单元5.2-基数和基数的比较与运算

第一编 集合论 第5章 基数

5.3 基数

5.4 基数的比较

5.5 基数运算

内容提要

- 基数
- 基数的比较
- 基数运算

基数的定义

(1) card A=card B $\Leftrightarrow$ A$\approx$B

(2) 对有穷集A, card A=n $\Leftrightarrow$ A$\approx$n

(3) 对自然数集N, card N=$\aleph_0$

(ℵ 读 阿列夫)

(4) 对实数集R, card R=$\aleph_1$=$\aleph$

(5) 0,1,2,…,$\aleph_0$,$\aleph$都称作基数.

说明

- 0,1,2,...称作有穷基数
- $\aleph_0$,$\aleph$称作无穷基数
- 若card A=$\aleph_i$, 则card P(A)=$\aleph_{i+1}$
- 用希腊字母κ,λ,μ等表示任意基数

- card A是对|A|的推广

K_κ

- 设κ是任意基数, 令

$$K_\kappa = \{x \mid x\text{是集合且card } x=\kappa\}$$

- 当$\kappa=0$时, $K_\kappa=\{\varnothing\}$是集合
- 当$\kappa\neq0$时, K_κ不是集合, 是类

例子

- $A=\{a,b,c\}$, $B=\{\{a\},\{b\},\{c\}\}$
- $N_{偶}=\{n \mid n\in N \wedge n$是偶数$\}$
- $N_{奇}=\{n \mid n\in N \wedge n$是奇数$\}$
- $[0,1]$, $(0,1)$

例子

- $A=\{a,b,c\}$, $B=\{\{a\},\{b\},\{c\}\}$
- $N_{偶}=\{n|n\in N\wedge n是偶数\}$
- $N_{奇}=\{n|n\in N\wedge n是奇数\}$
- $[0,1]$, $(0,1)$
- card A=card B=3,
- card $N_{偶}$=card $N_{奇}$=card $N=\aleph_0$
- card $[0,1]$=card $(0,1)$=card $R=\aleph$

优势, 劣势

- B比A优势 $\Leftrightarrow$ A比B劣势 $\Leftrightarrow$

  $\exists$ f:A→B 单射

  $\Leftrightarrow$ A≼•B

绝对优势, 绝对劣势

- B比A绝对优势 ⇔ A比B绝对劣势 ⇔

$$A \preccurlyeq\bullet B \ \wedge A \not\approx B$$

$$\Leftrightarrow A \prec\bullet B$$

定理5.7

定理5. 7 $A \preccurlyeq \bullet B \Leftrightarrow \exists C \subseteq B$, 使得$A \approx C$

证明: $(\Rightarrow)$ $A \preccurlyeq \bullet B \Rightarrow \exists\, f:A \to B$ 单射

$\Rightarrow \exists\, f:A \to \mathrm{ran} f$ 双射

$\Rightarrow A \approx \mathrm{ran}\, f \subseteq B$

$\Rightarrow$ 取$C=\mathrm{ran}\, f$ 即可.

$(\Leftarrow)$ $\exists C \subseteq B$, 使得$A \approx C \Rightarrow \exists\, g:A \to C$ 双射

$\Rightarrow \exists\, g:A \to B$ 单射

$\Rightarrow A \preccurlyeq \bullet B$. #

推论

(1) $A \subseteq B \Rightarrow A \preccurlyeq \bullet B$

(2) $A \approx B \Rightarrow A \preccurlyeq \bullet B \wedge B \preccurlyeq \bullet A$

#

定理5.8

(1) $A \preccurlyeq\!\bullet A$

(2) $A \preccurlyeq\!\bullet B \wedge B \preccurlyeq\!\bullet C \Rightarrow A \preccurlyeq\!\bullet C$

#

定理5.9

$A \preccurlyeq\bullet B \wedge C \preccurlyeq\bullet D \Rightarrow$

(1) $A \cup C \preccurlyeq\bullet B \cup D$ $(B \cap D = \varnothing)$

(2) $A \times C \preccurlyeq\bullet B \times D$

定理5.9证明(1)

- (1) $A \preccurlyeq\bullet B \wedge C \preccurlyeq\bullet D \wedge B \cap D = \varnothing \Rightarrow$

  $A \cup C \preccurlyeq\bullet B \cup D$

- 证明: $A \preccurlyeq\bullet B \wedge C \preccurlyeq\bullet D \wedge B \cap D = \varnothing$

