Lecture 11

Profit-Maximization

Economic Profit

A firm uses inputs $j = 1\ldots,m$ to make products $i = 1,\ldots n$.

Output levels are $y_1,\ldots,y_n$.

Input levels are $x_1,\ldots,x_m$.

Product prices are $p_1,\ldots,p_n$.

Input prices are $w_1,\ldots,w_m$.

The Competitive Firm

The competitive firm **takes** all output prices $p_1,\ldots,p_n$ and all input prices $w_1,\ldots,w_m$ as given constants.

完全竞争企业是产品和要素的价格接受者。

Economic Profit

The **economic profit** generated by the production plan $(x_1,\ldots,x_m,y_1,\ldots,y_n)$ is

$$\Pi = p_1y_1 + \cdots + p_ny_n - w_1x_1 - \cdots w_mx_m.$$

Economic Profit

- Output and input levels are typically **flows**.
- E.g. x_1 might be the number of labor units **used per hour**.
- And y_3 might be the number of cars **produced per hour**.
- Consequently, profit is typically a flow also; e.g. the number of dollars of profit earned per hour.

Economic Profit

- How do we value a firm?
- Suppose the firm's stream of periodic economic profits is $\Pi_0, \Pi_1, \Pi_2, \ldots$ and r is the rate of interest.
- Then the present-value of the firm's economic profit stream is

$$PV = \Pi_0 + \frac{\Pi_1}{1+r} + \frac{\Pi_2}{(1+r)^2} + \cdots$$

- A competitive firm seeks to maximize its **present-value**.

Economic Profit

Suppose the firm is in a short-run circumstance in which $x_2 \equiv \tilde{x}_2$.

Its short-run production function is

$$y = f(x_1, \tilde{x}_2).$$

x_1**: variable factor (可变要素)**

x_2**: fixed factor (不变要素)**

Economic Profit

Suppose the firm is in a short-run circumstance in which $x_2 \equiv \tilde{x}_2$.

Its short-run production function is

$$y = f(x_1, \tilde{x}_2).$$

The firm's fixed cost is $FC = w_2\tilde{x}_2$ and its profit function is

$$\Pi = py - w_1x_1 - w_2\tilde{x}_2.$$

Short-Run Iso-Profit Lines

- A $\Pi iso-profit line contains all the production plans that yield a profit level of $\Pi. (等利润线)
- A $\Pi iso-profit line's equation is

$$\Pi \equiv py - w_1x_1 - w_2\tilde{x}_2.$$

Short-Run Iso-Profit Lines

A $\Pi iso-profit line contains all the production plans that yield a profit level of $\Pi. (等利润线)

The equation of a $\Pi iso-profit line is

$$\Pi \equiv py - w_1 x_1 - w_2 \tilde{x}_2.$$

I.e.

$$y = \frac{w_1}{p} x_1 + \frac{\Pi + w_2 \tilde{x}_2}{p}.$$

Short-Run Iso-Profit Lines

$$y = \frac{w_1}{p} x_1 + \frac{\Pi + w_2 \tilde{x}_2}{p}$$

has a slope of

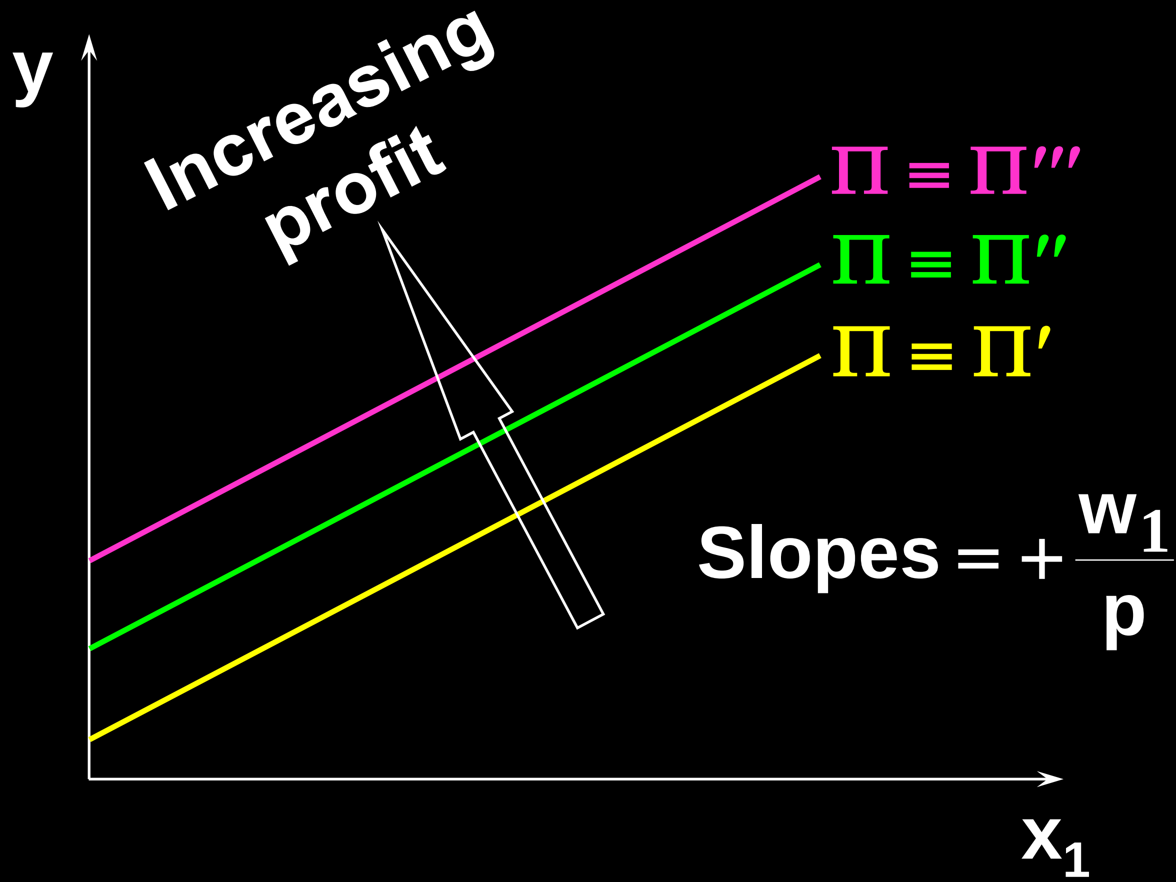
$$+\frac{w_1}{p}$$

and a vertical intercept of

$$\frac{\Pi + w_2 \tilde{x}_2}{p}.$$

Short-Run Iso-Profit Lines

Short-Run Iso-Profit Lines



纵截距越大的等利润线代表的利润越高。

Short-Run Profit-Maximization

The firm's problem is to locate the production plan that attains the highest possible iso-profit line, given the firm's constraint on choices of production plans.

Short-Run Profit-Maximization

The firm's problem is to locate the production plan that attains the **highest** possible iso-profit line, given the firm's **constraint** on choices of production plans.

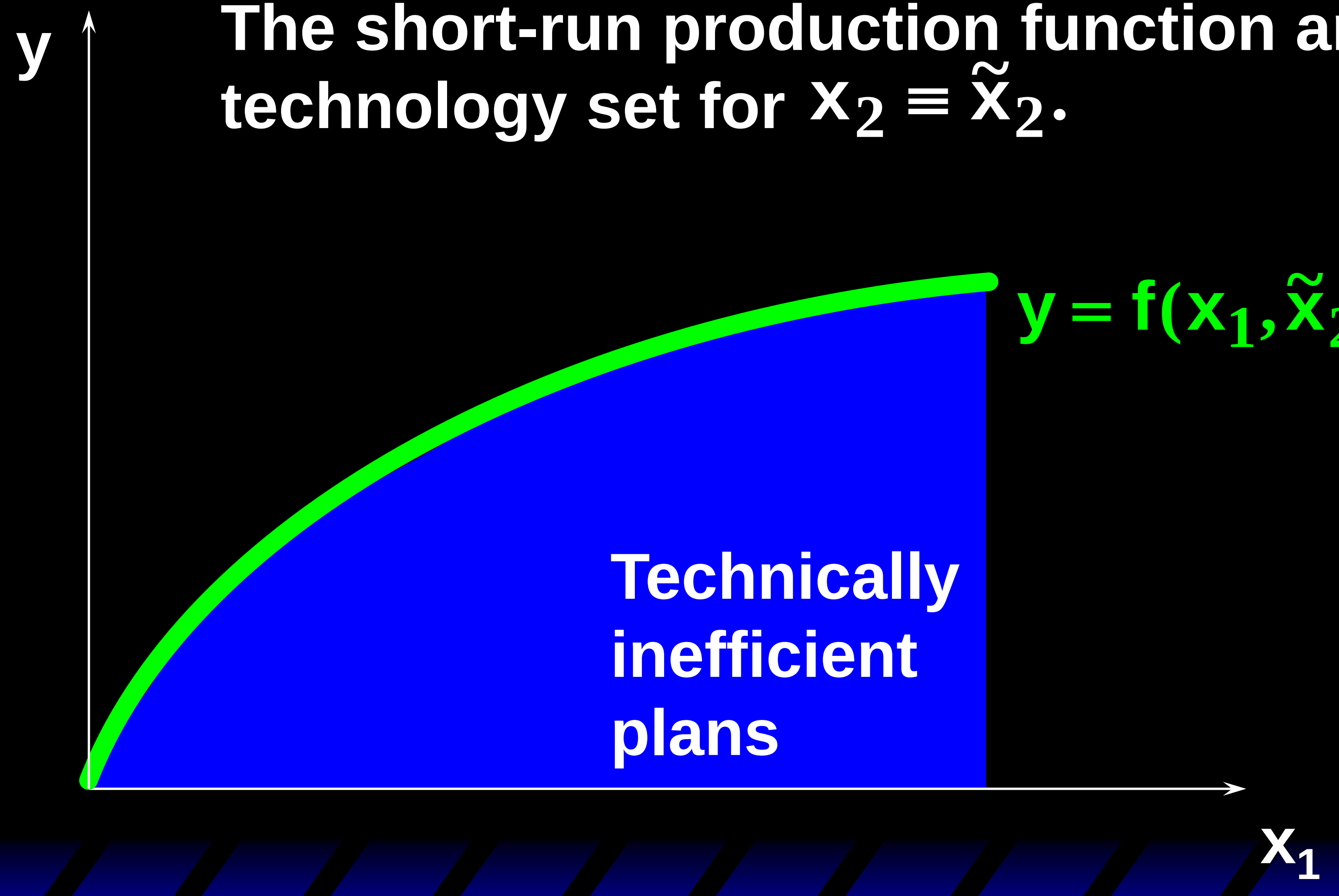
Q: What is this constraint?

A: The production function.

