单元6.1 图的基本概念

第二编 图论　第七章 图

7.1 图的基本概念

内容提要

- 图、无向图、有向图、简单图
- 相邻、关联
- 度、度序列、握手定理
- 图同构
- 图族
- 图运算

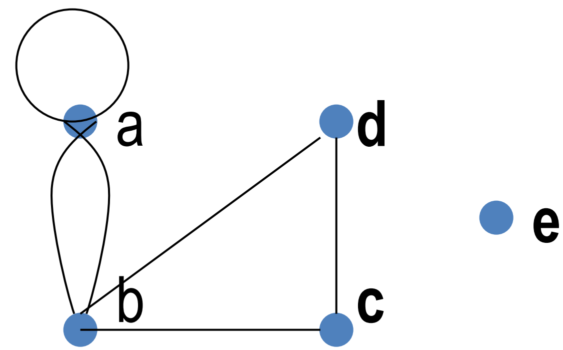
无序积

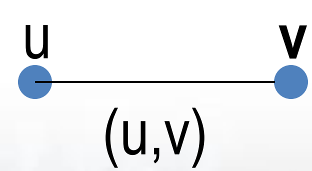
$$A\&B = \{ \{a,b\} \mid a \in A \wedge b \in B \}$$

- 记$\{a,b\}=(a,b)$
- 允许 $a=b$
- $(a,b)=(b,a)$

无向图

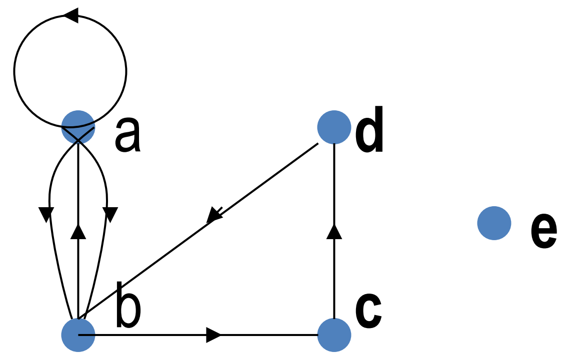
- 图 $G=<V,E>$
  - $V\neq\varnothing$　顶点集 $V(G)$
  - $E\subseteq V\&V$　边集(多重集) $E(G)$
- 例：$G=<V,E>$
  - $V=\{a,b,c,d,e\}$
  - $E=\{(a,a),(a,b),(a,b),(b,c),(c,d),(b,d)\}$
- 顶点、边



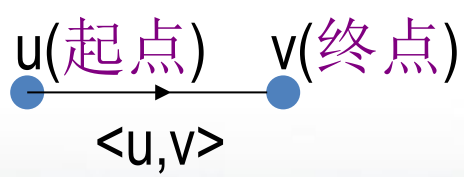


有向图

- 有向图 D=<V,E>
  - $V\neq\varnothing$, 顶点集 V(D)
  - $E\subseteq V\times V$, 边集(多重集) E(D)
- 例： D=<V,E>
  - V={a,b,c,d,e}
  - E={ <a,a>,<a,b>,<a,b>, <b,a>,<b,c>,<c,d>,<d,b> }



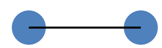
- 有向边



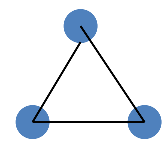
n阶图、有限图、零图、平凡图、空图

- n阶图： |V(G)|=n

1阶图



2阶图

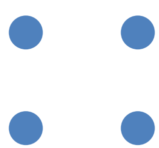


3阶图

4阶图

- 有限图： |V(G)|<∞
- 零图N_n： E=∅
- 平凡图： 1阶零图N_1
- 空图： V=E=∅

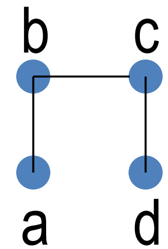
N_1



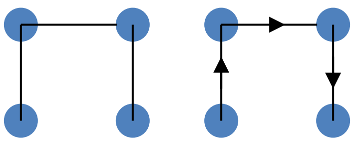
N_4

标定图、非标定图、底图

- 标定图：顶点或边带标记的图



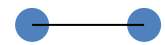
- 非标定图：顶点和边不带标记的图



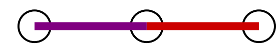
- 底图(基图)：有向图去掉边的方向后得到的无向图

相邻、邻接、关联

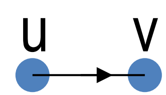
- 有边相连的两个顶点是相邻的



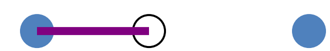
- 有公共顶点的两条边是相邻的



- u邻接到v, v邻接于u



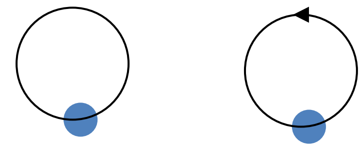
- 一条边的端点与这条边是关联的



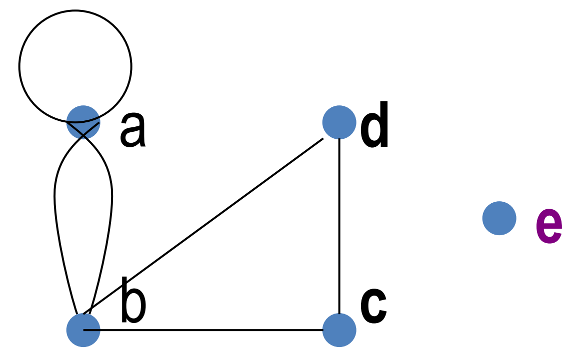
- 关联次数

环、孤立点、平行边

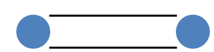
- 环：只与一个顶点关联的边



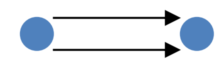
- 孤立点：不与任何边关联的顶点



- 平行边
  - 端点相同的两条无向边是平行边



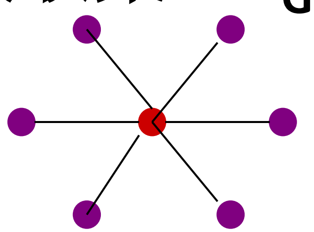
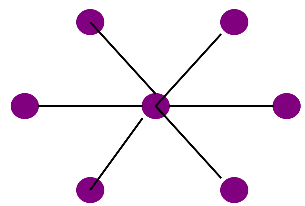
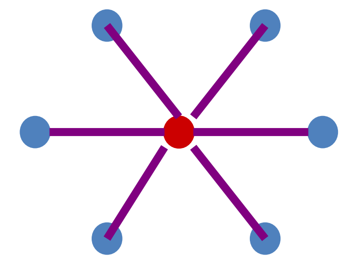
  - 起点与终点相同的两条有向边是平行边



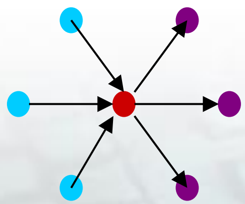
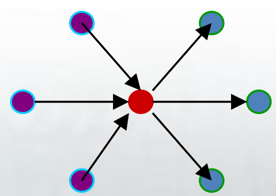
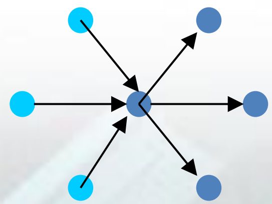
非平行边

邻域、闭邻域、关联集

- 邻域：$N_G(v)=\{\, u\in V(G) \mid (u,v)\in E(G)\wedge u\neq v \,\}$
- 闭邻域：$\overline{N_G(v)} = N_G(v) \cup \{v\}$
- 关联集：$I_G(v) = \{\, e \mid e$与v关联$\,\}$

- 后继：$\Gamma_D^+(v)=\{u\in V(D) \mid \langle v,u\rangle\in E(D)\wedge u\neq v\}$
- 前驱：$\Gamma_D^-(v)=\{u\in V(D) \mid \langle u,v\rangle\in E(D)\wedge u\neq v\}$
- (闭)邻域：$N_D(v)=\Gamma_D^+(v)\cup\Gamma_D^-(v)$　$\overline{N_D(v)} = N_D(v)\cup\{v\}$

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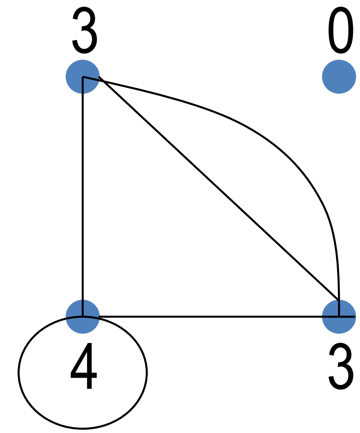
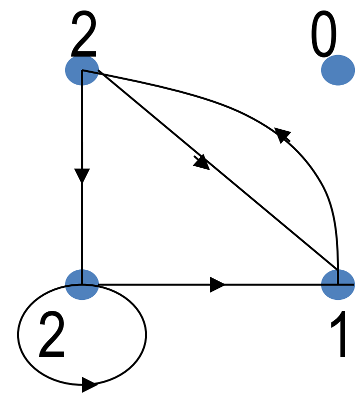
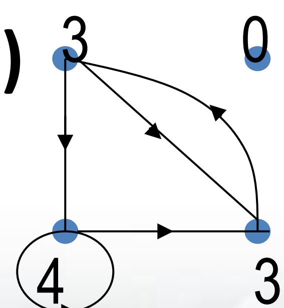
顶点的度

