Lecture 12

Cost Minimization

Cost Minimization

A firm is a **cost-minimizer** if it produces any given output level $y \geq 0$ at smallest possible total cost.

c(y) denotes the firm's **smallest** possible total cost for producing y units of output.

c(y) is the firm's **total cost function**.

成本函数$C(y)$度量的是当要素价格为(ω_1, ω_2)时，生产y单位产品所需的**最小成本**。

The Cost-Minimization Problem

- Consider a firm using two inputs to make one output.
- The production function is $f(x_1,x_2)$.
- Take the output level $y \geq 0$ as given.
- Given the input prices w_1 and w_2, the cost of an input bundle (x_1,x_2) is $w_1x_1 + w_2x_2$.

The Cost-Minimization Problem

For given w_1, w_2 and y, the firm's cost-minimization problem is to solve

$$\min_{x_1,x_2 \geq 0} w_1 x_1 + w_2 x_2$$

subject to $f(x_1, x_2) = y.$

The Cost-Minimization Problem

- The levels $x_1^*(\omega_1, \omega_2, y)$ and $x_2^*(\omega_1, \omega_2, y)$ in the least-costly input bundle are the firm's **conditional demands for inputs 1 and 2**. (使成本最小化的要素投入量被称为条件要素需求函数)
- The (**smallest possible**) total cost for producing y output units is therefore

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y).$$

Conditional Input Demands

Given w_1, w_2 and y, how is the least costly input bundle located?

And how is the total cost function computed?

给定(ω_1, ω_2, y)，如何用图示的方法找到成本最小化的要素组合?

Iso-cost Lines

A curve that contains all of the input bundles that cost the same amount is an **iso-cost curve**（等成本线）.

E.g., given w_1 and w_2, the $100 iso-cost line has the equation

$$w_1x_1 + w_2x_2 = 100.$$

具有相同成本的要素组合的集合被称为等成本线。

Iso-cost Lines

Generally, given w_1 and w_2, the equation of the \$c iso-cost line is

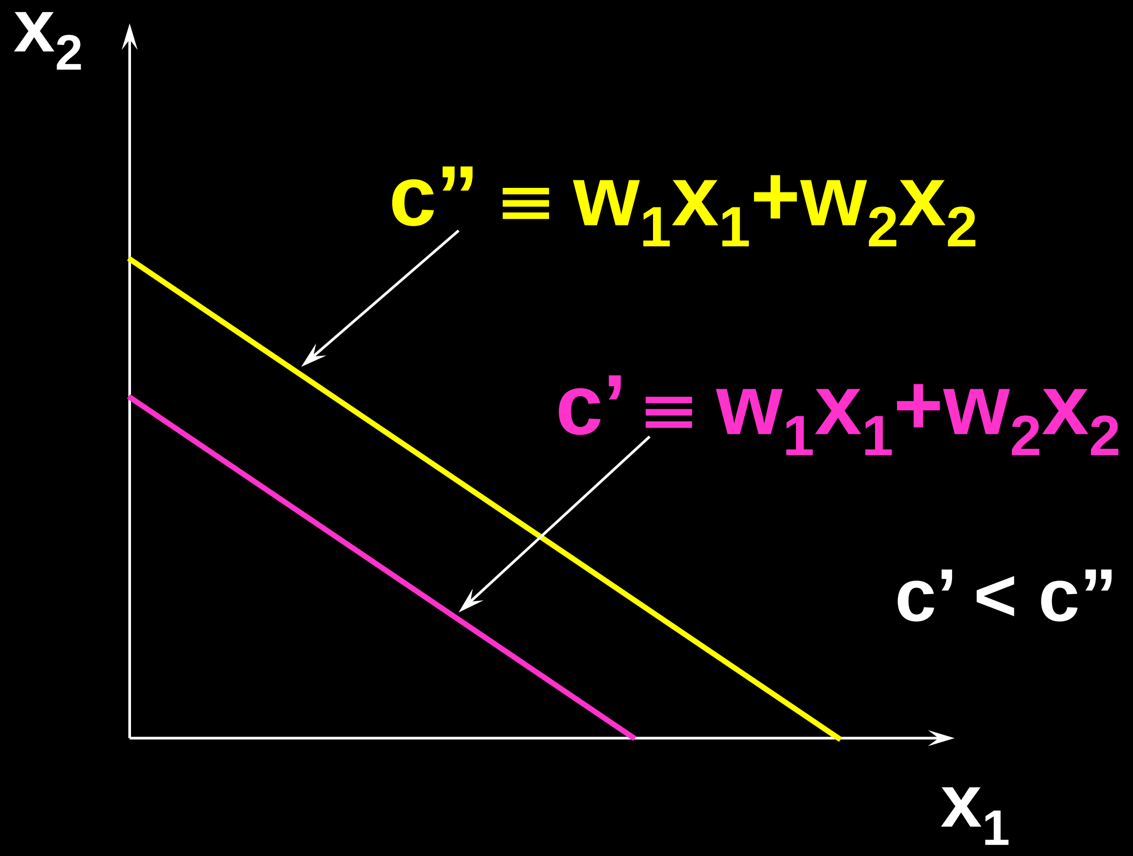
$$w_1 x_1 + w_2 x_2 = c$$

i.e.

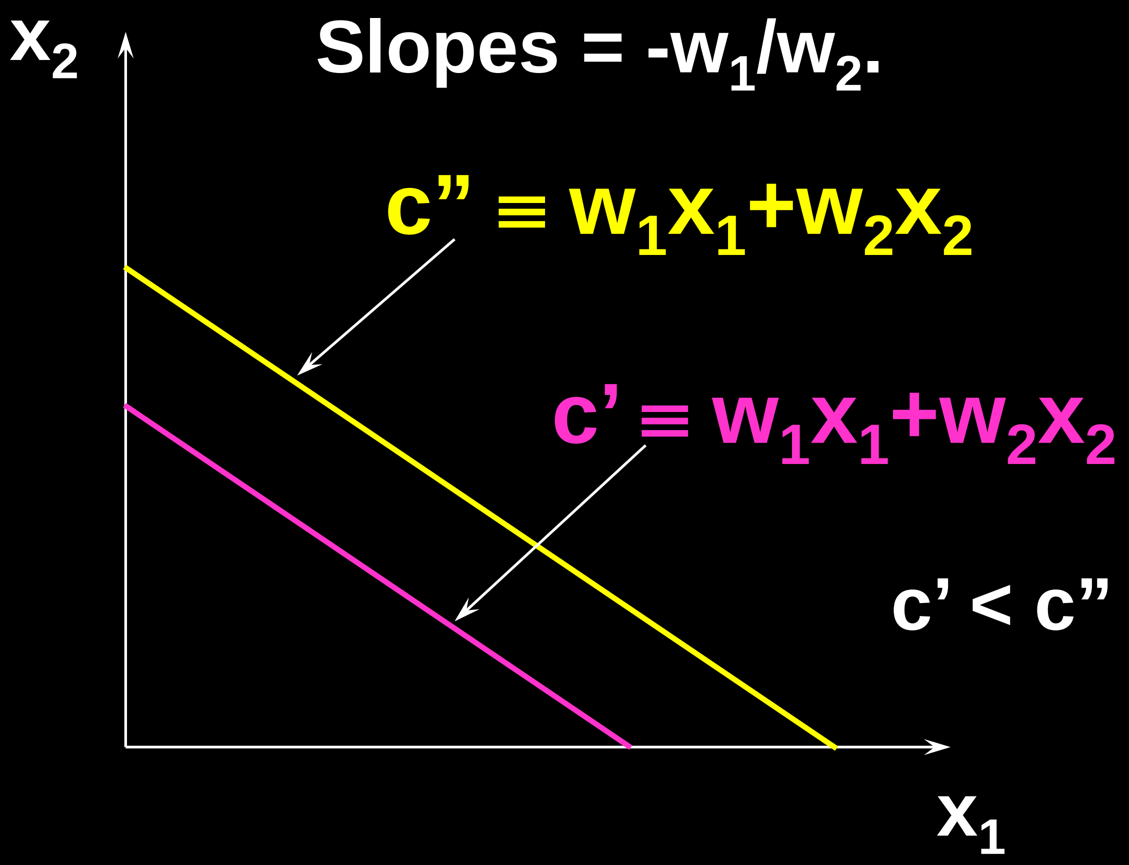
$$x_2 = -\frac{w_1}{w_2} x_1 + \frac{c}{w_2}.$$

Slope is - w_1/w_2.

Iso-cost Lines



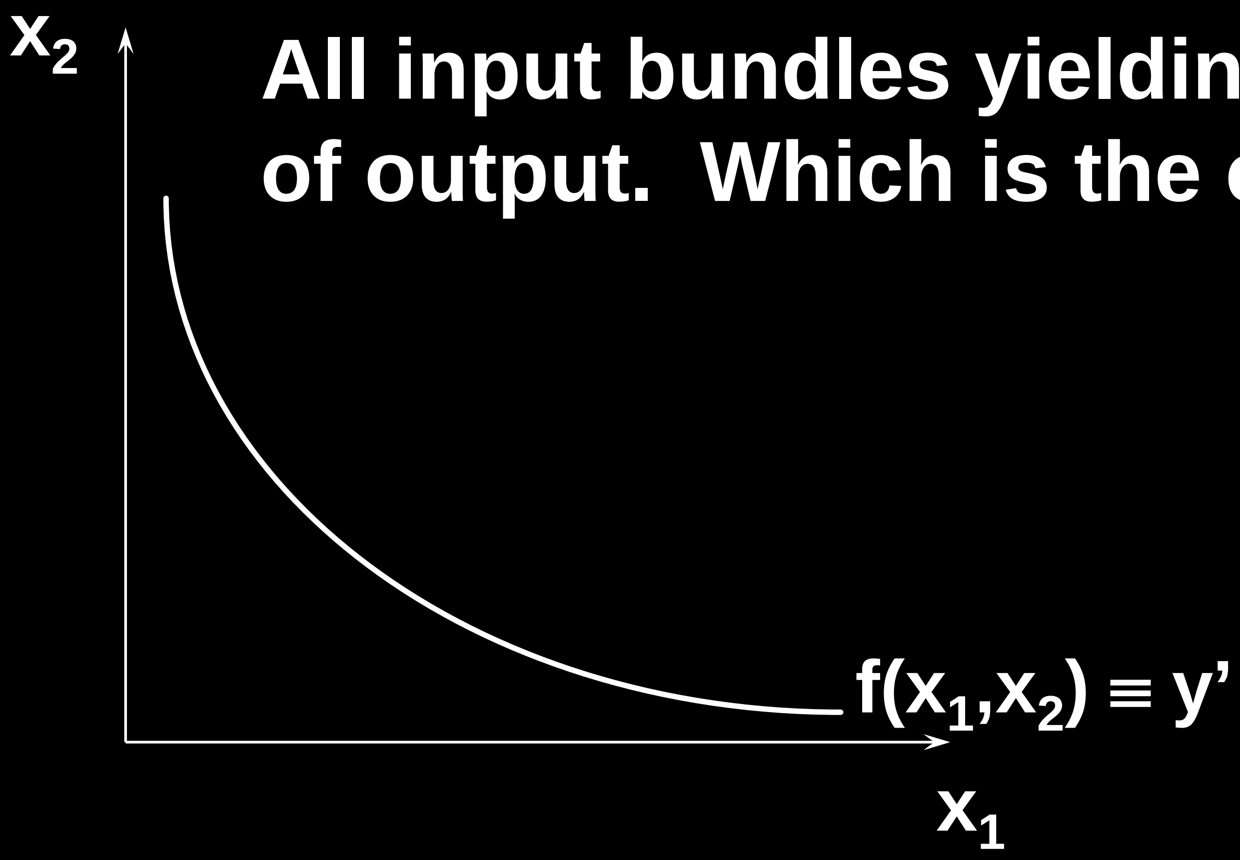
Iso-cost Lines



纵截距是c/ω_2;更高的等成本线对应着更高的成本

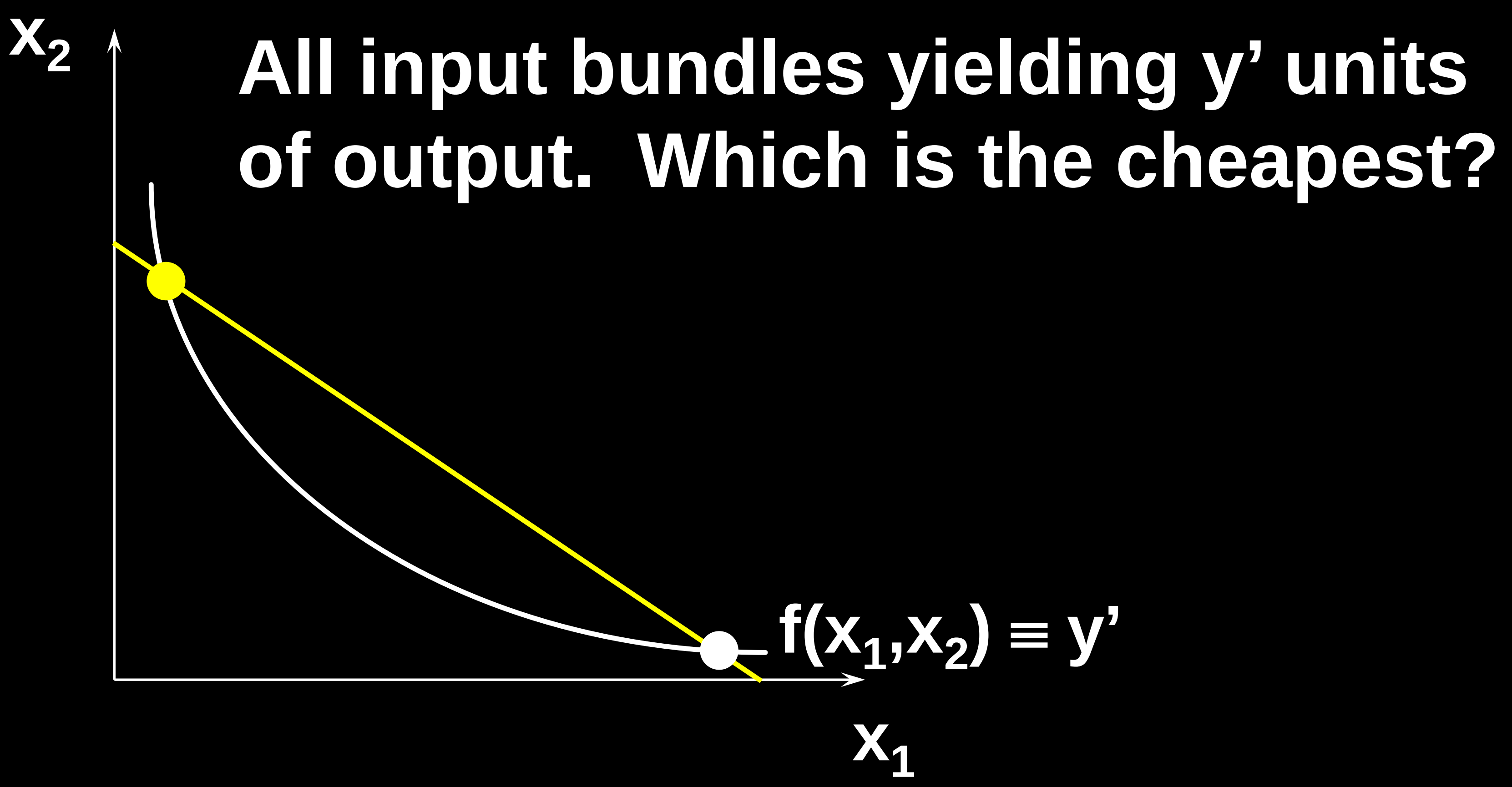
The y'-Output Unit Isoquant

All input bundles yielding y' units of output. Which is the cheapest?