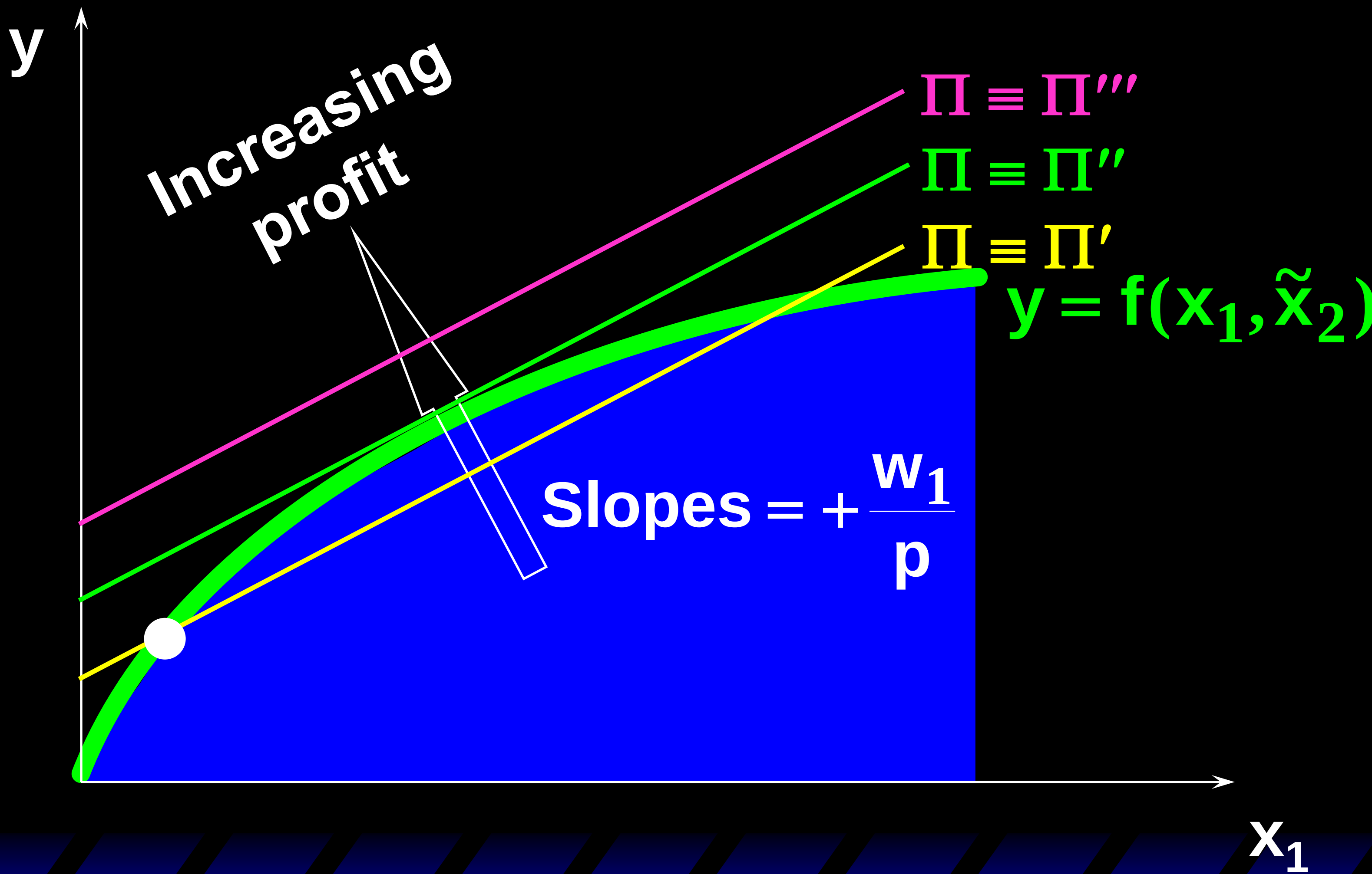
生产者问题：在可行的生产计划中选择位于最高等利润线上的那一个。

Short-Run Profit-Maximization

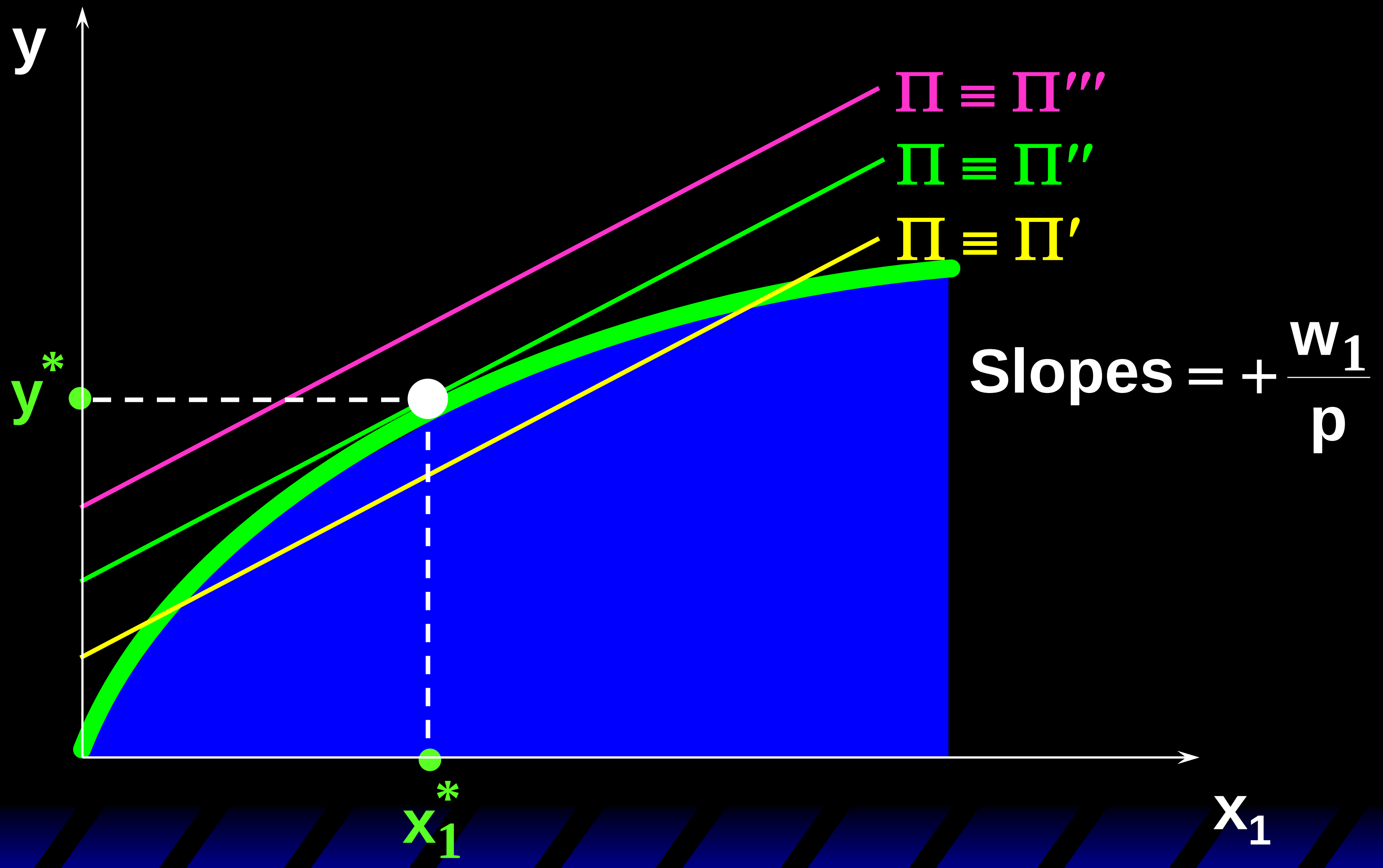
The short-run production function and technology set for $x_2 \equiv \tilde{x}_2$.



Short-Run Profit-Maximization



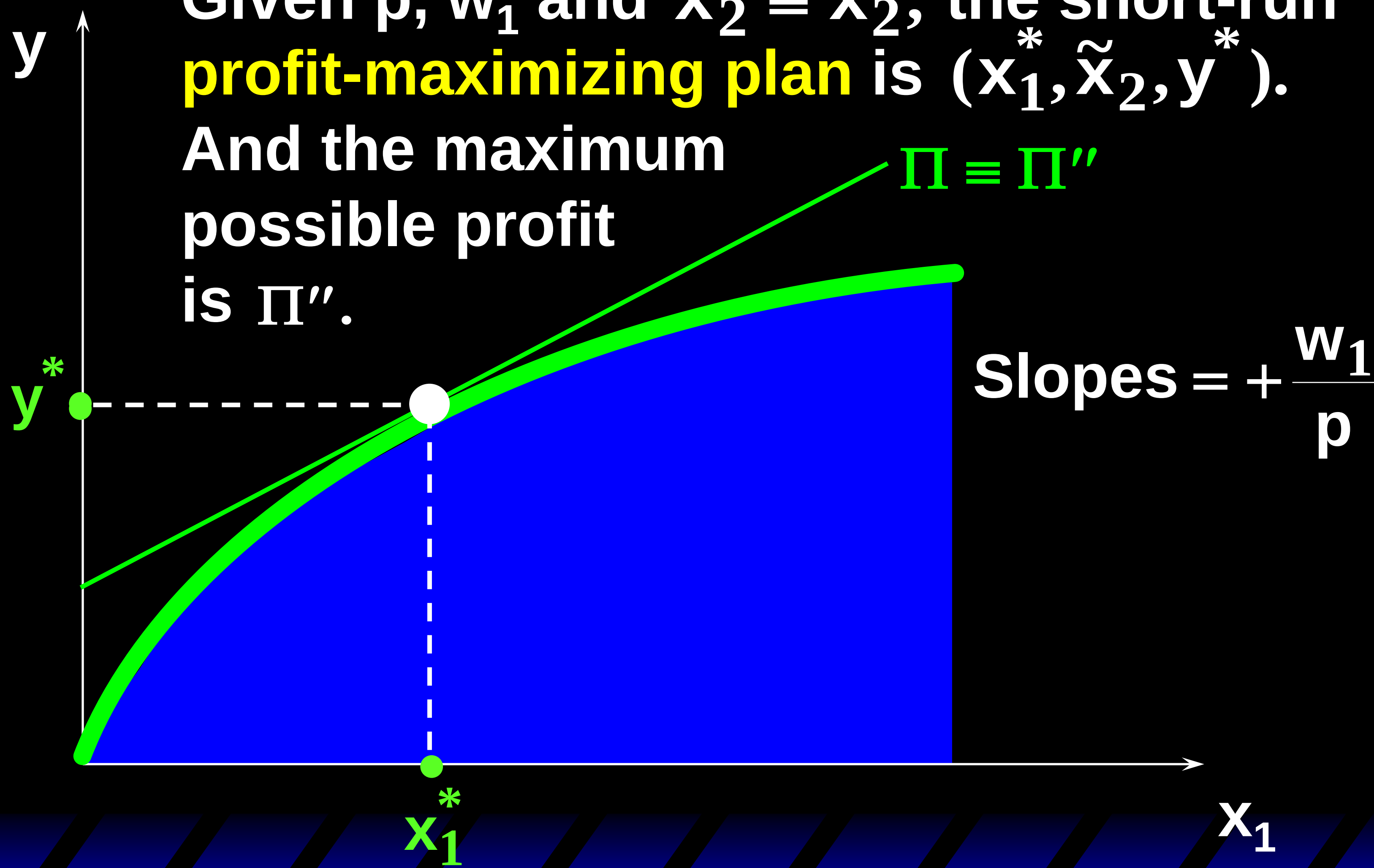
Short-Run Profit-Maximization



Short-Run Profit-Maximization

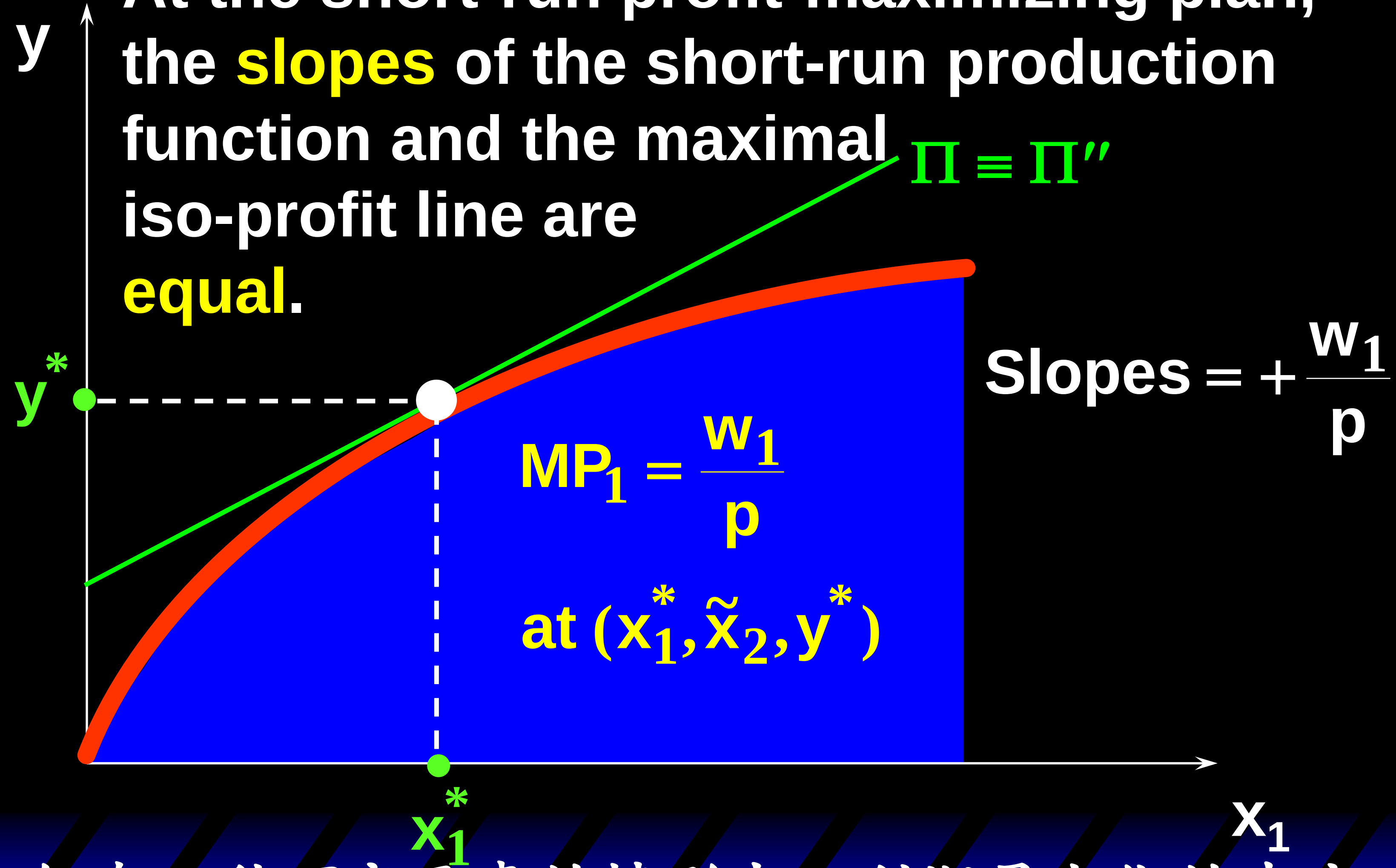
Given p, w_1 and $x_2 \equiv \tilde{x}_2$, the short-run **profit-maximizing plan** is $(x_1^*, \tilde{x}_2, y^*)$.

And the maximum possible profit is Π''.



Short-Run Profit-Maximization

At the short-run profit-maximizing plan, the slopes of the short-run production function and the maximal iso-profit line are equal.



在仅有一种可变要素的情形中，利润最大化的生产计划满足：等利润线与生产函数线相切。

Short-Run Profit-Maximization

$$MP_1 = \frac{w_1}{p} \quad \Leftrightarrow \quad p \times MP_1 = w_1$$

$p \times MP_1$ is the **marginal revenue product of input 1 (边际产值)**, the rate at which revenue increases with the amount used of input 1.

If $p \times MP_1 > w_1$ then profit increases with x_1.

If $p \times MP_1 < w_1$ then profit decreases with x_1.

Short-Run Profit-Maximization; A Cobb-Douglas Example

Suppose the short-run production function is $y = x_1^{1/3}\tilde{x}_2^{1/3}$.

The marginal product of the variable input 1 is $MP_1 = \dfrac{\partial y}{\partial x_1} = \dfrac{1}{3}x_1^{-2/3}\tilde{x}_2^{1/3}$.

The profit-maximizing condition is

$$MRP_1 = p \times MP_1 = \frac{p}{3}(x_1^*)^{-2/3}\tilde{x}_2^{1/3} = w_1.$$

Short-Run Profit-Maximization; A Cobb-Douglas Example

Solving $\frac{p}{3}(x_1^*)^{-2/3}\tilde{x}_2^{1/3} = w_1$ for x_1 gives

$$(x_1^*)^{-2/3} = \frac{3w_1}{p\tilde{x}_2^{1/3}}.$$

so $$x_1^* = \left(\frac{p\tilde{x}_2^{1/3}}{3w_1}\right)^{3/2} = \left(\frac{p}{3w_1}\right)^{3/2}\tilde{x}_2^{1/2}.$$

要素1的短期需求函数

Short-Run Profit-Maximization; A Cobb-Douglas Example

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2} \tilde{x}_2^{1/2}$$

is the firm's short-run demand for input 1 when the level of input 2 is fixed at $\tilde{x}_2$ units. （短期要素需求函数）

The firm's short-run output level is thus

$$y^* = (x_1^*)^{1/3} \tilde{x}_2^{1/3} = \left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2}.$$

（短期供给函数）

Comparative Statics of Short-Run Profit-Maximization

What happens to the short-run profit-maximizing production plan as the output price p changes?

