Lecture 2: Part 2

Utility

Utility Functions

$U(x_1, x_2, \ldots, x_n)$ is a function that assigns a value to each consumption bundle $(x_1, x_2, \ldots, x_n)$

e.g. U(1 corn, 2 eggs) = 5

U(2 beer, 5 chicken wings) = 10

效用函数是将每一个商品组合转化为一个数值的函数

Utility Functions

A utility function U(x) **represents** a preference relation $\succsim$ if and only if:

$x' \succ x''$ ⟺ $U(x') > U(x'')$

$x' \prec x''$ ⟺ $U(x') < U(x'')$

$x' \sim x''$ ⟺ $U(x') = U(x'')$.

若效用函数对任意两个商品组合的排序与消费者对它们的偏好次序一致，该效用函数代表了偏好关系

Utility Functions

Consider the bundles (4,1), (2,3) and (2,2).

Suppose (2,3) $\succ$ (4,1) ~ (2,2).

Assign to these bundles any numbers that **preserve the preference ordering**;

e.g. $U(x_1, x_2) = x_1 x_2$;

U(2,3) = 6 > U(4,1) = U(2,2) = 4.

Call these numbers **utility levels**.

Utility Functions

Utility is an **ordinal** (i.e. ordering) concept.

E.g. if U(x) = 6 and U(y) = 2 then bundle x is strictly preferred to bundle y. But x is not preferred three times as much as is y.

效用函数是一个次序的概念，绝对数值本身没有意义。

Utility Functions

There is no unique utility function representation of a preference relation.

代表同一个偏好关系的效用函数并不是唯一的。

Utility Functions

$U(x_1,x_2) = x_1x_2$, so

$U(2,3) = 6 > U(4,1) = U(2,2) = 4$;

that is, $(2,3) \succ (4,1) \sim (2,2)$.

Utility Functions

$U(x_1,x_2) = x_1x_2$ $\Rightarrow$ $(2,3) \succ (4,1) \sim (2,2).$

$U(2,3) = 6 > U(4,1) = U(2,2) = 4$

Define $V = U^2 = x_1^2x_2^2$

Utility Functions

$U(x_1,x_2) = x_1x_2 \Rightarrow (2,3) \succ (4,1) \sim (2,2).$

$U(2,3) = 6 > U(4,1) = U(2,2) = 4$

Define $V = U^2 = x_1^2x_2^2$

then $V(2,3) = 36 > V(4,1) = V(2,2) = 16.$

so again

$(2,3) \succ (4,1) \sim (2,2)$

V preserves the same order as U and so represents the same preferences.

Utility Functions

$U(x_1,x_2) = x_1x_2$ $\Rightarrow$ $(2,3) \succ (4,1) \sim (2,2)$.

$U(2,3) = 6 > U(4,1) = U(2,2) = 4$

Define $W = 2U + 10 = 2x_1x_2+10$

Utility Functions

$U(x_1,x_2) = x_1x_2 \Rightarrow (2,3) \succ (4,1) \sim (2,2).$

$U(2,3) = 6 > U(4,1) = U(2,2) = 4$

Define $W = 2U + 10 = 2x_1x_2+10$

then $W(2,3) = 22 > W(4,1) = W(2,2) = 18.$

Again,

$(2,3) \succ (4,1) \sim (2,2)$

W preserves the same order as U and V and so represents the same preferences.

Utility Functions

If

– U is a utility function that represents a preference relation $\succsim$ and

– f is a **strictly increasing** function,

then V = f(U) is also a utility function representing $\succsim$.

严格单增变换后的效用函数仍代表相同的偏好关系

Utility Functions

$\forall$ x and y, $x \succ y$,

$$U(x) > U(y)$$

Because $V = f(U)$ is a **strictly increasing** function,

$$f(U(x)) > f(U(y))$$

$$\text{i.e. } V(x) > V(y)$$

V(.) preserves the same preference order as U(.) => V and U represent the same preferences

Utility Functions

A preference relation that is complete, reflexive, transitive and **continuous** can be represented by a **continuous** utility function.

具备完备性、自反性、传递性和连续性的偏好关系总可以被一个连续的效用函数所描述。

Utility Functions

Continuity means that small changes to a consumption bundle cause only small changes to the preference level.

$\forall x \succ y$, $\exists \delta > 0$ **such that** $\forall x' \in U_\delta(x)$, $x' \succ y$

若x严格偏好于y，则非常接近x的商品组合也严格偏好于y

Utility Functions

- Continuity means that small changes to a consumption bundle cause only small changes to the preference level.
  - e.g. (1.001 corn, 2 eggs) can’t be “too” preferred over (1 corn, 2 eggs). Otherwise, there will be a “jump” in U(.) from (1,2) to (1.001, 2)

Utility Functions & Indiff. Curves

- An indifference curve contains equally preferred bundles.
- Equal preference $\Rightarrow$ same utility level.
- Therefore, all bundles on an indifference curve have the same utility level.