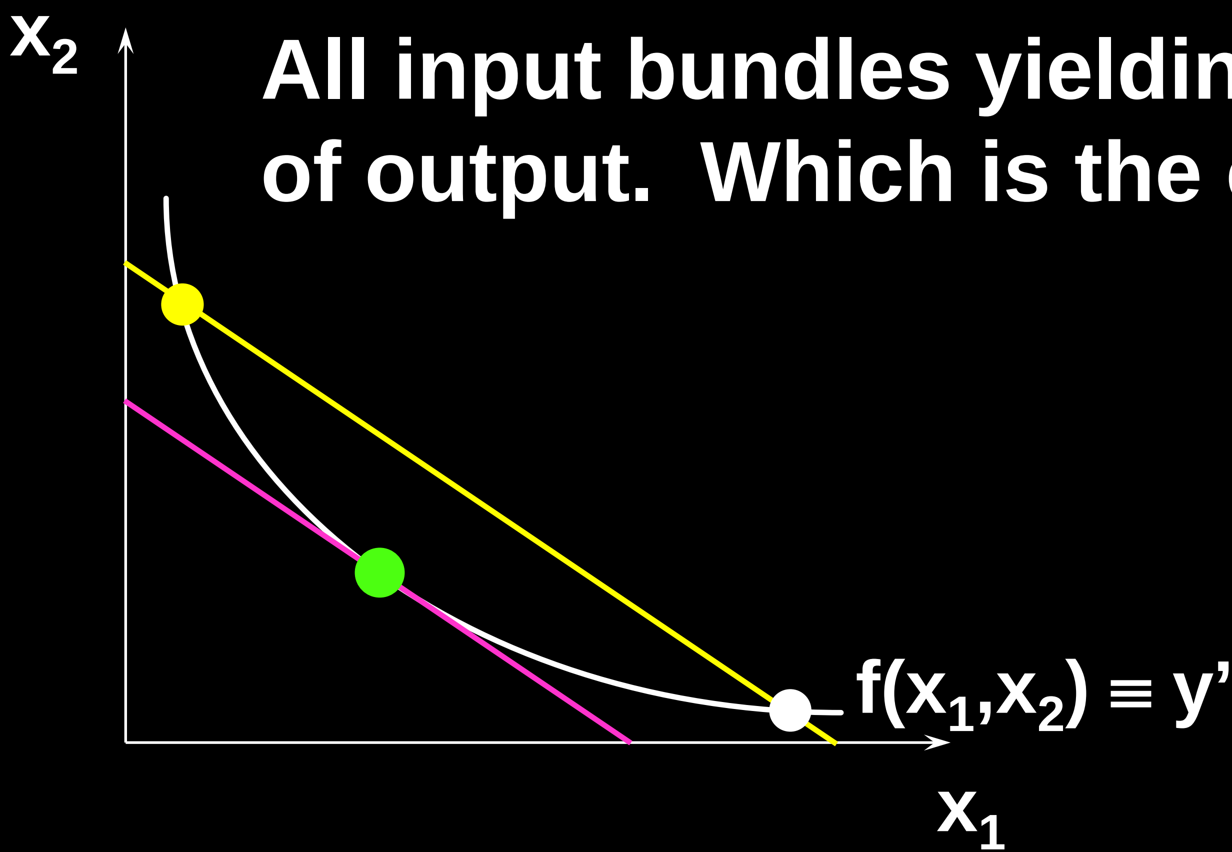
y-Isoquant 产出水平为y的等产量线

The Cost-Minimization Problem



The Cost-Minimization Problem

All input bundles yielding y’ units of output. Which is the cheapest?



The Cost-Minimization Problem

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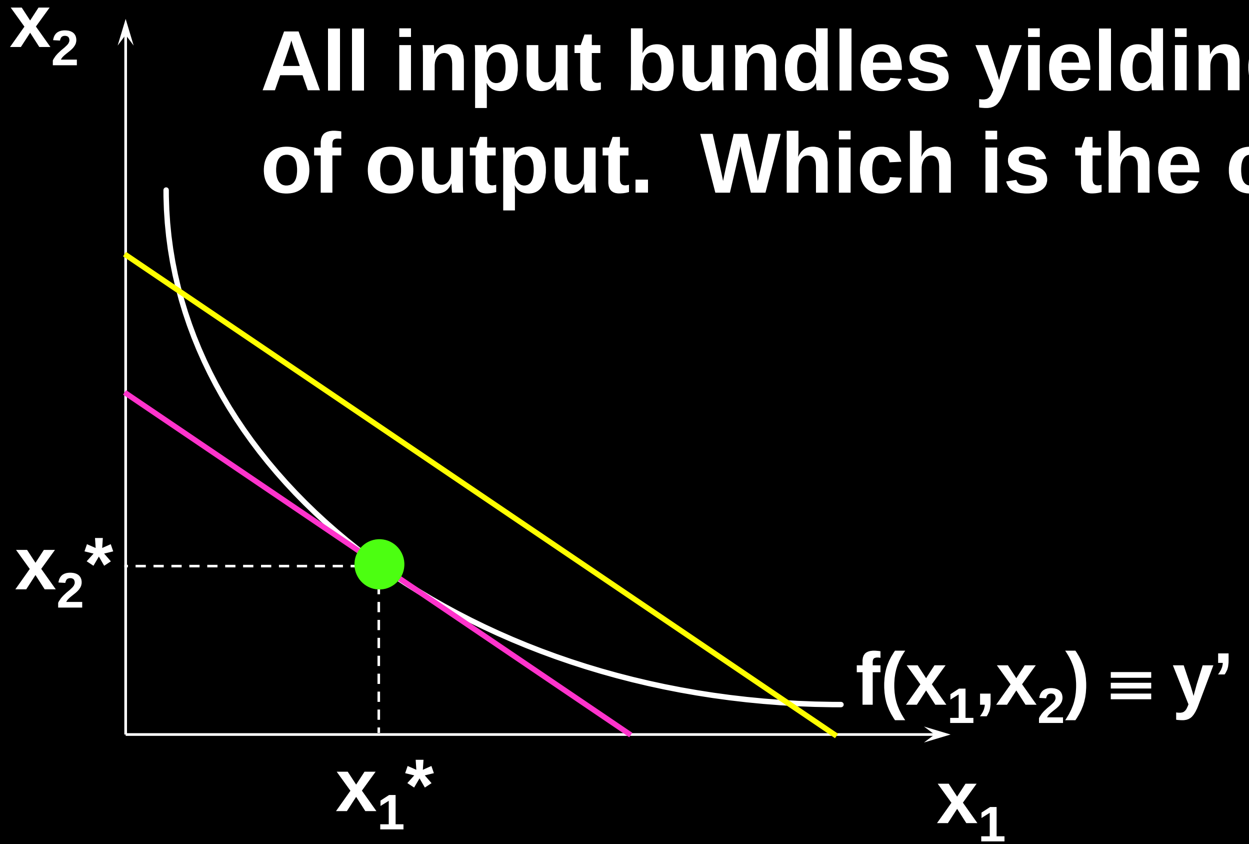
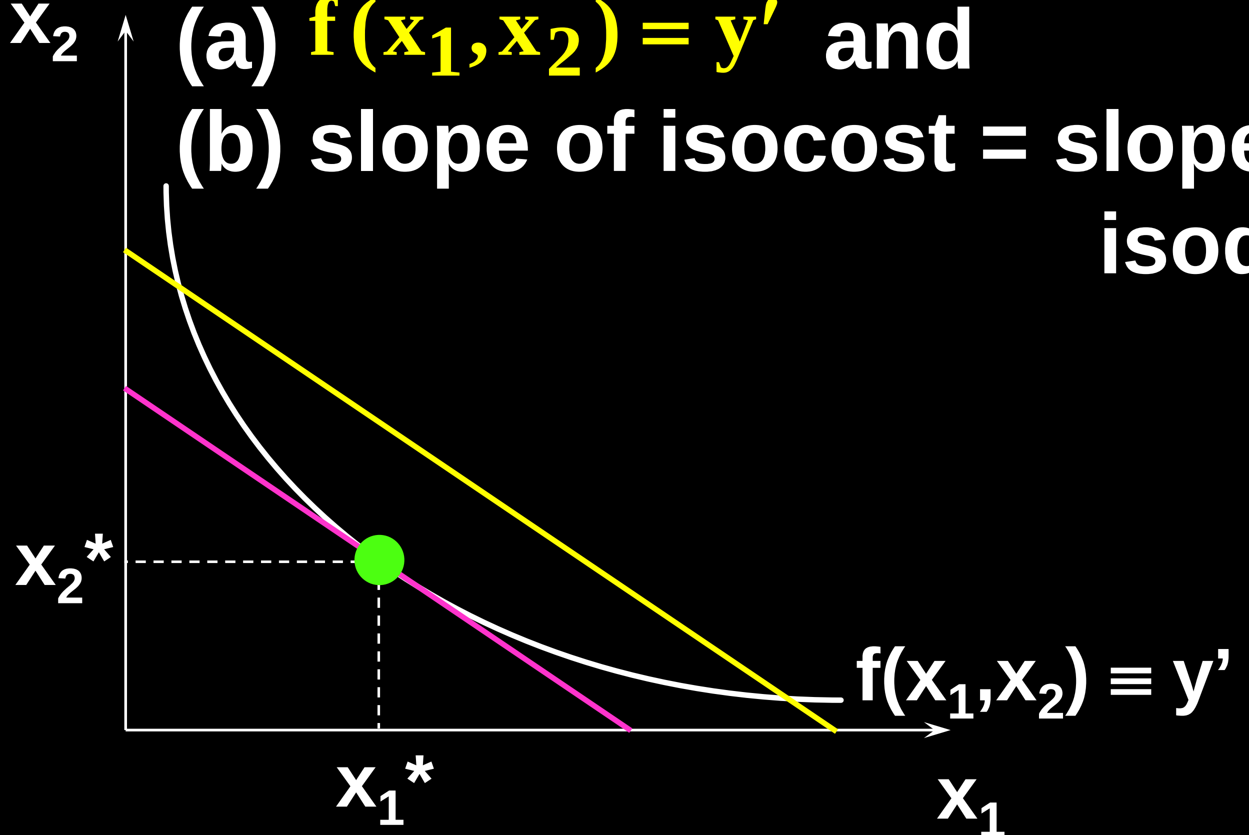


The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ and

(b) slope of isocost = slope of isoquant



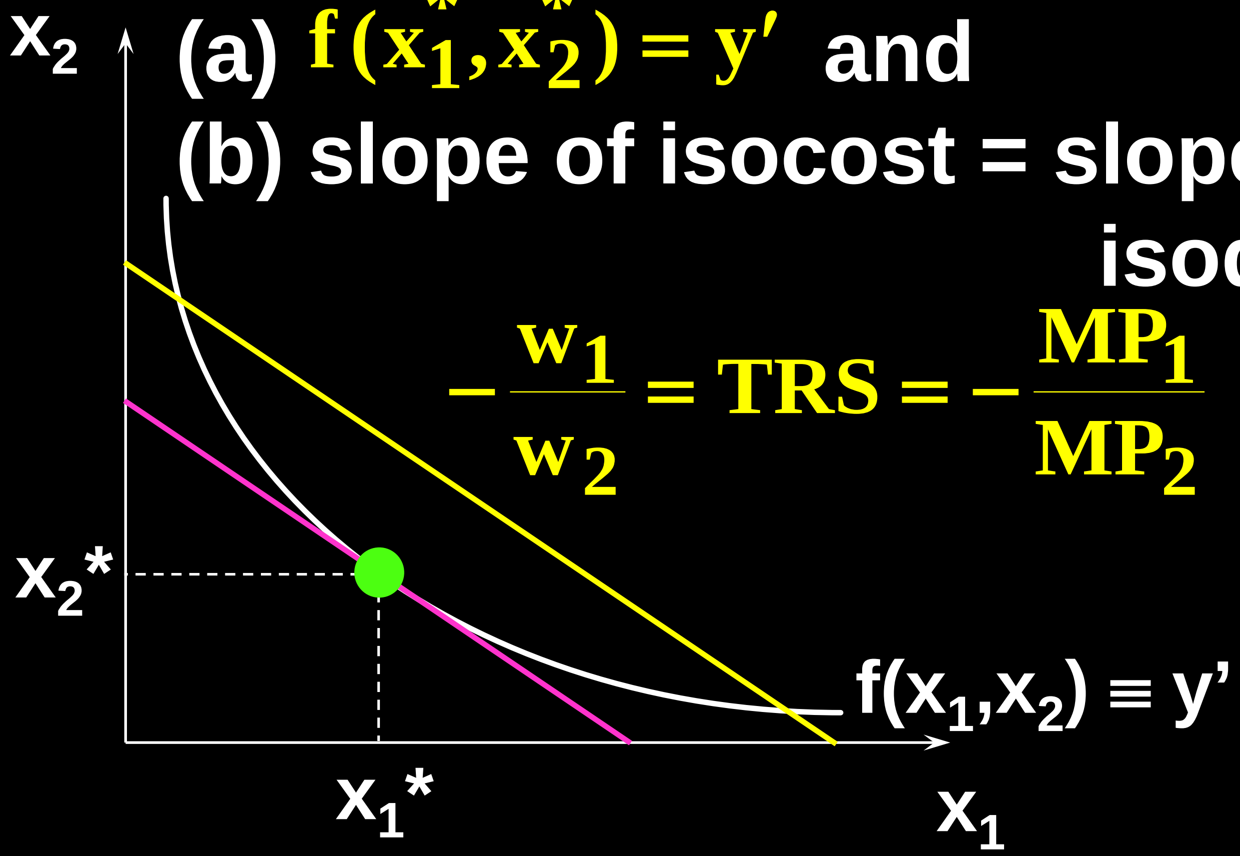
The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ and

(b) slope of isocost = slope of isoquant; i.e.

$$-\frac{w_1}{w_2} = TRS = -\frac{MP_1}{MP_2} \text{ at } (x_1^*, x_2^*).$$



A Cobb-Douglas Example of Cost Minimization

- A firm's Cobb-Douglas production function is

$$\mathbf{f(x_1, x_2) = x_1^{1/3} x_2^{1/3}}$$

- Input prices are w_1 and w_2.
- What are the firm's **conditional input demand functions**?

A Cobb-Douglas Example of Cost Minimization

At the input bundle (x_1*,x_2*) which minimizes the cost of producing y output units:

(a)

$$y = x_1^{1/3} x_2^{1/3}$$

and

(b)

$$-\frac{\omega_1}{\omega_2} = \text{TRS} = -\frac{MP_1}{MP_2}$$

$$-\frac{\omega_1}{\omega_2} = -\frac{1/3\mathrm{x}_1^{-2/3}\mathrm{x}_2^{1/3}}{1/3\mathrm{x}_1^{1/3}\mathrm{x}_2^{-2/3}} = -\frac{\mathrm{x}_2}{\mathrm{x}_1}$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{1/3}$ **(b)** $\dfrac{\omega_1}{\omega_2} = \dfrac{x_2^*}{x_1^*}$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{1/3}$ **(b)** $\dfrac{\omega_1}{\omega_2} = \dfrac{x_2^*}{x_1^*}$

From (b), $x_2^* = \dfrac{\omega_1}{\omega_2} x_1^*$

A Cobb-Douglas Example of Cost Minimization

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From (b), $x_2^* = \dfrac{\omega_1}{\omega_2} x_1^*$

Substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{\omega_1}{\omega_2} x_1^*\right)^{1/3} = \left(\frac{\omega_1}{\omega_2}\right)^{1/3}(x_1^*)^{2/3}$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{1/3}$ **(b)** $\dfrac{\omega_1}{\omega_2} = \dfrac{x_2^*}{x_1^*}$

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Substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{\omega_1}{\omega_2}x_1^*\right)^{1/3} = \left(\frac{\omega_1}{\omega_2}\right)^{1/3}(x_1^*)^{2/3}$$

$x_1^*(\omega_1, \omega_2, y) = \left(\dfrac{\omega_2}{\omega_1}\right)^{1/2} y^{3/2}$ **is the firm's conditional demand for input 1**

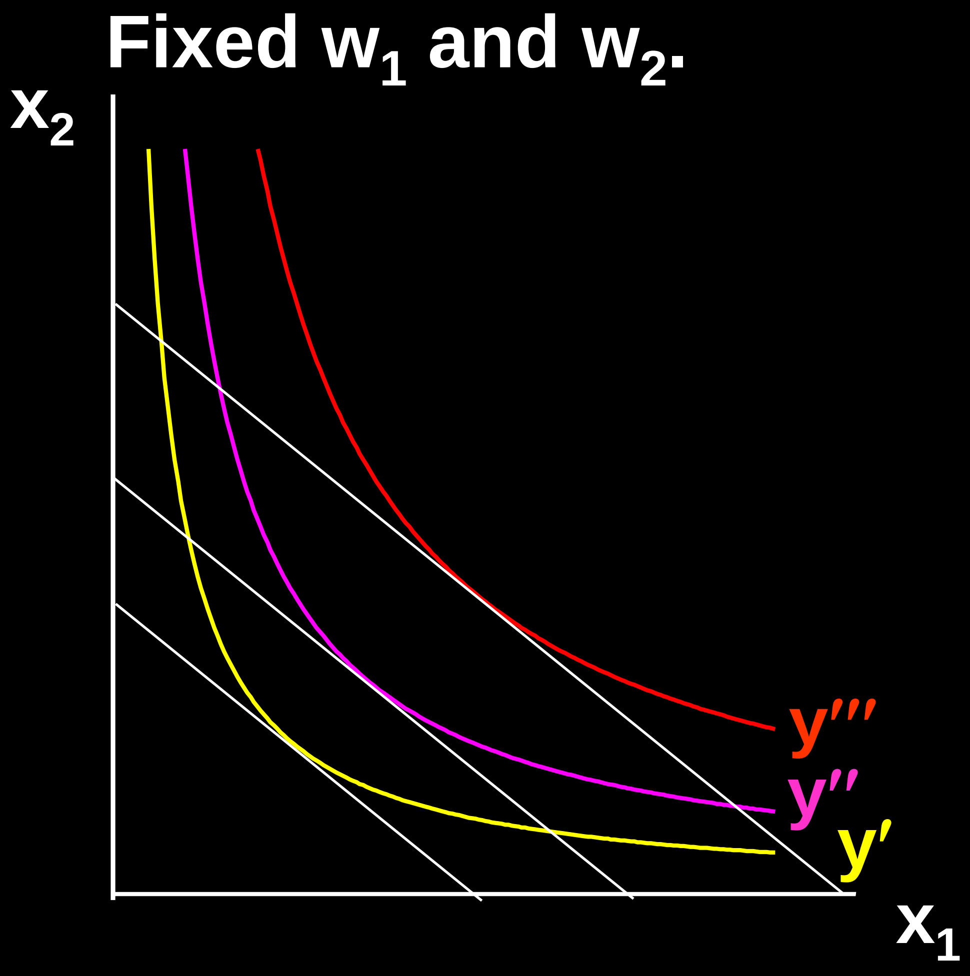
A Cobb-Douglas Example of Cost Minimization