  $\Rightarrow \exists\, f: A \to B$ 单射, $g: C \to D$ 单射

  $\Rightarrow \exists\, h: A \cup C \to B \cup D$ 单射

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in C - A \end{cases}$$

  $\Rightarrow A \cup C \preccurlyeq\bullet B \cup D$

定理5.9证明(2)

- (2) $A \preccurlyeq\bullet B \wedge C \preccurlyeq\bullet D \Rightarrow A\times C \preccurlyeq\bullet B\times D$

- 证明: $A \preccurlyeq\bullet B \wedge C \preccurlyeq\bullet D \wedge B\cap D=\varnothing$

  $\Rightarrow \exists\, f:A\to B$ 单射, $g:C\to D$ 单射

  $\Rightarrow \exists\, H: A\times C \preccurlyeq\bullet B\times D$ 单射

  $H(\langle x,y\rangle)=\langle f(x),g(y)\rangle$

  $\Rightarrow A\times C \preccurlyeq\bullet B\times D.$ #

定理5.10

- card A=card B=κ $\wedge$ card C=card D=λ $\Rightarrow$

  A$\preccurlyeq\bullet$C $\leftrightarrow$ B$\preccurlyeq\bullet$D

- 证: ($\rightarrow$) card A=card B $\Rightarrow$ $\exists$ f:A$\rightarrow$B 双射

  card C=card D $\Rightarrow$ $\exists$ g:C$\rightarrow$D 双射

  A$\preccurlyeq\bullet$C $\Rightarrow$ $\exists$ h:A$\rightarrow$C 单射

  于是, $\exists$ $j=(g\circ h)\circ f^{-1}$: B$\rightarrow$D 单射 $\Rightarrow$ B$\preccurlyeq\bullet$D.

  ($\leftarrow$) 类似. #

基数的比较

- 设card A=κ, card B=λ

$$\kappa \le \lambda \Leftrightarrow A \preccurlyeq \bullet B$$

$$\kappa < \lambda \Leftrightarrow A \prec \bullet B$$

例5.2

- $\kappa\le\lambda \Rightarrow \exists\ A\subseteq B$, card $A=\kappa \wedge$ card $B=\lambda$
- 证: $\kappa\le\lambda \Rightarrow \exists$集合K,L 使得

  card $K=\kappa \wedge$ card $L=\lambda \wedge K\preccurlyeq\bullet L$

  $K\preccurlyeq\bullet L \Rightarrow \exists\ f:K\to L$ 单射

  $\Rightarrow \exists\ f:K\to \mathrm{ran} f$ 双射

  $\Rightarrow K \approx \mathrm{ran}\, f \subseteq L$

  $\Rightarrow$ 取A=ranf, B=L即可. #

例5.3

- (1) $0\le\kappa$ (2) $n<\aleph_0$

- 证: (1) 设card $A=\kappa$, $\varnothing: \varnothing\to A$ 单射

  $\Rightarrow \varnothing\preccurlyeq\bullet A \Rightarrow 0=\mathrm{card}\,\varnothing \le \mathrm{card}\,A=\kappa$.

  (2) $n\subset N \land n\not\approx N \Rightarrow n\prec\bullet N$

  $\Rightarrow n=\mathrm{card}\,n < \mathrm{card}\,N=\aleph_0$. #

定理5.11

- 定理5.11 card A < card P(A)

- 证: 取$f:A\rightarrow P(A)$, $f(x)=\{x\}$,

  f单射 $\Rightarrow A\preccurlyeq\!\bullet P(A)$

  康托定理 $\Rightarrow A\not\approx P(A)$

于是,

$A\prec\!\bullet P(A) \Rightarrow$ card A < card P(A). #

例5.5

(1) $\kappa \le \kappa$

(2) $\kappa \le \lambda \wedge \lambda \le \mu \Rightarrow \kappa \le \mu$

#

定理5.12

- SchrÖder-Bernstein定理

(1) $A \preccurlyeq \bullet B \wedge B \preccurlyeq \bullet A \Rightarrow A \approx B$

(2) $\kappa \leq \lambda \wedge \lambda \leq \kappa \Rightarrow \kappa = \lambda$

定理5.12证明(1)

- (1) $A \preccurlyeq \!\bullet B \wedge B \preccurlyeq \!\bullet A \Rightarrow A \approx B$

