Lecture 10

Technology

Technologies

- **A technology is a process by which inputs are converted to an output.**

  E.g. **labor, a computer, a projector, electricity, and software are being combined to produce this lecture.**
- **Technologies describe the constraints faced by producers.**

投入与产出的转化关系被称为技术，它描述了厂商所面临的生产约束

Technologies

Today's lecture:

- **the description of technology**
- **the properties of technology**

Input Bundles

- x_i denotes the amount used of input i; *i.e.* the level of input i.
- An **input bundle** is a vector of the input levels; $(x_1, x_2, \ldots, x_n)$.
- *E.g.* $(x_1, x_2, x_3) = (6, 0, 9)$.

我们用一个包含不同种要素使用数量的坐标来表示 投入组合/要素组合

Production Functions

- y denotes the output level.
- The technology's **production function** states the **maximum** amount of output possible from an input bundle.

$$y = f(x_1, \cdots, x_n)$$

我们用**生产函数**来描述技术，它表示一个给定要素组合所能带来的最高产出

Production Functions

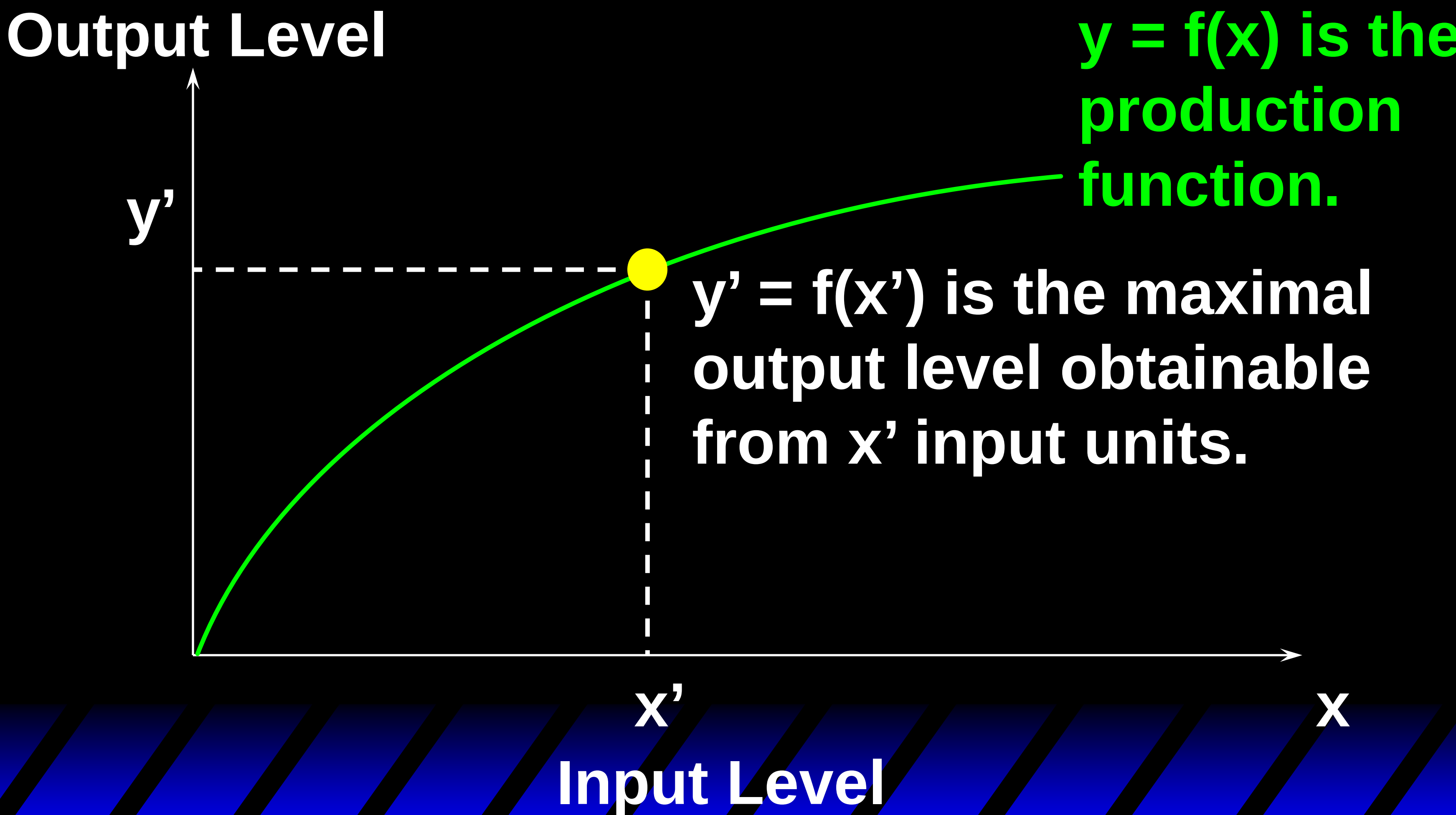
- **A simple case with only one input:**

$$y = f(x) = \sqrt{x}$$

i.e. x units of the input can produce at most $\sqrt{x}$ units of output

Production Functions

One-input case



Technology Sets

- A **production plan** is an input bundle and an output level; $(x_1, \ldots, x_n, y)$.
- A production plan is **feasible** if

$$y \le f(x_1, \cdots, x_n)$$

- The collection of all feasible production plans is the **technology set (生产集)**.

所有可行的生产计划的集合被成为生产集。

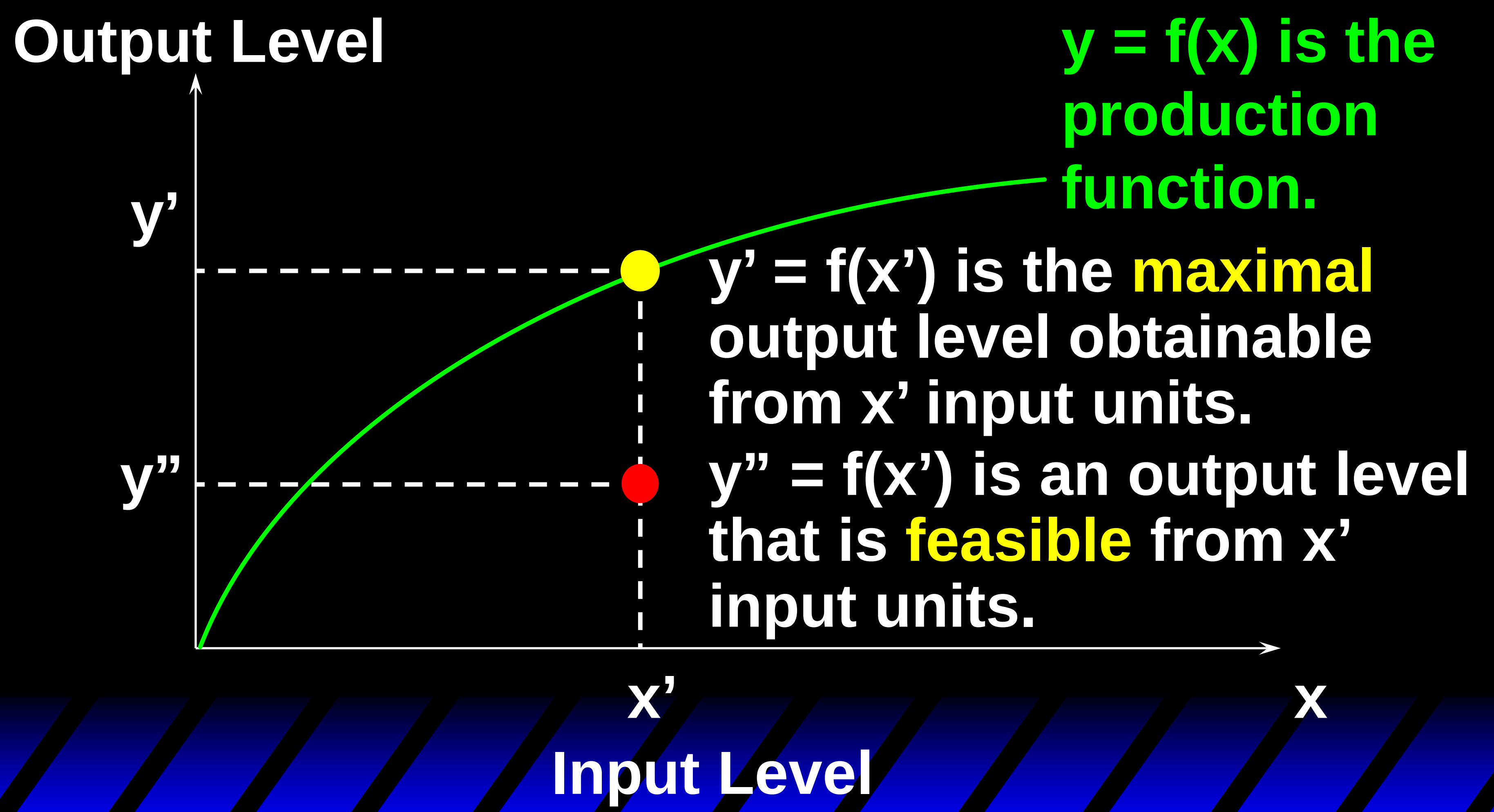
Technology Sets

The **technology set** is

$$T = \{(x_1, \cdots, x_n, y) \mid y \le f(x_1, \cdots, x_n) \text{ and } x_1 \ge 0, \ldots, x_n \ge 0\}.$$

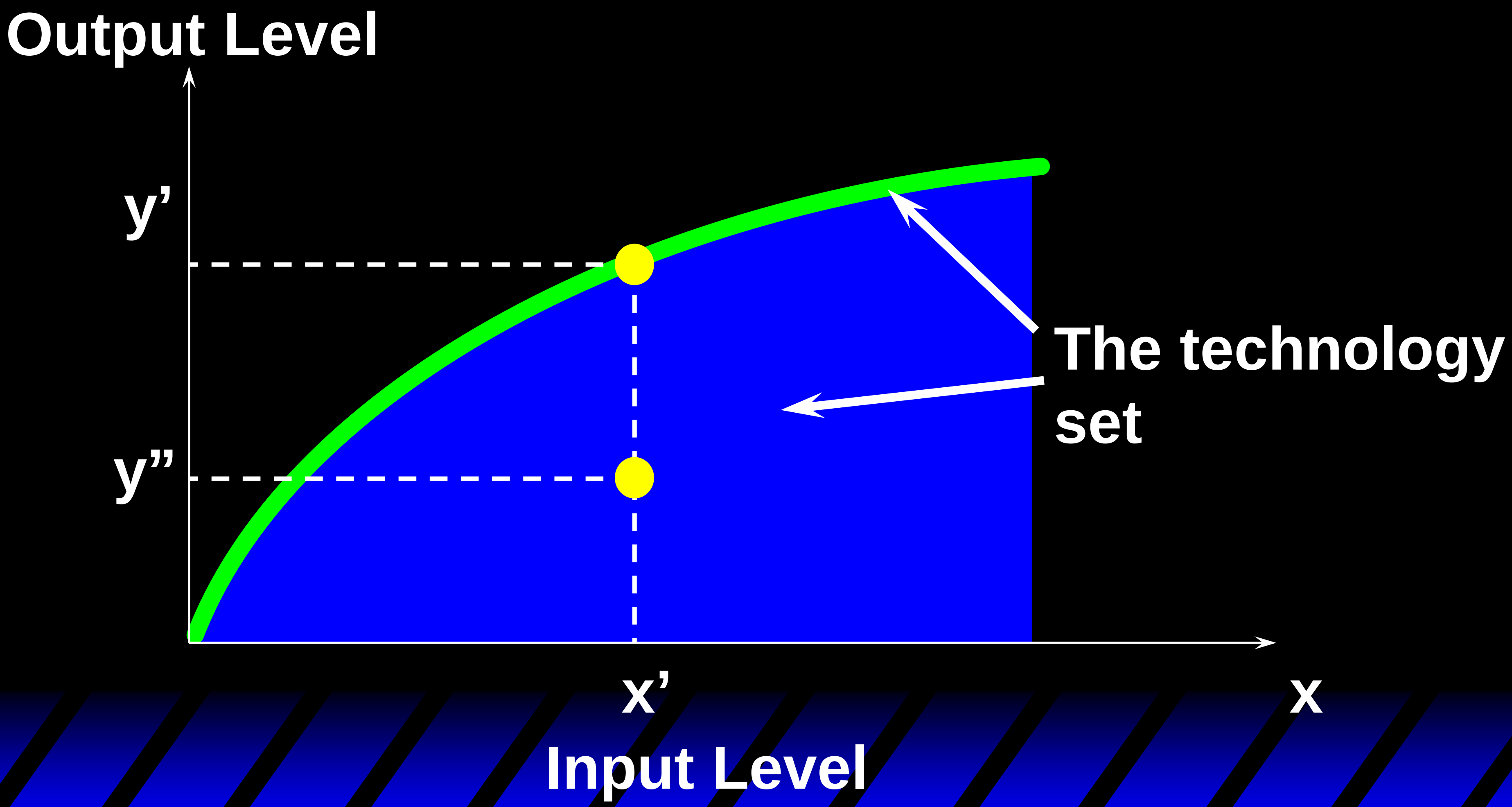
Technology Sets

one-input case



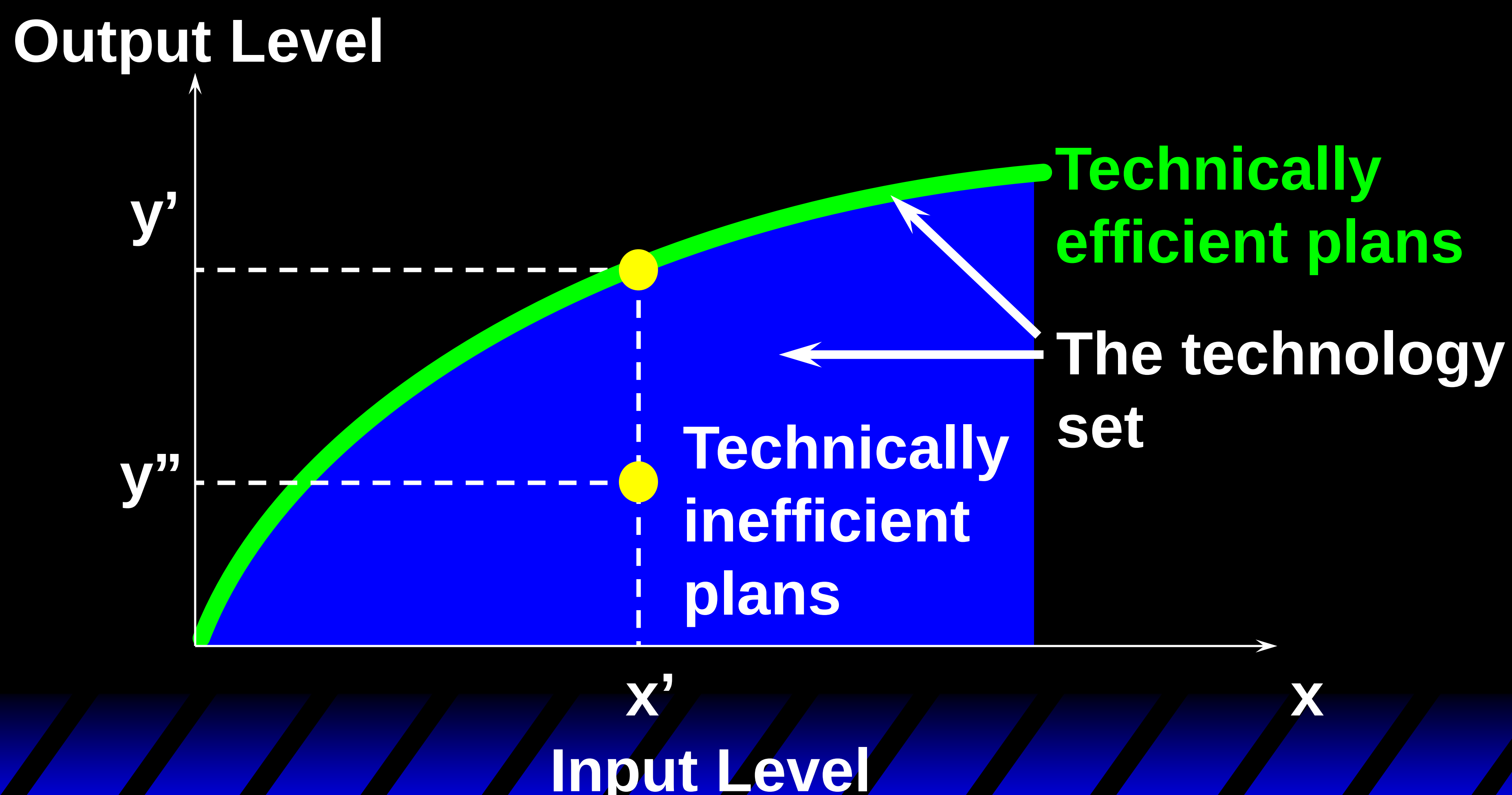
Technology Sets

A simple case with only one input



Technology Sets

A simple case with only one input



Technologies with Multiple Inputs

- **What does a technology look like when there is more than one input?**
- **The two-input case: Input levels are x_1 and x_2. Output level is y.**
- **Suppose the production function is**

$$y = f(x_1, x_2) = 2x_1^{1/3}x_2^{1/3}.$$

Technologies with Multiple Inputs

- **For the two-input case, we use isoquants（等产量线）to depict the technology.**
- **The y-output unit isoquant is the set of all input bundles that yield at most the same output level y.**