Since $\mathbf{x}_2^* = \frac{\omega_1}{\omega_2}\mathbf{x}_1^*$, and

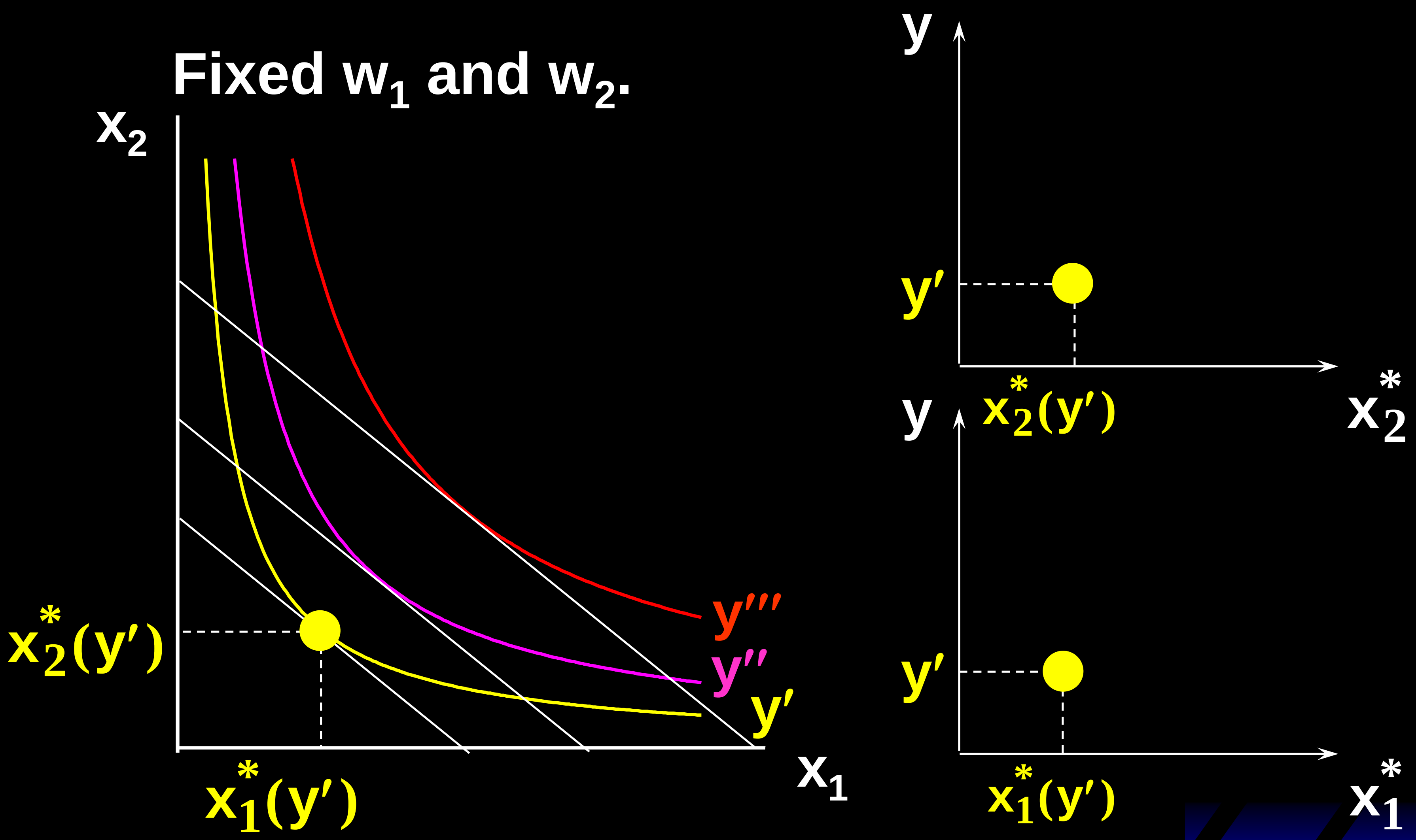
$x_1^*(\omega_1, \omega_2, y) = \left(\frac{\omega_2}{\omega_1}\right)^{1/2} y^{3/2}$ =>

$x_2^*(\omega_1, \omega_2, y) = \left(\frac{\omega_1}{\omega_2}\right)^{1/2} y^{3/2}$ is the firm's conditional demand for input 2

