\ 26\ 26 = 26^6$

Similarly.

$P_7 = 36^7 - 26^7$

$P_8 = 36^8 - 26^7$

Total no of valid passwords of diff lengths ⇒ $P_6 + P_7 + P_8$.

EXAMPLE 17:-

let x ⇒ no. of available addresses.

as we have 3 prominent classes of such addresses.

$\therefore\ x = x_A + x_B + x_C$.

netids.

$x_A ⇒ 2^7 - 1$ (-1 because 1111111 is unavailable)

↳ netids ⇒ 127

hostids.

$2^{24} - 2 = 16,777,214$ (as all 0s and 1s cannot be apart of it)

$x_A = (127).(16777214)$

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Class B
netids => 2^{14}
hostids => $2^{16} - 2$

class C.
netids => 2^{21}
hostids => $2^{8}-2$

XB = (16,384 × 65534)

XC = (2097152 × 254)

X = XA + XB + XC => 3737091842.

EXERCISE QUESTIONS :-

ONLY PRODUCT & SUM RULE:-

1.
(a)

let an event 'a' has possibilities a and another event has possibilities 'b', where both are a sequence of independant events.

total no. of outcomes for both events => a.b

Similarly;
to determine how many ways there are to pick 2 representatives so that one in math and other is comp sci major is;

18 . 325 = 5850.

(b)
now the goal is that to know how many ways one representative can be chosen who is either a mathematician or compsci major.

(if we have multiple mutually exclusive choices/events and want to determine the total no of outcomes, we add them together).

18 + 325 = 343

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SUBSTRACTION RULE:-

If a task can be done in either 'n_1' ways or 'n_2' ways, then the no. of ways to do the task $n_1 + n_2$ minus the no. of ways to do the task that are common to the different ways.

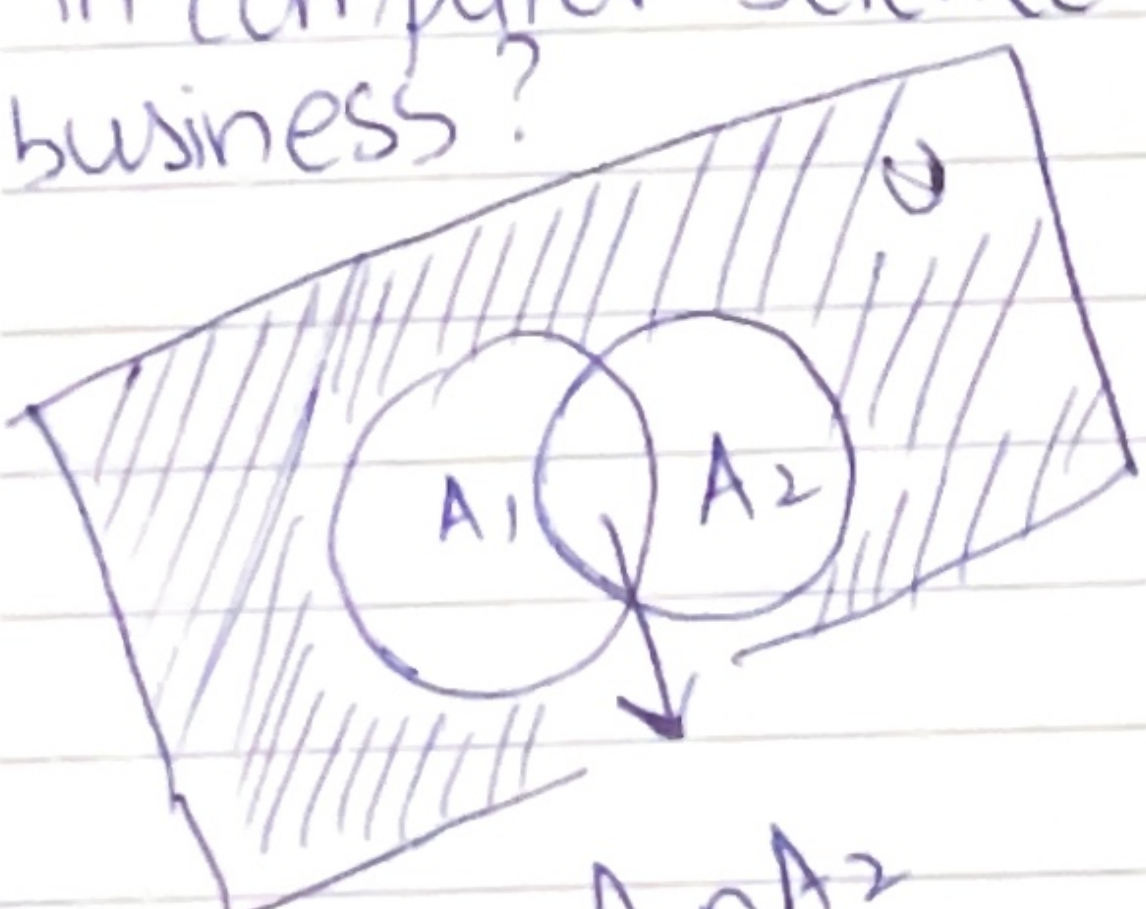
★ also known as the principle of inclusion-exclusion.