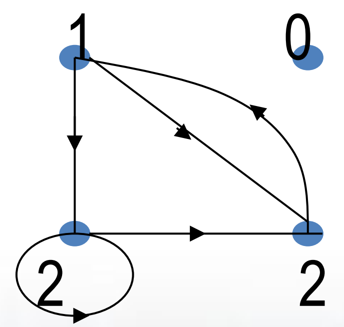


- 度：$d_G(v)$ = 与v关联的边的次数之和



- 出度：$d_D^+(v)$ =与v关联的出边的次数之和
- 入度：$d_D^-(v)$ =与v关联的入边的次数之和
- 度：$d_D(v) = d_D^+(v) + d_D^-(v)$





最大度、最小度

- 最大度 $\Delta(G) = \max\{ d_G(v) \mid v \in V(G) \}$

- 最小度 $\delta(G) = \min\{ d_G(v) \mid v \in V(G) \}$
- 最大出度 $\Delta^+(D) = \max\{ d_D^+(v) \mid v \in V(D)$
- 最小出度 $\delta^+(D) = \min\{ d_D^+(v) \mid v \in V(D) \}$
- 最大入度 $\Delta^-(D) = \max\{ d_D^-(v) \mid v \in V(D)$
- 最小入度 $\delta^-(D) = \min\{ d_D^-(v) \mid v \in V(D) \}$

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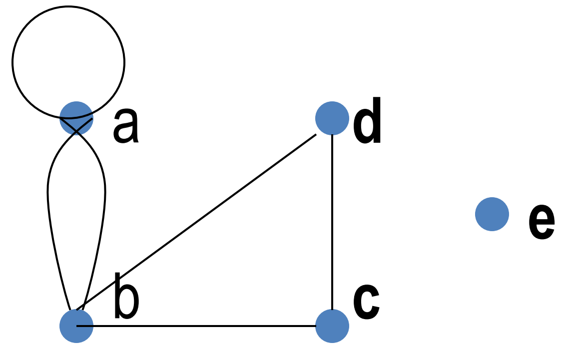
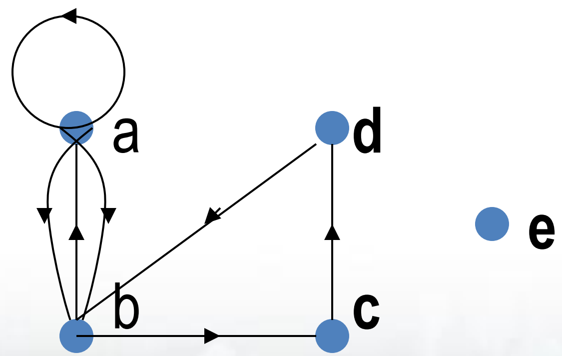
举例

- $\Delta=4, \delta=0$



- $\Delta^{+}=3, \delta^{+}=0$

  $\Delta^{-}=3, \delta^{-}=0$

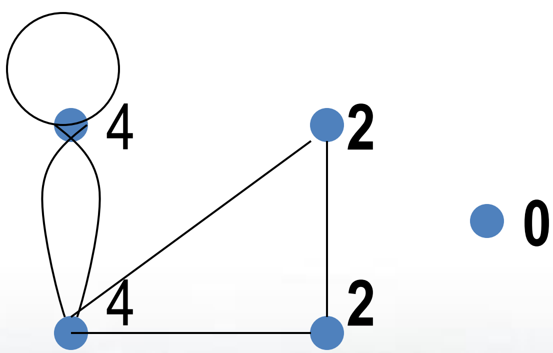


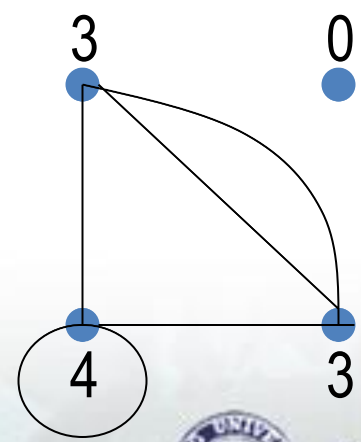
图论基本定理(握手定理)

- 设$G=<V,E>$是无向图,

  $V=\{v_1,v_2,\ldots,v_n\}$, $|E|=m$, 则

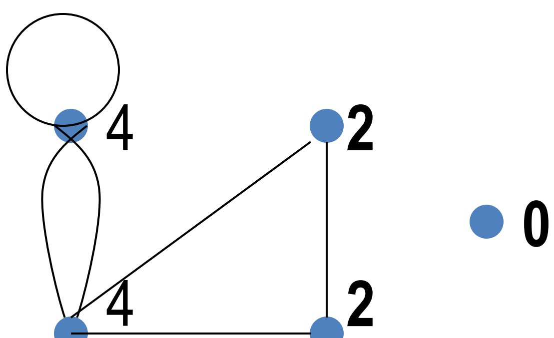
  $$d(v_1)+d(v_2)+\ldots+d(v_n)=2m. \quad \#$$

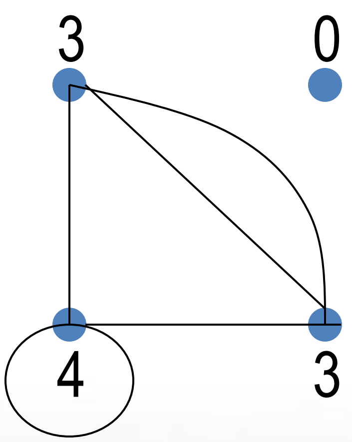




握手定理推论

任何图中奇度顶点的个数是偶数. #



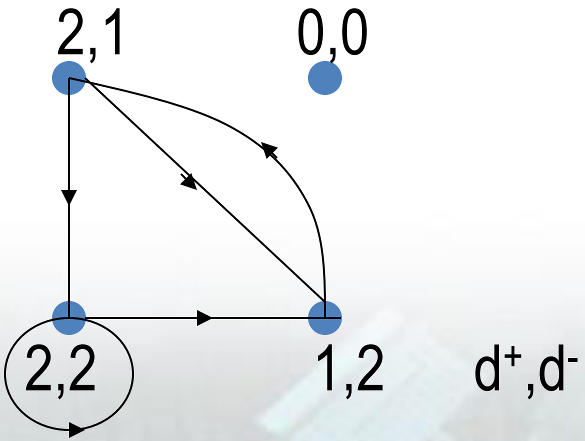


图论基本定理(握手定理)

- 设D=<V,E>是有向图, $V=\{v_1,v_2,\ldots,v_n\}$, $|E|=m$, 则

$$d^+(v_1)+d^+(v_2)+\ldots+d^+(v_n)$$

$$= d^-(v_1)+d^-(v_2)+\ldots+d^-(v_n) = m. \quad \#$$

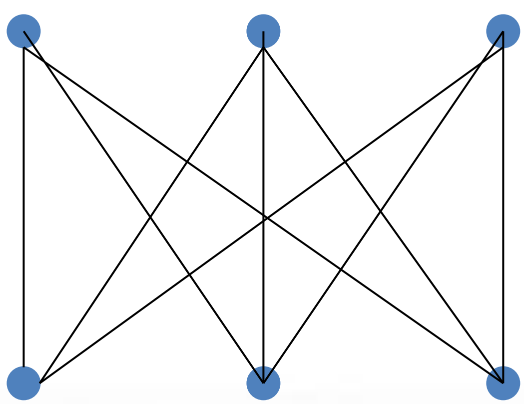


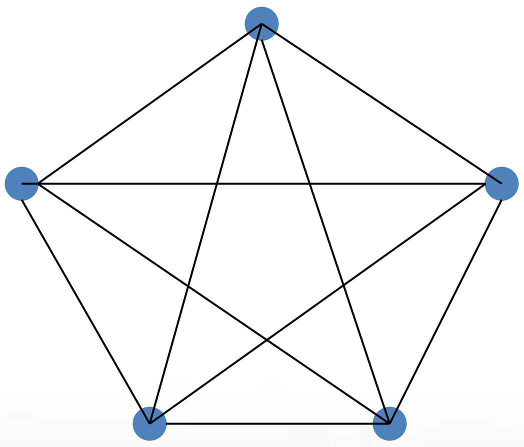
简单图、 k-正则图

- 简单图：既无环也无平行边的图

$$\Rightarrow 0 \le \Delta(G) \le n\text{-}1$$

- k-正则图：所有顶点的度都是k





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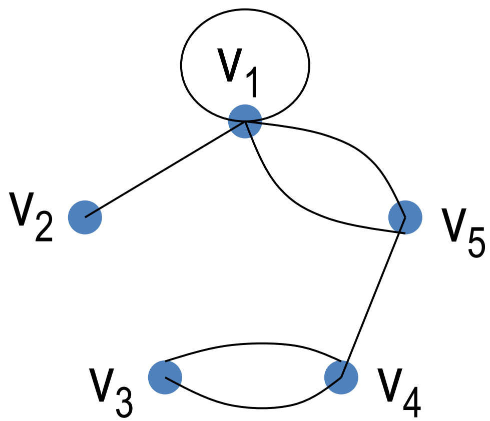
度数列

- 设G=<V,E>,V={$v_1,v_2,\ldots,v_n$}, 称

$$d = (\, d(v_1), d(v_2), \ldots, d(v_n) \,)$$

为G的度数列

- 例：d = (5, 1, 2, 3, 3)



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可图化

- 设非负整数列d=($d_1,d_2, \ldots, d_n$), 若存在图G, 使得G的度数列是d, 则称d为可图化的

- 可图化举例：度数列 d = (5, 3, 3, 2, 1)

可图化充要条件

非负整数列$d=(d_1,d_2,\ldots,d_n)$可图化 $\Leftrightarrow$

$d_1+d_2+\ldots+d_n \equiv 0 \pmod 2$.