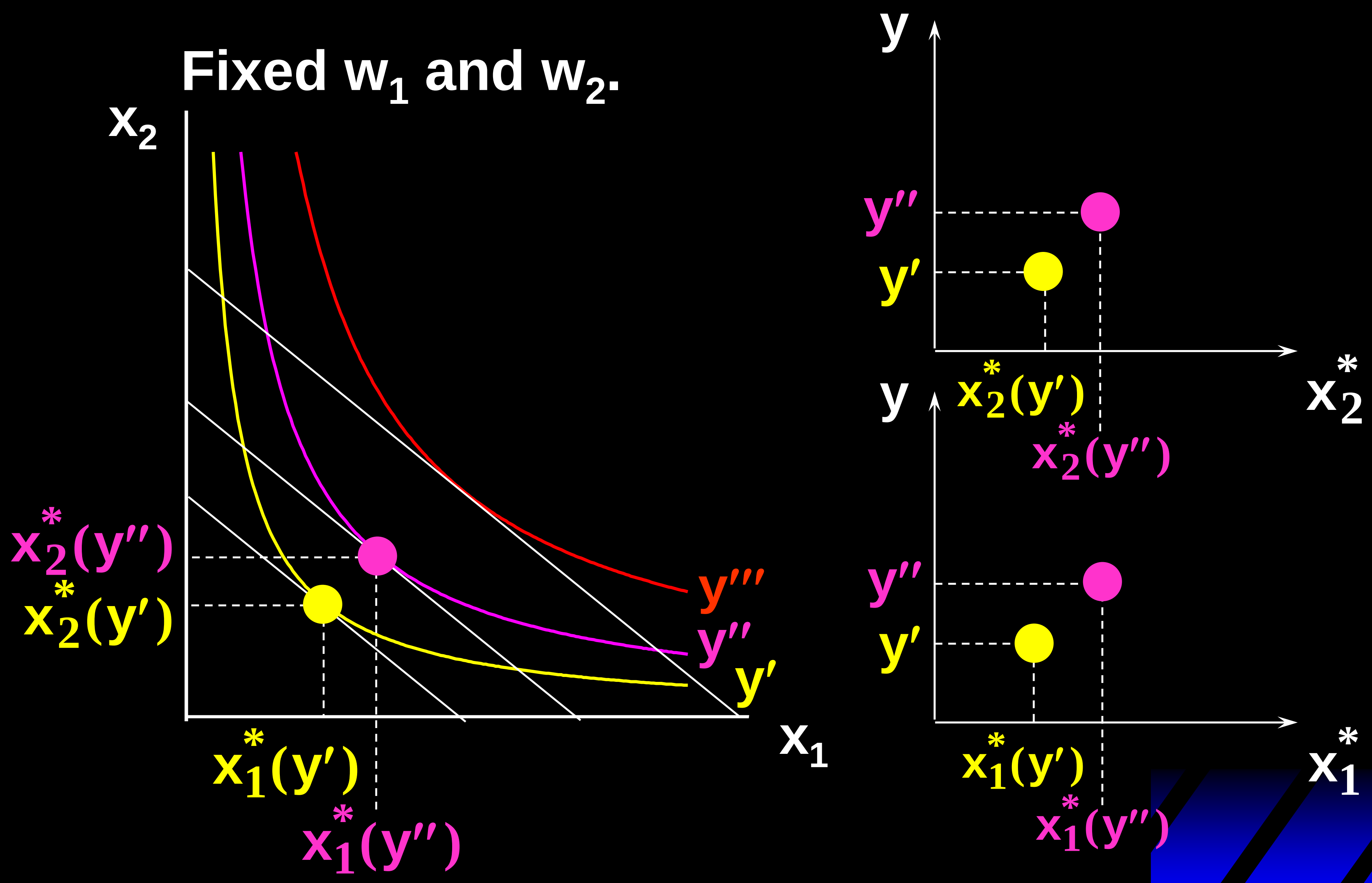
Conditional Input Demand Curves



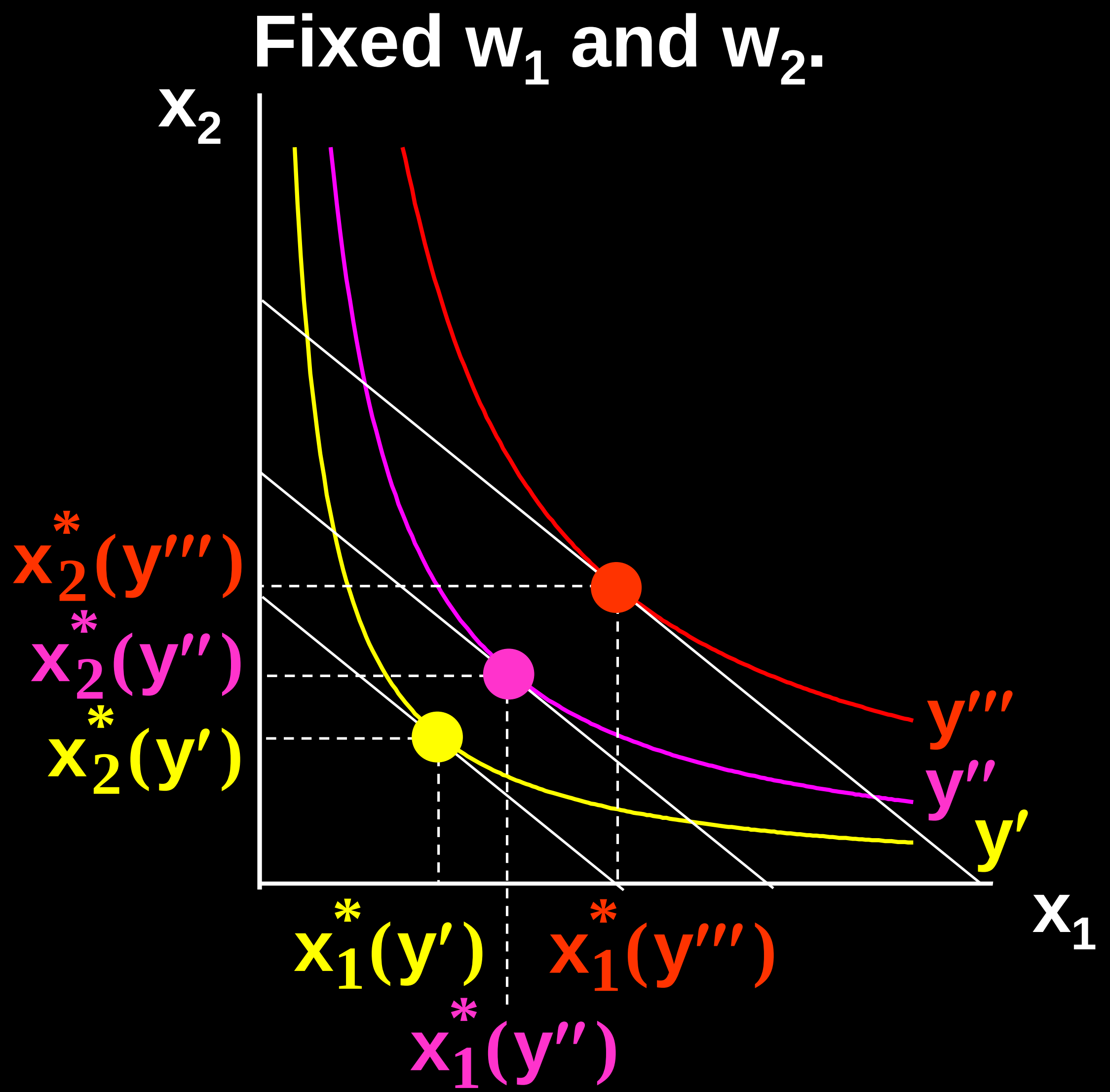
Conditional Input Demand Curves

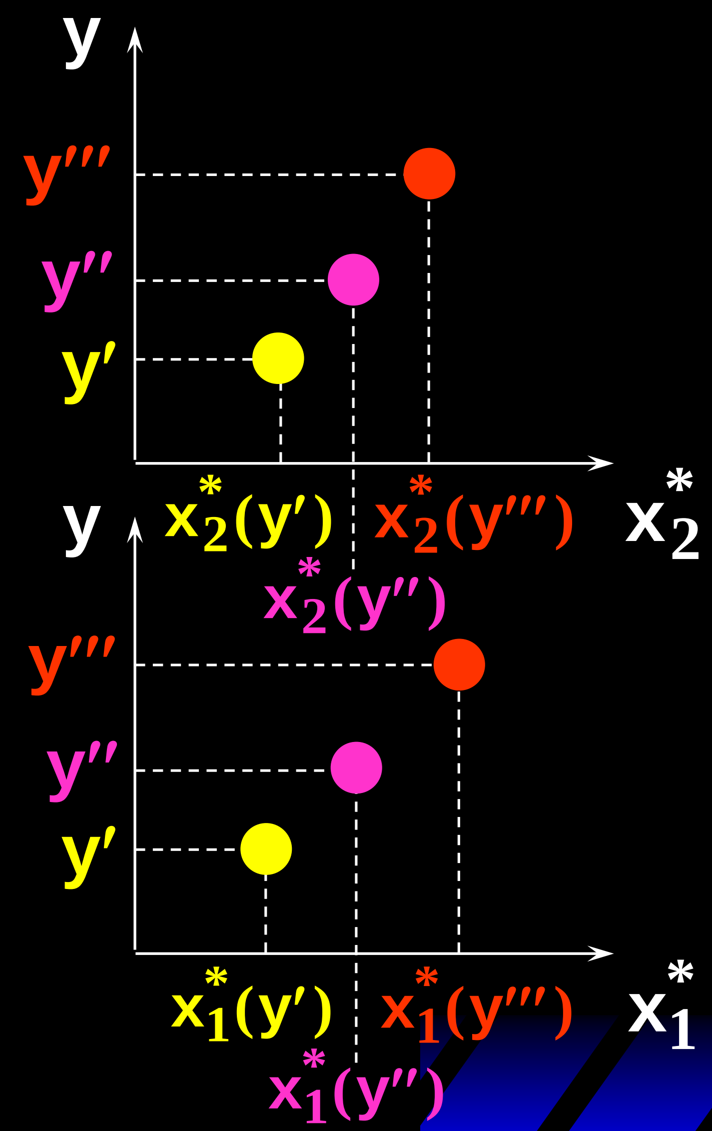


Conditional Input Demand Curves

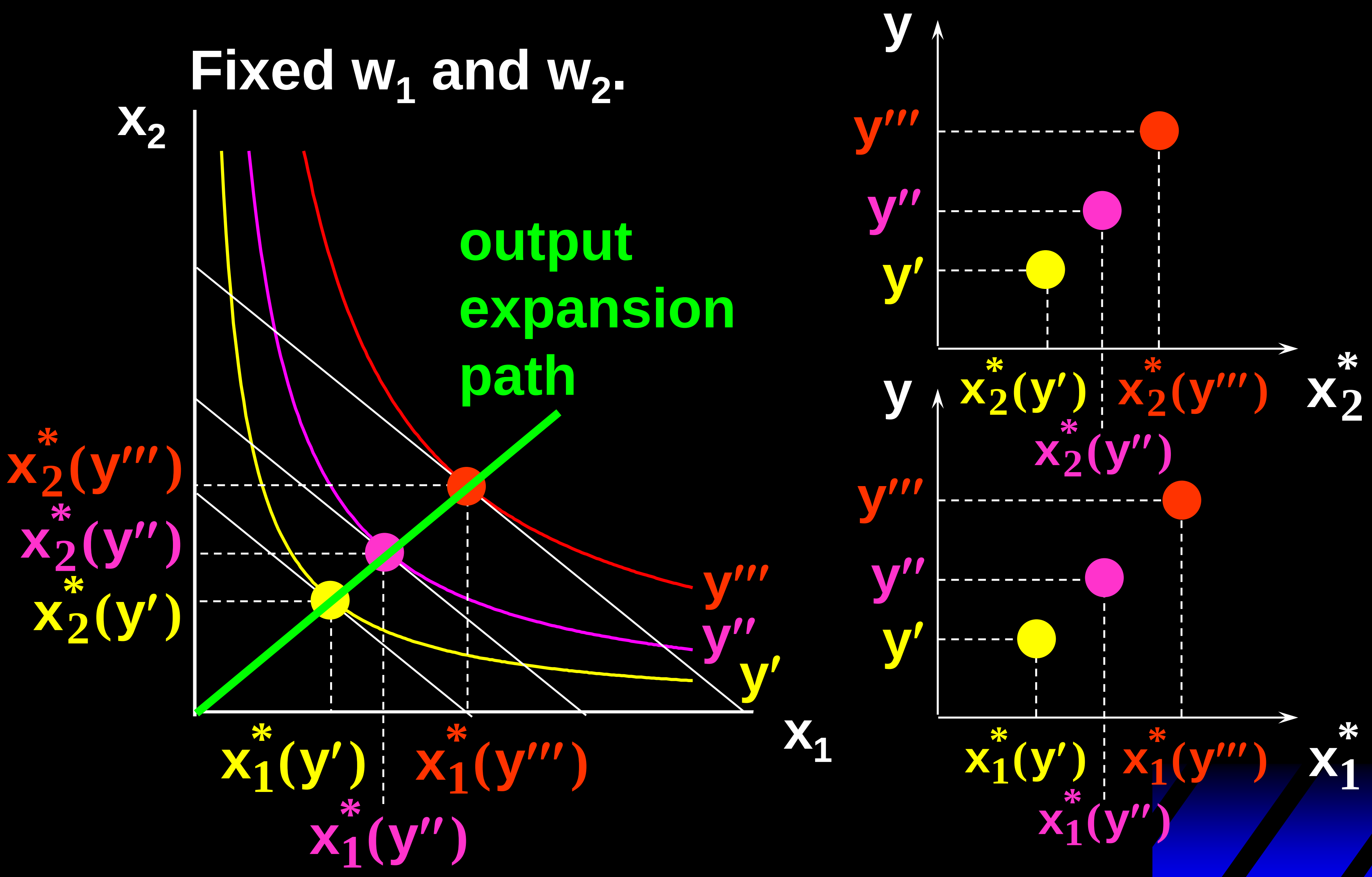


Conditional Input Demand Curves

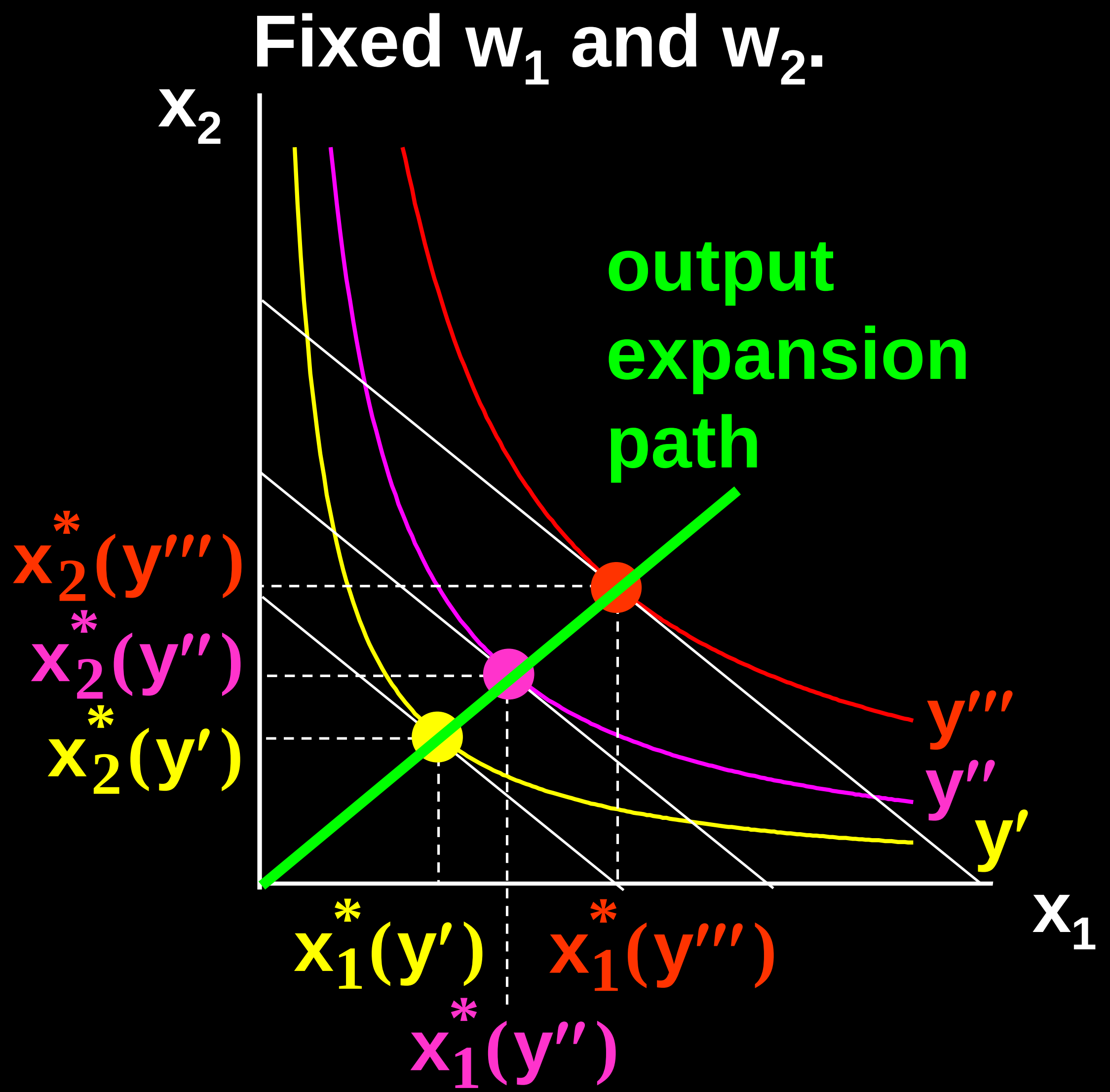


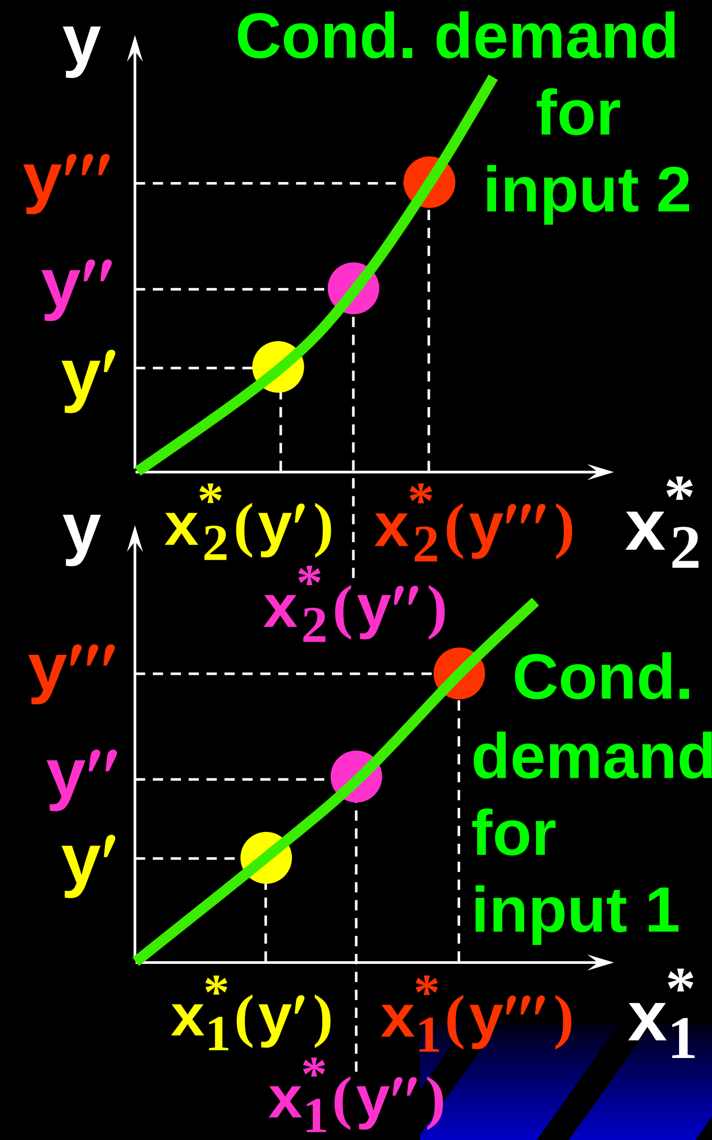


Conditional Input Demand Curves



Conditional Input Demand Curves





Conditional Input Demand Curves

$\forall t > 0$和(x_1, x_2),如果**TRS**在(x_1, x_2)和(tx_1, tx_2)相等，那么产出扩张曲线**(output expansion path)**是一条直线；两种要素的投入比例保持不变。

A Cobb-Douglas Example of Cost Minimization

For the production function

$$\mathbf{f(x_1, x_2) = x_1^{1/3} x_2^{1/3}}$$

the cheapest input bundle yielding **y** output units is

$$x_1^*(\omega_1, \omega_2, y) = \left(\frac{\omega_2}{\omega_1}\right)^{1/2} y^{3/2}$$

$$x_2^*(\omega_1, \omega_2, y) = \left(\frac{\omega_1}{\omega_2}\right)^{1/2} y^{3/2}$$

A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$C(\omega_1, \omega_2, y) = \omega_1 x_1^* + \omega_2 x_2^*$$

$$= \omega_1 \left(\frac{\omega_2}{\omega_1}\right)^{1/2} y^{3/2} + \omega_2 \left(\frac{\omega_1}{\omega_2}\right)^{1/2} y^{3/2}$$

$$= 2(\omega_1 \omega_2)^{\frac{1}{2}} y^{\frac{3}{2}}$$

A Cobb-Douglas Example of Cost Minimization

Given the cost function

$$\mathrm{C}(\omega_1, \omega_2, y) = 2(\omega_1\omega_2)^{\frac{1}{2}} y^{\frac{3}{2}}$$

The firm's profit can be written as

$$\Pi = py - C(y) = py - 2(\omega_1\omega_2)^{1/2} y^{3/2}$$

$$\frac{\partial \Pi}{\partial y} = p - 3(\omega_1\omega_2)^{\frac{1}{2}} y^{\frac{1}{2}} = 0$$

$$y^* = \frac{p^2}{9\omega_1\omega_2}$$

A Cobb-Douglas Example of Cost Minimization

Given the cost function

$$C(\omega_1, \omega_2, y) = 2(\omega_1\omega_2)^{\frac{1}{2}} y^{\frac{3}{2}}$$

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$$y^* = \frac{p^2}{9\omega_1\omega_2}$$

Same as what we found in Lec 11

Profit Max. vs. Cost Min.

- Profit maximization: choosing the optimal production plan (x_1^*, x_2^*, y^*) to maximize profit

$$\max_{(x_1, x_2, y)} \Pi = py - \omega_1 x_1 - \omega_2 x_2$$

- Equivalent to a two-step problem
  – Given y, choosing the optimal input bundle to minimize the costs
  – Choosing the optimal y to maximize the profit

Profit Max. vs. Cost Min.

Equivalent to a two-step problem

- **(1) Given y, choosing the optimal input bundle to minimize the costs**

$$\min_{x_1, x_2} C = \omega_1 x_1 + \omega_2 x_2$$

$$\text{s.t. } f(x_1, x_2) = y$$

The minimized C^* is a function of y, known as the cost function $C(y)$

Two-step problem

- Equivalent to a two-step problem
  - (2) Given the cost function, choosing the optimal y to maximize the profit

$$\max_{y} \Pi = py - C(y)$$

A Perfect Complements Example of Cost Minimization

The firm's production function is

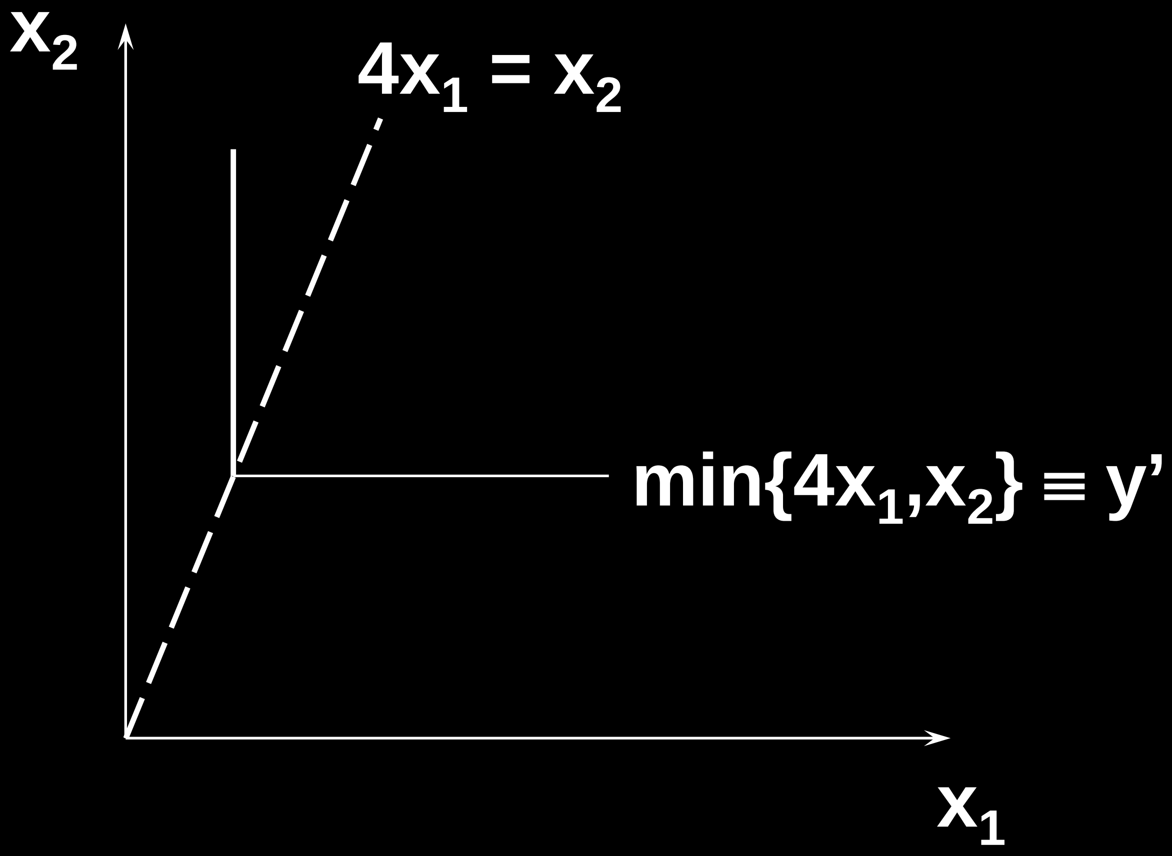
$$y = \min\{4x_1, x_2\}.$$

Input prices w_1 and w_2 are given.

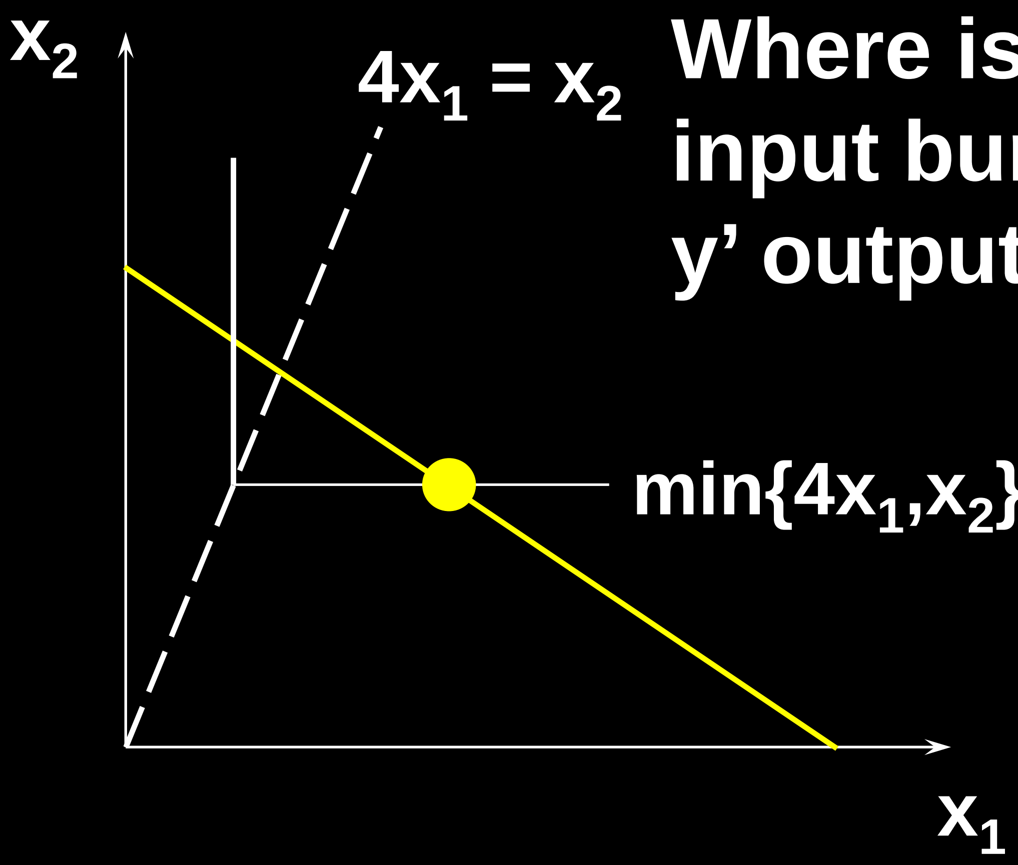
What are the firm's **conditional demands for inputs** 1 and 2?

