复习:

- 边缘密度:

  $p_X(x) = \int p(x,y)dy$

- 条件密度:

  $p_{Y|X}(y|x) = \frac{1}{p_X(x)} p(x,y)$

- 给定x, 视为y 的函数

  $p_{Y|X}(y|x) \propto p(x,y)$.

- 独立:

  $p_{(X,Y)}(x,y) = p_X(x) p_Y(y)$.

  $E(XY) = (EX)(EY)$.

- 独立同分布i.i.d.

§3.4 两个随机变量的函数

1. 求$Z = f(X,Y)$ 的分布：先求分布函数再求导

例1(定理4.1)　设(X,Y)的密度为$p(x,y)$, 求$Z = X+Y$ 的密度.

解 Z 的分布函数为

$$
\begin{aligned}
F(z) = \quad & P(Z \leqslant z) = P(X+Y \leqslant z) \\
= \quad & \iint_{x+y\leqslant z} p(x,y)dxdy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} p(x,y)dy.
\end{aligned}
$$

求导得Z 的密度为$\int_{-\infty}^{\infty} p(x,z-x)dx$.

2. 公式法(定理4.3)

例4.4, 4.5, 习题三、21. 假设X, Y 独立同分布, $X \sim N(0,1)$.

极坐标: $X = R\cos\Theta$, $Y = R\sin\Theta$.

则R, Θ 相互独立, 且$\Theta \sim U(0, 2\pi)$, $p_R(r) = re^{-\frac{r^2}{2}}$, $r > 0$.

- 将(x, y)用(r, θ)表示: $f : (x, y) \mapsto (r, \theta)$, $f^{-1} : (r, \theta) \mapsto (x, y)$

  $x = r\cos\theta$, $y = r\sin\theta$.

- 确定(R, Θ)的取值空间: (R, Θ)取值于$(0, \infty) \times (0, 2\pi)$.

- $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$

- $\rho_{R,\Theta}(r, \theta) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} |J| 1_{(0,\infty)\times(0.2\pi)}(r, \theta) =$

  $\frac{1}{2\pi} 1_{(0,2\pi)}(\theta) re^{-\frac{r^2}{2}} 1_{(0,\infty)}(r)$.

- 若$\rho_{R,\Theta}(r, \theta) = f(r)g(\theta)$.

  则R, Θ 相互独立, 且$p_R = C_1 f$, $p_\Theta = C_2 g$.

线性变换:

$$\frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} \longrightarrow Ce^{-ax^2-by^2+cxy+dx+ey}$$

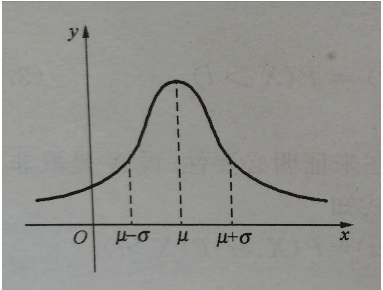

图 2.3.1

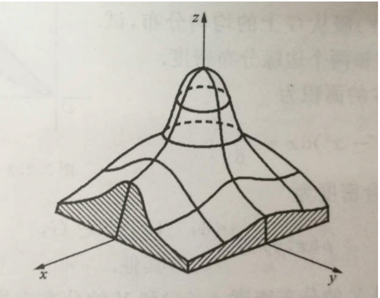

图 3.2.1 二维正态分布密度函数的图像

函数的分布:

- $P(f(\vec{X}) \leqslant y) = P(\vec{X} \in D), \cdots$.
  自习系4.1, 例4.3, 定理4.2, 例4.6.
- 若$\vec{\xi} \overset{d}{=} \vec{\eta}$, 则$f(\vec{\xi}) \overset{d}{=} f(\vec{\eta})$.

随机向量函数的期望(定理4.6):

- 离散型: $Ef(X,Y) = \sum_{i,j} f(x_i, y_j) p_{ij}$.
- 连续型: $Ef(X,Y) = \iint f(x,y)p(x,y)dxdy$.
- 假设X,Y独立且$E|X|, E|Y| < \infty$, 则$E(XY) = (EX)(EY)$ (定理4.4).

§3.5 二维随机向量的数字特征

定义

假设$EX^2, EY^2 < \infty$.

<u>协方差</u> *(covariance)*: $\operatorname{cov}(X,Y) := E((X-EX)(Y-EY))$.

<u>(线性)相关系数</u>: $\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)}\sqrt{\operatorname{var}(Y)}}$.

<u>(线性)不相关</u>: $\operatorname{cov}(X,Y) = 0$ ($\Leftrightarrow \rho_{X,Y} = 0$).

- $\operatorname{cov}(X,Y) = E(XY) - (EX)(EY)$.
- $|\operatorname{cov}(X,Y)|^2 \leqslant \operatorname{var}(X) \cdot \operatorname{var}(Y)$ (定理5.1).