- 证: (⇒) 握手定理

(⇐) 奇数度点两两之间连一边,

剩余度用环来实现. #

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可简单图化

- 设非负整数列$d=(d_1, d_2, \ldots, d_n)$, 若存在简单图G, 使得G的度数列是d, 则称d为可简单图化的

- 例：$d = (5, 3, 3, 2, 1)$ 不可简单图化
  - $\Delta=n \quad (\Delta\leq n-1)$

- 例：$d = (4, 4, 3, 2, 1)$ 不可简单图化
  - $(n-1, n-1, \ldots ,1)$

可简单图化充要条件(Havel定理)

- 设非负整数列$d=(d_1,d_2,\ldots,d_n)$满足:

$$d_1+d_2+\ldots+d_n \equiv 0 \pmod 2,$$

$$n-1 \geq d_1 \geq d_2 \geq \ldots \geq d_n \geq 0,$$

则d可简单图化$\Leftrightarrow$

$$d'=(d_2-1,d_3-1,\ldots,d_{d_1+1}-1,d_{d_1+2},\ldots,d_n)$$

可简单图化

- 例: $d=(4,4,3,3,2,2)$, $d'=(3,2,2,1,2)$

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Havel定理举例

- 判断下列非负整数列是否可简单图化. (1)(5,5,4,4,2,2)   (2)(4,4,3,3,2,2)
- 解: (1) (5,5,4,4,2,2), (4,3,3,1,1),

  (2,2,0,0), (1,-1,0), 不可简单图化.

(2) (4,4,3,3,2,2), (3,2,2,1,2), (3,2,2,2,1),

(1,1,1,1), (0,1,1), (1,1), 可简单图化. #

可简单图化充要条件

- 定理7.4(P.Erdös,T.Gallai,1960):

设非负整数列$d=(d_1,d_2,\ldots,d_n)$满足:

$$n-1\geq d_1\geq d_2\geq\ldots\geq d_n\geq 0,$$

则d可简单图化 $\Leftrightarrow$

$$d_1+d_2+\ldots+d_n=0 \pmod 2$$

并且对$r=1,2,\ldots,n-1$有

$$d_1+d_2+\ldots+d_r\leq r(r-1)+\min\{r,d_{r+1}\}+\min\{r,d_{r+2}\}+\ldots+\min\{r,d_n\}.$$

#

定理7.4等价形式

- 定理7.4’(P.Erdös,T.Gallai,1960):非负整数列 $d=(d_1,d_2,\ldots,d_n)$可简单图化 $\Leftrightarrow$ $d_1+d_2+\ldots+d_n=0 \pmod 2)$
  并且对$r=1,2,\ldots,n$有
  $d_1+d_2+\ldots+d_r \le r(r-1)+\min\{r,d_{r+1}\}+\min\{r,d_{r+2}\}+\ldots+\min\{r,d_n\}$. #
- 说明: $n-1\ge d_1\ge d_2\ge\ldots\ge d_n\ge 0 \Rightarrow$
  $d_1+d_2+\ldots+d_n \le n(n-1)$.

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定理7.4举例

- 下列非负整数列是否可简单图化?

(1) (5,4,3,2,2,1)

(2) (5,4,4,3,2)

(3) (3,3,3,1)

(4) (6,6,5,4,3,3,1)

(5) (5,5,3,3,2,2,2)

(6) $(d_1,d_2,\ldots,d_n)$, $d_1>d_2>\ldots>d_n\geq 1$

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定理7.4举例

- (1) $5+4+3+2+2+1=17\neq 0 \pmod 2$.

不可(简单)图化.

- (2) $5+4+4+3+2=18=0 \pmod 2$.

但是$d_1=5>n-1=4$,不满足$n-1\geq d_1$, 不可简单图化.

(或者: 但是r=1时, $d_1=5>1(1-1)+\min\{1,4\}+\min\{1,4\}+\min\{1,3\}+\min\{1,2\}=4$, 不可简单图化.)

- (3) $3+3+3+1=10=0 \pmod 2$. $d_1=3=n-1$,满足$n-1\geq d_1$,

但是r=2时, $d_1+d_2=6 > 2(2-1) + \min\{2,3\} + \min\{2,1\} = 5$, 不可简单图化.

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定理4举例(4)

- (4) $6+6+5+4+3+3+1=28=0(\mathrm{mod}\ 2)$.

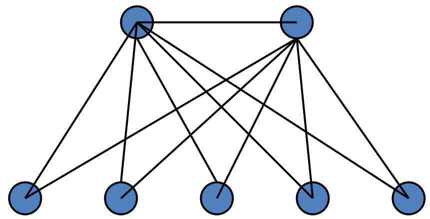
$d_1=6=n-1$,满足$n-1\geq d_1$. $r=1,2$时,

$d_1=6\leq 1(1-1)+\min\{1,6\}+\min\{1,5\}+\ldots=6$,

$d_1+d_2=12>2(2-1)+\min\{2,5\}+\ldots=11$,

不可简单图化.

- 或:6,6,*,*,*,*,1不可简单图化



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定理4举例(5)

- (5) $5+5+3+3+2+2+2=22=0 \pmod 2$.

$d_1=5<n-1$,满足$n-1 \ge d_1$. $r=1,2,\ldots,7$时,

$d_1=5<1(1-1)+\min\{1,5\}+\min\{1,5\}+\ldots=6$,

$d_1+d_2=10<2(2-1)+\min\{2,3\}+\ldots=12$,

$d_1+d_2+d_3=13<3(3-1)+\min\{3,3\}+\ldots=15$,

$d_1+d_2+d_3+d_4=16<4(4-1)+\min\{4,2\}+\ldots=18$,

定理4举例(5)

$d_1+d_2+d_3+d_4 =16<4(4-1)+\min\{4,2\}+\ldots =18$,

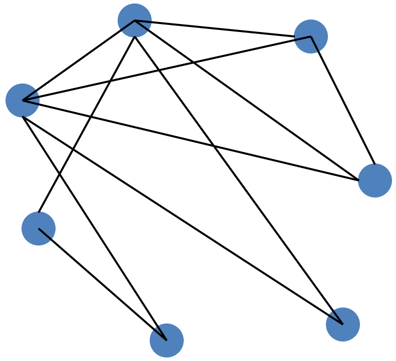
$d_1+d_2+\ldots+d_5 =18<5(5-1)+\min\{5,2\}+\ldots =24$,

$d_1+d_2+\ldots+d_6 =20<6(6-1)+\min\{6,2\}=32$,

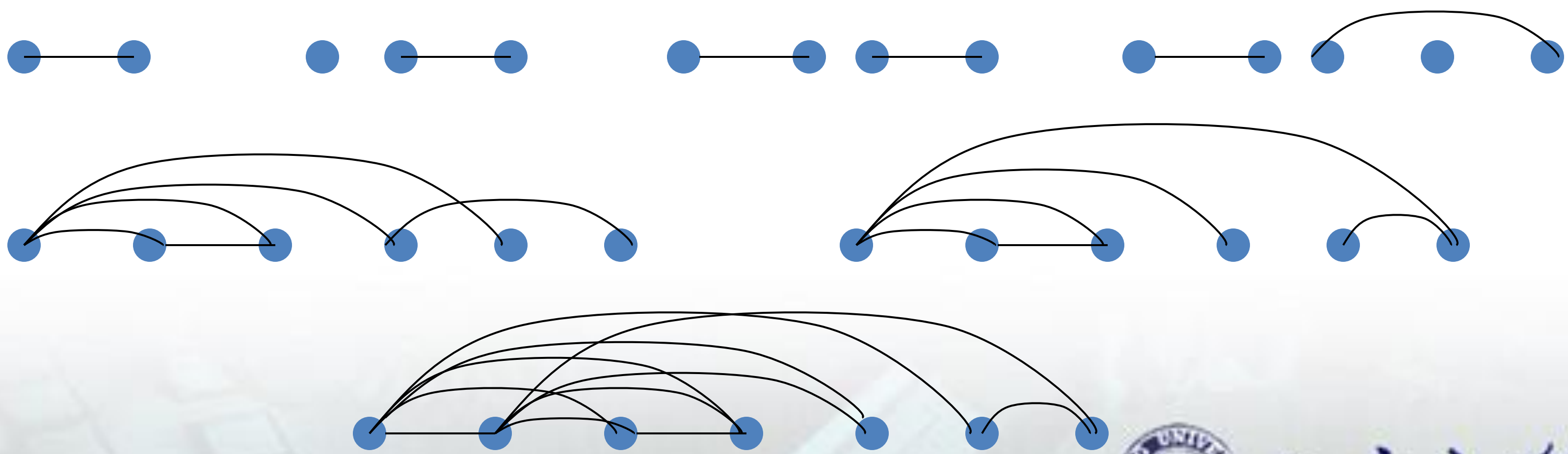
$d_1+d_2+\ldots+d_7 =22<7(7-1)=42$,

可简单图化.

定理4举例(5)



(5,5,3,3,2,2,2), (4,2,2,1,1,2), (4,2,2,2,1,1),
(1,1,1,0,1),(1,1,1,1), (0,1,1),(1,1)



定理4举例(6)

- (6) $d_1>d_2>\ldots>d_n\geq 1$,

  $d_{n-1}\geq 2$, $d_{n-2}\geq 3,\ldots$, $d_1\geq n$,

  不满足$n-1\geq d_1$,

  不可简单图化. #

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Paul Erdös(1913-1996)

- 保罗•爱尔特希, 匈牙利人
- 廿世纪数学界的传奇人物
- “Another roof, another proof.”

Paul Erdös

"My mother said, 'Even you, Paul, can be in only one place at one time.'

Maybe soon I will be relieved of this disadvantage.

Maybe, once I've left, I'll be able to be in many places at the same time.