What is the firm's **total cost** function?

A Perfect Complements Example of Cost Minimization

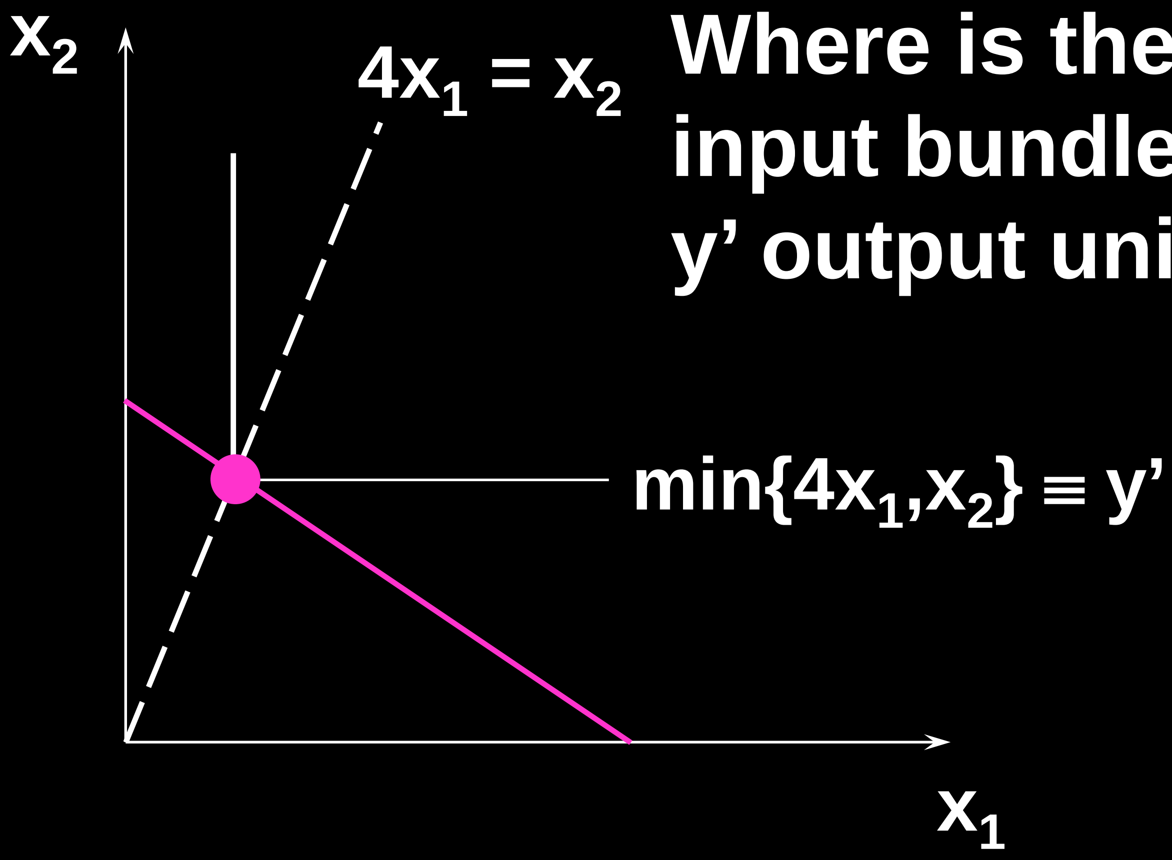


A Perfect Complements Example of Cost Minimization



Where is the least costly input bundle yielding y' output units?

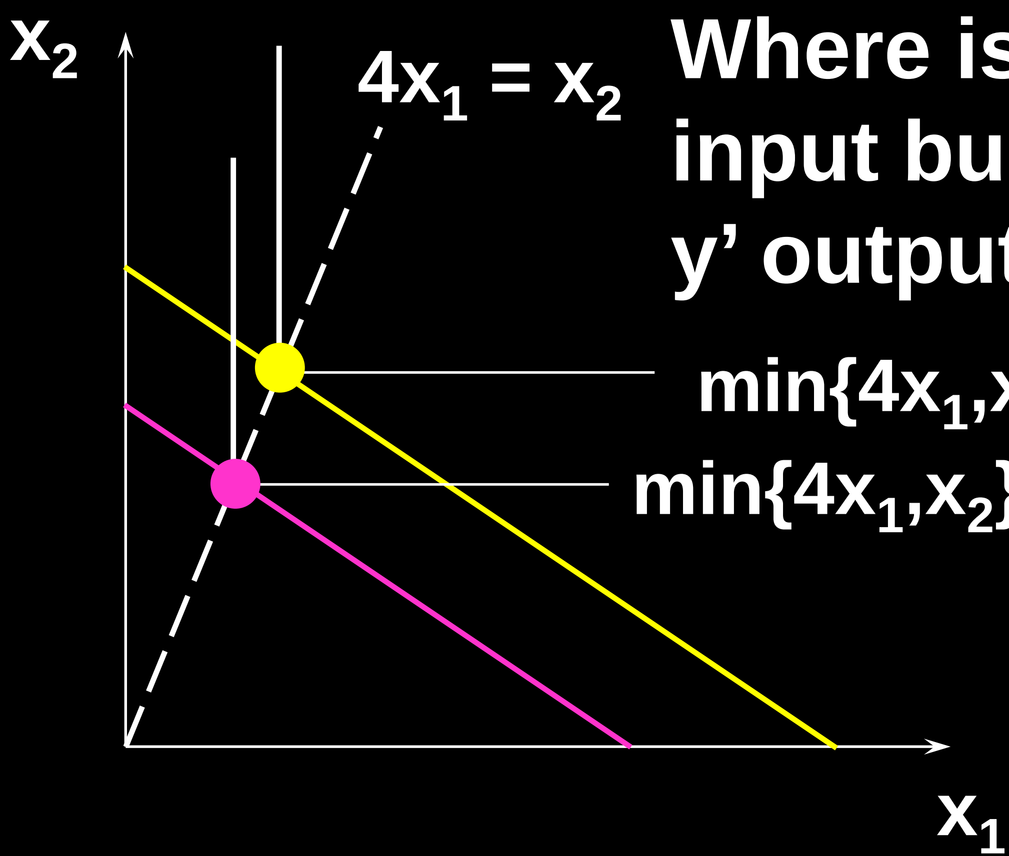
A Perfect Complements Example of Cost Minimization



Where is the least costly input bundle yielding y' output units?

A Perfect Complements Example of Cost Minimization

Where is the least costly input bundle yielding y' output units?



A Perfect Complements Example of Cost Minimization

- The firm's production function is

$$y = \min\{4x_1, x_2\}$$

and the conditional input demands are

$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

A Perfect Complements Example of Cost Minimization

The firm's production function is

$$y = \min\{4x_1, x_2\}$$

and the conditional input demands are

$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

$$= w_1 \frac{y}{4} + w_2 y = \left(\frac{w_1}{4} + w_2\right) y.$$

Average Total Production Costs

For positive output levels y, a firm's average total cost of producing y units is

$$AC(w_1, w_2, y) = \frac{c(w_1, w_2, y)}{y}.$$

平均成本

Returns-to-Scale and Av. Total Costs

- The returns-to-scale properties of a firm’s technology determine how **average production costs** change with output level（规模报酬决定平均成本如何随产出变化而变化).

Returns-to-Scale and Av. Total Costs

- The returns-to-scale properties of a firm's technology determine how **average production costs** change with output level（规模报酬决定平均成本如何随产出变化而变化）.
- Our firm is presently producing y' output units.
- How does the firm's average production cost change if it instead produces 2y' units of output?

Constant Returns-to-Scale and Average Total Costs

If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.

Total production cost doubles.

当规模报酬不变时，如要使产量加倍，仅需使要素投入量加倍，总成本加倍。

Constant Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits **constant** returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.
- Total production cost doubles.
- Average production cost does not change.

规模报酬不变时，平均成本不随产出变化而变化。

Increasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits **increasing** returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.
- Total production cost less than doubles.

当规模报酬递增时，如果要使产量加倍，要素投入量的增加小于2倍即可，总成本的增长小于2倍。

Increasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits **increasing** returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.
- Total production cost less than doubles.
- Average production cost **decreases**.

规模报酬递增时，平均成本随产出的上升而下降。

Decreasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits **decreasing** returns-to-scale then doubling its output level from y' to 2y' requires more than doubling all input levels.
- Total production cost more than doubles.

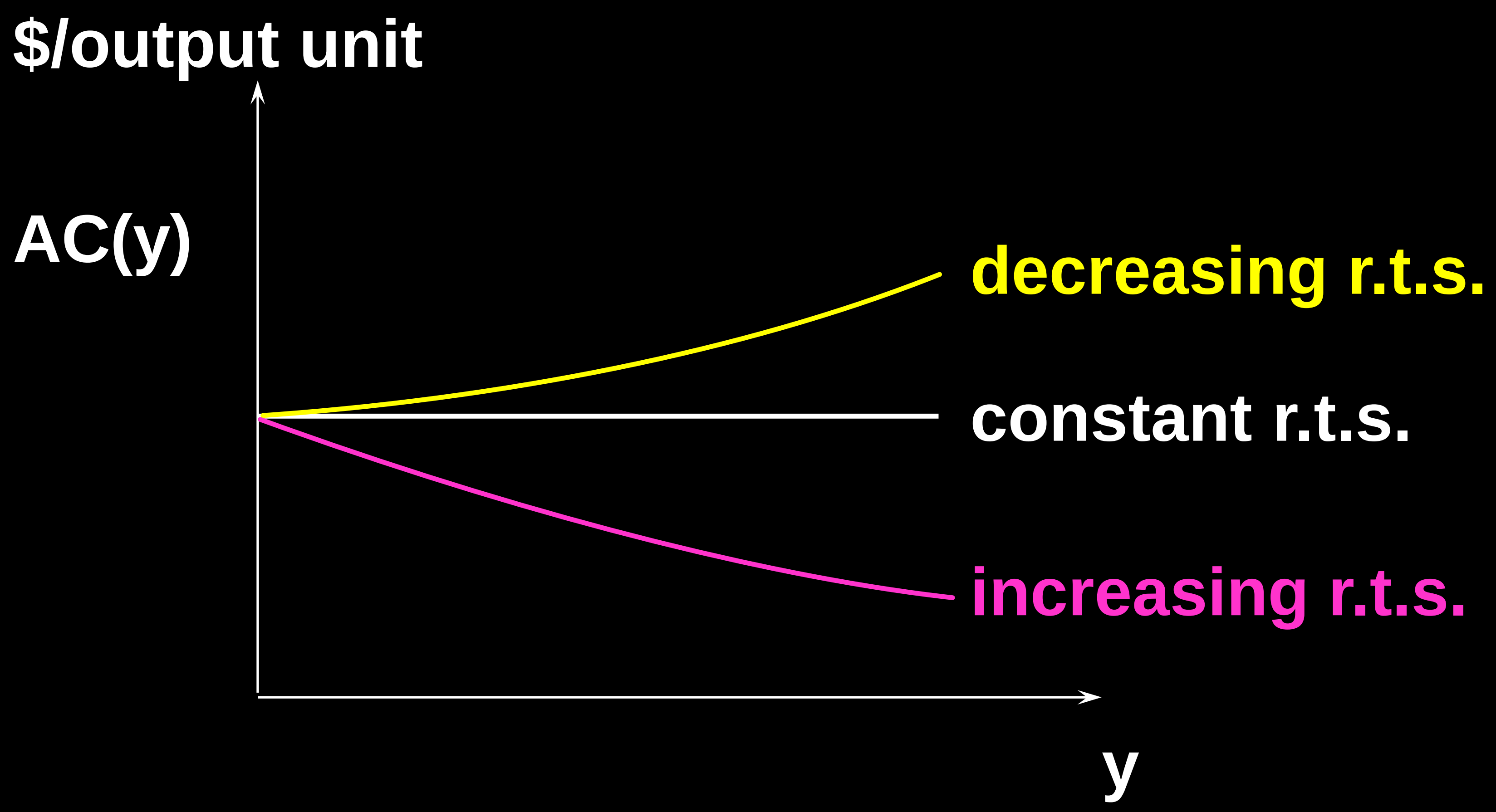
当规模报酬**递减**时，如要使产量加倍，需使要素投入量增加2倍以上，总成本增长2倍以上。

Decreasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits **decreasing** returns-to-scale then doubling its output level from y' to 2y' requires more than doubling all input levels.
- Total production cost more than doubles.
- Average production cost **increases**.

规模报酬递减时，平均成本随产出的上升而上升。

Returns-to-Scale and Av. Total Costs

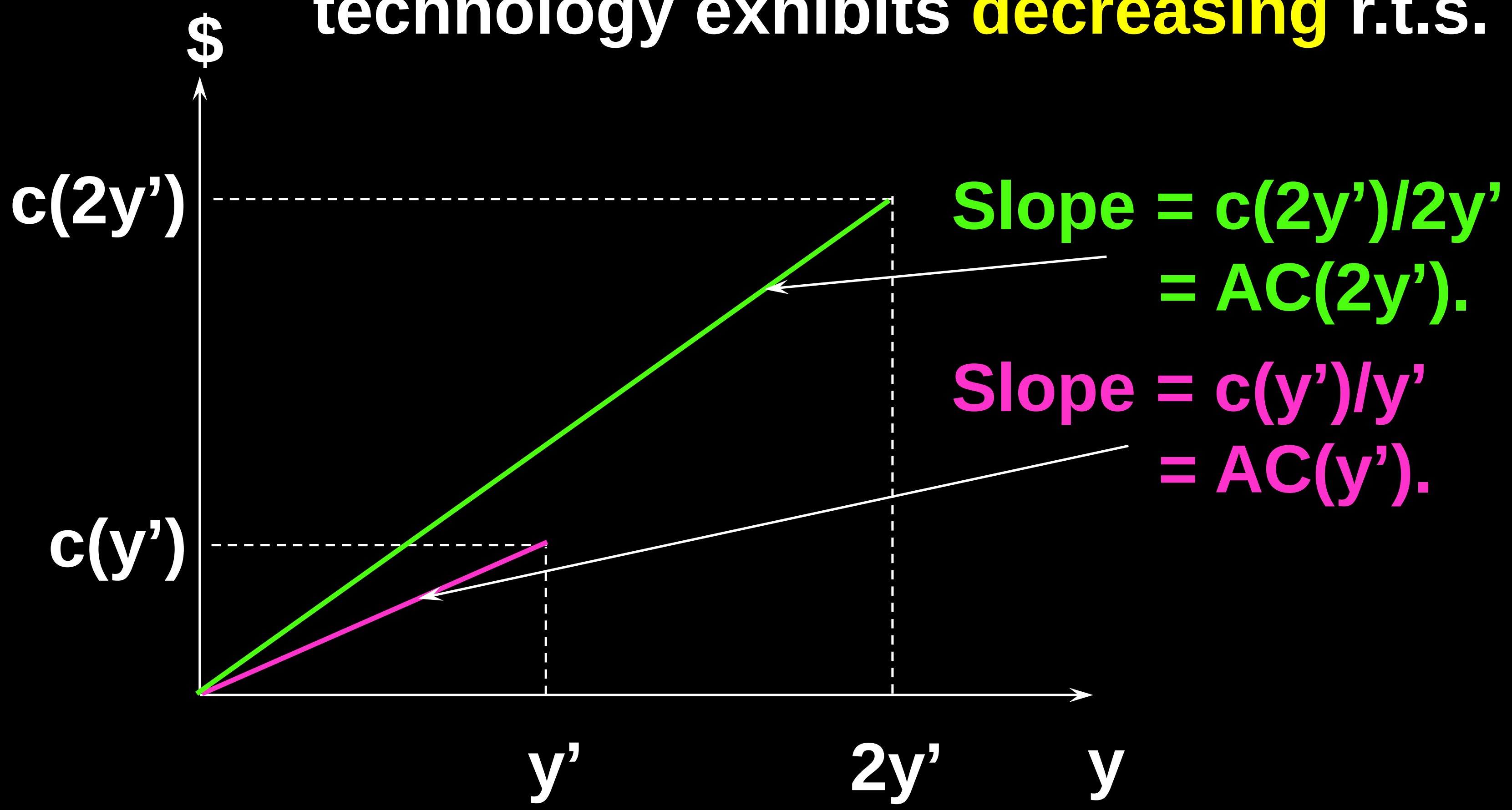


Returns-to-Scale and Total Costs

What does this imply for the shapes of total cost functions?

Returns-to-Scale and Total Costs

Av. cost increases with y if the firm's technology exhibits decreasing r.t.s.