Comparative Statics of Short-Run Profit-Maximization

The equation of a short-run iso-profit line is

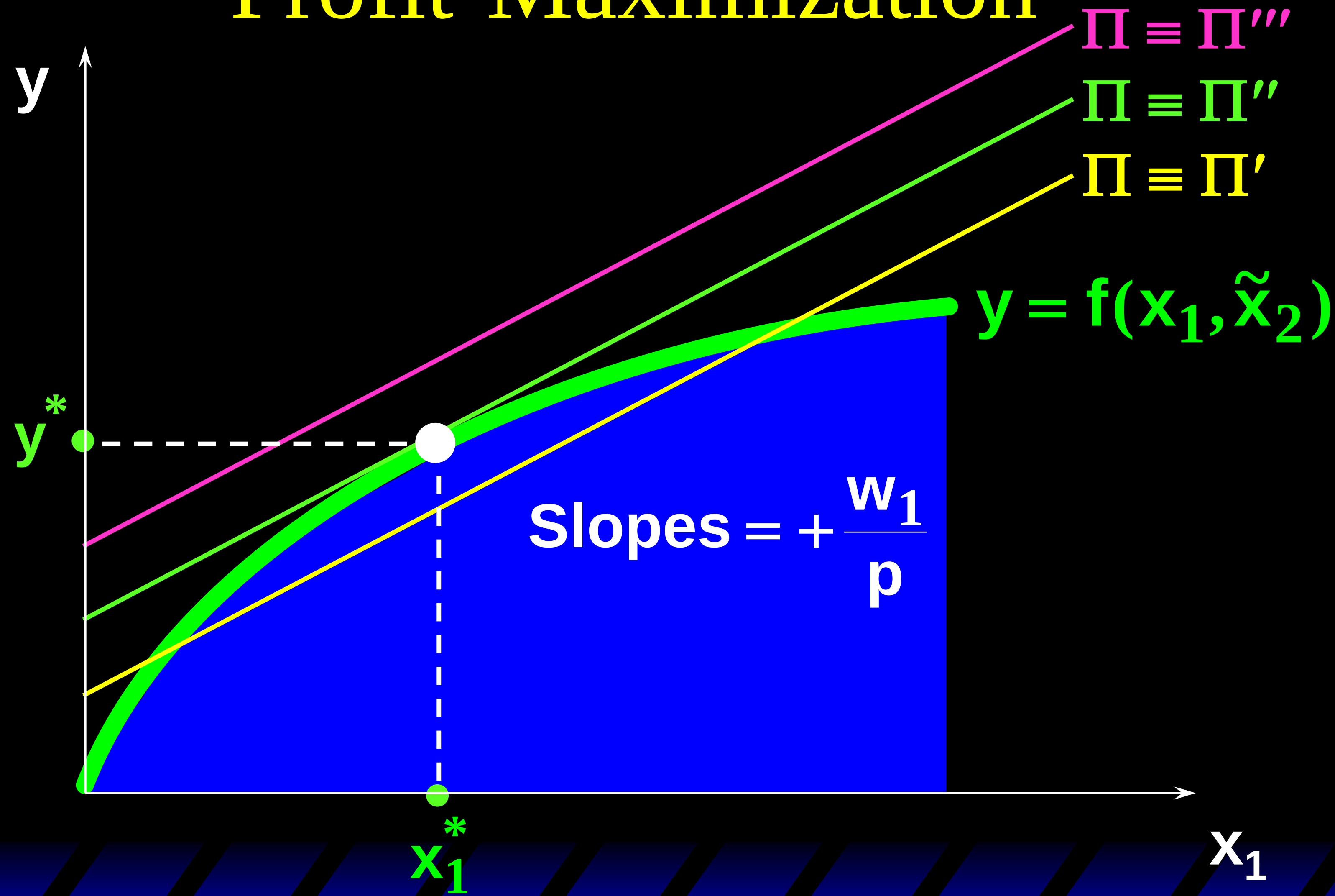
$$y = \frac{w_1}{p}x_1 + \frac{\Pi + w_2\tilde{x}_2}{p}$$

so an increase in p causes

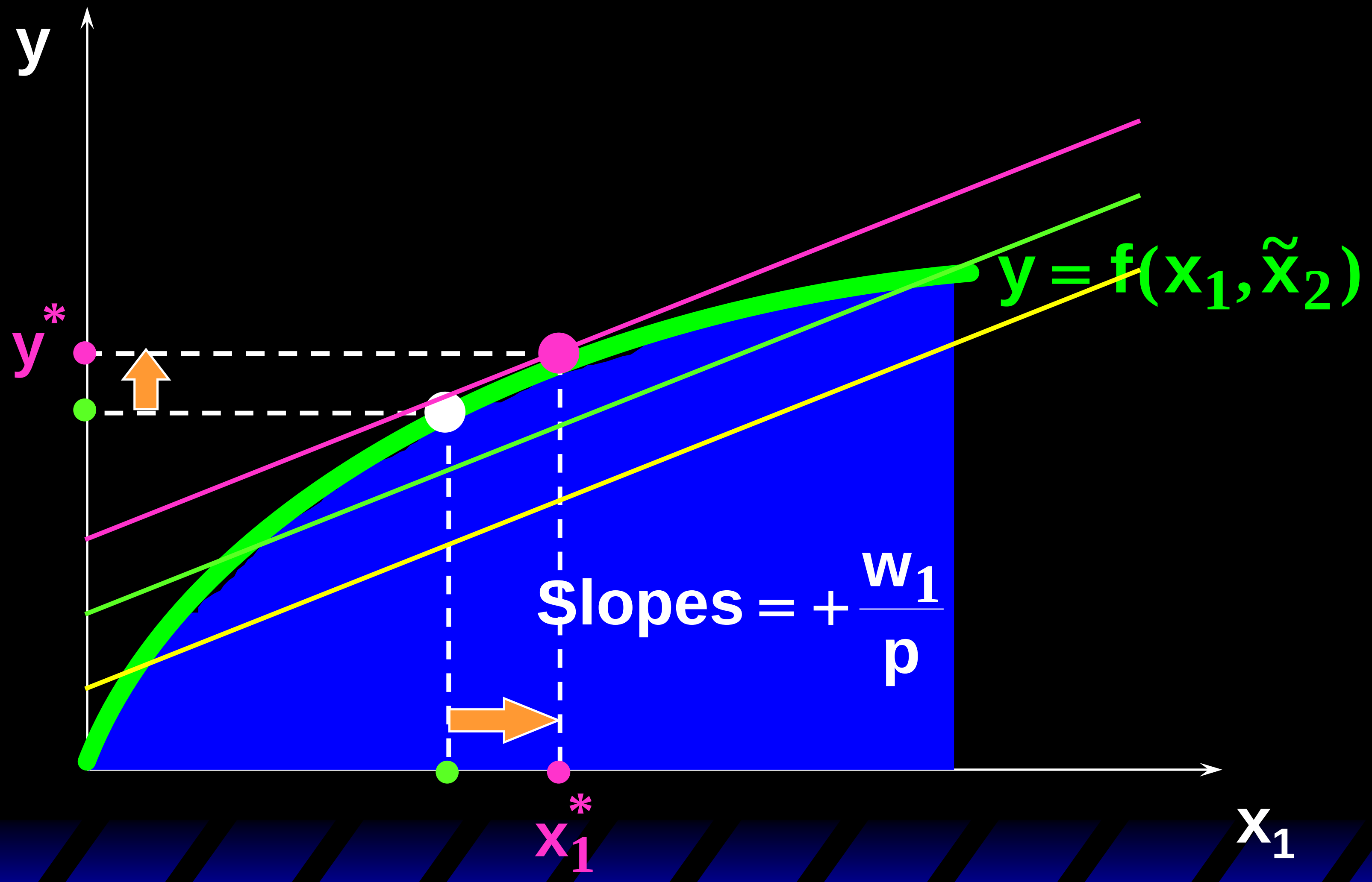
-- a reduction in the slope, and

-- a reduction in the vertical intercept.

Comparative Statics of Short-Run Profit-Maximization



Comparative Statics of Short-Run Profit-Maximization



Comparative Statics of Short-Run Profit-Maximization

- An **increase** in **p**, the price of the firm's output, causes
  - an **increase** in the firm's **output** level (the firm's supply curve slopes upward), and
  - an **increase** in the level of the firm's variable **input** (the firm's demand curve for its variable input shifts outward).

产品价格上升导致供给上升、要素需求上升。

Comparative Statics of Short-Run Profit-Maximization

The Cobb-Douglas example: When $y = x_1^{1/3}\tilde{x}_2^{1/3}$ then the firm's short-run demand for its variable input 1 is

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2}\tilde{x}_2^{1/2}$$

and its short-run supply is

$$y^* = \left(\frac{p}{3w_1}\right)^{1/2}\tilde{x}_2^{1/2}.$$

x_1^* increases as p increases.

y^* increases as p increases.

Comparative Statics of Short-Run Profit-Maximization

What happens to the short-run profit-maximizing production plan as the variable input price w_1 changes?

Comparative Statics of Short-Run Profit-Maximization

The equation of a short-run iso-profit line is

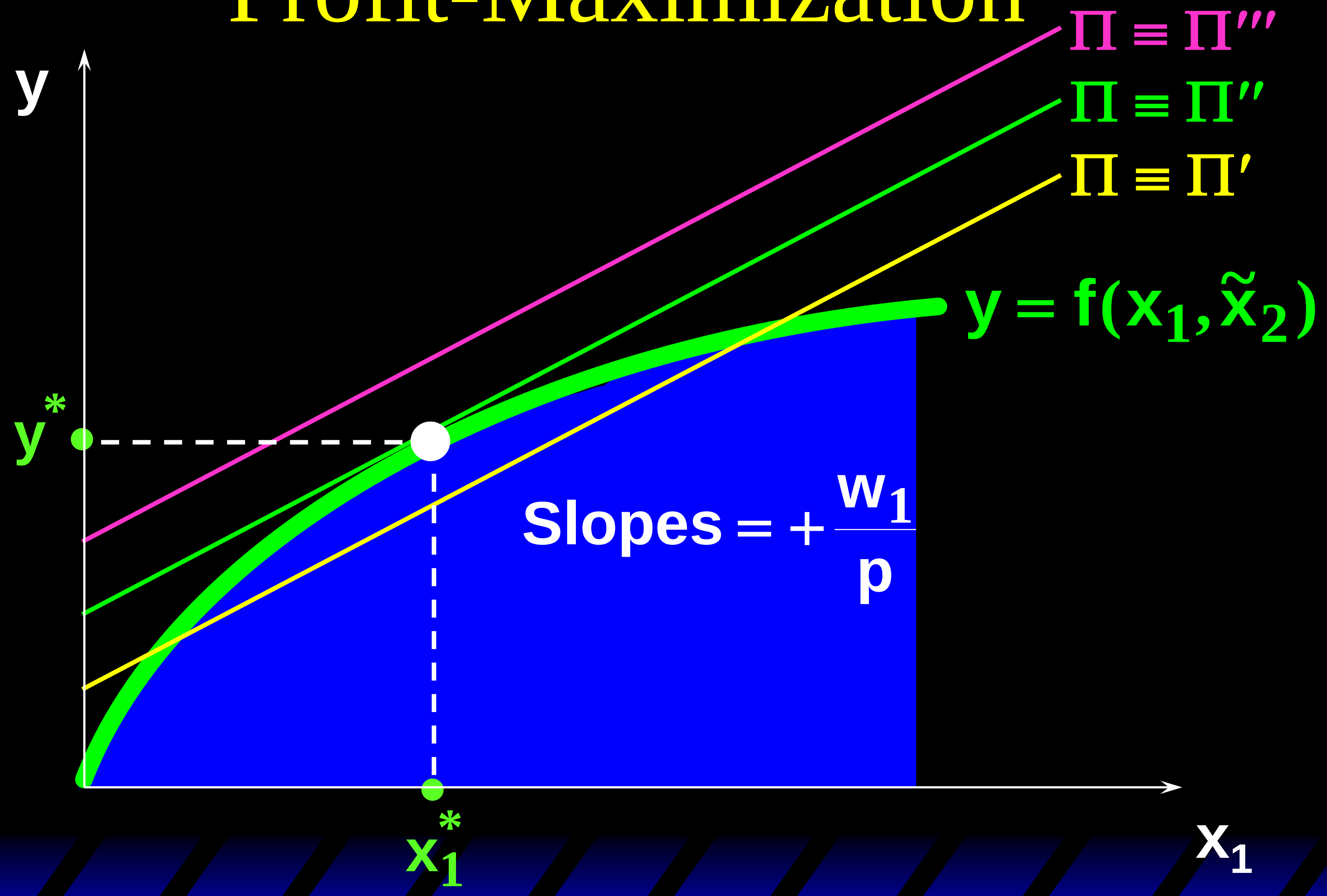
$$y = \frac{w_1}{p}x_1 + \frac{\Pi + w_2\tilde{x}_2}{p}$$

so an **increase in w_1** causes

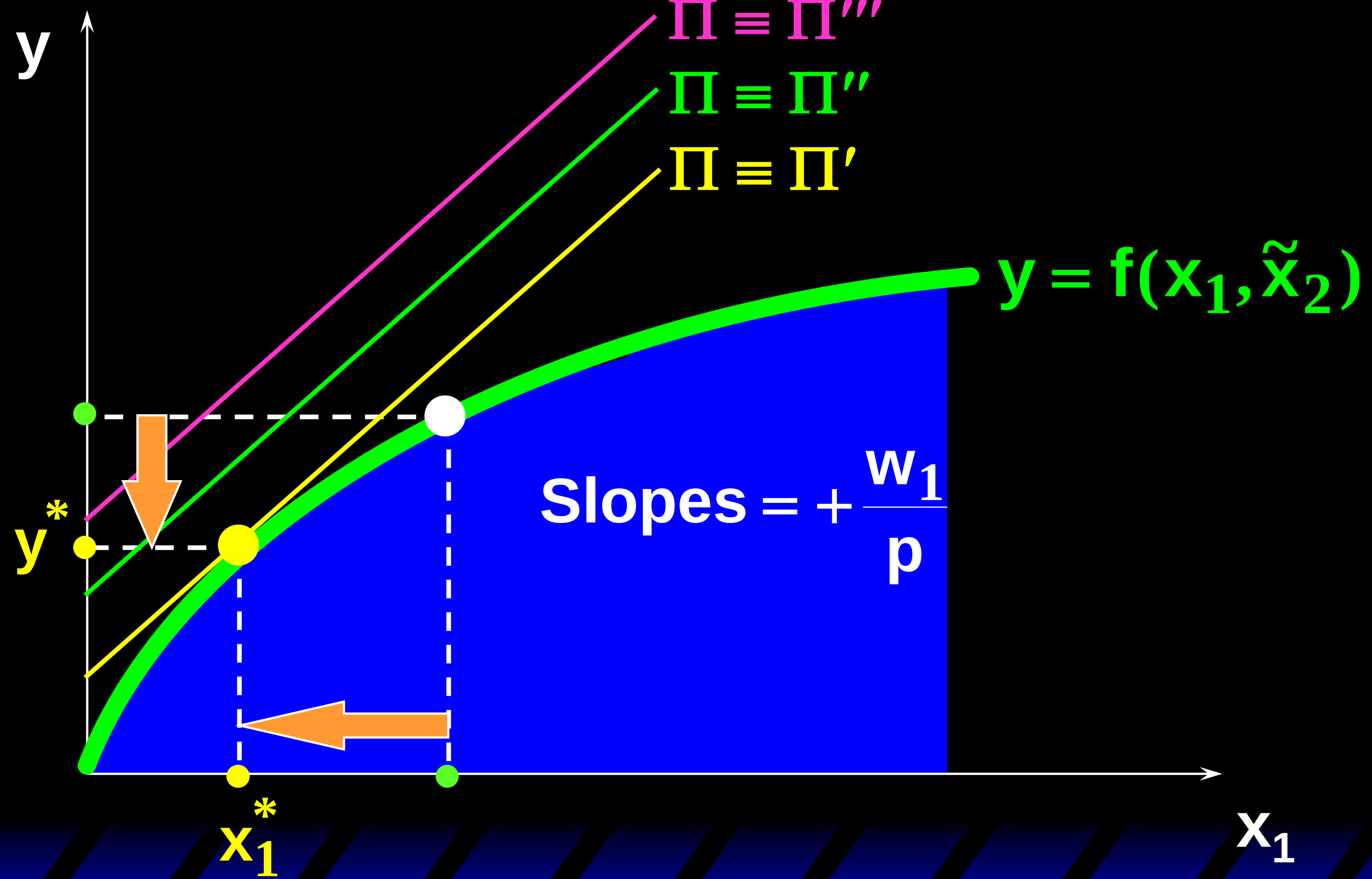
-- an **increase** in the slope, and

-- no change to the vertical intercept.