等产量线是具有相同最大产量的所有要素组合的集合

Isoquants with Two Inputs

E.g.

$$y = 2x_1^{1/3}x_2^{1/3}$$

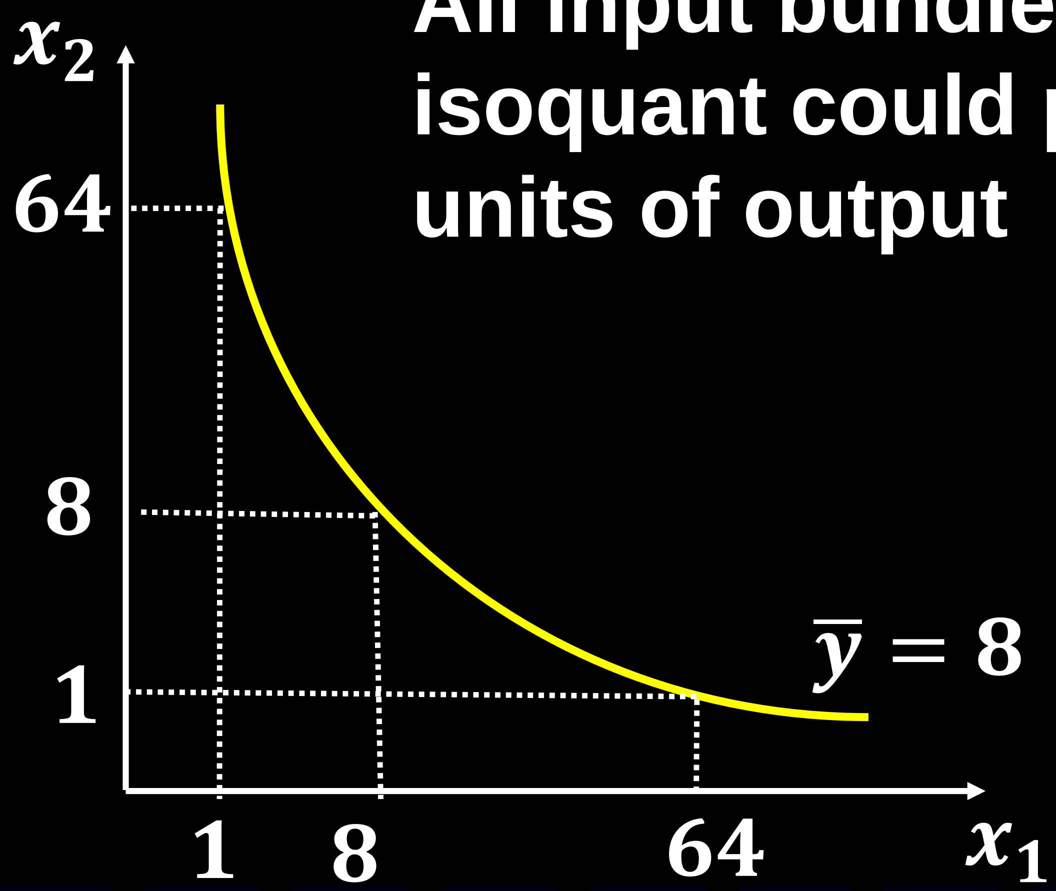
The isoquant with an output level y=8:

$$(x_1 = 1, x_2 = 64) \Rightarrow \mathrm{y} = 8$$
$$(x_1 = 64, x_2 = 1) \Rightarrow \mathrm{y} = 8$$
$$(x_1 = 8, x_2 = 8) \Rightarrow \mathrm{y} = 8$$
$$(x_1 = 2, x_2 = 32) \Rightarrow \mathrm{y} = 8$$

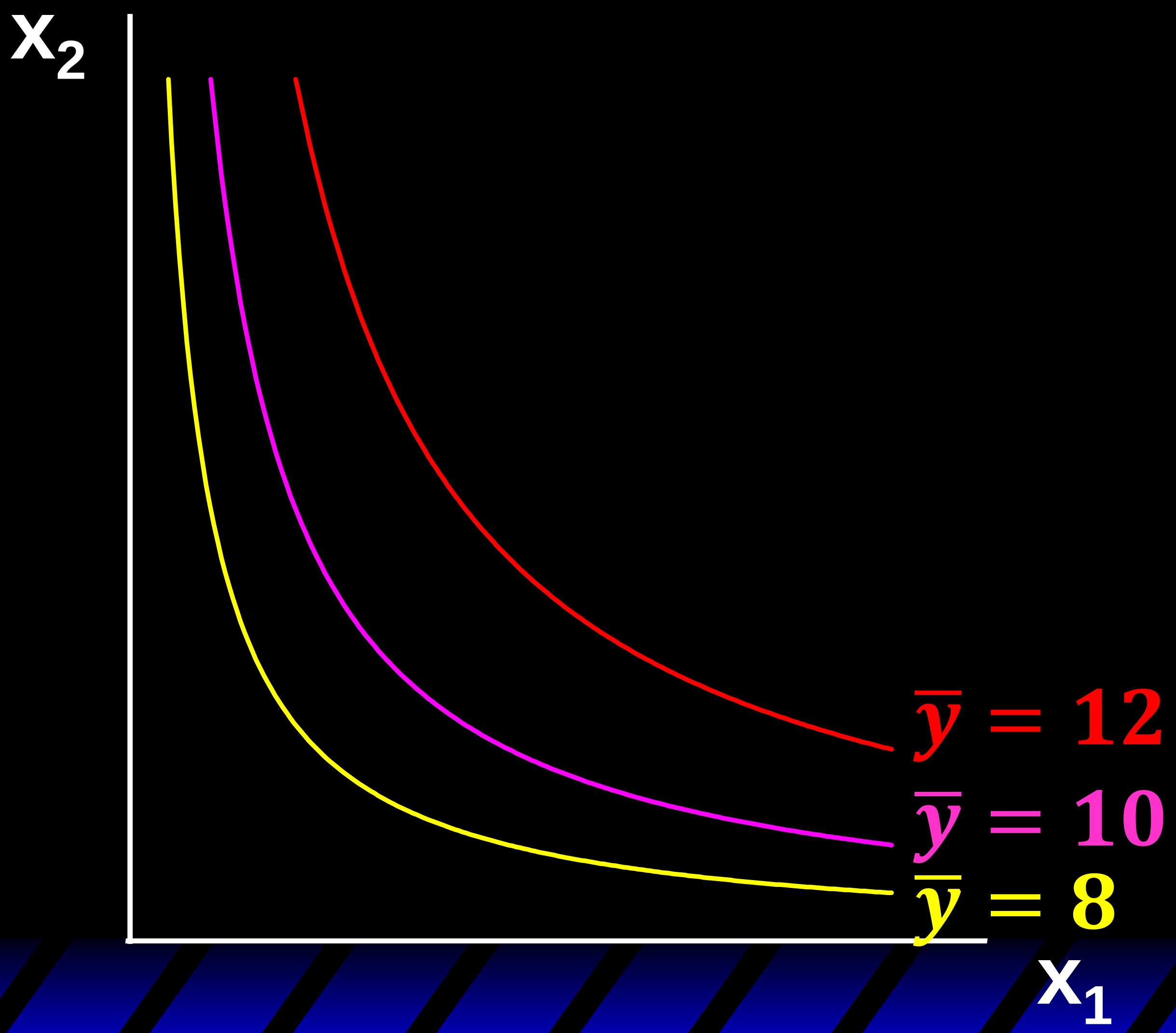
… …

Isoquants with Two Inputs

All input bundles on this isoquant could produce 8 units of output



Isoquants with Two Inputs



Cobb-Douglas Technologies

- A Cobb-Douglas production function is of the form

$$y = A x_1^{a_1} x_2^{a_2} \times \cdots \times x_n^{a_n}.$$

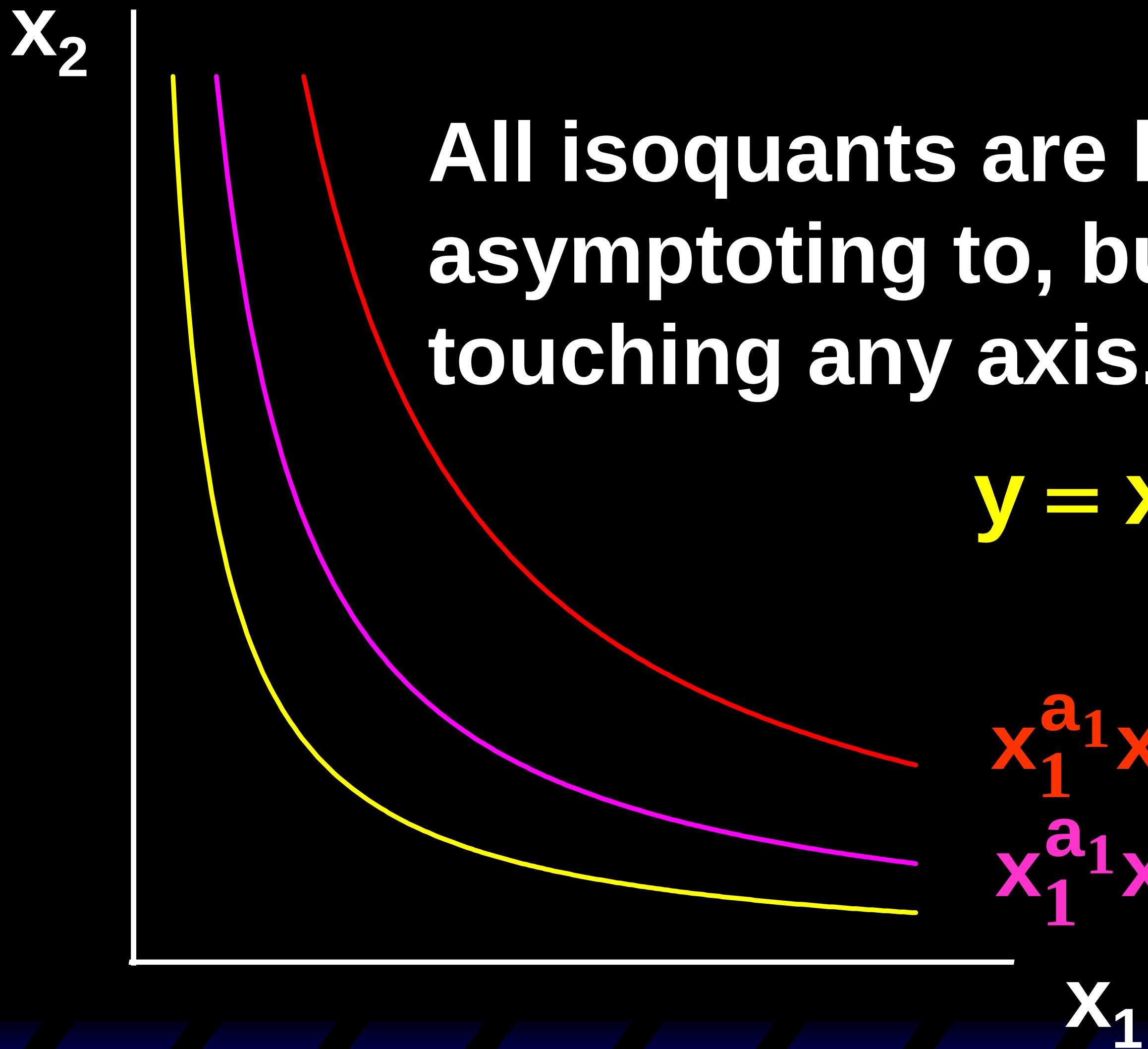
- E.g.

$$y = x_1^{1/3} x_2^{1/3}$$

with

$$n = 2,\ A = 1,\ a_1 = \frac{1}{3} \text{ and } a_2 = \frac{1}{3}.$$

Cobb-Douglas Technologies



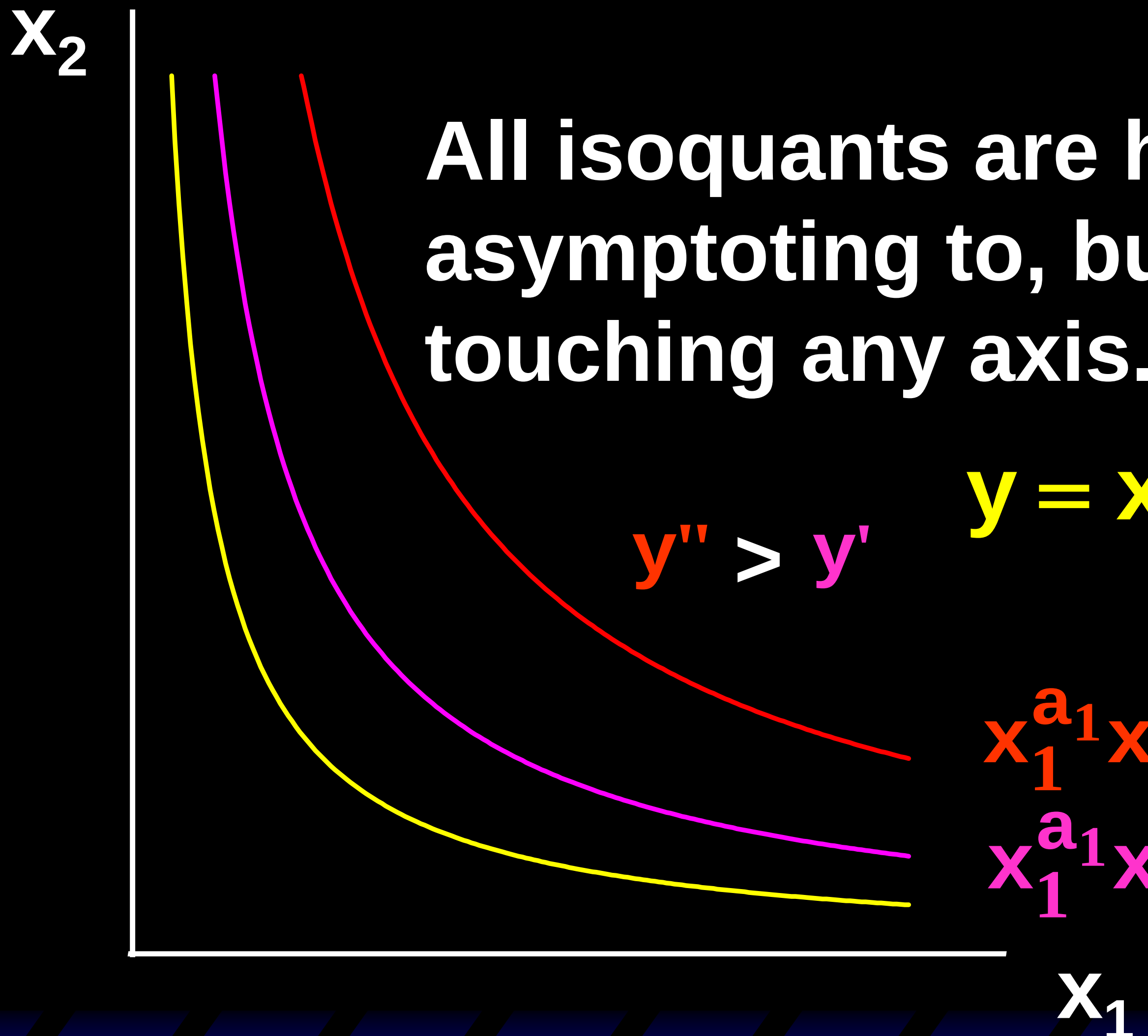
All isoquants are hyperbolic, asymptoting to, but never touching any axis.

$$y = x_1^{a_1} x_2^{a_2}$$

$$x_1^{a_1} x_2^{a_2} = y''$$

$$x_1^{a_1} x_2^{a_2} = y'$$

Cobb-Douglas Technologies



All isoquants are hyperbolic, asymptoting to, but never touching any axis.

$$y = x_1^{a_1} x_2^{a_2}$$

更靠外的等产量线对应更高的产出水平

Fixed-Proportions Technologies

- **A fixed-proportions production function is of the form**

$$y = \min\{a_1 x_1, a_2 x_2, \cdots, a_n x_n\}.$$

固定比例的生产函数

Fixed-Proportions Technologies

- E.g.