同一条无差异曲线上的所有商品组合具有同样的效用值

Utility Functions & Indiff. Curves

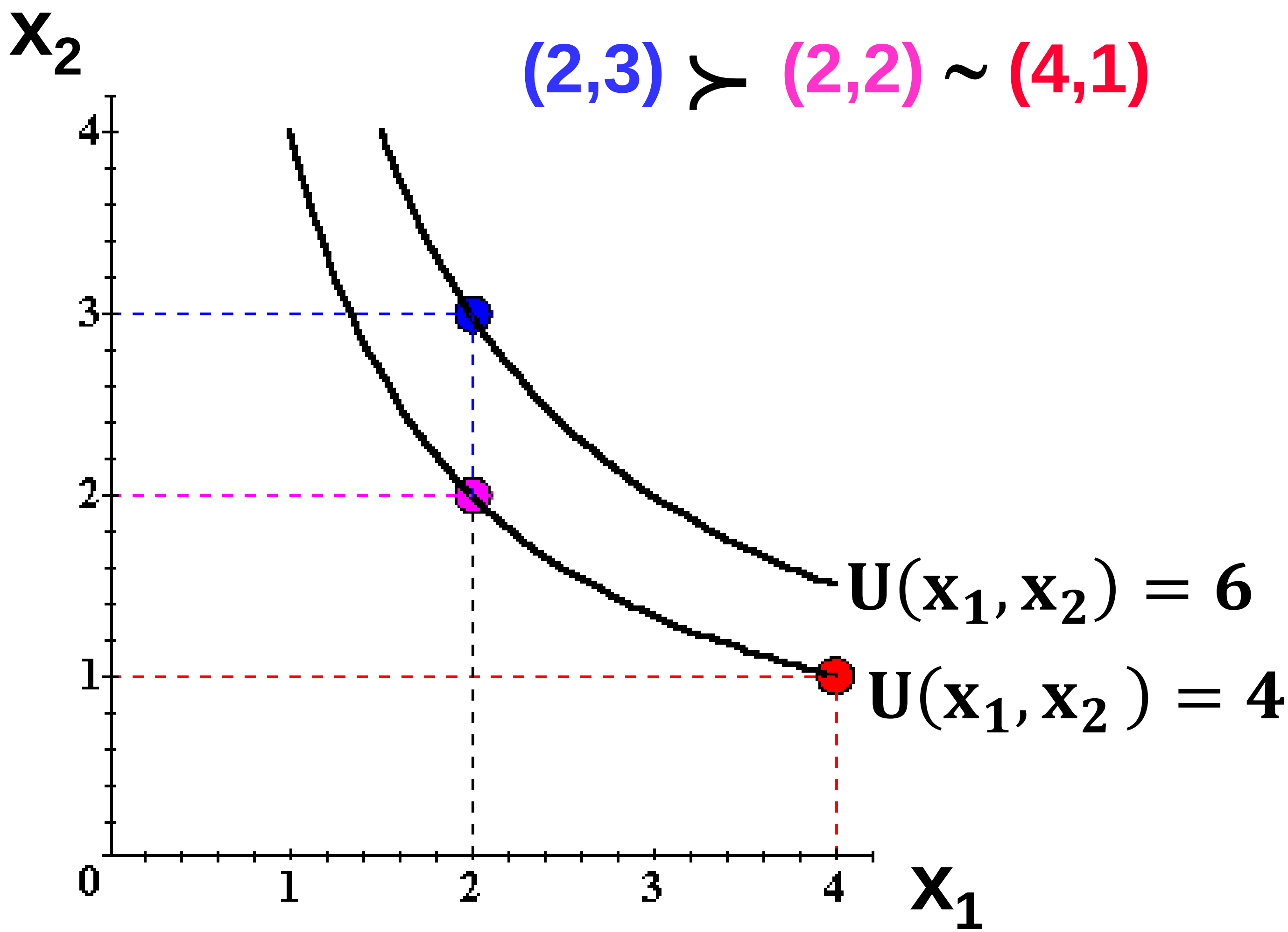
- Suppose $U(x_1, x_2) = x_1 x_2$

$$(2,3) \succ (4,1) \sim (2,2)$$

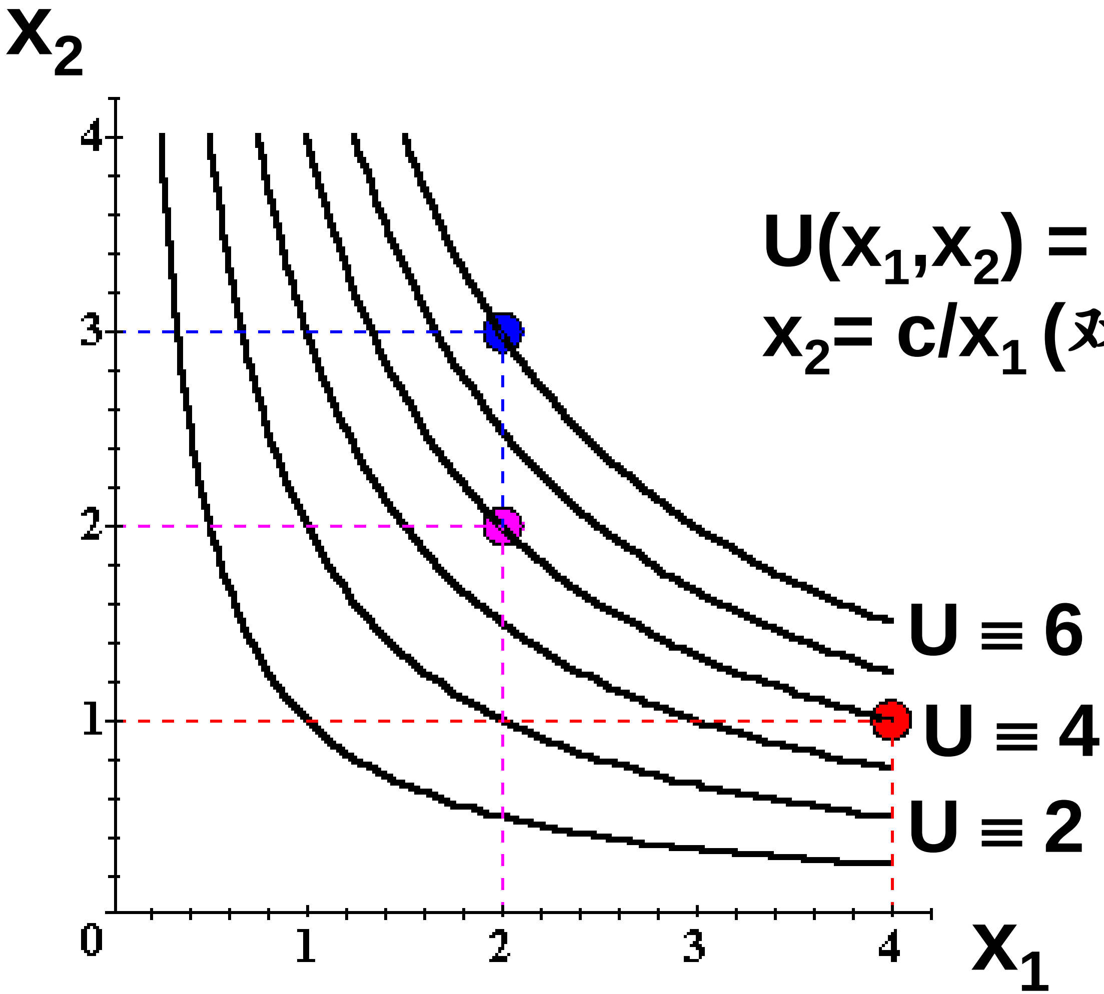
$$U(4,1) = U(2,2) = 4$$

- The indifference curve that passes through (4,1) and (2,2) is represented by $U(x_1, x_2) = x_1 x_2 = 4$
- On an indifference curve diagram, this preference information looks as follows:

Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves



$U(x_1,x_2) = x_1x_2 = c;$
$x_2 = c/x_1$ (双曲线)

Cobb-Douglas Utility Functions and Their Indifference Curves

Any utility function of the form

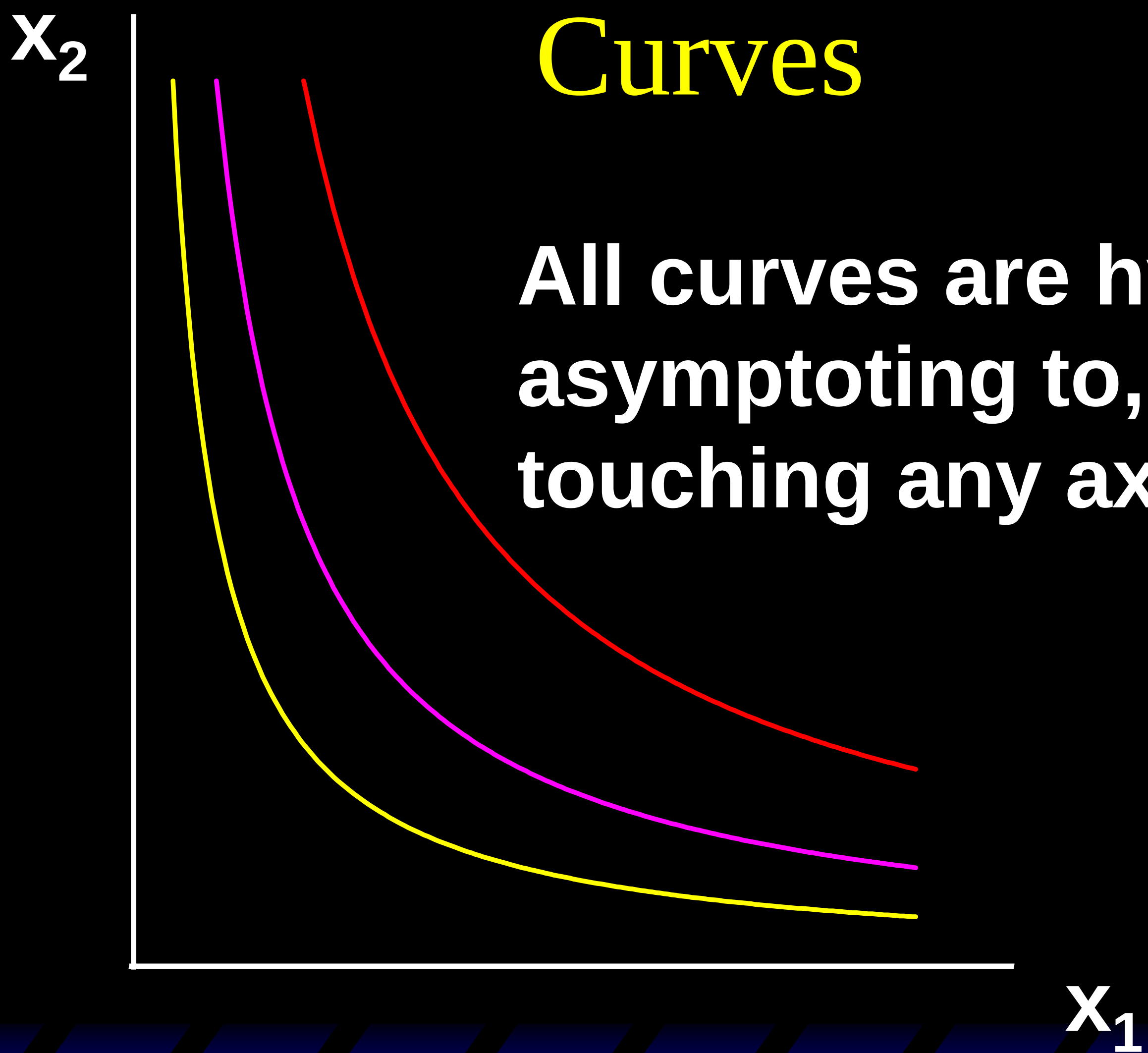
$$U(x_1,x_2) = x_1^a x_2^b$$

with $a > 0$ and $b > 0$ is called a Cobb-Douglas utility function.

E.g. $U(x_1,x_2) = x_1^{1/2} x_2^{1/2}$ **(a = b = 1/2)**

$V(x_1,x_2) = x_1 x_2^3$ **(a = 1, b = 3)**

Cobb-Douglas Indifference Curves



All curves are hyperbolic, asymptoting to, but never touching any axis.