Comparative Statics of Short-Run Profit-Maximization



Comparative Statics of Short-Run Profit-Maximization



Comparative Statics of Short-Run Profit-Maximization

An **increase** in w_1, the price of the firm's variable input, causes

- a **decrease** in the firm's output level (the firm's supply curve shifts inward), and
- a **decrease** in the level of the firm's variable input (the firm's demand curve for its variable input slopes downward).

要素价格上升导致供给下降、要素需求下降。

Comparative Statics of Short-Run Profit-Maximization

The Cobb-Douglas example: When $y = x_1^{1/3}\tilde{x}_2^{1/3}$ then the firm's short-run demand for its variable input 1 is

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2} \tilde{x}_2^{1/2}$$

and its short-run supply is

$$y^* = \left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2}.$$

x_1^* decreases as w_1 increases.

y^* decreases as w_1 increases.

Long-Run Profit-Maximization

- Now allow the firm to vary both input levels.
- Since no input level is fixed, there are no fixed costs.

Long-Run Profit-Maximization

- Both x_1 and x_2 are variable (可变要素).
- The optimality condition must now hold for **each** factor choice.

Long-Run Profit-Maximization

The input levels of the long-run profit-maximizing plan satisfy

$$\mathbf{p \times MP_1 - w_1 = 0} \quad \text{and} \quad \mathbf{p \times MP_2 - w_2 = 0.}$$

That is, marginal revenue equals marginal cost for all inputs.

Long-Run Profit-Maximization

The producer's problem:

$$\max_{x_1, x_2} \Pi = py - \omega_1 x_1 - \omega_2 x_2$$

$$\text{s.t. } y = f(x_1, x_2)$$

This can be written as:

$$\max_{x_1, x_2} \Pi = pf(x_1, x_2) - \omega_1 x_1 - \omega_2 x_2$$

Long-Run Profit-Maximization

$$\max_{x_1, x_2} \boldsymbol{\Pi} = \boldsymbol{pf(x_1, x_2) - \omega_1 x_1 - \omega_2 x_2}$$

F.O.C.

$$\mathbf{p \times MP_1 = \omega_1}$$

$$\mathbf{p \times MP_2 = \omega_2}$$

The Cobb-Douglas example

The Cobb-Douglas example:

$$\max_{x_1,x_2} \Pi = py - \omega_1 x_1 - \omega_2 x_2$$

$$\text{s.t. } y = x_1^{1/3} x_2^{1/3}$$

F.O.C.

$$\mathrm{p} \times \mathrm{MP}_1 = \frac{1}{3}\mathrm{px}_1^{-2/3} x_2^{1/3} = \omega_1$$

$$\mathrm{p} \times \mathrm{MP}_2 = \frac{1}{3}\mathrm{px}_1^{1/3} x_2^{-2/3} = \omega_2$$

The Cobb-Douglas example

$$\mathbf{p} \times \mathbf{MP_1} = \frac{1}{3}\mathbf{p}\mathbf{x}_1^{-2/3}\boldsymbol{x}_2^{1/3} = \boldsymbol{\omega}_1$$

$$\mathbf{p} \times \mathbf{MP_2} = \frac{1}{3}\mathbf{p}\mathbf{x}_1^{1/3}\boldsymbol{x}_2^{-2/3} = \boldsymbol{\omega}_2$$

=>

$$\frac{1/3\mathbf{p}\mathbf{x}_1^{-2/3}\mathbf{x}_2^{1/3}}{1/3\mathbf{p}\mathbf{x}_1^{1/3}\mathbf{x}_2^{-2/3}} = \frac{\mathbf{x}_2}{\mathbf{x}_1} = \frac{\boldsymbol{\omega}_1}{\boldsymbol{\omega}_2}$$

$$\mathbf{x}_2 = \frac{\boldsymbol{\omega}_1}{\boldsymbol{\omega}_2}\mathbf{x}_1$$

The Cobb-Douglas example

$$p \times MP_1 = \frac{1}{3} p x_1^{-2/3} x_2^{1/3} = \omega_1$$

$$p \times MP_2 = \frac{1}{3} p x_1^{1/3} x_2^{-2/3} = \omega_2$$

=>

$$\frac{1/3 p x_1^{-2/3} x_2^{1/3}}{1/3 p x_1^{1/3} x_2^{-2/3}} = \frac{x_2}{x_1} = \frac{\omega_1}{\omega_2}$$

$$x_2 = \frac{\omega_1}{\omega_2} x_1$$

The Cobb-Douglas example

$$\mathbf{p \times MP_2 = \frac{1}{3} p x_1^{1/3} \boldsymbol{x_2^{-2/3}} = \omega_2}$$

$$\mathbf{x_2 = \frac{\omega_1}{\omega_2} x_1}$$

$$\mathbf{\frac{1}{3} p x_1^{1/3} \left(\frac{\omega_1}{\omega_2} x_1\right)^{-2/3} = \omega_2}$$

$$\mathbf{x_1} = \frac{p^3}{27\omega_1^2 \omega_2}$$

The Cobb-Douglas example

$$\mathrm{x}_1 = \frac{p^3}{27\omega_1^2\omega_2}$$

$$\mathrm{x}_2 = \frac{\omega_1}{\omega_2}\mathrm{x}_1 = \frac{p^3}{27\omega_1\omega_2^2}$$

$$y = x_1^{1/3}x_2^{1/3} = \frac{p^2}{9\omega_1\omega_2}$$

长期要素需求函数和商品供给函数

Long-Run Profit-Maximization

So given the prices p, w_1 and w_2, and the production function $y = x_1^{1/3} x_2^{1/3}$

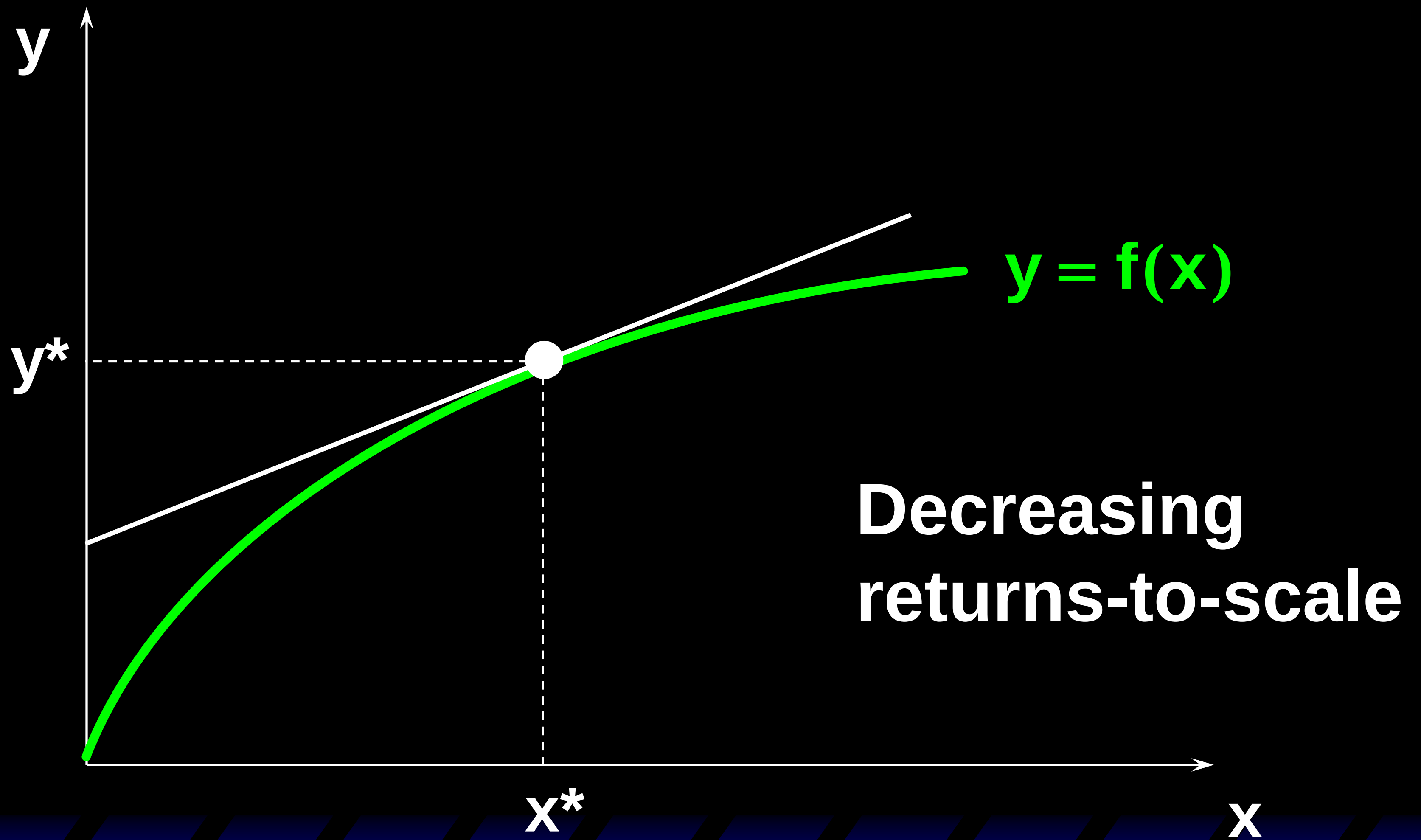
the long-run profit-maximizing production plan is

$$(x_1^*, x_2^*, y^*) = \left(\frac{p^3}{27 w_1^2 w_2}, \frac{p^3}{27 w_1 w_2^2}, \frac{p^2}{9 w_1 w_2} \right).$$

Returns-to-Scale and Profit-Maximization

If a competitive firm's technology exhibits decreasing returns-to-scale then the firm has a single long-run profit-maximizing production plan.

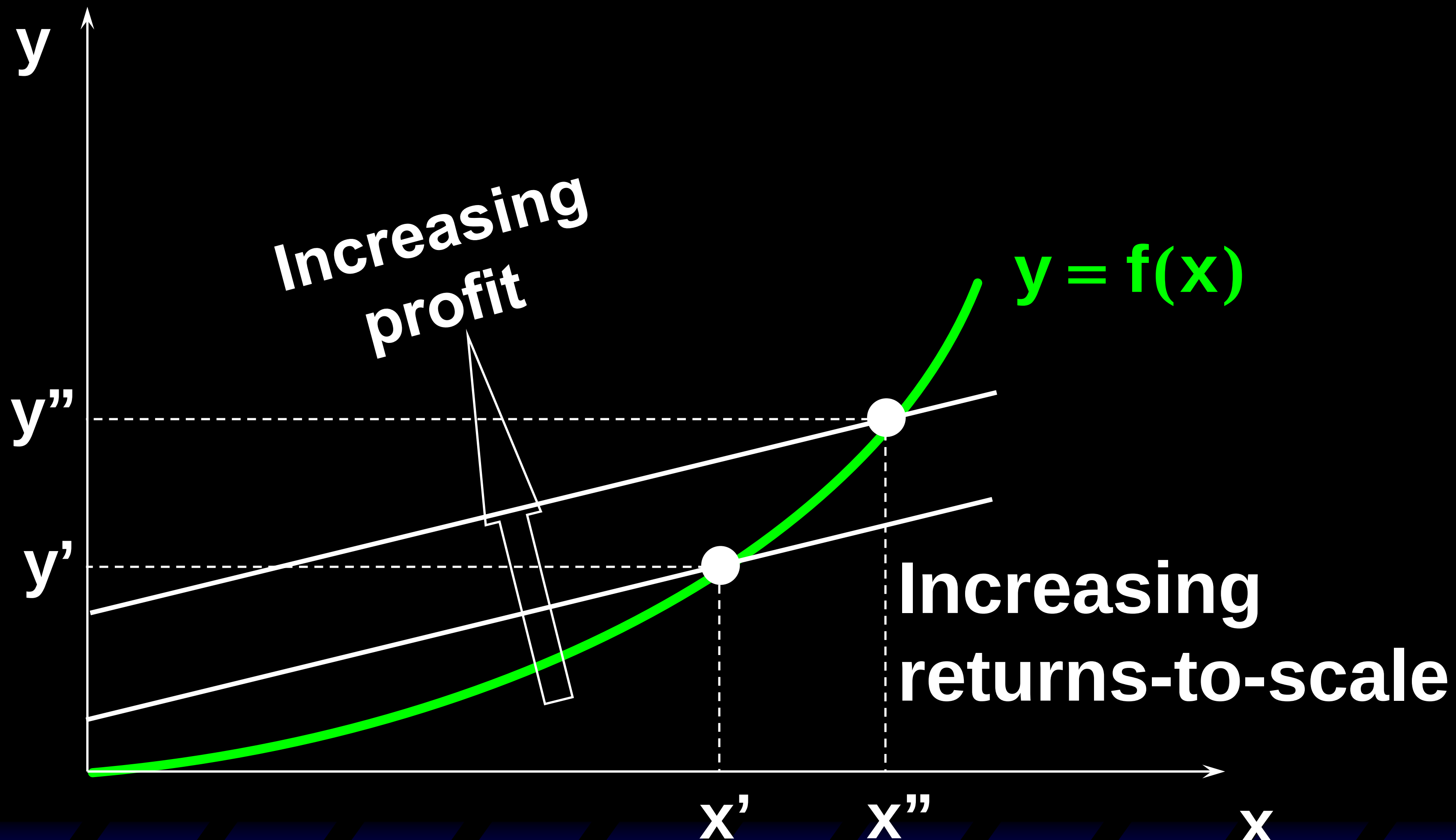
Returns-to Scale and Profit-Maximization



Returns-to-Scale and Profit-Maximization

If a competitive firm's technology exhibits increasing returns-to-scale then the firm does not have a profit-maximizing plan.