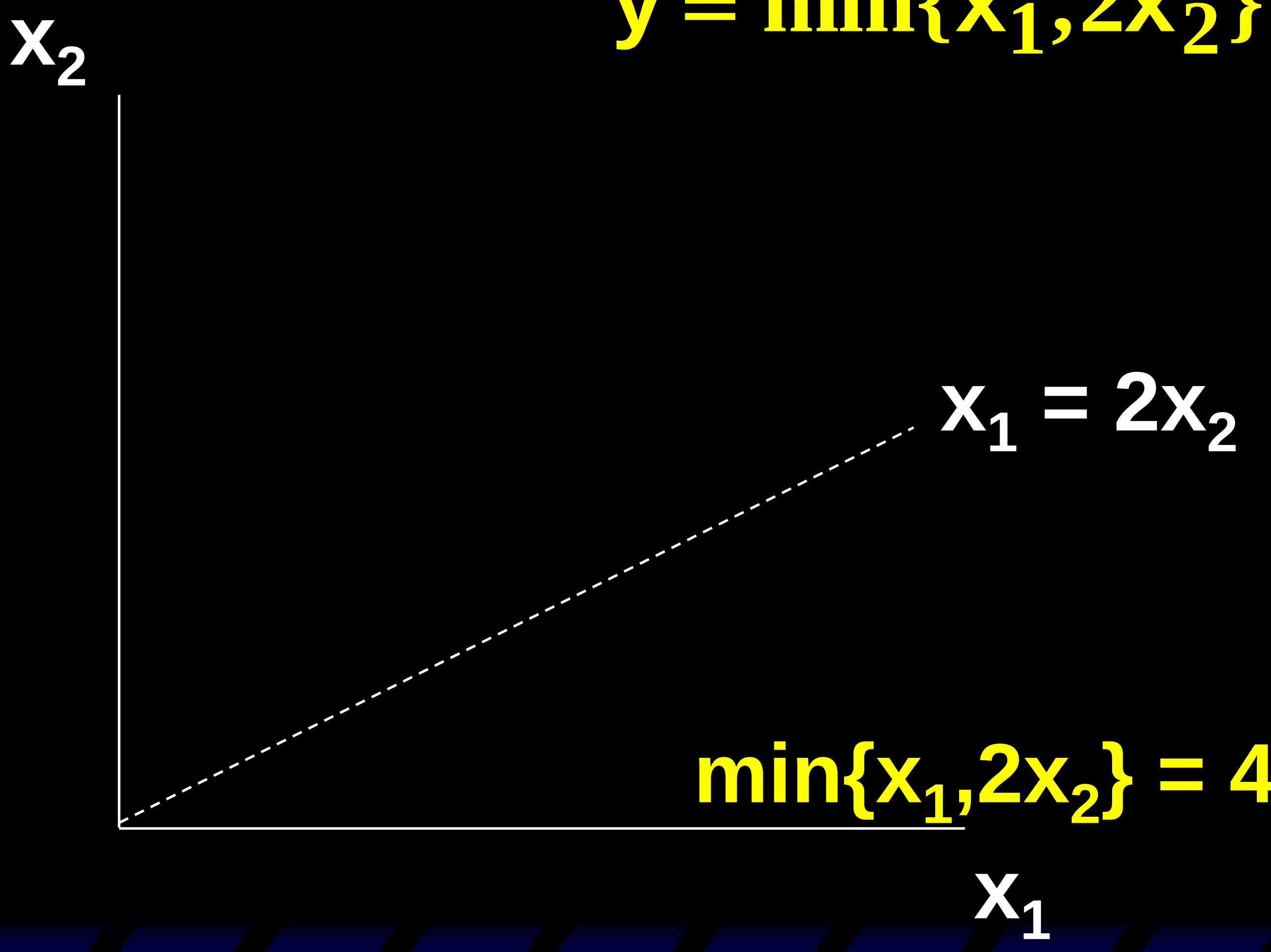
$$y = \min\{x_1, 2x_2\}$$

How to interpret this production function?

A: **2** units of input1 and **1** unit of input2 are always used together to produce **2** unit of output.

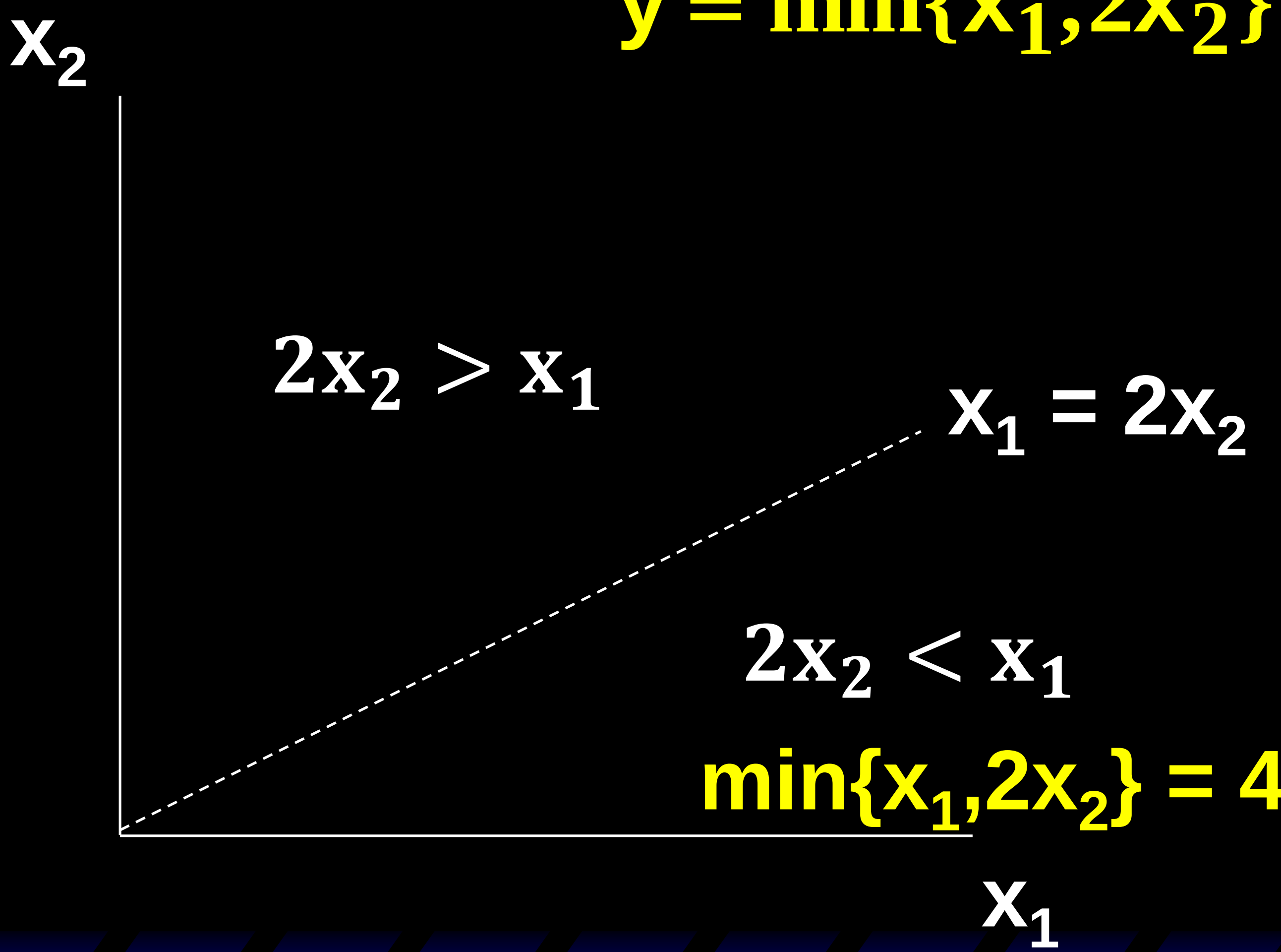
Fixed-Proportions Technologies

$y = \min\{x_1, 2x_2\}$



Fixed-Proportions Technologies

$$y = \min\{x_1, 2x_2\}$$



Fixed-Proportions Technologies

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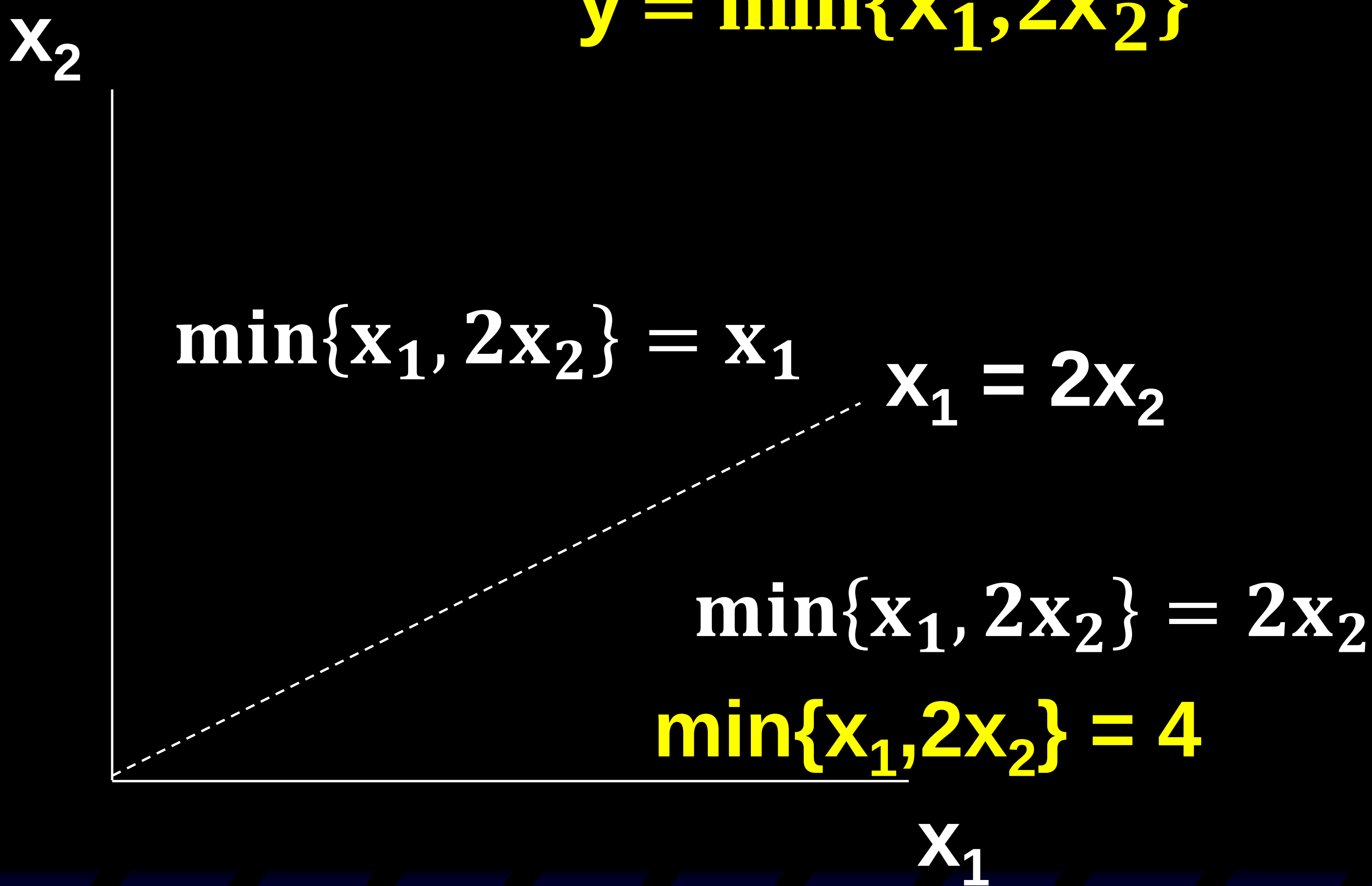

Fixed-Proportions Technologies

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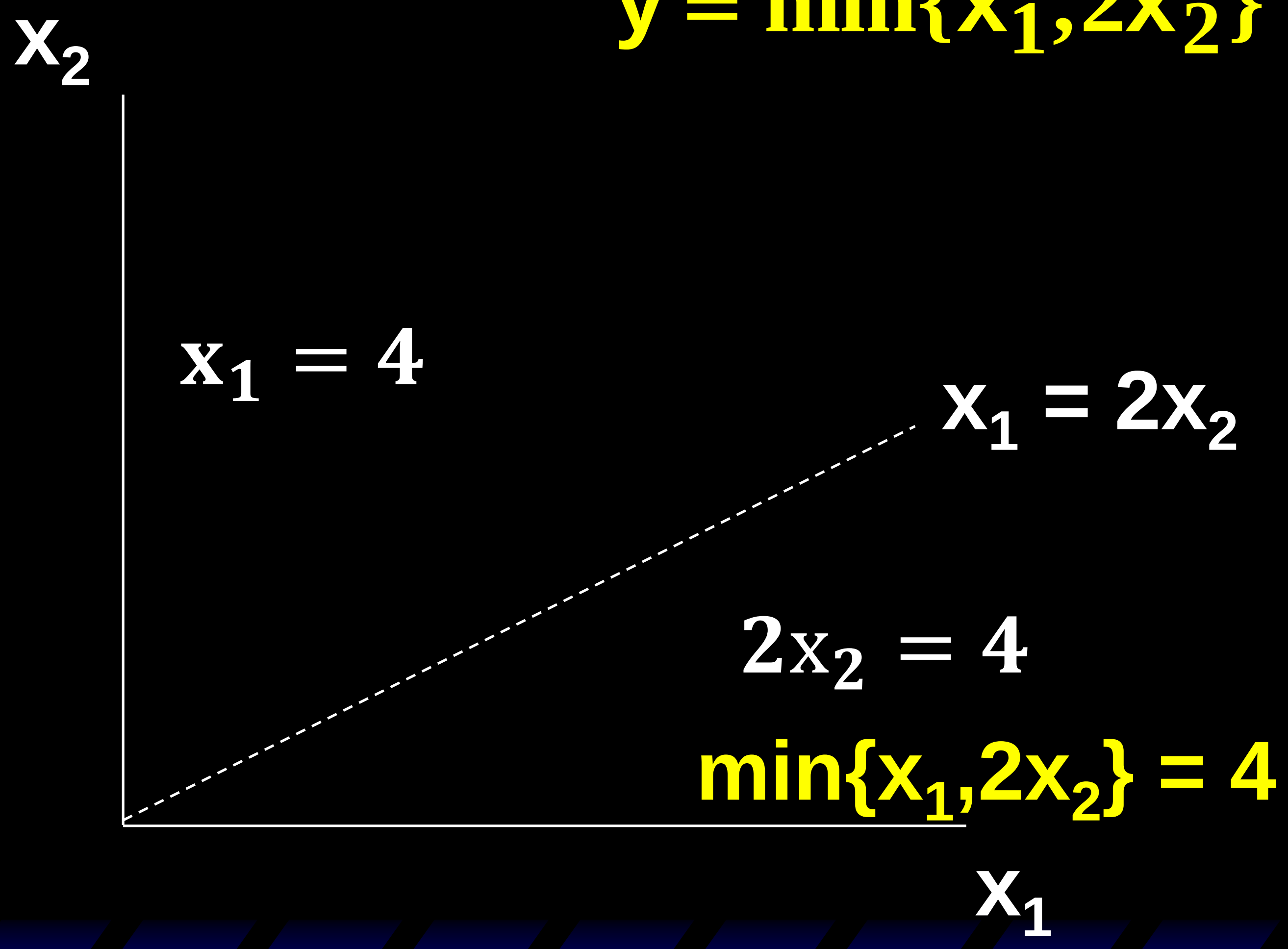