无限逼近坐标轴的双曲线

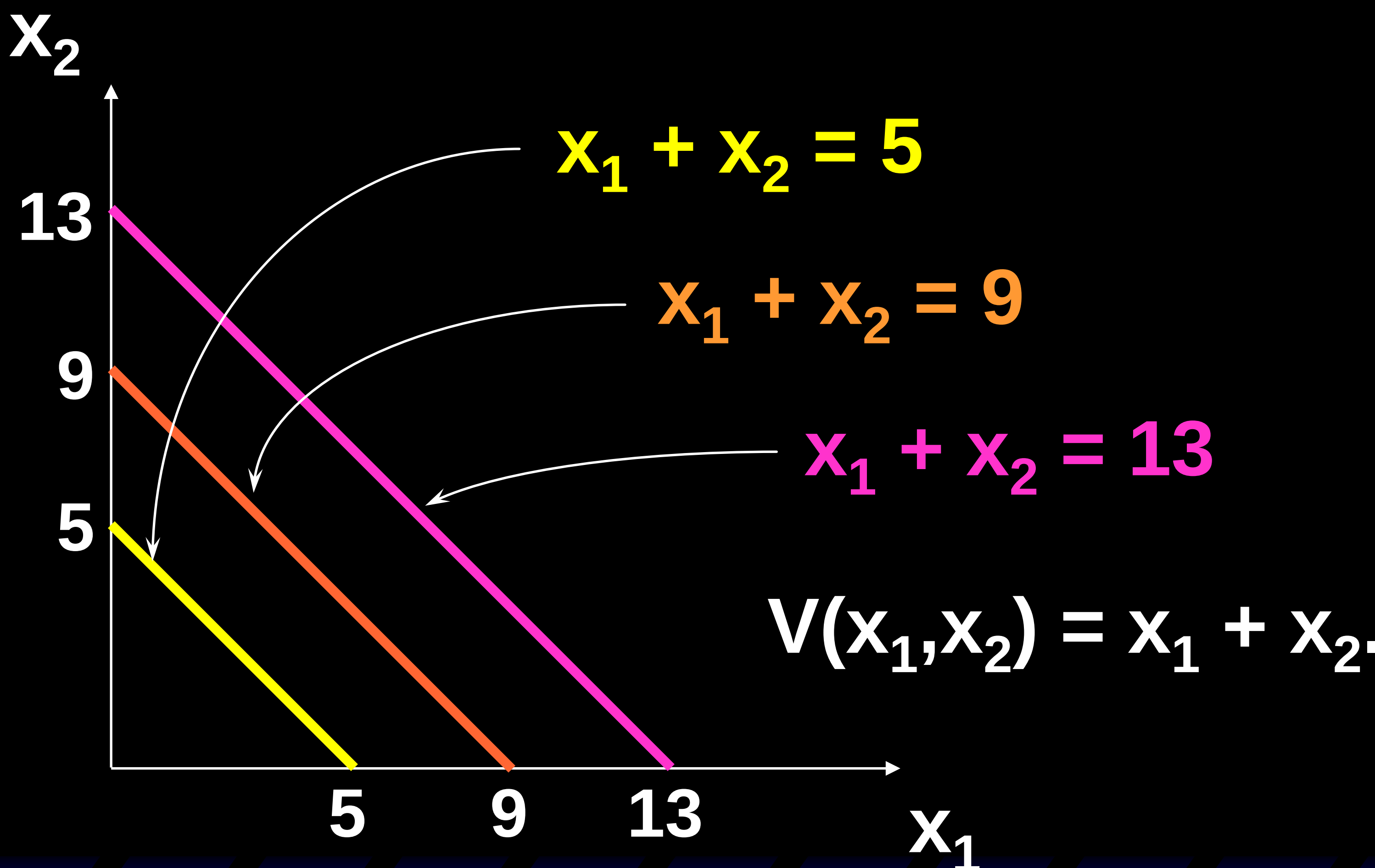
Some Other Utility Functions and Their Indifference Curves

Instead of $U(x_1,x_2) = x_1x_2$ consider

$$V(x_1,x_2) = x_1 + x_2.$$

What do the indifference curves for this utility function look like?

Perfect Substitution Indifference Curves



All curves are linear and parallel to each other (完全替代效用函数).

Some Other Utility Functions and Their Indifference Curves

$V(x_1,x_2) = x_1 + x_2$.

$\Delta x_1 = 1$, $\Delta x_2 = -1$ => same utility level

i.e. $1x_1 \sim 1x_2$

Perfect-substitutes

Some Other Utility Functions and Their Indifference Curves

- If **one** unit of x_1 is exactly as preferred as **two** units of x_2, what is the utility function?

Some Other Utility Functions and Their Indifference Curves

If 1 unit of x_1 is exactly as preferred as two units of x_2, the utility function can be given as

$$V(x_1,x_2) = 2x_1 + x_2.$$

e.g. good 1 - \$20 bill, good 2 - \$10 bill

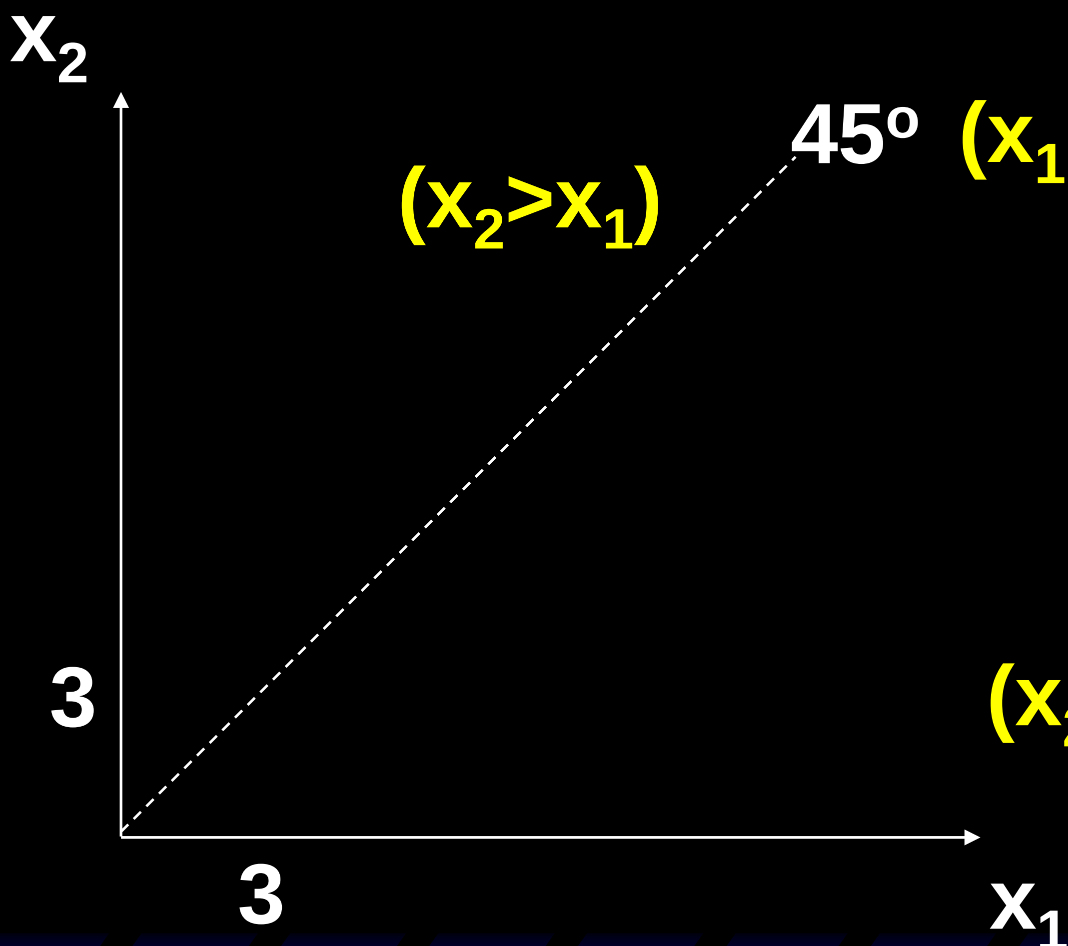
Some Other Utility Functions and Their Indifference Curves