Returns-to-Scale and Total Costs

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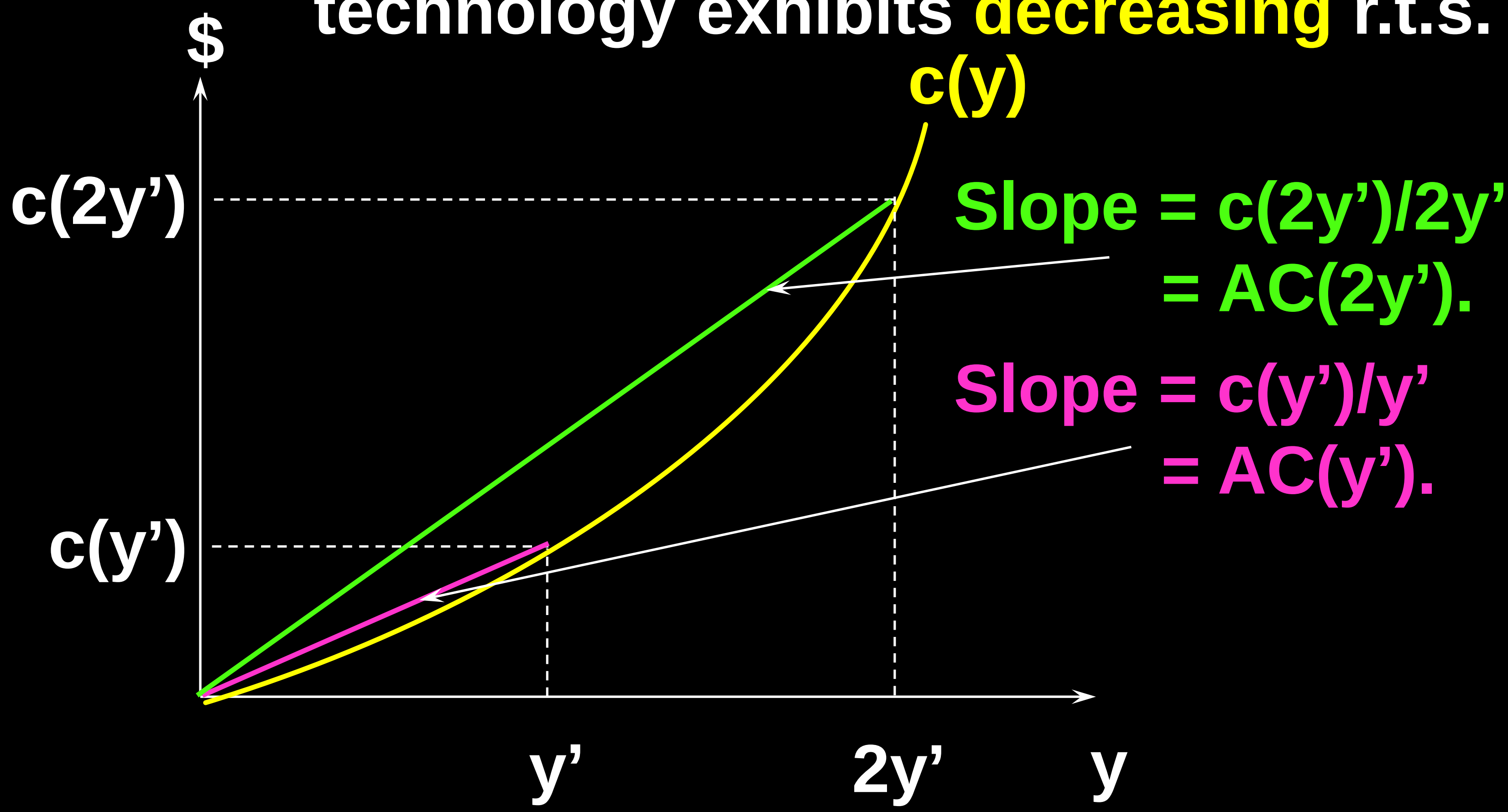
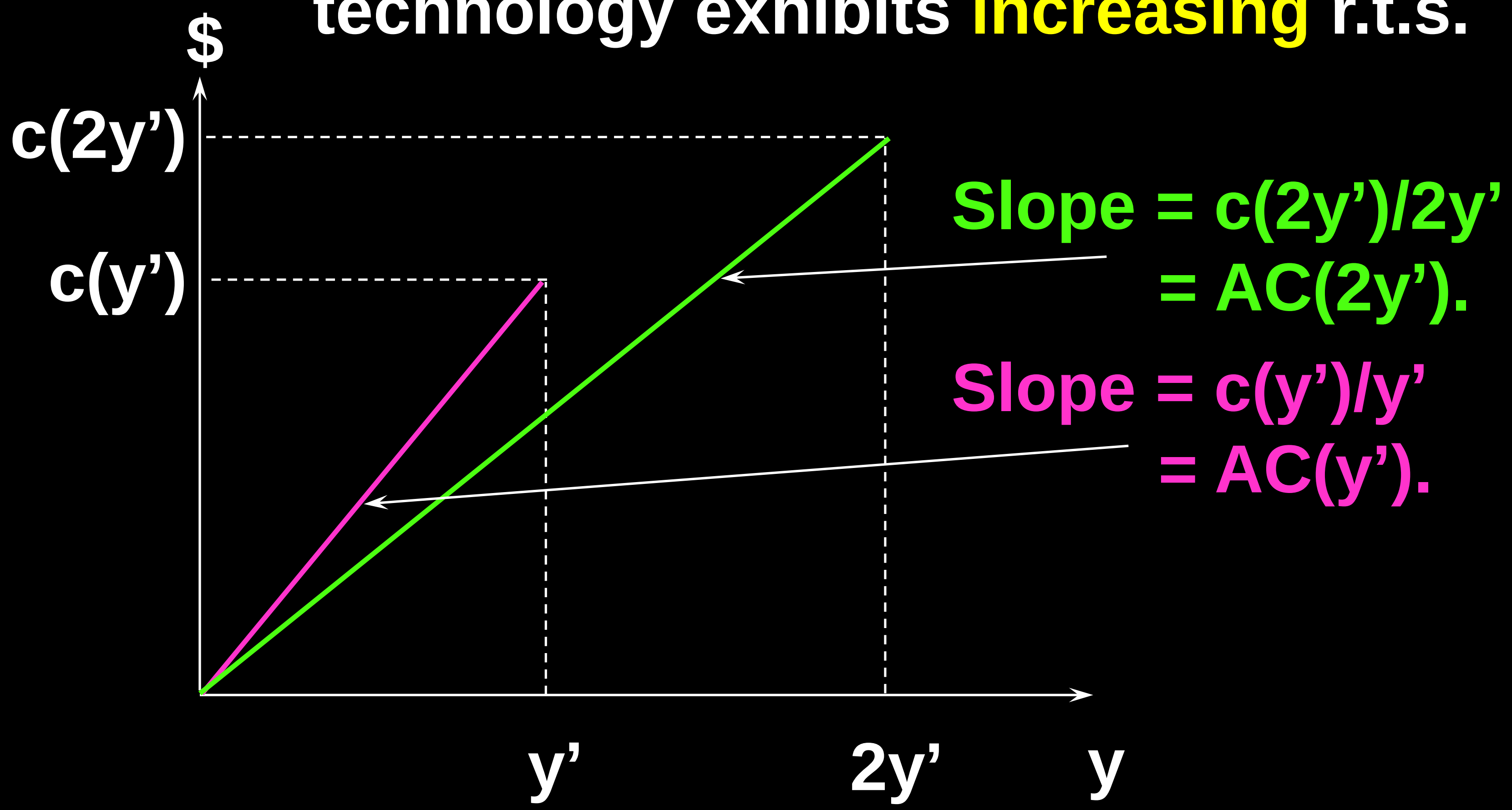


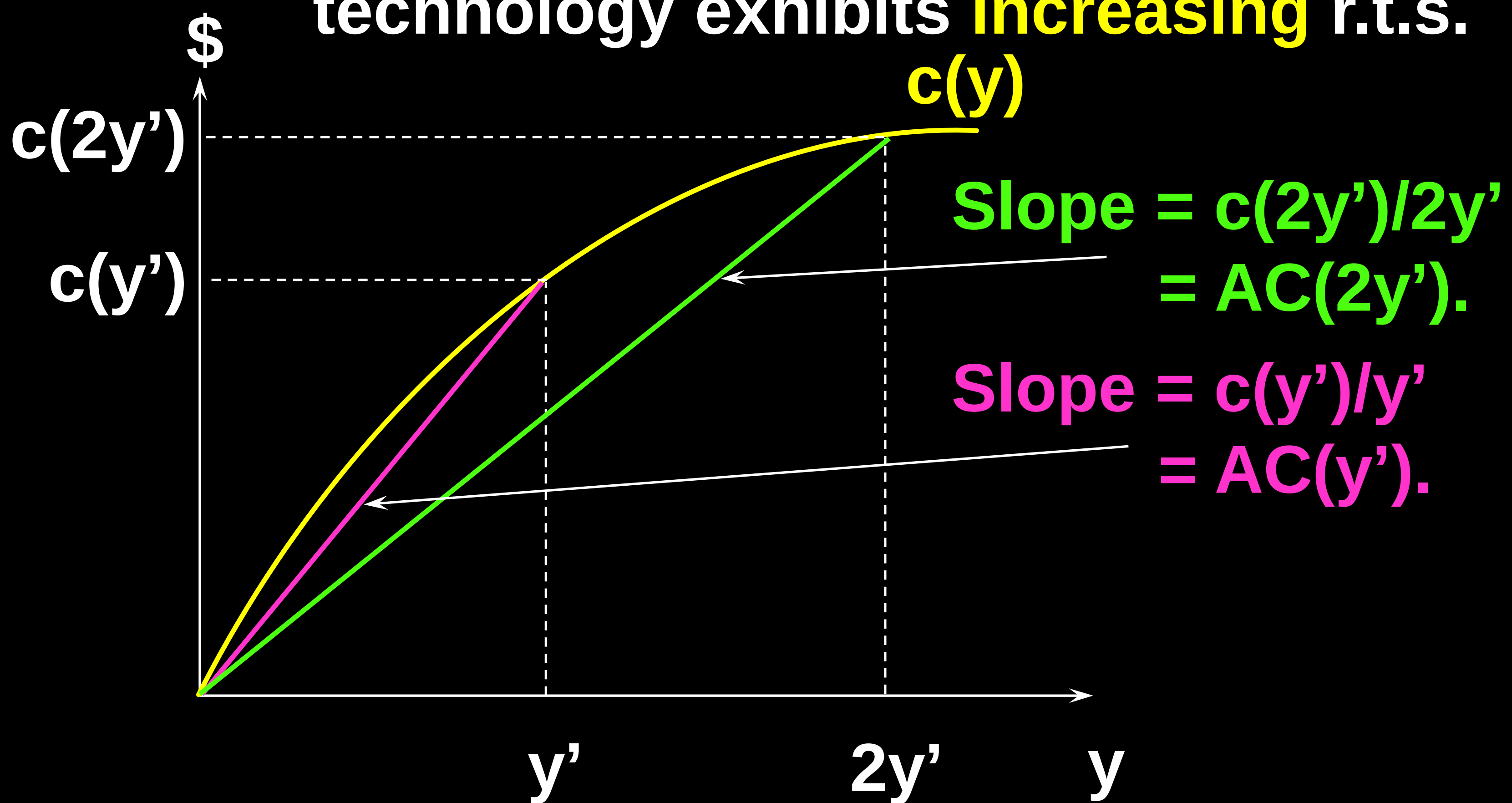
Returns-to-Scale and Total Costs

Av. cost decreases with y if the firm's technology exhibits **increasing** r.t.s.



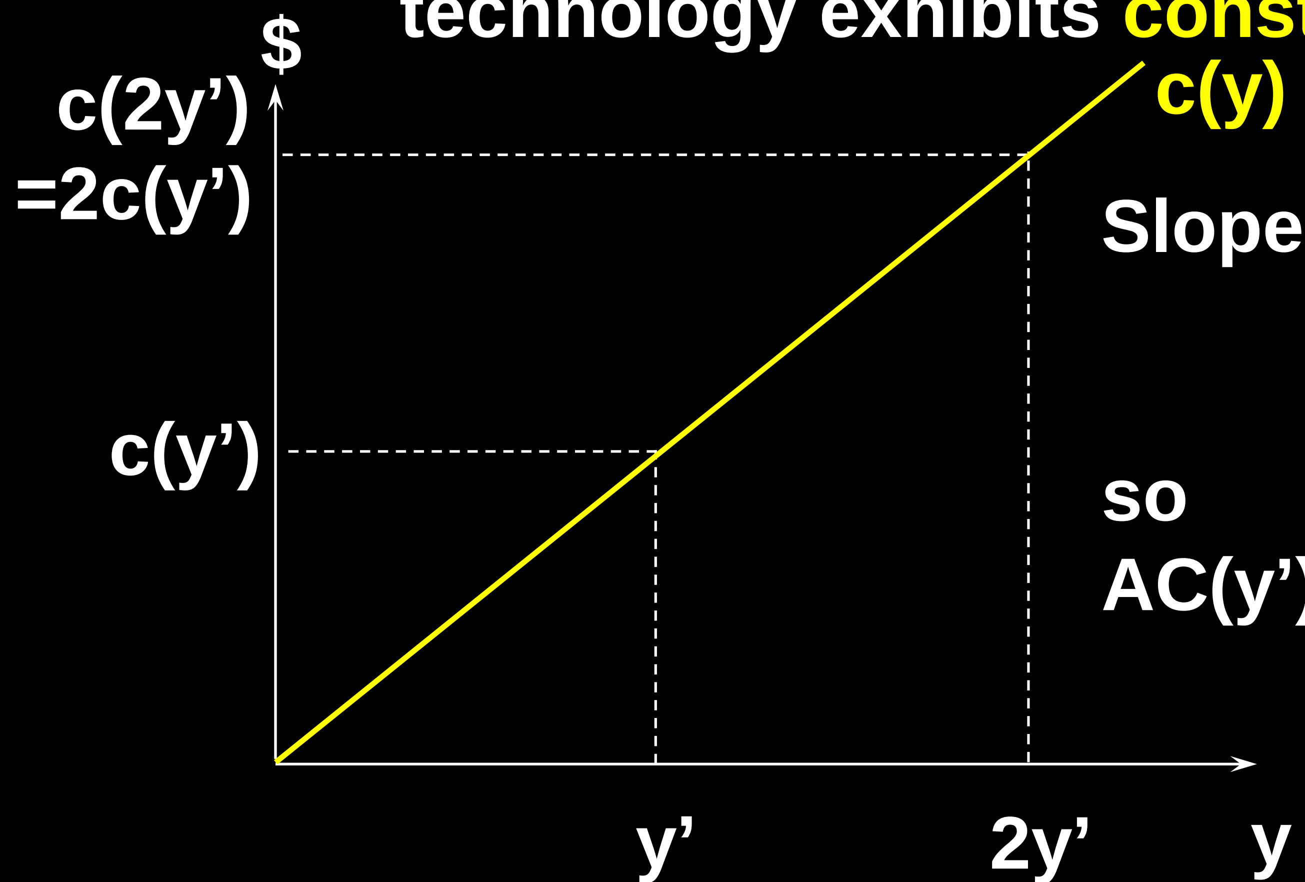
Returns-to-Scale and Total Costs

Av. cost decreases with y if the firm's technology exhibits increasing r.t.s.



Returns-to-Scale and Total Costs

Av. cost is constant when the firm's technology exhibits constant r.t.s.



Slope = c(2y')/2y'
= 2c(y')/2y'
= c(y')/y'
so
AC(y') = AC(2y').

A Perfect Complements Example of Cost Minimization

- The firm's production function is

$$y = \min\{4x_1, x_2\}$$

- The firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

$$= w_1 \frac{y}{4} + w_2 y = \left(\frac{w_1}{4} + w_2\right) y.$$

A Perfect Complements Example of Cost Minimization

The firm's production function is

$$y = \min\{4x_1, x_2\}$$

The firm's total cost function is

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$$= w_1 \frac{y}{4} + w_2 y = \left(\frac{w_1}{4} + w_2\right) y.$$

生产函数满足规模报酬不变，平均成本不变。

A Cobb-Douglas Example of Cost Minimization

For the production function

$$\mathbf{f(x_1, x_2) = x_1^{1/3} x_2^{1/3}}$$

The firm's total cost function is

$$\mathbf{C(\omega_1, \omega_2, y) = 2(\omega_1 \omega_2)^{\frac{1}{2}} y^{\frac{3}{2}}}$$

A Cobb-Douglas Example of Cost Minimization

For the production function

$$f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$$

The firm's total cost function is

$$C(\omega_1, \omega_2, y) = 2(\omega_1 \omega_2)^{\frac{1}{2}} y^{\frac{3}{2}}$$

$$AC(y) = 2(\omega_1 \omega_2)^{\frac{1}{2}} y^{\frac{1}{2}}$$

生产函数满足规模报酬递减，平均成本随y上升而上升。

Short-Run & Long-Run Total Costs

- In the long-run a firm can vary all of its input levels.
- Consider a firm that cannot change its input 2 level from x_2' units.
- How does the short-run total cost of producing y output units compare to the long-run total cost of producing y units of output?