Returns-to Scale and Profit-Maximization



Returns-to-Scale and Profit-Maximization

Increasing returns to scale: a multiple-factor case

$$\max_{x_1, x_2} \Pi = py - \omega_1 x_1 - \omega_2 x_2$$

$$\text{s.t. } y = f(x_1, x_2)$$

Assume $(x_1^*, x_2^*, f(x_1^*, x_2^*))$ maximizes the profit. Denote the maximized profit as Π^*.

$$\Pi^* = pf(x_1^*, x_2^*) - \omega_1 x_1^* - \omega_2 x_2^*$$

Returns-to-Scale and Profit-Maximization

$$\Pi^* = pf(x_1^*, x_2^*) - \omega_1 x_1^* - \omega_2 x_2^*$$

Multiplying all inputs by $k > 1$ generates the following profits

$$\Pi' = pf(kx_1^*, kx_2^*) - k\omega_1 x_1^* - k\omega_2 x_2^* > \mathrm{k}\Pi^*$$

since $f(kx_1^*, kx_2^*) > kf(x_1^*, x_2^*)$

=> Contradiction to profit maximization

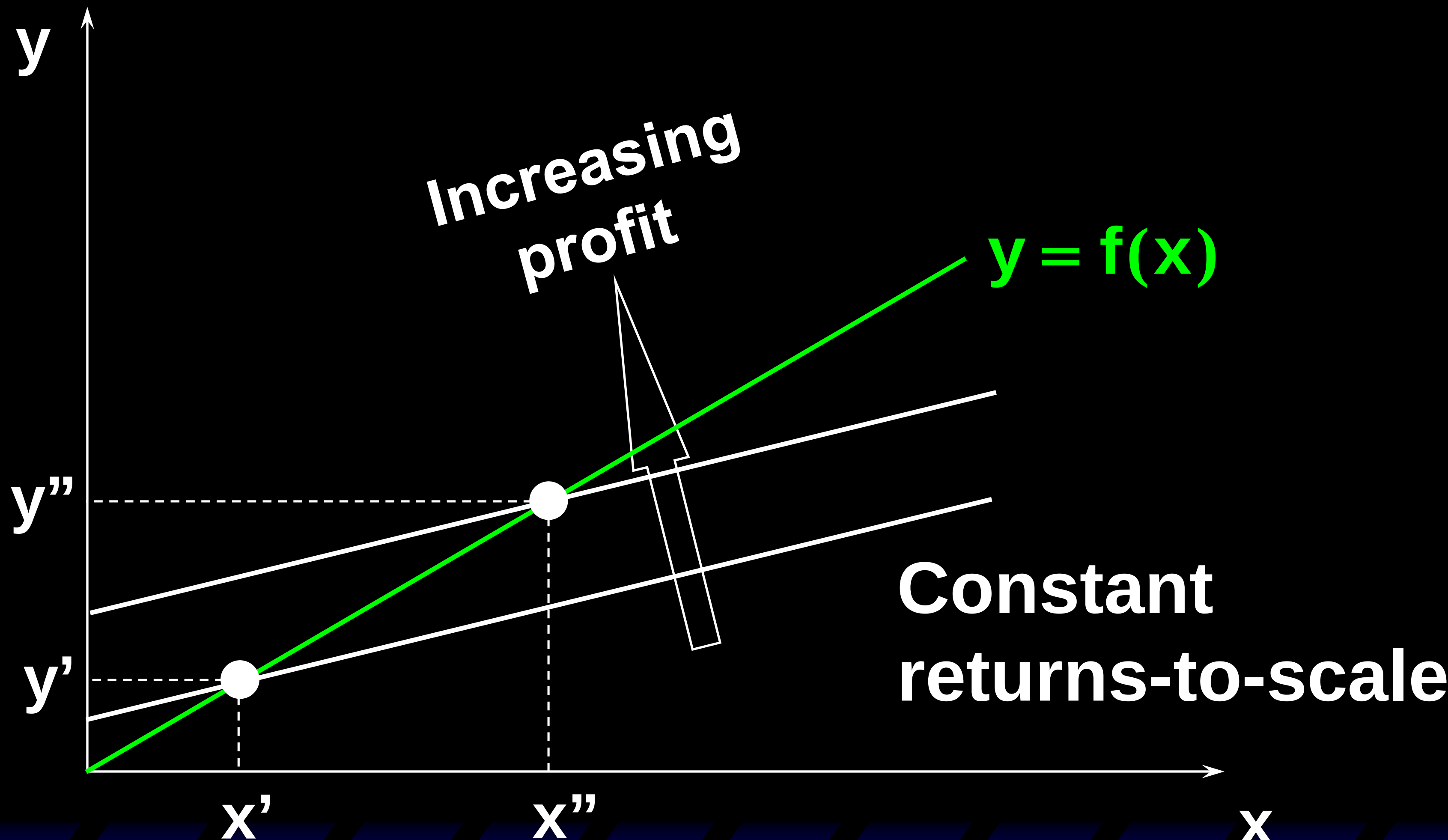
Returns-to-Scale and Profit-Maximization

An increasing returns-to-scale technology is inconsistent with firms being perfectly competitive.

Returns-to-Scale and Profit-Maximization

- What if the competitive firm's technology exhibits **constant** returns-to-scale?

Returns-to Scale and Profit-Maximization



Returns-to Scale and Profit-Maximization

So if any production plan earns a positive profit, the firm can double up all inputs to produce twice the original output and earn twice the original profit.

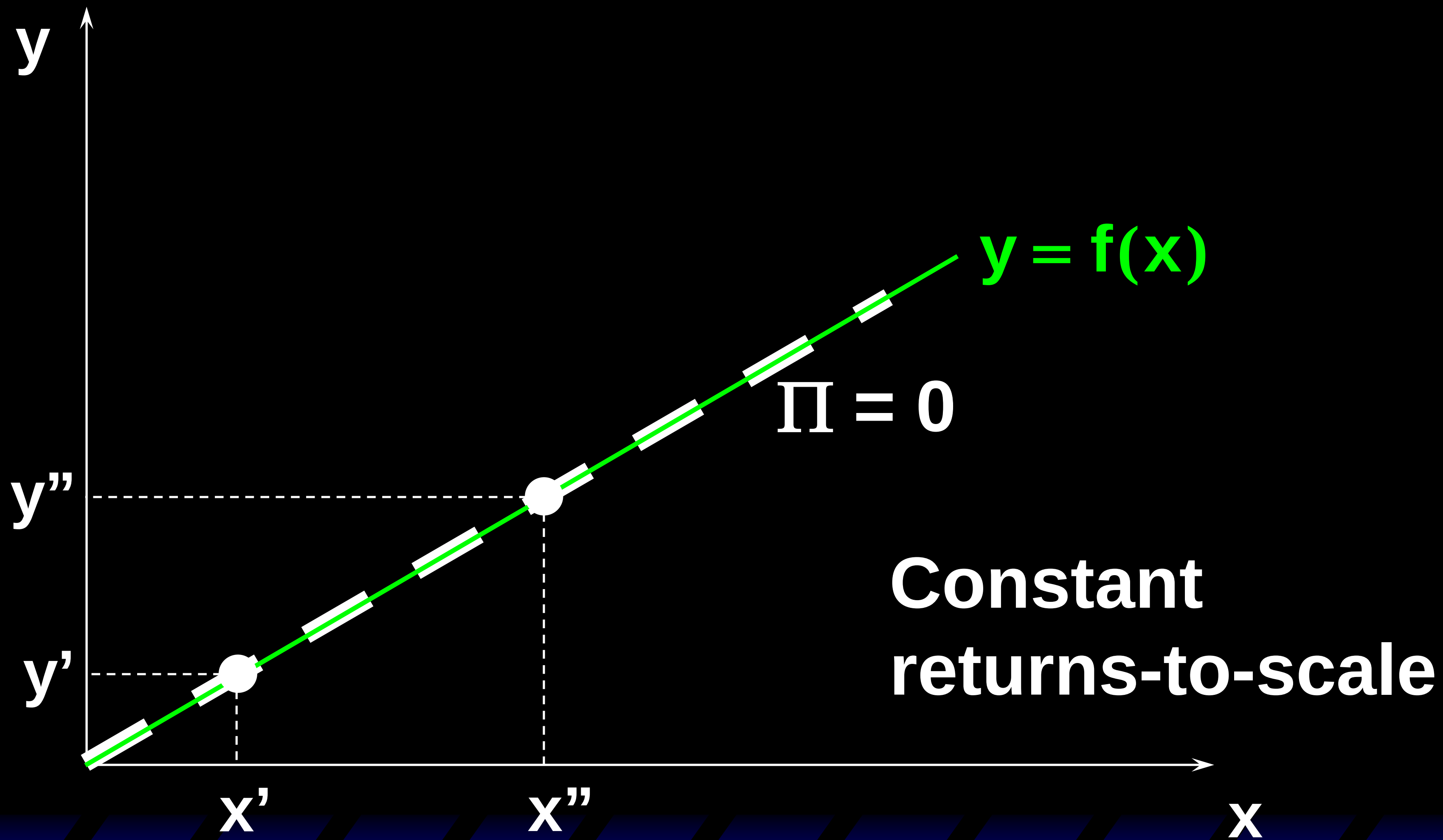
规模报酬不变的企业如果能够获得一个正利润，那么加倍使用要素会获得更多的利润，这与完全竞争市场相违背。

Returns-to Scale and Profit-Maximization

- Therefore, when a firm’s technology exhibits constant returns-to-scale, earning a positive economic profit is inconsistent with firms being perfectly competitive.
- Hence **constant returns-to-scale requires that competitive firms earn economic profits of zero.**

规模报酬不变的企业如果是完全竞争企业，那么它的(经济)利润一定是0

Returns-to Scale and Profit-Maximization



Revealed Profitability

Consider a competitive firm with a technology that exhibits **decreasing** returns-to-scale.

For a variety of output and input prices we observe the firm’s choices of production plans.

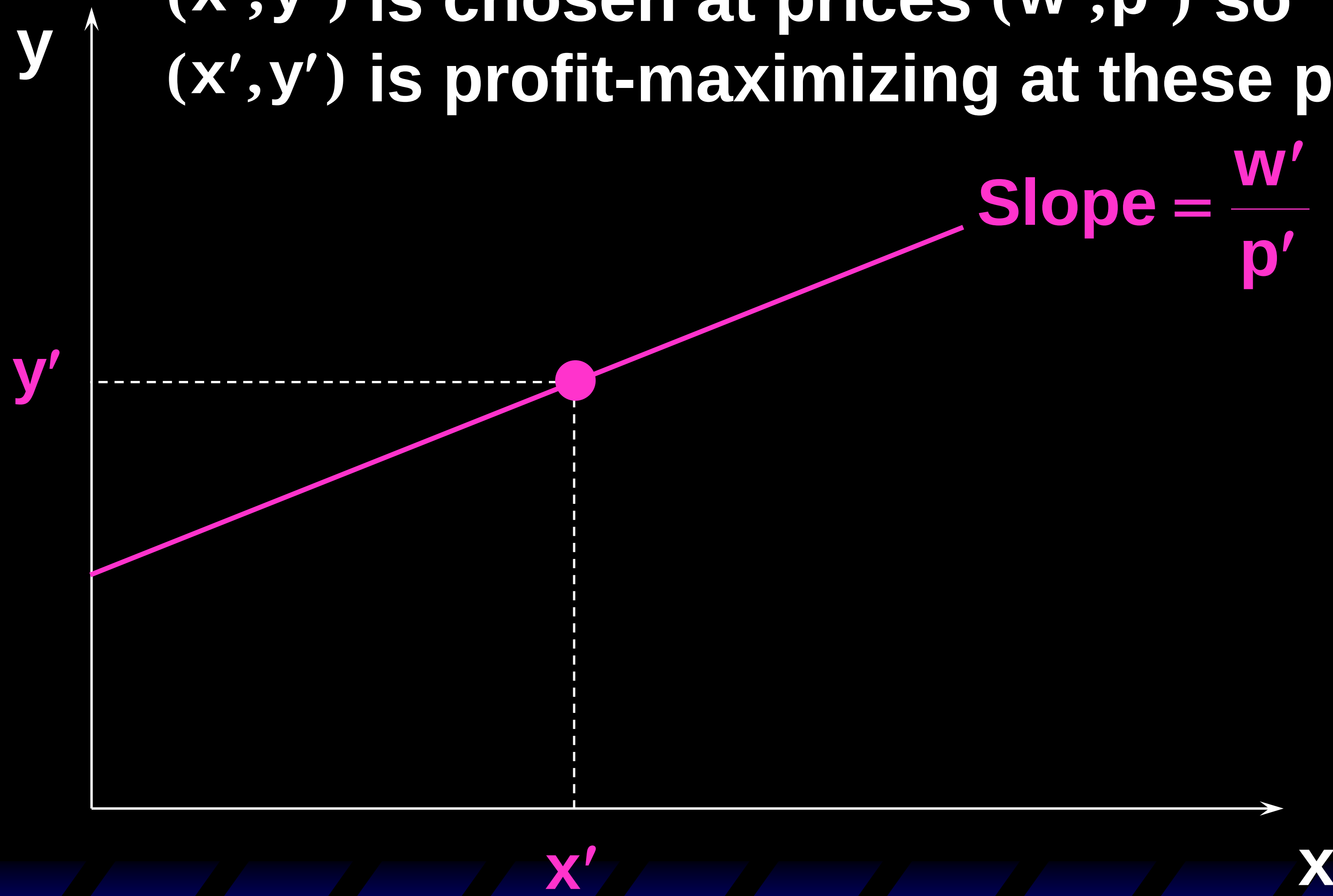
What can we learn from our observations?

Revealed Profitability

If a production plan (x’,y’) is chosen at prices (w’,p’) we deduce that the plan (x’,y’) is revealed to be profit-maximizing for the prices (w’,p’).

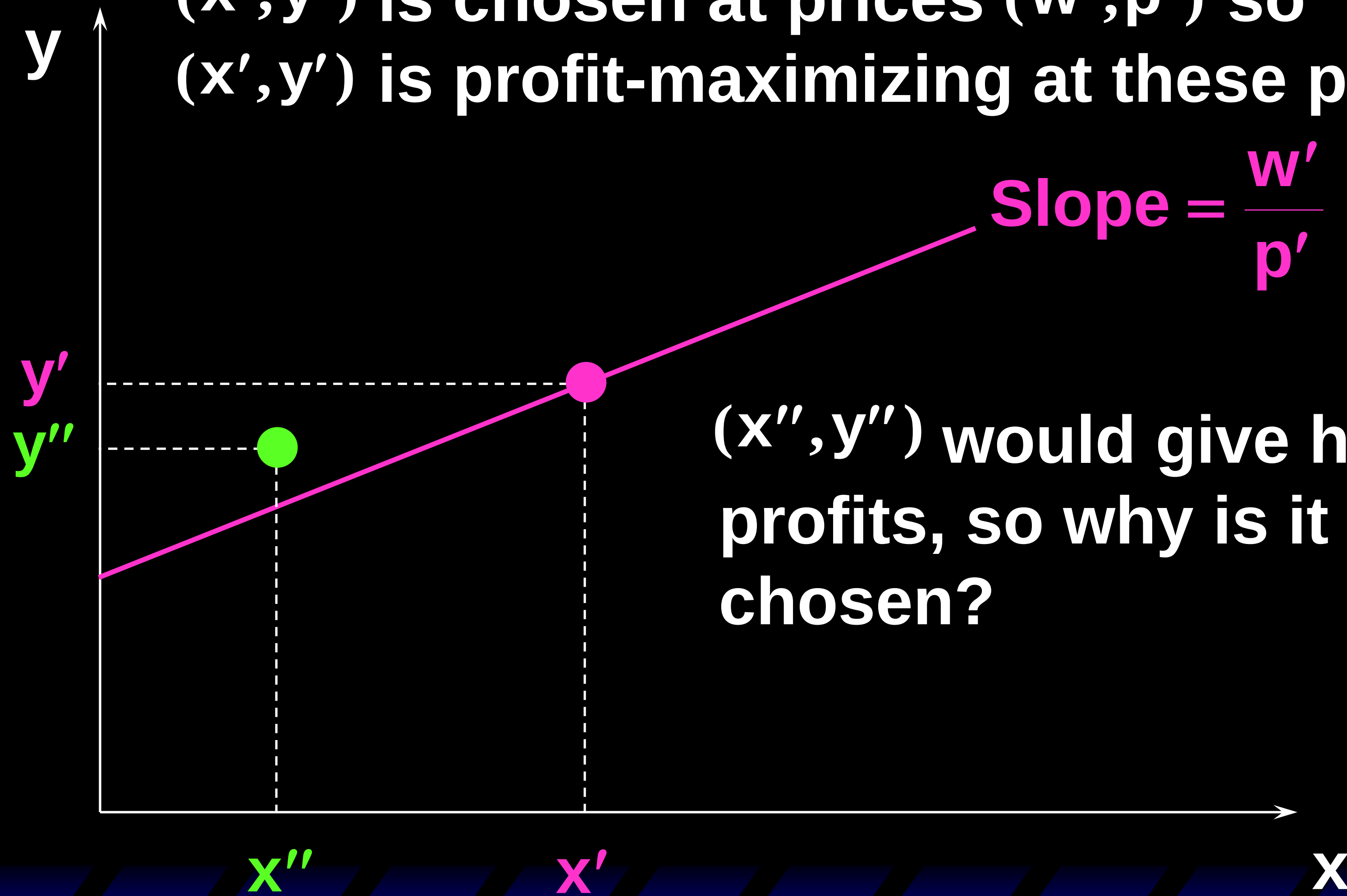
Revealed Profitability

(x', y') is chosen at prices (w', p') so
(x', y') is profit-maximizing at these prices.