Instead of $U(x_1,x_2) = x_1x_2$ or $V(x_1,x_2) = x_1 + x_2$, consider

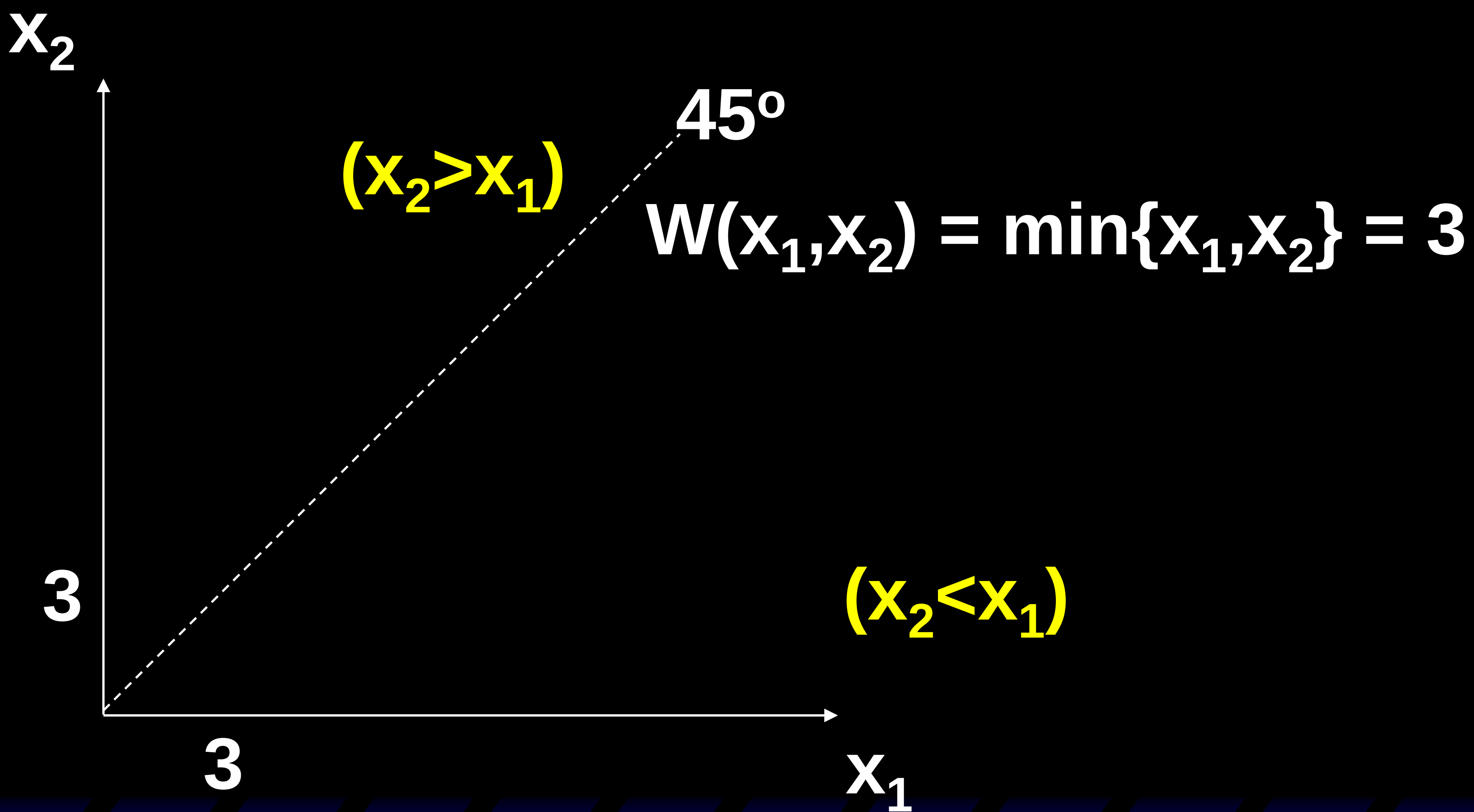
$$W(x_1,x_2) = \min\{x_1,x_2\}.$$

What do the indifference curves for this utility function look like?

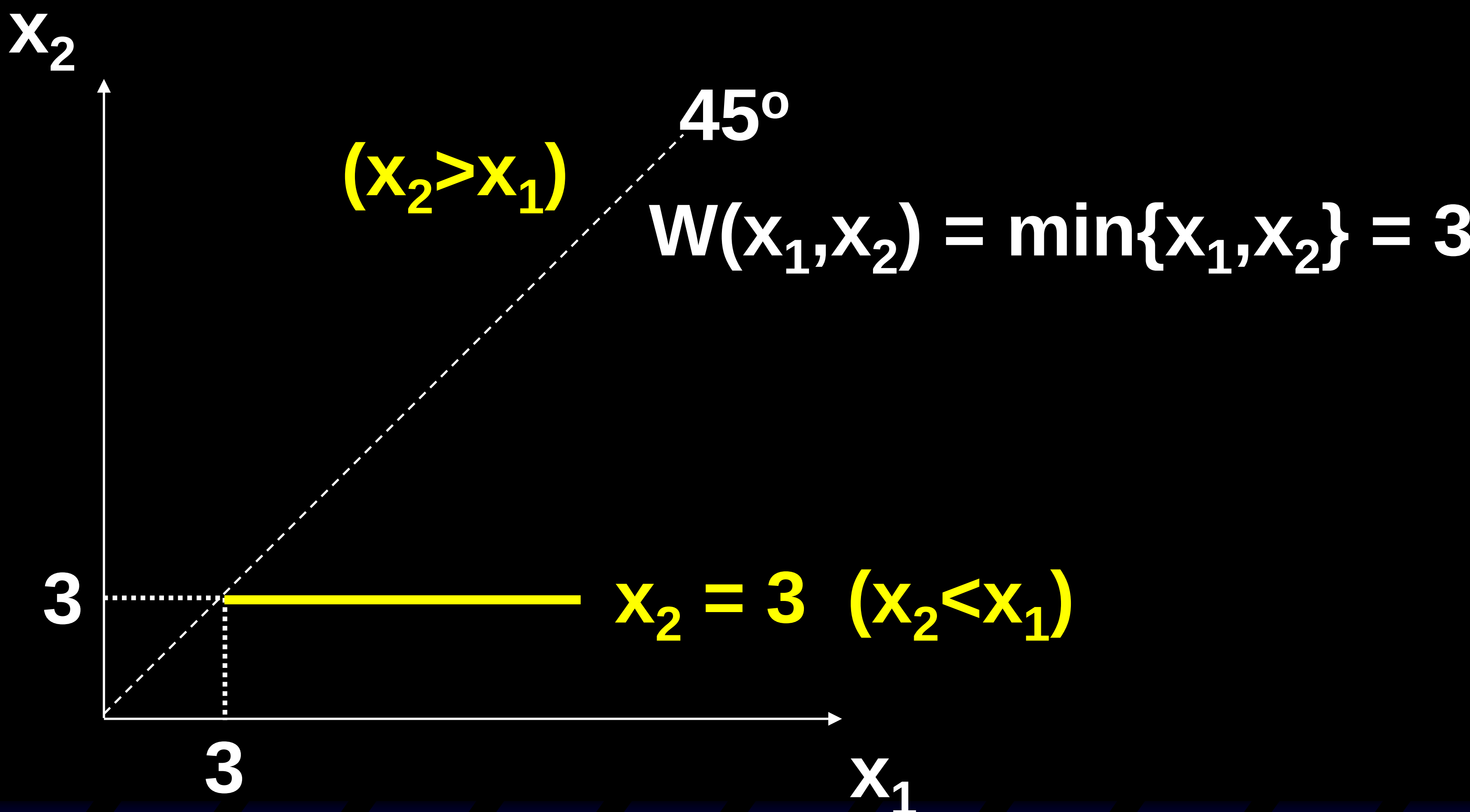
Perfect Complementarity Indifference Curves



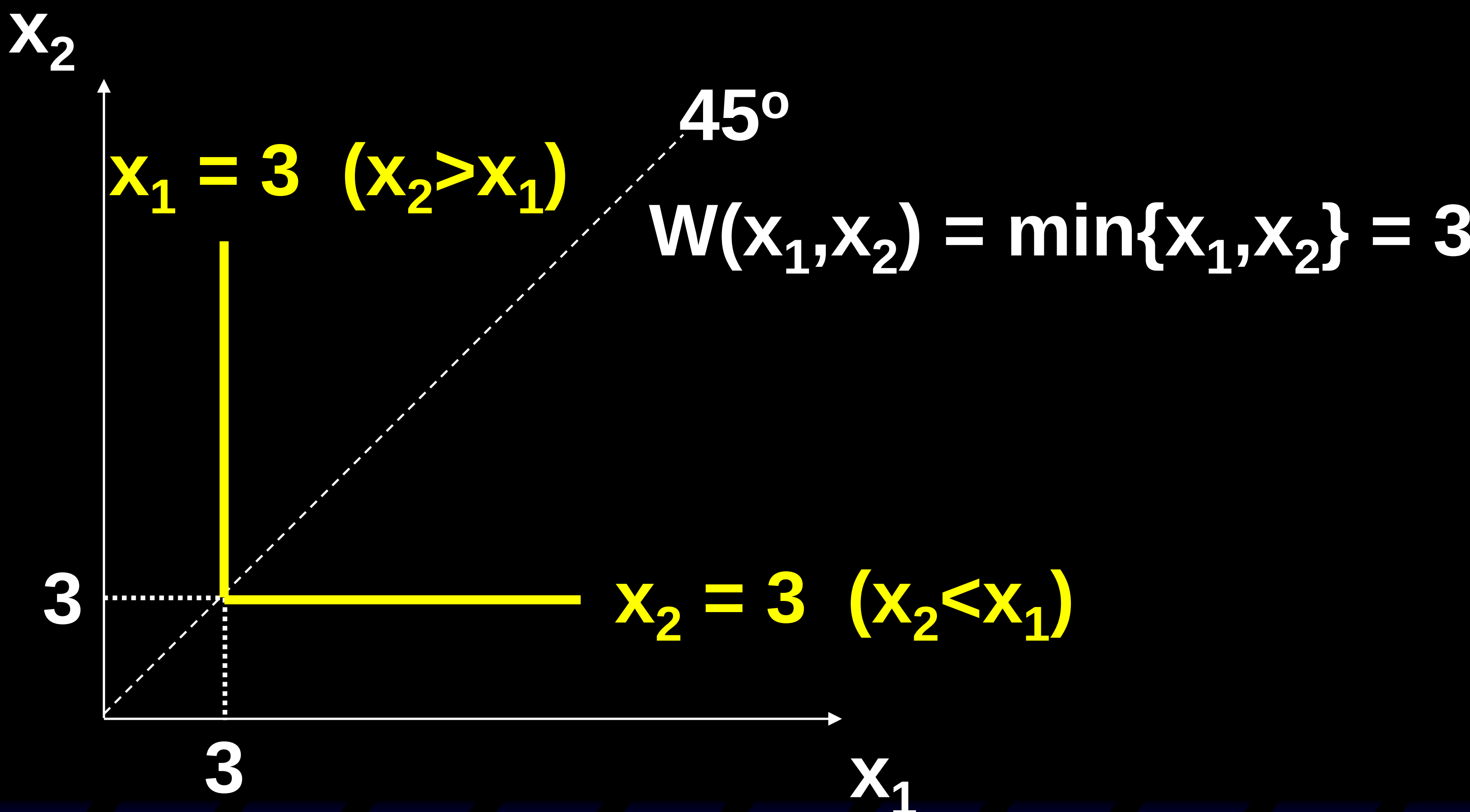
Perfect Complementarity Indifference Curves