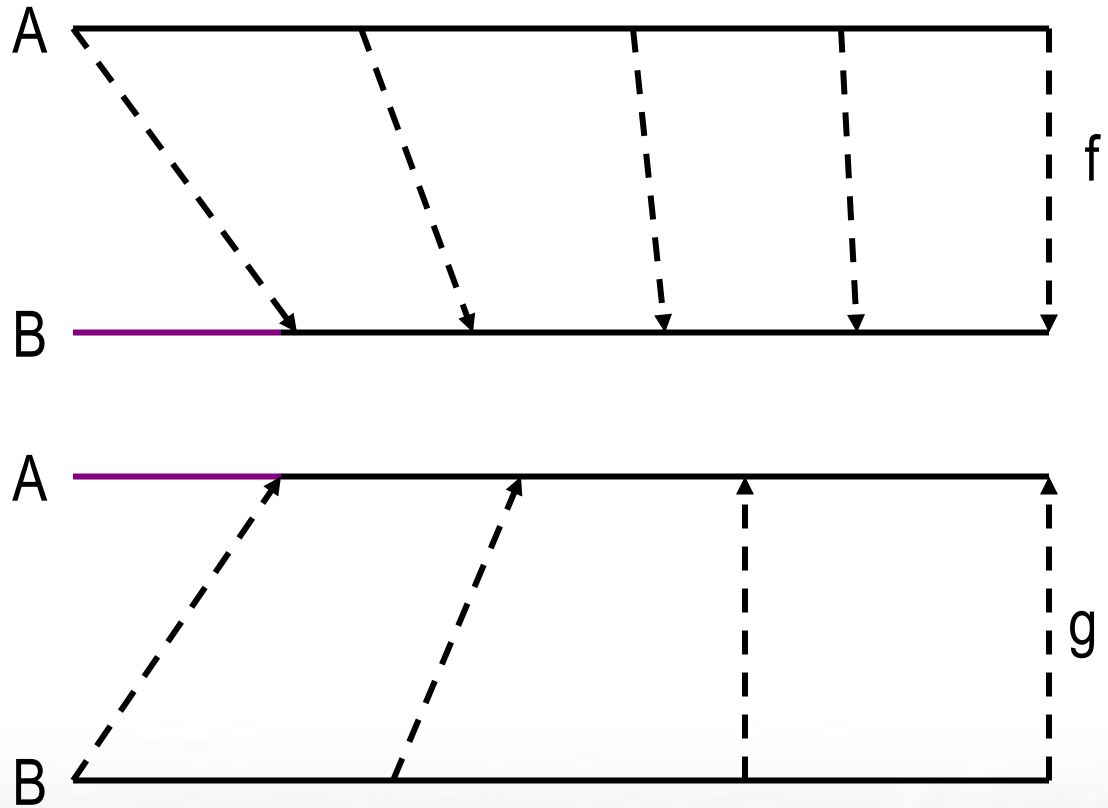
- 证: $A \preccurlyeq \!\bullet B \wedge B \preccurlyeq \!\bullet A$

  $\Rightarrow \exists$ $f:A \to B$ 单射, $g:B \to A$ 单射.