Maybe then I'll be able to collaborate with Archimedes and Euclid."

图同构

- 无向图 $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$，若存在双射 $f:V_1 \to V_2$ 满足$\forall u,v \in V_1$，$(u,v)\in E_1 \leftrightarrow (f(u),f(v))\in E_2$，
  则称G_1与G_2同构, 记作$G_1 \cong G_2$
- 同构的图,其图论性质完全一样
- NAUTY算法
- 有向图$D_1=<V_1,E_1>$, $D_2=<V_2,E_2>$, 若存在双射 $f:V_1 \to V_2$, 满足$\forall u,v \in V_1$, $<u,v>\in E_1 \leftrightarrow <f(u),f(v)>\in E_2$，
  则称D_1与D_2同构, 记作$D_1 \cong D_2$

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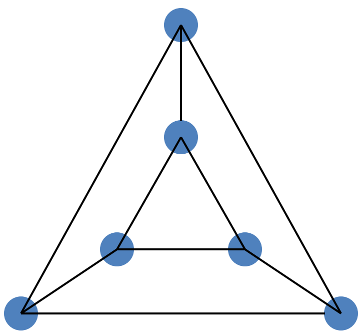
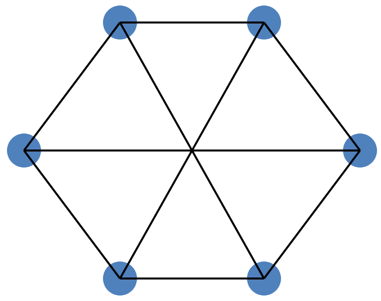
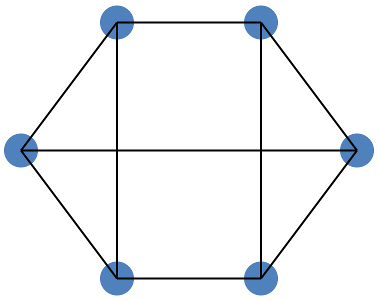
图同构(举例)



G_1



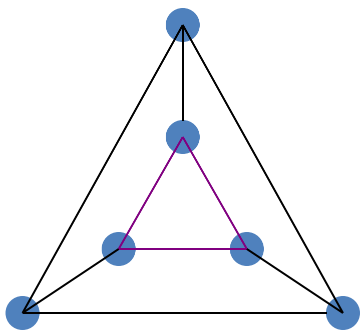
G_2



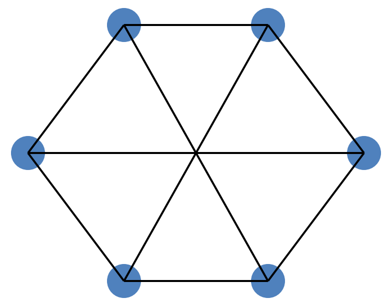
G_3

$G_1 \cong G_3$， $G_1 \ncong G_2$

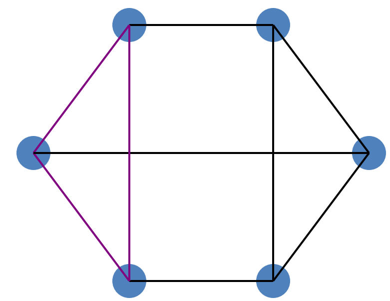
图同构(举例)



G_1

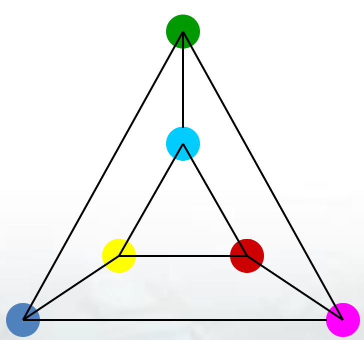


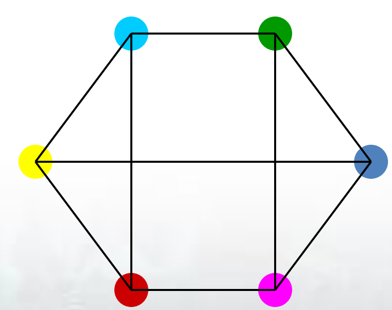
G_2



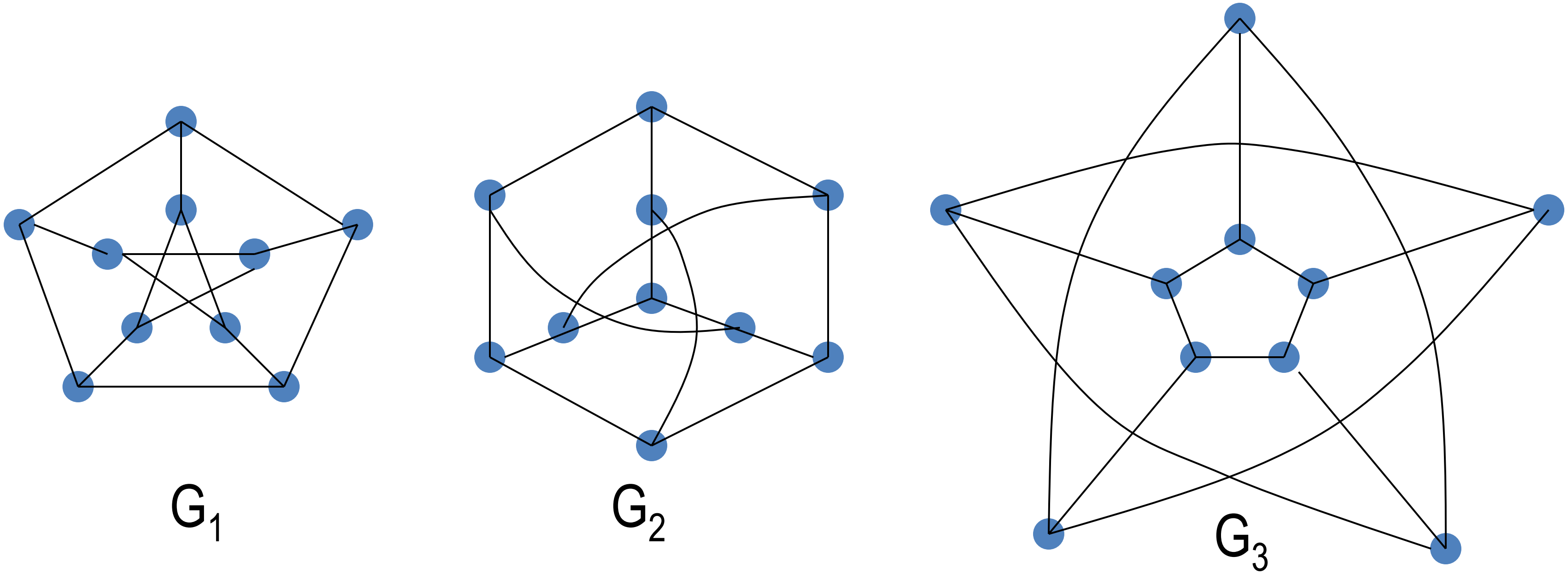
G_3

$G_1 \cong G_3$， $G_1 \ncong G_2$





图同构(举例)



G_1　G_2　G_3

$G_1 \cong G_2 \cong G_3$

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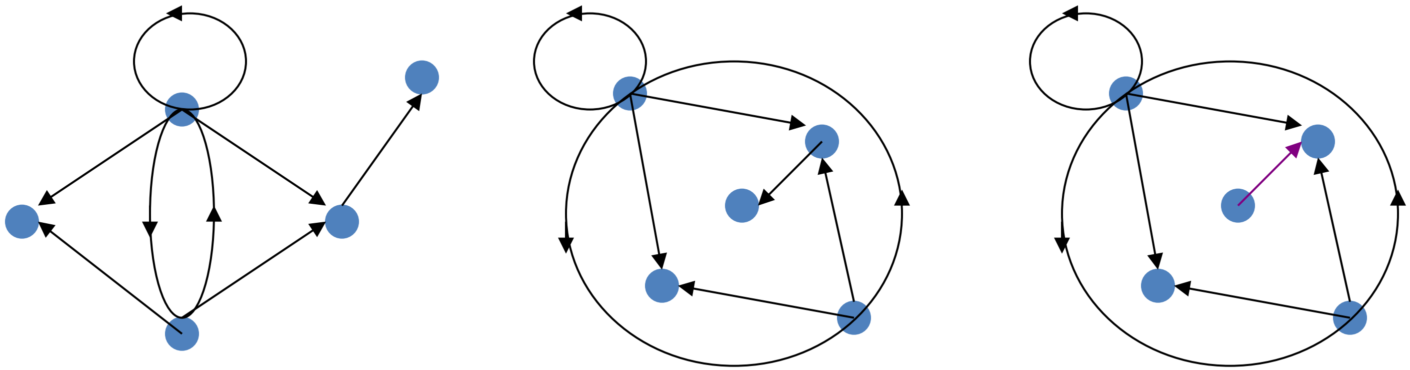
图同构(举例)