EXAMPLE - 19 :-

No. of total applicants = 350.
CS applicants = 220. (A_1)
147 applicants. ⇒ 147 (A_2)
("Business)

CS ∩ Business " ⇒ 51 ($A_1 \cap A_2$)

How many applicants are neither in computer Science nor in business?



$A_1 \cap A_2$

We require the shaded region

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$\Rightarrow 220 + 147 - 51$$

Now

$$|\overline{A_1 \cup A_2}| = U - |A_1 \cup A_2|$$
$$= 350 - 316$$
$$\Rightarrow 34 \text{ Ans!}$$

★ **Example 18:**

Bit strings that start with 1 (length: 8) —— (a)

$\underline{1} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2}$

2^7

Bit string that end with 00 similarly: 2^6 —— (b)

Now there can exist a common group of condition (a) & (b); for our sol we require either or 'or' but not together

$\therefore 2^7 + 2^6 - (2^5) \rightarrow$ if both (a) & (b) occur.

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Division Rule:-

If a task can be done in n different ways, but it turns out that for each way of doing the task, there are d equivalent ways of doing it. Then there are n/d inequivalent ways of doing the task.

Example:- How many d/f ways are there to seat 4 people around a circular table, when two seatings are considered the same when each person has the same left neighbour and the same right neighbour.

Consider the question in 2 parts:-

1^{st} part:-

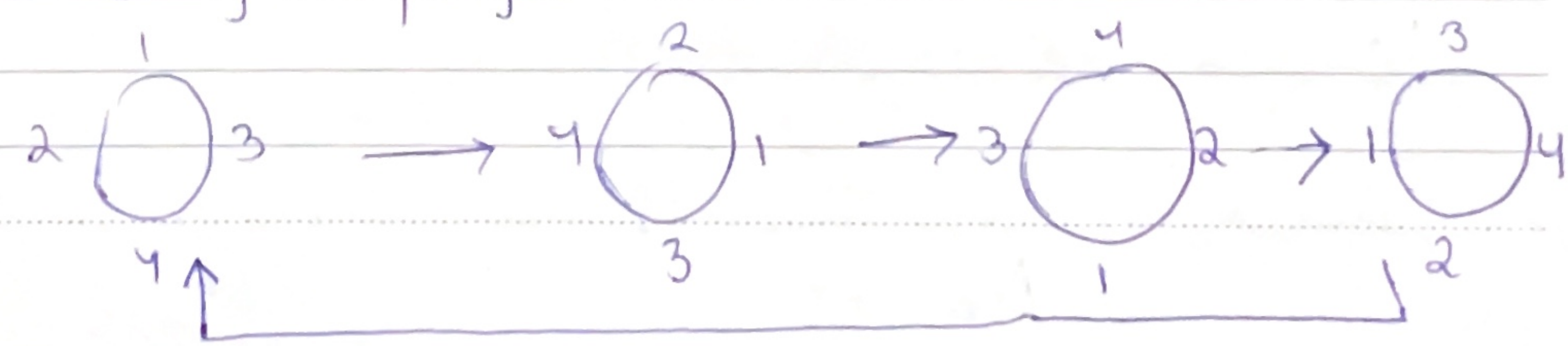
how many different ways to seat 4 people in a circular table?