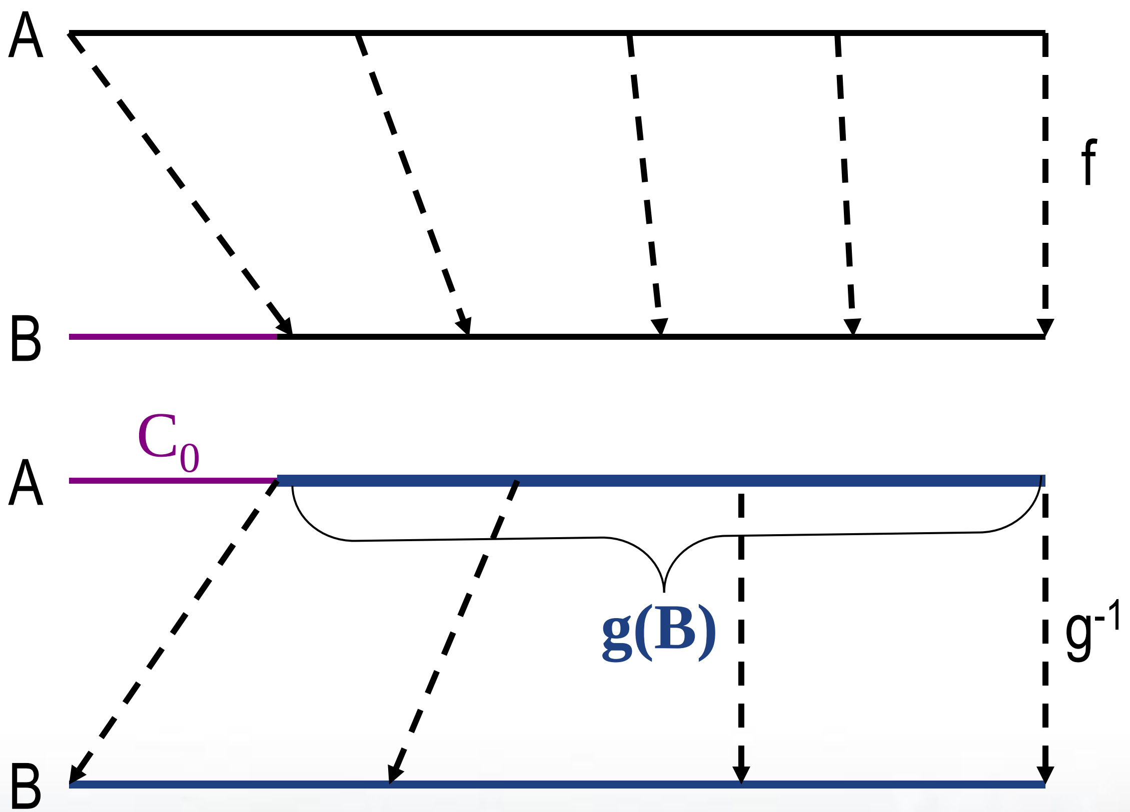
  若 ran $f=B$ 或 ran $g=A$, 则f或g是A和B之间的双射, 于是$A \approx B$.

  下面假设$A-\text{ran } g \neq \varnothing$且$B-\text{ran } f \neq \varnothing$.