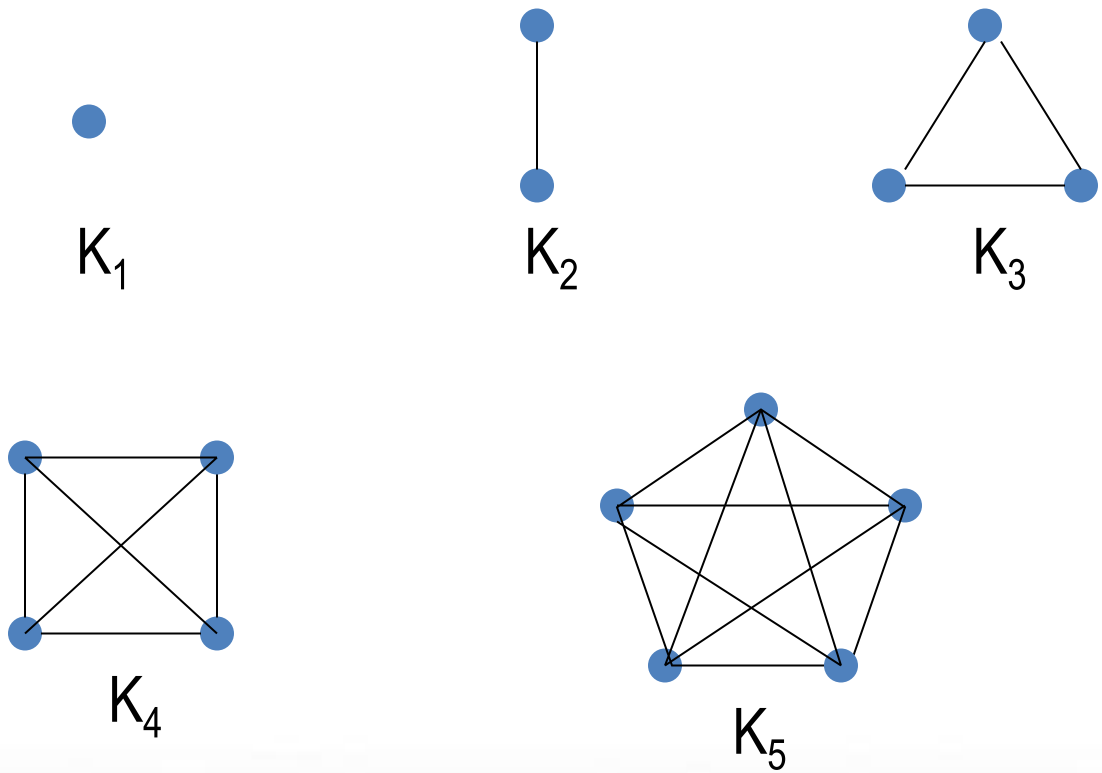
D_1　　　　D_2　　　　D_3

$D_1 \cong D_2$， $D_2 \ncong D_3$

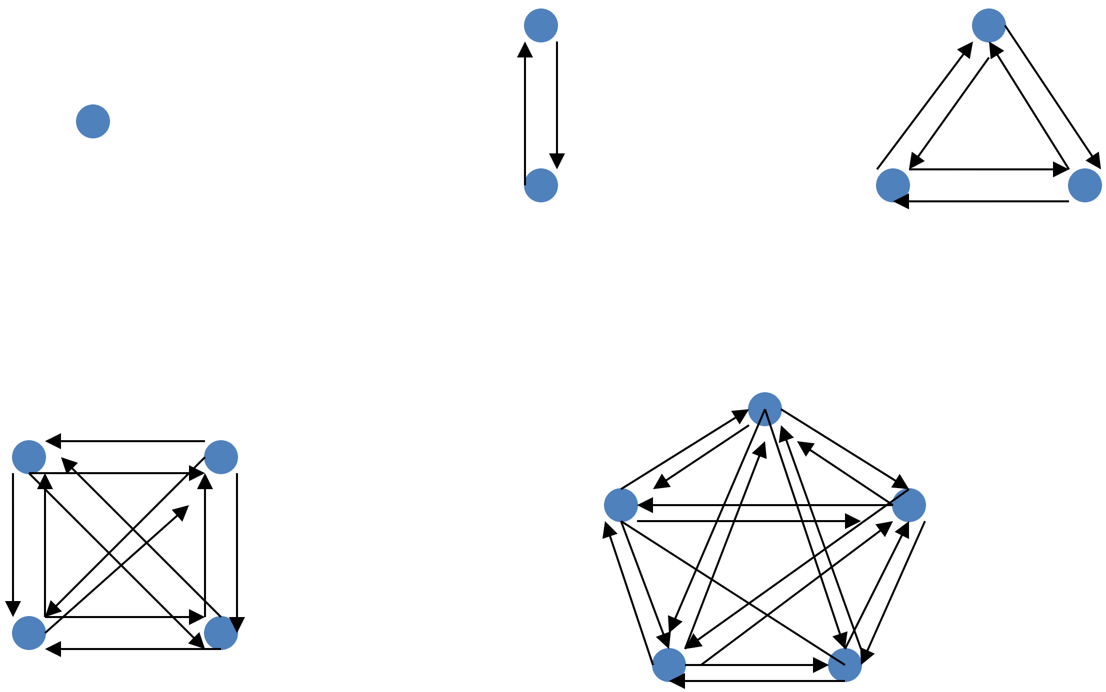
图族(graph class)