- 习题一, 17题: $|P(AB) - P(A)P(B)| \leqslant 1/4$.

- $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) + 2\mathrm{cov}(X,Y)$.
- $\mathrm{cov}(X,Y) = \mathrm{cov}(X+a, Y+b)$, $\rho_{X,Y} = \mathrm{cov}(X^*, Y^*)$.
- X, Y 若相互独立, 则不相关.

  $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$ (定理4.5).

  反之不然! 例: $X \sim N(0,1)$, $Y = X^2$.

最优线性预测(定理5.3). 假设$EX=0$, $EX^2=1$. 则

$$Q(a,b)=E(Y-(a+bX))^2$$

在$a_0=EY$, $b_0=\mathrm{cov}(X,Y)$ 达到最小值$(1-\rho_{X,Y}^2)\mathrm{var}(Y)$.

- $Q(a,b)=E((Y-bX)-a)^2$ 表明$a_0=E(Y-bX)=EY$.
- 当$a=EY$时,

$$\begin{aligned} Q(a,b)= & E((Y-EY)-bX)^2=\mathrm{var}(Y)+b^2-2b\mathrm{cov}(X,Y) \\ = & (b-\mathrm{cov}(X,Y))^2+\mathrm{var}(Y)-(\mathrm{cov}(X,Y))^2. \end{aligned}$$

  所以取$b_0=\mathrm{cov}(X,Y)=\rho_{X,Y}\sqrt{\mathrm{var}(Y)}$.
- $Q(a_0,b_0)=\mathrm{var}(Y)-(\mathrm{cov}(X,Y))^2=(1-\rho_{X,Y}^2)\mathrm{var}(Y)$.
- $\rho_{X,Y}=\pm 1 \Leftrightarrow \min_{a,b} Q(a,b)=0 \Leftrightarrow \exists a,b$ s.t. $Y=a+bX$.
- 自习例5.1, 5.3.

例5.2 二维正态的密度:

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left((\frac{x-\mu_1}{\sigma_1})^2+(\frac{y-\mu_2}{\sigma_2})^2-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right)},$$

其中$\mu_1 = EX$, $\mu_2 = EY$, $\sigma_1^2 = \mathrm{var}(X)$, $\sigma_2 = \mathrm{var}(Y)$, $\rho = \rho_{X,Y}$.

- $\rho_{X,Y} = \frac{E(X-\mu_1)(Y-\mu_2)}{\sigma_1\sigma_2} = \iint \frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}p(x,y)dxdy.$
- $\rho_{X,Y} = \frac{1}{2\pi\sqrt{1-\rho^2}}\iint xye^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}dxdy.$
- 先对y 积分.
  $e^{\frac{(\rho x)^2}{2(1-\rho^2)}}\int ye^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}dy = \sqrt{2\pi(1-\rho^2)}e^{\frac{(\rho x)^2}{2(1-\rho^2)}}\rho x.$
- 再对x 积分.
  $\frac{1}{\sqrt{2\pi}}\int \rho x e^{\frac{(\rho x)^2}{2(1-\rho^2)}}xe^{-\frac{x^2}{2(1-\rho^2)}}dx = \rho\int x^2\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = \rho.$

1. n维随机向量的分布.

2. n维随机向量的数字特征.

定义

假设$\xi = (X_1, \cdots, X_n)$是随机向量.

期望 : $E\xi := (EX_1, EX_2, \cdots, EX_n)$.

协方差阵Σ: $\Sigma = (\sigma_{ij})_{n\times n}$，其中$\sigma_{ij} = \mathrm{cov}(X_i, X_j)$.

相关阵R: $R = (\rho_{ij})_{n\times n}$，其中$\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma jj}$.

3. n维随机向量函数的期望(定理6.3).

假设随机向量$\xi = (X_1, \cdots, X_n)$的密度为$p(x_1, \cdots, x_n)$.

$Y = f(X_1, \cdots X_n)$ 则

$$EY = \int_{\mathbb{R}^n} f(x_1, \cdots, x_n)p(x_1, \cdots, x_n)dx_1 \cdots dx_n.$$

4. n **维正态分布(定义6.8).** $\xi = (X_1, \cdots, X_n) \sim N(\vec{\mu}, \mathbf{\Sigma})$.

$$p(\vec{x}) = \frac{1}{\sqrt{2\pi}^n \sqrt{|\mathbf{\Sigma}|}} \exp\{-\frac{1}{2}(\vec{x}-\vec{\mu})\mathbf{\Sigma}^{-1}(\vec{x}-\vec{\mu})^T\}.$$