Fixed-Proportions Technologies

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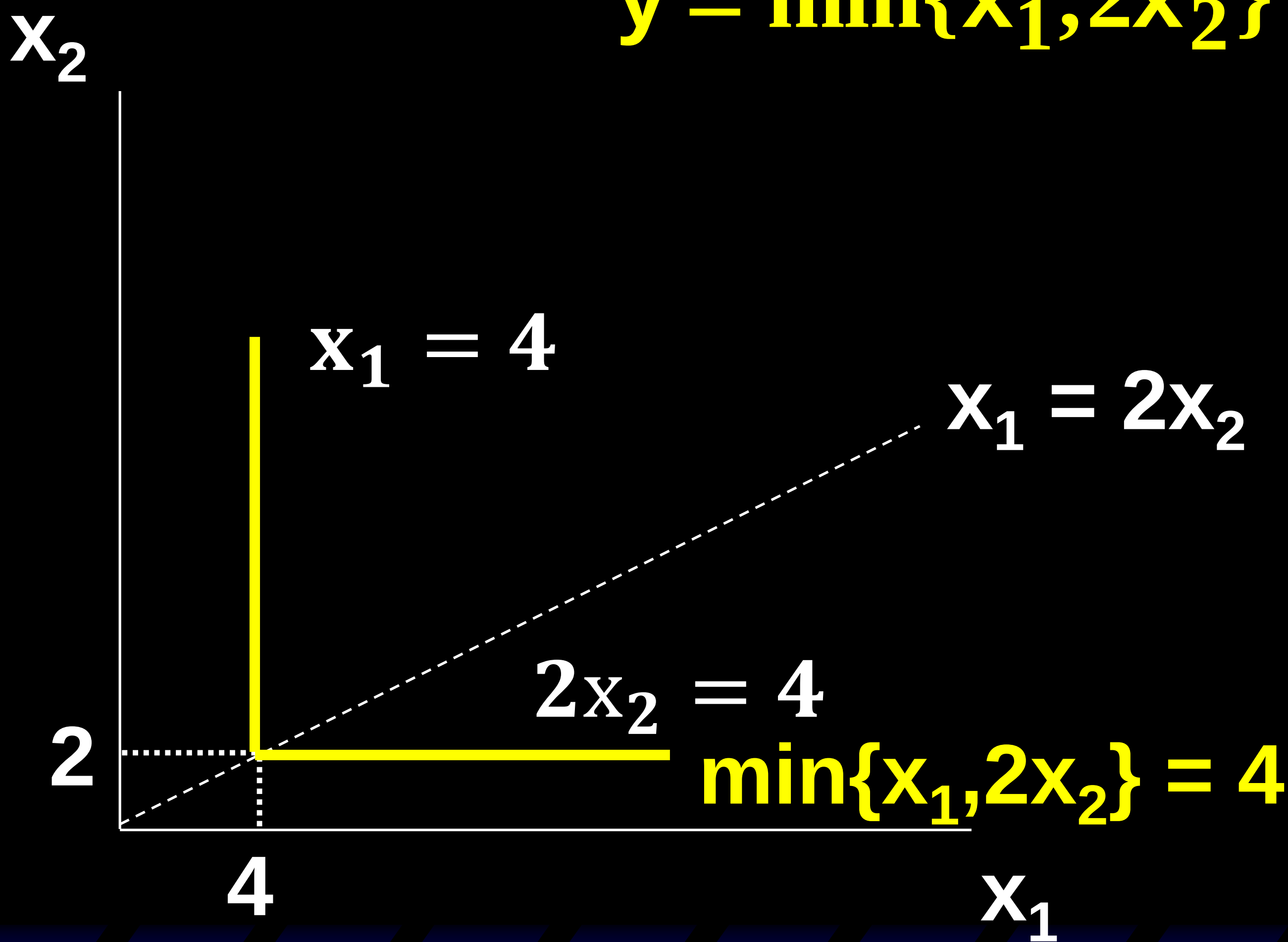
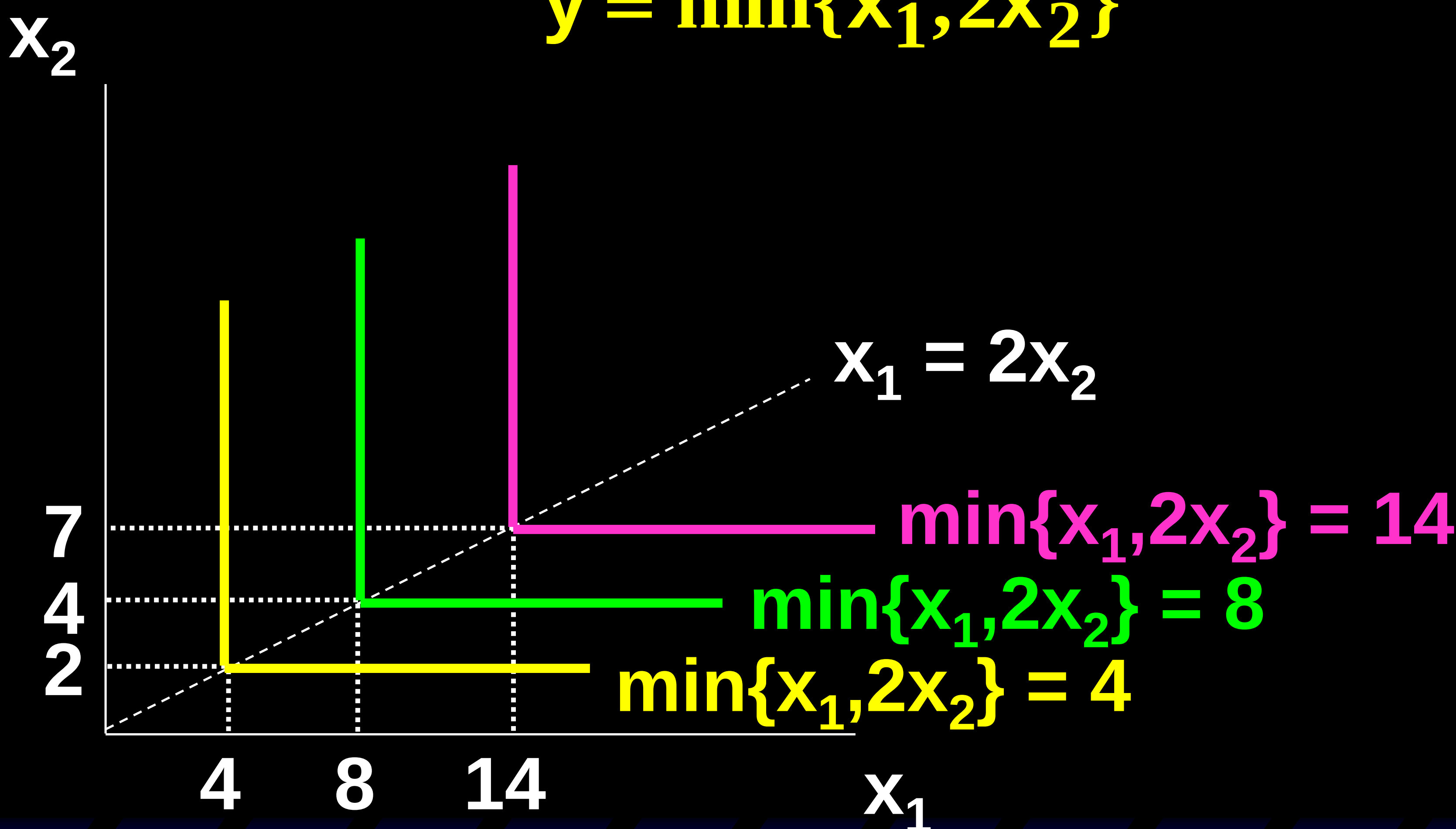


Fixed-Proportions Technologies

$$y = \min\{x_1, 2x_2\}$$



Fixed-Proportions Technologies

- Are

$$y = \min\{x_1, 2x_2\}$$

and

$$y = \min\{\frac{1}{2}x_1, x_2\}$$

representing the same technology?

Fixed-Proportions Technologies

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and

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representing the same technology?

单增变换(monotonic transformation)后的生产函数代表了一个不同的生产技术

Perfect-Substitutes Technologies

- A perfect-substitutes production function is of the form

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

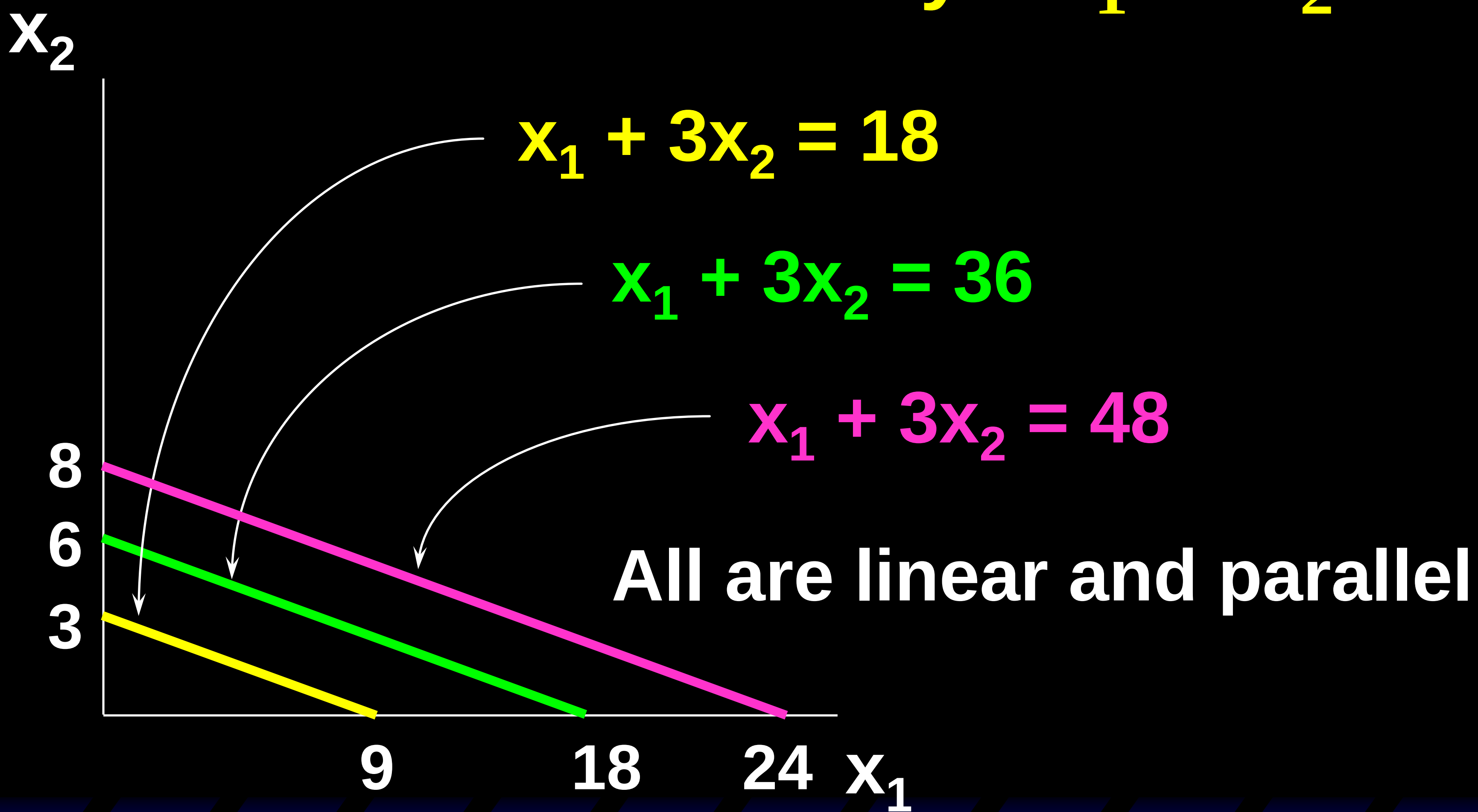
- E.g.

$$y = x_1 + 3x_2$$

当一单位要素1总是能够被固定单位的要素2所替代的时候，生产函数被称为完全替代生产函数

Perfect-Substitution Technologies

$$y = x_1 + 3x_2$$



Perfect-Substitution Technologies

- **Are**

$$y = x_1 + 3x_2$$

and

$$y = (x_1 + 3x_2)^2$$

representing the same technology?

Perfect-Substitution Technologies

- Does

$$y = (x_1 + 3x_2)^2$$

represent a perfect-substitution technology?

Marginal Products

$$y = f(x_1, \cdots, x_n)$$

- The **marginal product** of input i is the **rate-of-change** of the output level as the level of input i changes, **holding all other input levels fixed**.
- That is,

$$MP_i = \frac{\partial y}{\partial x_i}$$

某一要素的边际产量是当所有其它要素的投入量不变时，产出对该种要素的变动率

Marginal Products

E.g. if

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

then the marginal product of input 1 is

$$MP_1 = \frac{\partial y}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$$

and the marginal product of input 2 is

$$MP_2 = \frac{\partial y}{\partial x_2} = \frac{2}{3} x_1^{1/3} x_2^{-1/3}.$$

Marginal Products

E.g. if $y = x_1^{1/3} x_2^{2/3}$ then

$MP_1 = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$ and $MP_2 = \frac{2}{3} x_1^{1/3} x_2^{-1/3}$

so $$\frac{\partial MP_1}{\partial x_1} = -\frac{2}{9} x_1^{-5/3} x_2^{2/3} < 0$$

and $$\frac{\partial MP_2}{\partial x_2} = -\frac{2}{9} x_1^{1/3} x_2^{-4/3} < 0.$$

Both marginal products are diminishing.

Marginal Products

- The marginal product of input i is **diminishing** if it becomes smaller as the level of input i increases. That is, if

$$\frac{\partial MP_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial y}{\partial x_i}\right) = \frac{\partial^2 y}{\partial x_i^2} < 0.$$

当一种要素的**边际产量**随该种要素使用数量的增长而下降时（其它要素不变），我们称之为**边际产量递减**

Marginal Products

The marginal product of one input also depends on the amount used of other inputs.

$$y = x_1^{1/3} x_2^{2/3}$$

$$MP_1 = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$$

$$\frac{\partial MP_1}{\partial x_2} = \frac{2}{9} x_1^{-2/3} x_2^{-1/3} > 0$$

一种要素的边际产量随其它要素使用数量的增长而增长

Returns-to-Scale

- **Marginal products describe the change in output level as a single input level changes.**

边际产量描述了：其它要素数量不变，某一要素数量改变而造成的产出变化

- **Returns-to-scale describes how the output level changes as all input levels change in direct proportion (*e.g.* all input levels doubled, or halved).**

规模报酬描述了：所有要素同时、同比例变化而造成的产量变化

Returns-to-Scale

If, for any input bundle $(x_1,\ldots,x_n)$,

$$f(kx_1,kx_2,\cdots,kx_n) = kf(x_1,x_2,\cdots,x_n)$$

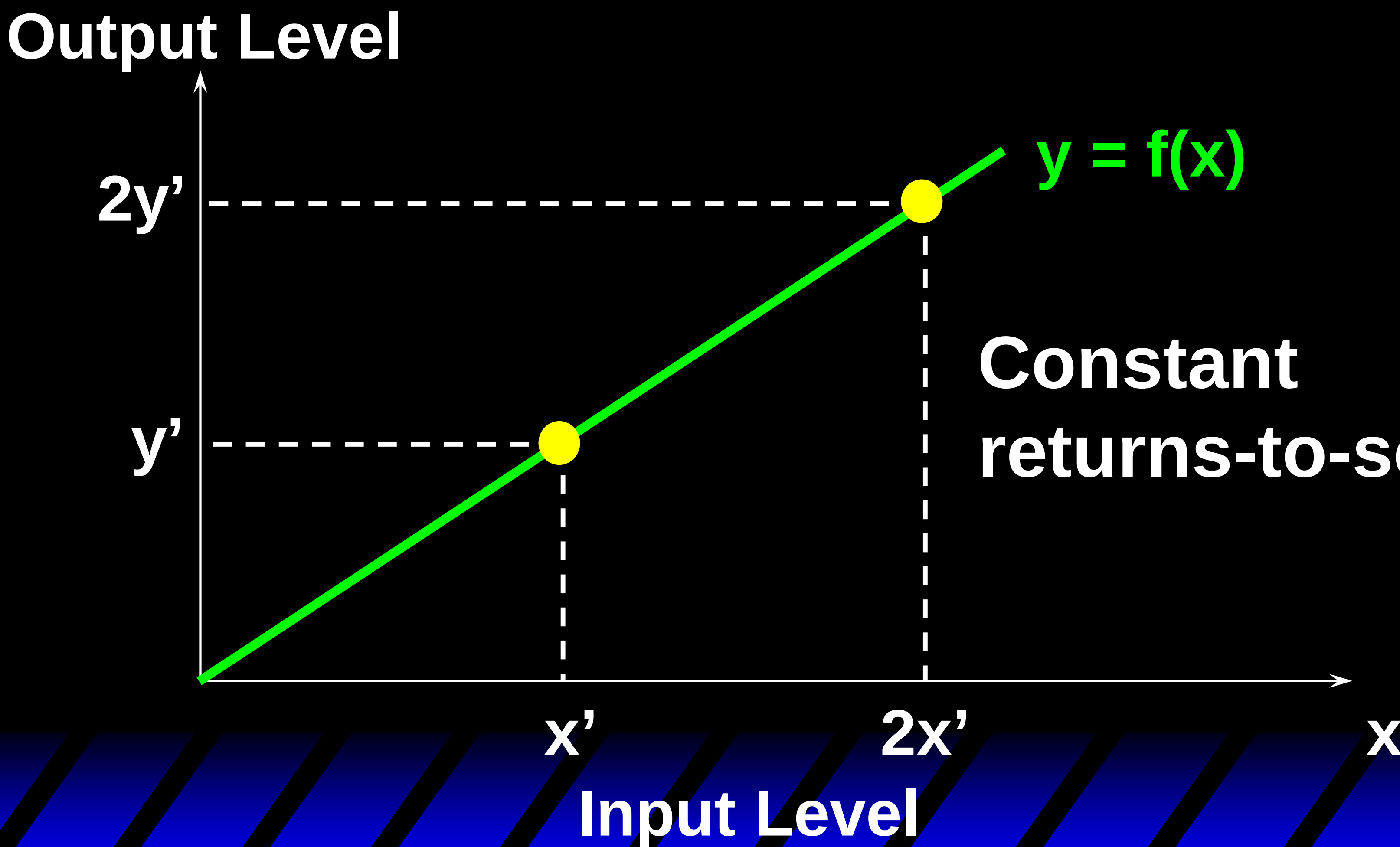
then the technology described by the production function f exhibits **constant returns-to-scale**.