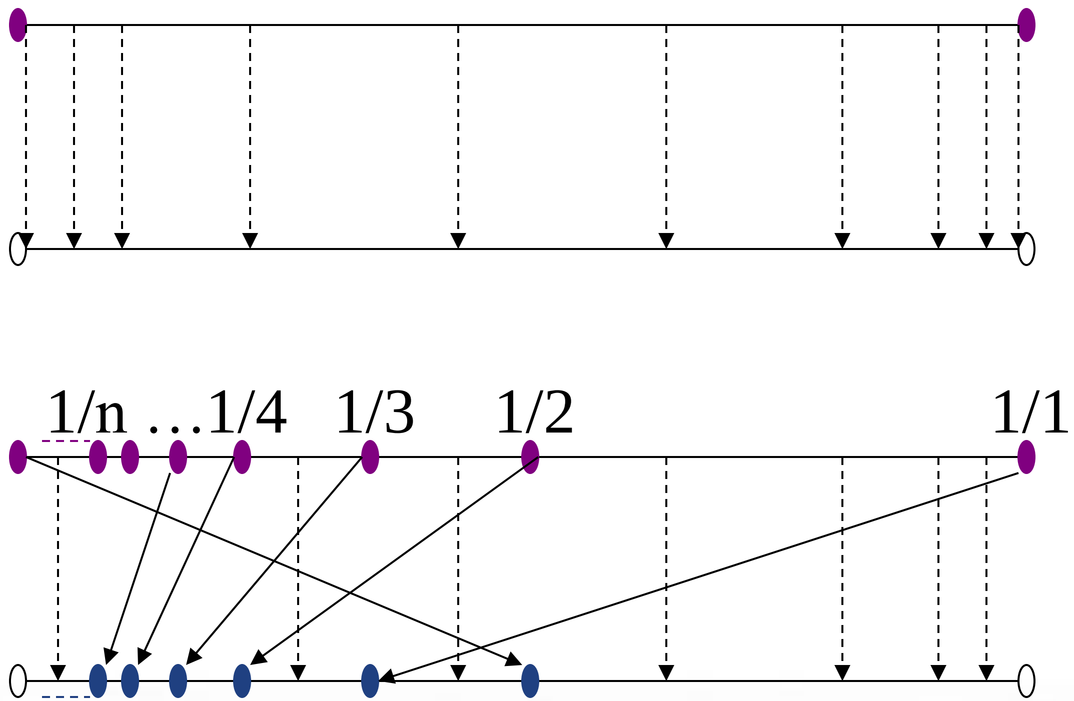
f和g是单射，不是满射



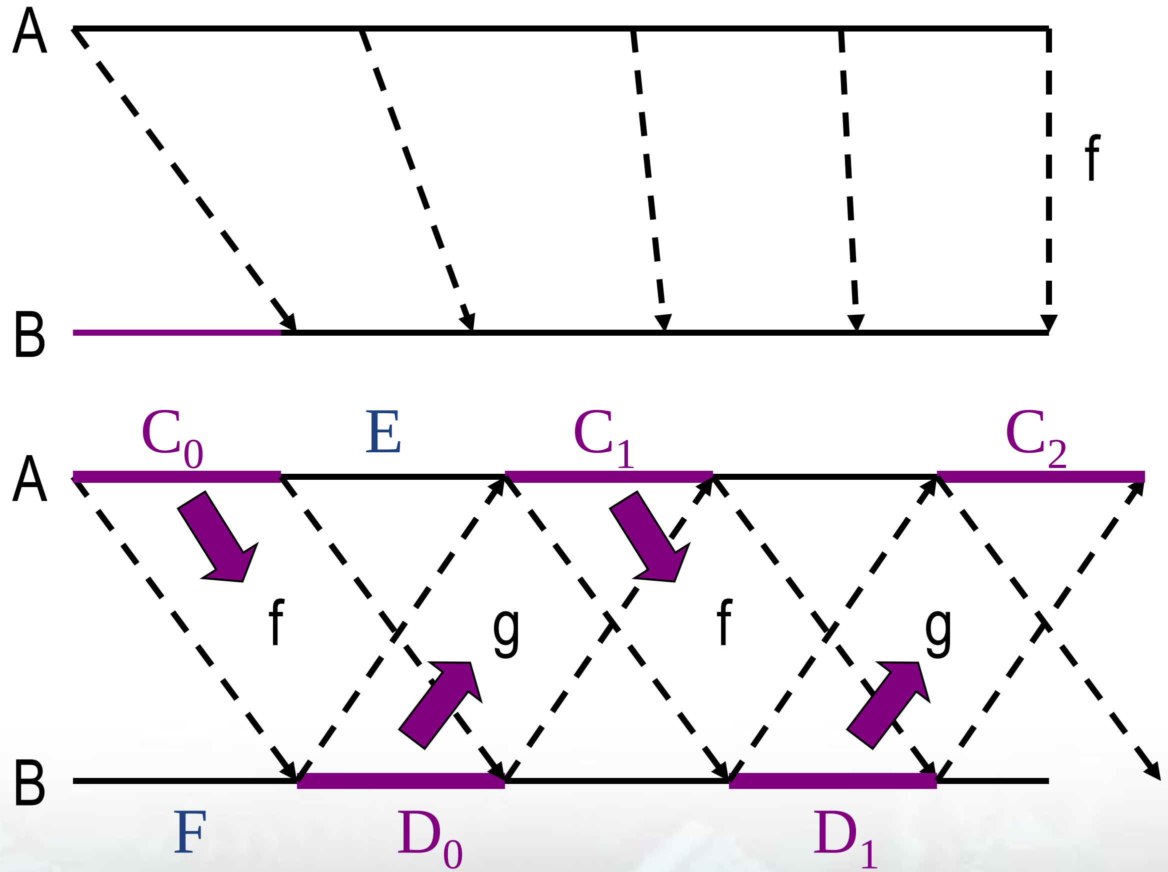
g^{-1}: g(B)→B 双射



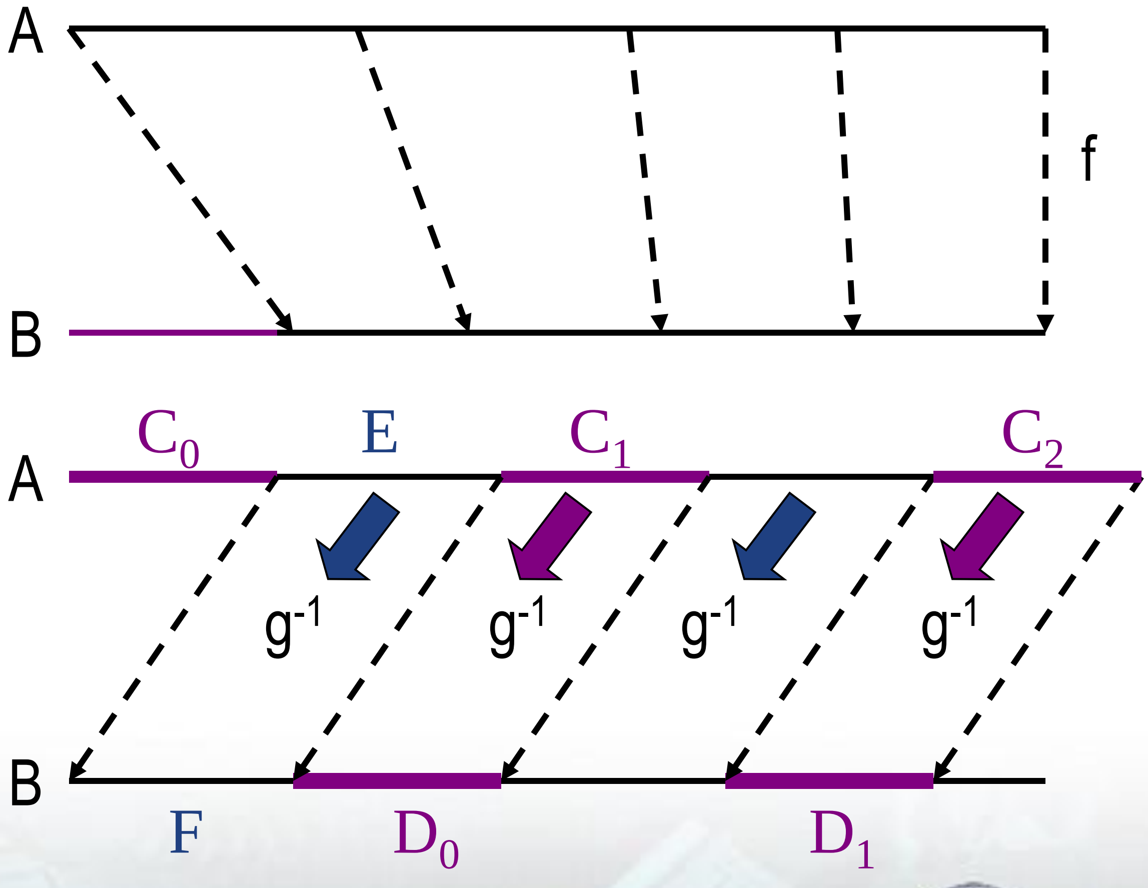
[0,1]≈(0,1): Hilbert旅馆



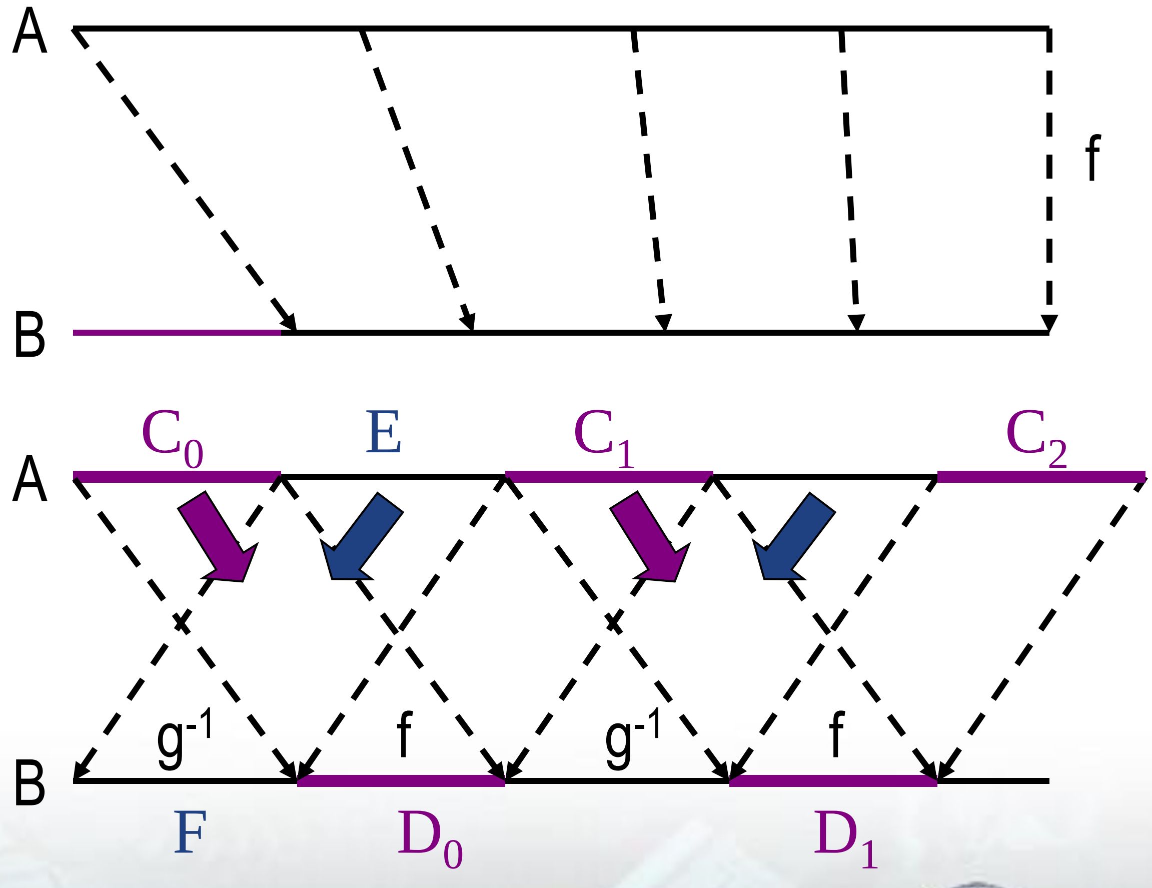
开始构造Hilbert旅馆



“搬家”之前



“搬家”之后



定理5.12证明(1)

- 令 C_0 = A-ran g $\neq \varnothing$, (前面假设)

  $D_n = f(C_n)$,

  $C_{n+1}=g(D_n)$, $n \in N$.

定理5.12证明(1)

- 记 $C = \cup\{C_n \mid n\in N\} \neq \varnothing$, $(C_0 \neq \varnothing)$

  $D = \cup\{D_n \mid n\in N\} = f(C) = g^{-1}(C-C_0)$,

  $E = A-C = (C_0\cup \mathrm{ran}\ g)-(C_0\cup(C-C_0))$

  $= \mathrm{ran}\ g-(C-C_0) \neq \varnothing$,

  $(C-C_0 \subseteq g(\mathrm{ran}\ f) \subset g(B) = \mathrm{ran}\ g)$