Revealed Profitability

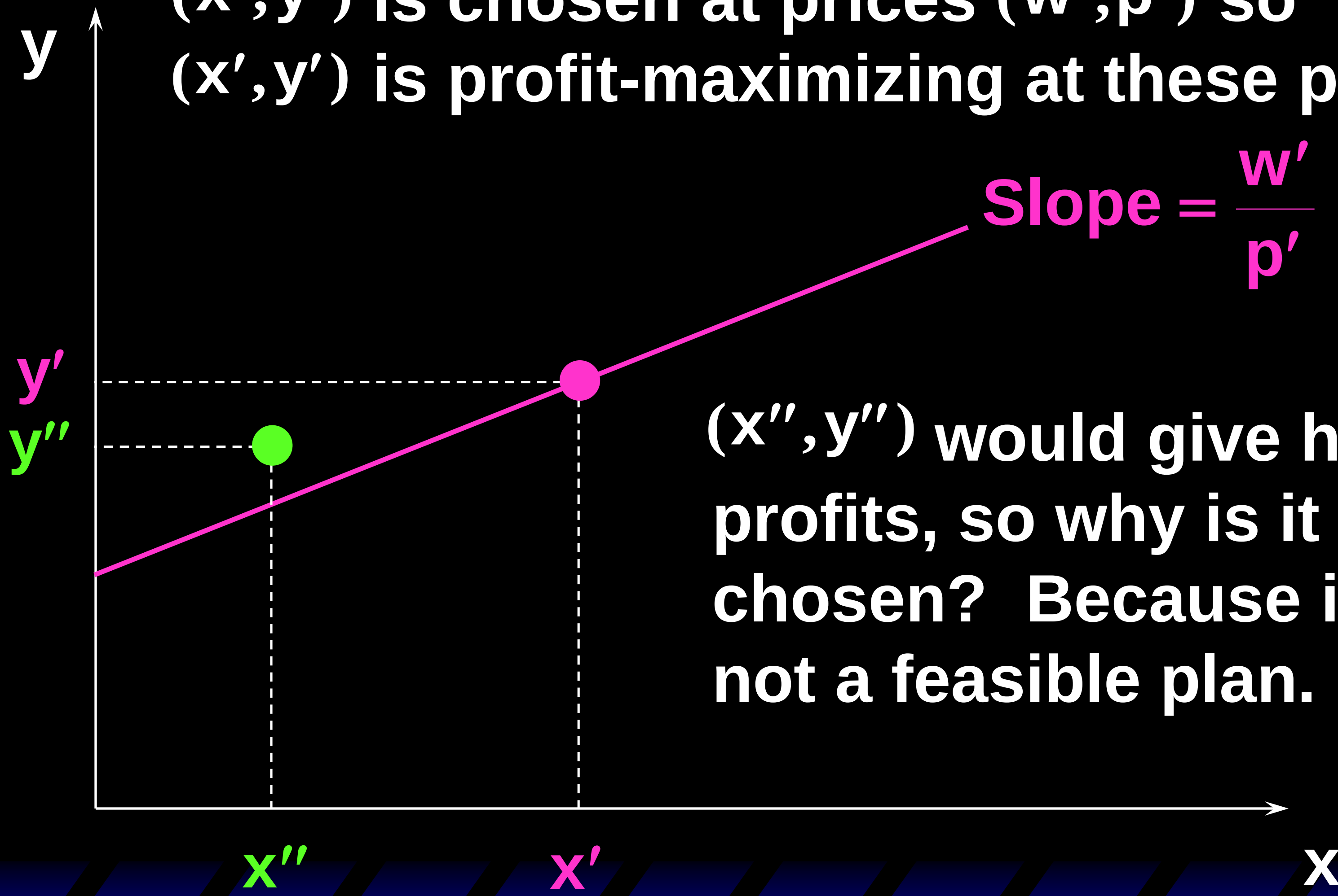
(x',y') **is chosen at prices** (w',p') **so**
(x',y') **is profit-maximizing at these prices.**



(x'',y'') **would give higher profits, so why is it not chosen?**

Revealed Profitability

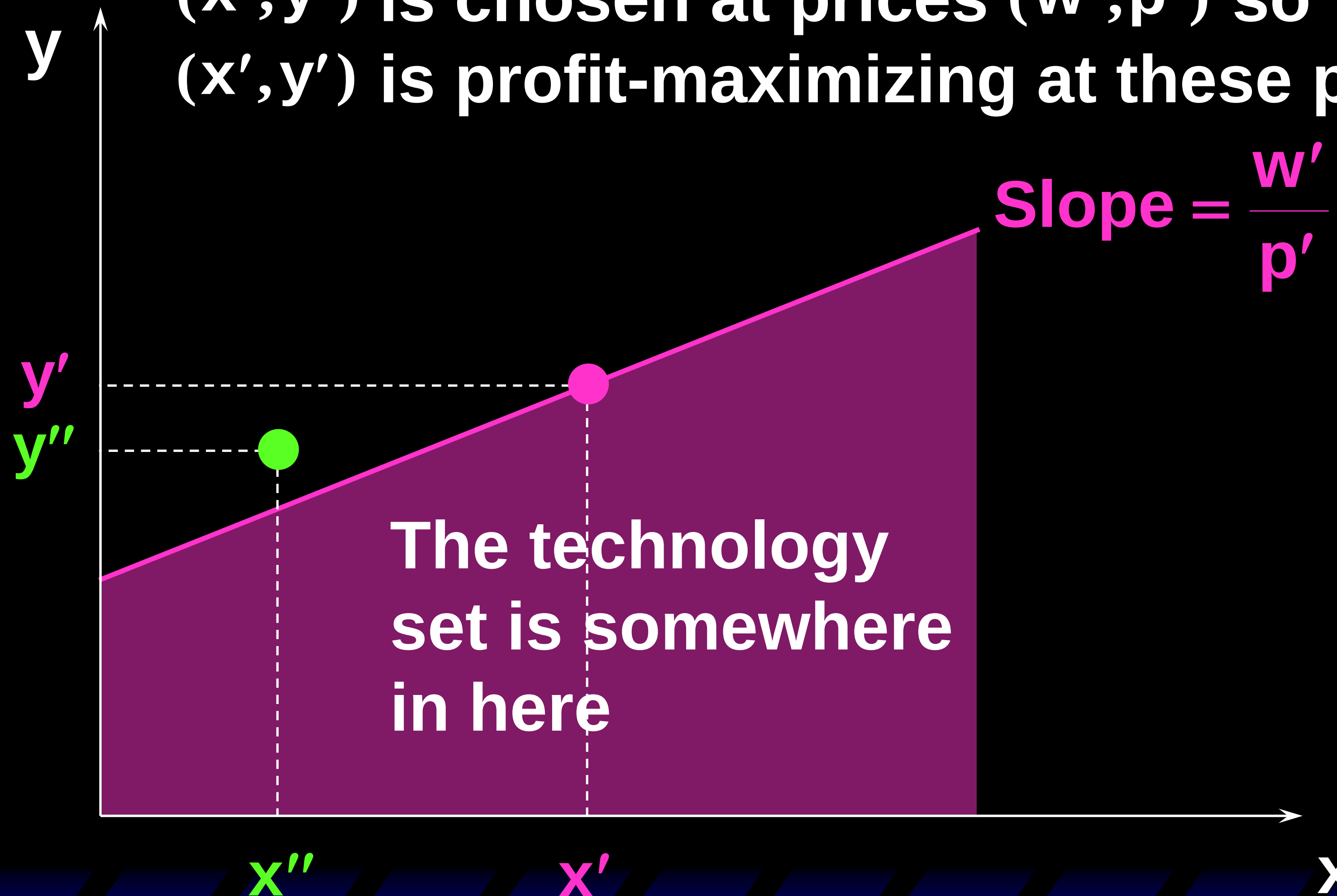
(x', y') is chosen at prices (w', p') so (x', y') is profit-maximizing at these prices.



(x'', y'') would give higher profits, so why is it not chosen? Because it is not a feasible plan.

Revealed Profitability

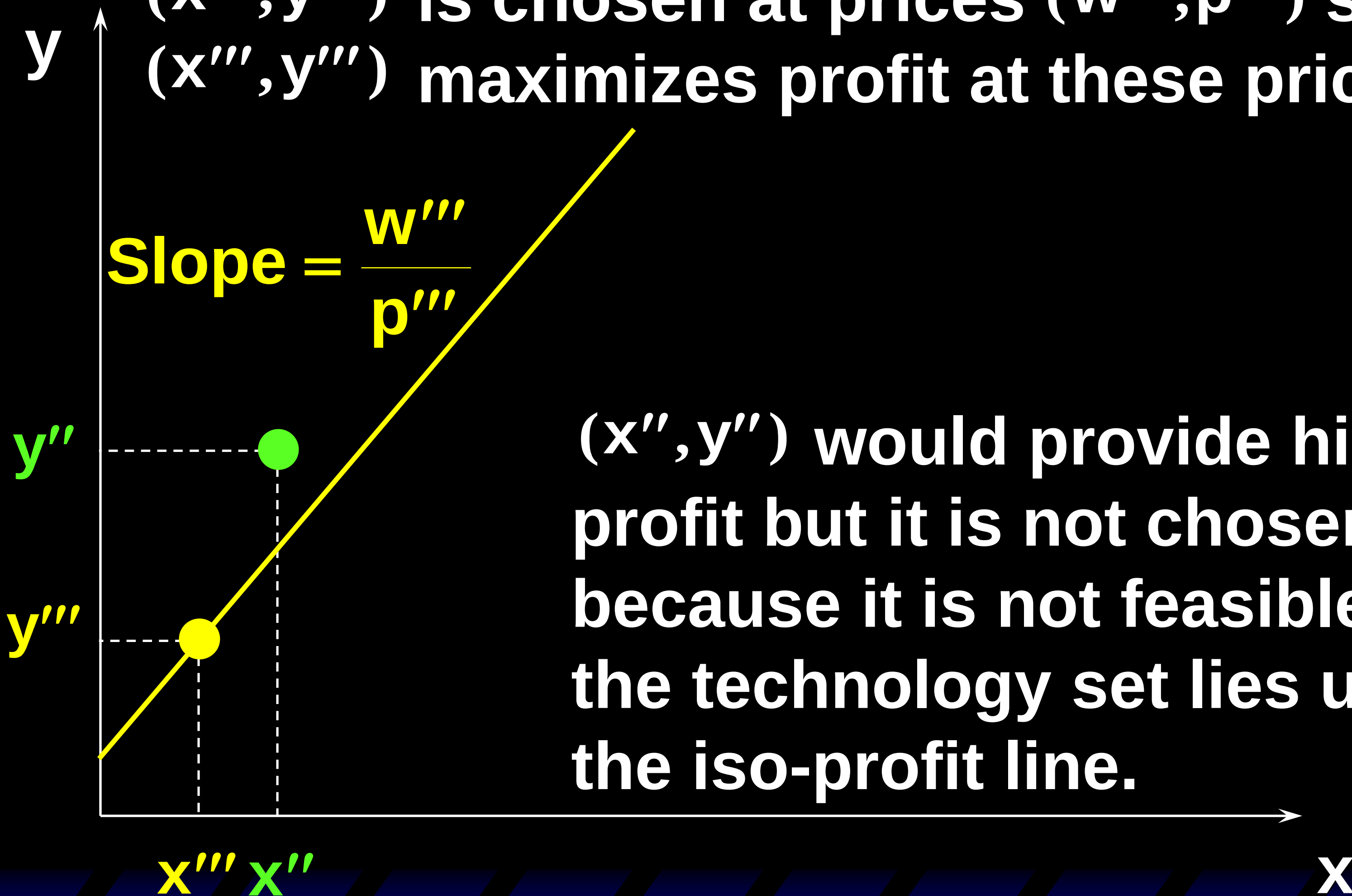
(x', y') is chosen at prices (w', p') so (x', y') is profit-maximizing at these prices.



So the firm's technology set must lie under the iso-profit line.