$\underline{4}\ \underline{3}\ \underline{2}\ \underline{1} \rightarrow 4! \Rightarrow 24.$

$n=24$

2^{nd} part.

Now considering 2 seatings the same when each person has the same left neighbor and the same right neigbor.



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thus d i.e equivalent ways of doing the task are 4

d = 4.

∴ the diff ways (inequivalent ways) ⇒ n/d ⇒ 24/4
⇒ 6.

PIGEONHOLE PRINCIPAL:-

"If k is a positive integer and k+1 or more objects are placed into k boxes; then there are/is atleast one box containing 2 or more objects.)

Example 1:-

Among any group of 367 people, there must be atleast two with the same birthday becauses there are only 366 Birthday's possible in an year.

alternate condition of feb makes 365+1.

Example no.2 :-

In any group of 27 English words; there must be atleast two that begin from the same letter, because there are 26 letters unique in English alphabets.

Example : How many students there should be in a class to garantee that atleast two students would have the same final score, if the exam is graded from 0 – 100 people?

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There should be atleast 101 students to receive each grade marks uniquely, and 102 to receive repeatedly one pair to.

Example no. 4. Show that forevery integer n there is amultiple of n that has 0s and 1s in its decimal expansion.

Generalizied Pigeonhole Principle.

If N objects are placed into k boxes, then there is atleast one box containing atleast $\lceil N/k \rceil$ object.

also

$$N = k(r-1)+1.$$

while

$$\lceil N/k \rceil \geq r.$$

Example 5

Among 100 people atleast how many are there that are born in the same month.

$$\left\lceil \frac{100}{12} \right\rceil = 9$$

Solve e.g 7, 8, 9, ~~10~~
~~11~~, ~~12~~, ~~13~~
(10 – 13) → not included

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Example 5:-

What are the minimum no. of students required in CS-231 class to ensure that atleast sixe receive the same grade if there are 5 possible grade.

$$\left\lceil \frac{N}{k} \right\rceil \Rightarrow 6$$

$N \to ?$, $k \to 5$

$\therefore N = (5)(6-1)+1$
$\Rightarrow (5)(5)+1$
$\Rightarrow 25$ Ans!

6.3
Permutations:-

A permutation of a set of distinct objects is an ordered arrangement of these objects.

r- permutations.

An ordered arrangement of r-elements of a set is known as r-permutations.

denoted by $P(n, r)$

Consider a set of 3 elements $\{a, b, c\}$ then 2-permutations of a 3 element set is given as.

$P(3,2) = 3.2 = 6.$

three ways to choose the first element

two ways to choose the 2nd element.

FORMULA :-

$$P(n,r) = \frac{n!}{(n-r)!} \qquad n, r \in \mathbb{Z} \quad 0 \le r \le n$$

also for $1 \le r \le n$

also if
$$P(n,n) = \frac{n!}{1} \Rightarrow n!$$

Example (4):-

3-permutations of 100.

$$P(100, 3) \Rightarrow \frac{100!}{97!}$$

$\Rightarrow (99 \times 98)100 \Rightarrow 970,200.$

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Example 6 :-

she travels 8 different cities, one city is fixed however she can visit 7 different cities from her own wish.

$\therefore$ $7! \Rightarrow 5040$ paths are to be considered.

Example 7.

ABC is a string that will always occur in our arrangement thus; it is considered

→ 1 block.

→ ordered arrangement

ABCDEFGH

1 2 3 4 5 6.

$6! = 720$ Ans!

COMBINATIONS :-

An r- combination of elements of a set is an unordered selection of r-elements from the set

Simply a subset of set (because the order does not matter).

Note:

$\{1,3,4\}$ is same as $\{4,1,3\}$.

The order in which elements are listed (esp in sets) does not matter.

** $C(n,r) = \dfrac{n!}{(n-r)!\,r!}$

$n \notin \mathbb{Z}^- \in \mathbb{Z}$

$r \in \mathbb{Z}$

$0 \leq r \leq n$

$C(n,r)$ also $\binom{n}{r} \Rightarrow$ Binomial Coefficient.

$P(n,r) = C(n,r) \times P(r,r)$

order doesnot matter.
(i) drawing card from a deck. (iii) Committee.
(ii) forming a match - team.

