单参数指数族.

- 总体密度为$f(x,\theta)=S(\theta)h(x)\exp\{C(\theta)T(x)\}$, $C(\theta)$ 严格增.

  $H_0:\theta\leqslant(\geqslant)\theta_0$. $\mathcal{W}=\{\vec{x}:\sum_{i=1}^n T(x_i)>(<)c\}$.

二、$X\sim N(\mu,\sigma^2)$，均值μ的检验.

- (1) σ^2 已知: $T(\vec{x})=\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}$.

  $H_0:\mu=(\leqslant,\geqslant)\mu_0$. $\mathcal{W}=\{\vec{x}:|T(\vec{x})|>c\}$, $P(|Z|>c)=\alpha$.

  $\mathcal{W}=\{\vec{x}:T(\vec{x})(>,<-)c\}$, $P(Z>c)=\alpha$,

- (2) σ^2 未知: $T(\vec{x})=\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^n(x_i-\bar{x})^2}}$.

  $H_0:\mu=(\leqslant,\geqslant)\mu_0$.

  $\mathcal{W}=\{\vec{x}:|T(\vec{x})|>c\}$, $P(|T_{n-1}|>c)=\alpha$.

  $\mathcal{W}=\{\vec{x}:T(\vec{x})(>,<-)c\}$, $P(T_{n-1}>(<-)c))=\alpha$.

三、正态总体$X \sim N(\mu, \sigma^2)$，方差σ^2 的检验

(1) 双边问题 $H_0 : \sigma^2 = \sigma_0^2 \leftrightarrow H_1 : \sigma^2 \neq \sigma_0^2$.

- 最大似然估计: $\hat{\mu} = \hat{\mu}_0 = \mu$ (μ 已知) 或$\bar{x}$ (μ 未知).
  $\hat{\sigma}^2 = \underline{\frac{1}{n}\sum_{i=1}^n (x_i - \hat{\mu})^2}$, $\hat{\sigma}_0^2 = \sigma_0^2$.
  若$\sigma^2 = \sigma_0^2$, 则$T(\vec{X}) := \frac{n\hat{\sigma}^2}{\sigma_0^2} \sim \chi^2(n)$ 或$\chi^2(n-1)$.
- 似然比: $\lambda(\vec{x}) = \frac{L(\hat{\theta})}{L(\hat{\theta}_0)} = \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{-\frac{n}{2}} \exp\{\frac{n\hat{\sigma}^2}{2\sigma_0^2}\} e^{-\frac{n}{2}}$. 因为
  $L(\hat{\theta}) = (\frac{1}{\sqrt{2\pi}\hat{\sigma}})^n \exp\{-\frac{1}{2\hat{\sigma}^2}\underline{\sum_{i=1}^n (x_i - \hat{\mu})^2}\} = (2\pi\hat{\sigma}^2)^{-\frac{n}{2}} e^{-\frac{n}{2}}$.
  $L(\hat{\theta}_0) = (2\pi\sigma_0^2)^{-\frac{n}{2}} \exp\{-\frac{n\hat{\sigma}^2}{2\sigma_0^2}\}$.
- <u>否定域</u>: $\mathcal{W} = \{\vec{x} : \left(\frac{T}{n}\right)^{-\frac{n}{2}} e^{\frac{T}{2}} > c\} = \underline{\{\vec{x} : T < c_1 \text{或} > c_2\}}$.
  $T = T(\vec{x})$, $\left(\frac{t}{n}\right)^{-\frac{n}{2}} e^{\frac{t}{2}}$ 关于t 先$\downarrow$ 后$\uparrow$, (导函数先负后正).
- 根据α 求c. $P_{\sigma_0^2}(T < c_1) + P_{\sigma_0^2}(T > c_2) = \alpha$. 但, $c \xrightarrow{?} c_1, c_2$
- 改用$\tilde{c}_1, \tilde{c}_2$, $P(T < \tilde{c}_1) = P(T > \tilde{c}_2) = \frac{\alpha}{2}$. $T = \chi_n^2$ 或χ_{n-1}^2

(2) 单边问题 $H_0: \sigma^2 \geqslant \sigma_0^2 \leftrightarrow H_1: \sigma^2 < \sigma_0^2$.