- 完全图,有向完全图,竞赛图
- 柏拉图图,彼德森图,库拉图斯基图
- r部图,二部图(偶图),完全r部图
- 路径,圈,轮,超立方体

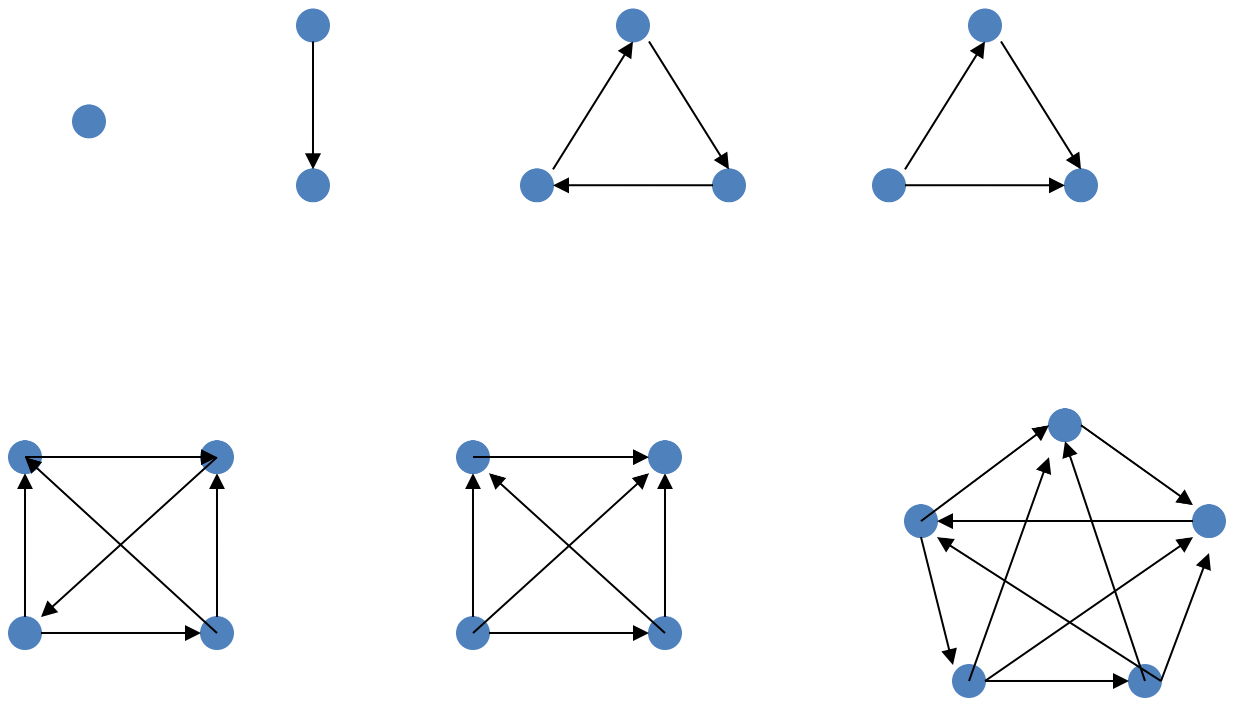
完全图



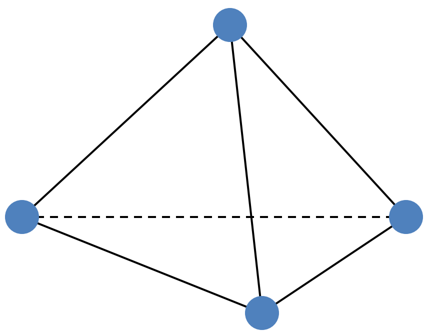
有向完全图



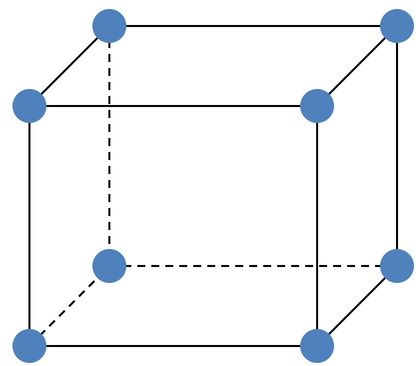
竞赛图



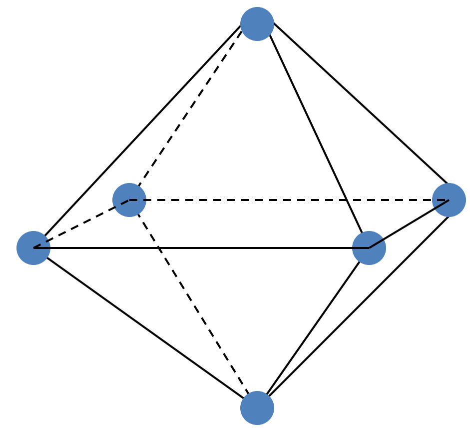
柏拉图图



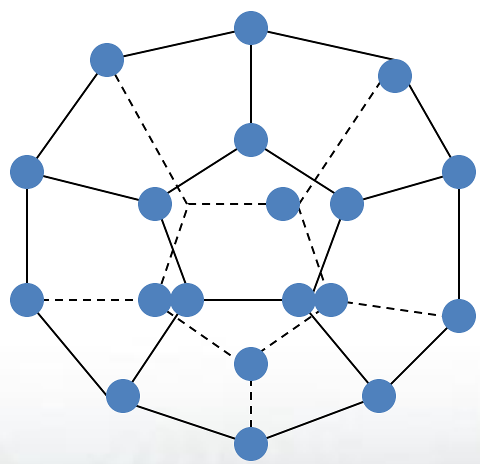
正四面体图



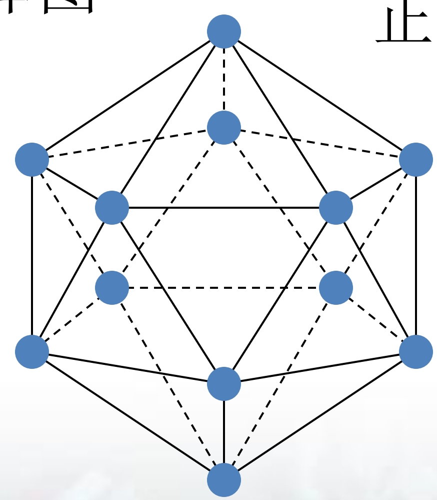
正六面体图



正八面体图

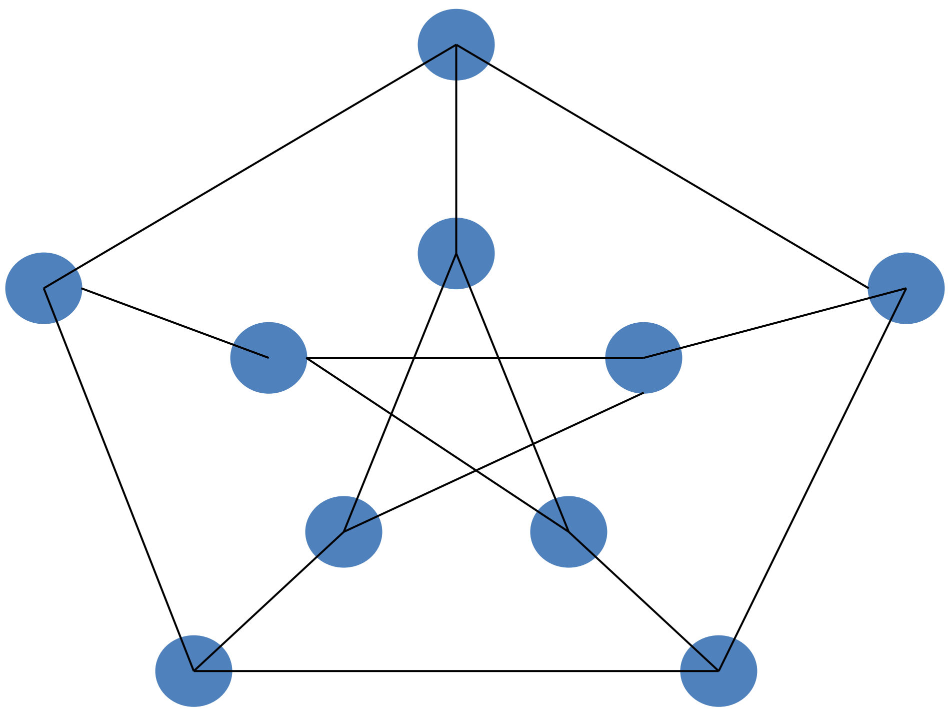


正十二面体图



正二十面体图

彼德森图



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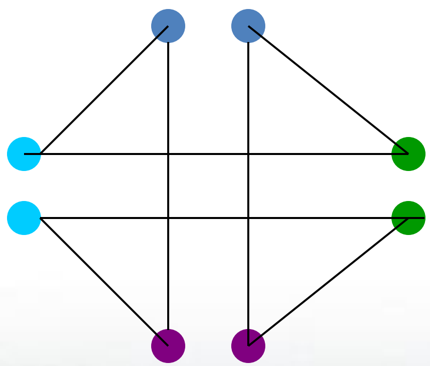
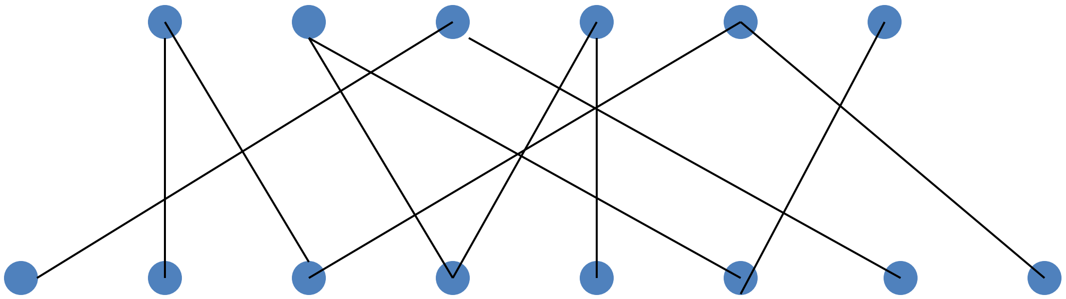
r部图

- r部图: $G=\langle V,E\rangle, V=V_1\cup V_2\cup\ldots\cup V_r$, $V_i\cap V_j=\varnothing\ (i\neq j)$, $E\subseteq \bigcup_{i\neq j}(V_i\&V_j)$,
- 也记作 $G=\langle V_1,V_2,\ldots,V_r\,;\,E\rangle$.

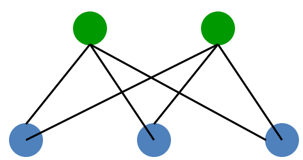


二部图(偶图)

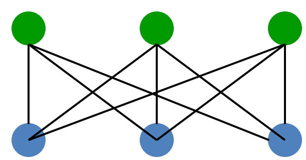
- 二部图: $G=<V_1,V_2; E>$, 也称为偶图



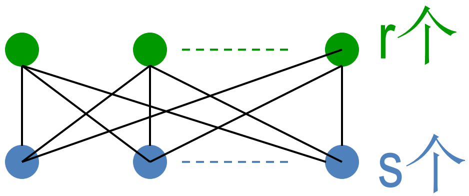
完全r部图



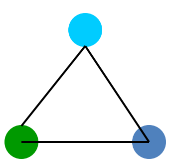
$K_{2,3}$



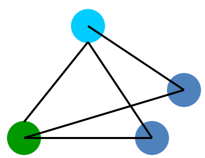
$K_{3,3}$



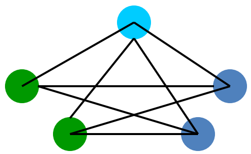
$K_{r,s}$



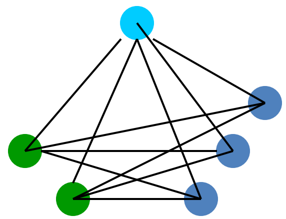
$K_{1,1,1}$



$K_{1,1,2}$



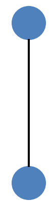
$K_{1,2,2}$



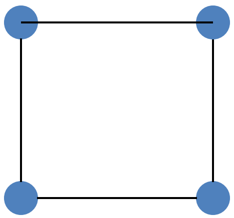
$K_{1,2,3}$

超立方体

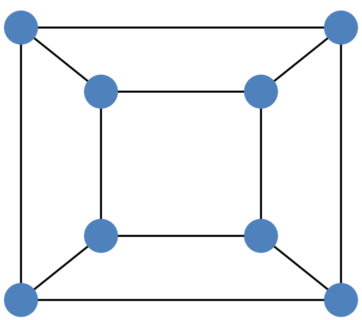
Q_0



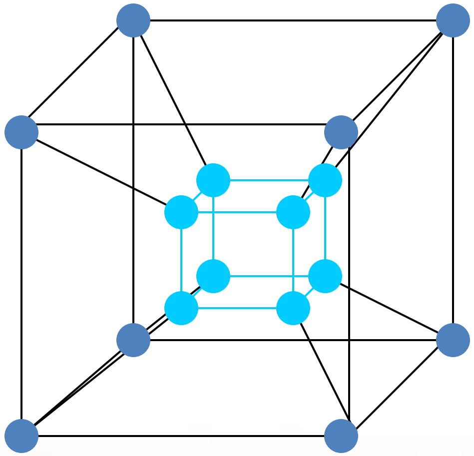
Q_1



Q_2



Q_3



Q_4

子图,生成子图

- 子图(subgraph): $G=<V,E>$, $G'=<V',E'>$,

$$G'\subseteq G \Leftrightarrow V'\subseteq V \wedge E'\subseteq E$$

- 真子图(proper subgraph):

$$G'\subset G \Leftrightarrow G'\subseteq G \wedge (V'\subset V \vee E'\subset E)$$