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*note

$$C(n,r) = C(n,n-r).$$

$$\frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-(n-r))!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!}$$

when n and r are non-negative integers.
hence proved!

Example 14 *
How many bit strings of length 'n' contain 'r' no. of 1s.

as the order of 1s in the bit strings is not specified, then.
$C(n,r)$ Ans!

Example 15.

$C(9,3) \cdot C(11,4) \Rightarrow 27,720$ Ans!

selecting members from mathematics

selecting members from computer science

multiplied both as both tasks are different from one another, to get the total no. of tasks ways.

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6.4 BINOMIAL COEFFICIENTS AND IDENTITIES

The binomial theorem gives a power of a binomial expression (such as $(a+b)^n$).
The binomial theorem gives the coefficient of expansion of powers of Binomial expressions.

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots$$

power of j effects on y.

$$\binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

Example 2

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$\Rightarrow \binom{4}{0} x^4 y^0 + \binom{4}{1} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^1 y^3$$

$$+ \binom{4}{4} x^0 y^4$$

$$\Rightarrow x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Example 3

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

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$\binom{25}{13} \Rightarrow \frac{25!}{(25-13)!\,(13)!} \Rightarrow 5,200,300$

Example 4

coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$.

Now as the Binomial expression is not in its base form.

$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$

$(2x-3y)^{25} \Rightarrow (2x + (-3y))^{25}$

$\Rightarrow \sum_{j=0}^{25} \binom{25}{0} 2x^{25-0} y^0.$

Now for $x^{12}y^{13}$.

$\Rightarrow \binom{25}{13} (2x)^{25-13} (3y)^{13}.$

$\Rightarrow -\binom{25}{13} 2^{12} 3^{13} (x^{12}y^{13})$

Ans!

Binomial expansion contains n+1 terms. $\binom{n}{0} = 1 = \binom{n}{n}$

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Note

(i) $\sum_{k=0}^{n} \binom{n}{k} \Rightarrow 2^n$

$(1+1)^n \Rightarrow 2^n$.

(ii) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

$(-1+1)^n \Rightarrow 0$.

(iii) $\sum_{k=0}^{n} 2k \binom{n}{k} = 3^n$

$(+1+2)^n \Rightarrow 3^n$

* also remember

$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5}$

VIDEO LECTURE:-
APPLICATION OF BINOMIAL THEOREM:

COUNTING PASSWORDS CONTAINING DIGITS /NO.S.

Example 16 (6.1).

The combination of digit and alphabets to form a password of length 6, can be given through the Binomial expansion as. (with combination of alphabets and atleast one digit)

$(26+10)^6 \Rightarrow \binom{6}{0} 26^6 10^0 + \binom{6}{1} 26^5 10 + \binom{6}{2} 26^4 10^2$

$36^{(6)}$.

$+ \binom{6}{3} 26^3 10^3 \binom{6}{4} 26^2 10^4 \binom{6}{5} 26\, 10^5 + \binom{6}{6} 10^6$

$\Rightarrow 26^6 + \binom{6}{1} 26^5 10 + \binom{6}{2} 26^4 10^2 + \binom{6}{3} 26^3 10^3$

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$+\binom{6}{4}26^4 10^2 + \binom{6}{5}26\ 10^5 + \binom{6}{6}26^0 10^6$

Now here each term represents a meaning :-

$\binom{6}{0}26^6 10^0 \Rightarrow 26^6$ ⇒ a password containing no digit and all six characters.

$\binom{6}{1}26^5 10^1 \Rightarrow$ a pass word containing one digit & 5 characters

$\binom{6}{2}26^4 10^2 \Rightarrow$ 4 characters and two digit.

and so on .....

Now as per the given condition.

we donot require 26^6.

$\therefore \quad (26+10)^6 - 26^6 \Rightarrow \binom{6}{1}26^5 10 + \binom{6}{2}26^4 10^2$

$+\binom{6}{3}26^3 10^3 + \binom{6}{4}26^2 10^4 + \binom{6}{5}26\ 10^5$

$+\ 10^6$

Now if someone asks for a specific condition like in a 6 length password of alphabets and digit give combination possible if A-Z = 2 and alphabet = 4 non-ve digits. then $\binom{6}{4}26^2 10^4$ Ans!