- 最大似然估计: $\hat{\mu} = \hat{\mu}_0 = \mu$ (μ已知) 或$\bar{x}$ (μ未知).
  $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \hat{\mu})^2$,
  $$\hat{\sigma}_0^2 = \begin{cases} \hat{\sigma}^2, & \sigma_0^2 < \hat{\sigma}^2; \\ \sigma_0^2, & \sigma_0^2 \geqslant \hat{\sigma}^2. \end{cases}$$
- 似然比: $T(\vec{x}) := n\hat{\sigma}^2/\sigma_0^2$.
  $$\lambda(\vec{x}) = \begin{cases} 1, & \sigma_0^2 < \hat{\sigma}^2 (\Leftrightarrow T(\vec{x}) > n); \\ \left(\frac{T(\vec{x})}{n}\right)^{-\frac{n}{2}} e^{\frac{T(\vec{x})}{2}} e^{-\frac{n}{2}}, & \sigma_0^2 \geqslant \hat{\sigma}^2 (\Leftrightarrow T(\vec{x}) \leqslant n). \end{cases}$$
- 否定域: 舍弃"$\sigma_0^2 < \hat{\sigma}^2$", 加限制"$\sigma_0^2 \geqslant \hat{\sigma}^2$" 即"$T(\vec{x}) \leqslant \cdots$".
  $\mathcal{W} = \{\vec{x} : \left(\frac{T(\vec{x})}{n}\right)^{-\frac{n}{2}} e^{\frac{T(\vec{x})}{2}} > c, T(\vec{x}) \leqslant n\} = \{\vec{x} : T(\vec{x}) < c_1\}$.
- 根据α 求c. $\forall \sigma^2 \geqslant \sigma_0^2$,
  $P_{\sigma^2}(T < c_1) = P_{\sigma^2}(\frac{n\hat{\sigma}^2}{\sigma^2} < c_1 \frac{\sigma_0^2}{\sigma^2}) \leqslant P(\chi_n^2$ 或$\chi_{n-1}^2 < c_1) = \alpha$.
  (等号在$\sigma^2 = \sigma_0^2$ 达到)

正态总体$X \sim N(\mu, \sigma^2)$ 方差σ^2 的假设检验问题总结:

- 检验统计量: $T(\vec{x}) = \frac{1}{\sigma_0^2}\sum_{i=1}^n (x_i - \mu)^2$ (μ 已知),
  $T(\vec{x}) = \frac{1}{\sigma_0^2}\sum_{i=1}^n (x_i - \bar{x})^2$ (μ 未知), χ^2 检验.
- 双边检验问题$H_0 : \sigma^2 = \sigma_0^2 \leftrightarrow H_1 : \sigma^2 \neq \sigma_0^2$.
  $\mathcal{W} = \{\vec{x} : T(\vec{x}) < c_1 \text{ 或 } > c_2\}$,
  $P(\chi_n^2 < c_1) = P(\chi_n^2 > c_2) = \frac{\alpha}{2}$.
  $P(\chi_{n-1}^2 < c_1) = P(\chi_{n-1}^2 > c_2) = \frac{\alpha}{2}$.
- 单边$H_0 : \sigma^2 \geqslant \sigma_0^2 \leftrightarrow H_1 : \sigma^2 < \sigma_0^2$.
  $\mathcal{W} = \{\vec{x} : T(\vec{x}) < c_1\}$, $P(\chi_n^2 < c_1) = P(\chi_{n-1}^2 < c_1) = \alpha$,
- 一般不用担心σ^2 小, 所以不必处理$H_0 : \sigma^2 \leqslant \sigma_0^2$.

四、两正态$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ 的参数检验

数据: $\vec{x} = (x_1, \cdots, x_{n_1})$, $\vec{y} = (y_1, \cdots, y_{n_2})$.

- $H_0 : \mu_1 = \mu_2 \leftrightarrow H_1 : \mu_1 \neq \mu_2$

  (1) σ_1^2, σ_2^2 已知;

  (2) σ_1^2, σ_2^2 未知, 但知道$\sigma_1^2 = \sigma_2^2$;

  (3) σ_1^2, σ_2^2 未知. (太过复杂, 略)

- $H_0 : \sigma_1^2 = \sigma_2^2 \leftrightarrow H_1 : \sigma_1^2 \neq \sigma_2^2$

  (4) μ_1, μ_2 已知. (自习, 略).

  (5) μ_1, μ_2 未知.

- 单边假设检验问题类似.

(1) σ_1^2, σ_2^2 已知

检验$H_0 : \mu_1 = \mu_2 \leftrightarrow H_1 : \mu_1 \neq \mu_2$..