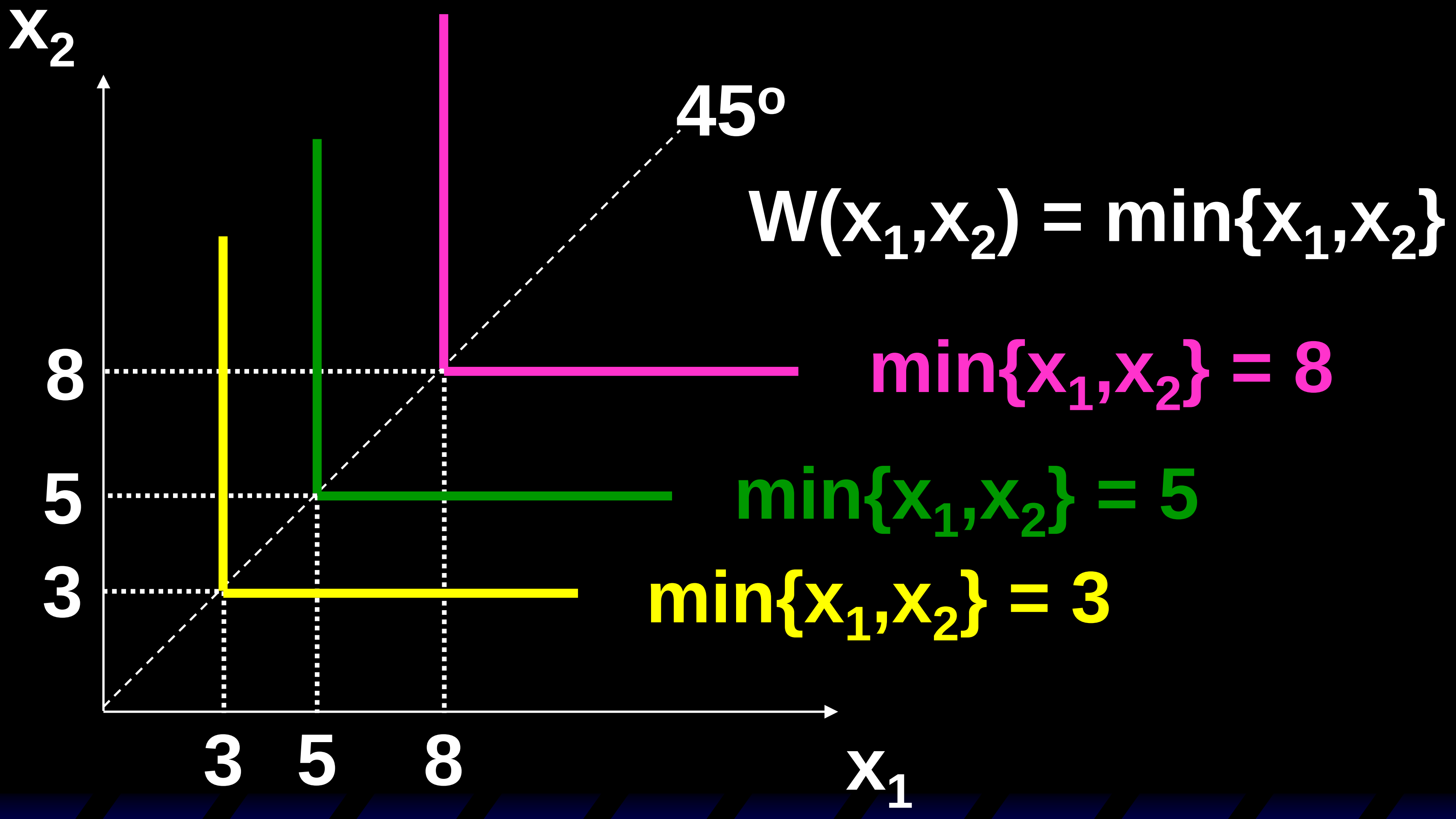
Perfect Complementarity Indifference Curves



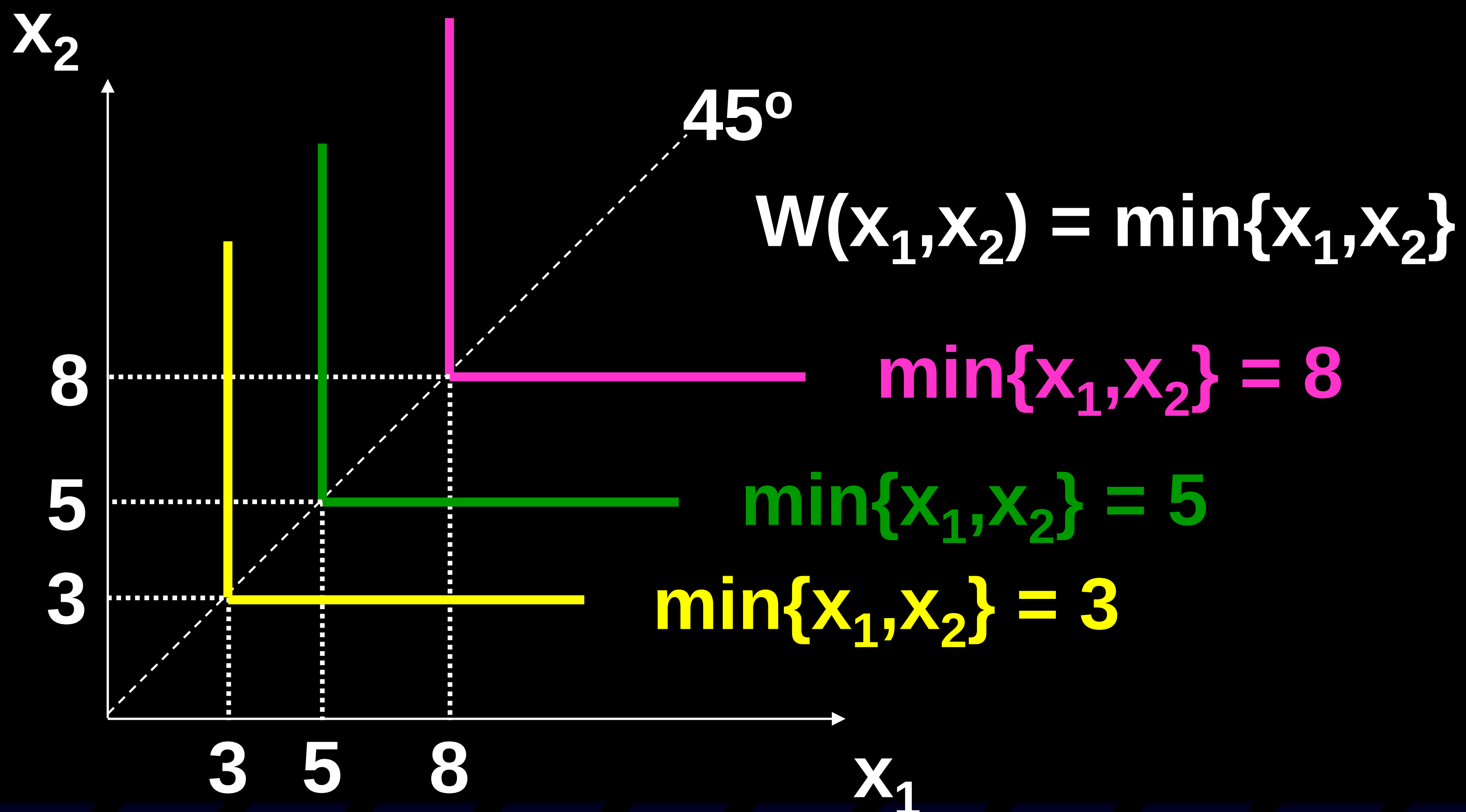
Perfect Complementarity Indifference Curves



Perfect Complementarity Indifference Curves



Perfect Complementarity Indifference Curves



All are right-angled with vertices on a ray from the origin.