  $F = B-D = g^{-1}(g(B))-g^{-1}(C-C_0)$

  $= g^{-1}(\mathrm{ran}\ g - (C-C_0)) = g^{-1}(E)$

定理5.12证明(1)

- 取 $h=f\upharpoonright C \cup g^{-1}\upharpoonright E$,

  则 $h: A \rightarrow B$ 双射.

$$h(x) = \begin{cases} f(x), & x \in C_n \\ g^{-1}(x), & \text{否则} \end{cases}$$

定理5.12证明(1)

- $h = f{\upharpoonright}C \cup g^{-1}{\upharpoonright}E$是单射.

(a) f和g都是单射 $\Rightarrow$

$f{\upharpoonright}C$和$g^{-1}{\upharpoonright}E$都是单射.

(b) $\mathrm{ran}(f{\upharpoonright}C) \cap \mathrm{ran}(g^{-1}{\upharpoonright}E)$

$= f(C) \cap g^{-1}(E) = g^{-1}(C-C_0) \cap g^{-1}(E)$

$\subseteq g^{-1}(C) \cap g^{-1}(E) = g^{-1}(C \cap E) = \varnothing$.

定理5.12证明(1)

- h是满射.

$$
\begin{aligned}
B &= D \cup F \\
&= f(C) \cup g^{-1}(E) \\
&= \mathrm{ran}(f{\upharpoonright}C) \cup \mathrm{ran}(g^{-1}{\upharpoonright}E) \\
&= \mathrm{ran}\, h.
\end{aligned}
$$

#

例5.6

- $A \subseteq B \subseteq C \wedge A \approx C \Rightarrow A \approx B \approx C$

- 证: $A \subseteq B \subseteq C \wedge A \approx C$

  $\Rightarrow A \preccurlyeq\bullet B \wedge B \preccurlyeq\bullet A$

  $\Rightarrow A \approx B$

  $\Rightarrow A \approx B \approx C$

定理5.13

- 定理5.13 $R \approx (N\to 2) = 2^N$

- 证: (1) H: $(0,1)\to(N\to 2)$ 单射,

  $\forall z\in(0,1)$, $H(z):N\to\{0,1\}$,

  $H(z)(n)=z$的二进制表示的第n+1位小数.