E.g. (k = 2) doubling all input levels doubles the output level.

当产量增加的比例等于生产要素增加的比例时，我们称之为规模报酬不变

Returns-to-Scale

One input, one output



Returns-to-Scale

If, for any input bundle $(x_1,\ldots,x_n)$ and k>1,

$$f(kx_1,kx_2,\cdots,kx_n) < kf(x_1,x_2,\cdots,x_n)$$

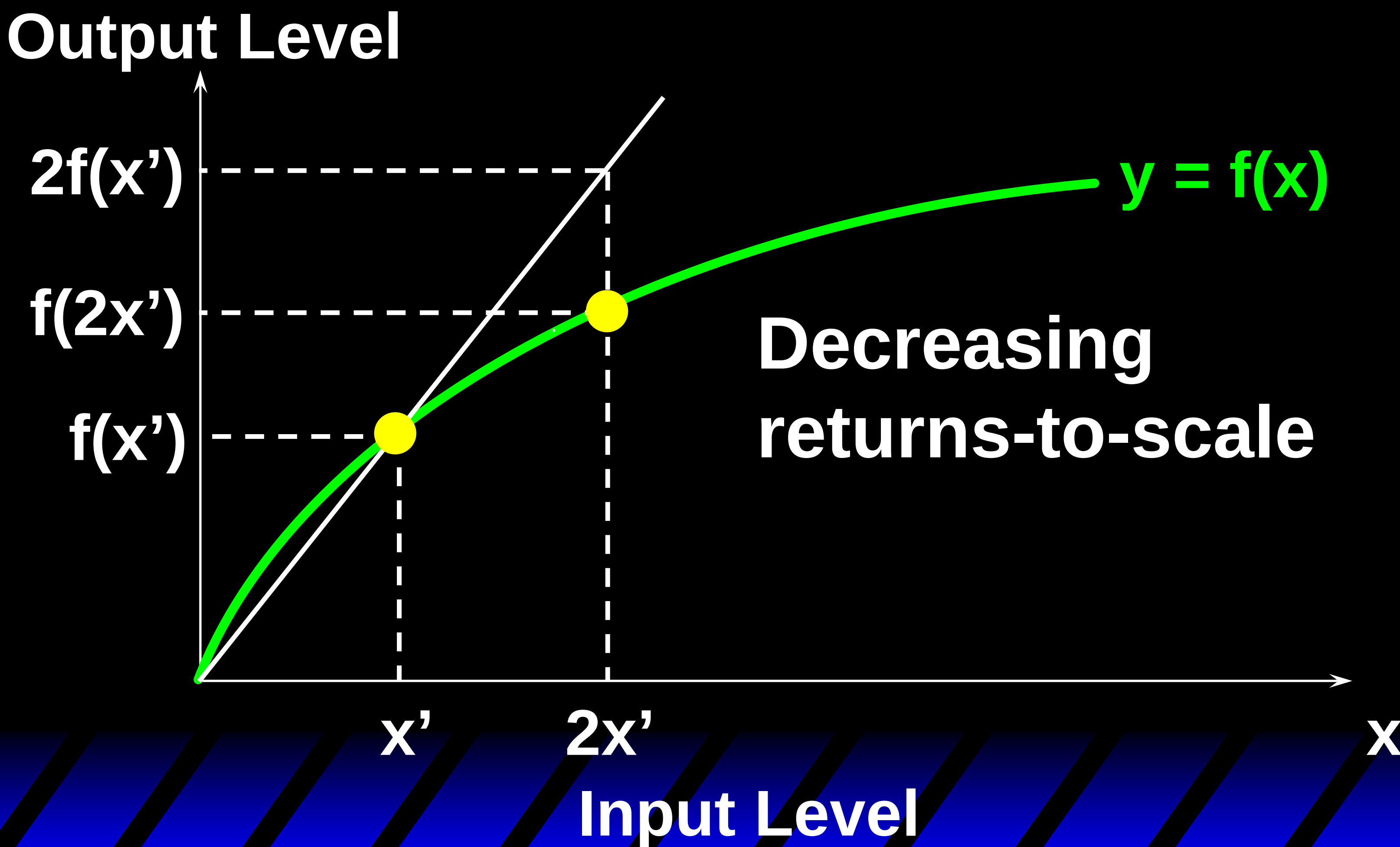
then the technology exhibits diminishing returns-to-scale.

***E.g.*. (k = 2) doubling all input levels less than doubles the output level.**

规模报酬递减

Returns-to-Scale

One input, one output



Returns-to-Scale

If, for any input bundle $(x_1,\ldots,x_n)$ and **k>1**,

$$f(kx_1,kx_2,\cdots,kx_n) > kf(x_1,x_2,\cdots,x_n)$$

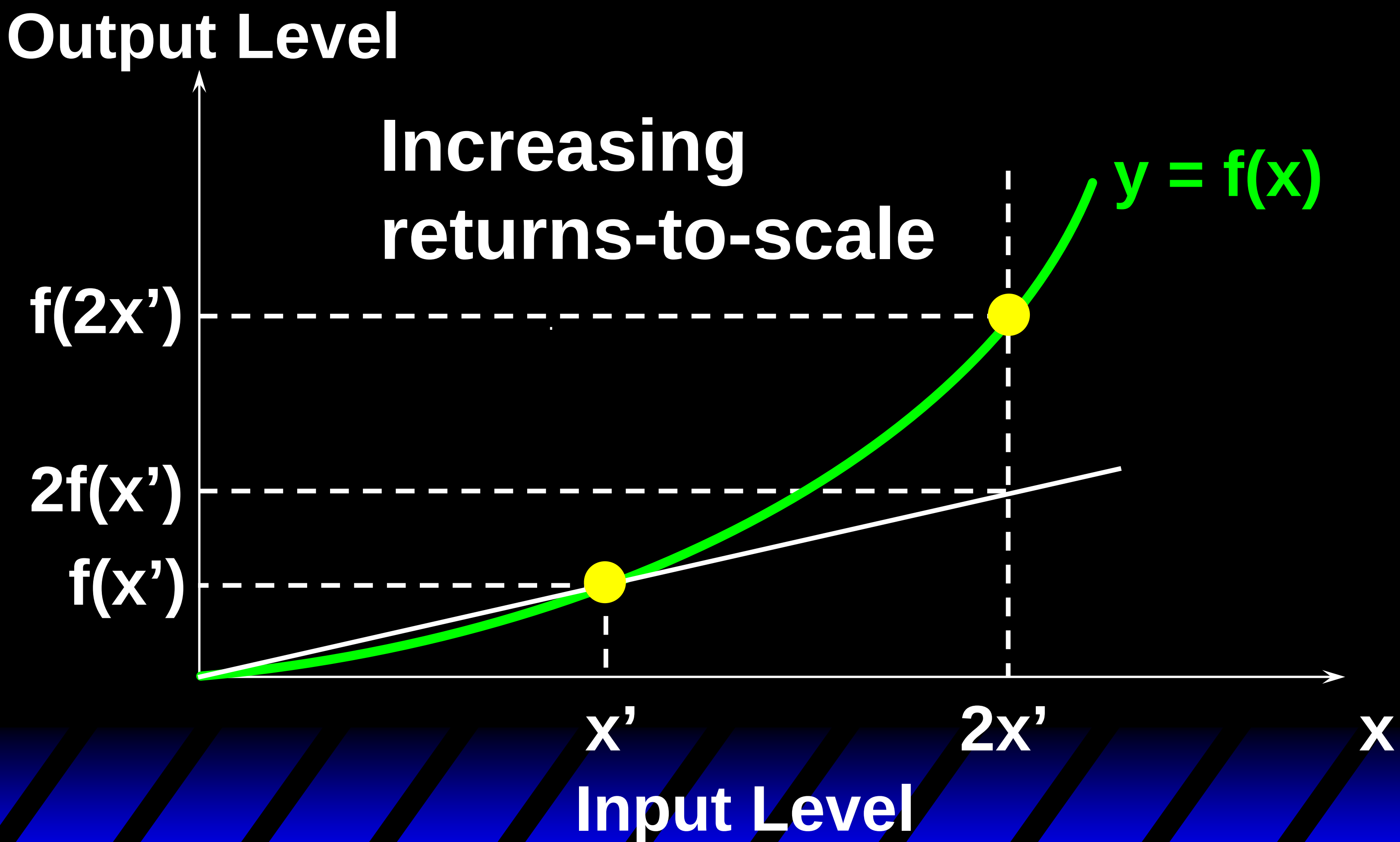
then the technology exhibits **increasing returns-to-scale**.

E.g. (k = 2) doubling all input levels more than doubles the output level.

规模报酬递增

Returns-to-Scale

One input, one output



Examples of Returns-to-Scale

The perfect-substitutes production function is

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

Expand all input levels proportionately by k. The output level becomes

$$a_1 (kx_1) + a_2 (kx_2) + \cdots + a_n (kx_n)$$

Examples of Returns-to-Scale

The perfect-substitutes production function is

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

Expand all input levels proportionately by k. The output level becomes

$$a_1(kx_1) + a_2(kx_2) + \cdots + a_n(kx_n)$$
$$= k(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)$$
$$= ky.$$

The perfect-substitutes production function exhibits constant returns-to-scale.

Examples of Returns-to-Scale

The perfect-complements production function is

$$y = \min\{a_1 x_1, a_2 x_2, \cdots, a_n x_n\}.$$

Expand all input levels proportionately by k. The output level becomes

$$\min\{a_1 (kx_1), a_2 (kx_2), \cdots, a_n (kx_n)\}$$

Examples of Returns-to-Scale

The perfect-complements production function is

$$y = \min\{a_1 x_1, a_2 x_2, \cdots, a_n x_n\}.$$

Expand all input levels proportionately by k. The output level becomes

$$\min\{a_1 (kx_1), a_2 (kx_2), \cdots, a_n (kx_n)\}$$

$$= k(\min\{a_1 x_1, a_2 x_2, \cdots, a_n x_n\})$$

$$= ky.$$

The perfect-complements production function exhibits constant returns-to-scale.

Examples of Returns-to-Scale

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Expand all input levels proportionately by k. The output level becomes

$$(kx_1)^{a_1}(kx_2)^{a_2} \cdots (kx_n)^{a_n}$$

Examples of Returns-to-Scale

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Expand all input levels proportionately by k. The output level becomes

$$(kx_1)^{a_1}(kx_2)^{a_2}\cdots(kx_n)^{a_n}$$

$$= k^{a_1} k^{a_2} \cdots k^{a_n} x^{a_1} x^{a_2} \cdots x^{a_n}$$

$$= k^{a_1+a_2+\cdots+a_n} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

$$= k^{a_1+\cdots+a_n} y.$$

Examples of Returns-to-Scale

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

$$(kx_1)^{a_1}(kx_2)^{a_2} \cdots (kx_n)^{a_n} = k^{a_1 + \cdots + a_n} y.$$

The Cobb-Douglas technology's returns-to-scale is

constant if $a_1 + \ldots + a_n = 1$

increasing if $a_1 + \ldots + a_n > 1$

decreasing if $a_1 + \ldots + a_n < 1$.

Examples of Increasing Returns-to-Scale

The Cobb-Douglas production function is

$$y = f(x_1, x_2) = x_1^{2/3} x_2^{2/3}$$

Expanding all input levels by k gives an output of

$$f(kx_1, kx_2) = (kx_1)^{\frac{2}{3}}(kx_1)^{\frac{2}{3}} = k^{4/3} x_1^{2/3} x_2^{2/3}$$

$$f(kx_1, kx_2) = k^{4/3} x_1^{2/3} x_2^{2/3} > f(x_1, x_2)$$

$$\forall k > 1$$

=> Increasing returns to scale

Returns-to-Scale

- Q: Can a technology exhibit increasing returns-to-scale even though all of its marginal products are diminishing?

Returns-to-Scale

- **Q: Can a technology exhibit increasing returns-to-scale even if all of its marginal products are diminishing?**
- **A: Yes.**
- **E.g.** $y = x_1^{2/3} x_2^{2/3}.$

Returns-to-Scale

$$y = x_1^{2/3} x_2^{2/3} = x_1^{a_1} x_2^{a_2}$$

$$a_1 + a_2 = \frac{4}{3} > 1$$ so this technology exhibits increasing returns-to-scale.

Returns-to-Scale

$$y = x_1^{2/3} x_2^{2/3} = x_1^{a_1} x_2^{a_2}$$

$a_1 + a_2 = \frac{4}{3} > 1$ so this technology exhibits increasing returns-to-scale.

But $MP_1 = \frac{2}{3} x_1^{-1/3} x_2^{2/3}$ diminishes as x_1 increases and

$MP_2 = \frac{2}{3} x_1^{2/3} x_2^{-1/3}$ diminishes as x_2 increases.

Returns-to-Scale

- **So a technology can exhibit increasing returns-to-scale even if all of its marginal products are diminishing. Why?**

Returns-to-Scale

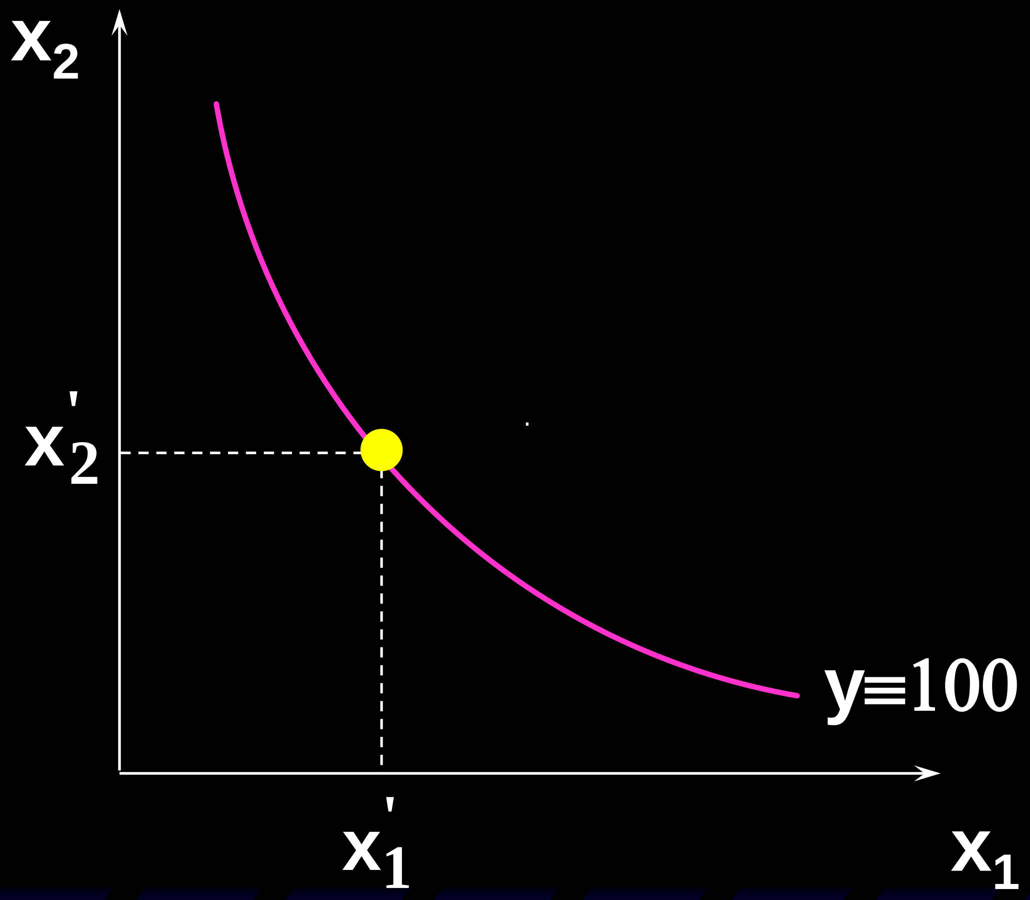
- So a technology can exhibit increasing returns-to-scale even if all of its marginal products are diminishing. Why?
  - Marginal product diminishes because the other input levels are fixed.
  - When all input levels are increased proportionately, other input levels are not held fixed. Input productivities need not fall and so returns-to-scale can be constant or increasing.