Some Other Utility Functions and Their Indifference Curves

If 1 unit of good 1 is always consumed with 2 units of good 2, which of the following utility function is correct?

$$W(x_1,x_2) = \min\{2x_1,x_2\}$$

$$W'(x_1,x_2) = \min\{x_1,2x_2\}.$$

Some Other Utility Functions and Their Indifference Curves

If 1 unit of good 1 is always consumed with 2 units of good 2, which of the following utility function is correct?

$$W(x_1,x_2) = \min\{2x_1,x_2\}$$

$$W'(x_1,x_2) = \min\{x_1,2x_2\}.$$

Some Other Utility Functions and Their Indifference Curves

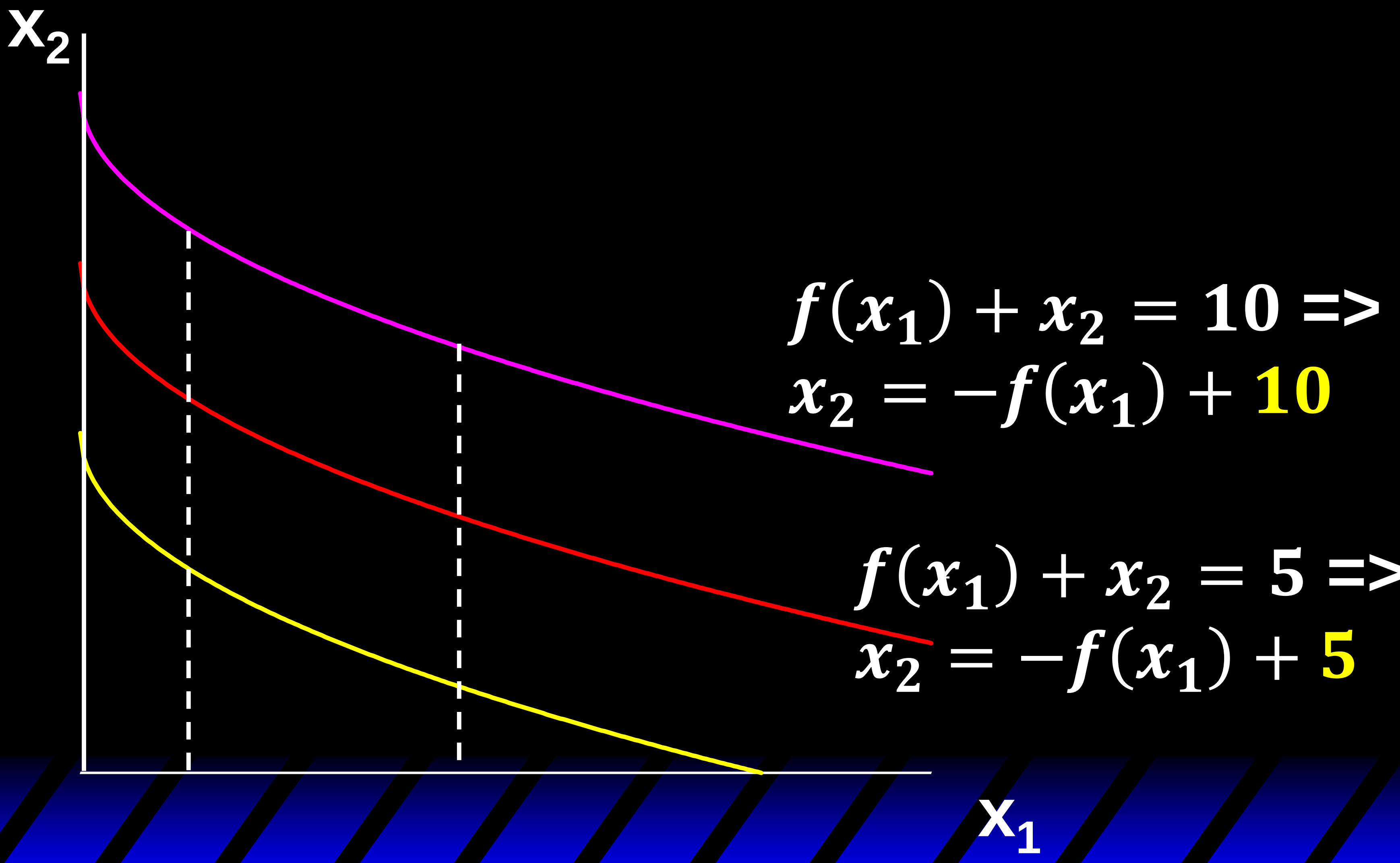
A utility function of the form

$$U(x_1,x_2) = f(x_1) + x_2$$

is non-linear in x_1 and linear in x_2 and is called quasi-linear.

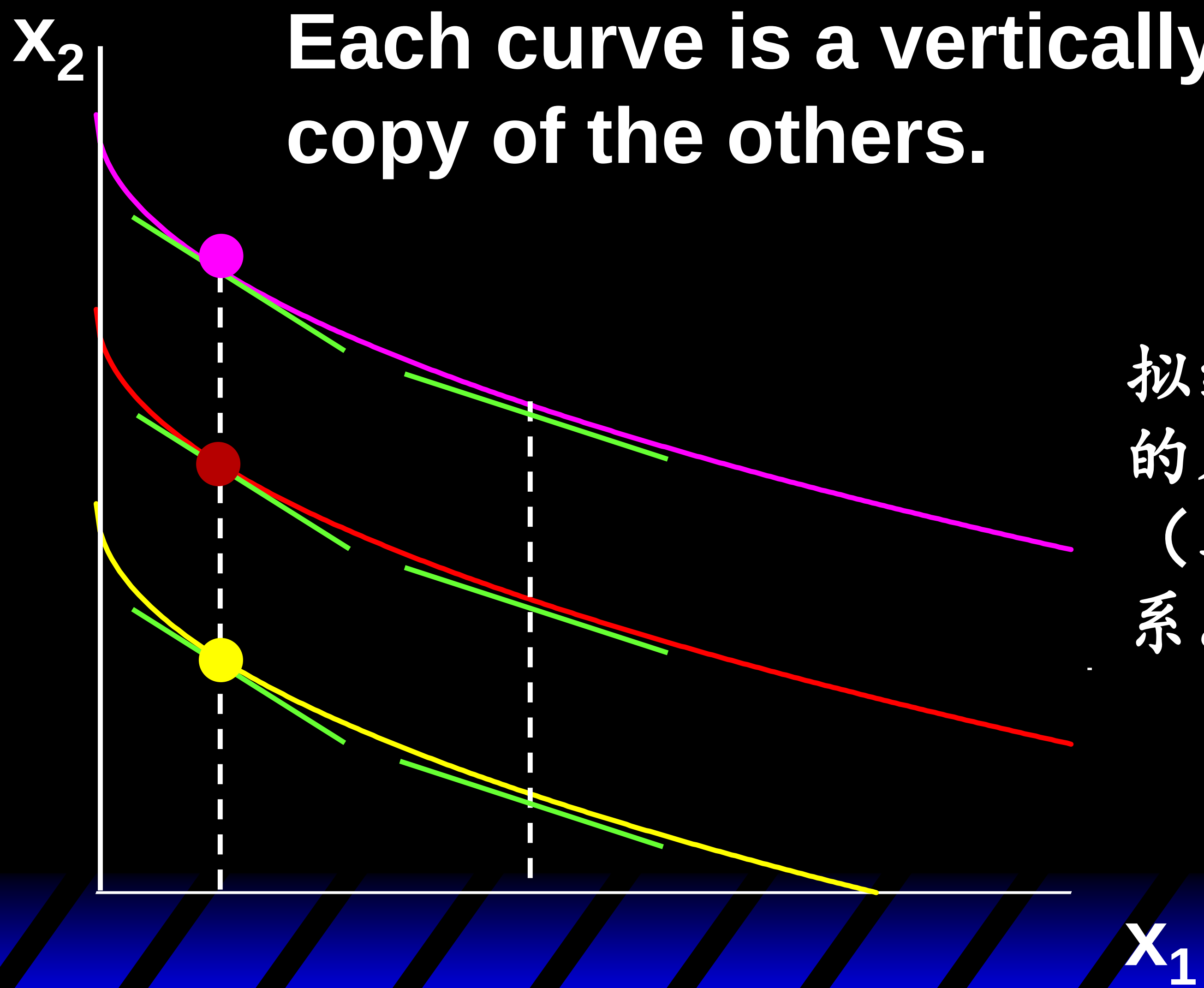
E.g. **$U(x_1,x_2) = 2x_1^{1/2} + x_2$.**

Quasi-linear Indifference Curves



Quasi-linear Indifference Curves

Each curve is a vertically shifted copy of the others.



拟线性效用函数下的无差异曲线互为（垂直）平移的关系。

Marginal Utilities

Marginal means “incremental”.

The marginal utility (边际效用) of commodity i is the rate-of-change of total utility as the quantity of commodity i consumed changes; *i.e.*