Revealed Profitability

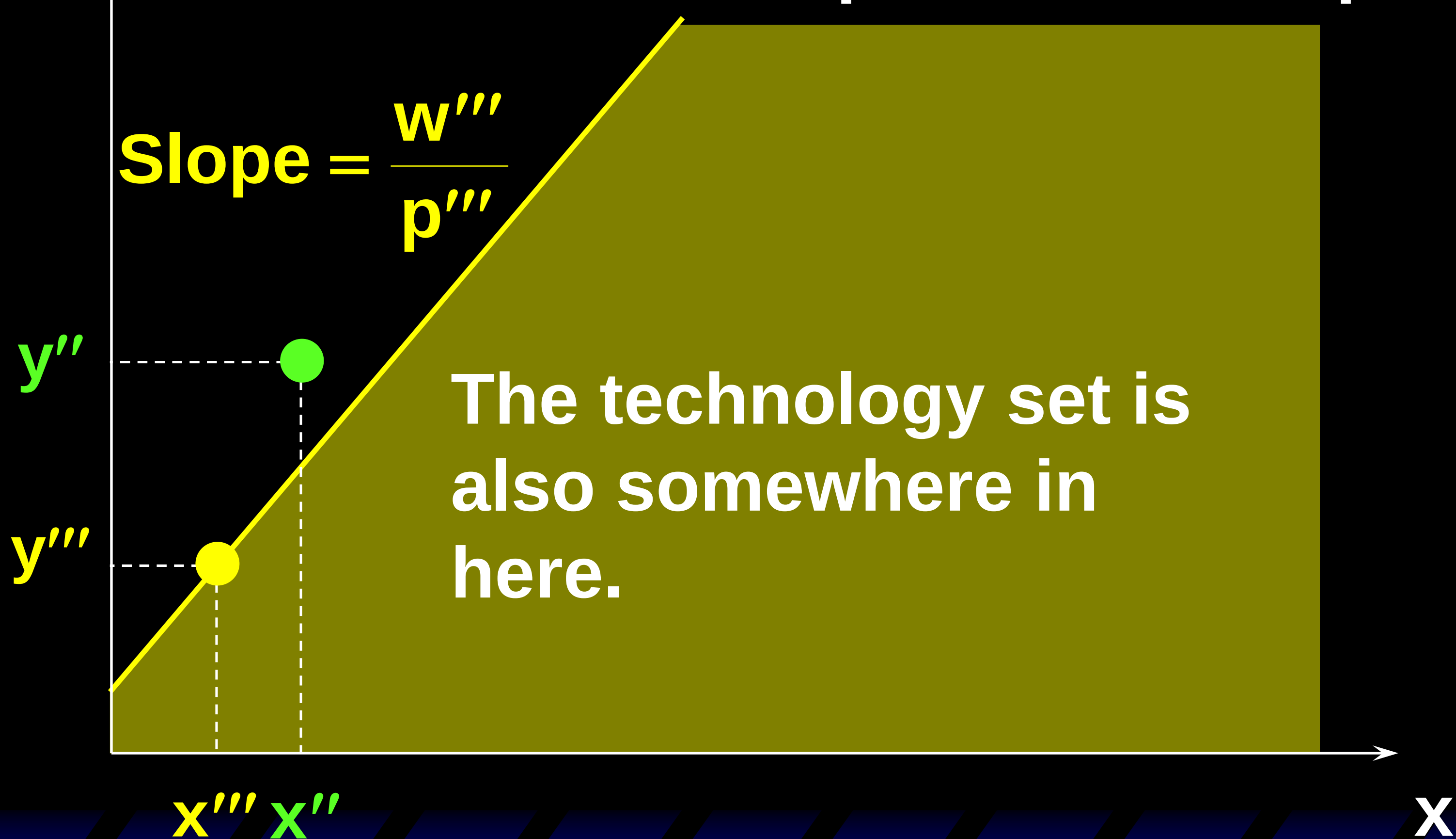
(x''',y''') is chosen at prices (w''',p''') so (x''',y''') maximizes profit at these prices.



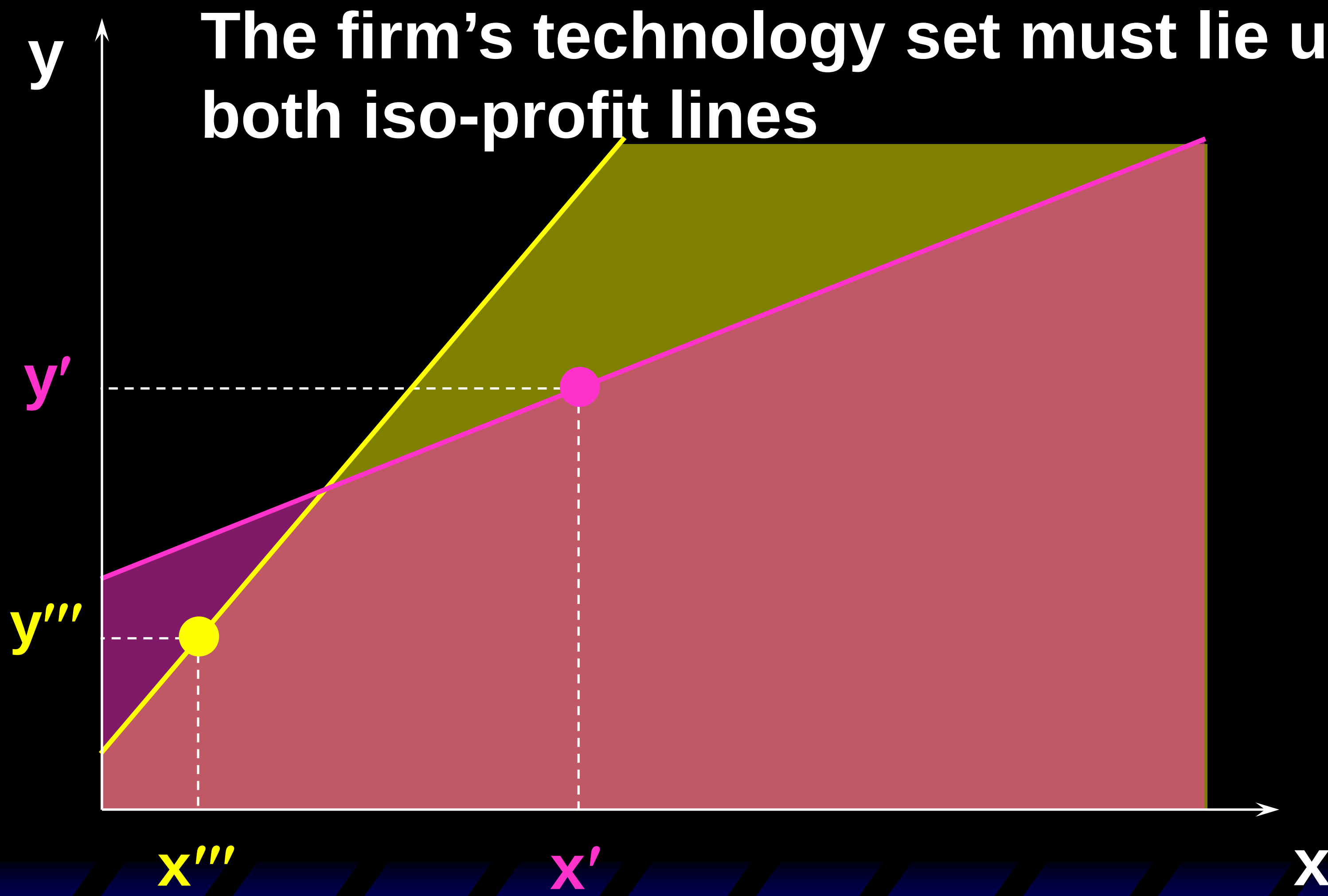
(x'',y'') would provide higher profit but it is not chosen because it is not feasible so the technology set lies under the iso-profit line.

Revealed Profitability

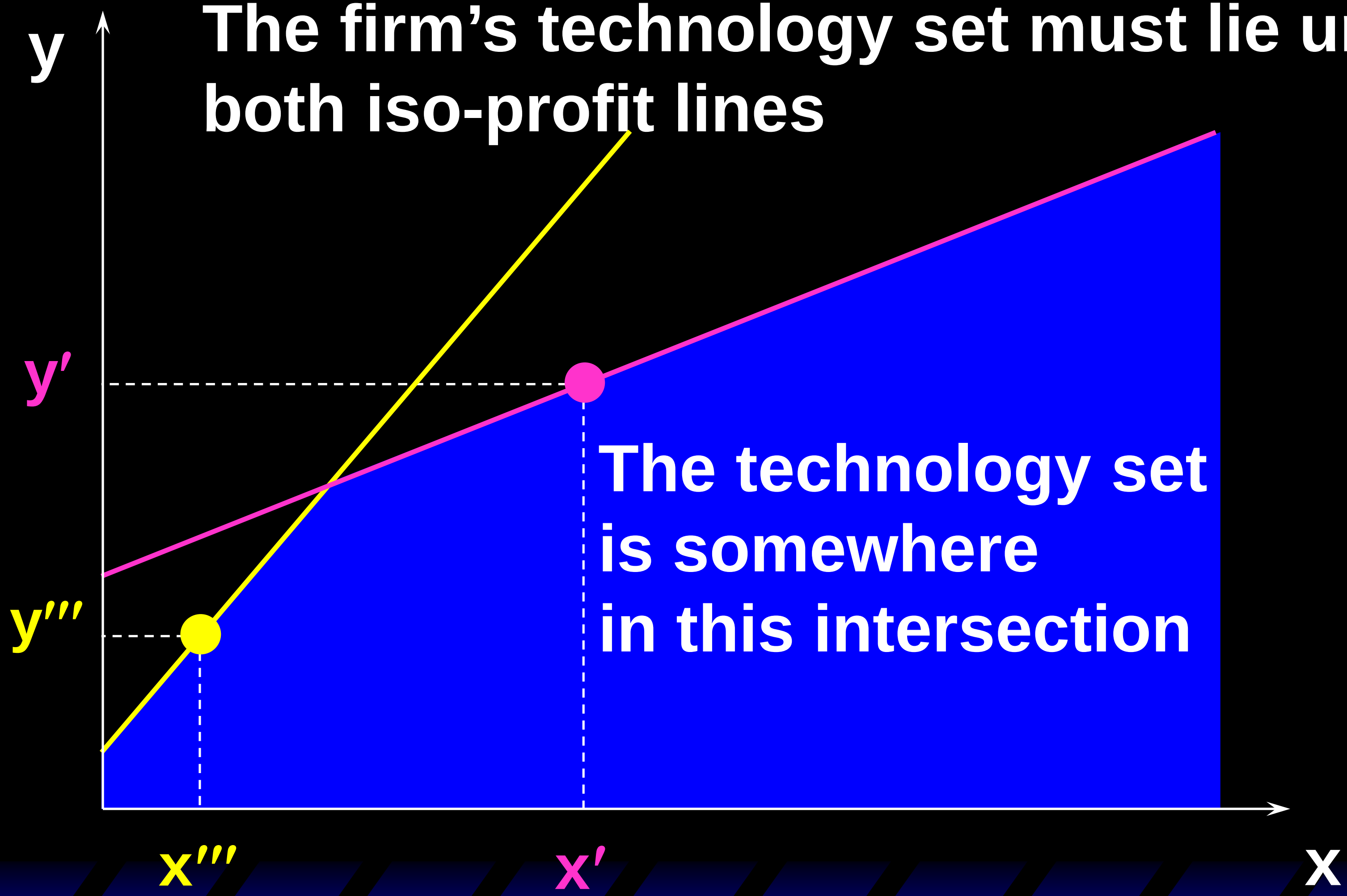
(x''', y''') is chosen at prices (w''', p''') so (x''', y''') maximizes profit at these prices.