- 生成子图(spanning subgraph):

$$G'\text{是}G\text{的生成子图} \Leftrightarrow G'\subseteq G \wedge V'=V$$

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导出子图

- 导出子图: $G=<V,E>$,
- 若$V_1 \subset V$, $E_1 = E \cap V_1 \& V_1$,则称

$$G[V_1] = <V_1,E_1>$$

为由V_1导出的子图

- 若$\emptyset \neq E_1 \subset E$, $V_1=\{v|v$与E_1中的边关联$\}$,

则称 $$G[E_1] = <V_1,E_1>$$

为由E_1导出的子图

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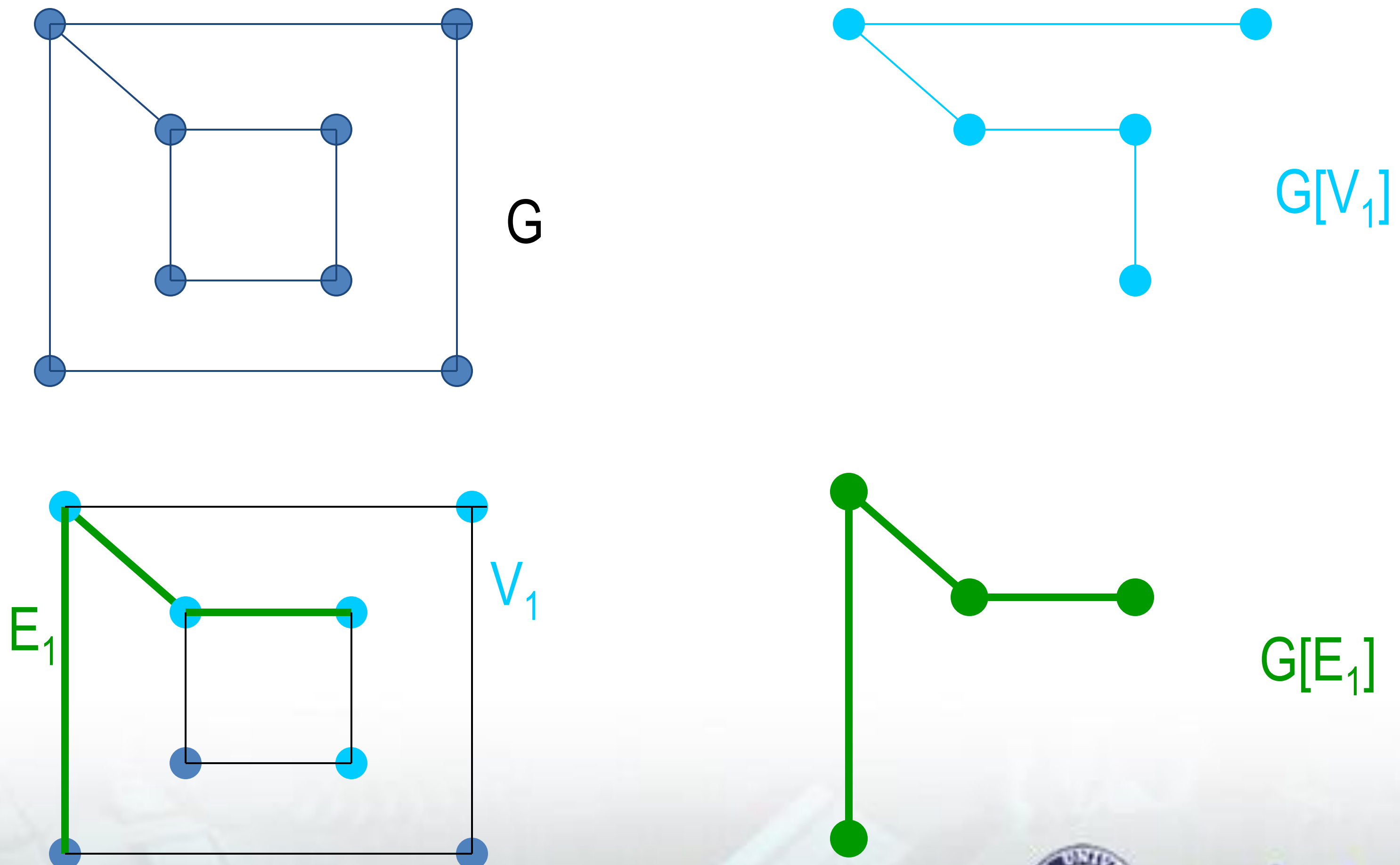
导出子图(举例)



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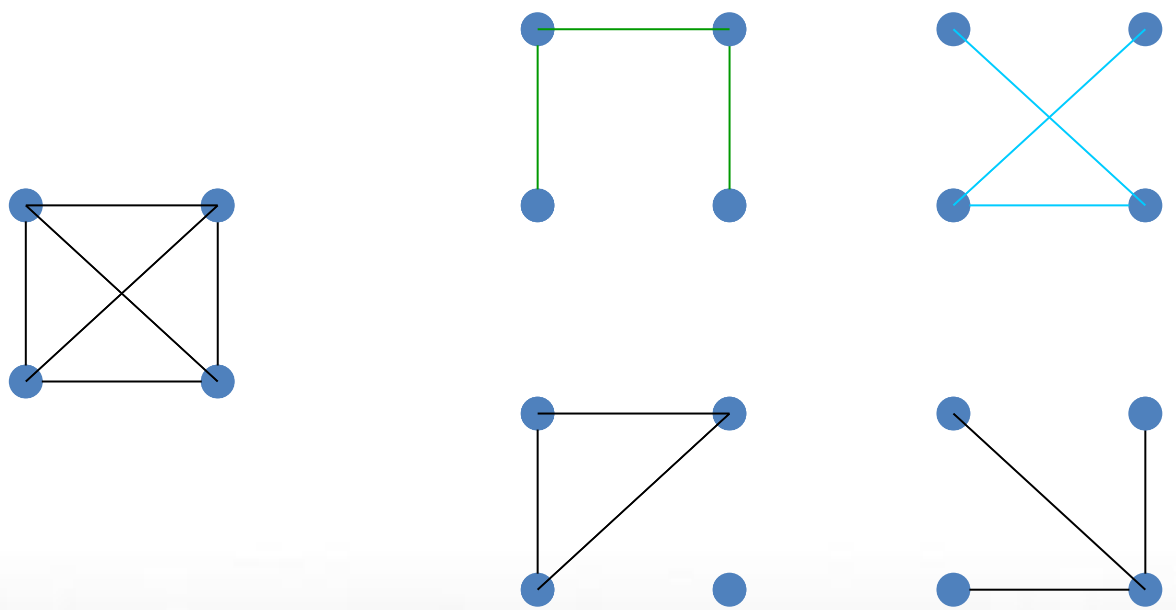
补图

- 补图: $G=<V,E>$, $\overline{G}=<V,E(K_n)-E>$
- 自补图(self-complement graph): $G \cong \overline{G}$



例7.5

- (1) 画出5阶4条边的所有非同构的无向简单图； (2) 画出4阶2条边的所有非同构的有向简单图.
- (1) $\Sigma d(v)=2m=8$, $\Delta\leq n-1=4$,

  (4,1,1,1,1),(3,2,1,1,1),(2,2,2,1,1),

  (3,2,2,1,0),(2,2,2,2,0)
- (2) $\Sigma d^+(v)=\Sigma d^-(v)=m=2$, $\Sigma d(v)=2m=4$,

  (2,1,1,0),(1,1,1,1),(2,2,0,0)

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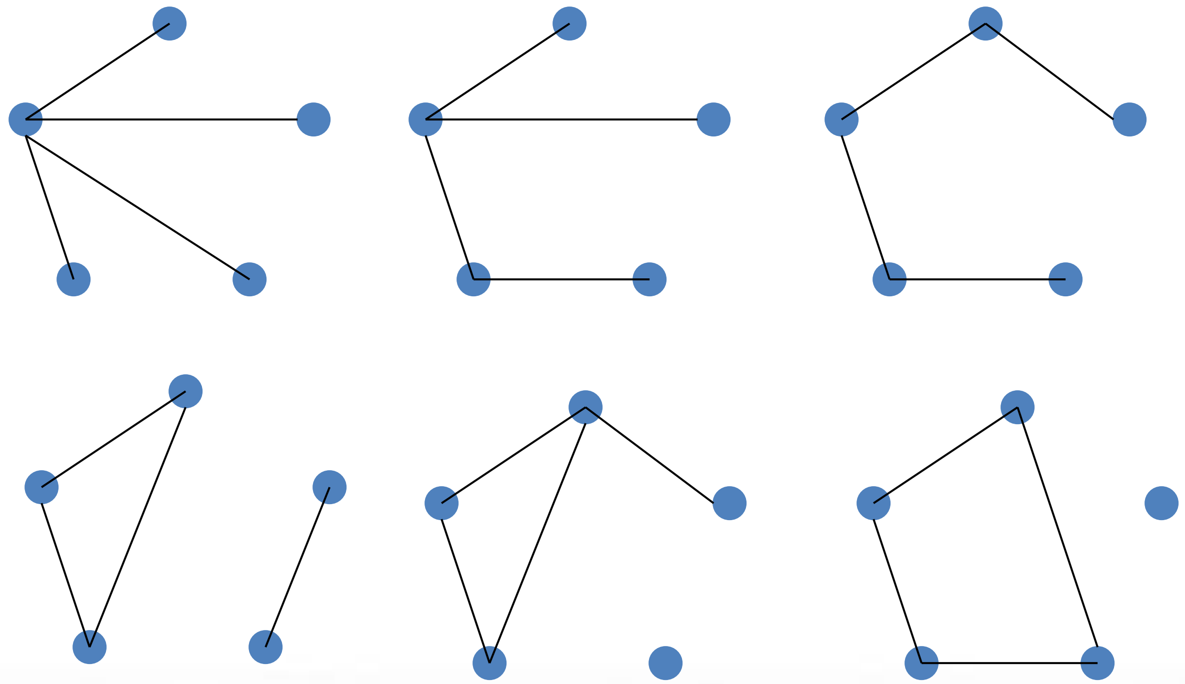
例7.5(1)

- 5阶4条边的所有非同构无向简单图



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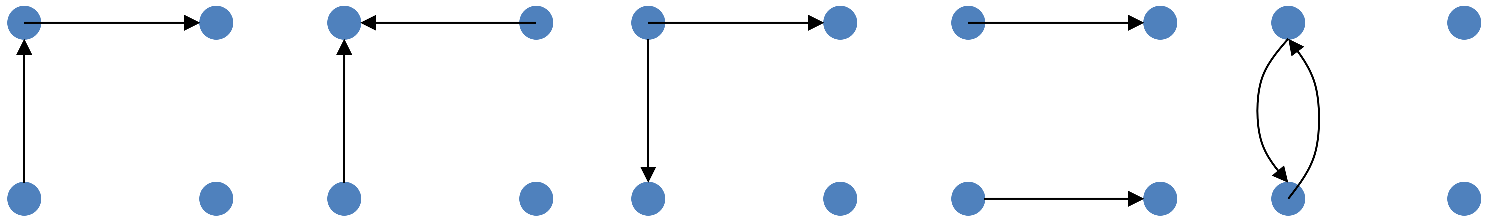
例7.5(2)

- 4阶2条边的所有非同构有向简单图.