$$MU_i = \frac{\partial U}{\partial x_i}$$

Marginal Utilities

E.g. if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

$$MU_1 = \frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-1/2} x_2^2$$

Marginal Utilities

E.g. if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

$$MU_2 = \frac{\partial U}{\partial x_2} = 2x_1^{1/2}x_2$$

Marginal Utilities

So, if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

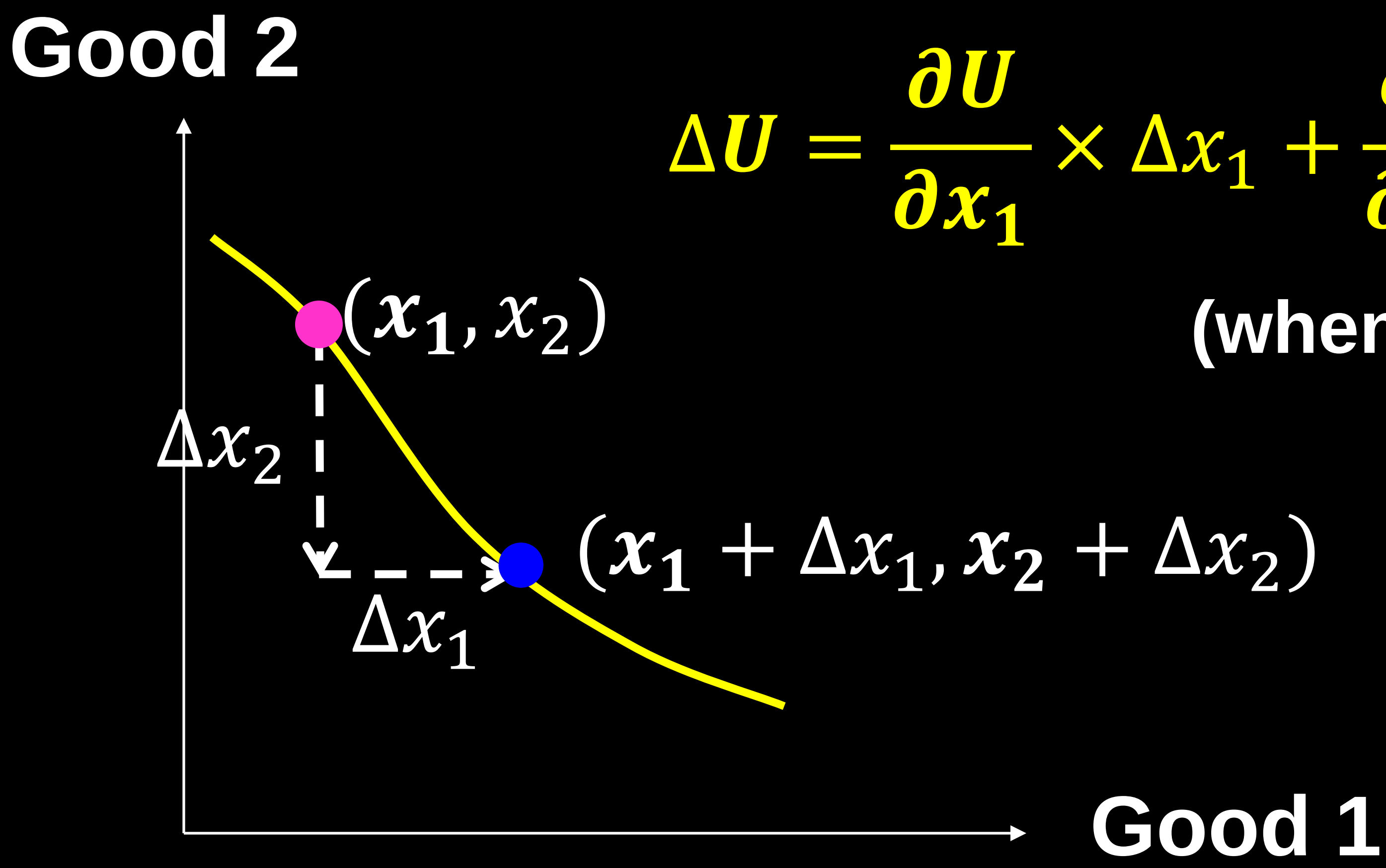
$$MU_1 = \frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-1/2} x_2^2$$

$$MU_2 = \frac{\partial U}{\partial x_2} = 2 x_1^{1/2} x_2$$

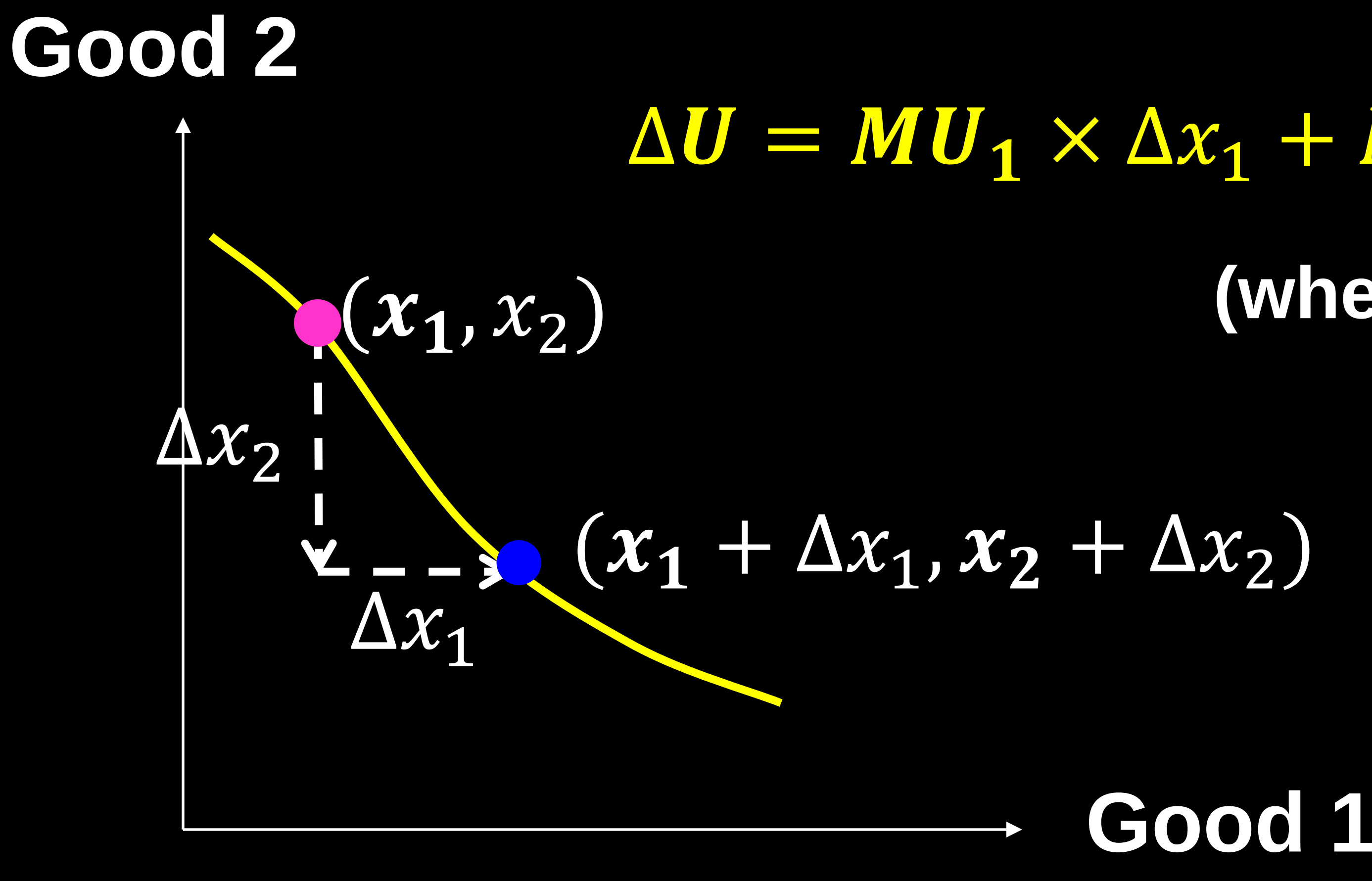
Marginal Utilities

$$\Delta U = \frac{\partial U}{\partial x_1} \times \Delta x_1 + \frac{\partial U}{\partial x_2} \times \Delta x_2$$

(when $\Delta x_1 \to 0$)



Marginal Utilities



$$\Delta U = MU_1 \times \Delta x_1 + MU_2 \times \Delta x_2$$

(when $\Delta x_1 \to 0$**)**

Marginal Utilities and Marginal Rates-of-Substitution

$$\Delta \boldsymbol{U} = \boldsymbol{MU_1} \times \Delta x_1 + \boldsymbol{MU_2} \times \Delta x_2 = 0$$

(when $\Delta x_1 \rightarrow 0$**)**

Marginal Utilities and Marginal Rates-of-Substitution

$$\Delta \boldsymbol{U} = \boldsymbol{MU_1} \times \Delta x_1 + \boldsymbol{MU_2} \times \Delta x_2 = 0$$

$$\frac{\Delta x_2}{\Delta x_1} = -\frac{\boldsymbol{MU_1}}{\boldsymbol{MU_2}}$$

Marginal Utilities and Marginal Rates-of-Substitution

$$\frac{\Delta x_2}{\Delta x_1} = -\frac{\boldsymbol{MU_1}}{\boldsymbol{MU_2}}$$

When $\Delta \boldsymbol{x_1} \to \mathbf{0}$,

$\frac{\Delta x_2}{\Delta x_1} =$ **slope of the indiff. curve at** (x_1, x_2)