- 检验统计量: $\bar{X} - \bar{Y} \overset{H_0}{\sim} N(0, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$.
  $T(\vec{X}, \vec{Y}) := \frac{|\bar{X} - \bar{Y}|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \overset{H_0}{\sim} N(0, 1)$.
- 否定域: $\mathcal{W} = \{(\vec{x}, \vec{y}) : T(\vec{x}, \vec{y}) > c\} = \{(\vec{x}, \vec{y}) : |\bar{x} - \bar{y}| > \tilde{c}\}$.
- 根据水平α 定c:
  $P_{H_0}((\vec{X}, \vec{Y}) \in \mathcal{W}) = P(|Z| > c) = \alpha$.
  假设$z_{\alpha/2}$满足: $P(|Z| > z_{\alpha/2}) = \alpha$, 则否定域为:

$$\mathcal{W} = \left\{ (\vec{x}, \vec{y}) : |\bar{x} - \bar{y}| > z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right\}.$$

(2) σ_1^2, σ_2^2 未知, 但知道$\sigma_1^2 = \sigma_2^2 (= \sigma^2)$.

检验$H_0 : \mu_1 = \mu_2 \leftrightarrow H_1 : \mu_1 \neq \mu_2$.

- 检验统计量:

  $\bar{X} - \bar{Y} \overset{H_0}{\sim} N(0, \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2})$,

  $S^2 = \sum_{i=1}^{n_1}(X_i - \bar{X})^2 + \sum_{i=1}^{n_2}(Y_i - \bar{Y})^2 = \sigma^2 \chi^2_{n_1-1+n_2-1}$.

  $T = \dfrac{\bar{X}-\bar{Y}}{\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\sqrt{\frac{S^2}{n_1+n_2-2}}} \overset{H_0}{\sim} t(n_1 + n_2 - 2)$.

- 否定域: $\mathcal{W} = \{(\vec{x}, \vec{y}) : |T| > c\}$. (广义似然比否定域)

- 根据水平α 定c:

  $P_{H_0}((\vec{X}, \vec{Y}) \in \mathcal{W}) = P(|t_{n_1+n_2-2}| > c) = \alpha$.

(5) μ_1, μ_2 未知. 检验$H_0: \sigma_1^2 = \sigma_2^2 \leftrightarrow H_1: \sigma_1^2 \neq \sigma_2^2$..

- 检验统计量:
  $$\left.\begin{array}{l} \frac{1}{\sigma_1^2}\sum_{i=1}^{n_1}(X_i - \bar{X})^2 \sim \chi^2(n_1-1) \\ \frac{1}{\sigma_2^2}\sum_{i=1}^{n_2}(Y_i - \bar{Y})^2 \sim \chi^2(n_2-1) \end{array}\right\} \text{相互独立.}$$
  F分布: 相互独立的χ^2 分布除以相应自由度后的比值.
  $$F = \frac{\frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_i-\bar{X})^2}{\frac{1}{n_2-1}\sum_{i=1}^{n_2}(Y_i-\bar{Y})^2} \overset{H_0}{\sim} F(n_1-1, n_2-1).$$
- 否定域: $\mathcal{W} = \{(\vec{x}, \vec{y}) : F < c_1 \text{ 或} F > c_2\}$.
- 根据水平α 定c:
  $$P(F_{n_1-1,n_2-1} < c_1) = P(F_{n_1-1,n_2-1} > c_2) = \frac{\alpha}{2}.$$
- 注：上面的否定域不是广义似然比否定域；广义似然比否定域也不是UMPU, 但能导出否定域的如上形式; 进一步研究知存在UMPU.

§8.6 拟合优度检验

$H_0: X \sim F_0 \leftrightarrow H_1: X \nsim F_0$. 假设样本量$n >> 1$.

一、χ^2 检验法

- $t_1 < \cdots < t_m$. $1 << m << n$. $m+1$ 个区间$I_1, \cdots, I_{m+1}$.
- 在I_i 中，有$V_i (= v_i)$ 个数据. $(\underline{v_1 + \cdots + v_{m+1} = n})$
  概率$\approx$ 频率: $\frac{v_i}{n} \approx P(X \in I_i) \overset{H_0}{=} F_0(t_i) - F_0(t_{i-1}) = p_i$.
- $\frac{V_i - np_i}{\sqrt{np_i}} \overset{1<<m \text{故} p_i<<1}{\approx} \frac{V_i - np_i}{\sqrt{np_i(1-p_i)}} \overset{m<<n, CLT, \text{近似}}{\sim} N(0,1)$.
- $V = \sum_{i=1}^{m+1} (\frac{V_i - np_i}{\sqrt{np_i}})^2 = \sum_{i=1}^{m+1} (\frac{V_i}{n} - p_i)^2 \frac{n}{p_i} \overset{\text{近似}}{\sim} \chi^2(\underline{m})$.
  <u>一个约束条件</u> $(\sum_{i=1}^{m+1} V_i = n)$, 自由度少1.
- 否定域:
  $\mathcal{W} = \{\vec{x} : V > c\}$, $P(\chi_m^2 > c) = \alpha$.