Technical Rate-of-Substitution

- **TRS (技术替代率) is the rate at which a firm can substitute one input for another without changing its output level?**

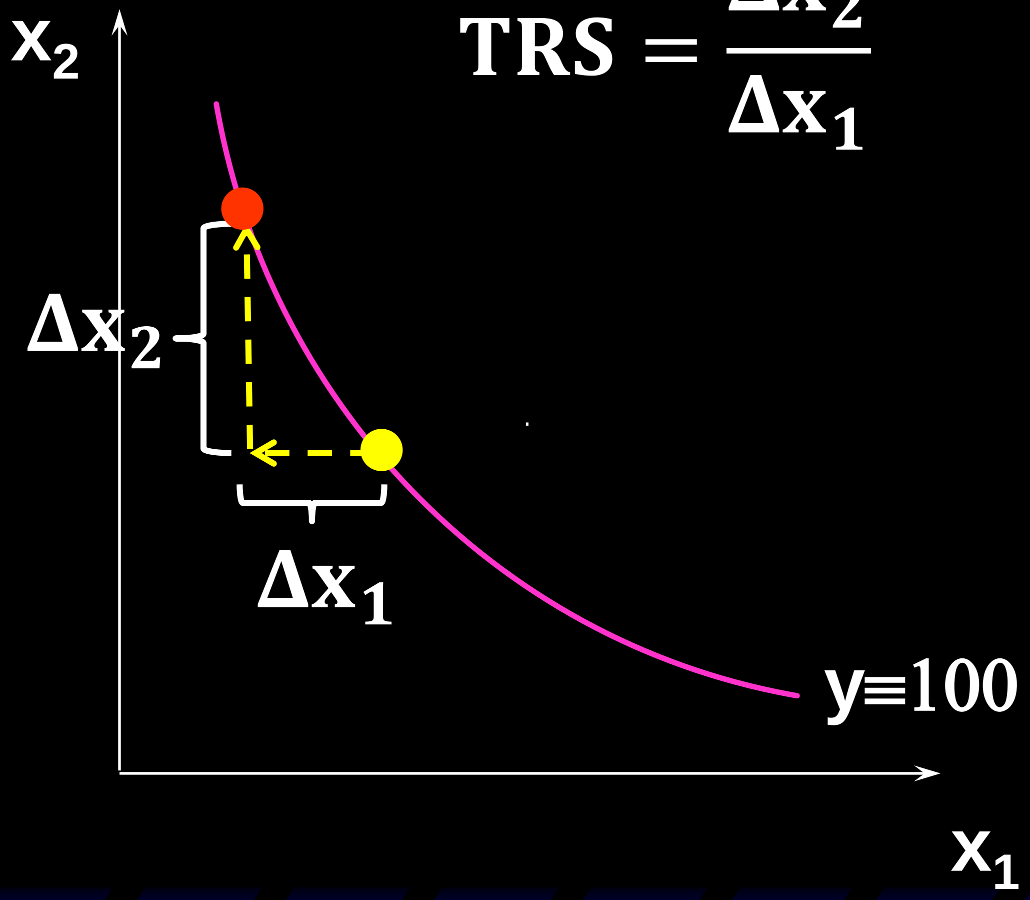
减少一单位x1时，为使产量不变而必须增加的x2的数量(i.e. 用x2来替换x1的比例)

Technical Rate-of-Substitution



Technical Rate-of-Substitution

$$\text{TRS} = \frac{\Delta x_2}{\Delta x_1}$$



Technical Rate-of-Substitution

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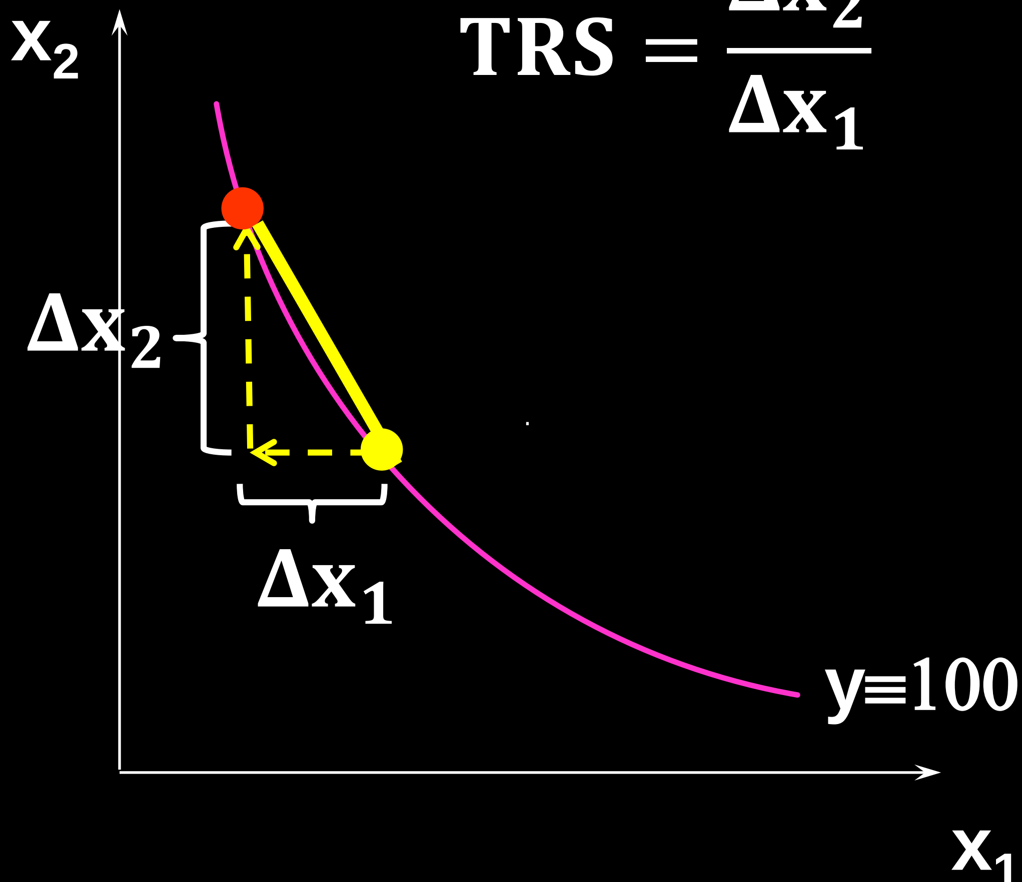

Technical Rate-of-Substitution

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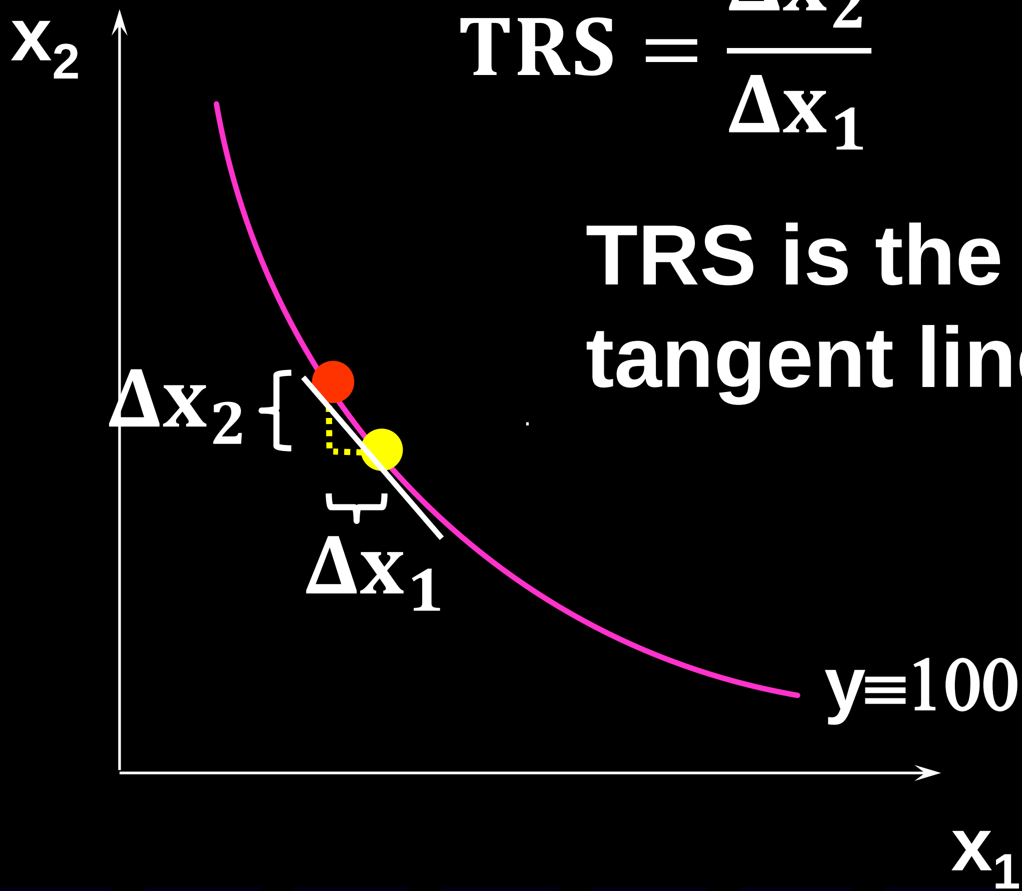


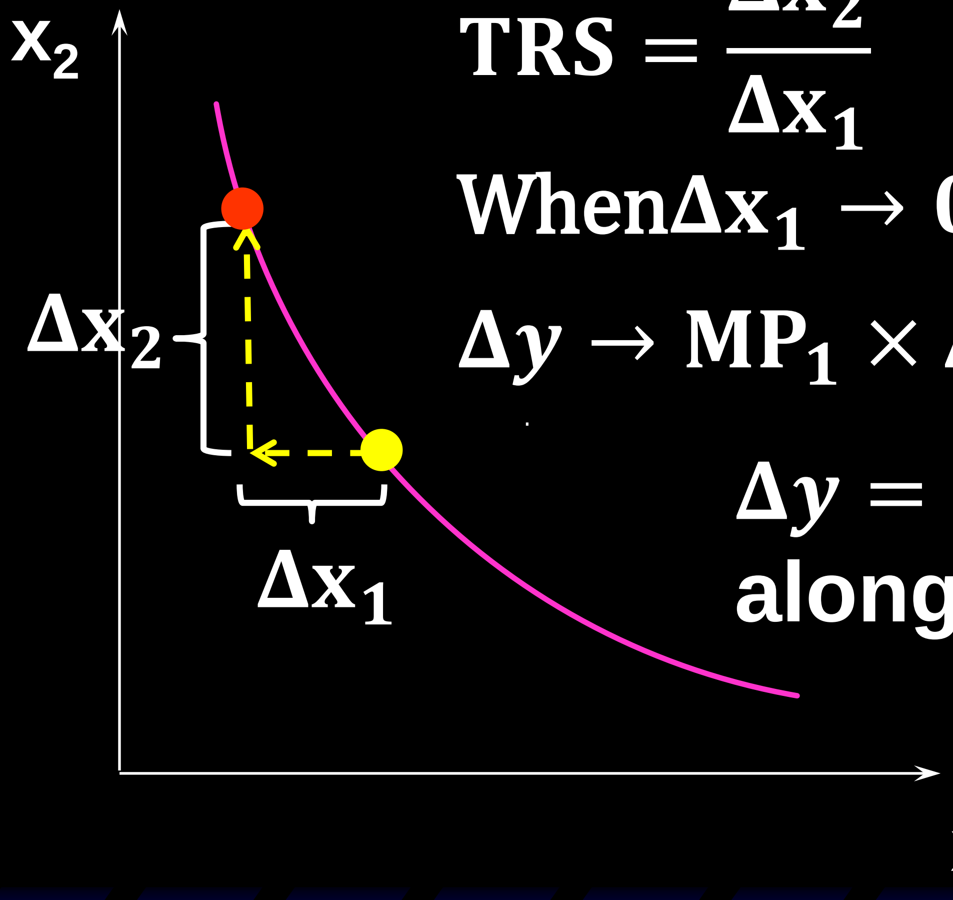
TRS is the slope of the tangent line as $\Delta x_1 \to 0$

某一要素组合处的TRS是等产量线在该要素组合处的斜率

Technical Rate-of-Substitution

- **How is a technical rate-of-substitution computed?**

Technical Rate-of-Substitution



$$\mathrm{TRS} = \frac{\Delta x_2}{\Delta x_1}$$

When $\Delta x_1 \to 0$,

$$\Delta y \to \mathrm{MP}_1 \times \Delta x_1 + \mathrm{MP}_2 \times \Delta x_2$$

$\Delta y = 0$ since it's a move along the isoquant

Technical Rate-of-Substitution

$$\mathbf{TRS} = \frac{\Delta \mathbf{x}_2}{\Delta \mathbf{x}_1}$$

$$\mathbf{When}\,\Delta \mathbf{x}_1 \to \mathbf{0},$$

$$\Delta y \to \mathbf{MP}_1 \times \Delta \mathbf{x}_1 + \mathbf{MP}_2 \times \Delta \mathbf{x}_2$$

$\Delta y = 0$ since it's a move along the isoquant

$$\mathbf{MP}_1 \times \Delta \mathbf{x}_1 + \mathbf{MP}_2 \times \Delta \mathbf{x}_2 = \mathbf{0}$$

$$\mathbf{TRS} = \frac{\Delta \mathbf{x}_2}{\Delta \mathbf{x}_1} = -\frac{\mathbf{MP}_1}{\mathbf{MP}_2}$$

Technical Rate-of-Substitution

$$\mathbf{TRS} = \frac{\Delta \mathbf{x}_2}{\Delta \mathbf{x}_1} = -\frac{\mathbf{MP}_1}{\mathbf{MP}_2}$$

$$= -\frac{\partial \mathbf{f}(\mathbf{x}_1, \mathbf{x}_2)/\partial \mathbf{x}_1}{\partial \mathbf{f}(\mathbf{x}_1, \mathbf{x}_2)/\partial \mathbf{x}_2}$$

Technical Rate-of-Substitution: A Cobb-Douglas Example

$$y = f(x_1, x_2) = x_1^a x_2^b$$

so $\dfrac{\partial y}{\partial x_1} = a x_1^{a-1} x_2^b$ and $\dfrac{\partial y}{\partial x_2} = b x_1^a x_2^{b-1}.$