Revealed Profitability



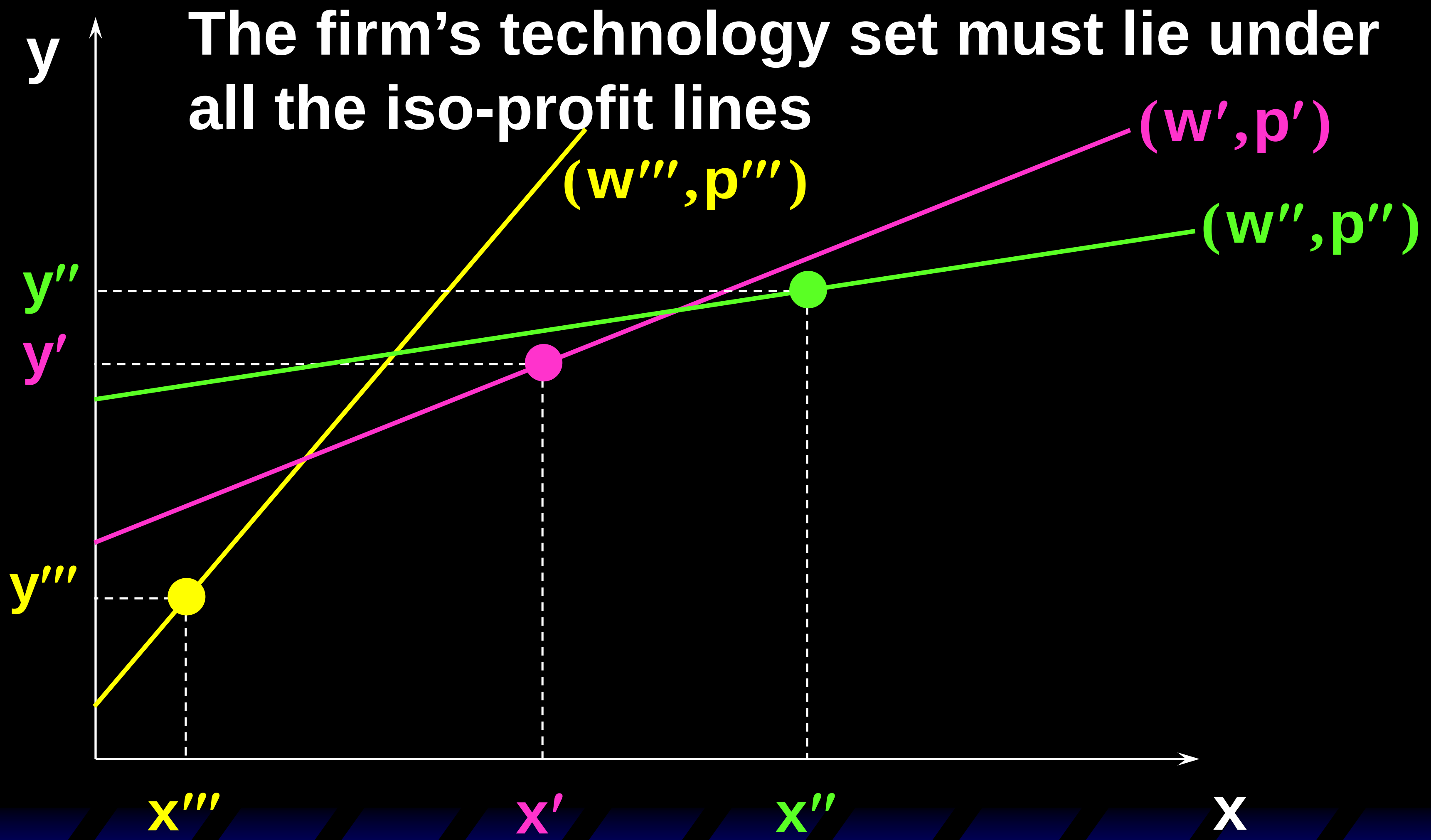
Revealed Profitability



Revealed Profitability

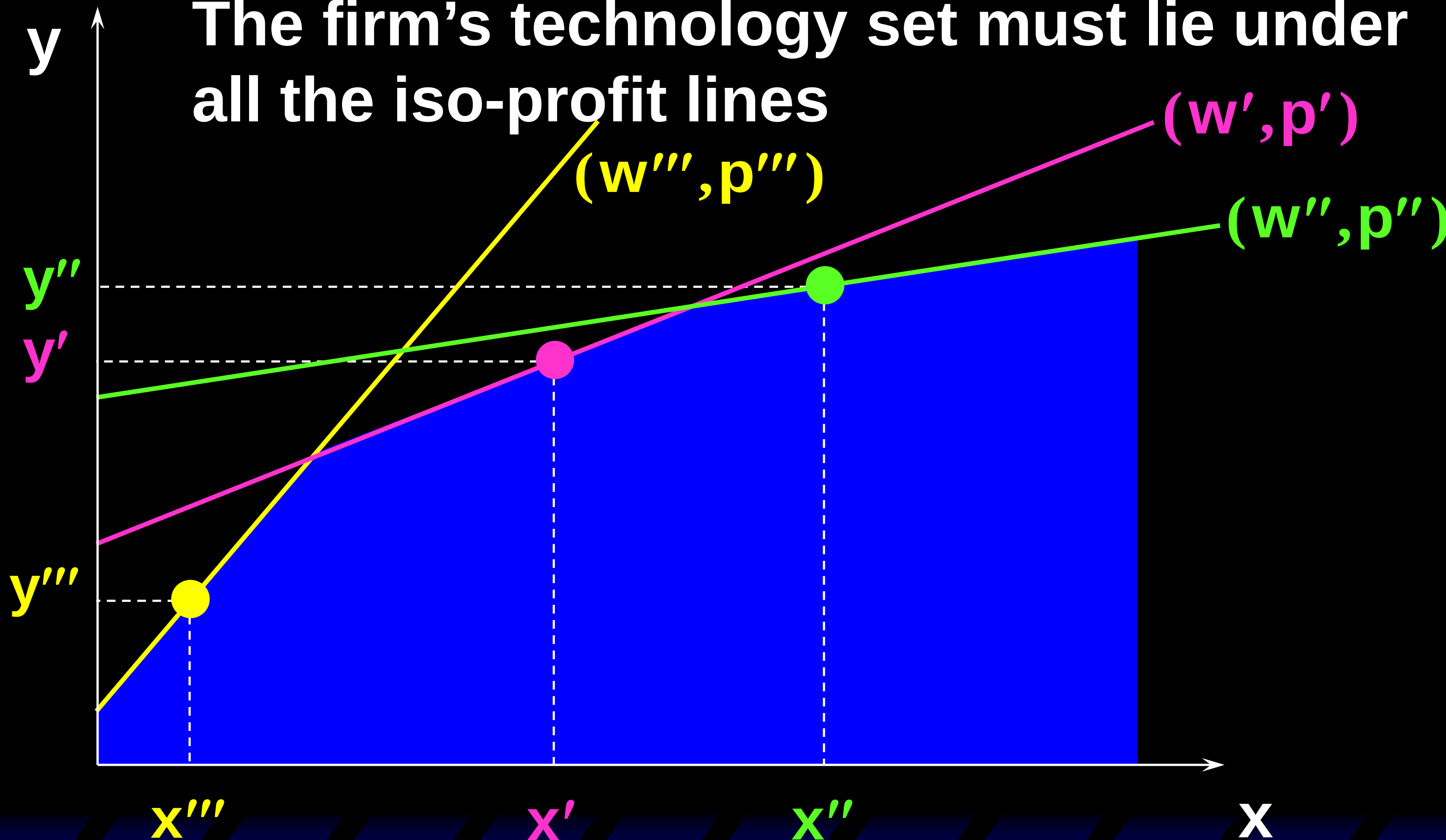
Observing **more choices** of production plans by the firm in response to different prices for its input and its output gives **more information** on the location of its technology set.

Revealed Profitability

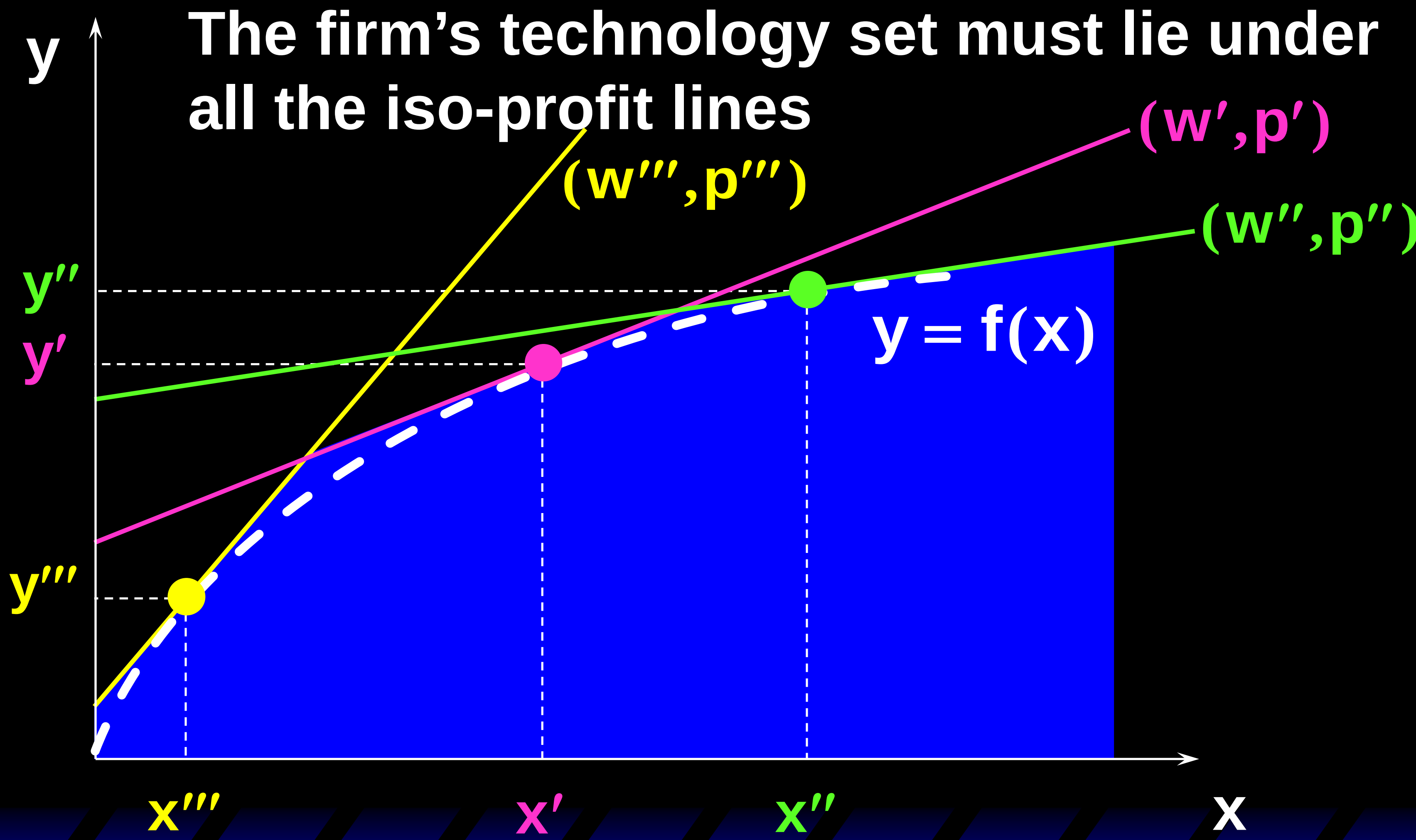


Revealed Profitability

The firm's technology set must lie under all the iso-profit lines



Revealed Profitability



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What else can be learned from the firm’s choices of profit-maximizing production plans?

Revealed Profitability

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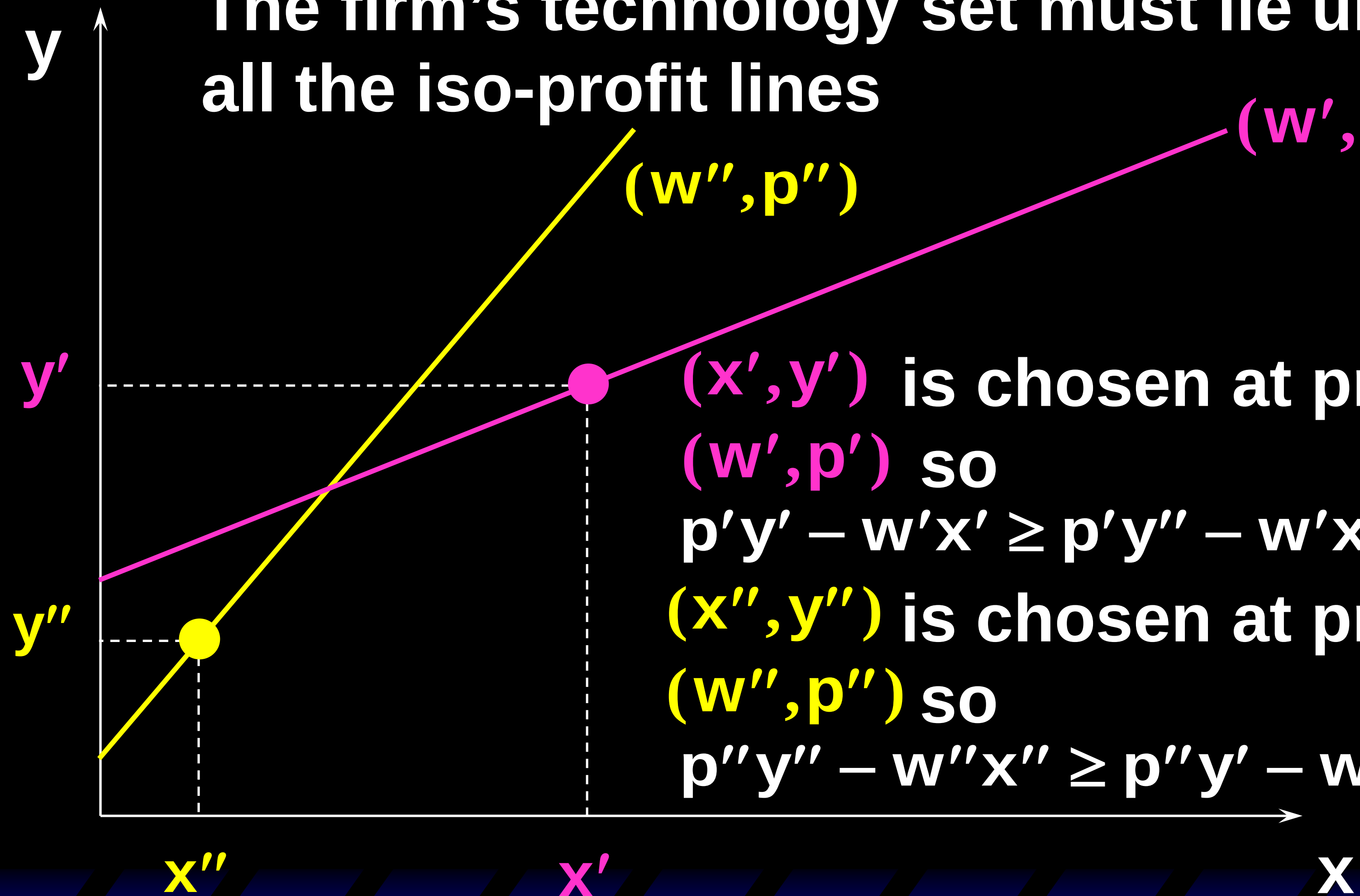

(x',y') is chosen at prices (w',p') so

$$p'y' - w'x' \ge p'y'' - w'x''.$$

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The production plan $(\mathbf{x}', \mathbf{y}')$ is chosen at prices $(\mathbf{w}', \mathbf{p}')$ when $(\mathbf{x}'', \mathbf{y}'')$ is also feasible.

$$\mathbf{p}'\mathbf{y}' - \mathbf{w}'\mathbf{x}' \geq \mathbf{p}'\mathbf{y}'' - \mathbf{w}''\mathbf{x}'' \qquad (1)$$

The production plan $(\mathbf{x}'', \mathbf{y}'')$ is chosen at prices $(\mathbf{w}'', \mathbf{p}'')$ when $(\mathbf{x}', \mathbf{y}')$ is also feasible.

$$\mathbf{p}''\mathbf{y}'' - \mathbf{w}''\mathbf{x}'' \geq \mathbf{p}''\mathbf{y}' - \mathbf{w}''\mathbf{x}' \qquad (2)$$

$$-\mathbf{p}''\mathbf{y}' + \mathbf{w}''\mathbf{x}' \geq -\mathbf{p}''\mathbf{y}'' + \mathbf{w}''\mathbf{x}'' \qquad (2')$$

Revealed Profitability

$$\mathbf{p}'\mathbf{y}' - \mathbf{w}'\mathbf{x}' \geq \mathbf{p}'\mathbf{y}'' - \mathbf{w}''\mathbf{x}'' \qquad (1)$$

$$-\mathbf{p}''\mathbf{y}' + \mathbf{w}''\mathbf{x}' \geq -\mathbf{p}''\mathbf{y}'' + \mathbf{w}''\mathbf{x}'' \qquad (2')$$

(1)+(2')=>

$$\mathbf{p}'\mathbf{y}' - \mathbf{w}'\mathbf{x}' + (-\mathbf{p}''\mathbf{y}' + \mathbf{w}''\mathbf{x}')$$
$$\geq \mathbf{p}'\mathbf{y}'' - \mathbf{w}''\mathbf{x}'' + (-\mathbf{p}''\mathbf{y}'' + \mathbf{w}''\mathbf{x}'')$$

Rearranging gives:

$$(\mathbf{p}' - \mathbf{p}'')\mathbf{y}' - (\mathbf{w}' - \mathbf{w}'')\mathbf{x}'$$
$$\geq (\mathbf{p}' - \mathbf{p}'')\mathbf{y}'' - (\mathbf{w}' - \mathbf{w}'')\mathbf{x}''$$

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Rearranging gives:

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$$\geq (\mathbf{p}' - \mathbf{p}'')\mathbf{y}'' - (\mathbf{w}' - \mathbf{w}'')\mathbf{x}''$$

$$(\mathbf{p}' - \mathbf{p}'')(\mathbf{y}' - \mathbf{y}'') \geq (\mathbf{w}' - \mathbf{w}'')(\mathbf{x}' - \mathbf{x}'')$$

$$\Delta\mathbf{p}\Delta\mathbf{y} \geq \Delta\mathbf{w}\Delta\mathbf{x}$$

is a necessary implication of profit-maximization.

Revealed Profitability

$$\Delta p \Delta y \geq \Delta w \Delta x$$

is a necessary implication of profit-maximization.
Suppose the input price does not change. Then $\Delta w = 0$ and profit-maximization implies $\Delta p \Delta y \geq 0$; *i.e.*, a competitive firm's output supply curve cannot slope downward.

利润最大化的一个推论/必要条件：当产品价格上升时，企业产出一定上升或不变。

Revealed Profitability

$$\Delta p \Delta y \geq \Delta w \Delta x$$

is a necessary implication of profit-maximization.
Suppose the output price does not change.
Then $\Delta p = 0$ and profit-maximization implies $0 \geq \Delta w \Delta x$; *i.e.*, a competitive firm's input demand curve cannot slope upward.
利润最大化的另一个推论/必要条件：当要素价格上升时，要素需求一定下降或不变。