推广: $\mathcal{F}_0=\{F(x,\theta),\theta\in\Theta\}$

$H_0: X\sim\mathcal{F}_0 \leftrightarrow H_1: X\nsim\mathcal{F}_0$. 假设样本量$n>>1$.

- 在H_0假设下，求出参数θ的ML估计$\hat\theta(x_1,\cdots,x_n)$
- $t_1<\cdots<t_m$. $1<<m<<n$. $m+1$ 个区间$I_1,\cdots,I_{m+1}$.
- 在I_i 中, 有$V_i(=v_i)$ 个数据. $(\underline{v_1+\cdots+v_{m+1}=n})$

  概率$\approx$ 频率: $\frac{v_i}{n}\approx P(X\in I_i)\overset{H_0}{\approx}F(t_i,\hat\theta)-F(t_{i-1},\hat\theta))=\hat p_i$.
- $V=\sum_{i=1}^{m+1}(\frac{V_i-n\hat p_i}{\sqrt{n\hat p_i}})^2=\sum_{i=1}^{m+1}(\frac{V_i}{n}-\hat p_i)^2\frac{n}{\hat p_i}\overset{近似}{\sim}\chi^2(\underline{m-k})$,

  其中k是参数θ的维数.
- 否定域:

  $\mathcal{W}=\{\vec x: V>c\}$, $P(\chi^2_{m-k}>c)=\alpha$.

二、柯氏(Kolmogorov)检验法

理论: 假设$X \sim F_0$, 即H_0 成立. 那么,

- $F_n(x) = \frac{1}{n}\sum_{i=1}^n 1_{\{X_i \leqslant x\}} \overset{LLN}{\approx} F_0(x)$.
- $\sqrt{n}(F_n(x) - F_0(x)) \overset{CLT}{\sim} N(0, \sigma^2)$.
- $D_n =: \sup_x |F_n(x) - F_0(x)| \xrightarrow{P} 0$(引理6.1).

  $\sqrt{n}D_n \overset{\text{近似}}{\sim} \xi$(见定理6.1), 即$D_n \overset{\text{近似}}{\sim} \frac{1}{\sqrt{n}}\xi$, 这里

  $F_\xi(x) = \sum_{k=-\infty}^{\infty}(-1)^k \exp(-2k^2x^2)1_{\{x>0\}} =: Q(x)$.

步骤:

- 将数据$x_1, \cdots, x_n$ 排序得到$x_{(1)} < \cdots < x_{(n)}$,

  则$D_n(\vec{x}) = \max_{1\leqslant k\leqslant n} \max\left\{\left|\frac{k}{n} - F_0(x_{(k)})\right|, \left|F_0(x_{(k)}) - \frac{k-1}{n}\right|\right\}$.
- 否定域: $\mathcal{W} = \{\vec{x} : D_n(\vec{x}) > c\}$.
- 根据α 定c. $P(D_n(\vec{X}) > c) = \alpha$.

$$F_n(x) = \begin{cases} 0, & x < X_{(1)}, \\ k/n, & X_{(k)} \leqslant x < X_{(k+1)}, k = 1, \cdots, n-1, \\ 1, & x \geqslant X_{(n)}. \end{cases}$$

推广: $\mathcal{F}_0 = \{F(x,\theta), \theta \in \Theta\}$

$H_0 : X \sim \mathcal{F}_0 \leftrightarrow H_1 : X \nsim \mathcal{F}_0$. 假设样本量$n >> 1$.

- 在H_0假设下，求出参数θ的ML估计$\hat{\theta}(x_1, \cdots, x_n)$, 确定一个分布$F_0 := F(\hat{\theta})$.
- 将数据$x_1, \cdots, x_n$ 排序得到$x_{(1)} < \cdots < x_{(n)}$,
  则$D_n(\vec{x}) = \max_{1 \leqslant k \leqslant n} \max\left\{\left|\frac{k}{n} - F_0(x_{(k)})\right|, \left|F_0(x_{(k)}) - \frac{k-1}{n}\right|\right\}$.
- 否定域: $\mathcal{W} = \{\vec{x} : D_n(\vec{x}) > c\}$.
- 根据α 定c. $P(D_n(\vec{X}) > c) = \alpha$.
- 注：H_0假设下D_n的极限分布是否存在? 以上检验只是近似检验.