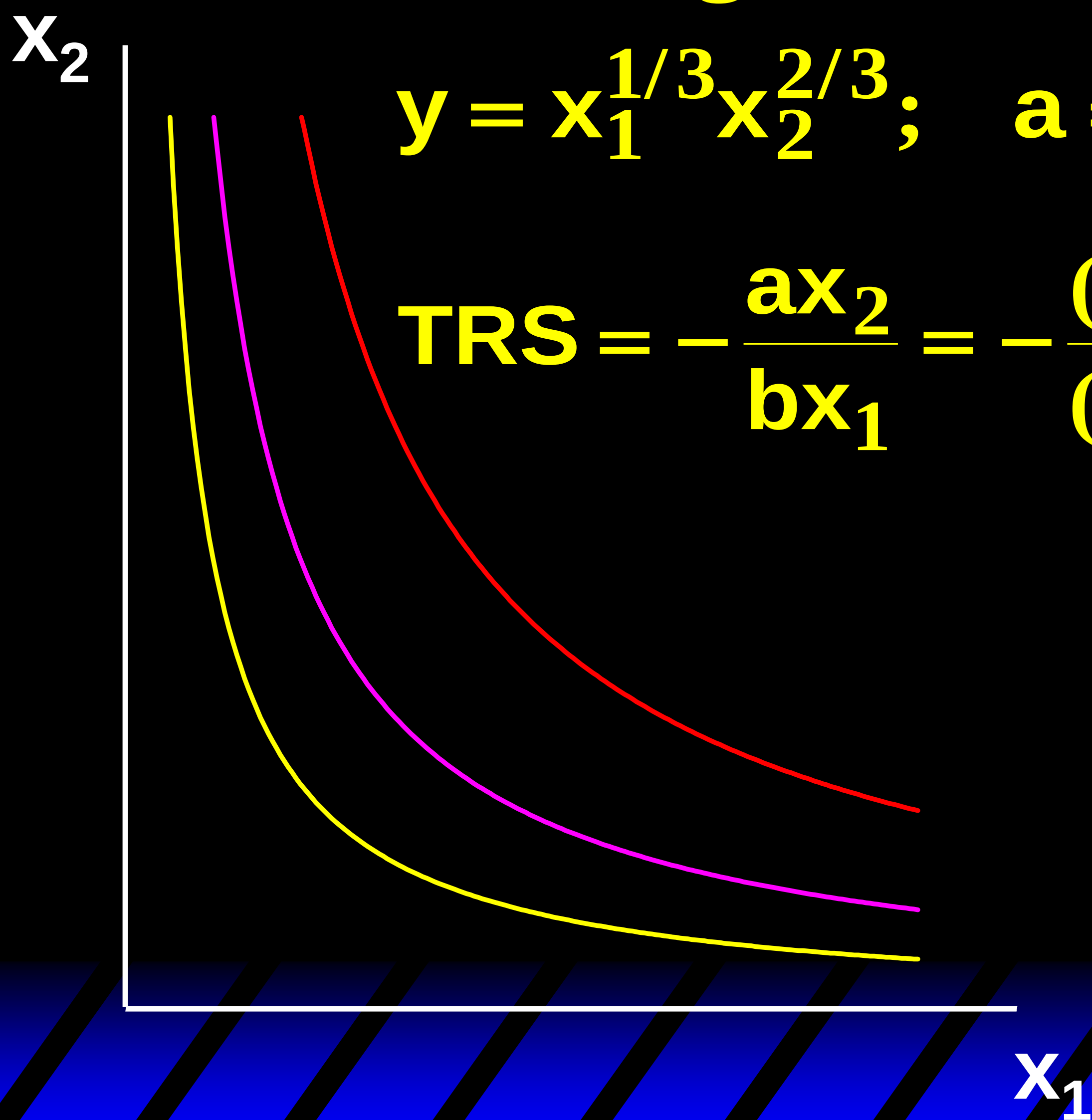
The technical rate-of-substitution is

$$\frac{dx_2}{dx_1} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2} = -\frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = -\frac{a x_2}{b x_1}.$$

Technical Rate-of-Substitution: A Cobb-Douglas Example

$$y = x_1^{1/3} x_2^{2/3}; \quad a = \frac{1}{3} \text{ and } b = \frac{2}{3}$$

$$\text{TRS} = -\frac{ax_2}{bx_1} = -\frac{(1/3)x_2}{(2/3)x_1} = -\frac{x_2}{2x_1}$$

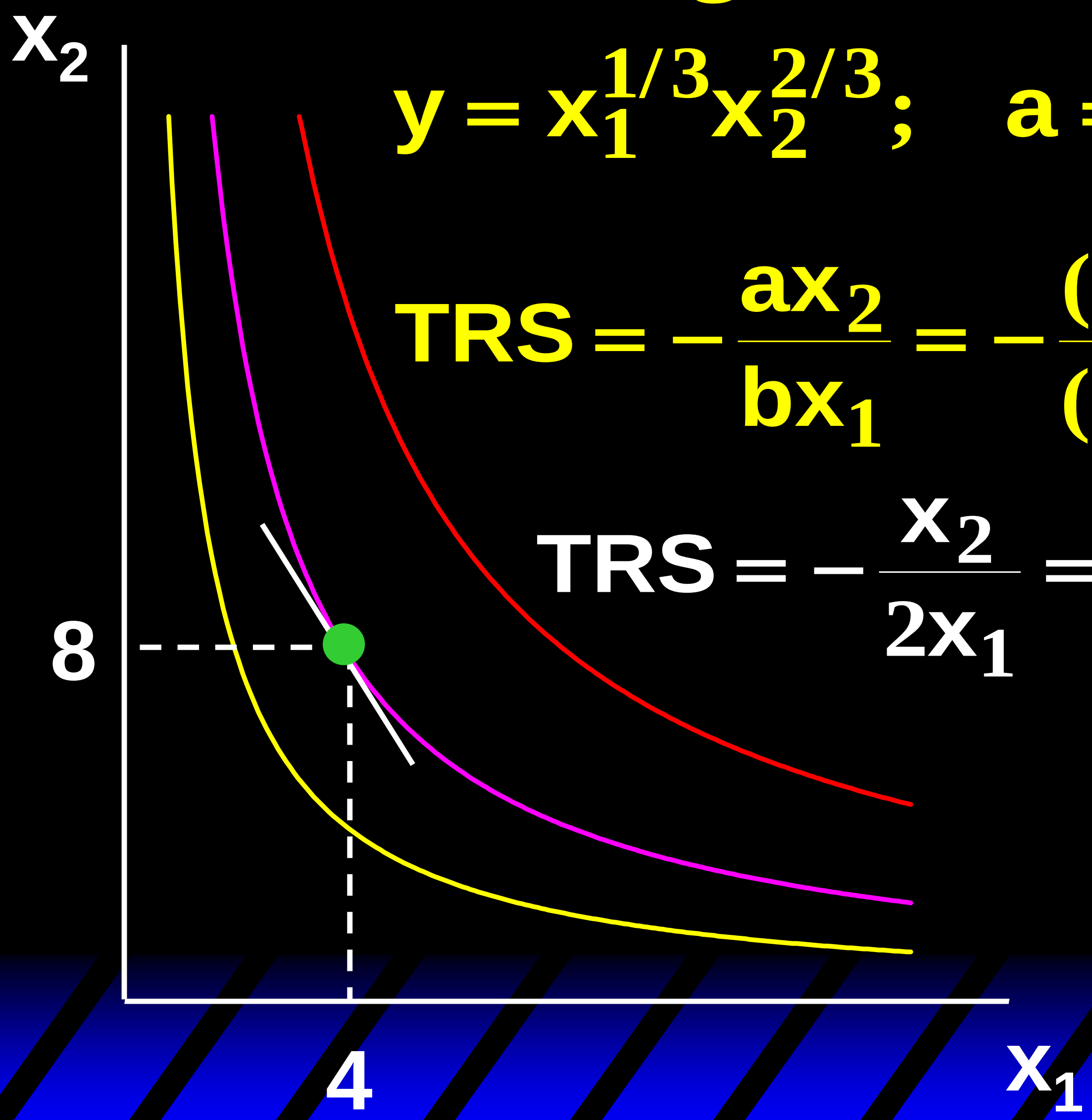


Technical Rate-of-Substitution; A Cobb-Douglas Example

$$y = x_1^{1/3} x_2^{2/3}; \quad a = \frac{1}{3} \text{ and } b = \frac{2}{3}$$

$$TRS = -\frac{a x_2}{b x_1} = -\frac{(1/3) x_2}{(2/3) x_1} = -\frac{x_2}{2 x_1}$$

$$TRS = -\frac{x_2}{2 x_1} = -\frac{8}{2 \times 4} = -1$$

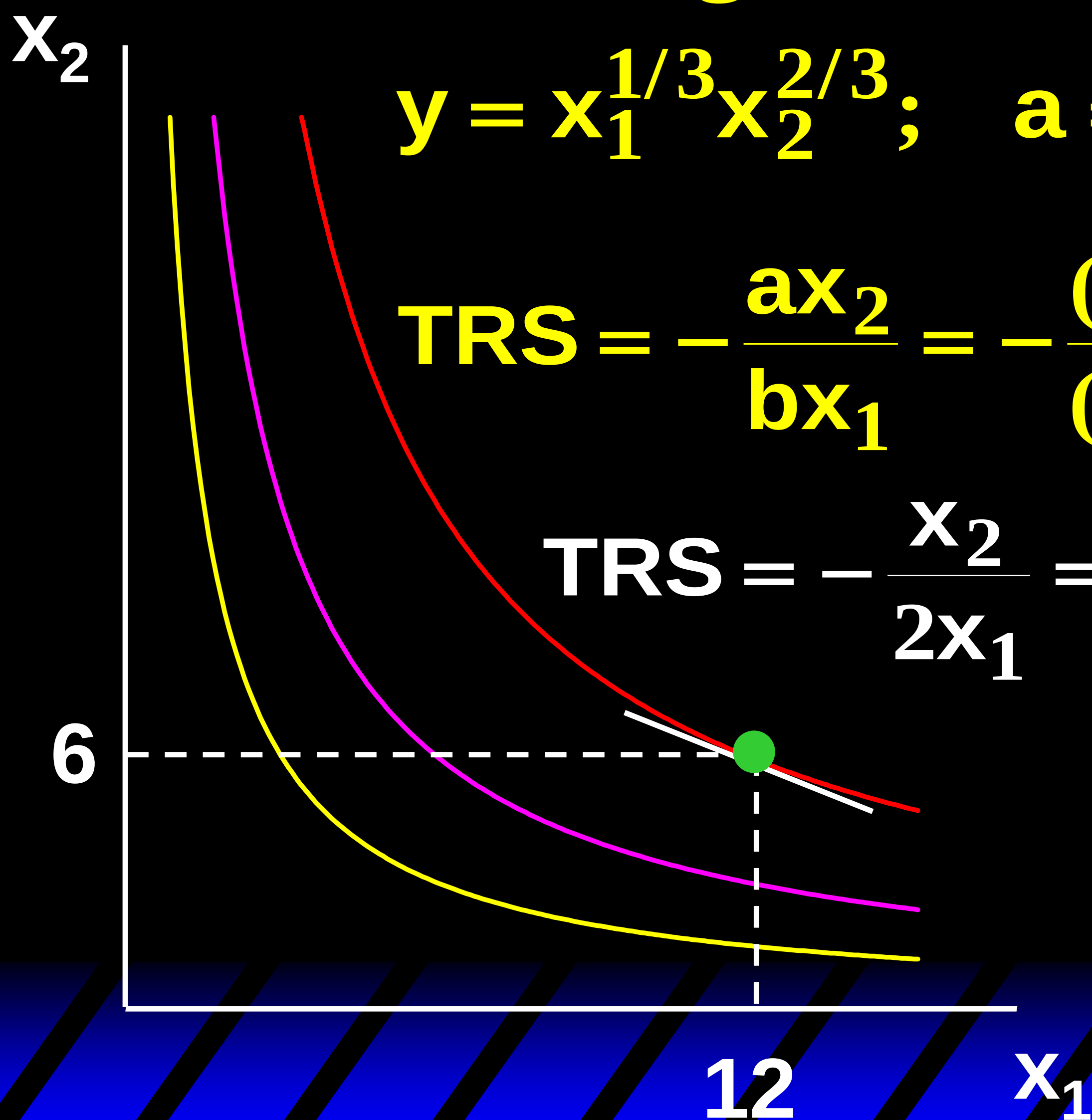


Technical Rate-of-Substitution; A Cobb-Douglas Example

$$y = x_1^{1/3} x_2^{2/3}; \quad a = \frac{1}{3} \text{ and } b = \frac{2}{3}$$

$$\text{TRS} = -\frac{a x_2}{b x_1} = -\frac{(1/3) x_2}{(2/3) x_1} = -\frac{x_2}{2 x_1}$$

$$\text{TRS} = -\frac{x_2}{2 x_1} = -\frac{6}{2 \times 12} = -\frac{1}{4}$$

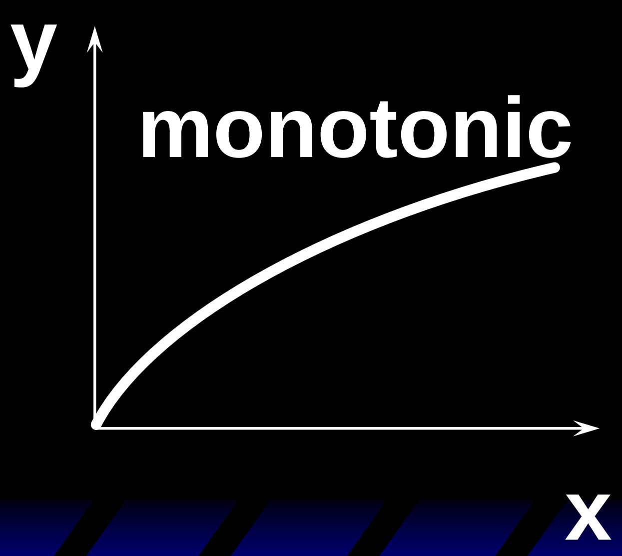


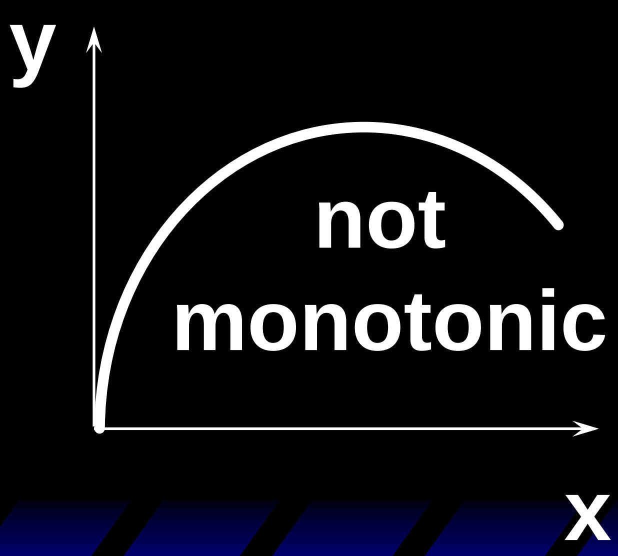
Well-Behaved Technologies

- A **well-behaved** technology is
  - **monotonic**, and
  - **convex**.