Short-Run & Long-Run Total Costs

The long-run cost-minimization problem is $\min\limits_{x_1,x_2 \geq 0} \; w_1x_1 + w_2x_2$

subject to $f(x_1,x_2) = y.$

The short-run cost-minimization problem is $\min\limits_{x_1 \geq 0} \omega_1 x_1 + \omega_2 \tilde{x}_2$

subject to $f(x_1,\tilde{x}_2) = y$

Short-Run & Long-Run Total Costs

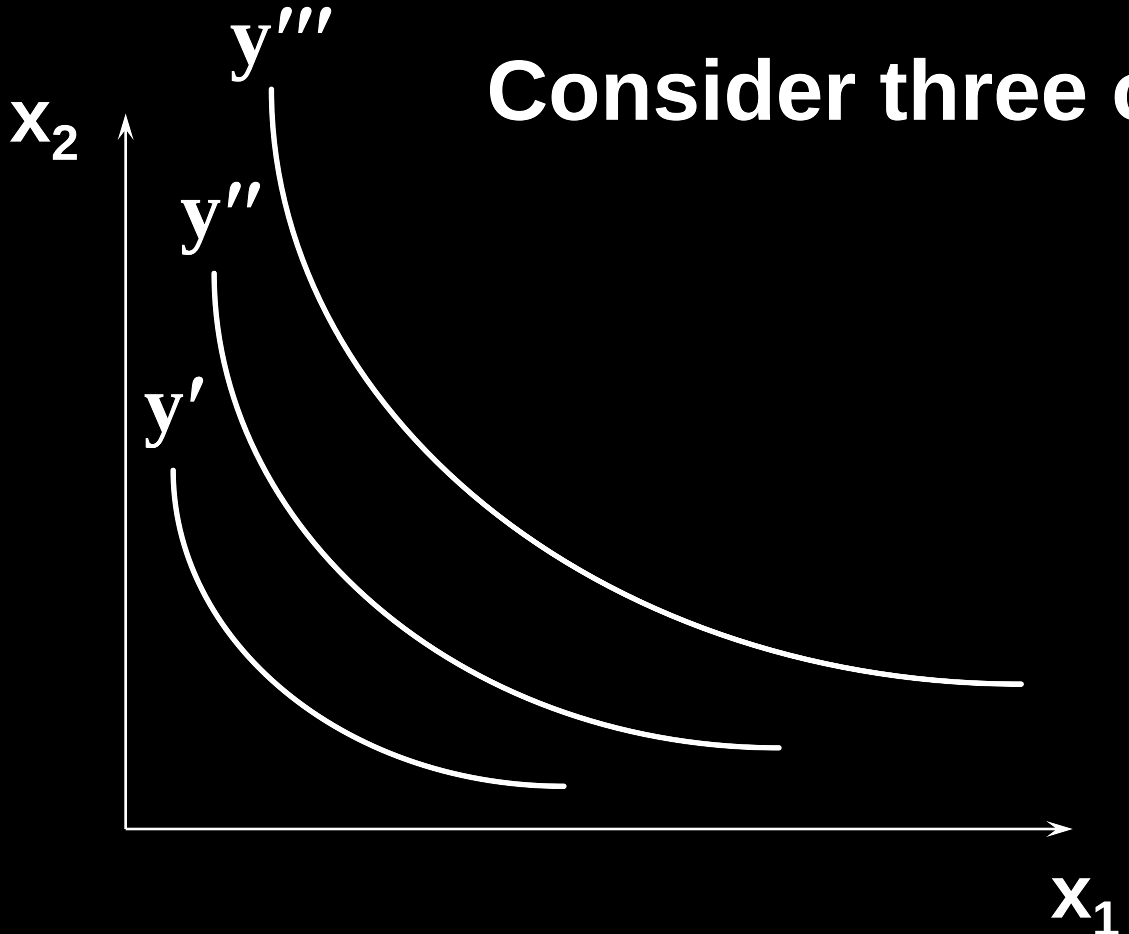
- The **short-run** cost-min. problem is the long-run problem subject to the extra constraint that $x_2 = \tilde{x}_2$
- If the long-run choice for x_2 was $\tilde{x}_2$ then the extra constraint $x_2 = \tilde{x}_2$ is not really a constraint at all and so the long-run and short-run total costs of producing y output units are the same.

Short-Run & Long-Run Total Costs

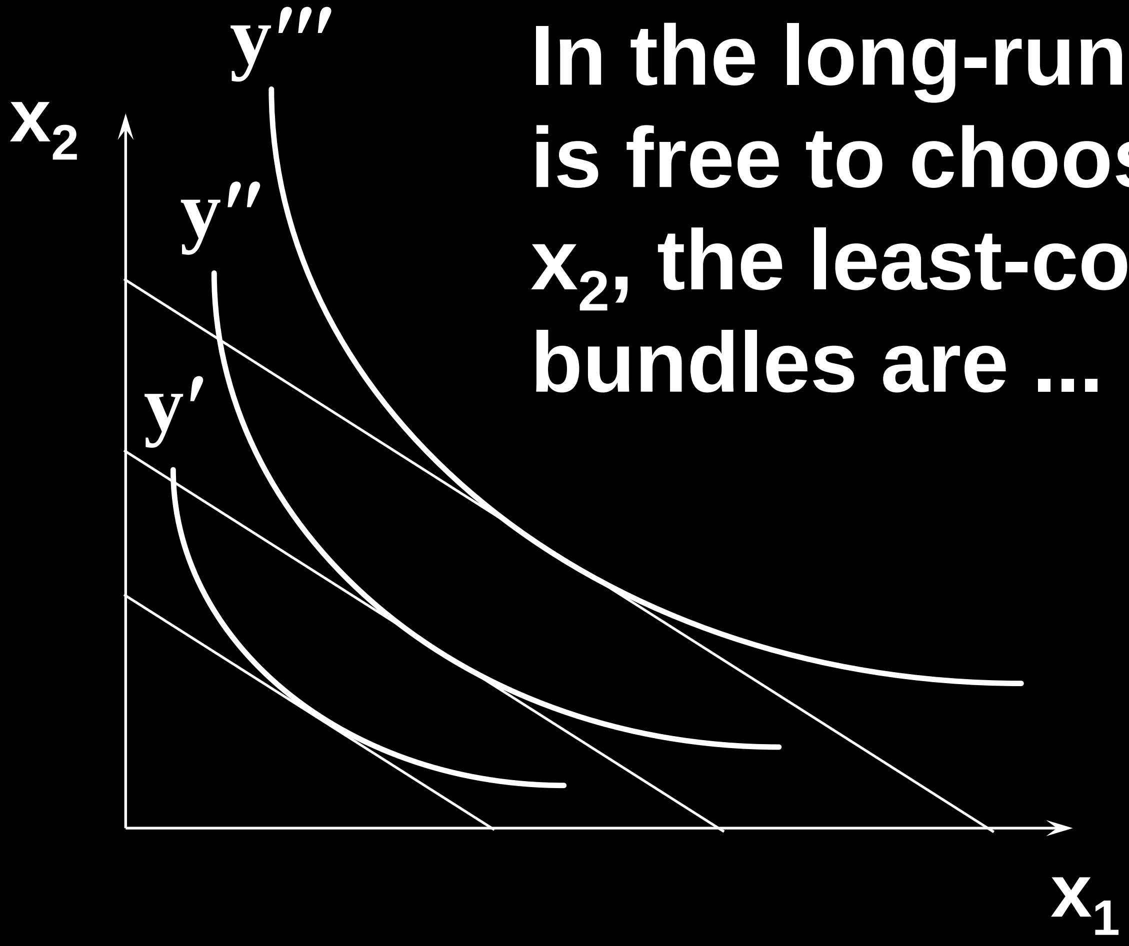
- The short-run cost-min. problem is the long-run problem subject to the extra constraint that $x_2 = \tilde{x}_2$
- But, if the long-run choice for $x_2 \neq \tilde{x}_2$ then the extra constraint $x_2 = \tilde{x}_2$ prevents the firm in this short-run from achieving its long-run production cost, causing the short-run total cost to **exceed** the long-run total cost of producing y output units.

Short-Run & Long-Run Total Costs

Consider three output levels.

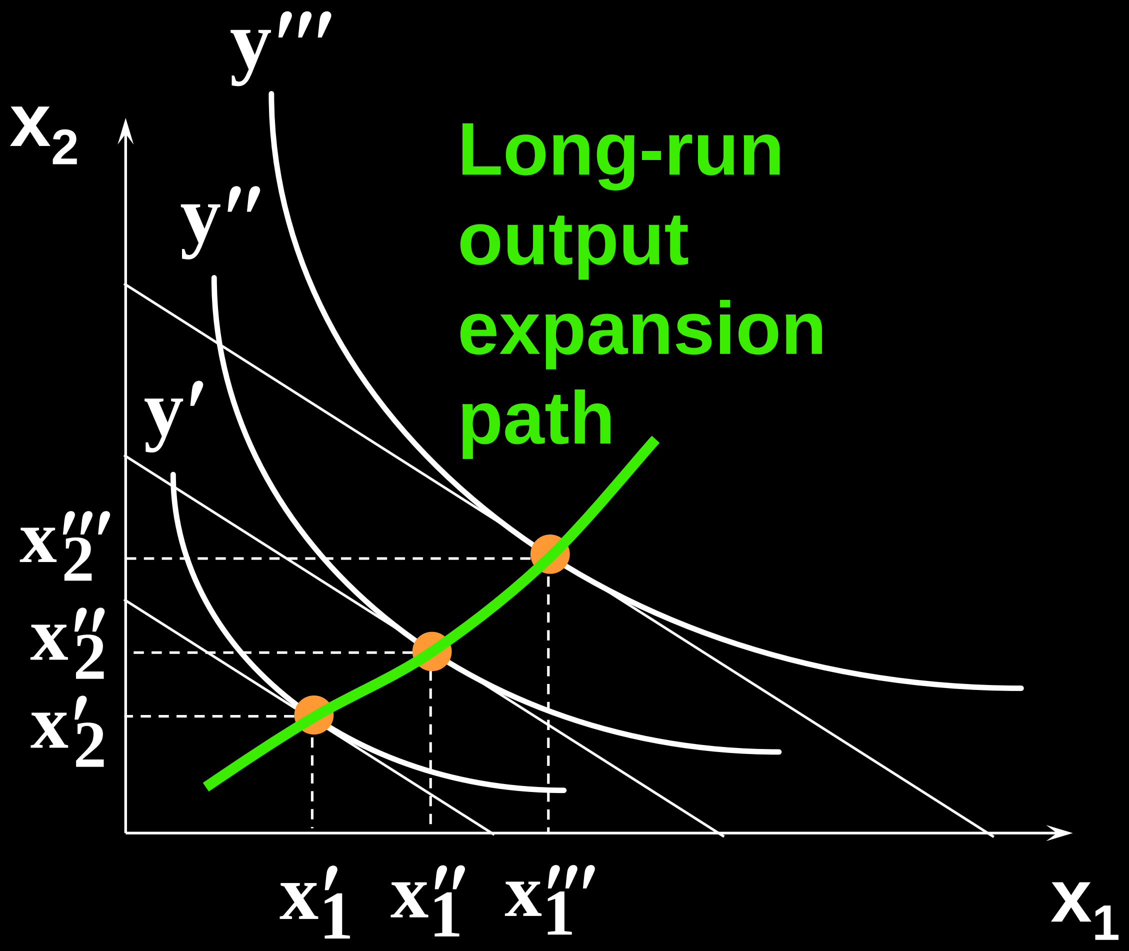


Short-Run & Long-Run Total Costs

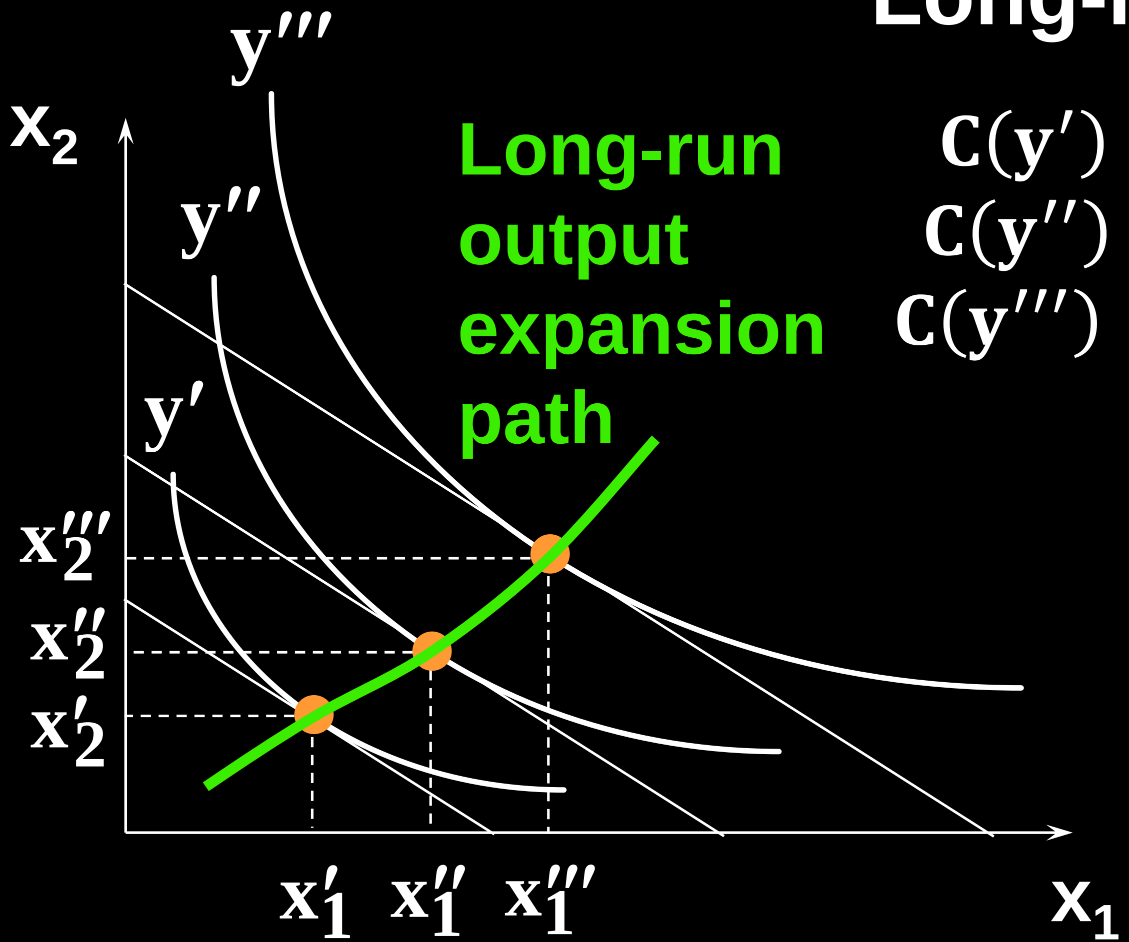


In the long-run when the firm is free to choose both x_1 and x_2, the least-costly input bundles are ...