= MRS

$$\boldsymbol{MRS = -\frac{MU_1}{MU_2}}$$

Marg. Utilities & Marg. Rates-of-Substitution; An example

Suppose $U(x_1,x_2) = x_1x_2$. Then

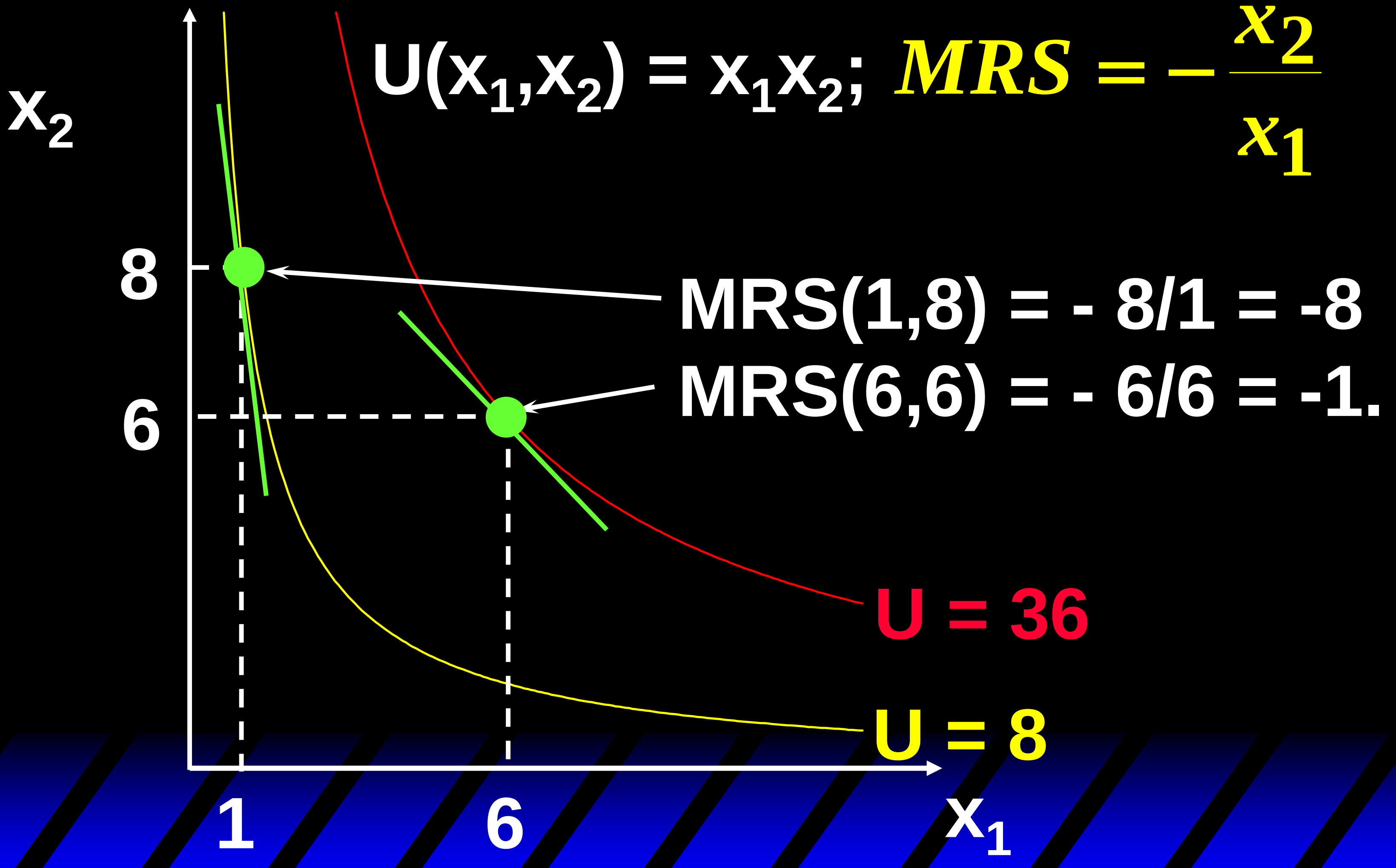
$$\frac{\partial U}{\partial x_1} = (1)(x_2) = x_2$$

$$\frac{\partial U}{\partial x_2} = (x_1)(1) = x_1$$

so $$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = -\frac{x_2}{x_1}.$$

Marg. Utilities & Marg. Rates-of-Substitution; An example

$U(x_1,x_2) = x_1x_2$; $MRS = -\frac{x_2}{x_1}$



Marg. Rates-of-Substitution for Quasi-linear Utility Functions

A quasi-linear utility function is of the form $U(x_1,x_2) = f(x_1) + x_2$.

$$\frac{\partial U}{\partial x_1} = f'(x_1) \qquad \frac{\partial U}{\partial x_2} = 1$$

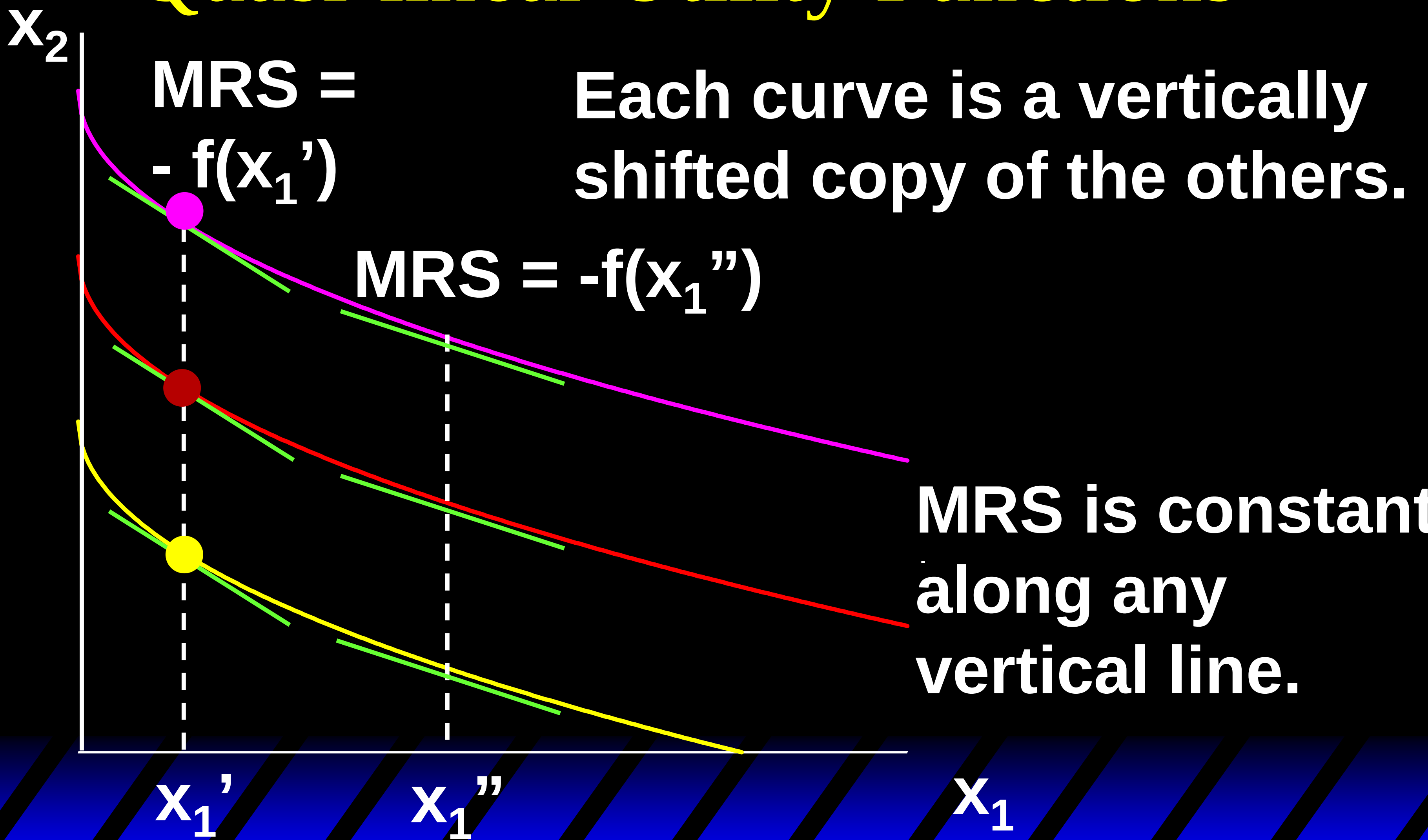
so $$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2} = -f'(x_1).$$

Marg. Rates-of-Substitution for Quasi-linear Utility Functions

MRS = - $f'(x_1)$ does not depend upon x_2 so the slope of indifference curves for a quasi-linear utility function is constant along any line for which x_1 is constant (i.e. vertical line).

拟线性效用函数的MRS只与x_1（非线性商品的数量）相关

Marg. Rates-of-Substitution for Quasi-linear Utility Functions



Monotonic Transformations & Marginal Rates-of-Substitution

Applying a **monotonic transformation** to a utility function simply creates another utility function representing the **same** preference relation.

What happens to marginal rates-of-substitution when a monotonic transformation is applied?

Monotonic Transformations & Marginal Rates-of-Substitution

When the preference is represented by $U(x_1, x_2)$,

$$\boldsymbol{MRS} = -\frac{\partial U/\partial x_1}{\partial U/\partial x_2}$$

Monotonic Transformations & Marginal Rates-of-Substitution

Suppose V = f(U) where f is a strictly increasing function, then

$$MRS = -\frac{\partial V/\partial x_1}{\partial V/\partial x_2} = -\frac{f'(U)\times\partial U/\partial x_1}{f'(U)\times\partial U/\partial x_2}$$

$$= -\frac{\partial U/\partial x_1}{\partial U/\partial x_2}.$$

MRS is unchanged by a positive monotonic transformation （单增变换不改变给定商品组合处的MRS)

Monotonic Transformations & Marginal Rates-of-Substitution

For $U(x_1,x_2) = x_1x_2$ the MRS $= - x_2/x_1$.

Create $V = U^2$; *i.e.* $V(x_1,x_2) = x_1^2x_2^2$.

What is the MRS for V?

$$MRS = -\frac{\partial V / \partial x_1}{\partial V / \partial x_2} = -\frac{2x_1x_2^2}{2x_1^2x_2} = -\frac{x_2}{x_1}$$

which is the same as the MRS for U.