图的运算

- 删除,收缩,加新边,不交
- 并图,交图,差图,环和
- 联图,积图

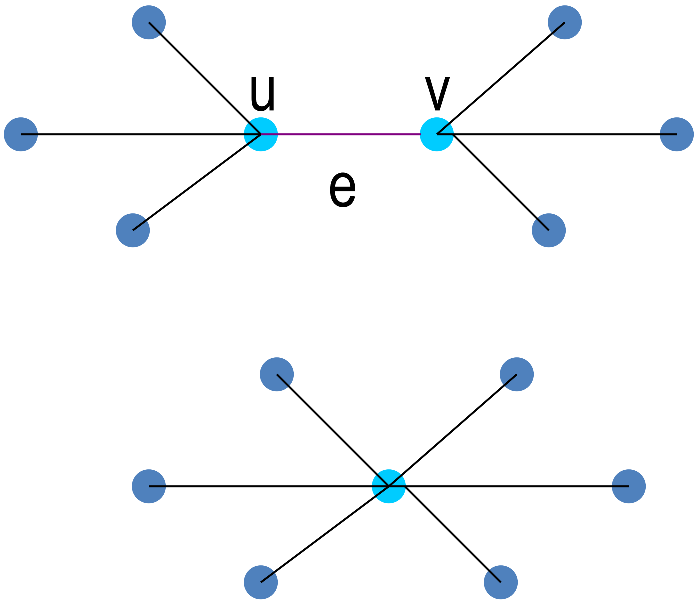
删除(delete)

- G-e = < V, E-{e} >，删除边e
- G-E’ = < V, E-E’ >，删除边集E’
- G-v = < V-{v}, E-I_G(v) >，删除顶点v以及v所关联的所有边
- G-V’ = < V-V’, E-I_G(V’) >,删除顶点集V’以及V’所关联的所有边

收缩、加新边

- G\e: e=(u,v), 删除e,合并u与v
- G∪(u,v) = <V,E∪{(u,v)}>
  - 在u与v之间加新边

也写成G+(u,v)



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不交

- $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$,
- G_1与G_2不交 $\Leftrightarrow V_1 \cap V_2=\varnothing$
- G_1与G_2边不交(边不重) $\Leftrightarrow E_1 \cap E_2=\varnothing$

并图,交图,差图,环和

- $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$, 都无孤立点
- 并图: $G_1 \cup G_2 = <V(E_1 \cup E_2), E_1 \cup E_2>$
- 交图: $G_1 \cap G_2 = <V(E_1 \cap E_2), E_1 \cap E_2>$
- 差图: $G_1 - G_2 = <V(E_1 - E_2), E_1 - E_2>$
- 环和: $G_1 \oplus G_2 = <V(E_1 \oplus E_2), E_1 \oplus E_2>$

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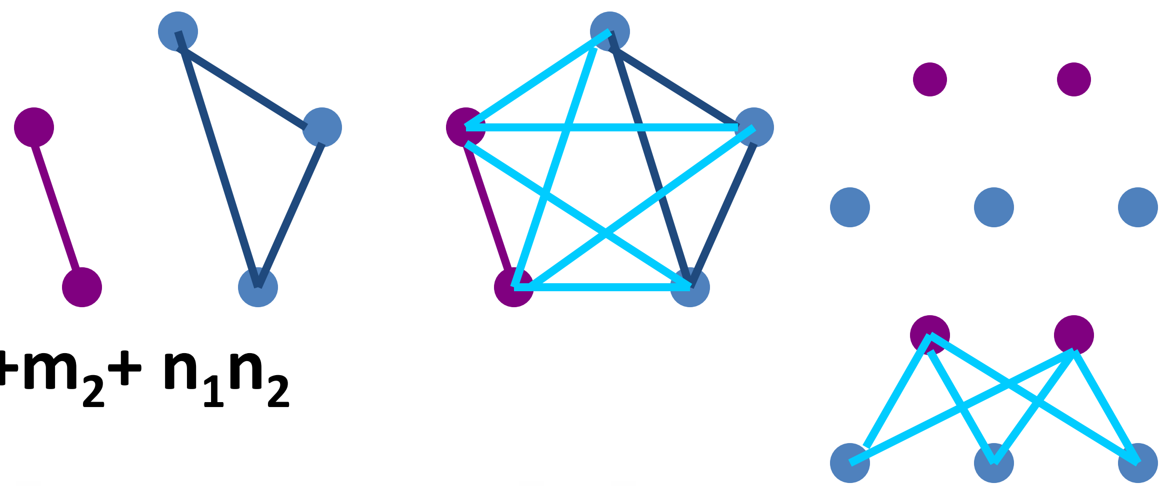
联图

- $G_1=<V_1,E_1>$, $G_2=<V_2,E_2>$, 不交无向图
- $G_1+G_2=<V_1\cup V_2, E_1\cup E_2\cup(V_1\&V_2)>$

- $K_r+K_s=K_{r+s}$
- $N_r+N_s=K_{r,s}$
- $n=n_1+n_2$, $m=m_1+m_2+n_1n_2$



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积图

- G_1=<V_1,E_1>, G_2=<V_2,E_2>, 无向简单图
- $G_1 \times G_2$=<$V_1 \times V_2$, E>, 其中

$E = \{ (\langle u_i,u_j\rangle,\langle u_k,u_s\rangle) \mid$

$(\langle u_i,u_j\rangle,\langle u_k,u_s\rangle \in V_1 \times V_2) \wedge$

$((u_i=u_k \wedge u_j$与u_s相邻$) \vee (u_j=u_s \wedge u_i$与$u_k$相邻$)) \}$

- $n=n_1n_2$, $m=n_1m_2+ n_2m_1$

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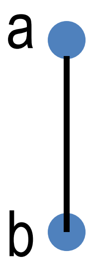
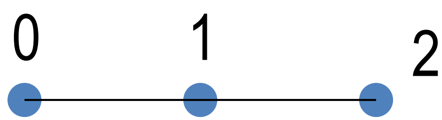
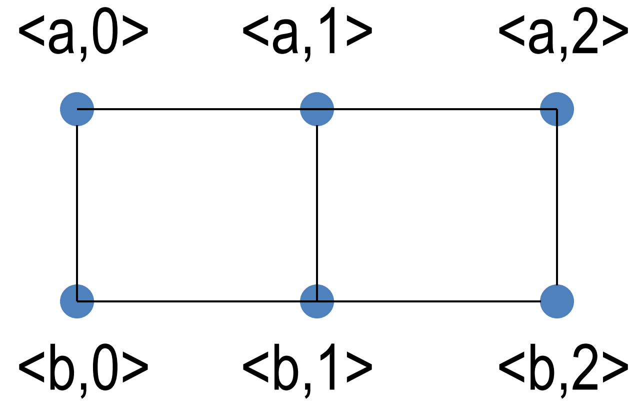
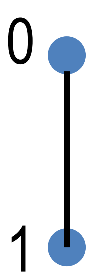
积图(举例)

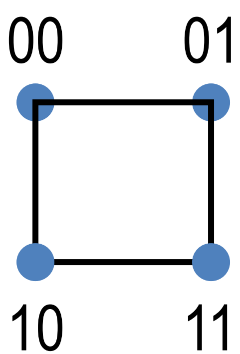




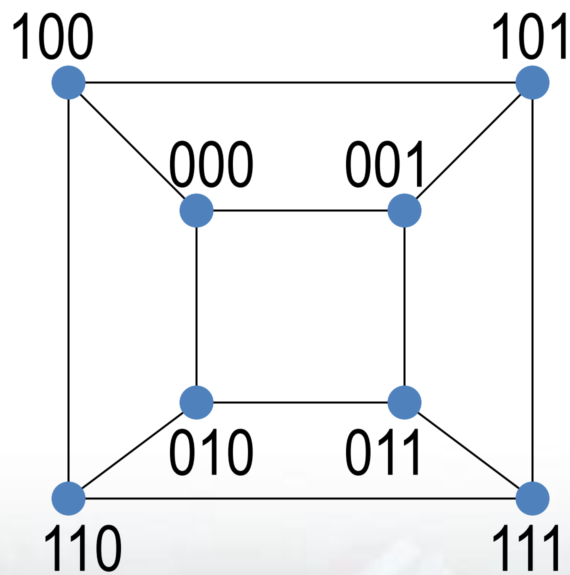




$Q_1=K_2$



$Q_2=K_2 \times Q_1$



$Q_k=K_2 \times Q_{k-1}$

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小结

- 无向图,有向图,简单图,相邻,关联
- 度, 握手定理
- 图同构
- 图族(完全图, 零图, ...)
- 图运算(并图, 联图, 积图, ...)