Well-Behaved Technologies - Monotonicity

- **Monotonicity: More of any input generates more output.**



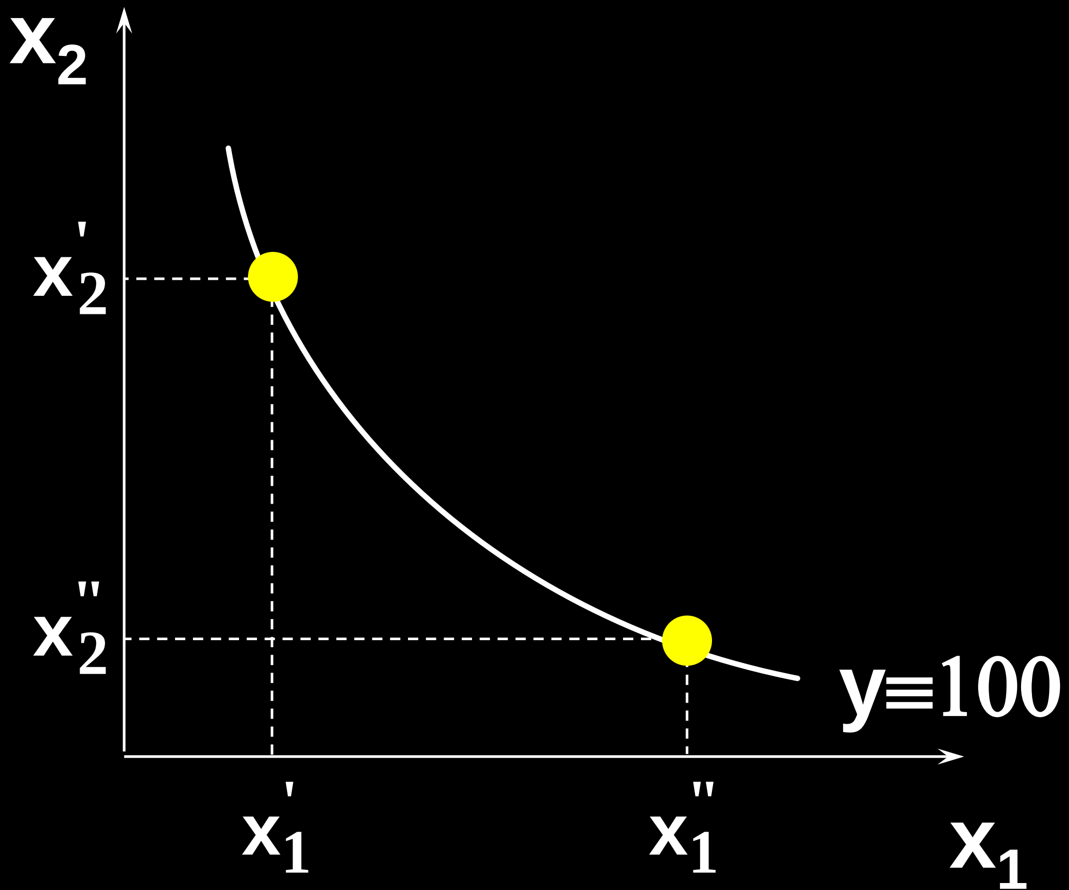


Well-Behaved Technologies - Convexity

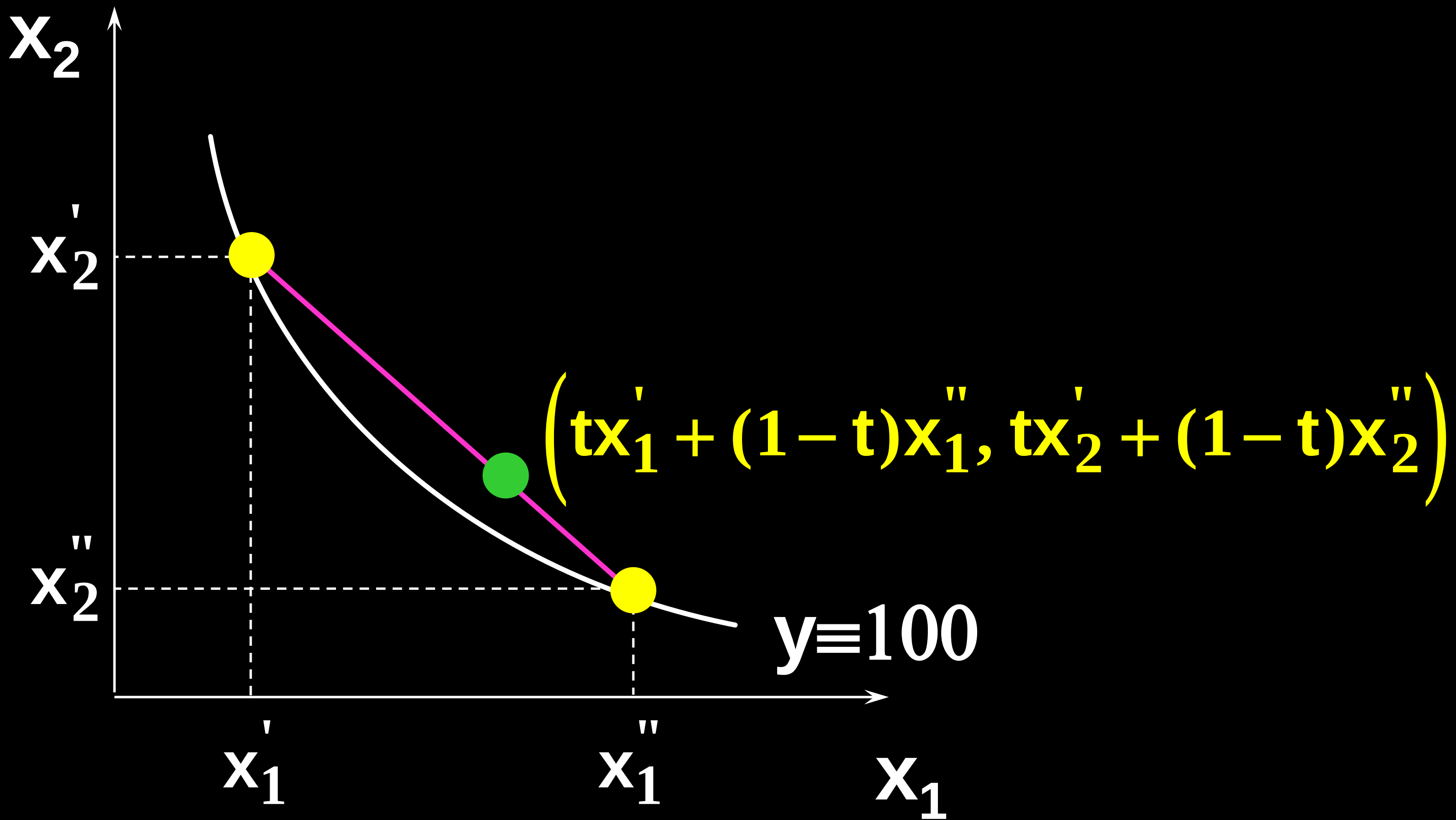
- **Convexity**: If the input bundles x' and x" both provide y units of output then the mixture tx' + (1-t)x" provides at least y units of output, for any 0 < t < 1.

“平均”优于“极端”

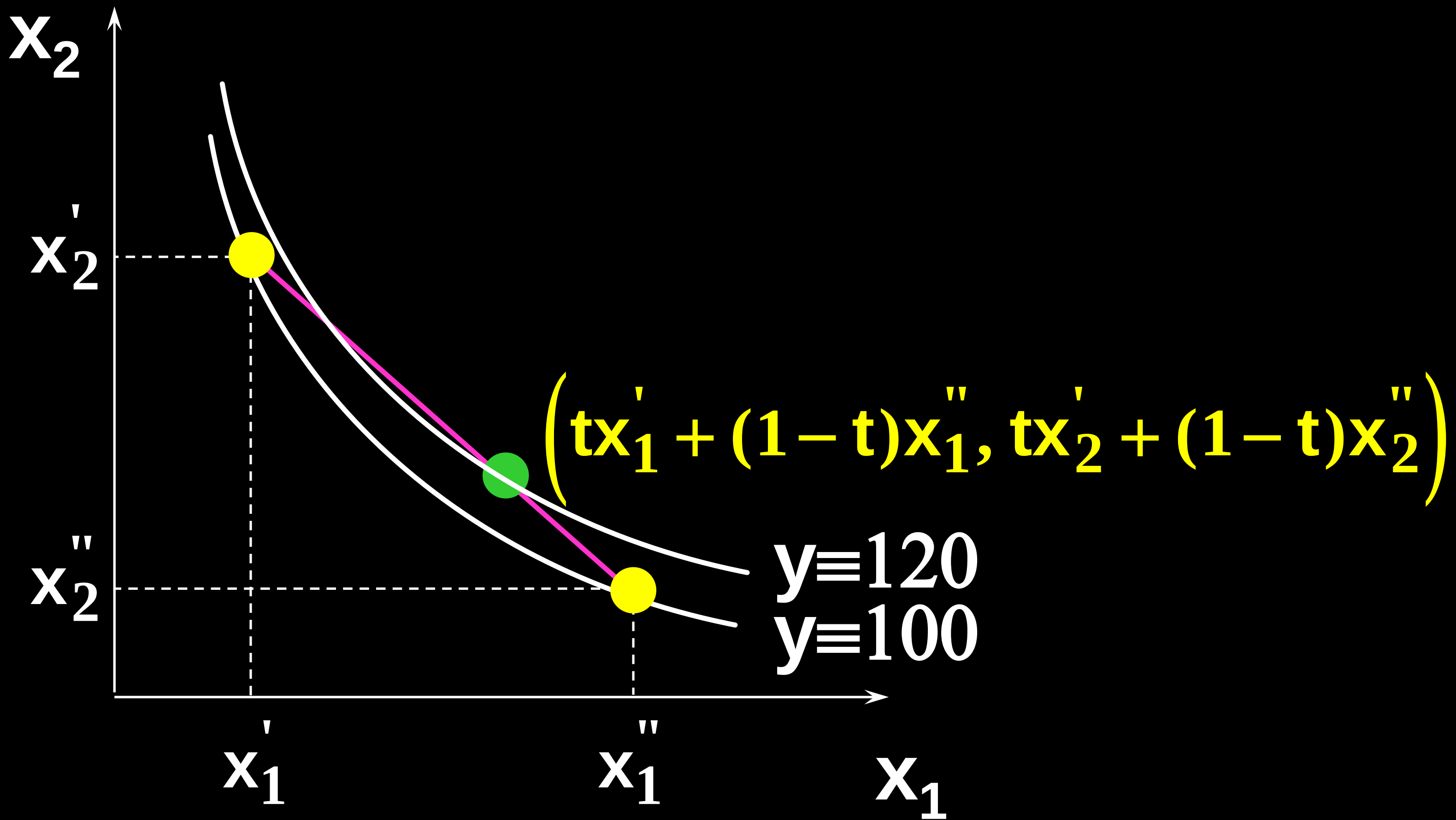
Well-Behaved Technologies - Convexity



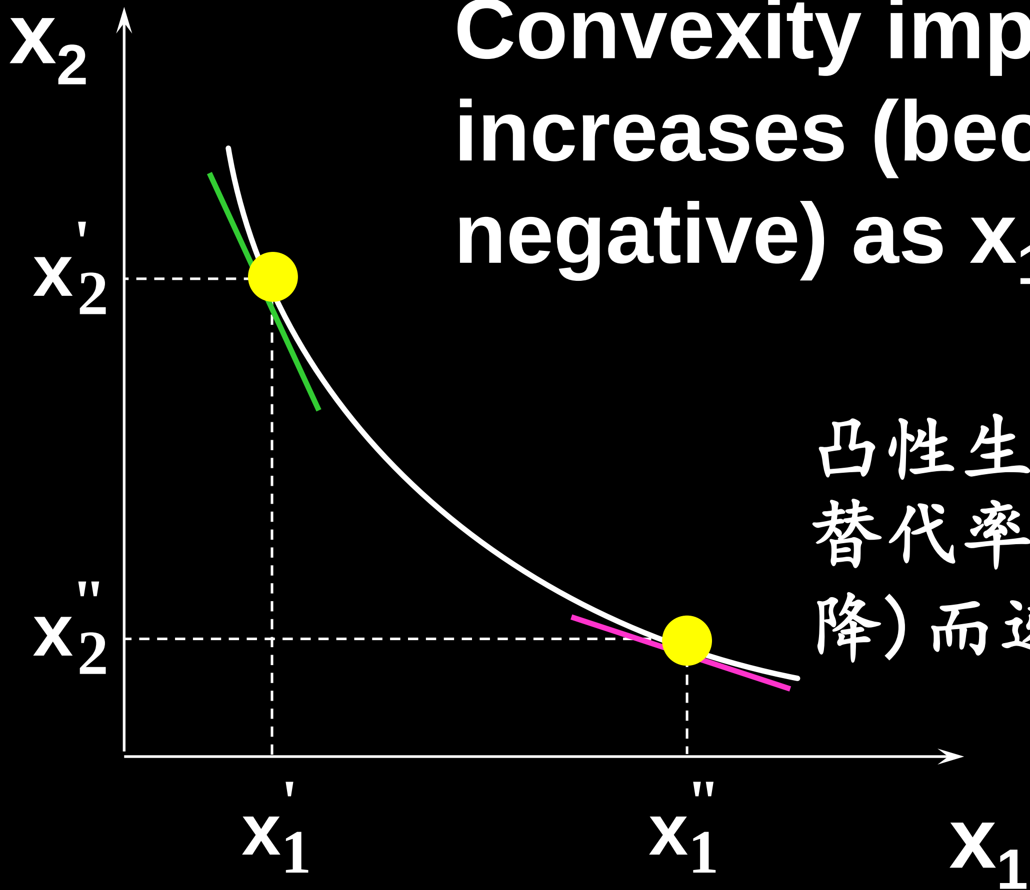
Well-Behaved Technologies - Convexity



Well-Behaved Technologies - Convexity



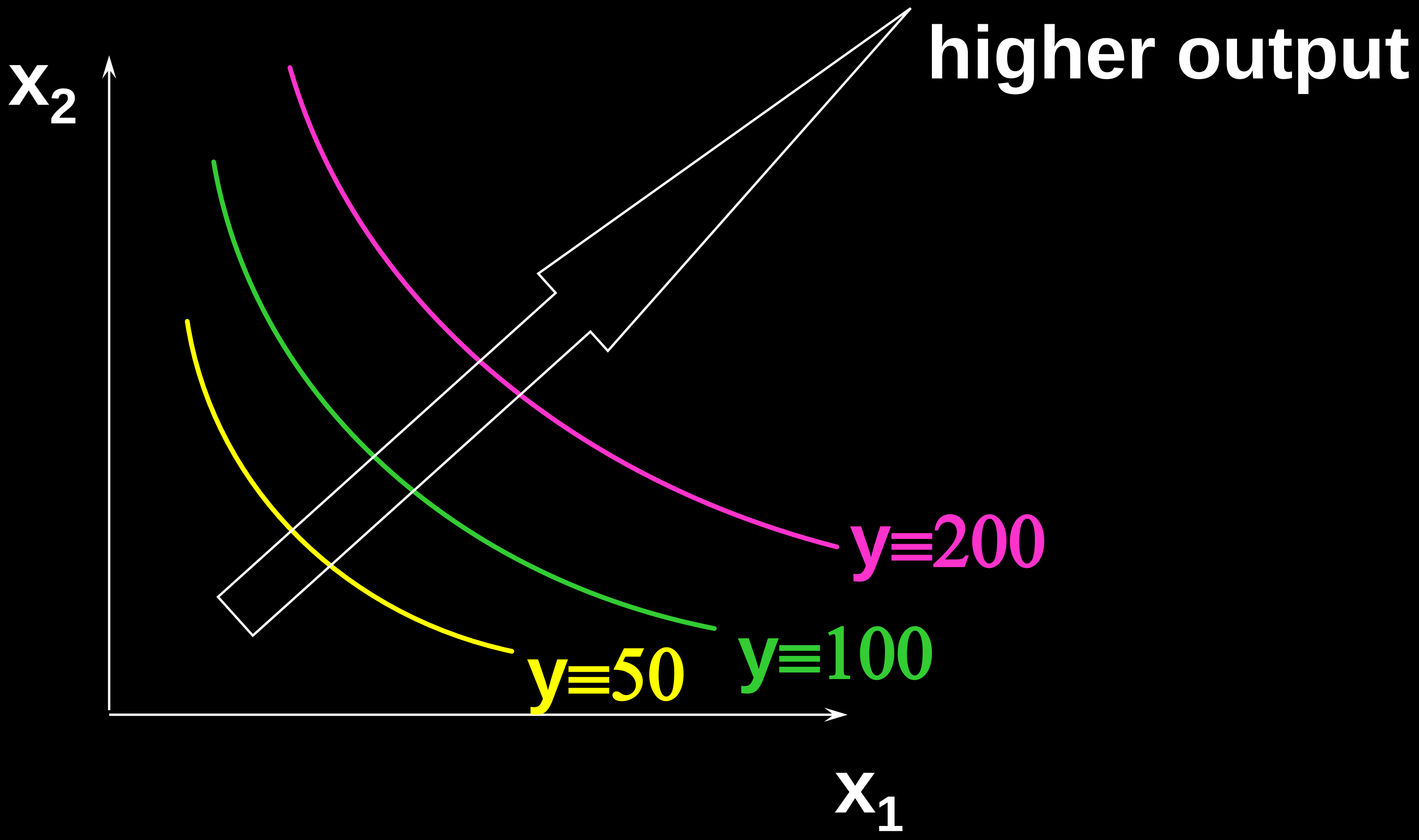
Well-Behaved Technologies - Convexity



Convexity implies that the TRS increases (becomes less negative) as x_1 increases.

凸性生产技术意味着技术替代率随x_1的增加(x_2的下降)而递减

Well-Behaved Technologies



The Long-Run and the Short-Runs

- **The long-run** is the circumstance in which a firm is **unrestricted** in its choice of all input levels.
- There are many possible short-runs.
- **A short-run** is a circumstance in which a firm is **restricted** in its choice of **at least one input level**.

The Long-Run and the Short-Runs

- Examples of restrictions that place a firm into a short-run:
  - temporarily being unable to install, or remove, machinery
  - being required by law to meet affirmative action quotas
  - having to meet domestic content regulations.

The Long-Run and the Short-Runs

- **A useful way to think of the long-run is that the firm can choose as it pleases in which short-run circumstance to be.**

可以将长期想象成“在不同的短期中任意挑选”的情况

The Long-Run and the Short-Runs

- What do short-run restrictions imply for a firm's technology?
- Suppose the short-run restriction is fixing the level of input 2.
- Input 2 is thus a **fixed input** in the short-run. Input 1 remains **variable**.

The Long-Run and the Short-Runs

$y = x_1^{1/3} x_2^{1/3}$ is the long-run production function (both x_1 and x_2 are variable).

The short-run production function when $x_2 \equiv 1$ is $y = x_1^{1/3} 1^{1/3} = x_1^{1/3}$.

The short-run production function when $x_2 \equiv 10$ is $y = x_1^{1/3} 10^{1/3} = 2 \cdot 15 x_1^{1/3}$.