- $\mu_i = EX_i$, $\sigma_{ij} = \operatorname{cov}(X_i, X_j)$, $\Sigma = (\sigma_{ij})$.
  $n = 1$时: $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\vec{\mu} = (\mu)$, $\Sigma = (\sigma^2)$.
- $p_{X,Y}(x,y) = Ce^{-ax^2 - by^2 + cxy + dx + ey}$.
- **例4.1, 4.2** 正态向量的(非退化)线性变换还是正态向量.
  $(U, V) = (X, Y)A$. $x = \alpha u + \beta v$, $y = \gamma u + \delta v$.
  $p_{U,V}(u,v) = \frac{1}{|\det A|} p_{X,Y}(x,y) = \tilde{C}e^{-\tilde{a}u^2 - \tilde{b}v^2 + \tilde{c}uv + \tilde{d}u + \tilde{e}v}$.
- 边缘分布, 条件分布都是正态.

例: $p(x,y) = Ce^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2+\left(\frac{y-\mu_2}{\sigma_2}\right)^2-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right)}$.

- $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$

$$\Sigma^{-1} = \frac{1}{\det\Sigma = (1-\rho^2)\sigma_1^2\sigma_2^2}\begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$
$$= \frac{1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}$$

- $p(x,y) = C\exp\{-\frac{1}{2}(\vec{x}-\vec{\mu})\mathbf{\Sigma}^{-1}(\vec{x}-\vec{\mu})^T\}$.
- 当$\mathrm{cov}(X,Y)=0$, 即$\rho=0$ 时, X,Y 相互独立.

正态情形: 独立$\Leftrightarrow$ 不相关.

X_i 与X_j 独立$\Leftrightarrow \sigma_{ij} = 0$.

$(X_{i_1}, \cdots, X_{i_k})$ 与$(X_{j_1}, \cdots, X_{j_\ell})$ 独立$\Leftrightarrow \sigma_{i_r, j_s} = 0, \forall r \leqslant k, s \leqslant \ell$.

例: 假设$(X, Y) \sim N(\vec{\mu}, \Sigma)$. 已知$\vec{\mu}, \Sigma$, 求条件密度$p_{Y|X}(y|x)$.

- 不妨设$\vec{\mu} = 0$. 做正交分解: $Y = aX + Z$, 其中$E(ZX) = 0$, 即$\mathrm{cov}(Y, X) = a \times \mathrm{cov}(X, X)$ $(\Rightarrow a = \rho\frac{\sigma_2}{\sigma_1})$.
- 取$a = \rho\frac{\sigma_2}{\sigma_1}$. 则$Z = Y - aX \sim N(0, \sigma^2)$, 其中$\sigma^2 = E(Y - aX)^2 = (1 - \rho^2)\sigma_2^2$.
- 在$X = x$ 的条件下, $Y = ax + Z \sim N(\rho\frac{\sigma_2}{\sigma_1}x, (1 - \rho^2)\sigma_2^2)$.

5. 随机数目的期望

匹配问题(第一章例3.10) n 封信, n 个信封, 随机装(一个信封装一封信), 装对了X 封, 求EX.

- 若第n 封信装入正确的信封, 令$X_i = 1$, 否则令$X_i = 0$.
- $X = X_1 + \cdots + X_n$. $EX = \sum_i EX_i = n \times \frac{1}{n} = 1$.
- 自习例6.5, 6.6.

6. 独立随机变量的最大、最小值

例4.8. 假设$X_1, \cdots, X_n$ 相互独立, $X_i \sim \mathrm{Exp}(\lambda_i)$, $\forall i$. 则$Y = \min_i X_i \sim \mathrm{Exp}(\lambda)$, 其中$\lambda = \lambda_1 + \cdots + \lambda_n$.

- $P(Y > x) = P(X_i > x, \forall i) = \prod_i P(X_i > x) = e^{-\lambda x}$.
- 例4.6: $P(\max_i X_i \leqslant x) = \prod_i P(X_i \leqslant x)$.
- 假设$Y = X_I$, 则$P(I = i) \propto \lambda_i$, 且$I$ 与Y 独立. 因为

$$\begin{aligned} & P(Y > x, I = i) = P(x < X_i < X_j, \forall j \neq i) \\ = & \int_x^{\infty} \lambda_i e^{-\lambda_i x_i} \left(\int_{x_j > x_i, \forall j \neq i} \prod_{j \neq i} (\lambda_j e^{-\lambda_j x_j} dx_j) \right) dx_i \\ = & \int_x^{\infty} \lambda_i e^{-\lambda_i x_i} e^{-\sum_{j \neq i} \lambda_j x_i} dx_i = \frac{\lambda_i}{\lambda} e^{-\lambda x}. \end{aligned}$$

(不要求)

- 自习次序统计量(定理6.7, 不要求)