Short-Run & Long-Run Total Costs



Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = \omega_1 x_1' + \omega_2 x_2'$$

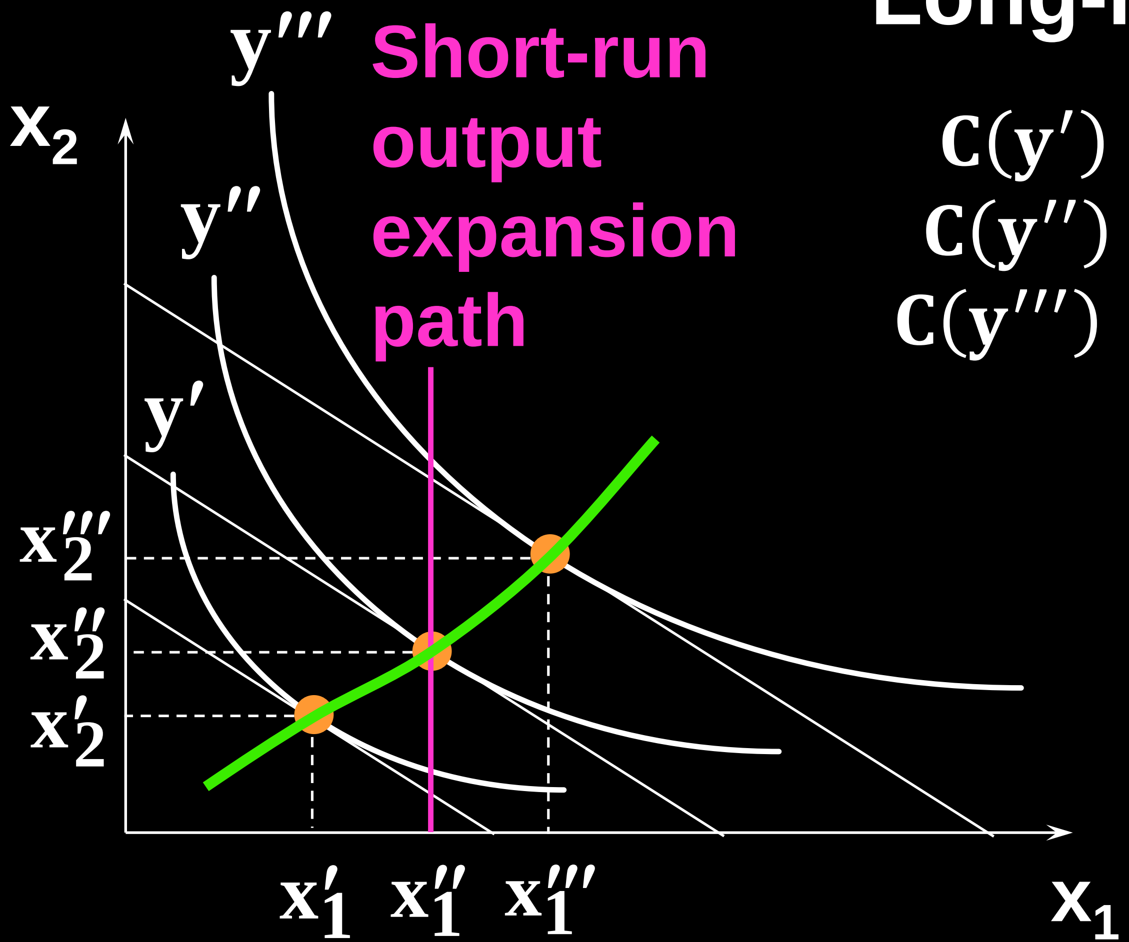
$$c(y'') = \omega_1 x_1'' + \omega_2 x_2''$$

$$c(y''') = \omega_1 x_1''' + \omega_2 x_2'''$$

Short-Run & Long-Run Total Costs

- Now suppose the firm becomes subject to the short-run constraint that $x_1 = x_1"$

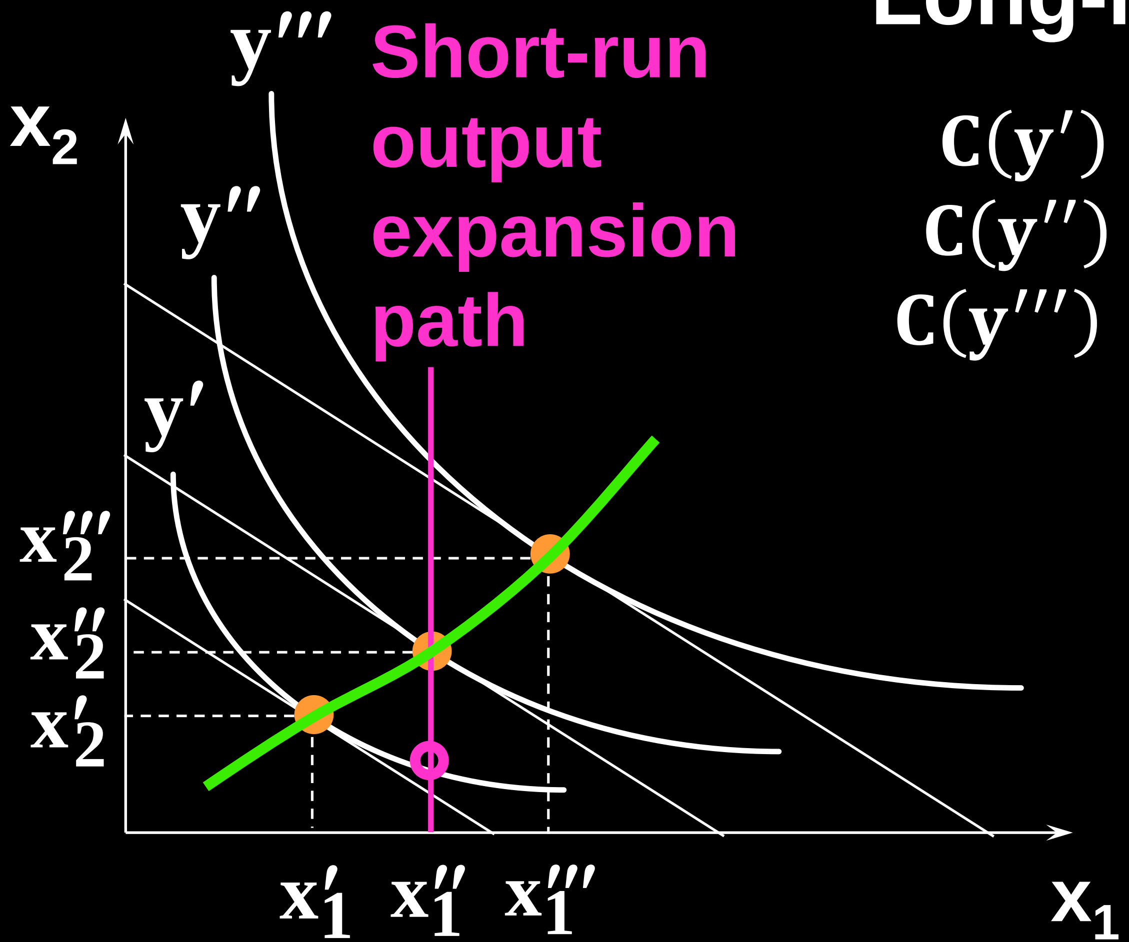
Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = \omega_1 x_1' + \omega_2 x_2'$$
$$c(y'') = \omega_1 x_1'' + \omega_2 x_2''$$
$$c(y''') = \omega_1 x_1''' + \omega_2 x_2'''$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = \omega_1 x_1' + \omega_2 x_2'$$
$$c(y'') = \omega_1 x_1'' + \omega_2 x_2''$$
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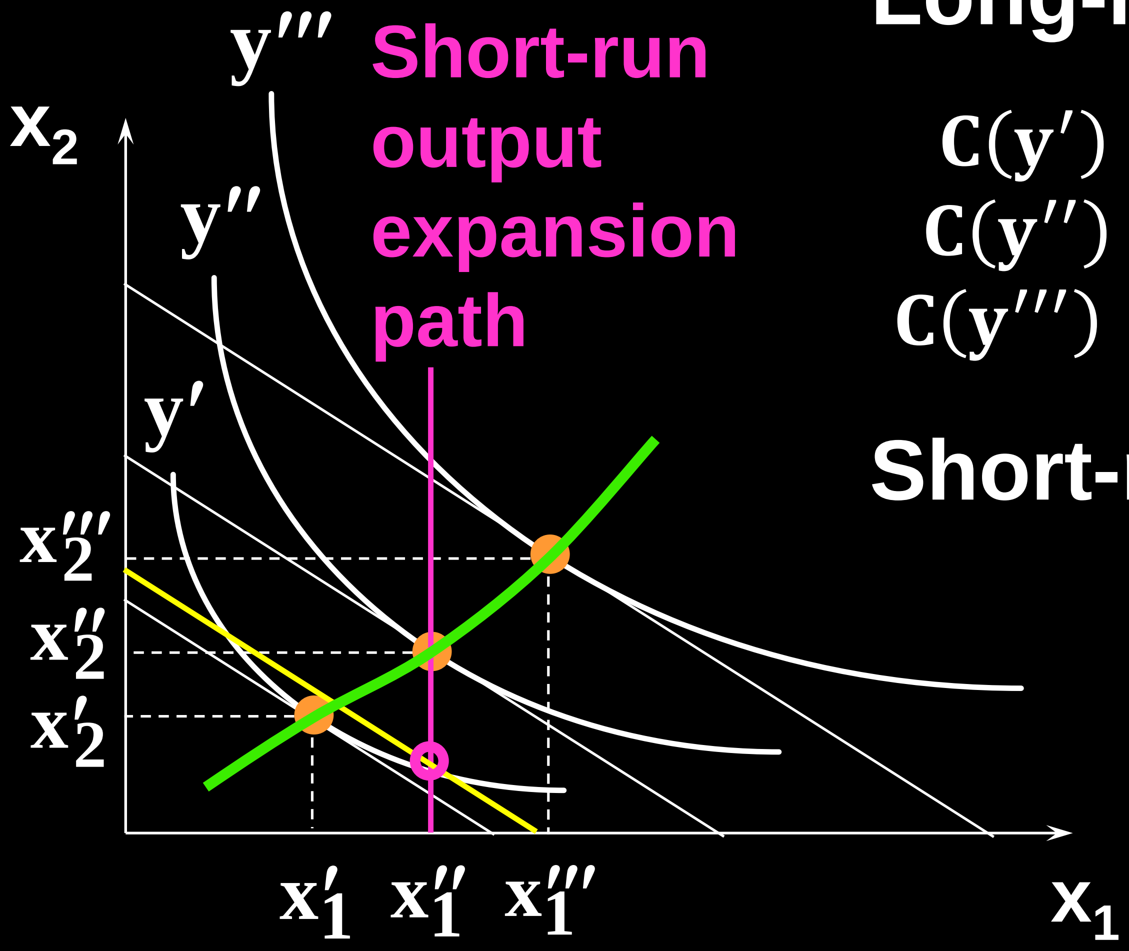
Short-Run & Long-Run Total Costs



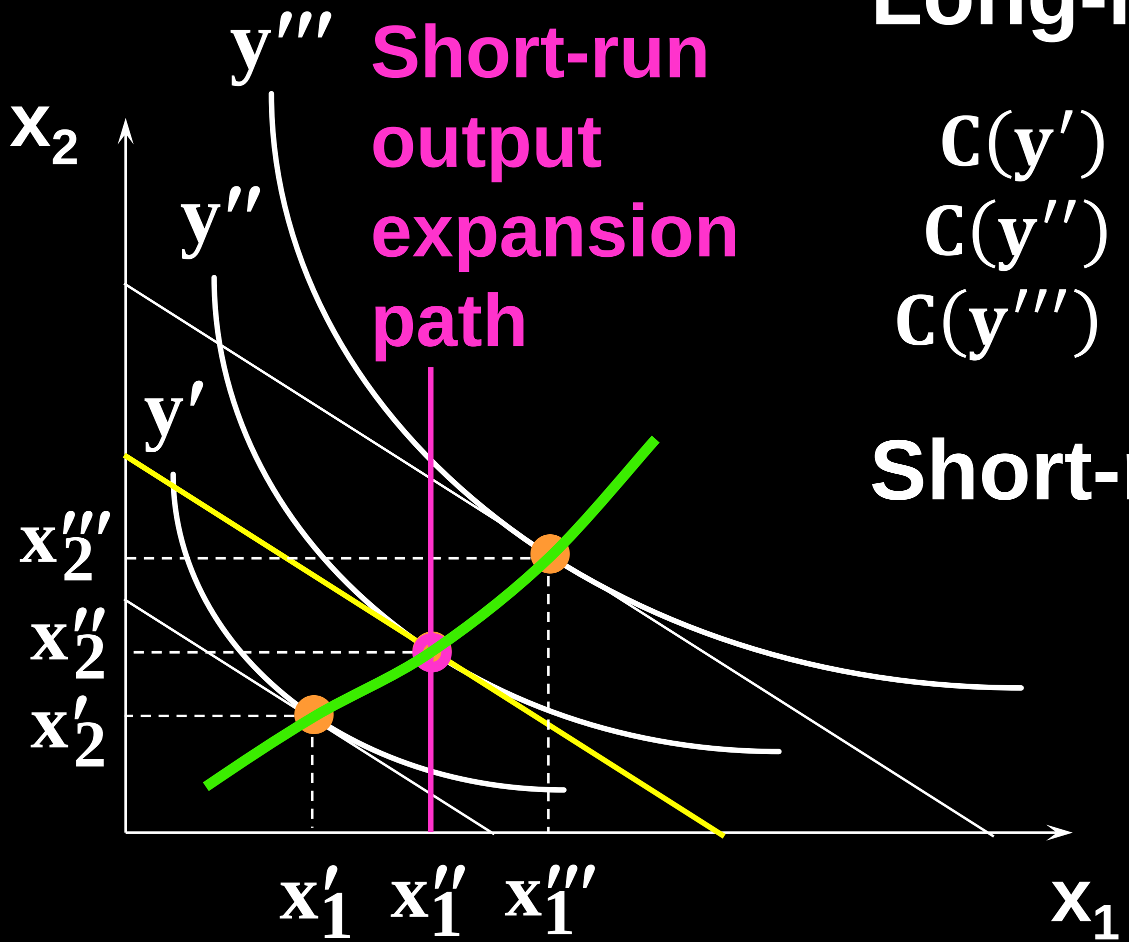
Long-run costs are:

$$c(y') = \omega_1 x_1' + \omega_2 x_2'$$
$$c(y'') = \omega_1 x_1'' + \omega_2 x_2''$$
$$c(y''') = \omega_1 x_1''' + \omega_2 x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$

Short-Run & Long-Run Total Costs



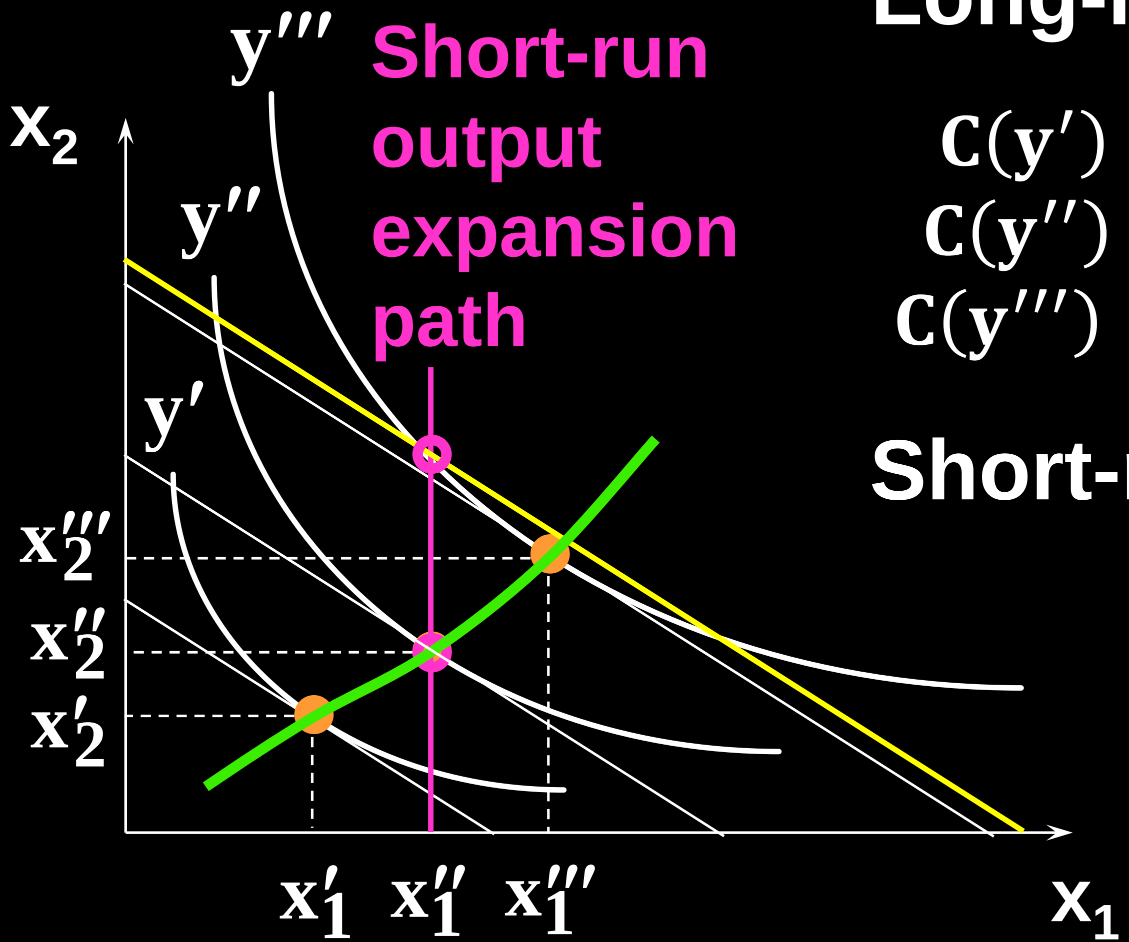
Long-run costs are:

$$c(y') = \omega_1 x_1' + \omega_2 x_2'$$
$$c(y'') = \omega_1 x_1'' + \omega_2 x_2''$$
$$c(y''') = \omega_1 x_1''' + \omega_2 x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$
$$c_s(y'') = c(y'')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = \omega_1 x_1' + \omega_2 x_2'$$
$$c(y'') = \omega_1 x_1'' + \omega_2 x_2''$$
$$c(y''') = \omega_1 x_1''' + \omega_2 x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$
$$c_s(y'') = c(y'')$$
$$c_s(y''') > c(y''')$$

Short-Run & Long-Run Total Costs

Short-run total cost exceeds long-run total cost except for the output level where the short-run input level restriction is the long-run input level choice.

给定某一个产量时，若短期要素的**固定投入量**恰好是长期最优的要素投入量，此时短期成本等于长期成本。其它情况下短期成本一定高于长期成本。

Short-Run & Long-Run Total Costs

This says that the long-run total cost curve always has one point in common with any particular short-run total cost curve.

这意味着，长期总成本曲线在某一个产量处和短期总成本曲线恰好相交，在其他产量处低于短期成本曲线。

Short-Run & Long-Run Total Costs

A short-run total cost curve always has one point in common with the long-run total cost curve, and is elsewhere higher than the long-run total cost curve.