  (2) G: $(N\to 2)\to[0,1]$ 单射, $\forall f:N\to 2$,

  $G(f)=0.f(0)f(1)f(2)\ \ldots$ (第n+1位小数是f(n)). #

可数集

- 可数集(可列集): card A $\leq \aleph_0$.
- 有穷可数集: n, $(\forall n \in N)$
- 无穷可数集: N

- 定理15: A是无穷可数集 $\Leftrightarrow$ A=$\{a_1,a_2,\ldots\}$
- 定理18: A是无穷集 $\Rightarrow$ P(A)不是可数集

基数运算的定义

- 设 κ,λ为基数, K,L为集合, card K = κ, card L = λ, 规定

(1) $\kappa+\lambda$ = card(K$\cup$L), 其中K$\cap$L=$\varnothing$

(2) $\kappa\times\lambda$ = $\kappa\bullet\lambda$ = $\kappa\lambda$ =card(K $\times$ L)

(3) κ^{λ} = card(L$\to$K)

定理5.19(定义的合理性)

- 设 K_1,K_2,L_1,L_2 为集合, $K_1\approx K_2$, $L_1\approx L_2$, 则

(1) 若$K_1\cap L_1=K_2\cap L_2=\varnothing$, 则

$$K_1\cup L_1\approx K_2\cup L_2$$

(2) $K_1\times L_1\approx K_2\times L_2$

(3) $K_1\to L_1\approx K_2\to L_2$. #

定理5.20及推论

- 定理5.20: (1) $2^{\mathrm{card}A} = \mathrm{card}\,P(A)$

  (2) $\kappa < 2^{\kappa}$.   #

- 推论: (1) $\mathrm{card}\,P(N) = 2^{\aleph_0}$

  (2) $\mathrm{card}\,P(R) = 2^{\aleph}$

  (3) $\aleph = 2^{\aleph_0}$.   #

基数运算性质

- 定理5.21:设 κ,λ,μ为基数

  (1) $\kappa+\lambda = \lambda+\kappa$

  (2) $(\kappa+\lambda)+\mu = \lambda+(\kappa+\mu)$

  (3) $\kappa\bullet(\lambda+\mu) = (\kappa\bullet\lambda)+(\kappa\bullet\mu)$

  (4) $\kappa^{\lambda+\mu} = \kappa^{\lambda}\bullet\kappa^{\mu}$

  (5) $(\kappa\bullet\lambda)^{\mu} = \kappa^{\mu}\bullet\lambda^{\mu}$

  (6) $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda\bullet\mu}$.   #

北京大学

基数运算性质

- 定理5.22:设 $\kappa\le\lambda$, μ为基数

  (1) $\kappa+\mu \le \lambda+\mu$

  (2) $\kappa\bullet\mu \le \lambda\bullet\mu$

  (3) $\kappa^{\mu} \le \lambda^{\mu}$

  (4) $\mu^{\kappa} \le \mu^{\lambda}$, 其中$\kappa$,$\mu$不同时为0. #

无穷基数运算性质

- 定理5.23:设 κ为无穷基数,则 $\kappa \bullet \kappa = \kappa$. #
- 定理5.24:设 κ为无穷基数, λ为基数,则

  $\kappa+\lambda = \kappa \bullet \lambda = \max\{\kappa,\lambda\}$, (其中$\lambda \neq 0$) #
- 推论: $\kappa+\kappa = \kappa \bullet \kappa = \kappa$. #
- 定理5.25:设 κ为无穷基数,则 $\kappa^\kappa = 2^\kappa$. #

基数小结

$$0,1,2,\cdots,\aleph_0,\aleph_1=2^{\aleph_0},\aleph_2=2^{2^{\aleph_0}},\aleph_3=2^{2^{2^{\aleph_0}}},\cdots$$

- **$\kappa < 2^{\kappa}$.**
- **$\kappa^{\kappa} = 2^{\kappa}$. ($\kappa$为无穷基数)**
- **$\kappa+\kappa = \kappa\bullet\kappa = \kappa$. ($\kappa$为无穷基数)**
- **连续统假设:**

$$\neg\exists\kappa(\aleph_0<\kappa<2^{\aleph_0})$$

小结

- 基数(势), $\aleph_0$, $\aleph$
- 等势, 优势, 劣势, 绝对优势, 绝对劣势
- Schröder-Bernstein定理
- 可数集(可列集)
- 基数运算